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Lefschetz fibrations with arbitrary signature

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Abstract. We develop techniques to construct explicit symplectic Lefschetz fibrations over the 2-sphere with any prescribed signature σ and any spin type when σ is divisible by 16. This solves a long-standing conjecture on the existence of such fibrations with positive signature. As applications, we produce symplectic 4-manifolds that are homeomorphic but not diffeomorphic to connected sums of $S^2 \times S^2$, with the smallest topology known to date, as well as larger examples as symplectic Lefschetz fibrations.

Keywords. 4-manifold, signature, Lefschetz fibration, monodromy factorization, exotic smooth structure

1. Introduction

An immense literature has been dedicated to the study of symplectic Lefschetz fibrations since the works of Donaldson and Gompf [19,31] established Lefschetz fibrations over the 2-sphere as topological counter-parts of closed symplectic 4-manifolds, after blow-ups. However, the possible values of one of the most fundamental invariants of 4-manifolds, the *signature* of a Lefschetz fibration over the 2-sphere, has not been quite understood, despite many effective ways of calculating it being available, due to the works of Endo, Ozbagci and others [17,20,21,42,45,51]. The aim of our article is to resolve this problem:

Theorem A. *There exist infinitely many relatively minimal symplectic Lefschetz fibrations over the* 2*-sphere, whose total spaces have any prescribed signature* $\sigma \in \mathbb{Z}$ *and any spin type when* σ *is divisible by* 16*. All the examples can be chosen to be simply-connected, minimal, and with fiber genus as small as* 9*, as well as arbitrarily high.*

By fiber summing with trivial fibrations over orientable surfaces, possibly with boundary, the main statements of the theorem carry over to Lefschetz fibrations over *any* given orientable surface.

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It was often conjectured that Lefschetz fibrations over the 2-sphere with positive signatures did not exist, which constituted a long-standing open problem; see e.g. [44, 45], [56], [57, Problems 6.3, 6.4], [39, Problems 7.4, 7.5]. Our examples settle this conjecture in the negative. In contrast, in the literature there are plenty of examples of Lefschetz fibrations over the 2-sphere with negative signatures, many coming from algebraic geometry. To the best of our knowledge, even in this case it was not known that every negative signature could be realized; see Remark 18.

We will prove the theorem by explicitly describing the Lefschetz fibrations in terms of their monodromy factorizations, which correspond to positive Dehn twist factorizations in the mapping class group of an orientable surface. Our constructions will heavily use variations of the *breeding technique* [10,32] for building Lefschetz pencils and fibrations out of lower genera ones. A great deal of our efforts will be spent on building monodromy factorizations for *spin* Lefschetz fibrations. We should note that, even though we effectively use the breeding technique to derive new symplectic 4-manifolds from smaller ones, this is not an inherently symplectic operation; we build pencils/fibrations with locally non-complex nodal singularities in intermediate steps, but then we match all the locally non-complex nodes with locally complex ones and remove these pairs at the end (which corresponds to a 5-dimensional 3-handle attachment, and at times to removing an $S^2 \times S^2$ summand from a connected sum); see Remark 16.

We should also note that more flexible variations of this *signature realization problem* are easier to address, even in the holomorphic category: For Lefschetz *pencils*, where one allows base points, examples with prescribed signatures are easy to obtain using very ample line bundles on suitable compact complex surfaces. Similarly, there are many compact complex surfaces admitting semistable fibrations over non-rational complex curves; in fact, when one allows the base surface of the fibration to be of higher genus, there are even smooth fibrations, often called Kodaira fibrations, whose total spaces have positive signatures. While the essential challenge is to describe positive signature Lefschetz fibrations *over the* 2*-sphere*, we point out that most, and possibly all, of our examples in Theorem A are non-holomorphic; see Remark 19. Since there are no separating vanishing cycles in our fibrations, by Endo's signature formula [20], none of our examples with non-negative signatures can be hyperelliptic either.

Recall that two 4-manifolds are said to be *stably diffeomorphic* if they become diffeomorphic after taking connected sums of each with copies of $S^2 \times S^2$. By a classical result of C. T. C. Wall, we have the following immediate corollary to our theorem:

Corollary B. *Any closed smooth oriented simply-connected* 4*-manifold is stably diffeomorphic to a symplectic Lefschetz fibration.*

Furthermore, by crafting the monodromies of our fibrations more carefully and invoking Freedman's celebrated work, we can show that any closed smooth oriented simply-connected 4-manifold X with signature σ, provided its holomorphic Euler characteristic $\chi_h(X) := \frac{1}{4}(\mathrm{e}(X) + \sigma(X))$ is integral and sufficiently large (depending solely on σ), is *homeomorphic* to a symplectic Lefschetz fibration. (Doing this for *all* integral $\chi_h(X)$ greater than a constant that depends on σ requires more work.) So we get symplectic

Lefschetz fibrations as exotic copies of standard 4-manifolds that are connected sums of copies of $S^2 \times S^2$, $\mathbb{CP}^2$ and the K3 surface, taken with either orientations. It is a very interesting problem to determine the smallest χ_h (or b_2) one can take for a given σ. For example, when $\sigma = 0$, we can show

Corollary C. *There exist symplectic Lefschetz fibrations, whose total spaces are pairwise non-diffeomorphic, but homeomorphic to* $\#_m(S^2 \times S^2)$*, where the examples can be chosen so that* m *is as small as* 127*, or is any odd* $m \geq 415$*.*

These are the first exampes of Lefschetz fibrations in the given homeomorphism classes. The spin examples are particularly interesting, and have been sought after for quite some time, in connection with the existence of exotically knotted *orientable* surfaces in the 4-sphere. Examples of exotically knotted *non-orientable* surfaces in the 4-sphere were constructed by Finashin, Kreck and Viro [24], and further examples were given by Finashin [23], all using involutions on genus-1 Lefschetz fibrations. In contrast, it is still unknown whether there are any examples of exotically knotted orientable surfaces in the 4-sphere, whereas double covers along topologically but not smoothly trivial ones would yield exotic $\#_m(S^2 \times S^2)$; see Remark 21.

The motivation for Theorem A in part comes from the *symplectic geography problem*, pioneered by Gompf [30], which asks to determine the pairs of integers that can be realized as χ_h and $c_1^2 = 2\mathrm{e} + 3\sigma$ of minimal symplectic 4-manifolds for a given spin type and fundamental group, akin to the geography problem for compact complex surfaces [18, 48, 49]. This is because of the extensive use of Lefschetz fibrations in the past four decades as building blocks for new complex, symplectic and smooth 4-manifolds. Employing symplectic surgeries, which do not preserve the fibration structure, we can get sharper results and realize new pairs of values as (χ_h, c_1^2) of simply-connected spin symplectic 4-manifolds:

Theorem D. *There exist symplectic Lefschetz fibrations over the* 2*-torus, which are equivalent via Luttinger surgeries to infinitely many, pairwise non-diffeomorphic symplectic* 4*-manifolds homeomorphic to* $\#_m(S^2 \times S^2)$*, where* m *can be chosen as small as* 23*, as well as arbitrarily large. They further yield examples of infinitely many, pairwise non-diffeomorphic symplectic* 4*-manifolds homeomorphic to* $\#_m(S^2 \times S^2)$ *for* any *odd* $m \geq 23$*.*

To the best of our knowledge, Theorem C contains examples with the smallest topology among the exotic $\#_m(S^2 \times S^2)$ discovered to date. Symplectic 4-manifolds homeomorphic but not diffeomorphic to $\#_m(S^2 \times S^2)$ were first constructed by J. Park [47] for unspecified large m, and this result was improved dramatically by Akhmedov and D. Park to $m \geq 275$ [4] and later to $m \geq 175$ [5]. (Recent talks by Sakallı announce further improvement to $m \geq 27$ in joint work with the same authors [6], but this preprint is not publicly available at the time of writing.) All these previous works use compact complex surfaces with positive signatures built by algebraic geometers as essential ingredients. Our main ingredients, the symplectic Lefschetz fibrations over the 2-torus we build, are not homotopy equivalent to any complex surfaces.

We discuss how to describe spin structures on Lefschetz pencils and fibrations using their monodromy factorizations in Section 2. There we present a way to calculate the divisibility of the fiber class from the monodromy factorization, and produce a very handy criterion for equipping the fibration with a spin structure, building on Stipsicz's work [54]; see Proposition 8 and Theorem 9. Both are employed repeatedly in our constructions of spin Lefschetz fibrations to follow. In the proof of our main theorem, Theorem A, the signature zero genus-9 Lefschetz fibration over the 2-sphere singled out in Theorem 14 will play a key role. The construction of this fibration, which spans the entire Section 3, is the most technically involved one in our paper, so we chose to present it in multiple steps, in each step producing positive factorizations for Lefschetz pencils that might be of particular interest themselves. Even more effort is spent on establishing whether this fibration has a primitive fiber, since we have not been able to identify any sections of this fibration. (We plan to examine the existence of (multi)sections of this fibration elsewhere.) We will finish the proof of Theorem A in Section 4, and prove Corollary C and Theorem D in Section 5, using the explicit monodromies of our signature zero Lefschetz fibrations, while now paying special attention to killing the fundamental group efficiently so as to produce symplectic 4-manifolds in the desired homeomorphism classes.

Conventions: Any Lefschetz fibration in this article is assumed to have non-empty critical set, and any Lefschetz pencil has non-empty base locus. All are assumed to be relatively minimal, i.e. no exceptional spheres contained in the fibers. Unless explicitly specified otherwise, the base of our Lefschetz fibrations is the 2-sphere. We denote by $\Sigma_{g,k}^b$ a compact orientable surface of genus g with b boundary components and k marked points in its interior, and we drop b or k from the notation when they are zero. The mapping class group $\mathrm{Mod}(\Sigma_{g,k}^b)$ then consists of orientation-preserving diffeomorphisms of $\Sigma_{g,k}^b$ which fix all the boundary points and marked points, modulo isotopies of the same type. A right-handed or positive Dehn twist about a simple closed curve or simple loop c on a surface Σ is denoted by t_c in $\mathrm{Mod}(\Sigma)$. Our products of mapping classes, and in particular of Dehn twists, act on curves starting with the rightmost factor. We express a monodromy factorization of a genus-g Lefschetz fibration by $t_{c_1} \cdots t_{c_n} = 1$ in $\mathrm{Mod}(\Sigma_g)$, and that of a genus-g Lefschetz pencil with b base points as $t_{c_1} \cdots t_{c_n} = t_{\delta_1} \cdots t_{\delta_b}$ in $\mathrm{Mod}(\Sigma_g^b)$, where $\{\delta_i\}$ are boundary parallel curves along distinct boundary components of Σ_g^b. We often refer to these as positive factorizations (of identity or boundary multi-twist). The reader can turn to [31] for general background on Lefschetz fibrations and pencils, monodromy factorizations and symplectic 4-manifolds, and to [21, 22] for more on relations in the mapping class group and signature calculations (while keeping in mind that in [22] the authors' convention is to use left-handed Dehn twists instead).

