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Bases for upper cluster algebras and tropical points

In memory of Kentaro Nagao

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Abstract. It is known that many (upper) cluster algebras possess different kinds of good bases which contain the cluster monomials and are parametrized by the tropical points of cluster Poisson varieties. For a large class of upper cluster algebras (injective-reachable ones with full rank coefficients), we describe all of their bases with these properties. Moreover, we show the existence of the generic basis for them. In addition, we prove that Bridgeland's representation-theoretic formula is effective for their theta functions (weak genteelness).

Our results apply to (almost) all known cluster algebras arising from representation theory or higher Teichmüller theory, including quantum affine algebras, unipotent cells, double Bruhat cells, skein algebras over surfaces, where we change the coefficients if necessary so that the full rank assumption holds.

Keywords. Cluster algebras, generic basis, canonical basis, scattering diagrams

Contents

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1. Introduction

1.1. Background: good bases for cluster algebras

Cluster algebras are commutative algebras equipped with extra combinatorial data. Fomin and Zelevinsky [26] invented these algebras as a combinatorial approach to the dual canonical bases of the quantized enveloping algebras [45, 55, 56]. They conjectured that the cluster monomials (certain monomials of generators) of some cluster algebras are elements of the dual canonical bases of quantized enveloping algebras. Similarly, an analogous conjecture due to Hernandez and Leclerc [42] expected that the cluster monomials of some other cluster algebras correspond to simple modules of quantum affine algebras. Inspired by these conjectures, there have been many works devoted to relating cluster algebras, their bases and representation theory: [1, 5, 7, 17, 31–33, 42, 44, 47, 52, 64, 67, 68, 71].

On the other hand, to each cluster algebra $\mathcal{A}$, one can associate geometric objects $\mathbb{A}$ and $\mathbb{X}$ called the cluster $K2$ variety and cluster Poisson variety respectively [24]. The upper cluster algebra $\mathcal{U}$ is defined to be the ring of regular functions over $\mathbb{A}$. Furthermore, (a weaker form[1] of) a conjecture by Fock and Goncharov predicts that $\mathcal{U}$ possesses a basis parametrized by the tropical points of $\mathbb{X}$ associated to the Langlands dual cluster algebra [23]. Gross–Hacking–Keel–Kontsevich [41] recently verified it in many cases and found that the conjecture does not always hold.

It is well known that the cluster algebra $\mathcal{A}$ is contained in the upper cluster algebra $\mathcal{U}$ (Laurent phenomenon [26]), and they coincide in many cases, e.g. for many cluster algebras arising from representation theory. In view of the above conjectures, it is natural to look for good bases of (upper) cluster algebras, where the meaning of "good" depends on the context. Good bases in the literature can be divided into the following three families (see Section 2 for the necessary definitions):

[1] Fock and Goncharov expect an additional stronger property that the basis should have positive structure constants. For the moment, we do not know how to pick out such positive bases from the candidates provided in our paper.

(1) The generic basis in the sense of [19]: If the collection of "generic" cluster characters from a certain triangulated category is a basis, it is called the generic basis. The existence of such bases is mostly known for the cluster algebras arising from unipotent cells [32], in which case the basis coincides with the dual semicanonical basis of Lusztig [57]. Also, its existence is preserved by source/sink extension [20]. Conjecturally, this family includes the bangle basis [21, 62] of cluster algebras arising from surfaces, with the no punctured case treated in [29, 30].

(2) The common triangular basis in the sense of [71]: It is defined using some triangular properties by [71] for "injective-reachable" quantum cluster algebras. Its existence is known for the quantum cluster algebras arising from quantum affine algebras, where it coincides with the basis consisting of the simple modules [71]. Also, its existence is known for cluster algebras arising from unipotent cells, where it coincides with the dual canonical basis [44, 46, 71]. Conjecturally, this family includes the band basis [76] of cluster algebras arising from surfaces and the Berenstein–Zelevinsky acyclic triangular bases [70, 72].

(3) The theta basis in the sense of [41]: It consists of the "theta functions" appearing in the associated scattering diagram. It turns out to be a basis for injective-reachable upper cluster algebras [41]. This family includes the greedy bases of cluster algebras of rank 2 [14,54]. For cluster algebras arising from surfaces [62], the bracelet bases in the sense of [62] are conjectured to be the theta bases. This conjecture will be verified in an upcoming work by Travis Mandel and the author [58].

The bases listed above appear naturally in their own backgrounds.[2] They are always parametrized by the tropical points and contain all cluster monomials [41, 68, 71]. But such good bases are known to be different even in easy toy models [75]. This surprising phenomenon is the main motivation of this paper. As there exist different bases parametrized by the tropical points (satisfying the Fock–Goncharov conjecture), the following question arises naturally.

Question 1.1.1. *How many bases are parametrized by the tropical points? How similar and how different are they?*

We shall give an answer for injective-reachable upper cluster algebras under the full rank assumption (see Remark 1.2.6).

1.2. Main results and comments

Let I be a given set of vertices with a partition $I = I_{\mathrm{uf}} \sqcup I_{\mathrm{f}}$ into unfrozen vertices and frozen vertices. A seed t is a collection $((b_{ij})_{i,j\in I}, (x_i)_{i\in I})$, where (b_{ij}) is a skew-

[2] The common triangular basis is related to the (dual) canonical basis in representation theory, which is often thought to be the best basis for quantized enveloping algebras. The theta basis was also said to be "canonical" in the original paper [41] and is very natural from a geometric point of view.

symmetrizable matrix and the x_i are the cluster variables in t (distinguished generators of $\mathcal{A}$). Throughout the paper, we often make the following assumption; see Remark 1.2.6.

Assumption (Full rank assumption). $\widetilde{B}(t) := (b_{ik})_{i\in I, k\in I_{\mathrm{uf}}}$ is of full rank.

We will work with a base ring $\Bbbk$, which will be $\Bbbk = \mathbb{Z}$ for classical (upper) cluster algebras and $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$ for the quantum case, where q is a formal quantum parameter.

We have the lattice $M^\circ(t) \simeq \mathbb{Z}^I$ of Laurent multidegrees with the natural basis f_i, the Laurent polynomial ring $\mathcal{LP}(t) = \Bbbk[x_i{}^{\pm}] = \Bbbk[M^\circ(t)]$, where $x^{f_i} := x_i$, and the (skew-)field of fractions $\mathcal{F}(t)$ (see Section 2.5 for the quantum case). In [71], the author introduced the dominance order $\preceq_t$ on $M^\circ(t)$ such that $g' \preceq_t g$ if and only if $g' = g + \widetilde{B}(t)\cdot n$ for some $n \in \mathbb{N}^{I_{\mathrm{uf}}}$.

On the one hand, for any unfrozen vertex $k \in I_{\mathrm{uf}}$, there is an algorithm μ_k called mutation which generates a new seed $t' = \mu_k(t)$ from t. We use Δ^+ to denote the set of seeds obtained by repeatedly applying mutations. In addition, there is a corresponding isomorphism $\mu_k^* : \mathcal{F}(t') \simeq \mathcal{F}(t)$ of (skew-)fields. We naturally extend these notions to seeds $t' = \overleftarrow{\mu} t$ related by a sequence $\overleftarrow{\mu}$ of mutations. Recall that the upper cluster algebra $\mathcal{U}$ equals $\bigcap_{t\in\Delta^+} \mathcal{LP}(t)$ where the fraction fields are identified.

On the other hand, on the tropical part, one has a tropical transformation (piecewise linear map) $\phi_{t',t} : M^\circ(t) \simeq M^\circ(t')$. By identifying Laurent degrees $g \in M^\circ(t)$ for all seeds $t \in \Delta^+$ via the tropical transformations, we define the set $\mathcal{M}^\circ$ of *tropical points*[3] to be the set of equivalence classes $[g]$. The set $\mathcal{M}^\circ$ is equipped with many dominance orders $\preceq_t$ by comparing the representatives in each seed t. Given any set S of seeds and any tropical point $[g] \in \mathcal{M}^\circ$, dominance orders cut out a subset of tropical points, $\mathcal{M}^\circ{}_{\preceq_S [g]} = \{[g'] \mid [g'] \preceq_t [g]\ \forall t \in S\}$

We say a Laurent polynomial $z \in \mathcal{LP}(t)$ is *pointed at degree* $\mathrm{deg}^t z = g \in M^\circ(t)$ (resp. *copointed at codegree* $\mathrm{codeg}^t z = g \in M^\circ(t)$) if z has a unique $\preceq_t$-maximal (resp. $\preceq_t$-minimal) Laurent monomial with degree g and coefficient 1. We say $z \in \mathcal{U}$ is *pointed at the tropical point* $[g]$ if it is pointed at the representatives of $[g]$ at all seeds $t \in \Delta^+$.

In this work, we restrict our attention to *injective-reachable seeds* t, which means that there is a seed $t[-1]$ such that, for some permutation σ of I_{uf}, the cluster variables $x_i(t)$ have degree $\mathrm{deg}^{t[-1]}(x_i(t)) = -f_{\sigma(i)}$ modulo the frozen part $\mathbb{Z}^{I_{\mathrm{f}}}$.

All bases. Our first main result is a description of all bases parametrized by the tropical points.

Theorem 1.2.1. *Consider the classical case $\Bbbk = \mathbb{Z}$. Let $\mathcal{U}$ be an upper cluster algebra with injective-reachable seeds $t = \overleftarrow{\mu} t[-1]$ subject to the full rank assumption.*

(1) *For any collection $\mathcal{S} = \{s_{[g]} \in \mathcal{U} \mid [g] \in \mathcal{M}^\circ\}$ such that each $s_{[g]}$ is pointed at the tropical point $[g]$, $\mathcal{S}$ must be a $\Bbbk$-basis of $\mathcal{U}$ containing all cluster monomials.*

[3] We remark that $\mathcal{M}^\circ$ should not be confused with the fixed abstract lattice M° used in [40]. The set $\mathcal{M}^\circ$ in our paper is viewed as the set of equivalence classes of Laurent degree lattices. In particular, it does not have an additive structure.

(2) *There exists at least one such basis, which we choose and denote by* $\mathcal{Z} = \{z_{[g]}\}$.

(3) *The set of all such bases* $\mathcal{S}$ *is parametrized as follows:*

$$\prod_{[g]\in\mathcal{M}^\circ} \Bbbk^{\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}} \simeq \{\mathcal{S}\},$$

$$((b_{[g],[g']})_{[g']\in\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}})_{[g]\in\mathcal{M}^\circ} \mapsto \mathcal{S} = \{s_{[g]} \mid [g] \in \mathcal{M}^\circ\},$$

where $s_{[g]} = z_{[g]} + \sum_{[g']\in\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}} b_{[g],[g']}z_{[g']}$. *In addition, each set* $\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}$ *is finite.*

By this result, the three families of good bases in previous literature correspond to three points in this (infinite) "moduli space" of bases. The quantum analog of Theorem 1.2.1 is discussed in Section 6.2. See also Remark 5.1.4 for bases that factor through frozen variables.

Remark 1.2.2 (Deformation factors). The main theorem shows that the set of bases $\{\mathcal{S}\}$ has a linear structure similar to that of the solution space of a non-homogeneous linear system, and a general basis could be obtained from a special one by a linear deformation controlled by the factors $\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}$, which we call the deformation factors.

These deformation factors are new mathematical objects, and further questions arise naturally; see Section 6.1. In particular, Conjecture 6.1.3 there would imply the open orbit conjecture for unipotent subgroups (see [31]); see Remark 6.1.4.

In practice, instead of using the set $\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}$, it would be easier to work with the larger finite sets $\mathcal{M}^\circ{}_{\prec_{\{t,t[-1]\}}[g]}$. These larger sets can be easily controlled by computing the difference between the degrees and codegrees (called support dimensions, or f-vectors following [28]) (Proposition 3.4.8). Correspondingly, in Theorem 5.1.2, we describe the bases subject to the weaker condition: we require the basis elements to be compatibly pointed at the seeds $t, t[-1]$ rather than compatibly pointed at all seeds (see Definition 3.4.2).

Next, we discuss how to choose one such basis for Theorem 1.2.1.

Generic bases. Assume that the seeds are skew-symmetric, i.e. their matrices are skew-symmetric. It is naturally expected that the generic cluster characters give rise to bases of many (upper) cluster algebras, called the generic bases. However, the existence of such bases has been verified in limited cases, such as in [32].

Our second main result gives the existence of the generic basis at a high level of generality, which provides a good choice for the special basis $\mathcal{Z}$ in Theorem 1.2.1.

Theorem 1.2.3 (Generic basis). *Consider the classical case* $\Bbbk = \mathbb{Z}$. *Let* t *be a skew-symmetric injective-reachable seed subject to the full rank assumption. Then the set of localized generic cluster characters is a basis of* $\mathcal{U}$, *called the generic basis.*

Theorem 1.2.3 is a consequence of Theorem 4.3.1. The latter result is a general criterion of independent interest, which states that if a collection of elements have well-behaved degrees under mutations, then they form a basis.

We refer the reader to Sections 5 and 6 for more precise statements, generalizations and more details. Our results apply to (almost) all well-known cluster algebras arising from representation theory or higher Teichmüller theory; see Remark 1.2.7. Note that a change of coefficients will be needed for punctured surfaces; see Remark 1.2.6.

In particular, we obtain the existence of the generic basis with high generality, covering all previously known cases such as [32]. This result will be used in an upcoming work [30] in studying generic bases of cluster algebras arising from surfaces.

Theta bases. For general seeds, a good choice for the special basis $\mathcal{Z}$ in Theorem 1.2.1 would be the theta basis [41] (see Section A.1).

Now, assume the seeds are skew-symmetric again. Our last result states that Bridgeland's representation-theoretic formula for many theta functions is effective (called weak genteelness, see Section 6.3), which can be viewed as a pleasant property predicted by Nagao's work [63].

Theorem 1.2.4 (Weak genteelness). *Take $\Bbbk = \mathbb{Z}$. Let t be a given skew-symmetric injective-reachable seed. Then Bridgeland's representation theoretic formula is effective for theta functions in the cluster scattering diagram. Moreover, the stability scattering diagram and the cluster scattering diagram are equivalent.*

Remark 1.2.5. [15] appeared soon after this work. Its results allow us to further understand and strengthen the present work.

First, an explicit topology was constructed for the Laurent polynomial ring $\mathcal{LP}(t)$ in [15, Section 2.2.2], which generalized the natural adic topology that we will use for seeds of principal coefficients in Section 4.2. We omit the details but point out that, in view of this topology, in Definition-Lemma 4.1.1, the dominance order decomposition is convergent and the pointed set $\mathcal{S}$ is a topological basis.

Secondly and most importantly, for any skew-symmetric seed under the full rank assumption, [15] constructed the quantum theta functions with strong properties. In particular, when the seed is injective-reachable, such functions form the quantum theta basis for the quantum upper cluster algebra. The existence of such a basis is crucial for describing more quantum bases; see Section 6.2.

Remark 1.2.6 (Full rank assumption). It is worth noting that, if an initial seed t_0 satisfies the full rank assumption, so do all the seeds obtained from t_0 by iterated mutations; see [61, Theorem 3.1.2], [71, Proposition 5.1.4].

But the full rank assumption does not hold true for an arbitrary seed $t = ((b_{ij})_{i,j\in I}, (x_i))$. Nevertheless, for studying many questions in cluster theory, one has the freedom to change the coefficients so that the assumption becomes true (i.e. one changes the set I_{f} of frozen vertices and the matrix (b_{ij}) but keep the principal part $(b_{ij})_{i,j\in I_{\mathrm{uf}}}$ unchanged).

A change of coefficients is justified by keeping important structures in cluster theory. For example, the exchange graphs remain the same [8, Proposition 3]. Moreover, if one knows the cluster expansion of cluster variables for some coefficients under the full rank assumption, then one can deduce the cluster expansion for all coefficients [27, Section 3].

Similarly, if a (quantum) cluster algebra subject to the full rank assumption possesses a good basis (as in Remark 5.1.4), one can construct a spanning set for the corresponding algebra with arbitrary coefficients, using the correction technique for pointed elements ([70, Section 9], [71, Section 4]). Moreover, under the full rank assumption, or the weaker assumption that $\widetilde{B}(t)\mathbb{R}_{\geq 0}^{I_{\mathrm{uf}}}$ is strictly convex (as used in [30]), the spanning set is again a basis.

It is natural to ask whether the spanning set constructed above is always a basis for all choices of coefficients. But, at this moment, very little is known about bases of (upper) cluster algebras without the full rank assumption or the convexity assumption above. Progress in this direction was made in [12], where it was shown that the set of cluster monomials (usually a proper subset of the basis) is linearly independent.

Finally, a seed can be quantized if and only if the full rank assumption holds. Except for punctured surfaces, the well-known cluster algebras listed in Remark 1.2.7 admit natural quantization and satisfy the full rank assumption. When the assumption fails, we have to choose appropriate coefficients so that the assumption becomes true, a quantization can be performed, and our results about bases become effective.

Remark 1.2.7 (Injective-reachable assumption). To derive the main results of this paper, the injective-reachable assumption is imposed.

This assumption implies that the associated Jacobian algebra is finite-dimensional. The converse is not necessarily true. A counterexample arising from a once-punctured torus was studied in [68, Example 4.3].

The injective-reachable assumption is satisfied by the following well-known cluster algebras:

- coordinate rings of unipotent cells [31, 33]: see [31, Section 13];
- level-l categories of representations of quantum affine algebras [42]: see [71, (52)];
- symmetric CGL extensions (including double Bruhat cells) [2, 39]: see [38, Main theorem III];
- equivariant perverse coherent sheaves over affine Grassmannians: see [10, Theorem 3.1, Proposition 6.2];
- cluster algebras over marked surfaces (except once-punctured closed surfaces) [22, 25]: see [25, Proposition 7.10];
- PGL_m (or SL_m) local systems on marked surfaces (except once-punctured closed surfaces) [36, 37]: see [37, Theorem 1.2].

Key points in the proofs. As an important part of the paper, we give a systematic analysis of the tropical properties of upper cluster algebra elements, by which we mean how their degrees and codegrees change under mutations. More precisely, we introduce the notions of codegrees and support dimensions (Definitions 3.2.2, 3.4.1, 3.4.4). We also introduce a linear map $\psi_{t[-1],t} : M^{\circ}(t) \to M^{\circ}(t[-1])$, which reverses the dominance orders and swaps degrees and codegrees at different seeds t, $t[-1]$ (Definition 3.3.1, Propositions 3.3.11, 3.3.12). Then we derive the equivalence between being compatibly pointed at $t, t[-1]$ (i.e., degrees are controlled by tropical transformations) and being bipointed at t with the "correct" support dimension (Proposition 3.4.8). We arrive at the following interesting observation.

Lemma 1.2.8 (Lemma 3.4.12). *If an upper cluster algebra element Z and a cluster monomial M share the same tropical property, then they are the same.*

The parametrization of the set of bases (Theorem 1.2.1 (2)) is an application of the above analysis.

As another important part of the paper, we prove a criterion for a given collection of elements of an upper cluster algebra to be a basis (Theorem 4.3.1), which says that good tropical properties suffice. This criterion immediately implies Theorem 1.2.1 (1) as well as Theorem 1.2.3, since the generic cluster characters are known to have good tropical properties [68, Theorem 1.3].

The criterion is proved by introducing and analyzing the dominance order decomposition into the given collection (Definition-Lemma 4.1.1). A priori, the (possibly infinite) decomposition depends on the chosen seed. We first show that the decomposition is independent of the seed (Proposition 4.2.1); the proof is based on natural adic topologies induced by principal coefficients in the sense of [27], and an application of the nilpotent Nakayama Lemma (we learned the usefulness of that lemma from the inspirational work [41]). Then we show that the decomposition is finite by using the injective-reachability condition and conclude that the given collection is a basis.

Finally, we give a quick proof of Theorem 1.2.4 based on cluster theory and the trick of constructing opposite scattering diagrams.

Remark 1.2.9. The analysis of tropical properties in this paper has turned out to be useful in [69, 73]. In particular, the dominance order decomposition is used in [69], and the codegrees are used in [73].

1.3. Contents

Section 2 contains the necessary preliminaries. A reader could skip the details and the content familiar to him/her. But it is still recommended to read Section 2.1 which merges symbols and notions of cluster algebras of two different styles [26, 41]. In addition, we verify the equivalence between injective-reachability and the existence of green to red sequences.

In Section 3, we define and study degrees, codegrees and support. These are the main tools that will be used in this paper, which we develop by elementary manipulations on Laurent polynomials/series.

In Section 4, we study the properties of $\prec_t$-decompositions based on Section 3 and the nilpotent Nakayama Lemma. This section provides direct proofs for Theorems 1.2.1 (1) and 1.2.3.

In Section 5, we present the main results, consequences and proofs based on Sections 3 and 4.

In Section 6, we discuss related topics such as deformation factors, quantized versions of our results, a representation-theoretic formula for theta functions (weak genteelness), and bases for partial compactification cases.

In Appendix A, we briefly review some content of [41] about scattering diagrams and theta functions. Then we present two proofs of weak genteelness (Theorem 1.2.4). One is conceptual following Nagao [63], and the other uses the construction of an opposite scattering diagram. This section is independent of most of the paper, but provides definitions and properties of theta functions.

2. Preliminaries

2.1. Basics of cluster mutations and tropicalization

Throughout this paper, we shall consider cluster algebras with geometric coefficients in the sense of [27]. We define the notion of cluster algebra as in [27], but we follow the nice presentation of [34]. Furthermore, our convention is compatible with the different formalism of [34, 40], so that we can easily use results and arguments from those works.

We will work with a base ring $\Bbbk$. We usually take $\Bbbk = \mathbb{Z}$ for classical (upper) cluster algebras and $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$ for quantum (upper) cluster algebras, where $q^{1/2}$ is a formal quantum parameter. Unless otherwise specified, our arguments will be equally effective for both the classical and quantum case.

Seeds and B-matrices. Given a set $I = I_{\mathrm{uf}} \sqcup I_{\mathrm{f}}$ of vertices, the vertices in I_{uf} and I_{f} are called *unfrozen* and *frozen* respectively. Suppose that there is a collection of integers $d_i > 0$ and a matrix $(b_{ij})_{i,j\in I}$ such that

$$b_{ij} \in \begin{cases} \mathbb{Q}, & i, j \in I_{\mathrm{f}}, \\ \mathbb{Z}, & \text{else}, \end{cases} \qquad b_{ij} d_j = -b_{ji} d_i.$$

Definition 2.1.1. A *seed* t is a collection $((b_{ij}(t))_{i,j\in I}, (x_i(t))_{i\in I}, d_i, I, I_{\mathrm{uf}})$ with each $x_i(t)$ an indeterminate. The matrix $\widetilde{B}(t) := (b_{ik}(t))_{i\in I, k\in I_{\mathrm{uf}}}$ is called the *B-matrix* associated to t and the $x_i(t)$ are the *cluster variables*.

For any $m = (m_i) \in \mathbb{N}^{I_{\mathrm{uf}}} \oplus \mathbb{Z}^{I_{\mathrm{f}}}$, we call $x(t)^m := \prod_{i\in I} x_i(t)^{m_i}$ a (*localized*) *cluster monomial* in the seed t.

We usually fix d_i and $I_{\mathrm{uf}} \subset I$, and write $t = ((b_{ij}(t)), (x_i(t)))$ for simplicity. The symbol t will be omitted when the context is clear.

Let d denote the least common multiple of $(d_i)_{i\in I}$ and define the *Langlands dual* $d_i^\vee := d/d_i$. Then $d_i^\vee b_{ij} = -d_j^\vee b_{ji}$, and we say (b_{ij}) is *skew-symmetrizable* by the diagonal matrix $\mathrm{diag}(d_i^\vee)$. It follows that the *principal part* $B := (b_{ij})_{i,j\in I_{\mathrm{uf}}}$ of (b_{ij}) is skew-symmetrizable as well.

Conversely, suppose that we are given an $I \times I_{\mathrm{uf}}$ integer matrix $\widetilde{B} = (b_{ij})_{i \in I, j \in I_{\mathrm{uf}}}$ with principal part B, such that B is skew-symmetrizable by some diagonal matrix $D = \mathrm{diag}(d'_k)_{k \in I_{\mathrm{uf}}}$, $d'_k \in \mathbb{Z}_{>0}$. We can make the following extension.

Lemma 2.1.2. *We can find strictly positive integers d'_f, $f \in I_{\mathrm{f}}$, and extend the matrix $\widetilde{B}(t)$ to an $I \times I$ integer matrix $(b_{ij}(t))$ such that $d'_i b_{ij}(t) = -d'_j b_{ji}(t)$.*

Proof. Let d' denote the least common multiple of $(d'_k)_{k \in I_{\mathrm{uf}}}$. We might choose $d'_f = d'$, $b_{kf}(t) = -\frac{d'}{d'_k} b_{fk}(t)$, $b_{ff'} = 0$, for all $f, f' \in I_{\mathrm{f}}$ and $k \in I_{\mathrm{uf}}$. ∎

Recall that a seed according to Fomin–Zelevinsky [26] takes the form $(\widetilde{B}, (x_i))$ with a skew-symmetrizable principal part B. By Lemma 2.1.2, their seed could be extended to our seed by choosing a matrix extension. The extra data in our definition arise from the construction in [23, 40, 41].

We say the seed t is *skew-symmetrizable* (resp. *skew-symmetric*) if the matrix $(b_{ij}(t))$ is.

Lattices and ϵ-matrices. Following [40, 41], let $M^\circ(t)$ denote a lattice with a $\mathbb{Z}$-basis $\{f_i(t) \mid i \in I\}$ and $N(t)$ a lattice with a $\mathbb{Z}$-basis $\{e_i(t) \mid i \in I_{\mathrm{uf}}\}$. Define a pairing $\langle\ ,\ \rangle$ between $M^\circ(t)$ and $N(t)$ by setting $\langle f_i(t), e_j(t) \rangle = \frac{1}{d_i} \delta_{ij}$. Let $N_{\mathrm{uf}}(t)$ denote the sublattice of $N(t)$ generated by $\{e_k(t) \mid k \in I_{\mathrm{uf}}\}$.

Consider the $\mathbb{Q}$-valued bilinear form $\{\ ,\ \}$ on $N(t)$ defined by $b_{ij} = \{e_j(t), e_i(t)\} d_i$. It turns out that $\{\ ,\ \}$ is skew-symmetric.

Definition 2.1.3. The ϵ-*matrix* is defined to be

$$(\epsilon_{ij})_{i,j \in I} = (\{e_i(t), e_j(t)\} d_j)_{i,j \in I}.$$

Let p^* denote the linear map from $N(t)$ to $M^\circ(t) \otimes \mathbb{Q}$ such that

$$p^*(n) = \{n,\ \}.$$

Denote $v_k(t) = p^*(e_k(t)) = \{e_k(t),\ \}$ for $k \in I_{\mathrm{uf}}$. Then $v_k(t) = \sum_{i \in I} b_{ik} f_i(t) \in M^\circ(t)$.

We always assume that $p^*|_{N_{\mathrm{uf}}(t)}$ is injective, or equivalently that $\widetilde{B}(t)$ satisfies the full rank assumption.

Let us consider the group ring (of *characters*) $\mathcal{LP}(t) = \Bbbk[M^\circ(t)] = \Bbbk[\chi^m]_{m \in M^\circ(t)}$ and the group ring (of *cocharacters*) $\Bbbk[N(t)] = \Bbbk[\lambda^n]_{n \in N(t)}$. We consider the x-variables $x_i(t) = \chi^{f_i(t)}$, the Laurent monomials $x(t)^m = \chi^m$, and the y-variables $y_i(t) = \lambda^{e_i(t)}$. Similarly, we can define $\overline{\mathcal{LP}}(t) = \Bbbk[x_f(t)]_{f \in I_{\mathrm{f}}}[x_i(t)^{\pm}]_{i \in I_{\mathrm{uf}}}$ and call it the (*partially*) *compactified Laurent polynomial ring*.

The commutative product in $\mathcal{LP}(t)$ will be denoted by $\cdot$ or omitted for simplicity. For the quantum case ($\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$), we also define the twisted product $*$ in Section 2.5.

Note that, for $\Bbbk = \mathbb{Z}$, $\mathcal{LP}(t) \otimes \mathbb{C}$ is the ring of regular functions on the split algebraic torus $(\mathbb{C}^*)^I$. And $\overline{\mathcal{LP}}(t) \otimes \mathbb{C}$ is the ring of regular functions on the partial compactification $(\mathbb{C}^*)^{I_{\mathrm{uf}}} \times (\mathbb{C})^{I_{\mathrm{f}}}$ of $(\mathbb{C}^*)^I$.

Mutations. Let $[\ ]_+$ denote $\max(\ ,0)$ and define $[(g_i)_{i\in I}]_+ = ([g_i]_+)_{i\in I}$ for any vector $(g_i)_{i\in I}$. For any $k \in I_{\mathrm{uf}}$, we can define a seed $t' = \mu_k t$ by the following procedure.