2. Spin structures on Lefschetz pencils and fibrations

In this section, we will first review some preliminary results on spin structures on Lefschetz pencils and fibrations, and present a few quintessential examples that will be used in our later constructions. We will then discuss how to calculate the divisibility of the homology class of the regular fiber of a Lefschetz fibration, and present a practical way to build a spin structure on its total space, solely using monodromy factorizations. The reader can turn to [31, 36, 54] for basic definitions and background results on spin structures on 3- and 4-manifolds, and to [16, 35, 41] for spin structures on surfaces, quadratic forms and spin mapping class groups. While we will focus on Lefschetz pencils and fibrations, we note that *everything* we discuss is applicable, mutatis mutandis, to achiral pencils and fibrations. In what follows, $H_*(X)$ denotes the *integral* homology group of a space X, while coefficient groups other than $\mathbb{Z}$ in our homology calculations will be explicitly specified.

Given a Lefschetz pencil (X, f), we can easily determine whether X admits a spin structure by studying the $\mathbb{Z}_2$-homology classes of the Dehn twist curves in the monodromy factorization of f:

Theorem 1 (Baykur–Hayano–Monden [14]). *Let (X, f) be a Lefschetz pencil with monodromy factorization $t_{c_1} \cdots t_{c_n} = t_{\delta_1} \cdots t_{\delta_b}$ in $\mathrm{Mod}(\Sigma_g^b)$, $b \geq 1$. Then X admits a spin structure if and only if there is a quadratic form $q\colon H_1(\Sigma_g^b; \mathbb{Z}_2) \to \mathbb{Z}_2$ with respect to the $\mathbb{Z}_2$-intersection pairing such that $q(c_i) = 1$ for all i, and $q(\delta_j) = 1$ for some j.*

Note that X cannot admit a spin structure if b is odd or if any c_i is separating.

Given a Lefschetz fibration (X, f), we can also determine whether X admits a spin structure or not from the monodromy factorization of f provided the integral homology class of the regular fiber F is primitive and there is information available on the self-intersection of an algebraic dual S of F in $H_2(X)$, i.e. $F \cdot S = 1$. (These extra conditions will be the focus of our discussions to follow.) A characterization in this case was given by Stipsicz [54], which motivates and predates Theorem 1. We rephrase it as follows:

Theorem 2 (Stipsicz [54]). *Let (X, f) be a genus-g Lefschetz fibration with monodromy factorization $t_{c_1} \cdots t_{c_n} = 1$. Assume that the homology class of the regular fiber has an algebraic dual S in $H_2(X)$. Then X admits a spin structure if and only if S has even self-intersection, and there is a quadratic form $q\colon H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ with respect to the $\mathbb{Z}_2$-intersection pairing such that $q(c_i) = 1$ for all i.*

In our revised statement of Theorem 2, we replaced the original algebraic condition for the vanishing cycles $\{c_i\}$ given in [54] with an equivalent condition in terms of a quadratic form, which we find to be handier for our calculations. (See below for an explanation of this condition.) Further, we have stated the existence of an algebraic dual as a hypothesis, which always holds when the homology class of the regular fiber is primitive in $H_2(X)$. In [54, 55] it was claimed that every Lefschetz fibration has a primitive fiber class, but there is a mistake in the proof of this claim (and there are counter-examples in the achiral case, such as Matsumoto's genus-1 achiral Lefschetz fibration on S^4); see e.g. [11, Appendix] for an explanation. In the case of a pencil, the induced handle decomposition removes the need for additional assumptions on the homological dual of the fiber class. Note that when X is spin, the condition on the vanishing cycles is implied regardless of the divisibility of the fiber class.

Recall that a function $q\colon H_1(\Sigma_g^b;\mathbb{Z}_2) \to \mathbb{Z}_2$ describes a quadratic form with respect to the $\mathbb{Z}_2$-intersection pairing if $q(a+b) = q(a) + q(b) + a \cdot b \pmod 2$ for every $a, b \in H_1(\Sigma_g^b;\mathbb{Z}_2)$. There is a bijection between the set of isomorphism classes of spin structures $\mathrm{Spin}(\Sigma_g)$ and the set of such quadratic forms on $H_1(\Sigma_g;\mathbb{Z}_2)$ [16, 35]—and a similar correspondence holds for Σ_g^b, where the quadratic form is no longer non-singular. From the proof of the above theorems, one can in fact deduce that *any* spin structure on X comes from one on Σ_g^b or Σ_g, for which the monodromy curves satisfy the spin property given in the statements.

An important invariant of a quadratic form q on $H_1(\Sigma_g;\mathbb{Z}_2)$ is its $\mathbb{Z}_2$-valued *Arf invariant* $\mathrm{Arf}(q)$, which can be most easily calculated as

$$\mathrm{Arf}(q) = \sum_{i=1}^{q} q(\alpha_i)q(\beta_i),$$

where $\{\alpha_i, \beta_i\}$ is *any* symplectic basis for $H_1(\Sigma_g;\mathbb{Z}_2)$. For q_s, $q_{s'}$ the quadratic forms corresponding to $s, s' \in \mathrm{Spin}(\Sigma_g)$ respectively, there exists a spin diffeomorphism $\phi : (\Sigma_g, s) \to (\Sigma_g, s')$, or equivalently $\phi^*(q_{s'}) = q_s$, if and only if $\mathrm{Arf}(q_s) = \mathrm{Arf}(q_{s'})$ [7,35].

For a fixed spin structure $s \in \mathrm{Spin}(\Sigma_g)$, let $\mathrm{Mod}(\Sigma_g, s)$ be the *spin mapping class group*, which consists of $\phi \in \mathrm{Mod}(\Sigma_g)$ such that $\phi^* q = q$, where q is the quadratic form corresponding to s. Since q respects the $\mathbb{Z}_2$-intersections, it follows from the Picard-Lefschetz formula that a non-separating Dehn twist t_c is in $\mathrm{Mod}(\Sigma_g, s)$ if and only if $q(c) = 1$. Now, if Y is a Σ_g-bundle over S^1 with monodromy ϕ, then it admits a spin structure $\mathfrak{s}$ that restricts to a spin structure s on the fibers if and only if $\phi \in \mathrm{Mod}(\Sigma_g, s)$. When this holds, attaching a Lefschetz 2-handle to $Y \times I$ along a simple loop c on a fiber prescribes a fibered cobordism from Y to Y', another Σ_g-bundle with monodromy $\phi' = t_c \circ \phi$, if and only if c is non-separating and $t_c \in \mathrm{Mod}(\Sigma_g, s)$ [50,54]. With these in mind, we will next illustrate how Dehn twists in $\mathrm{Mod}(\Sigma_g, s)$ play a key role in building spin Lefschetz fibrations.

Let (X, f) be a Lefschetz fibration with monodromy factorization $t_{c_1} \cdots t_{c_n} = 1$, and assume that there is a quadratic form q satisfying the hypothesis of Theorem 2. The Lefschetz fibration f on X induces a decomposition $X = (D^2 \times \Sigma_g) \cup W \cup (D^2 \times \Sigma_g)$, where $W = (S^1 \times \Sigma_g \times I) \cup \sum_{i=1}^{n} h_i$ consists of Lefschetz 2-handles h_i attached along the Dehn twist curves c_i on Σ_g. For $s \in \mathrm{Spin}(\Sigma_g)$ corresponding to the quadratic form q, take the spin structure on the first copy of $D^2 \times \Sigma_g$ which is the product of the unique spin structure on D^2 with s. The induced spin structure $\mathfrak{s}$ on the boundary $\partial D^2 \times \Sigma_g$ is obviously the product of the bounding spin structure on S^1 and s [54]. Now, by our discussion in the previous paragraph, W yields a spin cobordism from $\partial D^2 \times \Sigma_g$ to $\partial X'$ for $X' = (D^2 \times \Sigma_g) \cup W$. As $t_{c_1} \cdots t_{c_n} = 1$, there is a diffeomorphism $\partial X' \to S^1 \times \Sigma_g$ which maps the induced spin structure $\mathfrak{s}'$ on $\partial X'$ to a spin structure $\mathfrak{s}''$ on $S^1 \times \Sigma_g$. However, for $\mathfrak{s}''$ to extend over the remaining $D^2 \times \Sigma_g$, it should come from a product spin structure where the spin structure on the S^1 factor is the bounding one.

We note that when we double the monodromy factorization, the spin condition for the vanishing cycles alone is sufficient to conclude that the total space of the new fibration is spin:

Proposition 3. *Let (X, f) be a Lefschetz fibration with monodromy factorization $(t_{c_1}\cdots t_{c_n})^2 = 1$, where $t_{c_1}\cdots t_{c_n} = 1$ in $\mathrm{Mod}(\Sigma_g)$. Then X admits a spin structure if and only if there is a quadratic form $q\colon H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2$ with respect to the $\mathbb{Z}_2$-intersection pairing such that $q(c_i) = 1$ for all i.*

Proof. In this case, the Lefschetz fibration f with doubled monodromy induces a decomposition

$$X = (D^2 \times \Sigma_g) \cup W \cup W \cup (D^2 \times \Sigma_g) = X' \cup X',$$

where $W = (S^1 \times \Sigma_g \times I) \cup \sum_{i=1}^n h_i$, with 2-handles h_i attached along c_i on Σ_g, is a spin cobordism as we argued above. So $X' = (D^2 \times \Sigma_g) \cup W$ admits a spin structure per our hypothesis. As before, $t_{c_1}\cdots t_{c_n} = 1$ induces an identification $\Phi\colon \partial X' \to S^1 \times \Sigma_g$, and for Ψ the orientation-reversing diffeomorphism on $S^1 \times \Sigma_g$ which is the product of the complex conjugation on S^1 and identity on Σ_g, we have $X = X' \cup_{\Phi^{-1}\Psi\Phi} X'$. Clearly, this gluing matches the induced spin structures $\mathfrak{s}'$ on both copies of $\partial X'$, and the spin structures we have on the two copies of X' then extend to a spin structure on the whole of X. (So we see that if the cobordism W from $S^1 \times \Sigma_g$ to itself happens to flip the spin structure on the S^1 factor, attaching W for a second time, we still get back the bounding spin structure on the S^1 factor.) ∎

The doubled monodromy in Proposition 3 obviously amounts to taking an untwisted fiber sum $(X, f) = (X', f')\,\#\,(X', f')$, where f' has monodromy $t_{c_1}\cdots t_{c_n} = 1$. In [53] Stipsicz argues that even if f' does not have a section, f always has one with even self-intersection. One can then invoke Theorem 2 to obtain another proof of the proposition.

More generally, the essential information on an algebraic dual of the fiber in Theorem 2 would be readily available if the fibration (X, f) admits a section S. While this is equivalent to having a lift of the monodromy $t_{c_1}\cdots t_{c_n} = 1$ in $\mathrm{Mod}(\Sigma_g)$ to $t_{c'_1}\cdots t_{c'_n} = 1$ in $\mathrm{Mod}(\Sigma_{g,1})$, finding such a lift—which might require isotoping the original $\{c_i\}$ so they now have a lot more geometric intersections—is a challenging job; see e.g. [32,38,43,58]. Nonetheless, once we have this lift, determining the self-intersection of the section is fairly easy. We have the short exact sequence

$$1 \to \langle t_\delta \rangle \to \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_{g,1}) \to 1$$

where the epimorphism is induced by capping the boundary component δ of Σ_g^1 by a disk with a marked point. So there is always a further lift $t_{c''_1}\cdots t_{c''_n} = t_\delta^k$ in $\mathrm{Mod}(\Sigma_g^1)$, which one can easily obtain by removing a disk neighborhood of the marked point and then calculating the number of boundary twists needed to get back to identity, by looking at the action of the monodromy on an essential arc on Σ_g^1. The power k of the boundary twist t_δ then determines the negative of the self-intersection number of S. This can be easily seen as follows: The disk neighborhood of the marked point corresponds to a tubular neighborhood of S, and by picking a fixed point on the boundary of the disk, we get a push-off of S, so now by the action of t_δ^k, the intersection number of the two is $\pm k$. Running this calculation in a single example (like the ones below), once and for all we determine the sign, and conclude that the self-intersection of S is $-k$.

We continue with some quintessential examples:

Example 4. The *odd-chain relation* in $\mathrm{Mod}(\Sigma_g^2)$, with c_i as in Figure 1 (a),

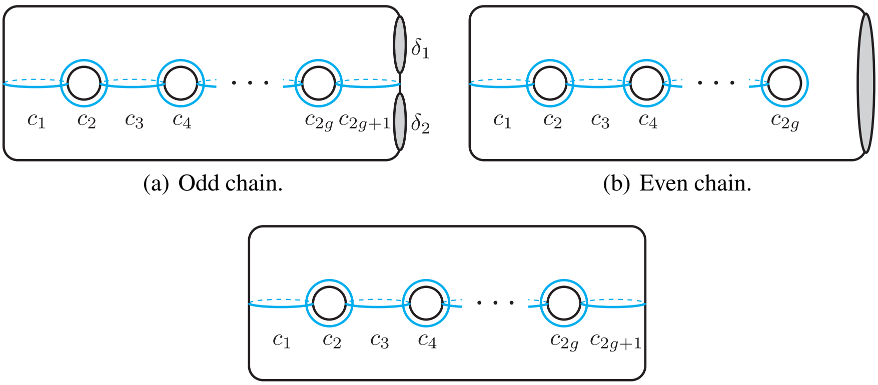
$$(t_{c_1} t_{c_2} \cdots t_{c_{2g+1}})^{2g+2} = t_{\delta_1} t_{\delta_2}, \tag{1}$$

is the monodromy factorization of a genus-g pencil on a simply-connected Kähler surface, which is the K3 surface when $g = 2$ [31]. Since the $\mathbb{Z}_2$-homology classes of $c_1, \ldots, c_{2g+1}$ generate $H_1(\Sigma_g^2; \mathbb{Z}_2)$, and the sum of all odd-indexed c_i is homologous to each δ_j, we see that there is in fact a unique quadratic form $q \colon H_1(\Sigma_g^2; \mathbb{Z}_2) \to \mathbb{Z}_2$ for which the monodromy curves satisfy the spin condition (corresponding to the unique spin structure on the 4-manifold) if and only if g is even.

Example 5. The *even-chain relation* in $\mathrm{Mod}(\Sigma_g^1)$, with c_i as in Figure 1 (b),

$$(t_{c_1} t_{c_2} \cdots t_{c_{2g}})^{4g+2} = t_{\delta}, \tag{2}$$

is the monodromy factorization of a genus-g pencil on a simply-connected Kähler surface. Since the $\mathbb{Z}_2$-homology classes of the curves $c_1, \ldots, c_{2g}$ generate $H_1(\Sigma_g^1; \mathbb{Z}_2)$, there is in fact a unique quadratic form $q \colon H_1(\Sigma_g^1; \mathbb{Z}_2) \to \mathbb{Z}_2$ for which $q(c_i) = 1$ for all i. While this pencil cannot be spin (here the number b of base points is 1), the double of this monodromy gives a Lefschetz fibration that *is* spin.



(a) Odd chain. (b) Even chain.

(c) Hyperelliptic relation.

Fig. 1. Dehn twist curves for standard relations.

Example 6. The *hyperelliptic relation* in $\mathrm{Mod}(\Sigma_g)$, with c_i as in Figure 1 (c),

$$(t_{c_1} t_{c_2} \cdots t_{c_{2g}} t_{c_{2g+1}}^2 t_{c_{2g}} \cdots t_{c_2} t_{c_1})^2 = 1, \tag{3}$$

is the monodromy factorization of a genus-g fibration on the rational surface $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$. The double of this factorization is known to yield a genus-g Lefschetz fibration on the elliptic surface $E(g+1)$ [31]. Once again, the $\mathbb{Z}_2$-homology classes of the curves $c_1, \ldots, c_{2g+1}$ generate $H_1(\Sigma_g; \mathbb{Z}_2)$. Since the odd-indexed c_i all together bound a subsurface, there is a unique quadratic form $q\colon H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2$ for which $q(c_i) = 1$ for all i if and only if g is odd.

Yet, even when g is odd, the total space $\mathbb{CP}^2 \,\#\, (4g+5)\overline{\mathbb{CP}}^2$ for the positive factorization (3) is obviously not spin, despite the fiber class being primitive (which is immediate by the particular case of Proposition 8 below, or by the fibration admitting sections). Indeed, this fibration arises as a blow-up of a pencil on a Hirzebruch surface, and admits as many as $4g+4$ exceptional sections, each of which can be taken as the dual S. As an instance of Proposition 3 however, when we double the monodromy factorization (3), we get a spin Lefschetz fibration, which yields the unique spin structure on $E(g+1)$ for g odd. This is also evident from Theorem 2 and the fact that by matching the exceptional sections, we see that the latter fibration has a section $S' := S \,\#\, S$ with self-intersection -2. Similarly, if we cap off the two boundary components in Example 4, we get a positive factorization of the identity, and doubling it, we get a monodromy factorization for a spin Lefschetz fibration provided g is odd. Note that in both examples, we have the same spin structure on Σ_g, and the corresponding quadratic form has Arf invariant 1 when $g \equiv 1 \pmod 4$ and 0 when $g \equiv 3 \pmod 4$.

On the other hand, if we cap off the boundary component in Example 5, the monodromy curves, which are in this case linearly independent in homology, already satisfy the spin condition for *any* genus g, and doubling it yields a spin Lefschetz fibration. The quadratic form for this unique spin structure on Σ_g now has Arf invariant 1 when $g \equiv 1$ or $2 \pmod 4$ and 0 when $g \equiv 3$ or $4 \pmod 4$.

Example 7. A less known relation in $\mathrm{Mod}(\Sigma_g)$ discovered by K.-H. Yun [63], which is a simultaneous generalization of the hyperelliptic relation and the Matsumoto–Cadavid–Korkmaz relation [37,42] (all coming from different involutions), is as follows:

$$(t_{A_{2n-2}} t_{A_{2n-3}} \cdots t_{A_2} t_{A_1}^2 t_{A_2} \cdots t_{A_{2n-3}} t_{A_{2n-2}}\, t_{B_0} \cdots t_{B_{2m}} t_{A_{2n-1}})^2 = 1, \tag{4}$$

where A_i, B_j are as in Figure 2, and $m, n > 0$. This is the monodromy factorization of a genus $g = 2m+n-1$ fibration on the ruled surface $(\Sigma_m \times S^2) \,\#\, 4n\overline{\mathbb{CP}}^2$, and importantly, a twisted fiber sum of two copies of this fibration is known to yield a genus-g Lefschetz fibration with several (-2)–sections, the total space of which is a knot surgered elliptic surface $E(n)_K$, where K is a fibered knot of genus m [27,63]. Since $E(n)_K$ are homeomorphic to $E(n)$ [26], in particular we deduce that there is a twisted fiber sum of two copies of this fibration that is simply-connected, and moreover spin with signature $-8n$ when n is even. We show that the untwisted fiber sum of two copies of the fibration, i.e. the double, is also spin when n is even.