We start by choosing a sign $\varepsilon \in \{+,-\}$, and define the $I \times I$ matrices $\widetilde{E}_\varepsilon$ and $\widetilde{F}_\varepsilon$ by

$$(\widetilde{E}_\varepsilon)_{ij} = \begin{cases} \delta_{ij}, & k \notin \{i,j\}, \\ -1, & i=j=k, \\ [-\varepsilon, b_{ik}]_+ & i \neq k,\ j=k, \end{cases} \qquad (\widetilde{F}_\varepsilon)_{ij} = \begin{cases} \delta_{ij}, & k \notin \{i,j\}, \\ -1, & i=j=k, \\ [\varepsilon b_{kj}]_+ & i = k,\ j \neq k. \end{cases}$$

Notice that $\widetilde{F}_\varepsilon^2 = \mathrm{Id}_{I_{\mathrm{uf}}}$ and $\widetilde{E}_\varepsilon^2 = \mathrm{Id}_I$. The $I_{\mathrm{uf}} \times I_{\mathrm{uf}}$ submatrix of $\widetilde{E}_\varepsilon$ (principal part) is denoted by E_ε and the $I_{\mathrm{uf}} \times I_{\mathrm{uf}}$ submatrix of $\widetilde{F}_\varepsilon$ is denoted by F_ε.

Next, define a lattice $M^\circ(t')$ with a basis $\{f_i' = f_i(t')\}_{i\in I}$ and a lattice $N(t')$ with a basis $\{e_i' = e_i(t')\}_{i\in I}$, where we omit the symbol t from now on. We define linear isomorphisms $\tau_{k,\varepsilon} : M^\circ(t') \to M^\circ(t)$ and $\tau_{k,\varepsilon} : N(t') \to N(t)$ such that $\tau_{k,\varepsilon}(e_i') = \sum_{j\in I} e_j \cdot (\widetilde{F}_\varepsilon)_{ji}$ and $\tau_{k,\varepsilon}(f_i') = \sum_{j\in I} f_j \cdot (\widetilde{E}_\varepsilon)_{ji}$, namely,

$$\tau_{k,\varepsilon}(e_i') = \begin{cases} e_i + [\varepsilon b_{ki}]_+ e_k, & i \neq k, \\ -e_k, & i = k, \end{cases} \tag{2.1}$$

$$\tau_{k,\varepsilon}(f_i') = \begin{cases} f_i, & i \neq k, \\ -f_k + \sum_j [-\varepsilon b_{jk}]_+ f_j, & i = k. \end{cases} \tag{2.2}$$

Clearly, $\tau_{k,\epsilon}$ preserves the pairing $\langle\ ,\ \rangle$. Further define a bilinear form $\{\ ,\ \}$ on $N(t')$ to be induced by that on $N(t)$ via $\tau_{k,\varepsilon}$. It is straightforward to check that the corresponding matrix $(b'_{ij})_{i,j\in I} = (\{e_j', e_i'\} d_i)_{i,j\in I}$ satisfies

$$b'_{ij} = \begin{cases} -b_{ij}, & k \in \{i,j\}, \\ b_{ij} + b_{ik}[\varepsilon b_{kj}]_+ + [-\varepsilon b_{ik}]_+ b_{kj}, & k \neq i, j. \end{cases}$$

Notice that the b'_{ij} are independent of the choice of the sign ε.

We define the mutated seed $t' = \mu_k t$ as $((b'_{ij})_{i,j\in I}, (x_i')_{i\in I})$. Let us now relate the cluster variables x_i and x_i'.

First, the maps $\tau_{k,\varepsilon}$ induce isomorphisms between Laurent polynomial rings, which are still denoted by $\tau_{k,\varepsilon}$, such that

$$\tau_{k,\varepsilon}(x_i') = \begin{cases} x_i, & i \neq k, \\ x_k^{-1} \prod_j x_j^{[-\varepsilon b_{jk}]_+}, & i = k, \end{cases} \qquad \tau_{k,\varepsilon}(y_i') = \begin{cases} y_i y_k^{[\varepsilon b_{ki}]_+}, & i \neq k, \\ y_k^{-1}, & i = k. \end{cases}$$

Now consider the classical case $\Bbbk = \mathbb{Z}$ for simplicity (see Section 2.5 for the quantum case). Define the automorphisms $\rho_{k,\varepsilon}$ on the fraction fields $\mathcal{F}(t) = \mathcal{F}(\mathcal{LP}(t))$ and $\mathcal{F}(\Bbbk[N(t)])$ respectively such that

$$\rho_{k,\varepsilon}(x_i) = \begin{cases} x_i, & i \neq k, \\ x_k(1 + x^{\varepsilon v_k})^{-1}, & i = k, \end{cases} \qquad \rho_{k,\varepsilon}(y_i) = \begin{cases} y_i(1 + y_k^{\varepsilon})^{-b_{ki}}, & i \neq k, \\ y_k, & i = k. \end{cases} \tag{2.3}$$

Then it turns out that

$$\rho_{k,\varepsilon}\circ\tau_{k,\varepsilon}(x_i') = \begin{cases} x_i, & i\neq k,\\ x_k^{-1}\prod_j x_j^{[-\varepsilon b_{jk}]_+}(1+\chi^{\varepsilon v_k}), & i=k,\end{cases}$$
$$\rho_{k,\varepsilon}\circ\tau_{k,\varepsilon}(y_i') = \begin{cases} y_i y_k^{[\varepsilon b_{ki}]_+}(1+y_k^{\varepsilon})^{-b_{ki}}, & i\neq k,\\ y_k^{-1}, & i=k.\end{cases} \tag{2.4}$$

We observe that the compositions $\rho_{k,\varepsilon}\circ\tau_{k,\varepsilon}$ are independent of the choice of ε. Let us call them the *mutation birational maps*, and denote by μ_k^*. The maps $\tau_{k,\varepsilon}$ are called their *monomial parts* and $\rho_{k,\varepsilon}$ their *Hamiltonian parts*. One can show that the μ_k^* give isomorphisms between the fraction fields, $\mathcal{F}(t')\simeq\mathcal{F}(t)$ and $\mathcal{F}(\Bbbk[N(t')])\simeq\mathcal{F}(\Bbbk[N(t)])$.

Given any two seeds t,t' such that $t'=\overleftarrow{\mu}t$ for some mutation sequence $\overleftarrow{\mu}$, let $\overleftarrow{\mu}^*$ denote the mutation map from the fraction field $\mathcal{F}(t')$ to $\mathcal{F}(t)$ defined by composing the corresponding mutation maps. Then we can denote $\mathcal{LP}(t)\cap\mathcal{LP}(t')=\mathcal{LP}(t)\cap(\overleftarrow{\mu}^*\mathcal{LP}(t'))$ and also $\mathcal{LP}(t)\cap\mathcal{LP}(t')=(\overleftarrow{\mu}^{-1})^*\mathcal{LP}(t)\cap\mathcal{LP}(t')$. Correspondingly, for any $z\in(\overleftarrow{\mu}^{-1})^*\mathcal{LP}(t)\cap\mathcal{LP}(t')$, the Laurent polynomial $\overleftarrow{\mu}^*z\in\mathcal{LP}(t)$ is sometimes also denoted by z for simplicity.

y-variables. Because p^* is linear and $\tau_{k,\epsilon}$ preserves $\{\ ,\ \}$ and $\langle\ ,\ \rangle$, we have

$$\tau_{k,\varepsilon}(v_i') = \begin{cases} v_i+[\varepsilon b_{ki}]_+v_k, & i\neq k,\\ -v_k, & i=k.\end{cases}$$

One can check that

$$\mu_{k,\varepsilon}^*(\chi^{v_i}) = \begin{cases} \chi^{v_i}\chi^{[\varepsilon b_{ki}]_+v_k}(1+\chi^{\varepsilon v_k})^{-b_{ki}}, & i\neq k,\\ \chi^{-v_k}, & i=k,\end{cases}$$

i.e. subject to the law given by (2.4). By abuse of notation, we define the Laurent monomial $y_k=\chi^{v_k}$, which equals $\prod_i x_i^{b_{ik}}$ in $\mathcal{LP}(t)$ under the commutative product. The y_k are still called the y-variables.

Tropicalization. We refer the reader to [23,40,41] for more details. Recall that $\langle f_i,e_j\rangle=\delta_{ij}/d_i$, $b_{ij}=\{e_j,e_i\}d_i$ and $b_{ji}\cdot d_j^{-1}=-b_{ij}\cdot d_i^{-1}$, $i,j\in I$. Let $M(t)$ denote the sublattice of $M^\circ(t)$ with basis $\{e_i^*=d_if_i\}$. Let $N^\circ(t)$ denote the sublattice of $N(t)$ with basis $\{d_ie_i\}$. Then $M(t)$ is dual to $N(t)$ and $N^\circ(t)$ is dual to $M^\circ(t)$ under the pairing $\langle\ ,\ \rangle$.

For any lattice L and its dual L^*, we consider the split algebraic torus $T_L=\operatorname{Spec}\mathbb{Z}[L^*]=\operatorname{Spec}\mathbb{Z}[\lambda^n]_{n\in L^*}$. Let $(P,\oplus,\otimes)$ be a given semifield and $P^\times$ the multiplicative group. Let $Q_{\mathrm{sf}}(L)$ denote the semifield of subtraction-free rational functions on T_L. A *tropical point* in T_L is defined to be a semifield homomorphism from $Q_{\mathrm{sf}}(L)$ to P. The set of tropical points in T_L is denoted by $T_L(P)$. One can show that $T_L(P)\simeq\operatorname{Hom}_{\mathrm{groups}}(L^*,P^\times)\simeq L\otimes_{\mathbb{Z}}P^\times$ so that any point $m\otimes_{\mathbb{Z}}p$ sends a subtraction-free Laurent polynomial $f=\sum_n\lambda^n\in Q_{\mathrm{sf}}(L)$ to $\oplus_n p^{\otimes\langle m,n\rangle}\in P$ (see [41]).

We usually work with $P = \mathbb{Z}^T = (\mathbb{Z}, \max(\ ,\ ), +)$ or $P = \mathbb{Z}^t = (\mathbb{Z}, \min(\ ,\ ), +)$, in which case $P^\times = \mathbb{Z}\backslash\{0\}$ and $T_L(P) \simeq L$. We have $-\max(a,b) = \min(-a,-b)$ for $a, b \in \mathbb{Z}$. It follows that the map $i : \mathbb{Z}^T \to \mathbb{Z}^t$ such that $i(a) = -a$ is an isomorphism between the semifields $\mathbb{Z}^T$ and $\mathbb{Z}^t$.

We will soon define the Langlands dual seed $t^\vee$. By taking the tropicalization of the corresponding mutation maps on $T_{M(t^\vee)} \simeq T_{M^\circ(t)}$ with the tropical semifield $P = \mathbb{Z}^T$ [23], we obtain the following definition.

Definition 2.1.4 (Tropical transformation). Let $t' = \mu_k t$ be given seeds. The *tropical transformation* $\phi_{t',t} : M^\circ(t) \to M^\circ(t')$ is the piecewise linear map such that, for any $g = \sum g_i f_i \in M^\circ(t)$, its image $g' = \sum g'_i f'_i = \phi_{t',t}(g)$ is given by

$$\begin{aligned} g'_k &= -g_k, \\ g'_i &= g_i + [b_{ik}(t)]_+[g_k]_+ - [-b_{ik}(t)]_+[-g_k]_+, \quad i \neq k. \end{aligned}$$

For any two seeds t', t related by a mutation sequence $\overleftarrow{\mu} = \mu_{k_r} \cdots \mu_{k_1}$ such that $t' = \overleftarrow{\mu} t$, define $\phi_{t',t}$ to be the composition of the corresponding tropical transformations. Then it is independent of the choice of $\overleftarrow{\mu}$ because it is the tropicalization of the mutation maps.

Langlands dual. Let us sketch the construction of the Langlands dual, although we will not investigate the duality in depth.

Let us define the *Langlands dual seed* $t^\vee = (b_{ij}(t^\vee), (x_i(t^\vee)))$ with strictly positive integers $d_i(t^\vee) = d_i^\vee = \frac{d}{d_i}$. We define $N(t^\vee)$ to be the lattice $N^\circ(t)$ with basis $\{e_i^\vee := d_i e_i\}$ endowed with the bilinear form $\{\ ,\ \}^\vee$ such that $\{\ ,\ \}^\vee = \frac{1}{d}\{\ ,\ \}$, which implies the definition $b_{ji}(t^\vee) := -b_{ij}$. Its dual lattice $M(t^\vee)$ is then defined to be $M^\circ(t)$ spanned by the basis $\{(e_i^\vee)^* = (d_i)^{-1} e_i^* = f_i\}$. Define $M^\circ(t^\vee)$ to be the lattice spanned by the basis $\{f_i^\vee := \frac{1}{d_i^\vee}(e_i^\vee)^* = \frac{1}{d} e_i^*\}$, and $N^\circ(t^\vee)$ the lattice spanned by the basis $\{d_i^\vee e_i^\vee = d e_i\}$.

By construction, we have $T_{M^\circ(t)} = T_{M(t^\vee)}$. Moreover, such identification commutes with mutations [23, Lemma 1.11].

Cluster algebras and cluster varieties. Choose an initial seed t_0. For any sequence $(k_1, \ldots, k_r)$ of unfrozen vertices, we have a sequence of sign-coherent vectors called c-vectors, whose construction is technical and will be postponed to Section 2.2. Correspondingly, we have a sequence $(\varepsilon_1, \ldots, \varepsilon_r)$ of signs and the corresponding sequence $\overleftarrow{\mu} = \mu_{k_r,\varepsilon_r} \cdots \mu_{k_1,\varepsilon_1}$ of mutations starting from t_0 (read from right to left); see Theorem 2.2.2. Unless otherwise specified, we always make this canonical choice of signs for mutations, and omit the sign symbols $\varepsilon_1, \ldots, \varepsilon_r$ for simplicity.

Let $\Delta^+ = \Delta^+_{t_0}$ denote the set of all seeds obtained from the initial seed by iterated mutations (with the canonical choice of signs). For any $t \in \Delta^+$, view its cluster variables $x_i(t)$ as elements in the (skew-)field of fractions $\mathcal{LP}(t_0)$ via the mutation maps.

In the following, we construct the classical cluster algebras using the commutative product, and the quantum cluster algebras using the twisted product (see Section 2.5).

Definition 2.1.5 (Cluster algebras). We define the (*partially*) *compactified cluster algebra* as $\overline{\mathcal{A}} = \Bbbk[x_i(t)]_{i \in I, t \in \Delta^+}$, and the (*localized*) *cluster algebra* as $\mathcal{A} = \overline{\mathcal{A}}(t_0)[x_f^{-1}]_{f \in I_{\mathrm{f}}}$. We define the (*localized*) *upper cluster algebra* as $\mathcal{U} = \bigcap_{t \in \Delta^+} \mathcal{LP}(t)$, where Laurent polynomials at different seeds are identified via mutation maps.

In this paper, we shall focus on the cluster algebras $\mathcal{A}$ and upper cluster algebras $\mathcal{U}$. Let us explain geometric objects associated to $\mathcal{U}$ with the choice $\Bbbk = \mathbb{Z}$.

Definition 2.1.6. We define the *cluster varieties* to be $\mathbb{A} = \bigcup_{t \in \Delta^+} T_{N^\circ(t)}$ and $\mathbb{X} = \bigcup_{t \in \Delta^+} T_{M(t)}$, where the tori are glued via mutation maps.

The *Fock–Goncharov dual* of a variety $V = \bigcup T_L$ is defined as $V^\vee = \bigcup T_{L^*}$. Therefore, the dual of $\mathbb{A}$ is given by $\mathbb{A}^\vee = \bigcup_{t \in \Delta^+} T_{M^\circ(t)}$ where the tori are glued by mutation maps. Then $\mathbb{A}^\vee$ coincides with the variety $\mathbb{X}(t_0^\vee)$ associated to the Langlands dual initial seed $t_0^\vee$. We observe that the ring of regular functions on $\mathbb{A}$ is just the upper cluster algebra $\mathcal{U}$ (with $\Bbbk = \mathbb{Z}$).

Recall that the gluing map between $T_{M^\circ(t)}$ and $T_{M^\circ(t')}$ tropicalizes to $\phi_{t,t'}$: $M^\circ(t) \simeq M^\circ(t')$. We define the set $\mathbb{A}(\mathbb{Z}^T)$ of tropical points to be the set of equivalence classes in $\bigsqcup_{t \in \Delta^+} M^\circ(t)$ under the identifications $\phi_{t,t'}$, which we also denote by $\mathcal{M}^\circ$. The elements in $\mathcal{M}^\circ$ are denoted by $[g]$ for the representatives $g \in M^\circ(t)$.

2.2. *Cluster expansions, c-vectors and g-vectors*

Cluster variables have been shown to enjoy the Laurent phenomenon [26]. They can be calculated by the Caldero–Chapoton type expansion formula for the classical case [6, 17] and for the quantum case [41, 77]. We summarize these properties using the commutative product.

Theorem 2.2.1. *For any seeds $t = \overleftarrow{\mu} t_0 \in \Delta_{t_0}^+$ and $i \in I$, $\overleftarrow{\mu}^*(x_i(t)) \in \mathcal{LP}(t_0)$. Moreover,*

$$\overleftarrow{\mu}^*(x_i(t)) = x(t_0)^{g_i(t)} \cdot \Big(\sum_{n \in N_{\mathrm{uf}}^{\geq 0}(t_0)} c_n x(t_0)^{\widetilde{B}(t_0) n} \Big)$$

where $g_i(t) \in M^\circ(t_0)$, $c_0 = 1$, and $c_n \in \Bbbk$ for all n.

The vector $g_i(t)$ is called the i-th (extended) *g-vector* of the seed t with respect to the initial seed t_0. Its *principal part* is $\mathrm{pr}_{I_{\mathrm{uf}}} g_i(t)$, where $\mathrm{pr}_{I_{\mathrm{uf}}}$ denotes the natural projection from $\mathbb{Z}^I$ to $\mathbb{Z}^{I_{\mathrm{uf}}}$. Let $\widetilde{G}$ denote the $I \times I_{\mathrm{uf}}$ matrix formed by the column vectors $g_k^{t_0}(t)$, $k \in I_{\mathrm{uf}}$, and $G(t) = G^{t_0}(t)$ its $I_{\mathrm{uf}} \times I_{\mathrm{uf}}$ submatrix called the *G-matrix*.

We extend the $I_{\mathrm{uf}} \times I_{\mathrm{uf}}$ matrix $B(t_0)$ to the $(I_{\mathrm{uf}} \sqcup I'_{\mathrm{uf}}) \times I_{\mathrm{uf}}$ matrix

$$\widetilde{B}(t_0)^{\mathrm{prin}} = \begin{pmatrix} B(t_0) \\ \mathrm{Id}_{I_{\mathrm{uf}}} \end{pmatrix}$$

with $I'_{\mathrm{uf}} = I_{\mathrm{uf}}$, called the *matrix of principal coefficients*. For any seed $t = \overleftarrow{\mu} t_0$, we apply the mutation sequence $\overleftarrow{\mu}$ to the initial matrix $\widetilde{B}(t_0)^{\mathrm{prin}}$ and the resulting matrix takes the form $\binom{B(t)}{C(t)}$. The $I'_{\mathrm{uf}} \times I_{\mathrm{uf}}$ matrix $C(t) = C^{t_0}(t)$ is called the *C-matrix*. The k-th column vector of $C(t)$, denoted by $c_k^{t_0}(t)$, is called the k-th *c-vector*.

Notice that the construction of c-vectors and g-vectors depends on the choice of the initial seed t_0. In addition, the c-vectors and principal g-vectors only depend on the principal part $B(t_0)$. When the context is clear, we often omit the symbol t_0.

The following result is a consequence of [41, Theorem 5.11]; see also [65], [51, Section 5.6].

Theorem 2.2.2. (1) *The c-vectors are sign coherent, i.e., for any seed t and any $k \in I_{\mathrm{uf}}$, we must have $c_k(t) \geq 0$ (all coordinates are non-negative) or $c_k(t) \leq 0$.*

(2) *For any given mutation sequence $\overleftarrow{\mu} = \mu_{i_r} \cdots \mu_{i_0}$, denote $t_s = \mu_{i_{s-1}} \cdots \mu_{i_0} t_0$. Choose ε_s to be the sign of the k-th c-vector $c_i(t_s)$. Then $C(t) = C(t_{r+1}) = F_{i_0,\varepsilon_0}(t_0) \cdots F_{i_r,\varepsilon_r}(t_r)$ and $G(t) = E_{i_0,\varepsilon_0}(t_0) \cdots E_{i_r,\varepsilon_r}(t_r)$.*

Recall that $\mathrm{pr}_{I_{\mathrm{uf}}}$ denotes the natural projection from $\mathbb{Z}^I$ to $\mathbb{Z}^{I_{\mathrm{uf}}}$.

Corollary 2.2.3. *Given seeds $t = \overleftarrow{\mu} t_0$ where t_0 is any chosen initial seed, the c-vectors $c_i(t)$ of t form a $\mathbb{Z}$-basis of $\mathbb{Z}^{I_{\mathrm{uf}}}$, and the principal g-vectors $\mathrm{pr}_{I_{\mathrm{uf}}} g_i(t)$ form a basis of $\mathbb{Z}^{I_{\mathrm{uf}}}$.*

We can view extended g-vectors as principal g-vectors in the following way. View the vertices I as unfrozen and add principal coefficients as in [27]. Then the previous extended g-vectors become principal g-vectors. Consequently, , by extending the matrix $\widetilde{G}(t)$ with unit column vectors f_j, $j \in I_{\mathrm{f}}$, the matrix $(\widetilde{G}(t) \mid f_j,\ j \in I_{\mathrm{f}})$ equals $\widetilde{E}_{i_0,\varepsilon_0}(t_0) \cdots \widetilde{E}_{i_r,\varepsilon_r}(t_r)$.

It is useful to collect some facts about the matrices $E_{k,\varepsilon}$ and $F_{k,\varepsilon}$ [51, Section 5.6].

Let $t^\vee$ denote the Langlands dual of t whose associated matrix satisfies $b_{ij}(t^\vee) = -b_{ji}(t)$. Let t^{op} denote the seed opposite to t such that $b_{ij}(t^{\mathrm{op}}) = -b_{ij}(t)$.

Proposition 2.2.4. *Let $t' = \mu_k t$ for some $k \in I_{\mathrm{uf}}$. Let ε be any sign.*

(1) $\widetilde{B}(t') = \widetilde{E}_{k,\varepsilon}(t) \cdot \widetilde{B}(t) \cdot F_{k,\varepsilon}(t)$ *for any sign ε.*

(2) $\widetilde{E}_{k,\varepsilon}^2 = \mathrm{Id}_I$ *and* $F_{k,\varepsilon}^2 = \mathrm{Id}_{I_{\mathrm{uf}}}$.

(3) *We have*

$$E_{k,-\varepsilon}(t') = E_{k,\varepsilon}^{-1}(t), \quad E_{k,\varepsilon}(t^{\mathrm{op}}) = E_{k,-\varepsilon}(t),$$
$$F_{k,-\varepsilon}(t') = F_{k,\varepsilon}^{-1}(t), \quad F_{k,\varepsilon}(t^{\mathrm{op}}) = F_{k,-\varepsilon}(t).$$

(4) *If D' denotes the diagonal matrix $\mathrm{diag}(d'_k)_{k \in I_{\mathrm{uf}}}$, then $E_{k,\varepsilon}^T D' F_{k,\varepsilon} = D'$.*

(5) $E_{k,\varepsilon}(t^\vee)^T = F_{k,\varepsilon}(t)$.

(6) *For any initial seed t_0, we have $G(t') = G(t) \cdot E_{k,\mathrm{sign}(c_k(t))}(t)$.*

Proof. Claim (6) is a consequence of Theorem 2.2.2. The other claims can be obtained from direct calculation. ∎

The following result shows that $\widetilde{B}(\overleftarrow{\mu}(t^{\mathrm{op}})) = \widetilde{B}((\overleftarrow{\mu} t)^{\mathrm{op}})$.

Lemma 2.2.5. *Let $t = \overleftarrow{\mu} t_0$ where $\overleftarrow{\mu} = \mu_{i_r} \cdots \mu_{i_0}$. Then $\overleftarrow{\mu}(-\widetilde{B}(t_0)) = -(\overleftarrow{\mu}\widetilde{B}(t_0))$.*

Proof. Denote $t_s = \mu_{i_{s-1}} \cdots \mu_{i_0} t_0$. Choose any signs ε_s for the seeds t_s.

We prove the claim by induction on the length of $\overleftarrow{\mu}$ which equals $r+1$. The case $r+1=0$ is trivial. Assume that we have shown the result for length r.

We have

$$\begin{aligned}-(\overleftarrow{\mu}\widetilde{B}(t_0)) &= -\widetilde{E}_{i_r,\varepsilon_r}(t_r)\widetilde{B}(t_r)F_{i_r,\varepsilon_r}(t_r) = \widetilde{E}_{i_r,\varepsilon_r}(t_r)\widetilde{B}(t_r^{\mathrm{op}})F_{i_r,\varepsilon_r}(t_r) \\ &= \widetilde{E}_{i_r,-\varepsilon_r}(t_r^{\mathrm{op}})\widetilde{B}(t_r^{\mathrm{op}})F_{i_r,-\varepsilon_r}(t_r^{\mathrm{op}}) = \mu_{i_r}\widetilde{B}(t_r^{\mathrm{op}}).\end{aligned}$$

By induction hypothesis,

$$\widetilde{B}(t_r^{\mathrm{op}}) := -\widetilde{B}(t_r) = -\mu_{i_{r-1}} \cdots \mu_{i_0}\widetilde{B}(t_0) = \mu_{i_{r-1}} \cdots \mu_{i_0}(-\widetilde{B}(t_0)).$$

Therefore, $-(\overleftarrow{\mu}\widetilde{B}(t_0)) = \mu_{i_r}\mu_{i_{r-1}} \cdots \mu_{i_0}(-\widetilde{B}(t_0)) = \overleftarrow{\mu}(-\widetilde{B}(t_0))$. ■

Finally, we have the following duality between c-vectors and g-vectors.

Theorem 2.2.6 ([65, Theorem 1.2], [41]). *For any seeds $t = \overleftarrow{\mu} t_0$, we have*

$$\begin{aligned}G^{t_0}(t)^T \cdot C^{t_0^\vee}(\overleftarrow{\mu} t_0^\vee) &= \mathrm{Id}_{I_{\mathrm{uf}}}, \\ C^{t_0}(t) \cdot C^{t^{\mathrm{op}}}(\overleftarrow{\mu}^{-1}(t^{\mathrm{op}})) &= \mathrm{Id}_{I_{\mathrm{uf}}}, \\ G^{t_0}(t)^T &= C^{(t^\vee)^{\mathrm{op}}}(\overleftarrow{\mu}^{-1}((t^\vee)^{\mathrm{op}})).\end{aligned}$$

When $B(t_0)^T = -B(t_0)$, we have $B(t_0^\vee) = B(t_0)$, and so $G^{t_0}(t)^T \cdot C^{t_0}(t) = \mathrm{Id}_{I_{\mathrm{uf}}}$.

The g-vectors of a seed t' obey the tropical transformation ϕ_{t,t_0} where t, t_0 are initial seeds. More precisely, we have the following result.

Theorem 2.2.7 ([17, 41]). *For any seeds t_0, t, t' related by mutations, we have $\widetilde{G}^t(t') = \phi_{t,t_0}\widetilde{G}^{t_0}(t')$.*

2.3. Injective-reachability and green to red sequences

Definition 2.3.1 ([71]). A seed t is said to be *injective-reachable* if there exists a seed $t' = \overleftarrow{\mu} t$ and a permutation σ of I_{uf} such that the principal g-vector $\mathrm{pr}_{I_{\mathrm{uf}}} g_k^t(t')$ equals $-\mathrm{pr}_{I_{\mathrm{uf}}} f_{\sigma(k)}$ for any $k \in I_{\mathrm{uf}}$, where f_i is the i-th unit vector of $M^\circ(t) \simeq \mathbb{Z}^I$.

In this case, we denote t' by $t[1]$, and t by $t'[-1]$.

Note that the mutation sequence $\overleftarrow{\mu}$ is not unique. We fix such a sequence once and for all.

For any permutation σ, let P_σ denote the $I_{\mathrm{uf}} \times I_{\mathrm{uf}}$ matrix such that $(\mathrm{P}_\sigma)_{ik} = \delta_{i,\sigma(k)}$. Then t is injective-reachable if and only if $G(t) = -\mathrm{P}_\sigma$ for some σ. Notice that $\mathrm{P}_{\sigma^{-1}} = \mathrm{P}_\sigma^T$.

Notice that the seed $t[1]$, if it exists, is determined by t up to a permutation of I_{uf}. Defining $t[d+1] = t[d][1]$ and $t[d-1] = t[d][-1]$, we obtain a chain $(t[d])_{d \in \mathbb{Z}}$ of seeds. In addition, if some $t \in \Delta^+$ is injective-reachable then all $t' \in \Delta^+$ are injective-reachable. See [71] for more details. We have the following notion following [50].