Let $\{\alpha_i, \beta_i\}_{i=1}^{n-1} \cup \{\alpha'_j, \beta'_j\}_{j=1}^{2m}$ be a symplectic basis of $H_1(\Sigma_g;\mathbb{Z}_2)$, represented by the same labeled curves in Figure 2. We calculate the $\mathbb{Z}_2$-homology classes of the vanishing cycles as

$$\begin{aligned} A_1 &= \beta_1, \\ A_{2i-1} &= \beta_{i-1} + \beta_i \quad (i = 2, \ldots, n-1), \\ A_{2n-1} &= \beta_{n-1}, \end{aligned}$$

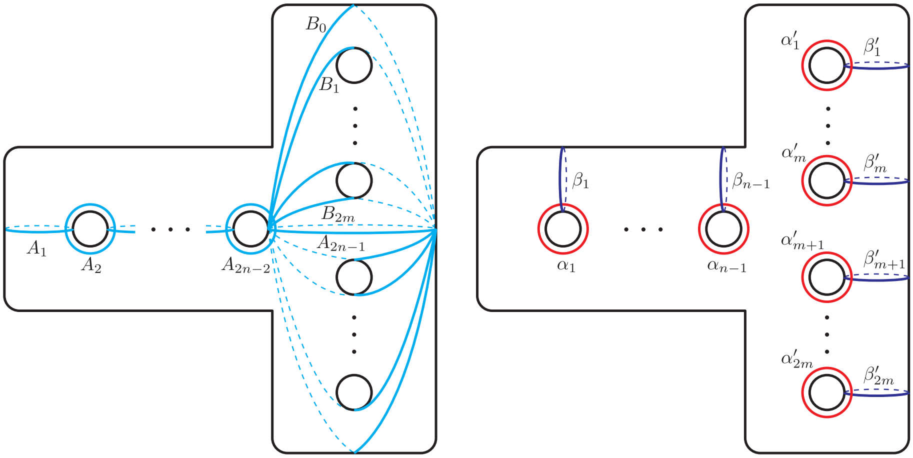


Fig. 2. Dehn twist curves of Yun's relation and the homology generators.

$$\begin{aligned}
A_{2i} &= \alpha_i \quad (i = 1, \dots, n-1), \\
B_0 &= \alpha'_1 + \cdots + \alpha'_{2m} + \beta_{n-1}, \\
B_{2j} &= \alpha'_{j+1} + \cdots + \alpha'_{2m-j} + \beta'_j + \beta'_{2m+1-j} + \beta_{n-1} \quad (j = 1, \dots, m-1), \\
B_{2m} &= \beta'_m + \beta'_{m+1} + \beta_{n-1}, \\
B_{2j-1} &= \alpha'_j + \cdots + \alpha'_{2m+1-j} + \beta'_j + \beta'_{2m+1-j} + \beta_{n-1} \quad (j = 1, \dots, m).
\end{aligned}$$

If n is odd then no quadratic form $q\colon H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ satisfies $q(A_i) = 1$ for all odd i since such A_i cobound a subsurface Σ^n_m. Suppose that n is even. Let $q(\alpha_i) = q(\alpha'_j) = q(\beta'_j) = 1$ for all i, j, and set $q(\beta_i) = 1$ for all odd $1 \le i \le n-1$ and $q(\beta_i) = 0$ for all even $2 \le i \le n-2$. This defines a quadratic form on $H_1(\Sigma_g; \mathbb{Z}_2)$ for which $q(A_i) = q(B_j) = 1$ for all i, j, and the Arf invariant of this quadratic form is 0 when $n \equiv 0 \pmod 4$ and 1 when $n \equiv 2 \pmod 4$.

We will next show, when there is no information available for a section, how, using the monodromy, we can still determine whether the fiber is primitive and thus has an algebraic dual as required in Theorem 2. Since the kernel of the forgetful homomorphism $\mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_g)$ is generated by point-pushing maps [22], if we take an arbitrary marked point on Σ_g avoiding all the c_i, we still get a lift $t_{c'_1} \cdots t_{c'_n} = \mathcal{P}_{\vec{\alpha}_1} \cdots \mathcal{P}_{\vec{\alpha}_m}$ in $\mathrm{Mod}(\Sigma_{g,1})$, where on the right-hand side we now have point-pushing maps along oriented simple loops $\vec{\alpha}_i$. This product of point-pushing maps can be expressed as one point-pushing map along a possibly immersed oriented loop $\vec{\alpha}$, and when α is null-homotopic, the marked point we picked in fact gives an honest section of the fibration.

Proposition 8. *Let (X, f) be a genus-g Lefschetz fibration which has a monodromy factorization $t_{c_1} \cdots t_{c_n} = 1$ with the following lift in* $\mathrm{Mod}(\Sigma_{g,1})$*:*

$$t_{c'_1} \cdots t_{c'_n} = \mathcal{P}_{\vec{\alpha}_1} \cdots \mathcal{P}_{\vec{\alpha}_m}.$$

Let γ_i be the homology class of c_i taken with either orientation for $i = 1, \dots, n$, and let α be the sum of the homology classes of all $\vec{\alpha}_j$ in $H_1(\Sigma_g)$. The divisibility of the homology class of the regular fiber F in $H_2(X)$ is the smallest positive integer d such that the image of $d\alpha$ under the homomorphism $H_1(\Sigma_g) \to H_1(\Sigma_g)/\langle\gamma_1, \dots, \gamma_n\rangle$ is trivial. In particular, if $\{\gamma_i\}_{i=1}^n$ generate $H_1(\Sigma_g)$, then $[F]$ is primitive.

Proof. We have $X = X' \cup (D^2 \times \Sigma_g)$, where $X' = (D^2 \times \Sigma_g) \cup \sum_{i=1}^n h_i$ and $\{h_i\}$ are the Lefschetz 2-handles attached along c_i on Σ_g. So X' contains all the 2-handles of X but one, which comes from the second copy of $D^2 \times \Sigma_g$ in the decomposition of X. The isotopy, which takes the product $t_{c_1} \cdots t_{c_n}$ to 1 in $\mathrm{Diff}^+(\Sigma_g)$, prescribes a fiber-preserving diffeomorphism Φ from the boundary fibration $f|_{\partial X'}$ to the trivial Σ_g-fibration on $\partial D^2 \times \Sigma_g$. The attaching circle c' of the last 2-handle h' is a section $s' := \Phi^{-1}(\partial D^2 \times p)$ of $f|_{\partial X'}$, for some point $p \in \Sigma_g$.

In the standard handle diagram of X', the attaching circle c' of h' links the 2-handle of the fiber geometrically once, and possibly goes over the 1-handles coming from Σ_g, and links with the attaching circles $\{c_i\}$ of the Lefschetz 2-handles $\{h_i\}$. While the section s' of $f|_{\partial X'}$ carries the first information, i.e. how c' would go over the 1-handles as it traverses around $\partial D^2 \times p$ once, there are still many ways h' can be attached along a circle c in the complement of the attaching circles $\{c_i\}$—where c, even if it is not isotopic to c' in the diagram, is still isotopic to s' on $\partial X'$. Equivalently, there are several ways to isotope all the attaching circles $\{c_i\}$ in the diagram to curves that will project on to Σ_g still as embedded curves $\{c_i'\}$, while now avoiding a marked point corresponding to c. Each of them prescribes a lift of the identity $t_{c_1} \cdots t_{c_n} = 1$ in $\mathrm{Mod}(\Sigma_g)$ to a factorization $t_{c_1'} \cdots t_{c_n'} = \mathcal{P}_{\vec{\alpha}}$ in $\mathrm{Mod}(\Sigma_{g,1})$, where $\vec{\alpha}$ is an oriented curve on Σ_g. However, c and c' only differ by handle slides over h_i, and the homology classes of their attaching circles on Σ_g will only differ by relations generated by the homology classes of $\{c_i\}$ in $H_1(\Sigma_g)$. Thus, they will all map to the same homology class under the projection $H_1(\Sigma_g) \to H_1(\Sigma_g)/\langle\gamma_1, \dots, \gamma_n\rangle$.

Corresponding to a given lift $t_{c_1'} \cdots t_{c_n'} = \mathcal{P}_{\vec{\alpha}_1} \cdots \mathcal{P}_{\vec{\alpha}_m}$ in $\mathrm{Mod}(\Sigma_{g,1})$ is such a choice c as above. To see this, let $\pi \colon S^1 \times \Sigma_g \to \Sigma_g$ be the projection, and let t parametrize $S^1 = [0,1]/\{0 \sim 1\}$. Take a partition $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_m = 1$. For each $\vec{\alpha}_i$, we can find an oriented arc β_i in $[t_{i-1}, t_i] \times \Sigma_g$, running from the point $t_{i-1} \times p$ to the point $t_i \times p$, so that $\pi|_{\beta_i} \colon \beta_i \to \vec{\alpha}_i$ is an immersion. Concatenating all of them, we get an oriented arc β in $[0,1] \times \Sigma_g$. Identifying the end points of either $+\beta$ or $-\beta$, we obtain c. Just like our reading of the self-intersection number of a section from the corresponding monodromy factorization, the correct sign of β here can be settled once and for all after calculating it in one example, but for our homology calculations to follow, the sign will not matter.

From the decomposition $X = X' \cup (D^2 \times \Sigma_g)$, it is clear that there is no class with a representative in X' which has non-trivial pairing with the homology class of the regular fiber F in $H_2(X)$. By our discussion above, up to relations generated by the homology classes γ_i of c_i on $H_1(\Sigma_g)$, the attaching circle c' of the last 2-handle h' is homologous to c we build from a given lift, which in turn is homologous to a concatenation of

$S^1 \times p$ and all $\vec{\alpha}_j$s in $H_1(S^1 \times \Sigma_g) = H_1(S^1) \times H_1(\Sigma_g)$. Since the $S^1 \times p$ curve already bounds $D^2 \times p$, when we attach h', its attaching circle becomes homologous to $\pm\alpha$ in $H_1(S^1 \times \Sigma_g)$ for $\alpha = \sum_{j=1}^{m} [\vec{\alpha}_j]$ in $H_1(\Sigma_g)$. It follows that if the image of α is trivial in the quotient $H_1(\Sigma_g)/\langle \gamma_1, \ldots, \gamma_n \rangle \cong H_1(X')$, then the attaching circle of h' bounds some surface S' in X'. We can then form a closed surface S in X by gluing S' and the core of h' to obtain a surface which intersects the fiber F once, and conclude that $[F]$ is primitive.

In general, if the image of $d\,\alpha$ is trivial in $H_1(\Sigma_g)/\langle \gamma_1, \ldots, \gamma_n \rangle$, we get a surface S' in X' that bounds d parallel copies of the attaching circle of h'. So we can build a closed surface S in X, which satisfies $S \cdot F = d$, by gluing S' and d parallel copies of the core of the 2-handle h'. Conversely, if $[F] = dF_0$ for a primitive class F_0, let S be a surface in X representing the dual of F_0, so $S \cdot F = d$. If necessary, by tubing—along F—between any pair of positive and negative intersections of S with F, we can replace S with a higher genus surface in the same homology class, which now intersects F geometrically d times. We can then isotope this S so that in a small neighborhood $N(F)$ of F identified as $N(F) \cong D^2 \times \Sigma_g$, we have $S \cap N(F) \cong D^2 \times (d \text{ points})$. Identifying $N(F)$ with $D^2 \times \Sigma_g$ in the decomposition $X = X' \cup (D^2 \times \Sigma_g)$, we conclude that any class dual to the primitive root F_0 of $[F]$ can in fact be represented by a surface S that splits into a surface S' in X' and d parallel copies of the core of h'. Hence, the smallest positive integer d, for which $d\,\alpha$ is trivial in the quotient, gives the divisibility of the fiber class $[F]$ in $H_2(X)$. ∎

In the case of pencils, the induced handle decomposition nullifies the need for the additional information on the attaching of the last 2-handle, and one can determine the divisibility of the fiber class of a genus-g Lefschetz *pencil* with b base points directly using a lift of the monodromy in $\mathrm{Mod}(\Sigma_g^b)$; see [33, Appendix].

We will call a surface S a *pseudosection* of (X, f) if it intersects a regular fiber F once. As seen from our proof of Proposition 8, when the lift of the monodromy factorization to $\mathrm{Mod}(\Sigma_{g,1})$ involves non-trivial point-pushing maps, one gets a pseudosection (possibly after some handle slides over the Lefschetz 2-handles) provided the class α determined by the point-pushing curves becomes trivial under the quotient homomorphism $H_1(\Sigma_g) \to H_1(\Sigma_g)/\langle \gamma_1, \ldots, \gamma_n \rangle$, and conversely any pseudosection yields such a lift. However, unlike in the case of a section, now a further lift to $\mathrm{Mod}(\Sigma_g^1)$ does not allow us to directly determine the self-intersection number of S.[1]

[1]In principle, one can of course attempt to access all this information by first drawing the standard handle diagram for $(X', f|_{X'})$, then finding a sequence of Kirby moves from the induced diagram on $\partial X'$ to the standard handle diagram of $S^1 \times \Sigma_g$ with $2g$ 1-handles and a 2-handle, and finally dragging back a 0-framed meridian to the 2-handle in the latter diagram to a 2-handle in the former, so as to calculate its framing, and so on. However, this not only proves to be a difficult exercise even for Kirby calculus aficionados, but also departs from our general approach to read off all the essential information from monodromy factorizations.

We will show that, under favorable conditions that apply to all the examples we will deal with in this article, knowing that the fiber class is primitive will be enough to build the spin structure:

Theorem 9. *Let (X, f) be a genus-g Lefschetz fibration with a monodromy factorization $t_{c_1} \cdots t_{c_n} = 1$, a primitive fiber class in $H_2(X)$, and signature $\sigma(X) \equiv 0 \pmod{16}$. Assume that there is a quadratic form $q\colon H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ with respect to the $\mathbb{Z}_2$-intersection pairing such that $q(c_i) = 1$ for all i, and $Arf(q) = 1$. Then* any *spin structure s' corresponding to a quadratic form q' on $H_1(\Sigma_g; \mathbb{Z}_2)$ with $q'(c_i) = 1$ for all i induces a spin structure on X.*

Proof. Let us first assume that there exists a genus-g *achiral* Lefschetz fibration (Y_g, f_g), with a monodromy factorization $t_{d_1}^{\epsilon_1} \cdots t_{d_m}^{\epsilon_m} = 1$ in $\mathrm{Mod}(\Sigma_g)$ (for some $\epsilon_i = \pm 1$) which satisfies the following four properties:

(i) $\sigma(Y_g) \equiv 8 \pmod{16}$,

(ii) f_g admits a section S_g of (possibly positive) odd self-intersection,

(iii) there is an $s_g \in \mathrm{Spin}(\Sigma_g)$ with a quadratic form q_g such that $q_g(d_i) = 1$ for all i,

(iv) $\mathrm{Arf}(q_g) = 1$.

We will show in a bit that there are such model fibrations for every genus $g \geq 1$.

Since (X, f) has a primitive fiber class, by our discussion above, it has a pseudosection S intersecting a regular fiber F geometrically once. Let $s \in \mathrm{Spin}(\Sigma_g)$ correspond to the quadratic form q in the hypothesis.

Removing a small neighborhood $N(F)$ of F, where S intersects F once, and $N(F_g)$ of the regular fiber F_g of (Y_g, f_g), we can take a fiber sum $(\hat{X}, \hat{f}) = (X, f) \# (Y_g, f_g)$, so that under the gluing $\partial(X \setminus N(F)) \to \partial(Y_g \setminus N(F_g))$ we have $S \cap \partial N(F)$ sent to $S_g \cap \partial N(F_g)$, which we can always achieve after an isotopy for any given fiber-preserving diffeomorphism. So this fibration has a pseudosection $\hat{S} = (S \setminus S \cap N(F)) \cup (S_g \setminus S_g \cap N(F_g)) = S \# S_g$, with self-intersection $\hat{S} \cdot \hat{S} = S \cdot S + S_g \cdot S_g \equiv S \cdot S + 1 \pmod 2$, as we have assumed S_g has odd self-intersection.

Identifying the boundaries $\partial N(F)$ and $\partial N(F_g)$ with $S^1 \times \Sigma_g$, the gluing diffeomorphism is prescribed by a self-diffeomorphism Φ of $S^1 \times \Sigma_g$, which is the product of the complex conjugation of the unit circle $S^1 \subset \mathbb{C}$, and some $\phi \in \mathrm{Diff}^+(\Sigma_g)$. Since both spin structures s and s_g have the same Arf invariant, we can choose ϕ to be a spin diffeomorphism $\phi\colon (\Sigma_g, s) \to (\Sigma_g, s_g)$. Now, the fibration $(\hat{X}, \hat{f})$ has monodromy curves $\{c_i, \phi^{-1}(d_j)\}$, with $q(c_i) = 1$ and $q(\phi^{-1}(d_j)) = q_g(d_j) = 1$ for all i, j. If the pseudosection S of (X, f) had odd self-intersection, then $\hat{S}$ would have even self-intersection, and by Theorem 2, $\hat{X}$ would be spin. However, by Novikov additivity, the signature satisfies

$$\sigma(\hat{X}) = \sigma(X) + \sigma(Y_g) \equiv 8 \pmod{16},$$

and Rokhlin's theorem implies that $\hat{X}$ cannot be spin. So we deduce that S has even self-intersection, and thus we can invoke Theorem 2 to conclude that our original Lefschetz fibration (X, f) is spin, in fact not only for q, but for any quadratic form q' with $q'(c_i) = 1$ for all i.

We are left with building the models (Y_g, f_g). Note that by Proposition 3, any achiral Lefschetz fibration satisfying (iii) should necessarily have signature $\sigma(Y_g) \equiv 0 \pmod 8$. Some of these models can be derived immediately from Examples 4–6. When we cap off the boundary components, all these relations yield hyperelliptic Lefschetz fibrations, so using Endo's signature formula for hyperelliptic fibrations [20], we can easily see that the even-chain relation yields Lefschetz fibrations with the desired properties when $g \equiv 2 \pmod 4$, whereas all three relations yield such examples when $g \equiv 1 \pmod 4$.

To get models for all g however, we will go about it a little differently, and build achiral fibrations out of the smallest chain relations instead—while remembering that everything we have discussed so far also applies to achiral fibrations. Below, let $\{\alpha_i, \beta_i\}$ be a symplectic basis for $H_1(\Sigma_g; \mathbb{Z}_2)$ generated by the same labeled curves given in Figure 3.

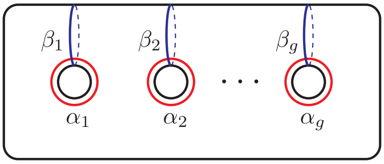


Fig. 3. The symplectic basis for $H_1(\Sigma_g; \mathbb{Z}_2)$.

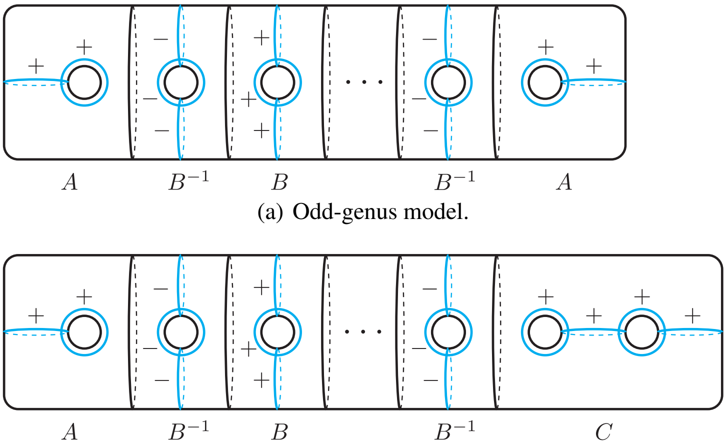
$g = 1$: The capped off even-chain, odd-chain and hyperelliptic relations all give Hurwitz equivalent positive factorizations, the monodromy curves of which evaluate as 1 under the quadratic form prescribed by $q_1(\alpha_1) = q_1(\beta_1) = 1$. Its signature is -8.

$g = 2$: The capped off even chain relation gives a positive factorization, the monodromy curves of which evaluate as 1 under the quadratic form prescribed by $q_2(\alpha_1) = q_2(\beta_1) = q_2(\alpha_2) = 1$ and $q_2(\beta_2) = 0$. Its signature is -24.

Every odd $g \geq 3$: We have a decomposition $\Sigma_g = \Sigma_1^1 \cup \Sigma_1^2 \cup \cdots \cup \Sigma_1^2 \cup \Sigma_1^1$, with $2k-1$ copies of Σ_1^2 for $g = 2k+1$. Let A denote the 2-chain relation $(t_{c_1} t_{c_2})^6 t_\delta^{-1}$ and B denote the 3-chain relation $(t_{c_1} t_{c_2} t_{c_3})^4 t_{\delta_1}^{-1} t_{\delta_2}^{-1}$. Embed A in both copies of Σ_1^1, and embed B^{-1} into the first copy of Σ_1^2, B into the second copy, and keep alternating the embeddings in the same fashion for all the copies of Σ_1^2. The embeddings of the boundary twists $t_\delta, t_{\delta_1}, t_{\delta_2}$ all cancel out with each other. So we get an achiral fibration (Y_g, f_g), for $g = 2k+1$, with only non-separating monodromy curves as in Figure 4 (a). Take the quadratic form q which maps $q_g(\alpha_i) = q_g(\beta_i) = 1$ for all i, so q maps all the monodromy curves to 1. By [21], taking the algebraic sum of the signatures of the relations $\sigma(A) = -7$ and $\sigma(B) = -6$, we calculate that $\sigma(Y_g) = 2(-7) + k(6) + (k-1)(-6) = -8$.