Definition 2.3.2. For any seeds $t' = \overleftarrow{\mu} t$, the mutation sequence $\overleftarrow{\mu}$ is said to be a *green to red sequence* starting from t if $c_k^t(t')$ has negative sign for all $k \in I_{\mathrm{uf}}$.

Proposition 2.3.3. *The injective-reachable condition is satisfied if and only if $c_k^t(t') = -e_{\sigma(k)}$ for any $k \in I_{\mathrm{uf}}$, where e_k is the k-th unit vector of $N_{\mathrm{uf}}(t) \simeq \mathbb{Z}^{I_{\mathrm{uf}}}$, or equivalently $C(t) = -\mathrm{P}_\sigma$. In addition, when $C(t) = -\mathrm{P}_\sigma$, we must have $d'_k = d'_{\sigma(k)}$ for any $k \in I_{\mathrm{uf}}$.*

Proof. Denote $t' = \overleftarrow{\mu} t$ where $\overleftarrow{\mu} = \mu_{i_r} \cdots \mu_{i_0}$. Define $t_s = \mu_{i_s} \cdots \mu_{i_0} t_0$, $\varepsilon_s = \mathsf{sign}(c_{i_s}(t_s))$, $D' = \mathrm{diag}(d'_k)_{k \in I_{\mathrm{uf}}}$ as before.

By Proposition 2.2.4 and Theorem 2.2.2, we have

$$
\begin{aligned}
D' &= E_{i_r,\varepsilon_r}^T \cdots E_{i_0,\varepsilon_0}^T D' F_{i_0,\varepsilon_0} \cdots F_{i_r,\varepsilon_r} = (E_{i_0,\varepsilon_0} \cdots E_{i_r,\varepsilon_r})^T D' F_{i_0,\varepsilon_0} \cdots F_{i_r,\varepsilon_r} \\
&= G^t(t')^T \cdot D' \cdot C^t(t').
\end{aligned}
$$

If the injective-reachable condition is satisfied, then $G^t(t') = -\mathrm{P}_\sigma$. Therefore, $D' = -\mathrm{P}_\sigma^T D' C^t(t')$, and consequently $C^t(t') = -D'^{-1} \mathrm{P}_\sigma D'$, $c_k^t(t') = -\frac{d'_k}{d'_{\sigma(k)}} e_{\sigma(k)}$. Because $c_k^t(t')$ are integer vectors, we must have $d'_k = d'_{\sigma(k)}$ and $c_k^t(t') = -e_{\sigma(k)}$ for any $k \in I_{\mathrm{uf}}$.

Conversely, if $C^t(t') = -\mathrm{P}_\sigma$ then we can similarly show $d'_k = d'_{\sigma(k)}$ for all $k \in I_{\mathrm{uf}}$ and $G^t(t') = -\mathrm{P}_\sigma$. ∎

Corollary 2.3.4. *For any seeds $t' = \overleftarrow{\mu} t$, we have $t' = t[1]$ if and only if $\overleftarrow{\mu}$ is a green to red sequence starting from t.*

Proof. The "only if" part is a consequence of Proposition 2.3.3.

On the other hand, if $\overleftarrow{\mu}$ is a green to red sequence, then $c_k^t(t') < 0$ for all k. It defines a chamber $\mathcal{C}^{t'} = \{m \in \mathbb{R}^I \mid c_k^t(t') \cdot \mathsf{pr}_{I_{\mathrm{uf}}} m \geq 0\}$ in the cluster scattering diagram associated to the initial seed t (Section A.1). But the chamber $\mathcal{C}^{t'}$ contains the negative chamber $\mathcal{C}^- = (\mathbb{R}_{\leq 0}^{I_{\mathrm{uf}}}) \oplus \mathbb{R}^{I_{\mathrm{f}}}$ of the scattering diagram. Therefore, one must have $\mathcal{C}^- = \mathcal{C}^{t'}$, and consequently $C^t(t') = -\mathrm{P}_\sigma$ for some σ. The claim follows from Proposition 2.3.3. ∎

2.4. *Cluster categories*

We refer the reader to [48, 67] for details of this section. A quiver $\widetilde{Q}$ is a finite oriented graph, which we assume to have no loops or 2-cycles throughout this paper. Denote its set of vertices by I and of arrows by E. An *ice quiver* $\widetilde{Q}$ is a quiver endowed with a partition of its vertices, $I = I_{\mathrm{uf}} \sqcup I_{\mathrm{f}}$ (unfrozen and frozen respectively). The full subquiver of $\widetilde{Q}$ supported on the unfrozen vertices I_{uf} is called the *principal part* and denoted by Q.

To any ice quiver $\widetilde{Q}$, we can associate an $I \times I$ skew-symmetric matrix (b_{ij}) such that b_{ij} is the difference between the number of arrows from j to i and that from i to j. Its $I \times I_{\mathrm{uf}}$ submatrix and $I_{\mathrm{uf}} \times I_{\mathrm{uf}}$ submatrix are denoted by $\widetilde{B}$ and B as before. Conversely, to any $I \times I$ skew-symmetric integer matrix (b_{ij}), we can associate an ice quiver $\widetilde{Q}$.

The *path algebra* $\mathbb{C}\widetilde{Q}$ is the $\mathbb{C}$-algebra generated by paths of $\widetilde{Q}$ whose multiplication is given by path composition. $\mathbb{C}\widetilde{Q}$ has the maximal ideal $\mathbf{m}$ generated by the arrows $a \in E$. Let $\widehat{\mathbb{C}\widetilde{Q}}$ denote the completion. Choosing a linear combination $\widetilde{W} \in \widehat{\mathbb{C}\widetilde{Q}}$ of oriented cycles called a *potential*, we can define its cyclic derivatives $\partial_a \widetilde{W}$ for any $a \in E$ (see [16]).

The ideal $\langle \partial_a \widetilde{W} \rangle_{a\in E}$ of $\widehat{\mathbb{C}\widetilde{Q}}$ has the closure $\overline{\langle \partial_a \widetilde{W} \rangle_{a\in E}} = \bigcap_{n>0}(\langle \partial_a \widetilde{W} \rangle_{a\in E} + \mathbf{m}^n)$. We define the *completed Jacobian algebra* associated to the quiver with potential to be $J_{(\widetilde{Q},\widetilde{W})} = \widehat{\mathbb{C}\widetilde{Q}}/\overline{\langle \partial_a \widetilde{W} \rangle_{a\in I}}$. By restricting the potential $\widetilde{W}$ to the full subquiver Q (arrows not contained in Q are sent to 0), we obtain the principal quiver with potential (Q, W) and the corresponding Jacobian algebra $J_{(Q,W)}$.

Let $\Gamma = \Gamma_{(\widetilde{Q},\widetilde{W})}$ denote the Ginzburg dg algebra (differential graded algebra) associated to $(\widetilde{Q}, \widetilde{W})$ [35]. Its homology is concentrated in negative degrees so that $H^{>0}\Gamma = 0$, $H^0\Gamma = J_{(\widetilde{Q},\widetilde{W})}$. Let $\mathsf{per}\,\Gamma$ denote the perfect derived category of Γ (smallest triangulated category containing Γ), and $D_{\mathrm{fd}}\Gamma$ the full subcategory consisting of objects with finite-dimensional total homology. Let Σ denote the shift functor.

The (generalized) *cluster category* $\mathcal{C} = \mathcal{C}_{(\widetilde{Q},\widetilde{W})}$ is defined to be the quotient category $\mathsf{per}\,\Gamma/D_{\mathrm{fd}}\Gamma$ [1]. Let π denote the natural projection. We further assume that $J_{(\widetilde{Q},\widetilde{W})} = H^0\Gamma$ is finite-dimensional. Then the category $\mathcal{C}$ is a Hom-finite 2-Calabi–Yau triangulated category, which means $\mathsf{Hom}(X, \Sigma Y) \simeq D\,\mathsf{Hom}(Y, \Sigma X)$. Furthermore, $\pi\Gamma$ is a cluster tilting object of $\mathcal{C}$, i.e., $\mathsf{Hom}_{\mathcal{C}}(\pi\Gamma, \Sigma(\pi\Gamma)) = 0$ and $\mathsf{Hom}_{\mathcal{C}}(\pi\Gamma, \Sigma X) = 0$ implies $X \in \mathsf{add}(\pi\Gamma)$. The subcategory of coefficient-free objects is defined to be the full subcategory

$$^{\perp}(\Sigma T_{\mathsf{f}}) = \{X \in \mathcal{C} \mid \mathsf{Hom}(X, \Sigma T_{\mathsf{f}}) = 0\}$$

where $T_{\mathsf{f}} = \bigoplus_{i\in I_{\mathsf{f}}} \pi\Gamma_i$ and Γ_i denotes the i-th indecomposable projective of Γ.

From now on, we always assume that the potential $\widetilde{W}$ is chosen to be non-degenerate [16]. Then we can mutate cluster tilting objects. The cluster category $\mathcal{C}$ associated to $(\widetilde{Q}, \widetilde{W})$ provides a categorification for the cluster algebra associated to the initial seed $t_0 = ((b_{ij}), (x_i))$ such that we associate to $t \in \Delta^+$ cluster tilting objects $T(t)$, with $T(t_0) = \pi\Gamma$, and quivers with potential $((\widetilde{Q}(t), \widetilde{W}(t))$ with $\widetilde{Q}(t)$ corresponding to $(b_{ij}(t))$. Notice that T_{f} is a common summand for all $T(t)$, $t \in \Delta^+$.

For any $M \in \mathcal{C}$ and $T = T(t)$, we have an $\mathsf{add}\,T$-approximation in $\mathcal{C}$

$$T^{(1)} \to T^{(0)} \to M \to \Sigma T^{(1)}.$$

Let us identify the Grothendieck ring of $\mathsf{add}\,T$ with $M^\circ(t) \simeq \mathbb{Z}^I$ so that the isoclass $[T_i]$ corresponds to the i-th unit vector f_i. The *index* of X is defined to be $\mathsf{Ind}^T M = [T^{(0)}] - [T^{(1)}]$.

For convenience, we consider right modules unless otherwise specified. We define the functor F such that

$$F : \mathcal{C} \to J_{(\widetilde{Q}(t),\widetilde{W}(t))}\text{-}\mathsf{mod}, \quad X \mapsto \mathsf{Hom}(T, \Sigma X).$$

Its restriction $^{\perp}(\Sigma T_{\mathsf{f}})$ has image in $J_{(Q(t),W(t))}\text{-}\mathsf{mod}$.

Definition 2.4.1 (Caldero–Chapoton formula). Consider the classical case $\Bbbk = \mathbb{Z}$. For any given skew-symmetric seed t, the corresponding cluster tilting object $T = T(t)$, and any coefficient-free object $M \in {}^{\perp}(\Sigma T_{\mathsf{f}})$, the cluster character of M is defined to be the Laurent polynomial in $\mathcal{LP}(t)$:

$$CC^t(M) = x(t)^{\mathsf{Ind}^T M}\Big(\sum_{n\in N_{\mathsf{uf}}^{\geq 0}(t)} \chi(\mathsf{Gr}_n\, FM) \cdot x(t)^{\widetilde{B}\cdot n}\Big)$$

where $\mathrm{Gr}_n\, FM$ is the submodule Grassmannian of the $J_{(Q(t),W(t))}$-module FM consisting of n-dimensional submodules, and χ denotes the topological Euler characteristic.

We also define $CC^t(FM) = CC^t(M)$.

Let us recall the Calabi–Yau reduction in the sense of [43]; see [68, Section 3.3] for a brief introduction.

Let (T_{f}) denote the ideal of all morphisms of the cluster category $\mathcal{C}_{(\tilde{Q},\tilde{W})}$ which factor through T_{f}. Then the quotient ${}^{\perp}(\Sigma T_{\mathrm{f}})/(T_{\mathrm{f}})$ is naturally endowed with a structure of triangulated category. Furthermore, ${}^{\perp}(\Sigma T_{\mathrm{f}})/(T_{\mathrm{f}})$ is equivalent to the cluster category $\mathcal{C}_{(Q,W)}$ associated to (Q, W).

Let us use $\underline{\Gamma}$ to denote the Ginzburg algebra $\Gamma_{Q,W}$ and let $\underline{T}$ denote the corresponding cluster tilting object in $\mathcal{C}_{(Q,W)}$. Then, under the above quotient and equivalence, any T_k with $k \in I_{\mathrm{uf}}$ is sent to $\underline{T}_k$.

Any object $M \in {}^{\perp}(\Sigma T_{\mathrm{f}})$ is sent to an object $\underline{M}$ in $\mathcal{C}_{(Q,W)}$. By [68], the index of $\underline{M}$ is given by projection,

$$\mathrm{Ind}^{\underline{T}}\,\underline{M} = \mathrm{pr}_{I_{\mathrm{uf}}}(\mathrm{Ind}^{T} M).$$

In particular, if we let I_k denote the indecomposable object in ${}^{\perp}(\Sigma T_{\mathrm{f}})$ which corresponds to $\Sigma(\underline{T}_k)$ in $\mathcal{C}_{(Q,W)}$, then $\mathrm{pr}_{I_{\mathrm{uf}}}\,\mathrm{Ind}^T I_k = -f_k$. Notice that $F\Sigma(\underline{T}_k)$ is the k-th injective module of $J_{(Q,W)}$, which we also denote by I_k.

By [68], for any $g \in \mathbb{Z}^{I_{\mathrm{uf}}}$, there exists some $m \in \mathbb{N}^{I_{\mathrm{f}}}$ depending on g such that, for a generic morphism $f \in \mathrm{Hom}(T^{[-g]_+}, T^{[g]_+ + m})$ (see [66]), cone f belongs to $\in {}^{\perp}(\Sigma T_{\mathrm{f}})$ and has no direct summand in add T_{f}. We define the *generic cluster character* associated to $g + m$ to be $\mathbb{L}_{g+m} = CC(\mathrm{cone}\, f)$.

Theorem 2.4.2 ([68, Theorem 1.3]). *For any skew-symmetric seeds $t' = \overleftarrow{\mu} t$, the generic cluster characters satisfy*

$$\overleftarrow{\mu}^* \mathbb{L}^t_{g+m} = \mathbb{L}^{t'}_{\phi_{t',t}\, g+m}.$$

2.5. *Quantization*

We briefly recall the necessary modification needed for the quantum case $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$. Assume that a seed t satisfies the full rank assumption as before.

First, we endow the seed t with a quantum seed structure by choosing a compatible $\mathbb{Z}$-valued skew-symmetric bilinear form λ on $M^\circ(t)$ and strictly positive integers d'_k, $k \in I_{\mathrm{uf}}$. By compatibility, we mean

$$\lambda(f_i, p^* e_k) = -\delta_{i,k} d'_k, \quad \forall i \in I,\ k \in I_{\mathrm{uf}}.$$

For any seed $t' = \mu_k t$, $k \in I_{\mathrm{uf}}$, the linear isomorphism $M^\circ(t') \simeq M^\circ(t)$ via (2.2) induces a bilinear form on $M^\circ(t)$, which we still denote by λ. It follows from [3] that λ is compatible with t' as well. Repeatedly, we assign quantum seed structures to all seeds obtained from t by iterated mutations.

For any quantum seed t, we endow the Laurent polynomial ring $\mathcal{LP}(t)$ with an extra multiplication, called *twisted product* $*$, such that

$$x^m * x^{m'} = q^{\frac{1}{2}\lambda(m,m')} x^{m+m'}, \quad \forall m, m' \in M^\circ(t).$$

Note that $*$ becomes the commutative product $\cdot$ when we specialize $q^{1/2}$ to 1.

Unless otherwise specified, we will choose this twisted product $*$ as the multiplication for the $\Bbbk$-algebra $\mathcal{LP}(t)$ instead of the commutative product $\cdot$.

Similarly, we endow $\Bbbk[N(t)]$ with the twisted product $*$ such that

$$y^n * y^{n'} = q^{\frac{1}{2}\lambda(p^*n, p^*n')} y^{n+n'}, \quad \forall n, n' \in N_{\mathrm{uf}}(t).$$

Then p^* induces a $\Bbbk$-algebra homomorphism from $\Bbbk[N(t)]$ to $\mathcal{LP}(t)$ commuting with the twisted products.

Using the twisted product, we construct the skew-fields of fractions of $\mathcal{LP}(t)$ and $\Bbbk[N(t)]$ and denote them by $\mathcal{F}(t) = \mathcal{F}(\mathcal{LP}(t))$ and $\mathcal{F}(\Bbbk[N(t)])$ respectively. The classical automorphisms in (2.3) are quantized to the automorphisms $\rho_{k,\varepsilon}$ such that, for $i \neq k$,

$$\begin{aligned}
\rho_{k,\varepsilon}(x_i) &= x_i, \\
\rho_{k,\varepsilon}(x_k^{-1}) &= x_k^{-1} + x^{-f_k + \varepsilon v_k}, \\
\rho_{k,\varepsilon}(y_i) &= y_i \cdot \sum_{s=0}^{|b_{ki}|} \binom{|b_{ki}|}{s}_{q_k} y_k^{\varepsilon s}, \qquad b_{ki} \leq 0, \\
\rho_{k,\varepsilon}(y_i^{-1}) &= y_i^{-1} \cdot \sum_{s=0}^{|b_{ki}|} \binom{|b_{ki}|}{s}_{q_k} y_k^{\varepsilon s}, \quad b_{ki} > 0, \\
\rho_{k,\varepsilon}(y_k) &= y_k.
\end{aligned} \tag{2.5}$$

where we denote $q_k = q^{\frac{1}{2}d'_k}$, $[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$ for $0 \neq a \in \mathbb{N}$, $[0]_q! = 1$, $[a]_q! = [a]_q[a-1]_q \cdots [1]_q$, and $\binom{a}{b}_q = \frac{[a]_q!}{[b]_q![a-b]_q!}$.

As before, define quantum mutations μ_k^* as the compositions $\rho_{k,\varepsilon} \circ \tau_{k,\varepsilon}$. Then they are independent of the choice of the sign ε, and such that, for $i \neq k$,

$$\begin{aligned}
\mu_k^*(x'_i) &= x_i, \\
\mu_k^*(x'_k) &= x^{-f_k + \sum_j [-b_{jk}]_+ f_j} + x^{-f_k + \sum_i [b_{ik}]_+ f_i}, \\
\mu_k^*(y'_i) &= y_i \cdot \sum_{s=0}^{|b_{ki}|} \binom{|b_{ki}|}{s}_{q_k} y_k^s, \qquad\qquad b_{ki} \leq 0, \\
\mu_k^*((y'_i)^{-1}) &= y_i^{-1} y_k^{-b_{ki}} \cdot \sum_{s=0}^{|b_{ki}|} \binom{|b_{ki}|}{s}_{q_k} y_k^s, \quad b_{ki} > 0, \\
\mu_k^*(y'_k) &= y_k^{-1}.
\end{aligned} \tag{2.6}$$

We define the quantum (upper) cluster algebras as in Definition 2.1.5, using the quantum mutations and twisted products in the construction.

3. Bidegrees and support of Laurent polynomials

Let $t = ((b_{ij}(t))_{i,j\in I}, (x_i(t))_{i\in I})$ be a seed such that the $I \times I_{\mathrm{uf}}$ matrix $\widetilde{B}(t)$ is of full rank. Recall that

$$M^\circ(t) \simeq \mathbb{Z}^I, \quad N(t) \simeq \mathbb{Z}^I, \quad N_{\mathrm{uf}}(t) \simeq \mathbb{Z}^{I_{\mathrm{uf}}},$$
$$N_{\mathrm{uf}}{}^{\geq 0}(t) \simeq \mathbb{N}^{I_{\mathrm{uf}}}, \quad N_{\mathrm{uf}}{}^{>0}(t) \simeq \mathbb{N}^{I_{\mathrm{uf}}} - \{0\},$$

where the natural bases of $M^\circ(t)$, $N(t)$ and $N_{\mathrm{uf}}(t)$ are denoted by $\{f_i \mid i \in I\}$, $\{e_i \mid i \in I\}$ and $\{e_k \mid k \in I_{\mathrm{uf}}\}$ respectively. The pairing $\langle\ ,\ \rangle$ between $M^\circ(t)$ and $N(t)$ is such that $\langle f_i, e_j\rangle = \frac{1}{d_i}\delta_{ij}$. In addition, $N(t)$ is endowed with the skew-symmetric bilinear form $\{\ ,\ \}$ such that $\{e_i, e_j\} = d_j^{-1} b_{ji}$. We also have the linear map $p^* : N_{\mathrm{uf}}(t) \to M^\circ(t)$ such that $p^*(n) = \{n,\ \}$, which turns out to be $p^*(n) = \widetilde{B}(t)\cdot n$ under the identification $M^\circ(t) \simeq \mathbb{Z}^I$ and $N_{\mathrm{uf}}(t) \simeq \mathbb{Z}^{I_{\mathrm{uf}}}$. Denote $v_k = p^*(e_k)$ for $k \in I_{\mathrm{uf}}$. The vectors $\{v_k\}_{k\in I_{\mathrm{uf}}}$ are linearly independent by the full rank assumption on $\widetilde{B}(t)$.

3.1. Dominance order

The *dominance order* is the following partial order defined on $M^\circ(t)$.

Definition 3.1.1 (Dominance order [71, Definition 3.1.1]). For any given seed t and $g, g' \in M^\circ(t)$, we say g' is *dominated* by g, denoted by $g' \preceq_t g$, if $g' = g + p^*(n)$ for some $n \in N_{\mathrm{uf}}{}^{\geq 0}(t)$. We write $g' \prec_t g$ if $g \neq g'$.

For any given $g, \eta \in M^\circ(t)$, we define the following subsets of $M^\circ(t)$:

$$M^\circ(t)_{\preceq_t g} = \{g' \in M^\circ(t) \mid g' \preceq_t g\} = g + p^* N_{\mathrm{uf}}^{\geq 0}(t),$$
$${}_{\eta\preceq_t} M^\circ(t) = \{g' \in M^\circ(t) \mid \eta \preceq_t g'\} = \eta - p^* N_{\mathrm{uf}}^{\geq 0}(t),$$
$${}_{\eta\preceq_t} M^\circ(t)_{\preceq_t g} = \{g' \in M^\circ(t) \mid \eta \preceq_t g' \preceq_t g\} = {}_{\eta\preceq_t} M^\circ(t) \cap M^\circ(t)_{\preceq_t g}.$$

Lemma 3.1.2 (Finite Interval Lemma, [71, Lemma 3.1.2]). *For any $\eta, g \in M^\circ(t)$, the set ${}_{\eta\preceq_t} M^\circ(t)_{\preceq_t g}$ is finite. In particular, if $\eta \preceq_t g$ and $g \preceq_t \eta$, we must have $\eta = g$ as elements in $M^\circ(t)$.*

Proof. The claim follows from the assumption that $\widetilde{B}(t)$ is of full rank. ■

Recall that, for any two seeds $t, t' \in \Delta^+$, we have the tropical transformation $\phi_{t',t} : M^\circ(t) \to M^\circ(t')$. By viewing $\phi_{t',t}$ as an identification, the set $\mathcal{M}^\circ$ of tropical points is the set of equivalence classes. Moreover, the dominance order $\prec_{t'}$ is transported to $M^\circ(t)$ and $\mathcal{M}^\circ$ so that, for any $g, h \in M^\circ(t)$, whenever $\phi_{t',t} h \prec_{t'} \phi_{t',t'} g$, we define $h \prec_{t'} g$ in $M^\circ(t)$ and $[g] \prec_{t'} [h]$ in $\mathcal{M}^\circ$.

In general, for any given sets S, S' of seeds, we define

$$\mathcal{M}^\circ{}_{\preceq_S [g]} = \{[g'] \in \mathcal{M}^\circ \mid [g'] \preceq_t [g],\ \forall t \in S\},$$
$${}_{[\eta]\preceq_{S'}} \mathcal{M}^\circ = \{[g'] \in \mathcal{M}^\circ \mid [\eta] \preceq_t [g'],\ \forall t \in S'\},$$
$${}_{[\eta]\preceq_{S'}} \mathcal{M}^\circ{}_{\preceq_S [g]} = {}_{[\eta]\preceq_{S'}} \mathcal{M}^\circ \cap \mathcal{M}^\circ{}_{\preceq_S [g]}.$$

We have similar definitions for $M^\circ(t)_{\preceq_S g}$, ${}_{\eta\preceq_{S'}} M^\circ(t)$, and ${}_{\eta\preceq_{S'}} M^\circ(t)_{\preceq_S g}$.

From now on, we use the symbols $M^\circ(t)$ and $g \in M^\circ(t)$ if we want to specify a special seed t, and $\mathcal{M}^\circ$ and $[g] \in \mathcal{M}^\circ$ otherwise.

3.2. *Formal Laurent series and bidegrees*

The monoid algebra $\Bbbk[N_{\mathrm{uf}}{}^{\geq 0}(t)] = \Bbbk[\lambda^n]_{n \in N_{\mathrm{uf}}{}^{\geq 0}(t)}$ has a maximal ideal $\mathbf{m} = \Bbbk[N_{\mathrm{uf}}{}^{>0}(t)]$. The corresponding completion is denoted by $\widehat{\Bbbk[N_{\mathrm{uf}}{}^{\geq 0}(t)]}$. The injective linear map $p^* : N_{\mathrm{uf}}(t) \to M^\circ(t)$ induces an embedding p^* from $\Bbbk[N_{\mathrm{uf}}{}^{\geq}(t)]$ to $\mathcal{LP}(t) = \Bbbk[M^\circ(t)] = \Bbbk[\chi^m]_{m \in M^\circ(t)}$ such that $p^*(\lambda^n) = \chi^{p^*(n)}$ for all $n \in N_{\mathrm{uf}}(t)$. We define the set of formal Laurent series to be

$$\widehat{\mathcal{LP}(t)} = \mathcal{LP}(t) \otimes_{\Bbbk[N_{\mathrm{uf}}{}^{\geq 0}(t)]} \widehat{\Bbbk[N_{\mathrm{uf}}{}^{\geq 0}(t)]}$$

where $\Bbbk[N_{\mathrm{uf}}{}^{\geq 0}(t)]$ is viewed as a subalgebra of $\Bbbk[M^\circ(t)]$ via the embedding p^*.

Then a formal Laurent series is a finite sum of elements of the type

$$a \cdot x(t)^g \cdot \sum_{n \in N_{\mathrm{uf}}{}^{\geq 0}(t)} b_n y(t)^n$$

where $a, b_n \in \Bbbk$, $g \in M^\circ(t)$, $x_i(t) = \chi^{f_i}$ and $y_k = \chi^{p^*(e_k)} = \prod_i x_i^{b_{ik}}$ by the embedding p^*.

Similarly, letting $\widehat{\Bbbk[-N_{\mathrm{uf}}{}^{\geq 0}(t)]}$ denote the completion of $\Bbbk[-N_{\mathrm{uf}}{}^{\geq 0}(t)]$ with respect to its maximal ideal $\Bbbk[-N_{\mathrm{uf}}{}^{>0}(t)]$, we can define

$$\widetilde{\mathcal{LP}(t)} = \mathcal{LP}(t) \otimes_{\Bbbk[-N_{\mathrm{uf}}{}^{\geq 0}(t)]} \widehat{\Bbbk[-N_{\mathrm{uf}}{}^{\geq 0}(t)]}$$

Then any formal series $z \in \widetilde{\mathcal{LP}(t)}$ is a finite sum of elements of the type

$$a \cdot x(t)^g \cdot \sum_{n \in -N_{\mathrm{uf}}{}^{\geq 0}(t)} b_n y(t)^n$$

where $a, b_n \in \Bbbk$ and $g \in M^\circ(t)$.

Let us postpone the discussion of the ring structure for the moment and give an intuitive definition of (co)degrees arising from the dominance order.

Definition 3.2.1 (Degree, pointed [71]). For any formal sum $z = \sum_{g \in M^\circ(t)} c_g x(t)^g$ where $c_g \in \Bbbk$, if the set $\{g \mid c_g \neq 0\}$ of Laurent degrees has a unique $\prec_t$-maximal element g, we say z has *degree g with respect to t*, and denote $\mathrm{deg}^t z = g$.

If $\mathrm{deg}^t z = g$ and $c_g = 1$, then z is said to be *pointed at g*.

A set is said to be *pointed* if it consists of elements pointed at various degrees.

We also need the following dual notion.

Definition 3.2.2 (Codegree, copointed). For any formal sum $z = \sum_{g \in M^\circ(t)} c_g x(t)^g$ where $c_g \in \Bbbk$, if the set $\{g \mid c_g \neq 0\}$ of Laurent degrees has a unique $\prec_t$-minimal element η, we say z has *codegree η with respect to t*, and denote $\mathrm{codeg}^t z = \eta$.

If $\mathrm{codeg}^t z = \eta$ and $c_\eta = 1$, then z is said to be *copointed at η*.