Every even $g \geq 4$: This time we have a decomposition $\Sigma_g = \Sigma_1^1 \cup \Sigma_1^2 \cup \cdots \cup \Sigma_1^2 \cup \Sigma_2^1$ with $2k-1$ copies of Σ_1^2 for $g = 2k+2$. Let A and B denote the 2-chain and 3-chain relations as above, and in addition, let C denote the 4-chain relation $(t_{c_1} t_{c_2} t_{c_3} t_{c_4})^{10}\, t_\delta^{-1}$. Embed A into Σ_1^1, then B^{-1} and B into the copies of Σ_1^2 in the same alternating fashion as above. This time we finish with embedding C into the last Σ_2^1 piece. Once again, the embeddings of the boundary twists $t_\delta, t_{\delta_1}, t_{\delta_2}$ all cancel out with each other, and we get an achiral fibration (Y_g, f_g), for $g = 2k + 2$, with non-separating monodromy curves as in Figure 4 (b). Take the quadratic form q_g which maps all $q_g(\alpha_i) = q_g(\beta_j) = 1$ except for $q_g(\beta_{g-1}) = 0$. One can see that all the monodromy curves are mapped to 1 under q_g. Using [21] again, by taking the algebraic sum of the signatures of the relations $\sigma(A) = -7$, $\sigma(B) = -6$ and $\sigma(C) = -23$, we calculate that $\sigma(Y_g) = (-7) + k(6) + (k-1)(-6) + (-23) = -24$.

It is easy to check that the quadratic form we described for each fibration (Y_g, f_g) has Arf invariant 1. (In fact, there is a unique quadratic form for each one, since the monodromy curves of each f_g generate $H_1(\Sigma_g; \mathbb{Z}_2)$.) Finally, note that every (Y_g, f_g) has exceptional sections: This is obvious for $g = 1, 2$, and the higher genus ones always have an exceptional section supported in the Σ_1^1 piece (for instance since the 2-chain is obtained from the 3-chain by capping off a boundary component which has a single Dehn twist). So each (Y_g, f_g) has a section S_g of odd self-intersection (in fact a lot of them). ∎



(a) Odd-genus model.

(b) Even-genus model.

Fig. 4. Model spin achiral Lefschetz fibrations.

Remark 10. The converse to the statement of Theorem 9 is not true. For instance, when we double the capped off even-chain relation on Σ_g with $g \equiv 0$ or 3 (mod 4), we get a spin Lefschetz fibration with signature divisible by 16, but the only possible quadratic form on Σ_g, under which all the monodromy curves are mapped to 1, has Arf invariant 0. On the other hand, when $\operatorname{rank}(H^1(X; \mathbb{Z}_2)) > 0$, one often finds spin structures on (X, f) coming from quadratic forms on the fiber Σ_g with either Arf invariant, which will be the case for our key example given in Theorem 14. (While this is often the case, it is not always true either; the double of the genus $g = 2m + 1$ Matsumoto–Cadavid–Korkmaz fibration on $(\Sigma_m \times S^2) \# 8\overline{\mathbb{CP}}^2$ has 2^{2m} distinct spin structures, but a calculation similar to ours in Example 7 shows that *all* of these spin structures come from a quadratic form on Σ_g with Arf invariant 1!)

Remark 11. While the achiral Lefschetz fibrations (Y_g, f_g) we built in the proof of Theorem 9 might be of some interest, the 4-manifolds Y_g themselves, as well as their spin doubles can all be seen to decompose into a connected sum of standard simply-connected 4-manifolds $S^2 \times S^2$, $\mathbb{CP}^2$ and K3 surface, taken with either orientations. To see this, first observe that they are all fiber sums of 4-manifolds that are such connected sums themselves, along embedded spheres of opposite self-intersections (coming from canceling boundary twists), then apply classical cobordism arguments due to Mandelbaum and Moishezon, as in [12, 29, 40]. In particular, none of them are symplectic when $g > 2$.

3. Lefschetz fibrations with signature zero

We will construct our examples of signature zero Lefschetz fibrations in three steps, split into the next three subsections, where we employ the breeding technique [10] in increasing complexity, to build signature zero Lefschetz pencils and fibrations out of lower genera pencils. This is done through careful embeddings of the corresponding positive factorizations into the mapping class group of higher genus surfaces, so that one can cancel all the negative Dehn twists against positive ones. Our 3-step construction will result in the signature zero genus-9 Lefschetz fibration (X, f) of Theorem 14.

The interested reader can see how the key signature zero example (X, f) comes to life, without getting bogged down in more technical details. Here is the outline of our construction: We first build relation (6) in $\mathrm{Mod}(\Sigma_2^4)$ for a genus-2 pencil, using the 2-chain and several lantern relations. We embed two copies of this particular relation into $\mathrm{Mod}(\Sigma_3^4)$, as shown in Figure 10, in order to obtain a new relation for a genus-3 pencil. Importantly, the resulting relation (8) contains positive Dehn twists along four bounding pairs given in Figure 11, each cobounding two copies of Σ_1^4 with a pair of negative Dehn twist curves corresponding to the base points of the pencil. We then embed two copies of this special relation into $\mathrm{Mod}(\Sigma_9)$ as in Figure 15. At this point we have four pairs of negative Dehn twists, which we will cancel one pair at a time, by carefully embedding four more copies of the same genus-3 relation, so that each time pairs of negative boundary twists match and cancel with positive bounding pair twists of the first two original genus-3 relations, while a positive bounding pair matches and cancels with a negative pair. Here one simply needs to see how the positive bounding pairs from the top and the bottom halves of Σ_9 in Figure 15 cobound a Σ_3^4, split into two copies of Σ_1^4, cobounded by these positive pairs and a negative pair from the first two embeddings of the genus-3 relation. Following the ingenious work of Endo and Nagami [21], which allows one to calculate the signature of a Lefschetz fibration via elementary relations in the mapping class group, a simple algebraic count of the total number of 2-chain and lantern relations we employed (as cancellations and braid relations do not affect the signature) confirms that the genus-2 and genus-3 pencils, and the genus-9 Lefschetz fibration we built at the end, all have signature zero.

During our 3-step construction we will also chase around an additional marked point on the fiber, the information on which we will need only later when calculating the divisibility of the fiber class of (X, f). The reader who is not interested in this particular calculation can safely ignore the additional point-pushing maps that appear in our monodromy factorizations. In fact, while the breeding technique plays an innovative role in the construction of (X, f), because we present this fibration with an explicit positive factorization, the more conservative reader may choose to skip the next three subsections and verify its monodromy given in Theorem 14 in a straightforward fashion, using the Alexander method. (This is a tedious but still manageable calculation, as we have observed while exercizing due diligence to test our monodromy using the same method.)

3.1. A signature zero genus-2 pencil

We begin by describing a genus-2 Lefschetz pencil whose topology and monodromy both have special features we need for the later steps of our construction. Namely, we would like the total space to have signature zero and be spin, and the monodromy curves to contain a bounding pair and two disjoint separating curves. (The need for all these properties will become clear as we move on to next steps.)

In order to make our construction as self-contained as possible, we will derive all our relations from elementary relations that are known to generate the mapping class group, namely, the 2-chain relation, the lantern relation, and the braid relation, along with commutativity, cancellation and conjugation. In the computations we will freely perform Hurwitz moves and cyclic permutations without stating explicitly.

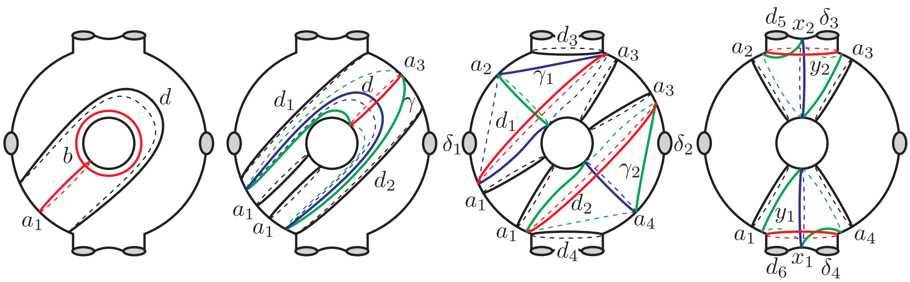
A variation of the 6-holed torus relation: We first present a variation of what is known as the 6-holed torus relation [38]. The curves involved in our construction to follow are given in Figure 5 (a).

We begin with the following 2-chain relation and lantern relation:

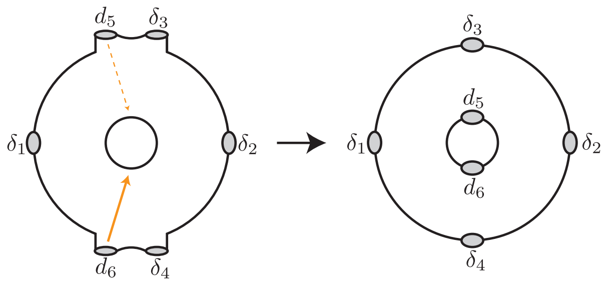
$$\begin{aligned} (t_{a_1} t_b)^6 &= t_d, \\ t_{a_3} t_d t_\gamma &= t_{a_1} t_{a_1} t_{d_1} t_{d_2}. \end{aligned}$$

We combine them as follows (here we underline the parts that we modify in that very step):

$$\begin{aligned} 1 &= t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_d^{-1} \cdot t_d t_\gamma t_{a_3} t_{a_1}^{-1} t_{a_1}^{-1} t_{d_1}^{-1} t_{d_2}^{-1} \\ &= t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_d^{-1} \cdot t_d t_\gamma t_{a_3} t_{a_1}^{-1} t_{a_1}^{-1} t_{d_1}^{-1} t_{d_2}^{-1} \cdot t_{a_1} t_b t_{a_1} t_b \\ &= t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_d^{-1} \cdot t_d t_\gamma t_{a_3} t_{a_1}^{-1} t_{a_1}^{-1} \cdot t_{a_1} t_b t_{a_1} t_b \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\ &= t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_\gamma t_{a_3} t_{a_1}^{-1} \underline{t_b t_{a_1} t_b} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\ &= t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_{a_1} t_b t_\gamma t_{a_3} t_{a_1}^{-1} t_{a_1} t_b t_{a_1} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\ &= t_{a_1} t_b t_{a_1} \underline{t_b t_{a_1} t_b t_{a_1} t_b t_\gamma} t_{a_3} t_b t_{a_1} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\ &= t_{a_1} t_b t_{a_1} t_{a_3} t_b t_{a_1} t_b t_{a_1} t_b t_{a_3} t_b t_{a_1} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \end{aligned}$$



(a) The 2-chain relation and lantern relations.



(b) Rearrangement of the boundary components.

Fig. 5. Construction of a 6-holed torus relation.

(here we have used the simple observation that $t_b t_{a_1} t_b t_{a_1} t_b(\gamma) = a_3$)

$$
\begin{aligned}
&= t_{a_1} t_b t_{a_1} t_{a_3} t_b t_{a_1} t_b t_{a_1} \underline{t_b t_{a_3} t_b} t_{a_1} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\
&= t_{a_1} t_b t_{a_1} t_{a_3} t_b t_{a_1} t_b t_{a_1} t_{a_3} t_b t_{a_3} t_{a_1} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\
&= t_{a_1} t_b t_{a_1} t_{a_3} t_b t_{a_1} t_b t_{a_3} \underline{t_{a_1} t_b t_{a_1}} t_{a_3} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\
&= t_{a_1} t_b t_{a_1} t_{a_3} t_b t_{a_1} \underline{t_b t_{a_3} t_b} t_{a_1} t_b t_{a_3} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\
&= t_{a_1} t_b t_{a_1} t_{a_3} t_b t_{a_1} t_{a_3} t_b t_{a_3} t_{a_1} t_b t_{a_3} \cdot t_{d_1}^{-1} t_{d_2}^{-1} \\
&= t_{a_1} t_b \cdot \underline{t_{a_1} t_{a_3} t_{d_1}^{-1}} \cdot t_b t_{a_1} t_{a_3} t_b \cdot \underline{t_{a_3} t_{a_1} t_{d_2}^{-1}} \cdot t_b t_{a_3}.
\end{aligned}
$$

Now we substitute the two lantern relations

$$
\begin{aligned}
t_{d_1} t_{\gamma_1} t_{a_2} &= t_{a_1} t_{a_3} t_{\delta_1} t_{d_3}, \\
t_{d_2} t_{a_4} t_{\gamma_2} &= t_{a_1} t_{a_3} t_{\delta_2} t_{d_4},
\end{aligned}
$$

to get

$$
\begin{aligned}
1 = t_{a_1} t_b \cdot t_{a_1} t_{a_3} t_{d_1}^{-1} \cdot t_{a_1}^{-1} t_{a_3}^{-1} t_{d_1} t_{\gamma_1} t_{a_2} t_{\delta_1}^{-1} t_{d_3}^{-1} \cdot t_b t_{a_1} t_{a_3} t_b \\
\cdot t_{\delta_2}^{-1} t_{d_4}^{-1} t_{a_4} t_{\gamma_2} t_{d_2} t_{a_1}^{-1} t_{a_3}^{-1} \cdot t_{a_3} t_{a_1} t_{d_2}^{-1} \cdot t_b t_{a_3}.
\end{aligned}
$$

Cancellation and commutativity yield

$$\begin{aligned}
t_{\delta_1} t_{\delta_2} &= \underline{t_{a_1} t_b t_{\gamma_1}} t_{a_2} t_b t_{a_1} \cdot t_{d_3}^{-1} t_{d_4}^{-1} \cdot t_{a_3} t_b t_{a_4} \underline{t_{\gamma_2} t_b t_{a_3}} \\
&= t_{B_0} t_{a_1} t_b t_{a_2} t_b t_{a_1} \cdot t_{d_3}^{-1} t_{d_4}^{-1} \cdot t_{a_3} t_b t_{a_4} t_b t_{a_3} t_{B'_0} \\
&= t_{B_0} t_{a_1} \underline{t_b t_{a_2} t_b} t_{a_1} \cdot t_{d_3}^{-1} t_{d_4}^{-1} \cdot t_{a_3} \underline{t_b t_{a_4} t_b} t_{a_3} t_{B'_0} \\
&= t_{B_0} \underline{t_{a_1} t_{a_2} t_b} t_{a_2} t_{a_1} \cdot t_{d_3}^{-1} t_{d_4}^{-1} \cdot t_{a_3} t_{a_4} \underline{t_b t_{a_4} t_{a_3}} t_{B'_0} \\
&= t_{B_0} t_{B_1} t_{a_1} t_{a_2} t_{a_2} t_{a_1} \cdot t_{d_3}^{-1} t_{d_4}^{-1} \cdot t_{a_3} t_{a_4} t_{a_4} t_{a_3} t_{B'_1} t_{B'_0} \\
&= t_{B_0} t_{B_1} t_{a_1} t_{a_2} \cdot \underline{t_{a_1} t_{a_4} t_{d_4}^{-1}} \cdot \underline{t_{d_3}^{-1} t_{a_2} t_{a_3}} \cdot t_{a_4} t_{a_3} t_{B'_1} t_{B'_0},
\end{aligned}$$

where $B_0 := t_{a_1} t_b(\gamma_1)$, $B'_0 := t_{a_3}^{-1} t_b^{-1}(\gamma_2)$, $B_1 := t_{a_1} t_{a_2}(b)$, and $B'_1 := t_{a_3}^{-1} t_{a_4}^{-1}(b)$. We further substitute two more lantern relations, specifically

$$\begin{aligned}
t_{d_4} t_{x_1} t_{y_1} &= t_{a_1} t_{a_4} t_{\delta_4} t_{d_6}, \\
t_{d_3} t_{x_2} t_{y_2} &= t_{a_2} t_{a_3} t_{\delta_3} t_{d_5},
\end{aligned}$$

so that

$$\begin{aligned}
t_{\delta_1} t_{\delta_2} = t_{B_0} t_{B_1} t_{a_1} t_{a_2} \cdot t_{a_1} t_{a_4} t_{d_4}^{-1} \cdot t_{a_1}^{-1} t_{a_4}^{-1} t_{d_4} t_{x_1} t_{y_1} t_{\delta_4}^{-1} t_{d_6}^{-1} \\
\cdot t_{d_5}^{-1} t_{\delta_3}^{-1} t_{x_2} t_{y_2} t_{d_3} t_{a_2}^{-1} t_{a_3}^{-1} \cdot t_{d_3}^{-1} t_{a_2} t_{a_3} \cdot t_{a_4} t_{a_3} t_{B'_1} t_{B'_0}.
\end{aligned}$$

This now gives

$$t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} t_{d_5} t_{d_6} = t_{B_0} t_{B_1} t_{a_1} t_{a_2} t_{x_1} t_{y_1} t_{x_2} t_{y_2} t_{a_4} t_{a_3} t_{B'_1} t_{B'_0}$$

or

$$t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} t_{d_5} t_{d_6} = t_{B_0} t_{B_1} \cdot t_{a_1} t_{a_2} t_{x_1} t_{x_2} \cdot t_{y_1} t_{y_2} t_{a_3} t_{a_4} \cdot t_{B'_1} t_{B'_0} \tag{5}$$

in $\mathrm{Mod}(\Sigma_1^6)$. Finally, we push the boundary components d_5 and d_6 as indicated in Figure 5 (b) so we get the Dehn twist curves as depicted in Figure 6. With this configuration of the curves in mind, relation (5) is the 6-holed torus relation we wanted. Note that we have used one 2-chain relation and five lantern relations to construct relation (5).

A genus-2 pencil: We now construct the genus-2 Lefschetz pencil that will become one of our main building blocks. Take the boundary components d_5, d_6 of the 6-holed torus Σ_1^6 in the previous step and connect them by a tube as shown in Figure 7 (a). This gives a Σ_2^4, a genus-2 surface with four boundary components. The curves d_5 and d_6 become identical, so we denote both by d.

Now for the quadruples of curves a_1, a_2, x_1, x_2 and y_1, y_2, a_3, a_4 each cobounding a Σ_0^4 as shown in Figure 7 (b), we have the lantern relations

$$\begin{aligned}
t_d t_{B_2} t_C &= t_{a_1} t_{a_2} t_{x_1} t_{x_2}, \\
t_{C'} t_{B'_2} t_d &= t_{y_1} t_{y_2} t_{a_3} t_{a_4},
\end{aligned}$$

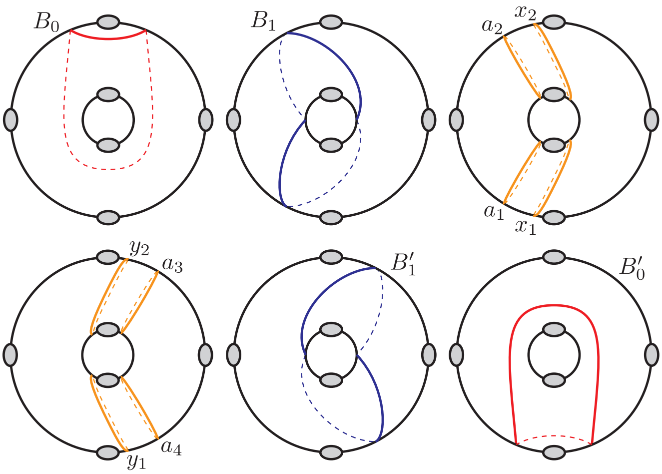
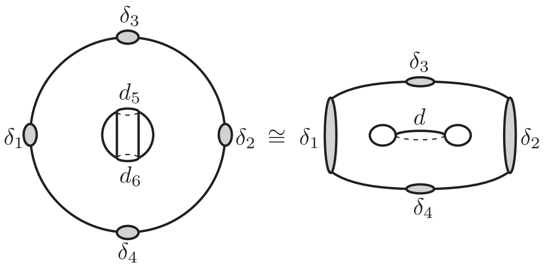
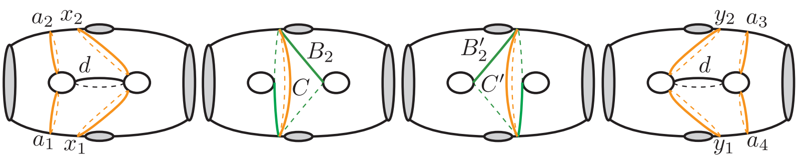


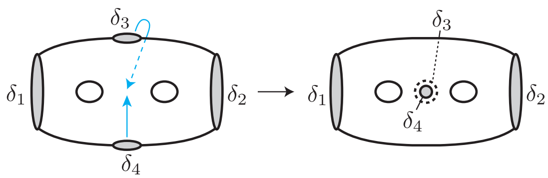
Fig. 6. The curve configuration for our 6-holed torus relation.



(a) The embedding of Σ_1^6 into Σ_2^4.