Definition 3.2.3 (Bidegree, bipointed). For any formal sum $z = \sum_{g \in M^\circ(t)} c_g x(t)^g$, if $\deg^t z = g$ and $\mathrm{codeg}^t z = \eta$ for some $g, \eta \in M^\circ(t)$, we say z has *bidegree* (η, g), denoted by $\mathrm{bideg}^t z = (\eta, g)$.

If z is further pointed at g and copointed at η, we say it is *bipointed at* (η, g).

We have the following easy observation.

Lemma 3.2.4. *If a formal sum $z = \sum_{g \in M^\circ(t)} c_g x(t)^g$ has bidegree (η, g), then the following claims are true:*

(1) $\eta \preceq_t g$.

(2) *z is a Laurent polynomial.*

(3) *z is a Laurent monomial if and only if $\eta = g$.*

Proof. The claim follows from the definitions and the finiteness of ${}_{\eta \preceq_t} M^\circ(t)_{\preceq_t g}$ (Lemma 3.1.2). ■

We will mainly be interested in Laurent polynomials. But sometimes our calculation will be carried out for formal series. Let us look at these series in more detail. Recall that we have identified $\Bbbk[N_{\mathrm{uf}}(t)]$ as a subalgebra of $\Bbbk[M^\circ(t)]$ via the embedding p^*.

For any $g \in M^\circ(t)$, the $\Bbbk$-submodule $x^g \cdot \Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)] \subset \Bbbk[M^\circ(t)]$ is a rank 1 free module over the algebra $\Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)]$. We define its completion to be the rank 1 free $\widehat{\Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)]}$-module $x^g \cdot \widehat{\Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)]}$.

The subset $\mathcal{PT}^t(g) := x^g \cdot (1 + \Bbbk[{N_{\mathrm{uf}}}^{>0}(t)])$ of $x^g \cdot \Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)]$ is the set of Laurent polynomials pointed at degree g. Let $\widehat{\Bbbk[{N_{\mathrm{uf}}}^{>0}(t)]}$ denote the subset of series in $\widehat{\Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)]}$ with vanishing constant terms. Then the subset $\widehat{\mathcal{PT}}^t(g) := x^g(1 + \widehat{\Bbbk[{N_{\mathrm{uf}}}^{>0}(t)]})$ of $x^g \cdot \widehat{\Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)]}$ is the set of formal Laurent series pointed at degree g. Notice that we have $\mathcal{PT}^t(g) \subset \widehat{\mathcal{PT}}^t(g) \subset \widehat{\mathcal{LP}(t)}$.

Similarly, the subset $\mathcal{CPT}^t(\eta) := x^\eta \cdot (1 + \Bbbk[-{N_{\mathrm{uf}}}^{>0}(t)])$ of $x^\eta \cdot \Bbbk[-{N_{\mathrm{uf}}}^{\geq 0}(t)]$ is the set of Laurent polynomials copointed at codegree η. In addition, we have the subset of of $x^\eta \cdot \widehat{\Bbbk[-{N_{\mathrm{uf}}}^{\geq 0}(t)]}$ consisting of the copointed formal Laurent series, $\widehat{\mathcal{CPT}}^t(\eta) = x^\eta \cdot (1 + \widehat{\Bbbk[-{N_{\mathrm{uf}}}^{>0}(t)]})$. Notice that we have $\mathcal{CPT}^t(g) \subset \widehat{\mathcal{CPT}}^t(g) \subset \widehat{\mathcal{LP}(t)}$.

Finally, the subset $\mathcal{BPT}^t(\eta, g) := \mathcal{PT}^t(g) \cap \mathcal{CPT}^t(\eta)$ of $\Bbbk[M^\circ(t)]$ is the set of Laurent polynomials bipointed at bidegree (η, g).

Lemma 3.2.5 (inverse). (1) *For any pointed formal Laurent series $u \in \widehat{\mathcal{PT}}^t(g)$, where $g \in M^\circ(t)$, u has a multiplicative inverse v in the ring of formal Laurent series $\widehat{\mathcal{LP}(t)}$. In addition, v belongs to $\widehat{\mathcal{PT}}^t(-g)$.*

(2) *For any copointed element $u' \in \widehat{\mathcal{CPT}}^t(\eta)$, where $\eta \in M^\circ(t)$, u' has a multiplicative inverse v' in $\widehat{\mathcal{LP}(t)}$. In addition, v' belongs to $\widehat{\mathcal{CPT}}^t(-\eta)$.*

Proof. (1) u takes the form $u = x(t)^g * F$, where $F \in 1 + \widehat{\Bbbk[{N_{\mathrm{uf}}}^{>0}(t)]}$, and $*$ denotes the twisted product. Notice that F has a unique inverse $F' \in 1 + \widehat{\Bbbk[{N_{\mathrm{uf}}}^{>0}(t)]}$ in $\widehat{\Bbbk[{N_{\mathrm{uf}}}^{\geq 0}(t)]}$. Then u has inverse $v = F' * x(t)^{-g}$.

(2) The proof is similar to (1). ■

Lemma 3.2.6 (product). (1) *For any given series z_g, z_η pointed at degree g and η respectively, their product is a well defined series pointed at degree $g+\eta$.*

(2) *For any given series z_g, z_η copointed at codegree g and η respectively, their product is a well defined series copointed at codegree $g+\eta$.*

Proof. (1) Notice that, to each Laurent degree g' in the product, only finitely many Laurent monomials of the pointed series z_g and z_η will contribute, because ${}_{g'\preceq}\mathcal{M}^\circ{}_{\preceq g}$ and ${}_{g'\preceq}\mathcal{M}^\circ{}_{\preceq \eta}$ are finite by Lemma 3.1.2. Therefore, the product is well defined. In addition, it is pointed at degree $g+\eta$ by direct computation.

(2) The proof is similar to that of (1). ■

3.3. *Degrees and codegrees under mutation*

Let t, t' be two seeds connected by a mutation sequence, $t' = \overleftarrow{\mu} t$. Recall that the lattice $M^\circ(t) \simeq \mathbb{Z}^I$ has a natural basis $\{f_i = f_i(t) \mid i \in I\}$.

Definition 3.3.1 (Degree transformation). We define the linear map $\psi_{t',t} : M^\circ(t) \to M^\circ(t')$ such that

$$\psi_{t',t}\Big(\sum_{i\in I} g_i f_i\Big) = \sum_{i\in I} g_i \phi_{t',t}(f_i)$$

for any $(g_i)_{i\in I} \in \mathbb{Z}^I$.

We have the following result (see Example 3.3.3).

Lemma 3.3.2. *Let $t' = \mu_k t \in \Delta^+$ for some $k \in I_{\mathrm{uf}}$. Denote $\phi = \phi_{t',t}$ and $\psi = \psi_{t',t}$. Let e'_k denote the k-th unit vector in $N_{\mathrm{uf}}(t')$. For any $i \neq k \in I$ and $g \in M^\circ(t)$, we have $\psi g - \phi g = [-g_k]_+ \widetilde{B}' e'_k$.*

Proof. Note that, in the lattice $M^\circ(t')$, we have $\psi(f_k) = \deg^{t'} x_k(t) = \phi(f_k) = -f'_k + [b_{ik}]_+ f'_i$; see Definition 2.1.4. Direct calculation shows that, for any $i \neq k \in I$,

$$\begin{aligned}
(\phi g - \psi g)_i &= (g_i + [b_{ik}]_+[g_k]_+ - [-b_{ik}]_+[-g_k]_+) - (g_i + [b_{ik}]_+ g_k) \\
&= ([b_{ik}]_+[g_k]_+ - [-b_{ik}]_+[-g_k]_+) - [b_{ik}]_+ g_k \\
&= ([b_{ik}]_+[g_k]_+ - [-b_{ik}]_+[-g_k]_+) - [b_{ik}]_+([g_k]_+ - [-g_k]_+) \\
&= -[-b_{ik}]_+[-g_k]_+ + [b_{ik}]_+[-g_k]_+ = b_{ik}[-g_k]_+ = -b'_{ik}[-g_k]_+.
\end{aligned}$$

Moreover, $(\phi g)_k = -g_k = (\psi g)_k$. We deduce that $\phi g - \psi g = -(\widetilde{B}') \cdot [-g_k]_+ e'_k$. ■

Example 3.3.3. Choose a seed t such that $I = I_{\mathrm{uf}} = \{1,2\}$, $\widetilde{B} = (b_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Take any $g = g_1 f_1 + g_2 f_2 \in M^\circ(t)$, $g_1, g_2 \in \mathbb{Z}$.

First, take $t' = \mu_1 t$. Then $\widetilde{B}' = -\widetilde{B}$. We have $\phi(g) = (-g_1) f'_1 + (g_2 + [g_1]_+) f'_2$ (see Definition 2.1.4). In particular, $\psi(f_1) = \phi(f_1) = -f'_1 + f'_2$ and $\psi(f_2) = \phi(f_2) = f'_2$. It follows that $\psi(g) = g_1\psi(f_1) + g_2\psi(f_2) = (-g_1) f'_1 + (g_1 + g_2) f'_2$. Therefore, $\psi g - \phi g = -[-g_1]_+ f'_2 = [-g_1]_+ \widetilde{B}' e'_1$.

Second, take $t' = \mu_2 t$. Then $\widetilde{B}' = -\widetilde{B}$. We have $\phi(g) = (g_1 - [-g_2]_+) f_1' + (-g_2) f_2'$ (see Definition 2.1.4). In particular, $\psi(f_1) = \phi(f_1) = f_1'$ and $\psi(f_2) = \phi(f_2) = -f_2'$. It follows that $\psi(g) = g_1\psi(f_1) + g_2\psi(f_2) = g_1 f_1' + (-g_2) f_2'$. Therefore, $\psi g - \phi g = [-g_2]_+ f_1' = [-g_2]_+ \widetilde{B}' e_2'$.

Remark 3.3.4 (Non-trivial monodromy). Recall that the maps $\phi_{t',t}$ are piecewise linear and $\phi_{t,t'}\phi_{t',t} = \phi_{t,t} = \mathrm{Id}_{M^\circ(t)}$. By contrast, the maps $\psi_{t',t}$ are linear, but at the cost that $\psi_{t,t'}\psi_{t',t} \neq \mathrm{Id}_{M^\circ(t)}$ in general.

It would be interesting to understand such non-trivial monodromy. We observe that this monodromy for adjacent seeds agrees with the monodromy of signed mutations.

More precisely, take t as the initial seed and assume that $t' = \mu_{k,+} t$ for some unfrozen vertex k. Note that $b'_{ik} = -b_{ik}$ for any $i \in I$. Direct computations show that, for $i \neq k$,

$$\begin{aligned}
\psi_{t,t'}\psi_{t',t}(f_i) &= f_i, \\
\psi_{t,t'}\psi_{t',t}(f_k) &= \psi_{t,t'}\Big(-f_k' + \sum_i [-b'_{ik}]_+ f_i'\Big) \\
&= f_k - \sum_j [-b_{jk}]_+ f_j + \sum_i [-b'_{ik}]_+ f_i' \\
&= f_k + \sum_i b_{ik} f_i.
\end{aligned} \tag{3.1}$$

On the other hand, if we apply signed mutations $\mu_{k,+}$ twice on the initial seed t, we obtain a seed $t'' = \mu_{k,+}\mu_{k,+}(t)$. Let us compute $\tau_{k,+}\tau_{k,+} : M^\circ(t'') \simeq M^\circ(t)$. For $i \neq k$, we have

$$\begin{aligned}
\tau_{k,+}\tau_{k,+}(f_i'') &= f_i, \\
\tau_{k,+}\tau_{k,+}(f_k'') &= \tau_{k,+}\Big(-f_k' + \sum_i [-b'_{ik}]_+ f_i'\Big) \\
&= f_k - \sum_j [-b_{jk}]_+ f_j + \sum_i [-b'_{ik}]_+ f_i' \\
&= f_k + \sum_i b_{ik} f_i.
\end{aligned} \tag{3.2}$$

We deduce that $\psi_{t,t'}\psi_{t',t} = \tau_{k,+}\tau_{k,+}$ if we identify $f_i'' = f_i$ for any i.

Note that the signed mutation monodromy $\mu_{k,+}\mu_{k,+}$ was discussed in [40, Remark 2.5].

Lemma 3.3.5. *$\psi_{t',t}$ is bijective.*

Proof. Identify $M^\circ(t') \simeq \mathbb{Z}^I$ so that f_i' is viewed as the i-th unit vector. Let $\mathrm{pr}_{I_{\mathrm{uf}}}$ denote the natural projection from $\mathbb{Z}^I$ to $\mathbb{Z}^{I_{\mathrm{uf}}}$.

Denote $g_k(t;t') = \phi_{t',t} f_k(t) = \psi_{t',t} f_k(t) \in M^\circ(t')$. It follows from Corollary 2.2.3 that the principal g-vectors $\mathrm{pr}_{I_{\mathrm{uf}}} g_k(t;t')$ with respect to the initial seed t', $k \in I_{\mathrm{uf}}$, form a basis of $\mathbb{Z}^{I_{\mathrm{uf}}}$. Note that $g_j(t;t') = f_j'$ for any frozen vertex j. It follows that $g_i(t;t')$, $i \in I$, is a basis for $\mathbb{Z}^I$. In particular, the linear map $\psi_{t',t}$ is bijective. ∎

Notice that we have two inclusions $\mathcal{LP}(t) \subset \mathcal{F}(t)$ and $\mathcal{LP}(t) \subset \widehat{\mathcal{LP}(t)}$. On the one hand, the mutation map $\overleftarrow{\mu}^*$ is an isomorphism from the rational function field $\mathcal{F}(t')$ to $\mathcal{F}(t)$. On the other hand, we have $\overleftarrow{\mu}^*(\Bbbk[x_i(t')]_i) \subset \mathcal{LP}(t) \subset \widehat{\mathcal{LP}(t)}$. In addition, $\overleftarrow{\mu}^*(x_i(t'))$, for each i, is a pointed Laurent polynomial in $\mathcal{LP}(t)$, which is invertible in $\widehat{\mathcal{LP}(t)}$ by Lemma 3.2.5. Consequently, the mutation map $\overleftarrow{\mu}^*$ induces an algebraic homomorphism $\iota : \mathcal{LP}(t') \to \widehat{\mathcal{LP}(t)}$.

Our next observation shows that the linear map $\psi_{t',t}$ tracks the degree of a Laurent monomial under change of seeds.

Lemma 3.3.6. *For any $t' = \overleftarrow{\mu}t$, $g' \in M^\circ(t')$ and $z = x(t')^{g'} \in \mathcal{LP}(t')$, we have $\iota(z) \in \widehat{\mathcal{PT}}^t(\psi_{t,t'}g')$.*

Proof. Notice that the map ι identifies $x_i(t')$ with a pointed Laurent polynomial in $\mathcal{PT}^t(\deg^t x_i(t'))$. Then Lemma 3.2.5 implies $\iota x_i(t')^{-1} \in \widehat{\mathcal{PT}}^t(-\deg^t x_i(t'))$. We obtain the claim by taking the product of these pointed formal series (Lemma 3.2.6). ∎

Lemma 3.3.7. (1) *The map ι is an embedding.*

(2) *If $z \in \mathcal{LP}(t') \cap (\overleftarrow{\mu}^*)^{-1}\mathcal{LP}(t)$, then $\iota(z) = \overleftarrow{\mu}^*(z) \in \mathcal{LP}(t)$.*

Proof. (1) For any Laurent polynomial $0 \neq z = \sum_{g' \in M^\circ(t')} b_{g'} x(t')^{g'} \in \mathcal{LP}(t)$, $b_{g'} \in \Bbbk$, the image $\iota(x(t')^{g'}) \in \widehat{\mathcal{LP}(t)}$ is pointed at degree $\psi_{t,t'}g'$. Since $\psi_{t,t'}$ is bijective, the image $\iota(z)$ is a finite sum of pointed elements with distinct leading degrees. In particular, $\iota(z) \neq 0$.

(2) Take any $z = (x')^{-d} * F$ for some $F \in \Bbbk[M^\circ(t')]$, $d \in \mathbb{N}^I$. On the one hand, we have $\iota((x')^d) * \iota(z) = \iota(F)$ in $\widehat{\mathcal{LP}(t)}$. On the other hand, we have $\overleftarrow{\mu}^*((x')^d) * \overleftarrow{\mu}^*(z) = \overleftarrow{\mu}^*(F)$ in $\mathcal{LP}(t)$. By definition of ι, we have $\iota((x')^d) = \overleftarrow{\mu}^*((x')^d)$ and $\iota(F) = \overleftarrow{\mu}^*(F)$ in $\mathcal{LP}(t)$. The claim follows. ∎

Using this embedding, we can identify any Laurent polynomial $z \in \mathcal{LP}(t')$ as a formal Laurent series $\overleftarrow{\mu}^*(z) := \iota(z)$ in $\widehat{\mathcal{LP}(t)}$, called the *formal Laurent series expansion* of z with respect to the seed t, or (formal) Laurent expansion for short.

Remark 3.3.8 (Different expansion using codegrees). Notice that the Laurent polynomials $\overleftarrow{\mu}^*(x_i(t))$, $i \in I$, are copointed (Proposition 3.4.13). Then we can construct a similar embedding ι' from $\mathcal{LP}(t')$ to $\widehat{\mathcal{LP}(t)}$ as a different formal series expansion.

Definition 3.3.9 (Tropical points as degrees). Given a formal Laurent series $z \in \widehat{\mathcal{LP}(t_0)}$ with degree $g \in M^\circ(t_0)$ such that, for any seeds $t_0 = \overleftarrow{\mu}t$, $\overleftarrow{\mu}^* z$ is a well defined formal Laurent series in $\widehat{\mathcal{LP}(t)}$ with degree $\deg^t \overleftarrow{\mu}^* z = \phi_{t,t_0} g \in M^\circ(t)$. Then we say z has *degree* $[g] \in \mathcal{M}^\circ$.

As before, denote $y_k(t) = y(t)^{e_k} = x(t)^{\sum_i b_{ik}(t) f_i}$, $k \in I_{\mathrm{uf}}$, where e_k is the k-th unit vector in $N_{\mathrm{uf}}(t) \simeq \mathbb{Z}^{I_{\mathrm{uf}}}$ and f_i the i-th unit vector in $M^\circ(t) \simeq \mathbb{Z}^I$. Apparently, $y_k(t)$ is a pointed Laurent polynomial in $\mathcal{LP}(t)$ and we have $\deg^t y_k(t) = \widetilde{B}(t) \cdot e_k = \sum_{i \in I} b_{ik}(t) f_i$. It follows that for any $n \in N_{\mathrm{uf}}(t)$, we have $\deg^t(y(t)^n) = \widetilde{B}(t) \cdot n$.

The next result shows how c-vectors appear when one calculates the degree of y-variables. This result is known for skew-symmetric seeds via the cluster category approach [49,51,63].

Proposition 3.3.10 ([27, Proposition 3.13]). *For any seeds $t' = \overleftarrow{\mu} t$ and any $k \in I_{\mathrm{uf}}$, we have $\deg^t \overleftarrow{\mu}^* y_k(t') = \deg^t (y(t)^{c_k^t(t')}) = \widetilde{B}(t) \cdot c_k^t(t')$, where $c_k^t(t')$ is the k-th c-vector of the seed t' with respect to the initial seed t.*

Proof. We use the description of c-vectors and g-vectors by Theorem 2.2.2. Let the mutation sequence $\overleftarrow{\mu}$ be $\mu_{i_r} \cdots \mu_{i_0}$, and consider the seeds $t_s = \mu_{i_{s-1}} \cdots \mu_{i_0} t_0$ where $t_0 = t$ and $t_{r+1} = t'$. Choose ε_s to be the sign of the k-th c-vector $c_i(t_s)$.

Recall that $\widetilde{B}(t') = E_{i_r,\varepsilon_r}(t_r) \cdots E_{i_0,\varepsilon_0}(t_0) \widetilde{B}(t_0) F_{i_0,\varepsilon_0}(t_0) \cdots F_{i_r,\varepsilon_r}(t_r)$. Starting with the product $\overleftarrow{\mu}^* y_k(t')$, we have

$$\begin{aligned}
\deg^t \overleftarrow{\mu}^* y_k(t') &= \sum_i \deg^t \overleftarrow{\mu}^* x_i(t') \cdot b_{ik}(t') = \widetilde{G}(t') \cdot \widetilde{B}(t') \cdot e_k \\
&= E_{i_0,\varepsilon_0}(t_0) \cdots E_{i_r,\varepsilon_r}(t_r) \cdot \big(E_{i_r,\varepsilon_r}(t_r) \cdots E_{i_0,\varepsilon_0}(t_0) \widetilde{B}(t_0) F_{i_0,\varepsilon_0}(t_0) \cdots F_{i_r,\varepsilon_r}(t_r)\big) \cdot e_k \\
&= \widetilde{B}(t_0) \cdot F_{i_0,\varepsilon_0}(t_0) \cdots F_{i_r,\varepsilon_r}(t_r) \cdot e_k = \widetilde{B}(t_0) \cdot C^{t_0}(t_{r+1}) \cdot e_k = \widetilde{B}(t) \cdot c_k^t(t'). \qquad \blacksquare
\end{aligned}$$

Assume the cluster algebra is injective-reachable. Then for any seed t we have seeds $t[1]$ and $t[-1]$ constructed from t by mutation sequences. The following crucial result tells us that the linear map $\psi_{t[-1],t}$ reverses the dominance order in t and $t[-1]$.

Proposition 3.3.11 (order reverse). *Let $t = \overleftarrow{\mu} t[-1]$ be an injective-reachable seed such that $C^{t[-1]}(t) = -\mathrm{P}_\sigma$ for some permutation σ of I_{uf}. Let $\eta, g \in M^\circ(t)$. Then $\eta = g + \widetilde{B}(t) \cdot n$ for some $n \in N_{\mathrm{uf}}(t)$ if and only if $\eta' = g' + \widetilde{B}(t[-1]) \cdot (-\mathrm{P}_\sigma \cdot n)$ where $\eta' = \psi_{t[-1],t} \eta$ and $g' = \psi_{t[-1],t} g$. In particular, $\eta \preceq_t g$ if and only if $\psi_{t[-1],t} \eta \succeq_{t[-1]} \psi_{t[-1],t} g$.*

Proof. Notice that $\psi_{t[-1],t}$ is a bijective linear map from $M^\circ(t)$ to $M^\circ(t[-1])$ by Lemma 3.3.5. The claim is equivalent to $\psi_{t[-1],t}(\widetilde{B}(t) \cdot n) = \widetilde{B}(t[-1]) \cdot (-\mathrm{P}_\sigma \cdot n)$. Also, recall that $\deg^t(y(t)^n) = \widetilde{B}(t) \cdot n$.

Applying the linear map $\psi_{t[-1],t} : M^\circ(t) \to M^\circ(t[-1])$ and using Lemma 3.3.6 and Proposition 3.3.10, we obtain

$$\begin{aligned}
\psi_{t[-1],t}(\widetilde{B}(t) \cdot n) = \psi_{t[-1],t} \deg^t(y(t)^n) &= \sum_k \psi_{t[-1],t} \deg^t(y_k(t)) \cdot n_k \\
\text{(Lemma 3.3.6)} \quad &= \sum_k \deg^{t[-1]} \overleftarrow{\mu}^*(y_k(t)) \cdot n_k \\
\text{(Proposition 3.3.10)} \quad &= \sum_k \deg^{t[-1]} y(t[-1])^{c_k^{t[-1]}(t)} \cdot n_k \\
&= \deg^{t[-1]} y(t[-1])^{C^{t[-1]}(t) \cdot n} = \widetilde{B}(t[-1]) \cdot (-\mathrm{P}_\sigma \cdot n). \qquad \blacksquare
\end{aligned}$$

We have the following consequence which tells us that the degree and codegree in t and $t[-1]$ swap.

Proposition 3.3.12 (Degree/codegree swap). *Let $t = \overleftarrow{\mu} t[-1]$ be an injective-reachable seed and let $z \in \mathcal{LP}(t)$ be such that $\overleftarrow{\mu}^* z \in \mathcal{LP}(t[-1])$. Then z is copointed in $\mathcal{LP}(t)$ at codegree $\mathsf{codeg}^t z = \eta$ if and only if $\overleftarrow{\mu}^* z$ is pointed in $\mathcal{LP}(t[-1])$ at degree $\mathsf{deg}^{t[-1]}(\overleftarrow{\mu}^* z) = \psi_{t[-1],t}\eta$.*

Proof. Let $z = \sum_{m \in M^\circ(t)} b_m x(t)^m$ be the Laurent expansion of z in $\mathcal{LP}(t)$, where only finitely many coefficients b_m are non-zero. Taking the formal Laurent expansion in $\widehat{\mathcal{LP}(t[-1])}$, we obtain $\overleftarrow{\mu}^* z = \sum_{m \in M^\circ(t)} \overleftarrow{\mu}^*(b_m x(t)^m)$.

Each formal Laurent series $\overleftarrow{\mu}^*(x(t)^m)$ in $\widehat{\mathcal{LP}(t[-1])}$ has degree $\psi_{t[-1],t} m$ by Lemma 3.3.6. On the one hand, z is copointed at η if and only if $\{m \mid b_m \neq 0\}$ has a unique $\prec_t$-minimal element η and $b_\eta = 1$. On the other hand, $\overleftarrow{\mu}^* z$ is pointed at some degree g if and only if $\{\psi_{t[-1],t} m \mid b_m \neq 0\}$ has a unique $\prec_{t[-1]}$-maximal element $g = \psi_{t[-1],t}\eta$ and $b_\eta = 1$. Because $\psi_{t[-1],t}$ reverses the order $\preceq_t$ and $\preceq_{t[-1]}$ by Proposition 3.3.11, these two conditions are equivalent. ∎

3.4. Support of bipointed Laurent polynomials

Definition 3.4.1 (Support). The *support* of any $n = \sum n_k e_k \in N_{\mathsf{uf}}(t)$ is defined to be the set $\mathsf{supp}\, n = \{i \in I_{\mathsf{uf}} \mid n_i \neq 0\}$.

For any Laurent polynomial $z \in \mathcal{LP}(t)$ with bidegree (η, g), its *support dimension* $\mathsf{suppDim}^t z$ is defined to be the unique element $n \in N_{\mathsf{uf}}{}^{\geq 0}(t)$ such that $\eta = g + p^* n$. We define its *support* to be $\mathsf{supp}^t z = \mathsf{supp}(n)$.

Recall that, for any seeds $t' = \overleftarrow{\mu} t$, the mutation map $\overleftarrow{\mu}^*$ identifies $\mathcal{F}(t')$ and $\mathcal{F}(t)$, and $\mathcal{LP}(t') \cap \mathcal{LP}(t)$ denotes $\mathcal{LP}(t') \cap (\overleftarrow{\mu}^*)^{-1} \mathcal{LP}(t)$.

Definition 3.4.2. Let S be any given set of seeds connected by mutations. A Laurent polynomial $z \in \bigcap_{t_i \in S} \mathcal{LP}(t_i)$ is said to be *compatibly pointed at the seeds in* S if we have $z \in \bigcap_{t \in S} \mathcal{PT}^t(g(t))$ for some degrees $g(t) \in M^\circ(t)$ such that $g(t') = \phi_{t',t} g(t)$ for all $t, t' \in S$.

Similarly, given any formal Laurent series $z \in \widehat{\mathcal{LP}(t_0)}$, $t_0 \in S$, such that its formal Laurent expansion in $\widehat{\mathcal{LP}(t)}$ is well defined for all $t \in S$ (NOT always true). We can say z is compatibly pointed at the seeds in S if z is pointed at degrees $g(t) \in M^\circ(t)$ in $\widehat{\mathcal{LP}(t)}$ such that $g(t') = \phi_{t',t} g(t)$ for all $t, t' \in S$.

Example 3.4.3. Let us give an example of an element z in the upper cluster algebra which is NOT compatibly pointed at all seeds.

Consider the classical case $\Bbbk = \mathbb{Z}$. Take a type A_2 cluster algebra whose initial seed t consists of the initial cluster variables x_1, x_2 and the initial B-matrix $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Denote $y_1 = x_2$ and $y_2 = x_1^{-1}$.

Applying the mutation μ_1 to t, we obtain a new seed $t' = \mu_1 t$ with new variables $x_1' = x_1^{-1}(1 + y_1) = x_1^{-1}(1 + x_2)$ and $x_2' = x_2$, $B' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $y_1' = (x_2')^{-1} = y_1^{-1}$, $y_2' = x_1'$.

Define $z := x_1 \cdot x_1' = 1 + x_2 = 1 + y_1$. Then z lies in the upper cluster algebra. It is 0-pointed at the seed t_0, but its leading term comes from the contribution of $x_2 = x_2'$ in the seed t'. In particular, it is not 0-pointed at the seed t', i.e. not compatibly pointed at the seeds $\{t, t'\}$.

Next, we define the "correct" support dimension for bipointed Laurent polynomials, as we shall show in Proposition 3.4.8.

Definition 3.4.4. Let t be an injective-reachable seed and $g \in M^\circ(t)$. If there exists $n \in N_{\mathsf{uf}}{}^{\geq 0}(t)$ such that

$$\eta = g + \widetilde{B}(t) \cdot n$$

where $\eta = \psi^{-1}_{t[-1],t} \phi_{t[-1],t} g$, we define the *support dimension* associated to g to be

$$\mathsf{suppDim}\, g = n$$

and the *bidegree interval* associated to g to be the following subset of $M^\circ(t)$:

$$\mathrm{BI}_g = {}_{\eta \preceq_t} M^\circ(t)_{\preceq_t g}.$$

Let $[g] \in \mathcal{M}^\circ$ be a tropical point. If for all $t \in \Delta^+$, $g \in M^\circ(t)$ has a support dimension, where $[g] = g$ under the identification $\mathcal{M}^\circ \simeq M^\circ(t)$, then we say that $[g]$ *has support dimensions*.

Notice that the support dimension $\mathsf{suppDim}\, g$ is well defined if and only if $\psi^{-1}_{t[-1],t} \phi_{t[-1],t} g \preceq_t g$. It will turn out that it is always well defined by Proposition 5.1.5 and the existence of generic cluster characters (for skew-symmetric cases) or the existence of theta functions (for skew-symmetrizable cases).

Remark 3.4.5. We claim that $\mathsf{suppDim}\, g$ and BI_g do not depend on the choice of $t[-1]$ up to permutations σ of I_{uf}. To see this, for any permutation σ, we introduce the index relabelling operation σ on the seed t which generates a new seed $\sigma t = ((b_{\sigma i, \sigma j})_{i,j \in I}, (x_{\sigma i}(t)))$. Then σ commutes with $\phi_{t,t'}$, $\psi_{t,t'}$, and induces automorphisms on fraction fields which commute with mutations. The claim follows from direct comparison between different choices of $t[-1]$ via the relabelling σ.

The following result tells us that the subset $\mathcal{M}^\circ{}_{\preceq_{\{t,t[-1]\}}[g]}$ of tropical points could be described by the inclusion of bidegree intervals. Notice that the inclusion gives a natural partial order bounded from below, and it will be crucial when we construct bases later.

Proposition 3.4.6 (Inclusion property). *Let $t = \overleftarrow{\mu} t[-1]$ be an injective-reachable seed and suppose $g, g' \in M^\circ(t)$ have support dimensions.*

(1) *If $g' \prec_t g$ and $\phi_{t[-1],t} g' \prec_{t[-1]} \phi_{t[-1],t} g$, then $\mathrm{BI}_{g'} \subsetneq \mathrm{BI}_g$. The converse is also true.*

(2) *Under the assumption in* (1)*, we have $M^\circ(t)_{\prec_{\{t,t[-1]\}} g} = \{g' \in M^\circ(t) \mid \mathrm{BI}_{g'} \subsetneq \mathrm{BI}_g\}$ for any $g \in M^\circ(t)$. In addition, $M^\circ(t)_{\prec_{\{t,t[-1]\}} g}$ is finite.*

Proof. (1) By Proposition 3.3.11, the condition $\phi_{t[-1],t} g' \prec_{t[-1]} \phi_{t[-1],t} g$ is equivalent to $\psi^{-1}_{t[-1],t} \phi_{t[-1],t} g' \succ_t \psi^{-1}_{t[-1],t} \phi_{t[-1],t} g$. Because g, g' have support dimensions, we have $g \succ \psi^{-1}_{t[-1],t} \phi_{t[-1],t} g$ and $g' \succ_t \psi^{-1}_{t[-1],t} \phi_{t[-1],t} g'$. The claim follows from the definition of the bidegree intervals $\mathrm{BI}_{g'}$, BI_g.

(2) The first claim follows from (1). Noticing that BI_g is finite by Lemma 3.1.2 and $g' \in \mathrm{BI}_g$ for any $g' \in M^\circ(t)_{\prec_{\{t,t[-1]\}} g}$, the second claim follows. ∎

Remark 3.4.7. It might be possible to generalize the notion of support dimensions by removing the restriction $n \in N_{\mathrm{uf}}^{\geq 0}(t)$. It is also an interesting question to write down the mutation rule of these dimensions; see [28] for a formula for the support dimensions for cluster variables (called f-vectors).

The following result gives an equivalence between being bipointed with the "correct" support dimension and being compatibly pointed at $t, t[-1]$.

Proposition 3.4.8 (Compatibility and support dimensions). *Consider seeds $t = \overleftarrow{\mu} t[-1]$ and a pointed Laurent polynomial $z \in \mathcal{PT}^t(g)$, $g \in M^\circ(t)$.*

(1) *If z is compatibly pointed at seeds $t, t[-1]$, then g has a support dimension. Moreover, z is bipointed with $\mathrm{suppDim}\, z = \mathrm{suppDim}\, g$ in this case.*

(2) *If g has a support dimension and $z \in \mathcal{LP}(t)$ is bipointed with $\mathrm{suppDim}\, z = \mathrm{suppDim}\, g$, then z is compatibly pointed at the seeds $t, t[-1]$.*

Proof. (1) By Proposition 3.3.12, we know that $z \in \mathcal{LP}(t)$ is copointed at codegree $\psi^{-1}_{t[-1],t} \deg^{t[-1]} \overleftarrow{\mu}^* z$, which equals $\psi^{-1}_{t[-1],t} \phi_{t[-1],t} g$ because z is compatibly pointed at the seeds $t, t[-1]$. The claims follow.

(2) By definition, z is bipointed at bidegree $(g, \psi^{-1}_{t[-1],t} \phi_{t[-1],t} g)$. By Proposition 3.3.12, we know that $\overleftarrow{\mu}^* z$ is pointed at degree $\phi_{t[-1],t} g$. ∎

Recall that we have the following result which tells us that a finite decomposition of pointed Laurent series is unitriangular.

Lemma 3.4.9 ([71, Lemma 3.1.10 (iii)]). *Consider any finite linear decomposition of pointed formal Laurent series u, z_j in $\widehat{\mathcal{LP}(t)}$, where z_j have distinct degrees:*

$$u = \sum_{0 \leq j \leq r} b_j z_j,$$

with $r \in \mathbb{N}$ and $b_j \in \Bbbk$. Then the decomposition must be $\prec_t$-unitriangular, i.e., we can reindex z_j so that $u = z_0 + \sum_{1 \leq j \leq r} b_j z_j$, with $b_0 = 1$, $\deg^t z_0 = \deg^t u$ and $\deg^t z_j \prec_t \deg^t u$ for all $j \geq 1$.

We have better control of a finite decomposition of Laurent polynomials compatibly pointed at $t, t[-1]$ (or, equivalently, bipointed with correct support dimensions by Proposition 3.4.8).

Proposition 3.4.10 (decomposition). *Consider seeds $t = \overleftarrow{\mu} t[-1]$ and any finite decomposition of pointed Laurent polynomials u, z_j in $\mathcal{LP}(t)$, where z_j have distinct degrees:*

$$u = \sum_{0 \le j \le r} b_j z_j,$$

with $\mathrm{deg}^t z_0 = \mathrm{deg}^t u$ and all coefficients b_j non-zero. Further assume that all u, z_j are compatibly pointed at $t, t[-1]$. Then the following claims are true:

(1) *All u, z_j are bipointed.*

(2) $\mathrm{deg}^t u = \mathrm{deg}^t z_0$ *and* $\mathrm{deg}^t z_0 \succ_t \mathrm{deg}^t z_j$ *for all* $j > 0$.

(3) $\mathrm{codeg}^t u = \mathrm{codeg}^t z_0$ *and* $\mathrm{codeg}^t z_j \succ_t \mathrm{codeg}^t z_0$ *for all* $j > 0$.

(4) $\mathrm{BI}_{\mathrm{deg}^t z_j} \subsetneq \mathrm{BI}_{\mathrm{deg}^t z_0}$ *for all* $j > 0$.

(5) $\mathrm{suppDim}\,\mathrm{deg}^t z_j < \mathrm{suppDim}\,\mathrm{deg}^t z_0$ *for all* $j > 0$ *in* $N_{\mathrm{uf}}^{\ge 0}(t)$.

Proof. (1) Because u, z_j are compatibly bipointed at $t, t[-1]$, we can apply Proposition 3.4.8. As consequences, $\mathrm{deg}^t u$ has $\mathrm{suppDim}\, u = \mathrm{suppDim}\,\mathrm{deg}^t u$, u is bipointed at bidegree $(\mathrm{deg}^t u, \psi^{-1}_{t[-1],t}\phi_{t[-1],t}\,\mathrm{deg}^t u)$, all $\mathrm{deg}^t z_j$ have $\mathrm{suppDim}\, z_j = \mathrm{suppDim}\,\mathrm{deg}^t z_j$, and all z_j are bipointed at bidegree $(\mathrm{deg}^t z_j, \psi^{-1}_{t[-1],t}\phi_{t[-1],t}\,\mathrm{deg}^t z_j)$.

(2) This claim follows from Lemma 3.4.9.

(3) As $\mathrm{deg}^t u = \mathrm{deg}^t z_0$, u and z_0 must have the same codegree $\psi^{-1}_{t[-1],t}\phi_{t[-1],t}\,\mathrm{deg}^t u = \psi^{-1}_{t[-1],t}\phi_{t[-1],t}\,\mathrm{deg}^t z_0$. Because $u = \sum b_j z_j$ is a finite decomposition, the $\prec_t$-minimal Laurent degree $\mathrm{codeg}^t u$ of u must be the $\prec_t$-minimal element of $\{\mathrm{codeg}^t z_j, \forall j\}$. Therefore, $\mathrm{codeg}^t z_j \succ_t \mathrm{codeg}^t z_0$ for all $j > 0$.

(4) The claim follows from (2)–(3).

(5) By (4), for any $j > 0$, we have

$$\mathrm{deg}^t z_0 \succ_t \mathrm{deg}^t z_j \succeq_t \mathrm{codeg}^t z_j \succ_t \mathrm{codeg}^t z_0.$$

Therefore, there exist $n_1, n_2, n_3 \in N_{\mathrm{uf}}^{\ge 0}(t)$ with $n_1, n_3 \ne 0$ such that

$$\begin{aligned} \mathrm{deg}^t z_j &= \mathrm{deg}^t z_0 + \widetilde{B}(t) n_1, \\ \mathrm{codeg}^t z_j &= \mathrm{deg}^t z_j + \widetilde{B}(t) n_2, \\ \mathrm{codeg}^t z_0 &= \mathrm{codeg}^t z_j + \widetilde{B}(t) n_3. \end{aligned}$$

We obtain $\mathrm{suppDim}\, z_j = n_2 < n_1 + n_2 + n_3 = \mathrm{suppDim}\, z_0$. ■

Conversely, by slightly changing the statement in Proposition 3.4.10, we describe a finite sum of pointed Laurent polynomials with well controlled bidegrees.

Proposition 3.4.11 (combination). *Consider seeds $t = \overleftarrow{\mu} t[-1]$ and any Laurent polynomials u, z_j in $\mathcal{LP}(t)$ such that we have*

$$u = \sum_{0 \le j \le r} b_j z_j$$

with coefficients $b_j \neq 0$. Further assume that all z_j are compatibly pointed at $t, t[-1]$ and their bidegrees satisfy $\mathrm{BI}_{\mathsf{deg}^t z_j} \subsetneq \mathrm{BI}_{\mathsf{deg}^t z_0}$ for all $j > 0$. Then u is compatibly pointed at t, $t[-1]$, bipointed at $\mathcal{LP}(t)$ with bidegree $(\mathsf{codeg}^t z_0, \mathsf{deg}^t z_0)$, and has support dimension $\mathsf{suppDim}\, u = \mathsf{suppDim}\, z_0$.

Proof. By the inclusion assumption on bidegrees of z_j, u must be bipointed at bidegree $(\mathsf{codeg}^t z_0, \mathsf{deg}^t z_0)$ with $\mathsf{suppDim}\, u = \mathsf{suppDim}\, z_0$. Because z_0 is compatibly pointed at $t, t[-1]$, $\mathsf{deg}^t z_0$ has $\mathsf{suppDim}\, \mathsf{deg}^t z_0 = \mathsf{suppDim}\, z_0$ by Proposition 3.4.8. Consequently, u is compatibly pointed at the seeds $t, t[-1]$ by Proposition 3.4.8 (2). ∎

Finally, we discuss properties of localized cluster monomials. Consider seeds $t' = \overleftarrow{\mu} t$ and a localized cluster monomial $x(t')^d$ where $d \in \mathbb{N}^{I_{\mathrm{uf}}} \oplus \mathbb{Z}^{I_{\mathrm{f}}}$. Recall that its Laurent expansion in $\mathcal{LP}(t)$ is computed as $\overleftarrow{\mu}^* x(t')^d$.

Lemma 3.4.12. *If any $z \in \mathcal{LP}(t)$ has degree $\mathsf{deg}^t z = \mathsf{deg}^t \overleftarrow{\mu}^* x(t')^d$ and is compatibly pointed at $\{t, t', t'[-1]\}$, then $z = \overleftarrow{\mu}^* x(t')^d$.*

Proof. We have $\mathsf{deg}^{t'} (\overleftarrow{\mu}^{-1})^* z = \phi_{t',t}\, \mathsf{deg}^t z = \phi_{t',t}\, \mathsf{deg}^t \overleftarrow{\mu}^* x(t')^d = d$. Therefore, $(\overleftarrow{\mu}^{-1})^* z$ and $x(t')^d$ have the same degree in $\mathcal{LP}(t')$. Because they are compatibly pointed at $\{t', t'[-1]\}$, by Proposition 3.4.8, they have the same support dimension, which is given by $\mathsf{suppDim}\, x(t')^d = 0$. Consequently, $(\overleftarrow{\mu}^{-1})^* z = x(t')^d$. ∎

It is natural to ask if we can extend the above property without the injective-reachability assumption.

The following property is known without this assumption.

Proposition 3.4.13 ([27, Proposition 5.3]). *For every initial seed t_0, the Laurent expansion $\overleftarrow{\mu}^* x_i(t')^d \in \mathcal{LP}(t)$ is bipointed.*

4. Properties of $\prec_t$-decompositions

4.1. $\prec_t$-decompositions

Consider a seed $t = ((b_{ij})_{i,j \in I}, (x_i)_{i \in I})$ and a collection $\mathcal{S} = \{s_g \mid g \in M^\circ(t)\} \subset \widehat{\mathcal{LP}(t)}$ such that s_g is pointed at g. By definition, any $z = \sum_{g \in M^\circ(t)} b_g x^g \in \widehat{\mathcal{LP}(t)}$ has finitely many $\prec_t$-maximal Laurent degrees. Similar to [71, Lemma 3.1.10 (i), Remark 3.1.8], we can decompose z in terms of the pointed elements in S inductively via the partial order $\prec_t$.

Definition-Lemma 4.1.1 (Dominance order decomposition). There exists a unique decomposition

$$z = \sum_{g \in M^\circ(t)} \alpha_t(z)(g) \cdot s_g, \quad \alpha_t(z) \in \mathsf{Hom}_{\mathrm{set}}(M^\circ(t), \Bbbk), \tag{4.1}$$

in $\widehat{\mathcal{LP}(t)}$ for some coefficient function $\alpha_t(z)$ such that the support $\mathsf{supp}(\alpha_t(z)) := \{g \mid \alpha_t(z)(g) \neq 0\}$ has finitely many $\prec_t$-maximal elements. We call it the *$\prec_t$-decomposition* of z into elements of $\mathcal{S}$.

Proof. Let $g^{(j)}$, $1 \le j \le l$, $0 \ne j \in \mathbb{N}$, denote the $\prec_t$-maximal Laurent degrees of z. If (4.1) holds, by comparing the Laurent monomials with $\prec_t$-maximal degrees on both sides, we deduce that the $\prec_t$-maximal elements of $\mathsf{supp}(\alpha_t(z))$ are exactly $g^{(j)}$, $1 \le j \le l$, and their coefficients must be $\alpha_t(z)(g^{(j)}) = b_{g^{(j)}}$.

Let us draw a directed graph G with vertices $\bigcup_{1\le j\le l} M^\circ(t)_{\preceq_t g^{(j)}}$ and, whenever $g' = g + \widetilde{B} \cdot e_k$ for some $k \in I_{\mathrm{uf}}$, we draw an arrow from g to g'. Then there is a (probably length 0) path from g to g' if and only if $g' \preceq_t g$.

Notice that the source points of G are the leading degrees $g^{(j)}$. Moreover, for any vertex g', there exist finitely many vertices g in G such that $g' \preceq_t g$ by the Finite Interval Lemma 3.1.2. Then the decomposition coefficients for general vertices $g \in G$ are inductively determined by travelling further away from the source points [71, Remark 3.1.8]. ∎

4.2. *Change of seeds*

We want to prove the desired property that the $\prec_t$-decomposition is independent of the seed t provided $\mathcal{S}$ satisfies some tropical properties. We learned from the inspirational paper [41, Section 6] how to give a proof based on the nilpotent Nakayama Lemma.

The idea of the proof is straightforward for the principal coefficient cases in the sense of [27]. Endow such (partially compactified) cluster algebras with natural adic topologies. Then the nilpotent Nakayama Lemma provides a method to verify that a given collection of elements is a basis. Our proof looks more technical because it treats general cases, and we need to modify the calculation for the principal coefficient cases in the spirit of the correction technique ([70, Section 9] or [71, Section 4]).

Let $k \in I_{\mathrm{uf}}$. We denote the mutated seed $t' = \mu_k t = ((b'_{ij}), (x'_i))$. Recall that we have the tropical transformation $\phi = \phi_{t',t} : M^\circ(t) \simeq M^\circ(t')$. For any $g \in M^\circ(t)$, denote $g' = \phi_{t',t}\, g$ for brevity.

For simplicity, let us assume $z \in \mathcal{LP}(t) \cap \mathcal{LP}(t')$ and $\mathcal{S} \subset \mathcal{LP}(t) \cap \mathcal{LP}(t')$, which is sufficient for this paper. Further, assume that the collection $\mathcal{S} = \{s_g \mid g \in M^\circ(t)\}$ is compatibly pointed at the seeds t, t', i.e., s_g is pointed at g' in $\mathcal{LP}(t')$. Then we have a (possibly infinite) $\prec_{t'}$-decomposition in $\mathcal{LP}(t')$:

$$z = \sum_{g' \in M^\circ(t')} \alpha_{t'}(z)(g') \cdot s_g, \quad \alpha_{t'}(z) \in \mathsf{Hom}_{\mathrm{set}}(M^\circ(t'), \Bbbk). \tag{4.2}$$

The aim of this section is to prove the following result.

Proposition 4.2.1. *We have*

$$\alpha_t(z)(g) = \alpha_{t'}(z)(g') \quad \textit{for all } g \in M^\circ(t).$$

In particular, $\phi\, \mathsf{supp}(\alpha_t(z)) = \mathsf{supp}(\alpha_{t'}(z))$.

Our strategy is to use the nilpotent Nakayama Lemma [59, Theorem 8.4] as in [41], and compare the collection $\mathcal{S}$ with the natural basis of the type A_1 cluster algebra $\mathcal{LP}(t) \cap \mathcal{LP}(t')$ using the tropical properties (Lemma 3.4.12).

Lemma 4.2.2 (Nilpotent Nakayama Lemma). *Let A denote a ring, $\mathbf{m}$ its nilpotent 2-sided ideal such that $\mathbf{m}^r = 0$, and U a left A-module. For any subset S of U, if its image in $U/\mathbf{m}U$ generates $U/\mathbf{m}U$ as an $A/\mathbf{m}$-module, then S generates U as an A-module.*

Proof. We learned the following proof from Matthew Emerton. By assumption, $U = AS + \mathbf{m}U$. Repeating the substitution, we get

$$\begin{aligned} U &= AS + \mathbf{m}(AS + \mathbf{m}U) \\ &= AS + \mathbf{m}(AS + \mathbf{m}(AS + \mathbf{m}U)) \\ &= \cdots \\ &= AS + \mathbf{m}S + \mathbf{m}^2 S + \cdots + \mathbf{m}^{r-1} S + \mathbf{m}^r U \\ &= AS + \mathbf{m}S + \mathbf{m}^2 S + \cdots + \mathbf{m}^{r-1} S. \end{aligned}$$

The claim follows. ∎

To apply the Nakayama Lemma, we want to work with the $\mathbf{m}$-adic topology where the ideal $\mathbf{m}$ is generated by the y-variables. Correspondingly, it is convenient to add extra principal framing frozen vertices $I' = \{i' \mid i \neq k,\ i \in I_{\mathrm{uf}}\}$, extending the vertex set I to $\widetilde{I} = I \sqcup I'$. Extend the matrix $(b_{ij})_{i,j\in I}$ to $(b_{ij})_{i,j\in\widetilde{I}}$ such that, for $i \neq k$, $i \in I_{\mathrm{uf}}$,

$$b_{i',i} = 1, \quad b_{i,i'} = -1,$$

and other entries are extended by zero. We obtain the principal framing seed $t^{\mathrm{prin}} = ((b_{ij})_{i,j\in\widetilde{I}}, (x_i)_{i\in\widetilde{I}})$, which is said to have (a modified version of) the *principal coefficients* in the sense of [27]. Then its mutated seed $(t^{\mathrm{prin}})' := \mu_k(t^{\mathrm{prin}})$ agrees with the principal framing $(t')^{\mathrm{prin}}$ of t'.

When working with the quantum case $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$, we extend the compatible bilinear form λ on $M^\circ(t)$ to $M^\circ(t^{\mathrm{prin}})$ by zero. The resulting bilinear form on $M^\circ(t^{\mathrm{prin}})$, still denoted by λ, is compatible with t^{prin}.

We have the natural embedding $M^\circ(t) \simeq M^\circ(t) \oplus 0 \subset M^\circ(t^{\mathrm{prin}})$. Conversely, for any $\widetilde{g}$ from the extended degree lattice $M^\circ(t^{\mathrm{prin}})$, denote its projection to $M^\circ(t)$ by g. Denote $\phi_{(t')^{\mathrm{prin}},t^{\mathrm{prin}}}\widetilde{g} = \widetilde{g}'$.

Notice that the y-variables in t^{prin} and t satisfy

$$y_i(t^{\mathrm{prin}}) = \begin{cases} x_{i'} \cdot y_i, & i \neq k \in I_{\mathrm{uf}}, \\ y_k, & i = k, \end{cases}$$

and the same formula holds for $(t')^{\mathrm{prin}}$ and t'. Define the grading $\mathrm{gr}(\ )$ on $M^\circ(t^{\mathrm{prin}})$ such that

$$\mathrm{gr}(f_i) = \begin{cases} 1, & i \in I', \\ 0, & i \notin I', \end{cases}$$

and similarly

$$\mathrm{gr}'(f_i') = \begin{cases} 1, & i \in I', \\ 0, & i \notin I', \end{cases}$$

on $M^\circ((t')^{\mathrm{prin}})$. Then $\phi : M^\circ(t^{\mathrm{prin}}) \simeq M^\circ((t')^{\mathrm{prin}})$ is homogeneous, i.e., $\mathrm{gr}(\widetilde{g}) = \mathrm{gr}'(\widetilde{g}')$. We have the following observation.

Lemma 4.2.3. *If $\widetilde{\eta} = \widetilde{g} + \widetilde{B} \cdot n$ in $M^\circ(t^{\mathrm{prin}})$ for some $n \in N_{\mathrm{uf}}^{\geq 0}(t^{\mathrm{prin}})$, then $\mathrm{gr}(\widetilde{\eta}) \geq \mathrm{gr}(\widetilde{g})$. Moreover, $\mathrm{gr}(\widetilde{\eta}) > \mathrm{gr}(\widetilde{g})$ if and only if $n_i > 0$ for some $i \neq k$, $i \in I_{\mathrm{uf}}$.*

We have an induced grading gr on $\mathcal{LP}(t^{\mathrm{prin}})$ such that $\mathrm{gr}(x_i) := \mathrm{gr}(f_i)$ and similarly gr' on $\mathcal{LP}((t')^{\mathrm{prin}})$.

The intersection $\mathcal{U}_k := \mathcal{LP}(t^{\mathrm{prin}}) \cap \mathcal{LP}((t')^{\mathrm{prin}})$ is the (type A_1) upper cluster algebra obtained from the initial seed t^{prin} such that k is the only unfrozen vertex. It is well known that it has the basis $\{m_{\widetilde{g}} \mid \widetilde{g} \in M^\circ(t^{\mathrm{prin}})\}$ where $m_{\widetilde{g}}$ are its localized cluster monomials with degree $\widetilde{g}$. Recall that, for the classical case $\mathbb{k} = \mathbb{Z}$, $m_{\widetilde{g}} = x^{\widetilde{g}}(1 + y_k)^{[-g_k]_+}$ for this type A_1 upper cluster algebra (see Section 2.5 for the quantum case $\mathbb{k} = \mathbb{Z}[q^{\pm 1/2}]$). In particular, $m_{\widetilde{g}}$ has homogeneous grading $\mathrm{gr}(\widetilde{g})$ in $\mathcal{LP}(t^{\mathrm{prin}})$. Similarly, $m_{\widetilde{g}}$ has homogeneous grading $\mathrm{gr}'(\widetilde{g}') = \mathrm{gr}(\widetilde{g})$ in $\mathcal{LP}((t')^{\mathrm{prin}})$. Therefore, the two gradings in $\mathcal{LP}(t^{\mathrm{prin}})$ and $\mathcal{LP}((t')^{\mathrm{prin}})$ give the same grading on the algebra $\mathcal{U}_k$.

Lemma 4.2.4. *Let $z \in \mathcal{LP}((t')^{\mathrm{prin}})$ and decompose $z = \sum z_i$ into homogeneous parts $z_i \in \mathcal{LP}((t')^{\mathrm{prin}})$ of different gradings. Then $\mu_k^* z \in \mathcal{LP}(t^{\mathrm{prin}})$ if and only if all $\mu_k^* z_i$ are in $\mathcal{LP}(t^{\mathrm{prin}})$.*

Proof. If $z \in \mathcal{LP}(t^{\mathrm{prin}}) \cap \mathcal{LP}((t')^{\mathrm{prin}})$, then we can decompose it into a finite sum $z = \sum \alpha_{\widetilde{g}} m_{\widetilde{g}}$. Since $m_{\widetilde{g}}$ are homogeneous, we find that $z_i = \sum_{\mathrm{gr}(\widetilde{g})=\mathrm{gr}(z_i)} \alpha_{\widetilde{g}} m_{\widetilde{g}}$. In particular, $z_i \in \mathcal{LP}(t^{\mathrm{prin}}) \cap \mathcal{LP}((t')^{\mathrm{prin}})$. The converse statement is trivial. ■

Take any $\widetilde{g} \in M^\circ(t^{\mathrm{prin}})$. Since $s_g \in \mathcal{S}$ is pointed at g, it takes the form $s_g = x^g \cdot F_g((y_i)_{i \in I_{\mathrm{uf}}})$ where $F_g(\ )$ is a multivariate polynomial with constant 1 and we use the commutative product. Correspondingly, define $s_{\widetilde{g}} := x^{\widetilde{g}} \cdot F_g((y_i(t^{\mathrm{prin}}))_{i \in I_{\mathrm{uf}}})$ and $\widetilde{\mathcal{S}} := \{s_{\widetilde{g}} \mid \widetilde{g} \in M^\circ(t^{\mathrm{prin}})\}$. Note that $s_{\widetilde{g}}$ belongs to $\mathcal{LP}(t^{\mathrm{prin}})$ and to $\mathcal{LP}((t')^{\mathrm{prin}})$ [70, Theorem 9.2].

Lemma 4.2.5. *If $\widetilde{g}' = \widetilde{g} + \widetilde{B}(t^{\mathrm{prin}}) \cdot n$ in $M^\circ(t^{\mathrm{prin}})$ for some $0 \neq n \in \mathbb{N}^{I_{\mathrm{uf}}}$, then $g' = g + \widetilde{B}(t) \cdot n$.*

Proof. The claim follows by taking the projection $M^\circ(t^{\mathrm{prin}}) \to M^\circ(t)$. ■

Lemma 4.2.6. *$s_{\widetilde{g}}$ is compatibly pointed at the $\widetilde{g}$ and $\widetilde{g}'$ at seeds t^{prin} and $(t')^{\mathrm{prin}}$ respectively.*

Proof. (i) Denote $\psi = \psi_{t',t}$, $\widetilde{\psi} = \psi_{(t')^{\mathrm{prin}},t^{\mathrm{prin}}}$ for simplicity. By Lemma 3.3.2, we have $\psi g - g' = (\widetilde{B}') \cdot [-g_k]_+ e'_k$, where e'_k denote the k-th unit vector in $N_{\mathrm{uf}}(t')$. Similarly, we have $\widetilde{\psi}\widetilde{g} - \widetilde{g}' = \widetilde{B}((t')^{\mathrm{prin}})) \cdot [-g_k]_+ e'_k$.

(ii) Let μ_k^* denote the mutation map from $\mathcal{LP}(t)$ to $\mathcal{LP}(t')$. For any $Z = \sum c_n y^n \in \widehat{\mathbb{k}[N_{\mathrm{uf}}^{\geq 0}(t)]}$, we denote its evaluation $Z|_{y^n = x^{\widetilde{B}n}}$ by $Z(x^{\widetilde{B}n})$. Similarly, we denote $Z((x')^{\widetilde{B}'n}) = Z|_{(y')^n=(x')^{\widetilde{B}'n}}$ for $Z \in \widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t')]}$. Note that we have $\widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t)]} = \widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t^{\mathrm{prin}})]}$ and $\widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t')]} = \widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}((t')^{\mathrm{prin}})]}$.

By assumption, s_g is compatibly pointed at t and t'. Then there exist $F \in \widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t)]}$ and $G \in \widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t')]}$ with constant term 1 such that $s_g = x^g * F(x^{\widetilde{B}n})$ and $\mu_k^* s_g = (x')^{g'} * G((x')^{\widetilde{B}'n})$.

Note that $\mu^* F$ and G^{-1} are well defined in $\widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t')]}$. By (2.4) and (2.6), we can write $\mu_k^*(x^g)$ as $(x')^{\psi g} * Q(x^{\widetilde{B}n})$, where $Q \in \widehat{\Bbbk[N_{\mathrm{uf}}^{\geq 0}(t')]}$ is a formal series in y'_k. Moreover, the mutation rules for $x^{\widetilde{B}n}$ and y^n are the same. We deduce that $\mu_k^*(F(x^{\widetilde{B}n})) = (\mu_k^* F)((x')^{\widetilde{B}'n})$. Then

$$\mu_k^*(s_g) = (x')^{\psi g} * Q((x')^{\widetilde{B}'n}) * \mu_k^*(F)((x')^{\widetilde{B}'n}) = (x')^{g'} * G(x^{\widetilde{B}'n})$$

(iii) It follows that

$$\begin{aligned}(Q * \mu_k^*(F) * G^{-1})((x')^{\widetilde{B}'n}) &= (x')^{-\psi g} * (x')^{g'} = q^{\alpha}(x')^{-\psi g + g'} \\ &= q^{\alpha}(x')^{-\widetilde{B}'[-g_k]_+ e'_k}.\end{aligned}$$

Here, $q = 1$ for the classical case $\Bbbk = \mathbb{Z}$. For the quantum case $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$, we have $\alpha = \frac{1}{2}\lambda(-\psi g, g')$.

We explicitly compute that

$$2\alpha = \lambda(g' - \psi g, g') = \lambda(g', \psi g - g') = -g'_k[-g_k]_+ d'_k = g_k[-g_k]_+ d'_k.$$

Similarly, we have $\lambda(\widetilde{g}' - \widetilde{\psi}\widetilde{g}, \widetilde{g}') = \widetilde{g}_k[-\widetilde{g}_k]_+ d'_k$. Note that $\widetilde{g}_k = g_k$. We deduce that

$$\begin{aligned}(Q * \mu_k^*(F) * G^{-1})((x')^{\widetilde{B}((t')^{\mathrm{prin}}n)}) &= q^{\alpha}(x')^{-\widetilde{B}((t')^{\mathrm{prin}})[-g_k]_+ e'_k} \\ &= (x')^{-\widetilde{\psi}\widetilde{g}} * (x')^{\widetilde{g}'}.\end{aligned} \tag{4.3}$$

(iv) Let us apply the mutation μ_k^* to $s_{\widetilde{g}} = x^{\widetilde{g}} * F(x^{\widetilde{B}(t^{\mathrm{prin}})n})$. Since Q only depends on $g_k = \widetilde{g}_k$, we deduce that $\mu_k^*(x^{\widetilde{g}}) = (x')^{\widetilde{\psi}\widetilde{g}} * Q((x')^{\widetilde{B}((t')^{\mathrm{prin}})n})$. Then (4.3) implies

$$\begin{aligned}\mu_k^*(s_{\widetilde{g}}) &= (x')^{\psi\widetilde{g}} * Q((x')^{\widetilde{B}((t')^{\mathrm{prin}})n}) * \mu_k^*(F)((x')^{\widetilde{B}((t')^{\mathrm{prin}})n}) \\ &= (x')^{\widetilde{g}'} * G((x')^{\widetilde{B}((t')^{\mathrm{prin}})n}).\end{aligned}$$

In particular, $\mu_k^*(s_{\widetilde{g}})$ is $\widetilde{g}'$-pointed. ■

Consider the following subalgebra of $\mathcal{U}_k$:

$$U_k := \{z \in \mathcal{U}_k \mid z \text{ has no pole at } x_{i'} = 0,\ \forall i \in I_{\mathrm{uf}},\ i \neq k\}.$$

In fact, U_k is a locally compactified version of the cluster algebra where the frozen variables $x_{i'}$, $i' \in I_{\mathrm{uf}}$, are not invertible, and thus allows us to use the nilpotent Nakayama Lemma. Define

$$C := \{\widetilde{g} \in M^{\circ}(t^{\mathrm{prin}}) \mid (\widetilde{g})_{i'} \geq 0,\ \forall i \in I_{\mathrm{uf}},\ i \neq k\}.$$

Lemma 4.2.7. *If $\widetilde{g} \in C$, then any $\widetilde{\eta} \preceq_t \widetilde{g}$ is contained in C.*

Proof. Notice that $\widetilde{\eta} = \widetilde{g} + \widetilde{B}(t^{\text{prin}}) \cdot n$ for $n \in \mathbb{N}^{I_{\text{uf}}}$, and all column vectors of $\widetilde{B}(t^{\text{prin}})$ have non-negative coordinates at I'. The claim follows. ■

As a consequence, $s_{\widetilde{g}} = x^{\widetilde{g}} \cdot F_{\widetilde{g}}((y_i(t^{\text{prin}})_{i \in I_{\text{uf}}}) \in U_k$ if and only if $\widetilde{g} \in C$.

Proposition 4.2.8. *The set $\{m_{\widetilde{g}} \mid \widetilde{g} \in C\}$ is a basis of U_k.*

Proof. We have $m_{\widetilde{g}} = x^{\widetilde{g}} \cdot (1 + y_k)^{[-g_k]_+}$ for the classical case $\Bbbk = \mathbb{Z}$. See Section 2.5 for the quantum case $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$. We deduce that $m_{\widetilde{g}}$ has a pole at some $x_{i'} = 0$ if and only if $\widetilde{g} \notin C$.

For any $z \in U_k \subset \mathcal{U}_k$, we have a finite decomposition $z = \sum b_{\widetilde{g}} m_{\widetilde{g}}$ in the basis $\{m_{\widetilde{g}} \mid \widetilde{g} \in M^\circ(t)\}$. Define the support $G = \{\widetilde{g} \mid b_{\widetilde{g}} \neq 0\}$.

Assume that $G \backslash C \neq \emptyset$. Let η denote a $\prec_{t^{\text{prin}}}$-maximal element in $G \backslash C$. Then m_η contributes a Laurent monomial $b_{\widetilde{\eta}} x^{\widetilde{\eta}}$ with a pole at some $x_{i'} = 0$. Since the $\widetilde{g}$ from C do not have a pole here, they do not contribute to the Laurent degree η. Since η is maximal, other $m_{\widetilde{g}}$ appearing with $\widetilde{g} \notin C$ do not contribute to this degree either. Therefore, z has a pole here and does not belong to U_k. This contradiction shows that every $z \in U_k$ is a finite linear combination of $m_{\widetilde{g}}$, $\widetilde{g} \in C$. The claim follows. ■

Define the graded polynomial ring $A = \Bbbk[x_{i'}]_{i' \in I'}$ with the grading $\text{gr}(x_{i'}) = 1$ (endowed with the twisted product in the quantum case). Take its homogeneous decomposition $A = \bigoplus_{r \in \mathbb{N}} A^r$. It has the maximal ideal $\mathbf{m} := \bigoplus_{r > 0} A^r$. Then $\mathbf{m}$ gives a nilpotent ideal $\mathbf{m}$ in the quotient ring $A^{\leq r} := A / \bigoplus_{d \geq r+1} A^d$.

We take the homogeneous decomposition $U_k = \bigoplus_{r \in \mathbb{N}} U_k^r$. It is an A-module such that the action is given by multiplication. The quotient algebra $U_k^{\leq r} = U_k / \bigoplus_{d \geq r+1} U_k^d$ is an $A^{\leq r}$-module, and it equals $\bigoplus_{0 \leq d \leq r} U_k^d$ as a $\Bbbk$-module. We have the natural projections $\pi^r : U_k \to U_k^r$ as $\Bbbk$-modules and $\pi^{\leq r} : U_k \to U_k^{\leq r}$ as algebras.

Lemma 4.2.9. *For any $\widetilde{g} \in C$, we have $\pi^{\leq \text{gr}(\widetilde{g})} s_{\widetilde{g}} = \pi^{\text{gr}(\widetilde{g})} s_{\widetilde{g}} = m_{\widetilde{g}}$.*

Proof. By Lemma 4.2.3, the homogeneous part of $s_{\widetilde{g}}$ in $\mathcal{LP}(t^{\text{prin}})$ with **minimal** grading has grading $\text{gr}(\widetilde{g})$ and contains the leading term $x^{\widetilde{g}}$. Similarly, the homogeneous part of $\mu_k^* s_{\widetilde{g}}$ in $\mathcal{LP}((t')^{\text{prin}})$ with **minimal** grading has grading $\text{gr}'(\widetilde{g}') = \text{gr}(\widetilde{g})$ and contains the leading term $(x')^{\widetilde{g}'}$. By Lemma 4.2.4, these homogeneous parts of $s_{\widetilde{g}}$ in $\mathcal{LP}(t^{\text{prin}})$ and $\mathcal{LP}((t')^{\text{prin}})$ respectively are related by mutation. We conclude that $\pi^{\text{gr}(\widetilde{g})} s_{\widetilde{g}}$ is pointed at $\widetilde{g}, \widetilde{g}'$ in $\mathcal{LP}(t^{\text{prin}})$ and $\mathcal{LP}((t')^{\text{prin}})$ respectively.

Because $\pi^{\text{gr}(\widetilde{g})} s_{\widetilde{g}} \in \mathcal{LP}(t^{\text{prin}})$ is pointed at $\widetilde{g}$ and has homogeneous grading, we have $\pi^{\text{gr}(\widetilde{g})} s_{\widetilde{g}} = x^{\widetilde{g}} * F(y_k(t^{\text{prin}}))$ for some polynomial F with constant term 1. Similarly, in $\mathcal{LP}(t')$ we have $\mu_k^*(\pi^{\text{gr}(\widetilde{g})} s_{\widetilde{g}}) = \pi^{\text{gr}'(\widetilde{g}')}(\mu_k^* s_{\widetilde{g}}) = (x')^{\widetilde{g}'} * G((y_k((t')^{\text{prin}}))$ for some polynomial G with constant term 1. Therefore, it is pointed at $\widetilde{g}$ and $\widetilde{g}'$ for the dominance orders associated to the seeds of the (type A_1) upper cluster algebra $\mathcal{U}_k$, where k is the only unfrozen vertex. By using Lemma 3.4.12, we deduce that $\pi^{\text{gr}(\widetilde{g})} s_{\widetilde{g}}$ agrees with the localized cluster monomial $m_{\widetilde{g}}$ of $\mathcal{U}_k$. ■

Lemma 4.2.10. *For any* $r \in \mathbb{N}$, $\{\pi^{\leq r} s_{\tilde{g}} \mid \tilde{g} \in C, \operatorname{gr}(\tilde{g}) \leq r\}$ *is a* $\Bbbk$*-basis of* $U_k^{\leq r}$.

Proof. First consider the case $r = 0$. For any $\tilde{g} \in C$, we have $\pi^{\leq 0} s_{\tilde{g}} = \pi^{\leq 0}(\pi^{\operatorname{gr}(\tilde{g})} s_{\tilde{g}}) = \pi^{\leq 0} m_{\tilde{g}}$. The claim follows from the fact that $\{m_{\tilde{g}} \mid \operatorname{gr}(\tilde{g}) = 0, \tilde{g} \in C\}$ is a $\Bbbk$-basis of the homogeneous component U_k^0 of U_k.

By the nilpotent Nakayama Lemma 4.2.2, $\{\pi^{\leq r} s_{\tilde{g}} \mid \tilde{g} \in C\}$ generates $U_k^{\leq r}$ over $A^{\leq r}$. Notice that $A^{\leq r}$ acts on $s_{\tilde{g}}$ by multiplication. We observe that $\{\pi^{\leq r} s_{\tilde{g}} \mid \tilde{g} \in C\}$ in fact generates $U_k^{\leq r}$ over $\Bbbk$. Because its non-zero elements have different leading terms, they are linearly independent and form a $\Bbbk$-basis. ∎

Proof of Proposition 4.2.1. Denote $C^{\leq r} = \{\tilde{g} \in C \mid \operatorname{gr}(\tilde{g}) \leq r\}$. Given any $z \in \mathcal{U}_k$, there exists some $c \in \mathbb{N}^{I'}$ such that $z \cdot x^c \in U_k$. Then, up to any order $r \in \mathbb{N}$, we have a finite decomposition inside the $\Bbbk$-module $U^{\leq r}$ by Lemma 4.2.10:

$$\pi^{\leq r}(z \cdot x^c) = \sum_{\tilde{g} \in C^{\leq r}} \alpha^{\leq r}(z \cdot x^c)(\tilde{g}) \cdot \pi^{\leq r} s_{\tilde{g}}. \tag{4.4}$$

By letting r tend to $+\infty$, the decomposition (4.4) becomes a possibly infinite decomposition (which converges in the $\mathbf{m}$-adic topology on the A-module U_k):

$$z \cdot x^c = \sum_{\tilde{g} \in C} \alpha(z \cdot x^c)(\tilde{g}) \cdot s_{\tilde{g}}. \tag{4.5}$$

Meanwhile, we have a $\prec_{t^{\mathrm{prin}}}$-decomposition with finitely many $\prec_{t^{\mathrm{prin}}}$-leading terms in $\mathcal{LP}(t^{\mathrm{prin}})$:

$$z \cdot x^c = \sum_{\tilde{g} \in C} \alpha_{(t^{\mathrm{prin}})}(z \cdot x^c)(\tilde{g}) \cdot s_{\tilde{g}}, \tag{4.6}$$

and a $\prec_{(t')^{\mathrm{prin}}}$-decomposition with finitely many $\prec_{(t')^{\mathrm{prin}}}$-leading terms in $\mathcal{LP}((t')^{\mathrm{prin}})$:

$$z \cdot x^c = \sum_{\tilde{g} \in C} \alpha_{((t')^{\mathrm{prin}})}(z \cdot x^c)(\tilde{g}') \cdot s_{\tilde{g}}. \tag{4.7}$$

Recall that $\preceq_{t^{\mathrm{prin}}}$ and $\preceq_{(t')^{\mathrm{prin}}}$ imply the grading order by Lemma 4.2.3. It follows that both decompositions (4.6), (4.7) agree with (4.5). To be more precise, we can compare the decompositions as follows: taking the restrictions of both decompositions (4.6), (4.7) to grading $\leq r$, they agree with the finite decomposition (4.4) by Lemma 4.2.10. If we let r tend to $+\infty$, then the restrictions grow to the triangular decompositions (4.6), (4.7) by Lemma 4.2.3, while (4.4) grows to (4.5).

Notice that $s_{\tilde{g}-c} = s_{\tilde{g}} \cdot x^{-c}$ by construction. Dividing both sides of the decomposition (4.5) by x^c, we obtain $\alpha(z)(\tilde{g} - c) := \alpha(z \cdot x^c)(\tilde{g})$ and

$$z = \sum_{\tilde{g}-c \in M^\circ(t^{\mathrm{prin}})} \alpha(z)(\tilde{g} - c) \cdot s_{\tilde{g}-c} = \sum_{\tilde{g} \in M^\circ(t)} \alpha(z)(\tilde{g}) \cdot s_{\tilde{g}}, \tag{4.8}$$

which gives simultaneously the $\prec_{t^{\mathrm{prin}}}$-decomposition in $\mathcal{LP}(t^{\mathrm{prin}})$ and the $\prec_{(t')^{\mathrm{prin}}}$-decomposition in $\mathcal{LP}((t')^{\mathrm{prin}})$. We obtain $\alpha_{(t^{\mathrm{prin}})}(z)(\tilde{g}) = \alpha_{((t')^{\mathrm{prin}})}(z)(\tilde{g}') = \alpha(z)(\tilde{g})$ for any $\tilde{g} \in M^\circ(t^{\mathrm{prin}})$.

Finally, let us return to the seeds t, t'. Let pr denote the natural projection from $\mathbb{Z}^{I\sqcup I'}$ to $\mathbb{Z}^I$. It induces the $\Bbbk$-linear map pr from $\mathcal{LP}(t^{\mathrm{prin}})$ to $\mathcal{LP}(t)$ such that $\mathrm{pr}(x^{\widetilde{g}}) = x^g$, and similarly the $\Bbbk$-linear map pr from $\mathcal{LP}((t')^{\mathrm{prin}})$ to $\mathcal{LP}(t')$ such that $\mathrm{pr}(x^{\widetilde{g}'}) = x^{g'}$. We deduce the claim by applying the linear maps pr to the decomposition (4.8) and by invoking Lemma 4.2.5. ■

4.3. Bases with tropical properties

We show that tropical properties of a collection $\mathcal{S}$ implies that it is a basis. Assume that t is injective-reachable and denote $t = \overleftarrow{\mu} t[-1]$.

As in Section 4.2, we restrict ourselves to elements in upper cluster algebras to avoid the difficulty of defining mutations for formal Laurent series.

Theorem 4.3.1. *Assume that the full rank assumption holds. If a subset $\mathcal{S}$ of the upper cluster algebra $\mathcal{U}(t)$ is compatibly pointed at the seeds appearing along the mutation sequence $\overleftarrow{\mu}$ from $t[-1]$ to t, then $\mathcal{S}$ is a basis of $\mathcal{U}(t)$.*

Proof. Let $z \in \mathcal{U}$. Working with the seed t, we have a $\prec_t$-decomposition in $\widehat{\mathcal{LP}(t)}$:

$$z = \sum \alpha_t(z)(g) \cdot s_g.$$

Notice that $\mathcal{S}$ remains pointed at the seed $t[-1]$ by our assumption. Similarly, working with the seed $t[-1]$, we have a $\prec_{t[-1]}$-decomposition in $\widehat{\mathcal{LP}(t[-1])}$:

$$z = \sum \alpha_{t[-1]}(z)(\phi_{t[-1],t}\, g) \cdot s_g.$$

Since z and $\mathcal{S}$ are contained in the upper cluster algebra $\mathcal{U}(t)$, the above decompositions take place in $\mathcal{LP}(t)$ and $\mathcal{LP}(t[-1])$ respectively.

By applying Proposition 4.2.1 for adjacent seeds along the sequence $\overleftarrow{\mu}$ from $t[-1]$ to t, we find that $\alpha_t(z)(g) = \alpha_{t[-1]}(z)(\phi_{t[-1],t}\, g)$, and $\phi_{t,t[-1]}\, \mathrm{supp}(\alpha_{t[-1]}(z)) = \mathrm{supp}(\alpha_t(z))$ $= \{g \mid \alpha_t(z)(g) \neq 0\}$.

Notice that $\mathrm{supp}(\alpha_t(z))$ has finitely many $\prec_t$-maximal elements which we denote by $g^{(i)}$, $1 \leq i \leq l$, $0 \neq l \in \mathbb{N}$. Then any s_g appearing satisfies $\deg^t s_g = g \preceq_t g^{(i)}$ for some i. Similarly, $\mathrm{supp}(\alpha_{t[-1]}(z))$ has finitely many $\prec_{t[-1]}$-maximal elements which we denote by $\phi_{t[-1],t} h^{(j)}$, $1 \leq j \leq r$, $0 \neq r \in \mathbb{N}$, for some $h^{(j)} \in M^\circ(t)$. Then any s_g appearing satisfies $\deg^{t[-1]} s_g = \phi_{t[-1],t}\, g \preceq_{t[-1]} \phi_{t[-1],t} h^{(j)} = \deg^{t[-1]} s_{h^{(j)}}$ for some j. By Proposition 3.3.11, this is equivalent to $\psi^{-1}_{t[-1],t} \deg^{t[-1]} s_g \succeq_t \psi^{-1}_{t[-1],t} \deg^{t[-1]} s_{h^{(j)}}$, i.e. $\mathrm{codeg}^t\, s_g \succeq_t \mathrm{codeg}^t\, s_{h^{(j)}}$ by Definition 3.4.4 and Proposition 3.4.8 (1). It follows that $g \succeq_t \mathrm{codeg}^t\, s_g \succeq_t \eta^{(j)} := \mathrm{codeg}^t\, s_{h^{(j)}}$.

Therefore, $\mathrm{supp}(\alpha_t(z))$ is contained in $\bigcup_{i,j} ({}_{\eta^{(j)}\preceq_t} \mathcal{M}^\circ{}_{\preceq_t g^{(i)}})$. In particular, it is a finite set by the Finite Interval Lemma 3.1.2. ■

Theorem 4.3.1 immediately implies Theorem 1.2.1 (1) and the existence of the generic basis for an injective-reachable skew-symmetric seed t (Theorem 1.2.3); see Section 5.2 for more details.

Remark 4.3.2. When we take $\mathcal{S}$ to be the collection of theta functions, this result recovers Theorem A.1.5, originally proved in [41]. Their proof is based on a thorough study of global monomials, tropical functions, convexity, boundedness of polytopes and EGM arguments [41, Section 7 8]. Our proof is specific to the injective-reachable case, but more direct and elementary.

Note that both works need the full rank assumption to obtain bases for the (upper) cluster algebra.

5. Main results

As before, we assume that the seeds satisfy the full rank assumption throughout this section.

5.1. Bases parametrized by tropical points

Lemma 5.1.1. *Let $t = \overleftarrow{\mu} t[-1]$ be an injective-reachable seed subject to the full rank assumption, $\Theta \subset M^\circ(t)$, and $\mathcal{Z} = \{z_g \in \mathcal{LP}(t) \mid g \in \Theta\}$ is a collection of Laurent polynomials such that the z_g are compatibly pointed at $t, t[-1]$ with $\deg^t z_g = g$. Let $\mathcal{A}^\Theta$ denote the free $\Bbbk$-module $\bigoplus_{g\in\Theta} \Bbbk \cdot z_g$. Then the following claims are true:*

(1) *Let $\mathcal{S} = \{s_g \in \mathcal{A}^\Theta \mid g \in \Theta\}$ be such that the s_g satisfy $\deg^t s_g = g$ and are compatibly pointed at $t, t[-1]$. Then $\mathcal{S}$ is a $\Bbbk$-basis of $\mathcal{A}^\Theta$.*

(2) *If $g \in \Theta$, $s_g \in \mathcal{A}^\Theta$ with $\deg^t s_g = g$ and s_g is compatibly pointed at $t, t[-1]$, then s_g has the following decomposition relative to $\{z_g \mid g \in \Theta\}$:*

$$s_g = z_g + \sum_{g' \in \Theta \cap M^\circ(t)_{\prec_{\{t,t[-1]\}} g}} b_{g,g'} z_{g'} \tag{5.1}$$

with $b_{g,g'} \in \Bbbk$. In addition, $\Theta \cap M^\circ(t)_{\prec_{\{t,t[-1]\}} g}$ is finite for all $g \in \Theta$.

(3) *If $\mathcal{S} = \{s_g \in \mathcal{A}^\Theta \mid g \in \Theta\}$ is such that the s_g have decompositions into $\{z_g\}$ as in (5.1), then the s_g satisfy $\deg^t s_g = g$ and are compatibly pointed at $t, t[-1]$. In particular, $\{s_g \mid g \in \Theta\}$ is a $\Bbbk$-basis of $\mathcal{A}^\Theta$ by (1).*

Notice that Lemma 5.1.1 gives a complete description of the bases $\mathcal{S}$ in (1) using the special chosen basis $\mathcal{Z}$ and the transition rule in claim (2).

Proof. (2) For any $g \in \Theta$, because $s_g \in \mathcal{A}^\Theta$ and $\{z_g \mid g \in \Theta\}$ is a basis of $\mathcal{A}^\Theta$, s_g has a finite decomposition into $\{z_g\}$:

$$s_g = \sum_{0 \le i \le r} b_i z_{g_i}$$

with coefficients $b_i \neq 0$. By assumption, s_g, z_{g_i} are compatibly pointed at $t, t[-1]$. Then we can apply Proposition 3.4.10 to deduce that, by reindexing z_{g_i}, we have $g_0 = g$, $b_0 = 1$, $g_i \in \Theta$, $\mathrm{BI}_{g_i} \subsetneq \mathrm{BI}_g$ for any $i > 0$. Notice that the last condition is equivalent to $g_i \in M^\circ(t)_{\prec_{t,t[-1]} g}$ by Proposition 3.4.6. Therefore, we obtain the claim on the decomposition of s_g. Finally, the sets $\Theta \cap M^\circ(t)_{\prec_{t,t[-1]} g}$ are finite by Proposition 3.4.6.

(1) Because the s_g are pointed at different degrees, they are linearly independent by Lemma 3.4.9. It suffices to verify that each z_g, $g \in \Theta$, is a finite sum of elements from $\{s_g \mid g \in \Theta\}$.

We use induction on the cardinality of the finite set $\Theta \cap M^\circ(t)_{\prec_{\{t,t[-1]\}} g}$. If it is an empty set, we have $z_g = s_g$ by (2).

Assume that the claim has been verified for all cardinalities no larger than $d \in \mathbb{N}$. Let us check the case $|\Theta \cap M^\circ(t)_{\prec_{\{t,t[-1]\}} g}| = d + 1$. Take any $g' \in \Theta \cap M^\circ(t)_{\prec_{\{t,t[-1]\}} g}$. By Proposition 3.4.6, we have

$$\begin{aligned} M^\circ(t)_{\prec_{t,t[-1]} g'} &= \{g'' \in M^\circ(t) \mid \mathrm{BI}_{g''} \subsetneq \mathrm{BI}_{g'}\} \subset \{g'' \in M^\circ(t) \mid \mathrm{BI}_{g''} \subsetneq \mathrm{BI}_{g}\} \\ &= M^\circ(t)_{\prec_{\{t,t[-1]\}} g} \end{aligned}$$

and in addition $\Theta \cap M^\circ(t)_{\prec_{t,t[-1]} g'} \neq \Theta \cap M^\circ(t)_{\prec_{t,t[-1]} g}$ because only the right hand side contains g'. Therefore, $|\Theta \cap M^\circ(t)_{\prec_{t,t[-1]} g'}| \leq d$ and $z_{g'}$ is a finite sum of elements of $\{s_g \mid g \in \Theta\}$ by our induction hypothesis. By (2), z_g is a finite linear combination of s_g and $z_{g'}$, $g' \in \Theta \cap M^\circ(t)_{\prec_{\{t,t[-1]\}} g}$, and the claim follows.

(3) follows from Proposition 3.4.11. ■

By applying Lemma 5.1.1 to injective-reachable upper cluster algebras, we obtain the following consequences.

Theorem 5.1.2. *Let $t = \bar{\mu} t[-1]$ be an injective-reachable seed subject to the full rank assumption. Consider the classical case $\Bbbk = \mathbb{Z}$.*

(1) *Any collection $\mathcal{S} = \{s_g \in \mathcal{U} \mid g \in M^\circ(t)\}$ such that the s_g satisfy $\deg^t s_g = g$ and are compatibly pointed at t, $t[-1]$ must be a $\Bbbk$-basis of $\mathcal{U}$.*

(2) *There exists at least one such basis, which we choose and denote by $\mathcal{Z} = \{z_{[g]}\}$.*

(3) *The set of all such bases $\mathcal{S}$ is parametrized as follows:*

$$\prod_{g \in M^\circ(t)} \Bbbk^{M^\circ(t)_{\prec_{\{t,t[-1]\}} g}} \simeq \{\mathcal{S}\},$$

$$((b_{g,g'})_{g' \in M^\circ(t)_{\prec_{\{t,t[-1]\}} g}})_{g \in M^\circ(t)} \mapsto \mathcal{S} = \{s_g \mid g \in M^\circ(t)\},$$

where $s_g = z_g + \sum_{g' \in M^\circ(t)_{\prec_{\{t,t[-1]\}} g}} b_{g,g'} z_{g'}$. In addition, each set $M^\circ(t)_{\prec_{\{t,t[-1]\}} g}$ is finite.

Proof. It suffices to show that there exists a collection $\mathcal{Z} = \{z_g \mid g \in M^\circ(t)\}$ in $\{\mathcal{S}\}$ that is a basis of $\mathcal{U}$. Then the claim follows from Lemma 5.1.1 where we take $\Theta = M^\circ(t)$.

If t is skew-symmetric, we can choose $\mathcal{Z}$ to be the collection of localized generic cluster characters, which are known to be compatibly pointed at $t' \in \Delta^+$ by [68, Theorem 1.3]. Then, by Theorem 4.3.1, it is a basis. See Section 5.2 for more details.

For general t, we have the theta functions $\theta^t_{t,g}$ for any $g \in M^\circ(t)$, which are compatibly pointed at $t \in \Delta^+$ by [9, 41] (see Theorem A.1.4). Therefore, the set $\{\theta^t_{t,g} \mid g \in M^\circ(t)\}$ is a basis of $\mathcal{U}$ by Theorem 4.3.1 (alternatively, see Theorem A.1.5). ■

Recall that $s_{[g]} \in \mathcal{U}$, $[g] \in \mathcal{M}^\circ$, is said to be *pointed at* $[g]$ if $s_{[g]}$ is pointed at the representative $g \in M^\circ(t)$ of $[g]$ in $\mathcal{LP}(t)$ for all seeds $t \in \Delta^+$.

Theorem 5.1.3 (Theorem 1.2.1). *Let $t = \overleftarrow{\mu} t[-1]$ be an injective-reachable seed subject to the full rank assumption. Consider the classical case $\Bbbk = \mathbb{Z}$.*

(1) *For any collection $\mathcal{S} = \{s_{[g]} \in \mathcal{U} \mid [g] \in \mathcal{M}^\circ\}$ such that the $s_{[g]}$ are pointed at the tropical points $[g]$, $\mathcal{S}$ must be a $\Bbbk$-basis of $\mathcal{U}$ containing all cluster monomials.*

(2) *There exists at least one such basis, which we choose and denote by $\mathcal{Z} = \{z_{[g]}\}$.*

(3) *The set of all such bases $\mathcal{S}$ is parametrized as follows:*

$$\prod_{g \in \mathcal{M}^\circ} \Bbbk^{\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}} \simeq \{\mathcal{S}\},$$

$$((b_{[g],[g']})_{[g'] \in \mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}})_{[g] \in \mathcal{M}^\circ} \mapsto \mathcal{S} = \{s_{[g]} \mid [g] \in \mathcal{M}^\circ\},$$

where $s_{[g]} = z_{[g]} + \sum_{[g'] \in \mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}} b_{[g],[g']} z_{[g']}$. In addition, each set $\mathcal{M}^\circ{}_{\prec_{\Delta^+}[g]}$ is finite.

Proof. Notice that being compatibly pointed at Δ^+ is a stronger property than being compatibly pointed at $t, t[-1]$. Theorem 5.1.2 gives a complete description of the bases $\{s_g \mid g \in M^\circ(t)\}$ such that the s_g are compatibly pointed at $t, t[-1]$. Choose a basis $\mathcal{Z}$ that is compatibly pointed at Δ^+, where possible candidates include the theta basis or the generic basis for skew-symmetric seeds (see the proof of Theorem 5.1.2).

Then the basis $\{s_g \mid g \in M^\circ(t)\}$, where $s_g = z_g + \sum_{g' \in M^\circ(t)_{\prec_{\{t,t[-1]\}}g}} b_{g,g'} z_{g'}$, satisfies this stronger property if and only if $\deg^{t'} s_g = \phi_{t',t} g = \deg^{t'} z_g$ for all t', i.e. if and only if $\deg^{t'} z_g \succ_{t'} \deg^{t'} z_{g'}$ for any t' and $g' \in M^\circ(t)_{\prec_{\{t,t[-1]\}}g}$ with non-vanishing coefficient $b_{g,g'}$. This condition is equivalent to requiring all $z_{g'}$ appearing to satisfy $g' \in M^\circ(t)_{\prec_{\Delta^+}g}$. The parametrization of $\{\mathcal{S}\}$ follows.

Finally, $\mathcal{S}$ contains all cluster monomials by Lemma 3.4.12. ■

We can understand the bijection in Theorem 1.2.1 as a statement that the set of bases with a choice of a special one is parametrized by the transition matrices, which are all nilpotent lower $\prec_{\Delta^+}$-triangular matrices with indices given by the tropical points.

Remark 5.1.4 (Basis and frozen factors). In cluster theory, it is often natural to look for pointed bases that factor through the frozen variables, i.e. $s_g \cdot x^c = s_{g+c}$ for $c \in \mathbb{Z}^{I_f}$ (see Definition 5.2.1). To adapt Theorem 1.2.1 to this, we simply impose the restriction that the special basis $\mathcal{Z}$ factors through the frozen variables, and that the transition matrix satisfies $b_{g+c,g'+c} = b_{g,g'}$. Possible candidates include the theta basis and the generic basis; see the proof of Theorem 5.1.2.

Finally, let us give a description of the bases in terms of "correct" support dimensions, which is more natural from the view of representation theory.

Proposition 5.1.5. *Let $t = \overleftarrow{\mu} t[-1]$ be an injective-reachable seed and $g \in M^\circ(t)$.*

(1) $\mathsf{suppDim}\, g$ *is well defined in $N_{\mathrm{uf}}^{\geq 0}(t)$.*

(2) $\mathsf{suppDim}\, g$ *only depends on the principal part $\mathrm{pr}_{I_{\mathrm{uf}}} g$ and $B(t)$.*

Proof. (1) By Proposition 3.4.8, it suffices to find a Laurent polynomial $z_g \in \mathcal{LP}(t)$ with degree g and compatibly pointed at $t, t[-1]$. One can take z_g to be the theta function $\theta^t_{t,g}$ or the localized generic cluster character $\mathbb{L}_g$ in Section 5.2 for skew-symmetric t.

(2) We have seen in (1) that $\mathsf{suppDim}\, g$ can be realized as the support dimension of the corresponding theta function or the localized generic cluster character. Then (2) follows from the properties of such elements. ∎

Theorem 5.1.6. *Consider the classical case $\Bbbk = \mathbb{Z}$. Let t be an injective-reachable seed subject to the full rank assumption and let $\mathcal{S} = \{s_g \mid g \in M^\circ(t)\}$ be a collection of bi-pointed elements of $\mathcal{U}$. Then $\mathcal{S}$ is a basis of $\mathcal{U}$ whose elements s_g are compatibly pointed at $t, t[-1]$ if and only if $\mathsf{suppDim}\, s_g = \mathsf{suppDim}\, g$ for all g.*

Proof. This follows from Theorem 5.1.2 and Proposition 3.4.8. ∎

5.2. *The generic basis and its analog*

Let us investigate the generic basis and analogous bases constructed from cluster characters. At this moment, generic quantum cluster characters are not defined in general, so we restrict to the classical case $\Bbbk = \mathbb{Z}$.

Definition 5.2.1. Let t be a seed and Θ a subset of $M^\circ(t)$. A set $\mathcal{Z} = \{z_g \mid g \in \Theta\}$ of pointed formal Laurent series, where $\deg^t z_g = g$, is said to *factor through the frozen variables x_j, $j \in I_{\mathrm{f}}$*, if for any $g, g' \in \Theta$ such that $g' = g + f_j$, we have $z_{g'} = z_g \cdot x_j$.

In this case, we define the *localization* of $\mathcal{Z}$ to be the set $\mathcal{Z}[x_j^{-1}]_{j\in I_{\mathrm{f}}} = \{z_g \cdot x^m \mid g \in \Theta, m \in \mathbb{Z}^{I_{\mathrm{f}}}\}$.

Let t be an injective-reachable skew-symmetric seed. Take T to be the corresponding cluster tilting object and identify $K_0(\mathsf{add}\, T) \simeq M^\circ(t) \simeq \mathbb{Z}^I$. For any $g \in \mathbb{Z}^I$, there exists some $m \in \mathbb{Z}^{I_{\mathrm{f}}}$ depending on g such that $\mathbb{L}_{g+m}$ is the *generic cluster character* of [68, Section 3.4] (see Section 2.4). Define the *localized generic cluster character* $\mathbb{L}_g$ to be the localization $\mathbb{L}_{g+m} \cdot x(t)^{-m}$.

Theorem 5.2.2 (Theorem 1.2.3). *Let t be an injective-reachable skew-symmetric seed. Then the set $\{\mathbb{L}_g \mid g \in M^\circ(t)\}$ of localized generic cluster characters is a basis of $\mathcal{U}$, called the generic basis.*

Proof. Recall that the generic cluster characters are known to be compatibly pointed at all seeds by Plamondon [68, Theorem 1.3]. So are the localized generic cluster characters. Thus, Theorem 4.3.1 provides a direct proof for the statement.

Alternatively, as an indirect proof, we use the fact that the theta basis exists ([41], Theorem A.1.5) and choose it to be the special basis in the main theorem (Theorem 1.2.1). Then the collection of the generic cluster characters is also a basis by the main theorem. ∎

Let us discuss an analog of the generic basis, where the objects chosen are not necessarily generic.

Lemma 5.2.3. *For any injective-reachable seeds $t = \overleftarrow{\mu} t[-1]$, assume that some $g \in M^\circ(t)$ has a support dimension* $\mathsf{suppDim}\, g$. *Then for any $m \in \mathbb{Z}^{I_\mathrm{f}}$, $g + m$ has support dimension* $\mathsf{suppDim}(g + m) = \mathsf{suppDim}\, g$.

Proof. For any $k \in I_\mathrm{uf}$, we have $\phi_{\mu_k t,t}(g+m) = \phi_{\mu_k t,t}(g) + \phi_{\mu_k t,t}(m) = \phi_{\mu_k t,t}(g)+m$. Repeatedly applying tropical transformations along $\overleftarrow{\mu}^{-1}$ from t to $t[-1]$, we find that $\phi_{t[-1],t}(g + m) = \phi_{t[-1],t}(g) + m$. Because the map $\psi^{-1}_{t[-1],t}$ is linear, we obtain

$$\begin{aligned}\psi^{-1}_{t[-1],t}\phi_{t[-1],t}(g+m) - (g+m) &= \psi^{-1}_{t[-1],t}\phi_{t[-1],t} g + \psi^{-1}_{t[-1],t} m - g - m\\ &= \psi^{-1}_{t[-1],t}\phi_{t[-1],t} g - g\\ &= \widetilde{B}(t)\cdot \mathsf{suppDim}\, g.\end{aligned}$$

The claim follows from the definition of support dimension. ∎

Proposition 5.2.4. *For any injective-reachable skew-symmetric seed t, every $g \in M^\circ(t)$ has support dimension given by that of the localized generic cluster character:* $\mathsf{suppDim}\, g = \mathsf{suppDim}\, \mathbb{L}_g$.

Proof. It follows from [68, Theorem 1.3] that the generic cluster characters $\mathbb{L}_{g+m}$, $g \in \mathbb{Z}^{I_\mathrm{uf}}$, $m \in \mathbb{Z}^{I_\mathrm{f}}$, are compatibly pointed at t, $t[-1]$. This implies the claim for such $g + m$ by Proposition 3.4.8. Finally, the claim holds for all $g \in M^\circ(t)$ by Lemma 5.2.3. ∎

Theorem 5.2.5. *Let t be an injective-reachable skew-symmetric seed. Denote $\Theta = \{\mathsf{Ind}^T M \mid M \in {}^\perp(\Sigma T_\mathrm{f})\}$ where T is the cluster tilting object corresponding to t. Let $\{M_g \mid g \in \Theta\}$ denote the set of any given objects in ${}^\perp(\Sigma T_\mathrm{f})$ such that $\mathsf{Ind}^T M_g = g$ and* $\dim FM_g = \mathsf{suppDim}(g)$. *Then the set $\{CC(M_g) \mid g \in \Theta\}[x_j^{-1}]_{j\in I_\mathrm{f}}$ of localized cluster characters is a basis of the upper cluster algebra $\mathcal{U}$.*

Proof. By Lemma 5.2.6 below, for any $g \in M^\circ(t)$ there is a localized cluster character $CC(M_{g+m}) \cdot x^{-m}$ pointed at g such that $g + m \in \Theta$. The claim follows from Proposition 5.2.4 and Theorem 5.1.6. ∎

Lemma 5.2.6. *For any $g \in M^\circ(t)$, there exists $m \in \mathbb{N}^{I_\mathrm{f}}$ such that $g + m = \mathsf{Ind}^T X$ for some $X \in {}^\perp(\Sigma T_\mathrm{f})$*

Proof. Consider the object $Y = (\bigoplus_{k\in I_\mathrm{uf}} T_k^{[g_k]_+}) \oplus (\bigoplus_{k\in I_\mathrm{uf}} I_k^{[-g_k]_+})$. It follows that $\mathsf{Ind}^T Y = \mathsf{pr}_{I_\mathrm{uf}} g + m'$ for some $m' \in \mathbb{Z}^{I_\mathrm{f}}$. Then we can take $m = ([m'_j]_+)_{j\in I_\mathrm{f}}$ and $X = Y \oplus (\bigoplus_{j\in I_\mathrm{f}} T_j^{[-m'_j]_+})$. ∎

By [2, Theorem 1.18], the cluster algebra $\mathcal{A}$ coincides with the upper cluster algebra $\mathcal{U}$ when the initial quiver $Q(t_0)$ is acyclic. The following result shows that a basis consisting of cluster characters can be constructed quite easily in this case.

Corollary 5.2.7. *Let t be a skew-symmetric seed and suppose the corresponding principal quiver $Q(t)$ is acyclic. Let T denote the corresponding cluster tilting object.*

(1) *Denote* $\Theta = \{\mathsf{Ind}^T M \mid M \in {}^{\perp}(\Sigma T_{\mathsf{f}})\}$. *Then for any choice of objects* $M_g \in {}^{\perp}(\Sigma T_{\mathsf{f}})$ *with* $\mathsf{Ind}^T M_g = g$, *the set* $\{CC(M_g) \mid g \in \Theta\}[x_j^{-1}]_{j \in I_{\mathsf{f}}}$ *of localized cluster characters is a basis of the cluster algebra* $\mathcal{A} = \mathcal{U}$.

(2) *Choose a pair* (V_d, m) *for each dimension vector* $d \in \mathbb{N}^{I_{\mathsf{uf}}}$ *and* $m \in \mathbb{N}^I$ *such that* V_d *is a* d*-dimensional* $\mathbb{C}Q(t)$*-module and* $\mathsf{supp}\, m \cap \mathsf{supp}\, d = \emptyset$. *Then the set* $\{x^m CC(V_d) \mid (V_d, m) \text{ as above}\}[x_j^{-1}]_{j \in I_{\mathsf{f}}}$ *of localized cluster characters is a basis of the cluster algebra* $\mathcal{A} = \mathcal{U}$.

Proof. (1) Notice that ${}^{\perp}(\Sigma T_{\mathsf{f}})$ is a full subcategory of $\mathcal{C}_{(\tilde{Q},\tilde{W})}$ and no morphism from $M \in {}^{\perp}(\Sigma T_{\mathsf{f}})$ to ΣT_k factors through T_{f}. We obtain, for any $k \in I_{\mathsf{uf}}$,

$$\begin{aligned}\mathsf{Hom}_{\mathcal{C}_{(\tilde{Q},\tilde{W})}}(M, \Sigma T_k) &= \mathsf{Hom}_{{}^{\perp}(\Sigma T_{\mathsf{f}})/(T_{\mathsf{f}})}(M, \Sigma T_k) \\ &= \mathsf{Hom}_{\mathcal{C}_{(Q,W)}}(\underline{M}, \Sigma \underline{T}_k).\end{aligned}$$

Therefore, the support dimension of $CC(M)$ equals that of $CC(\underline{M})$.

Let us work in $\mathcal{C}_{(Q,W)}$. Any object $\underline{M}$ has an $\mathsf{add}\, \underline{T}$-approximation $\underline{T}^{(1)} \to \underline{T}^{(0)} \to \underline{M}$. By applying the functor $F = \mathsf{Hom}(\underline{T}, \Sigma(\,))$, we obtain a long exact sequence

$$0 \to F\underline{M} \to F\Sigma\underline{T}^{(1)} \to F\Sigma\underline{T}^{(0)} \to \cdots .$$

Notice that $\Sigma\underline{T}^{(1)}$, $\Sigma\underline{T}^{(0)}$ are injective modules over the Jacobian algebra $J_{(Q,W)}$. Because Q is acyclic, we have $W = 0$ and $J_{(Q,W)}$ agrees with the hereditary path algebra $\mathbb{C}Q$. As a consequence, we obtain a short exact sequence

$$0 \to F\underline{M} \to F\Sigma\underline{T}^{(1)} \to F\Sigma\underline{T}^{(0)} \to 0.$$

It turns out that $\mathsf{suppDim}\, CC(\underline{M}) = \dim F\underline{M}$ only depends on the index $\mathsf{Ind}^{\underline{T}}\, \underline{M}$.

Therefore, for any $M \in {}^{\perp}(\Sigma T_{\mathsf{f}})$, $\dim FM = \dim FM_g = \mathsf{suppDim}\, g$ where M_g is a generic object of index $\mathsf{Ind}^T M$. The claim follows from Theorem 5.2.5.

(2) In the proof of (1), set $V_d = F\underline{M}$ and $d = \dim V_d$. Let R denote the matrix whose column vectors are the dimension vectors of the injectives $F(\Sigma T_k)$, $k \in I_{\mathsf{uf}}$. Then $d = -R \cdot \mathsf{pr}_{I_{\mathsf{uf}}} g$. Since Q is acyclic, R is a unitriangular matrix after relabelling the vertices. In particular, R is invertible. We can then deduce (2) from (1). ∎

6. Related topics and discussion

As before, we assume that the seeds satisfy the full rank assumption in the following discussion.

6.1. Deformation factors

Definition 6.1.1. The subset $\mathcal{M}^{\circ}{}_{\prec_{\Delta^+}[g]}$ is called the *deformation factor* associated to $[g]$.

We have seen in the main theorem (Theorem 1.2.1) that basis deformations are controlled by the deformation factors $\mathcal{M}^{\circ}{}_{\prec_{\Delta^+}[g]}$, $[g] \in \mathcal{M}^{\circ}$. These factors are important for constructing the bases. It is therefore a natural question to understand them. One might want to interpret these deformation factors in terms of homology in cluster category, or representation theory (such as quiver representations or Lie theory), or tropical geometry.

As a first step, one might ask when the deformation factors are empty sets, i.e., one cannot do a deformation. Recall that all bases in the construction share the localized cluster monomials by Lemma 3.4.12. This immediately implies the following property.

Proposition 6.1.2. *If $g \in M^\circ(t)$ is the maximal $\prec_t$-degree of any localized cluster monomial, then $M^\circ(t)_{\prec_{\Delta^+} g} = \emptyset$.*

This property is a supporting evidence for the following natural expectation.

Conjecture 6.1.3. *Assume that t is skew-symmetric. If a generic object M_g for some $g \in M^\circ(t)$ in the cluster category is rigid, then $M^\circ(t)_{\prec_{\Delta^+} g} = \emptyset$.*

Remark 6.1.4 (Open orbit conjecture). If Conjecture 6.1.3 is true, then all bases parametrized by tropical points must share the same elements for the g-vectors corresponding to rigid modules. In particular, if we consider the cluster algebras arising from the coordinate rings of unipotent subgroups, then the generic bases (dual semicanonical bases) and the dual canonical bases share such elements. Then we obtain the open orbit conjecture for these coordinate rings (see [31]).

One might also study the cardinality $|M^\circ(t)_{\prec_{\Delta^+} g}|$.

Example 6.1.5 (Bases for Kronecker type). Take $\Bbbk = \mathbb{Z}$, $I = I_{\mathrm{uf}} = \{1, 2\}$, and the initial seed t_0 such that $B(t_0) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$. Then $y_1 = x_2^2$ and $y_2 = x_1^{-2}$, which in particular have even degrees. Denote $\delta = (1, -1)$, $z = x^\delta(1 + y_2 + y_1 y_2)$. It is well known that the corresponding upper cluster algebra $\mathcal{U}$ has the generic basis which consists of the cluster monomials and z^d, $d \geq 1$.

Notice that δ is invariant under tropical transformations. Then any pointed element $s_{d\delta} \in \mathcal{U}$ parametrized by the tropical point $d\delta$ must always have leading degree $d\delta$ in all seeds. One can deduce that a deformation from z^d to $s_{d\delta}$ cannot involve any cluster monomials. Also notice that $s_{d\delta}$ is pointed and $\eta - d\delta$ has even degree whenever $\eta \prec_t d\delta$. We obtain

$$s_{d\delta} = z^d + \sum_{k \geq 0,\, d-2k \geq 0} b_{d-2k} z^{d-2k}, \quad b_{d-2k} \in \mathbb{Z}.$$

Therefore, the deformation factor has cardinality $|M^\circ(t_0)_{\prec_t d\delta}| = [d/2]$ where [ ] denotes the integer part.

An infinite family of bases in this Kronecker example is also found in [74] by using Lie theory.

Finally, still working with the Kronecker Example 6.1.5, it is known that the triangular basis (dual canonical basis) and theta basis (greedy basis) differ by taking the usual quiver Grassmannians or the transverse quiver Grassmannians [11,18]. We expect that one might relate the deformation factor to such a difference.

6.2. *Quantum bases*

Theorems 1.2.1, 5.1.2, 5.1.6 are stated for the classical case $\Bbbk = \mathbb{Z}$. Let us consider their analogs for the quantum case $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$.

Theorem 6.2.1. *Consider the quantum case $\Bbbk = \mathbb{Z}[q^{\pm 1/2}]$. Assume the quantum seeds are injective-reachable and satisfy the full rank assumption.*

(1) *The analog of Theorem* 1.2.1 (1) *remains true.*

(2) *If the analog of Theorem* 1.2.1 (2) *is true, then the analog of Theorem* 1.2.1 (3) *is true.*

(3) *If the analog of Theorem* 5.1.2 (2) *is true, then the analogs of Theorems* 5.1.2 *and* 5.1.6 *are true.*

Proof. The analog of Theorem 1.2.1 (1) is a direct consequence of Theorem 4.3.1.

Assume that a basis has been given by the analog of Theorem 1.2.1 (2) (resp. 5.1.2 (2)); the proof for the analog of Theorem 1.2.1 (3) (resp. 5.1.2) is the same as before. More precisely, we use Lemma 5.1.1 by setting $\Theta = M^\circ(t)$ and $\mathcal{A}^\Theta = \mathcal{U}$ the free $\Bbbk$-module spanned by the given basis.

As before, the analog of Theorem 5.1.6 is a consequence of Proposition 3.4.8 and the analog of Theorem 5.1.2. ■

The obstruction appears in the analogs of Theorems 5.1.2 (2) and 1.2.1 (2), i.e. we do not know a quantum basis $\mathcal{Z}$ inside a quantum upper cluster algebra. Thanks to [15], the quantum theta functions provide such a basis for an injective-reachable skew-symmetric seed t subject to the full rank assumption (see Remark 1.2.5). By [69], the dual canonical basis provides another such basis when t arises from a quantum unipotent cell with symmetrizable Cartan datum.

6.3. *Weak genteelness*

For skew-symmetric injective-reachable seeds, we have seen the existence of the generic basis, which is constructed using representation theory. It is natural to ask if we can also interpret the theta basis using representation theory in this case.

For a finite-dimensional Jacobian algebra $J_{(Q,W)}$, Bridgeland has defined a representation-theoretic version of the scattering diagram called the stability scattering diagram, for which some theta functions have a representation-theoretic formula [4]. This formula is effective for theta functions appearing in upper cluster algebras if the stability scattering diagram is equivalent to the cluster scattering diagram of [41]. If the latter condition holds, we say the quiver with potential is *weakly genteel*.

We refer the reader to Sections A.1–A.2 for the necessary definitions for the statements below.

Theorem 6.3.1 (Theorem 1.2.4). *Take $\Bbbk = \mathbb{Z}$. Let t be a skew-symmetric injective-reachable seed. Then Bridgeland's representation-theoretic formula is effective for theta func-*

tions in the cluster scattering diagram. Moreover, the stability scattering diagram and the cluster scattering diagram are equivalent.

The proof is given in Section A.2.

Conjecture 6.3.2. *Let (Q, W) be any quiver with a generic potential such that the Jacobian algebra $J_{(Q,W)}$ is finite-dimensional. Then it is weakly genteel.*

Here, by a *generic potential*, we mean a generic point in the space of all potentials in the sense of [16]. In particular, it is assumed to be non-degenerate.

Conjecture 6.3.3. *The Jacobian algebra $J_{(Q,W)}$ in Conjecture 6.3.2 is genteel.*

Here, we take a generic potential from the space of all potentials [16]. It might be possible to only assume that the potential W is non-degenerate. We can also generalize the conjectures to the case when $J_{(Q,W)}$ has infinite dimension, for which we need to modify the stability scattering diagram by working with nilpotent modules [63].

6.4. *Partial compactification*

In representation theory, it is often natural to work with a partially compactified upper cluster algebra $\overline{\mathcal{U}}$, defined as the ring of regular functions over some partial compactification $\overline{\mathbb{A}}$ of the cluster variety $\mathbb{A}$. Correspondingly, it is natural to ask if the basis of $\mathcal{U}$ gives rise to a basis of $\overline{\mathcal{U}}$, defined by choosing those basis elements without poles on the boundary $\overline{\mathbb{A}}\backslash\mathbb{A}$.

For example, for some important cluster algebras arising from representation theory, $\overline{\mathcal{U}}$ coincides with the compactified cluster algebra $\overline{\mathcal{A}}$, and the boundary condition demands the functions in $\mathcal{U}$ to have no pole at the frozen variables $x_j = 0$, $j \in I_{\mathrm{f}}$. Moreover, in such examples, for any $j \in I_{\mathrm{f}}$, there exists a seed $t \in \Delta^+$ such that $b_{jk}(t) \geq 0$ for any $k \in I_{\mathrm{uf}}$, called a seed optimized for x_j following [41].

This is a difficult and widely open question in general. Consider the classical case $\Bbbk = \mathbb{Z}$. [41, Section 9] gives an affirmative answer when one has enough optimized seeds, for which a subset of theta functions forms a basis of $\overline{\mathcal{U}}$. Let Θ denote the set of tropical points parametrizing this subset.

Then $\overline{\mathcal{U}}$ is a $\mathbb{Z}$-module spanned by the basis $\{\theta_g \mid g \in \Theta\}$. We can apply our Lemma 5.1.1 and obtain many bases of $\overline{\mathcal{U}}$. As in the proof of Theorem 1.2.1, we deduce that the set of bases of $\overline{\mathcal{U}}$ compatibly pointed at seeds in Δ^+ is in bijection with $\prod_{[g]\in\Theta} \mathbb{Z}^{\Theta\cap(\mathcal{M}^\circ \prec_{\Delta^+}[g])}$. Again, the restriction of the generic basis $\{\mathbb{L}_{\tilde{g}} \mid \tilde{g} \in \Theta\}$ is such a basis.

Appendix A. Scattering diagrams

For simplicity, we assume that the seeds satisfy the full rank assumption so that the scattering diagrams and theta functions can be easily constructed, except in the proof of Theorem

1.2.4. The construction for an arbitrary seed can be obtained by taking a projection from the construction for the corresponding seed with principal coefficients [41].

A.1. Basics of scattering diagrams and theta functions

We refer the reader to the original paper of Gross–Hacking–Keel–Kontsevich [41] for more details.

Let t_0 be any chosen initial seed. Recall that we have an isomorphism $N(t_0) \simeq \mathbb{Z}^I$ with the natural basis $\{e_i\}$ which endows $\mathbb{Z}^I$ with the bilinear form $\{\ ,\ \}$, and an isomorphism $M^\circ(t_0) \simeq \mathbb{Z}^I$ with the natural basis $\{f_i\}$. Define the ${N_{\mathrm{uf}}}^{\geq 0}(t_0)$-graded Poisson algebra $A = \mathbb{Z}[{N_{\mathrm{uf}}}^{\geq 0}(t_0)] = \bigoplus_{n \in {N_{\mathrm{uf}}}^{\geq 0}(t_0)} y(t_0)^n$ such that $\{y(t_0)^n, y(t_0)^{n'}\} = -\{n, n'\} y(t_0)^{n+n'}$. Let $|n| := \sum n_i$. Then $\mathfrak{g} = A_{>0}$ is naturally a graded Lie algebra via its Poisson bracket. Its completion $\widehat{\mathfrak{g}}$ is defined to be the inverse limit of $\mathfrak{g} / \bigoplus_{n:\,|n|>k} \mathfrak{g}_n$, $k > 0$. Let G denote the group $\exp \widehat{\mathfrak{g}}$ defined via the Baker–Campbell–Hausdorff formula.

Recall that the matrix $\widetilde{B}(t_0)$ gives us an embedding $p^* : \mathbb{Z}^{I_{\mathrm{uf}}} \to \mathbb{Z}^I$ such that $p^*(e_k) = \sum_{i \in I} b_{ik} f_i$. Let A act linearly on $\widehat{\mathcal{LP}(t_0)}$ via the derivation $\{A,\ \}$ such that $\{y(t_0)^n, x(t_0)^m\} = \langle m, n \rangle x(t_0)^{m+p^*(n)}$. In particular, $\{y(t_0)^n, x(t_0)^{p^* n'}\} = -\{n, n'\} x(t_0)^{p^*(n'+n)}$, which explains the minus sign in the definition of A. By the injectivity of p^*, this induces a faithful action of G on $\widehat{\mathcal{LP}(t_0)}$.

A *wall* in $M(t_0)_{\mathbb{R}} = M(t_0) \otimes \mathbb{R}$ is a pair $(\mathfrak{d}, \mathfrak{p}_{\mathfrak{d}})$ such that $\mathfrak{d}$ is a codimension 1 rational polyhedral cone, $\mathfrak{d} \subset n_0^{\perp}$ for some primitive normal direction $n_0 \in \mathbb{N}^{I_{\mathrm{uf}}}$, and the *wall crossing operator* $\mathfrak{p}_{\mathfrak{d}}$ is in $\exp(y(t_0)^{n_0} \mathbb{Z}[[y(t_0)^{n_0}]]) \subset G$. The wall is said to be *non-trivial* if $\mathfrak{p}_{\mathfrak{d}}$ is. A *scattering diagram* $\mathfrak{D}$ is a collection of walls subject to some finiteness condition in [41]. $\mathfrak{D}$ cuts out many chambers in $M(t_0)_{\mathbb{R}}$, among which we have two special ones, $\mathcal{C}^{\pm} := (\pm \mathbb{R}_{\geq 0}^{I_{\mathrm{uf}}}) \oplus \mathbb{R}^{I_{\mathrm{f}}}$.

Let $\mathcal{C}^1, \mathcal{C}^2$ be two chambers and $\gamma : [0,1] \to \mathbb{R}^I$ any smooth path from the interior of $\mathcal{C}^1$ to that of $\mathcal{C}^2$. We first assume that γ intersects transversely the interior of finitely many walls $\mathfrak{d}_i$ with normal direction $n_i \in \mathbb{N}^I$, $1 \leq i \leq r$, at times $t_1 \leq \cdots \leq t_r$, and we define the *wall crossing operator* along γ to be $\mathfrak{p}_\gamma := \mathfrak{p}_{\mathfrak{d}_r}^{\varepsilon_r} \cdots \mathfrak{p}_{\mathfrak{d}_1}^{\varepsilon_1}$ where $\varepsilon_i = -\operatorname{sign} \langle \gamma'(t_i), n_i \rangle$. Let $\gamma^{-1} : [0,1] \to \mathbb{R}^I$ denote the inverse path $\gamma^{-1}(t) = \gamma(1-t)$. Then $\mathfrak{p}_{\gamma^{-1}} = \mathfrak{p}_\gamma^{-1}$. We further define $\mathfrak{p}_\gamma$ for the case of infinitely many intersections as an inverse limit [41].

We say $\mathfrak{D}$ is *consistent* if $\mathfrak{p}_\gamma$ is always independent of the choice of γ, which we can denote by $\mathfrak{p}_{\mathcal{C}^2, \mathcal{C}^1}$. Two scattering diagrams are *equivalent* if they give the same $\mathfrak{p}_\gamma$ for any γ. The equivalence class of a consistent $\mathfrak{D}$ is determined by $\mathfrak{p}_{\mathcal{C}^-, \mathcal{C}^+}$ [41, Theorem 1.17], [53, 2.1.6].

A wall $(\mathfrak{d}, \mathfrak{p}_{\mathfrak{d}})$ with primitive normal direction $n_0 \in \mathbb{N}^{I_{\mathrm{uf}}}$ is said to be *incoming* if $p^*(n_0) \in \mathfrak{d}$. Up to equivalence, for any collection $\mathfrak{D}_{\mathrm{in}}$ of incoming walls, there exists a unique consistent scattering diagram $\mathfrak{D}$ such that $\mathfrak{D}_{\mathrm{in}} \subset \mathfrak{D}$ and there is no incoming wall in $\mathfrak{D} \backslash \mathfrak{D}_{\mathrm{in}}$.

For any chosen base point $Q \in M(t_0)_{\mathbb{R}}$ not contained in any non-trivial wall, the theta function $\theta_{Q,g}^{t_0}$, $g \in \mathbb{Z}^I$, is a certain formal Laurent series in $\widehat{\mathcal{LP}(t_0)}$ which takes the form $x(t_0)^g (1 + \sum_{n>0} c_n y(t_0)^n)$ with $c_n \in \mathbb{Z}$. It has the property $\theta_{Q',g}^{t_0} = \mathfrak{p}_\gamma \theta_{Q,g}^{t_0}$ for any path γ from Q to Q'. If Q is a generic point in some chamber $\mathcal{C}$, then $\theta^{t_0}_{Q,g}$ only depends on the chamber, and we write $\theta^{t_0}_{\mathcal{C},g} = \theta^{t_0}_{Q,g}$. We write $\theta_g = \theta^{t_0}_{\mathcal{C}^+,g}$ for simplicity.

Notice that to each seed $t \in \Delta^+$ one can associate a chamber $\mathcal{C}^t$. In particular, we have $\mathcal{C}^{t_0} = \mathcal{C}^+$, and when $t_0[1]$ exists, $\mathcal{C}^{t_0[1]} = \mathcal{C}^-$. So we can write $\theta^t_{t,g} = \theta^t_{\mathcal{C}^t,g}$.

Let $\mathrm{Li}_2(\,)$ denote the dilogarithm function [41].

Definition A.1.1. Let t_0 be an initial seed. The consistent scattering diagram $\mathfrak{D}$ whose incoming walls are $(e_k^\perp, \exp(-d_k \mathrm{Li}_2(-y(t_0)_k)))$, $k \in I_{\mathrm{uf}}$, is called the *cluster scattering diagram associated to t_0*.

Consider cluster scattering diagrams from now on. Let us compare our tropical transformations with those of [41]. By [41], for any $k \in I_{\mathrm{uf}}$, we have the tropical transformation which preserves the theta functions,

$$T_k : \mathbb{Z}^I \to \mathbb{Z}^I, \quad m = \sum m_i f_i \mapsto m + [m_k]_+ \sum_i b_{ik} f_i.$$

Consider the seed $t' = \mu_k t_0$. We identify $M^\circ(t') \simeq \mathbb{Z}^I \simeq M^\circ(t_0)$ so that the basis elements $f_i' = f_i(t')$ are given by (2.2) with the sign $\varepsilon = +$:

$$f_i' = \begin{cases} f_i, & i \neq k, \\ -f_k + \sum_j [-b_{jk}]_+ f_j, & i = k. \end{cases}$$

Lemma A.1.2. *For any $m = \sum m_i f_i$, the coordinates of its image $m' = T_k m = \sum m_i' f_i'$ are given by the tropical transformation ϕ_{t',t_0} (Definition 2.1.4):*

$$m_i' = \begin{cases} -m_k, & i = k, \\ m_i + m_k [b_{ik}]_+, & i \neq k,\ m_k > 0, \\ m_i + m_k [-b_{ik}]_+, & i = k,\ m_k < 0. \end{cases}$$

Proof. By the mutation rule of f_i', we have

$$\begin{aligned} m' &= \sum m_i' f_i' = m_k' f_k' + \sum_{i:i\neq k} m_i' f_i' \\ &= m_k' \Big(-f_k + \sum_i [-b_{ik}]_+ f_i\Big) + \sum_{i\neq k} m_i' f_i \\ &= (-m_k') f_k + \sum_{i:i\neq k} (m_i' + [-b_{ik}]_+ m_k') f_i. \end{aligned}$$

First assume $m_k \geq 0$. By the transformation T_k, we have

$$m' = m + m_k \sum_i b_{ik} f_i = m_k f_k + \sum_{i:\, i\neq k} (m_i + b_{ik} m_k) f_i.$$

Therefore, we obtain

$$\begin{aligned} m_k' &= -m_k, \\ m_i' &= m_i + (b_{ik} + [-b_{ik}]_+) m_k = m_i + [b_{ik}]_+ m_k. \end{aligned}$$

Next, assume that $m_k < 0$. By the transformation T_k, we have

$$m' = m = m_k f_k + \sum_{i:i \neq k} m_i f_i.$$

Therefore, we obtain

$$m'_k = -m_k, \quad m'_i = m_i + [-b_{ik}]_+ m_k. \qquad \blacksquare$$

Theorem A.1.3 ([41]). *For any $t \in \Delta^+$ and $g \in \mathcal{C}^t$, the theta function θ_g is a localized cluster monomial in the seed t. In particular, the cluster variable $x_i(t)$ equals $\theta_{g_i(t)}$ in $\widehat{\mathcal{LP}(t_0)}$.*

Theorem A.1.4. *For any seeds $t = \overleftarrow{\mu} t_0$, we have $\overleftarrow{\mu}^* \theta^t_{t,g} = \theta^{t_0}_{t_0, \phi_{t_0,t} g}$ for all $g \in M^\circ(t)$.*

Proof. It seems that [41] does not present this result exactly in this way. Nevertheless, it is known that theta functions are pointed at the tropical points by [41], and the claim follows.

To prove the statement, one will need the "CPS Lemma" [9, Section 4] which says that theta functions are sent to theta functions by wall crossings, as well as [41, Theorem 3.5, Proposition 3.6, Proposition 4.3, Theorem 4.4]. These results together tell us the construction of theta functions is compatible with monomial automorphisms $\tau_{k,\epsilon}$, Hamiltonian automorphisms (wall-crossings) $\rho_{k,\epsilon}$, and the tropical transformation $T_k = \phi_{\mu_k t_0, t_0}$ associated to the mutation of the initial seeds. Then it is also compatible with mutations because $\mu_k^* = \rho_{k,\epsilon} \tau_{k,\epsilon}$. $\blacksquare$

Theorem A.1.5 ([41, Proposition 8.25]). *Let t_0 be an injective-reachable initial seed. Then the theta functions θ_g, $g \in M^\circ(t_0)$, are pointed Laurent polynomials in $\mathcal{LP}(t_0)$. In addition, they form a basis of the upper cluster algebra $\mathcal{U}$, called the theta basis.*

Proof. Because t_0 is injective-reachable, the cluster algebra has large cluster complex in the sense of [41, Definition 8.23]. In particular, it satisfies the EGM condition (enough global monomials). The claims follow from [41, Proposition 8.25]. $\blacksquare$

A.2. Weak genteelness and the proofs

We shall show that, by combining known results from cluster theory, the scattering diagrams and some theta functions for skew-symmetric injective-reachable seeds have a representation-theoretic description due to Bridgeland [4, Theorem 1.4]; see also [13]. Related definitions can be found in Section A.1.

Let t_0 be an injective-reachable skew-symmetric initial seed subject to the full rank assumption. We take the corresponding principal quiver with a non-degenerate potential (Q, W). We omit the symbol t_0 for simplicity.

We take the stability scattering diagram $\mathfrak{D}^{(\mathrm{st})}_{\mathrm{uf}}$ constructed by integrating moduli of semistable modules of $J_{(Q,W)}$ introduced in [4, Section 11] . The walls $(\mathfrak{d}, \mathfrak{p}_{\mathfrak{d}})$ of $\mathfrak{D}^{(\mathrm{st})}_{\mathrm{uf}}$ live in $N_{\mathrm{uf}}(t_0)^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(N_{\mathrm{uf}}(t_0), \mathbb{R})$. We define the stability scattering diagram $\mathfrak{D}^{(\mathrm{st})}$ to be the collection of walls $(\mathfrak{d} \oplus \mathbb{R}^{I_f}, \mathfrak{p}_{\mathfrak{d}})$ which live in $M(t_0)_{\mathbb{R}}$. As in Section A.1, we define the action[4] of the Poisson algebra $A = \mathbb{Z}[\mathbb{N}^{I_{uf}}]$ on $\widehat{\mathcal{LP}(t_0)}$ such that $\{y^n, x^m\} = \langle m, n\rangle x^{m+\widetilde{B}n}$. Then the corresponding group G and its action on $\widehat{\mathcal{LP}(t_0)}$ are given as in Section A.1.

The scattering diagram $\mathfrak{D}^{(\mathrm{st})}$ can be described via representation theory [4, Theorems 1.1, 1.3]. Moreover, Bridgeland has the following description of theta functions in $\mathfrak{D}^{(\mathrm{st})}$ [4, Theorem 1.4]:

$$\theta^{(\mathrm{st}),t_0}_{Q,m} = x^m \cdot \Big(\sum K(n,m,Q) \cdot x^{\widetilde{B}\cdot n}\Big)$$

where the base point Q does not belong to any non-trivial wall, $m \in \mathbb{N}^{I_{uf}}$, and $K(n,m,Q)$ is the Euler characteristic of the quotient module Grassmannian $\mathrm{Quot}_n U(m,Q)$ consisting of n-dimensional quotient modules of a certain module $U(m,Q)$ in a tilted heart; see [4, Section 8.4] for details. A representation-theoretic formula for other theta functions is unknown at the moment. In particular, by taking Q to be a generic point in $\mathcal{C}^-$ and $m = f_i$, the formula reads

$$\theta^{(\mathrm{st}),t_0}_{t_0[1],f_i} = \begin{cases} x_i \cdot (\sum \chi(\mathrm{Quot}_n P_i) \cdot x^{\widetilde{B}\cdot n}), & i \in I_{uf}, \\ x_i, & i \in I_f, \end{cases}$$

where P_i, $i \in I_{uf}$, corresponds to the i-th projective module of $J_{(Q,W)}$.

Definition A.2.1 (Genteelness, [4]). We say the Jacobian algebra $J_{(Q,W)}$ is *genteel* if the only modules V that are $p^*(\dim V)$-stable are the simples S_k, $k \in I_{uf}$.

Let $\mathfrak{D}$ denote the cluster scattering diagram associated to t_0 (see Section A.1). The following property is a weaker version of genteelness.

Definition A.2.2 (Weak genteelness). We say the Jacobian algebra $J_{(Q,W)}$ is *weakly genteel* if $\mathfrak{D}^{(\mathrm{st})}$ and $\mathfrak{D}$ are equivalent.

Given a consistent scattering diagram $\mathfrak{D}$ in $\mathbb{R}^I$, let us construct the opposite scattering diagram $\mathfrak{D}^{\mathrm{op}}$ in $\mathbb{R}^I$ (see Example A.2.7).

Recall that $A = \mathbb{Z}[y^n]_{n\in\mathbb{N}^{I_{uf}}}$ is a Poisson algebra such that $\{y^n, y^{n'}\} = -\{n, n'\} y^{n+n'}$ and $\mathfrak{g} = A_{>0}$ (see Section A.1). Define the opposite Poisson algebra $A^{\mathrm{op}} = \mathbb{Z}[y^n]$ with the Poisson bracket $\{\ ,\ \}^{\mathrm{op}} = -\{\ ,\ \}$ and Lie algebra $\mathfrak{g}^{\mathrm{op}} = A^{\mathrm{op}}_{>0}$. We have $\iota : A \simeq A^{\mathrm{op}}$ as $\mathbb{Z}$-modules such that $\iota(y^n) = y^n$.

Lemma A.2.3. *For any $u, v, w \in \mathfrak{g}$ such that* $\exp w = \exp u \cdot \exp v$ *we have* $\exp \iota w = \exp \iota v \cdot \exp \iota u$ *in* $G^{\mathrm{op}} := \exp \widehat{\mathfrak{g}}^{\mathrm{op}}$.

Proof. The claim follows from the Baker–Campbell–Hausdorff formula which defines the group multiplication on G and G^{op}. ∎

[4] Our action is slightly different from the one in [4, Section 10.3] so that it is faithful.

Let κ denote the isomorphism $m \mapsto -m$ on $\mathbb{R}^I$ as well as the induced automorphism $\kappa(x^m) = x^{\kappa m}$ on $\widehat{\mathbb{Z}[x^m]_{m\in\mathbb{Z}^I}}$. The opposite scattering diagram $\mathfrak{D}^{\mathrm{op}}$ in $\mathbb{R}^I$ is defined to be the collection of walls $(\kappa\mathfrak{d} \subset n_{\mathfrak{d}}^{\perp}, \exp \iota u)$ for any wall $(\mathfrak{d} \subset n_{\mathfrak{d}}^{\perp}, \exp u) \in \mathfrak{D}$. Given any path γ, let $\mathfrak{p}_{\gamma}^{\mathrm{op}}$ denote the corresponding wall crossing operator in $\mathfrak{D}^{\mathrm{op}}$.

Lemma A.2.4. (1) *If* $\mathfrak{p}_{\gamma^{-1}} = \exp w$, *then* $\mathfrak{p}_{\kappa\gamma}^{\mathrm{op}} = \exp \iota w$.

(2) $\mathfrak{D}^{\mathrm{op}}$ *is consistent.*

Proof. (1) For any given generic path γ such that $\mathfrak{p}_{\gamma} = \mathfrak{p}_{\mathfrak{d}_r}^{\varepsilon_r} \cdots \mathfrak{p}_{\mathfrak{d}_1}^{\varepsilon_1}$, the wall crossing operator in $\mathfrak{D}^{\mathrm{op}}$ along $\kappa\gamma$ is

$$\mathfrak{p}_{\kappa\gamma}^{\mathrm{op}} = \exp(-\varepsilon_r \iota \log \mathfrak{p}_{\mathfrak{d}_r}) \cdots \exp(-\varepsilon_1 \iota \log \mathfrak{p}_{\mathfrak{d}_1}).$$

The claim follows from the equality $\mathfrak{p}_{\mathfrak{d}_1}^{-\varepsilon_1} \cdots \mathfrak{p}_{\mathfrak{d}_r}^{-\varepsilon_r} = \mathfrak{p}_{\gamma}^{-1} = \mathfrak{p}_{\gamma^{-1}}$ and Lemma A.2.3.

(2) The claim follows from (1) by taking all paths. ∎

Proposition A.2.5. *Let t_0^{op} denote the seed opposite to t_0 such that $\widetilde{B}(t_0^{\mathrm{op}}) = -\widetilde{B}$ and we take the same strictly positive integers d_i. Let $\mathfrak{D}(t_0)$ and $\mathfrak{D}(t_0^{\mathrm{op}})$ denote the cluster scattering diagrams associated to t_0, t_0^{op} respectively. Then $\mathfrak{D}(t_0^{\mathrm{op}})$ is equivalent to the opposite scattering diagram $\mathfrak{D}(t_0)^{\mathrm{op}}$, where we identify $M^\circ(t_0) \simeq \mathbb{Z}^I \simeq M^\circ(t_0^{\mathrm{op}})$ so that $f_i(t_0) \mapsto f_i(t_0^{\mathrm{op}})$ and $N(t_0) \simeq \mathbb{Z}^I \simeq N(t_0^{\mathrm{op}})$ so that $e_i(t_0) \mapsto e_i(t_0^{\mathrm{op}})$.*

Proof. Notice that the bilinear form on $N(t_0^{\mathrm{op}})$ is opposite to that of $N(t_0)$ under the identification. So we can view A^{op} and $\mathfrak{g}^{\mathrm{op}}$ in the construction of $\mathfrak{D}(t_0)^{\mathrm{op}}$ as $A(t_0^{\mathrm{op}})$ and $\mathfrak{g}(t_0^{\mathrm{op}})$ associated to t_0^{op}. Furthermore, $\mathfrak{D}(t_0)^{\mathrm{op}}$ is consistent with the incoming walls being $(e_k^{\perp}, \exp(-d_k \mathrm{Li}_2(-y_k)))$. Therefore, $\mathfrak{D}^{\mathrm{op}}(t_0)$ is equivalent to the cluster scattering diagram $\mathfrak{D}(t_0^{\mathrm{op}})$. ∎

The actions of A and A^{op} on $\widehat{\mathbb{Z}[x^m]_{m\in\mathbb{Z}^I}}$ are defined as in Section A.1 using the scattering diagrams associated to the seeds t_0, t_0^{op} respectively.

Lemma A.2.6. *We have $\mathfrak{p}_{\kappa\gamma}^{\mathrm{op}}(\kappa x^m) = \kappa \mathfrak{p}_{\gamma} x^m$ for any path γ.*

Proof. Recall that the action of A satisfies $\{y^n, x^m\} := \langle m, n\rangle x^{m+\widetilde{B}n}$ and the action of A^{op} satisfies $\{y^n, x^m\}^{\mathrm{op}} := \langle m, n\rangle x^{m-\widetilde{B}\cdot n}$. Then we have $\{\iota y^n, \kappa x^m\}^{\mathrm{op}} = -\langle m, n\rangle x^{-m-\widetilde{B}n} = \kappa\{-y^n, x^m\}$. Therefore, $\exp(\iota w)(\kappa x^m) = \kappa(\exp(-w)(x^m))$. The claim follows from Lemma A.2.4 (1). ∎

Example A.2.7. Let $I = I_{\mathrm{uf}} = \{1, 2\}$ and $B(t_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\epsilon = -B(t_0)$. The cluster scattering diagram $\mathfrak{D} = \mathfrak{D}(t_0)$ in $M(t_0)_{\mathbb{R}} = \mathbb{R} f_1 \oplus \mathbb{R} f_2 \simeq \mathbb{R}^2$ is given by

$$\mathfrak{D} = \{(e_1^{\perp}, \exp(-\mathrm{Li}_2(-y_1)\}, (e_2^{\perp}, \exp(-\mathrm{Li}_2(-y_2)), (\mathbb{R}_{\geq 0}(1, -1), \exp(-\mathrm{Li}_2(-y_1 y_2))\}$$

where the Poisson bracket on $A = \mathbb{Z}[y_1, y_2]$ satisfies $\{y_i, y_j\} = -\epsilon_{ij} y_i y_j$. By [41], we have $\exp(-\mathrm{Li}_2(y^n))(x^m) = x^m(1 + y^n)^{\langle m,n\rangle}$. Let γ denote a path from $\mathcal{C}^+$ to $\mathcal{C}^-$. One checks that, for $v_i = \widetilde{B} e_i$,

$$\mathfrak{p}_\gamma x_1 = x_1(1 + x^{v_1} + x^{v_1+v_2}), \quad \mathfrak{p}_{\gamma^{-1}} x_1^{-1} = x_1^{-1}(1 + x^{v_1}),$$
$$\mathfrak{p}_\gamma x_2 = x_2(1 + x^{v_2}), \quad \mathfrak{p}_{\gamma^{-1}} x_2^{-1} = x_2^{-1}(1 + x^{v_2} + x^{v_1+v_2}).$$

The opposite scattering diagram is given by

$$\mathfrak{D}^{\mathrm{op}} = \{(e_1^\perp, \exp(-\mathrm{Li}_2(-y_1)\}, (e_2^\perp, \exp(-\mathrm{Li}_2(-y_2)), (\mathbb{R}_{\geq 0}(-1,1), \exp(-\mathrm{Li}_2(-y_1 y_2))\}$$

and the Poisson bracket on $A^{\mathrm{op}} = \mathbb{Z}[y_1, y_2]$ satisfies $\{y_i, y_j\} = \epsilon_{ij} y_i y_j$. The opposite seed t_0^{op} has $B(t_0^{\mathrm{op}}) = -B(t_0)$, $\epsilon(t_0^{\mathrm{op}}) = -B(t_0^{\mathrm{op}})$. The corresponding cluster scattering diagram is just $\mathfrak{D}^{\mathrm{op}}$. One checks that

$$\mathfrak{p}^{\mathrm{op}}_{\kappa\gamma} x_1^{-1} = x_1^{-1}(1 + x^{-v_1} + x^{-v_1-v_2}), \quad \mathfrak{p}^{\mathrm{op}}_{\kappa\gamma} x_2^{-1} = x_2^{-1}(1 + x^{-v_2}).$$

Proof of Theorem 1.2.4. We refer the reader to [4,41,63] for details of the related notions below.

As in [41], replacing t_0 by a principal coefficient seed t_0^{prin} by adding principal framing vertices i' for all $i \in I$ if necessary, we first assume that the seed t_0 satisfies the full rank assumption.

Recall that the equivalence classes of the consistent scattering diagrams $\mathfrak{D}$, $\mathfrak{D}^{(\mathrm{st})}$ are determined by the respective wall crossing operators $\mathfrak{p}_{t_0[1],t_0}$ and $\mathfrak{p}^{(\mathrm{st})}_{t_0[1],t_0}$. Because G acts faithfully on $\widehat{\mathcal{LP}(M^\circ(t_0))}$, it suffices to show that $\mathfrak{p}_{t_0[1],t_0}$ and $\mathfrak{p}^{(\mathrm{st})}_{t_0[1],t_0}$ have the same action.

Because $\mathfrak{D}$ is a cluster scattering diagram, for any index i, θ_{-f_i} agrees with the localized cluster variable

$$\theta_{-f_i} = \begin{cases} x_i^{-1} \cdot \sum_n (\chi(\mathsf{Gr}_n\, I_i) \cdot x^{\widetilde{B} \cdot n}), & i \in I_{\mathrm{uf}}, \\ x_i^{-1}, & i \in I_{\mathrm{f}}, \end{cases}$$

where $I_i \in \mathcal{C}_{(Q,W)}$ corresponds to the i-th injective module of the Jacobian algebra $J_{(Q,W)}$ and f_i denotes the i-th unit vector.

As a conceptual proof, we observe that the theta functions in $\mathfrak{D}^{(\mathrm{st})}(t)$ can be calculated by using tilting theory as in the work of Nagao [4, Section 8.3], [63]. Moreover, the main result of Nagao [63] is the deduction of the Caldero–Chapoton type formula for cluster monomials from tilting theory. By the main result of Nagao, the theta function $\theta^{(\mathrm{st})}_{-f_i}$ in $\mathfrak{D}^{(\mathrm{st})}$ must agree with the localized cluster variable with degree $-f_i$. Therefore, we obtain $\mathfrak{p}_{t_0,t_0[1]}(x_i^{-1}) = \theta_{-f_i} = \theta^{(\mathrm{st})}_{-f_i} = \mathfrak{p}^{(\mathrm{st})}_{t_0,t_0[1]}(x_i^{-1})$ for any $i \in I_{\mathrm{uf}}$. The faithfulness of G implies $\mathfrak{p}_{t_0,t_0[1]} = \mathfrak{p}^{(\mathrm{st})}_{t_0,t_0[1]}$, and consequently $\mathfrak{p}_{t_0[1],t_0} = \mathfrak{p}^{(\mathrm{st})}_{t_0[1],t_0}$. We refer the reader to Mou's upcoming work [60] for a detailed treatment (and a quantized version) in terms of the Hall algebras.

Instead of re-examining the arguments of [63] in the setting of [4], we give an alternative proof by using the scattering diagram $\mathfrak{D}^{\mathrm{op}}$ opposite to $\mathfrak{D}$.

Choose any generic smooth path γ from $\mathcal{C}^+$ to $\mathcal{C}^-$ in $\mathbb{R}^I$. Assume $\mathfrak{p}_\gamma x_k = x_k \cdot f$. Then $\mathfrak{p}^{\mathrm{op}}_{\kappa\gamma} x_k^{-1} = x_k^{-1} \cdot \kappa f$ by Lemma A.2.6. Because $\mathfrak{D}^{\mathrm{op}} \simeq \mathfrak{D}(t_0^{\mathrm{op}})$ and $\kappa\gamma$ is a path from $\mathcal{C}^-$ to $\mathcal{C}^+$, we obtain the cluster expansion formula for cluster variables associated to $t_0^{\mathrm{op}}[1]$:

$$\mathfrak{p}^{\mathrm{op}}_{\kappa\gamma} x_k^{-1} = x_k^{-1} \sum_n \chi(\mathsf{Gr}_n\, I_k^{\mathrm{op}}) \cdot x^{-\widetilde{B} n}$$

where $k \in I_{\mathrm{uf}}$ and I_k^{op} is the k-th injective module associated to the opposite algebra $J^{\mathrm{op}}_{(Q,W)}$. By the natural isomorphism $\mathrm{Quot}_n(P_k) \simeq \mathsf{Gr}_n\, I_k^{\mathrm{op}}$, we obtain

$$\mathfrak{p}_{t_0[1],t_0} x_k = \mathfrak{p}_\gamma x_k = x_k \Big(\sum_n \chi(\mathrm{Quot}^n\, P_k) x^{\widetilde{B}n} \Big).$$

In addition, obviously $\mathfrak{p}_{t_0[1],t_0} x_i = x_i$ for any $i \in I_{\mathrm{f}}$. Therefore, $\mathfrak{p}_{t_0[1],t_0}$ and $\mathfrak{p}^{(\mathrm{st})}_{t_0[1],t_0}$ have the same action on $\widehat{\mathcal{LP}(t_0)}$.

Finally, if the original seed t_0 does not satisfy the full rank assumption and we have worked with its principal coefficient seed t_0^{prin} as in [41], we can consider the natural projection proj from $\mathbb{Z}^{I(t_0^{\mathrm{prin}})}$ to $\mathbb{Z}^{I(t_0)}$ and the induced $\mathbb{Z}$-linear projection proj from $\mathcal{LP}(t_0^{\mathrm{prin}})$ to $\mathcal{LP}(t)$. By applying the projections, we recover the theta functions and scattering diagrams for t_0 from those for t_0^{prin}; see [41] for details. The desired claim follows. ∎

Remark A.2.8. By [71], a seed is injective-reachable if and only if it is "projective-reachable". Recall that projective modules over $J = J_{(Q,W)}$ can be identified with injective modules over $J^{\mathrm{op}} = J_{(Q^{\mathrm{op}},W^{\mathrm{op}})}$. We deduce that if t is injective-reachable, then so is t^{op}. Consequently, if J is weakly genteel, then so is J^{op}.

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