(b) Two configurations of lantern curves in Σ_2^4.



(c) Rearranging the boundary components δ_3 and δ_4. On the right, δ_4 is on the front and δ_3 is on the back of the surface.

Fig. 7. Construction of a genus-2 pencil.

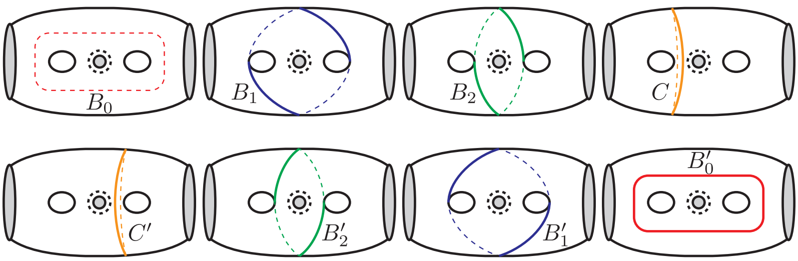


Fig. 8. Vanishing cycles of a genus-2 pencil. Notice that B_0, B_0' is a bounding pair and the two separating curves C, C' are disjoint.

where the curves B_2, C, C', B_2' are as in Figure 7 (b). Combining the two with our 6-holed torus relation (5) yields

$$\begin{aligned} t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4}t_d t_d &= t_{B_0}t_{B_1}\cdot \underline{t_{a_1}t_{a_2}t_{x_1}t_{x_2}}\cdot \underline{t_{y_1}t_{y_2}t_{a_3}t_{a_4}}\cdot t_{B_1'}t_{B_0'} \\ &= t_{B_0}t_{B_1}\cdot t_d t_{B_2}t_C\cdot t_{C'}t_{B_2'}t_d\cdot t_{B_1'}t_{B_0'}. \end{aligned}$$

By canceling the t_d factors we obtain the relation

$$t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4} = t_{B_0}t_{B_1}t_{B_2}t_C t_{C'}t_{B_2'}t_{B_1'}t_{B_0'} \tag{6}$$

in $\mathrm{Mod}(\Sigma_2^4)$. Moreover, after pushing the boundary components δ_3 and δ_4 as indicated in Figure 7 (c), we get a neater presentation of the Dehn twist curves as illustrated in Figure 8. Pairs of curves labeled with the same letters, but one decorated with a prime and one without it, are all symmetric under the obvious involution.

Relation (6) is the monodromy factorization for our genus-2 Lefschetz pencil. Observe that in the monodromy factorization (6) the pair B_0, B_0' cobounds two copies of Σ_1^4 with boundary components, and we also have two disjoint separating curves C and C', each cobounding a copy of Σ_1^2 with a boundary component.

Let us compute the signature σ of the total space of this genus-2 pencil. As we noted earlier, the 6-holed torus relation (5) is derived by using a single 2-chain relation and five lantern relations. We have then used two more lantern relations to obtain relation (6). Since the 2-chain relation and the lantern relation contribute -7 and $+1$ to the signature, respectively, we compute the signature of the pencil as $\sigma = 1(-7) + 7(+1) = 0$.

We can moreover describe an explicit spin structure on this pencil (which we will make use of in Section 5.1). Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \delta_1, \delta_2, \delta_4\}$ be a basis for $H_1(\Sigma_2^4; \mathbb{Z}_2)$, given by the same labeled curves in Figure 9.

We can then compute the $\mathbb{Z}_2$-homology classes of the vanishing cycles as

$$\begin{aligned} B_0 &= \alpha_1 + \alpha_2 + \delta_1 + \delta_2 + \delta_4, \\ B_1 &= \alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \delta_1 + \delta_2 + \delta_4, \\ B_2 &= \beta_1 + \beta_2 + \delta_1 + \delta_2 + \delta_4, \\ C &= \delta_1, \end{aligned}$$

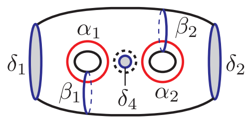


Fig. 9. A basis for $H_1(\Sigma_2^4;\mathbb{Z}_2)$.

$$
\begin{aligned}
C' &= \delta_2,\\
B_2' &= \beta_1 + \beta_2 + \delta_4,\\
B_1' &= \alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \delta_4,\\
B_0' &= \alpha_1 + \alpha_2 + \delta_4.
\end{aligned}
$$

If we now define a quadratic form q on $H_1(\Sigma_2^4;\mathbb{Z}_2)$ such that

$$
q(\alpha_i) = q(\beta_i) = q(\delta_j) = 1 \quad \text{for all } i = 1,2 \text{ and } j = 1,\dots,4, \tag{7}
$$

then it maps each vanishing cycle to 1, and by Theorem 1, we get a spin structure on the total space of the pencil.

Equipping the pencil with a Gompf–Thurston symplectic form, we get a symplectic 4-manifold. Then, observing that the fiber of the pencil violates the adjunction inequality, we conclude that the total space has to be a rational or a ruled surface. As it quickly follows from the above calculation, the rank of the first $\mathbb{Z}_2$-homology of this 4-manifold is 2, thus it should be the ruled surface $T^2 \times S^2$.

Remark 12. There are other explicit monodromy factorizations for genus-2 pencils with four base points on the ruled surface $T^2 \times S^2$; they were obtained by the second author [32] as lifts of Matsumoto's well-known relation [42]. It is natural to ask whether the one we discovered here is Hurwitz equivalent to any of those, and we will show that this is indeed the case in Appendix A.

3.2. A signature zero genus-3 pencil

We will next describe a genus-3 pencil, the total space of which has signature zero and is spin, whereas its monodromy has four (in fact five) pairs of Dehn twists about certain bounding pairs.

We breed two copies of the genus-2 pencil constructed in the previous subsection to obtain the desired genus-3 pencil. The curves $\delta_1, \delta_3, C_3, C_4$ on Σ_3^4 in Figure 10 cobound a subsurface diffeomorphic to Σ_2^4. The same goes for the curves $\delta_2, \delta_4, C_1, C_2$. Thus, we can embed two copies of relation (6) in $\mathrm{Mod}(\Sigma_2^4)$ into $\mathrm{Mod}(\Sigma_3^4)$ as

$$
\begin{aligned}
t_{a'} t_x t_b t_{C_1} t_{C_2} t_{d'} t_w t_a &= t_{\delta_1} t_{\delta_3} t_{C_3} t_{C_4},\\
t_{c'} t_z t_d t_{C_3} t_{C_4} t_{b'} t_y t_c &= t_{\delta_2} t_{\delta_4} t_{C_1} t_{C_2},
\end{aligned}
$$

where the curves are as shown in Figure 11. Note that the second embedding is obtained from the first embedding followed by a rotation by π about the horizontal line indicated

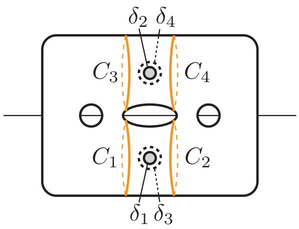


Fig. 10. The curves C_i, δ_i on Σ_3^4.

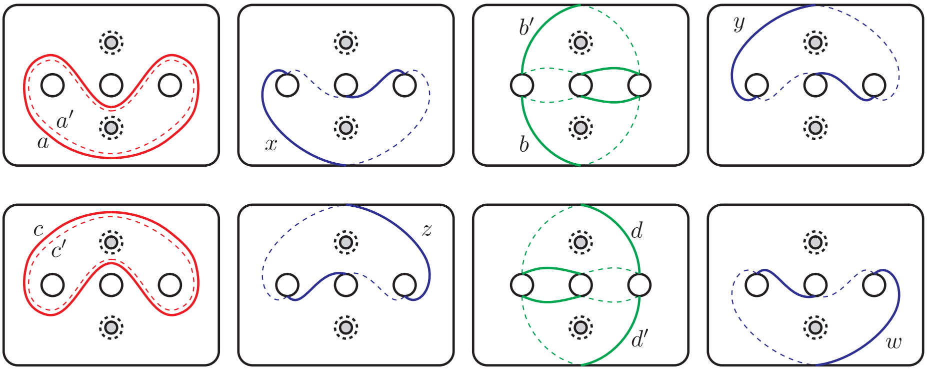


Fig. 11. Vanishing cycles of the genus-3 pencil.

in Figure 10. Dehn twist curves in the two relations above are the images, under the respective embeddings, of the curves of relation (6) in the same order.

By cyclic permutations, we can rewrite the relations as

$$t_{d'}t_w t_a t_{a'} t_x t_b = t_{\delta_1} t_{\delta_3} t_{C_3} t_{C_4} t_{C_2}^{-1} t_{C_1}^{-1},$$
$$t_{b'} t_y t_c t_{c'} t_z t_d = t_{\delta_2} t_{\delta_4} t_{C_1} t_{C_2} t_{C_4}^{-1} t_{C_3}^{-1}.$$

Combining them we get

$$t_{d'}t_w t_a t_{a'} t_x t_b \cdot t_{b'} t_y t_c t_{c'} t_z t_d = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4},$$

which, using cyclic permutation and commutativity, can be expressed as

$$t_a t_{a'} t_x t_b t_{b'} t_y t_c t_{c'} t_z t_d t_{d'} t_w = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}. \tag{8}$$

This is a positive factorization in $\mathrm{Mod}(\Sigma_3^4)$, which prescribes a genus-3 pencil.

In relation (8), we have the two bounding pairs a, a' and c, c', which are inherited from the genus-2 pencil. There are two more bounding pairs b, b' and d, d'; see Figure 11. (The existence of these four bounding pairs is the most essential feature for our constructing of the signature zero Lefschetz fibration in the next subsection.) There is in fact a fifth bounding pair we can get after Hurwitz moves: We can move the factor t_x over the subword $t_b t_{b'} t_y t_c t_{c'}$ (i.e. we conjugate this subword with t_x) to get the subword $t_x t_z$ in the monodromy, where the pair x, z also splits the surface into two genus-1 subsurfaces. (Note that the pair y, w is yet another bounding pair, but this property is destroyed when we bring t_x and t_z together above.)

The total space Y of our genus-3 pencil has signature zero since relation (8) is the combination of two copies of relation (6) with signature zero. The Euler characteristic is easily calculated as $\mathrm{e}(Y) = 4 - 4g + l - b = 4 - 4 \cdot 3 + 12 - 4 = 0$ (where g is the fiber genus, l is the number of Lefschetz critical points and b is the number of base points).

To pin down the topology of Y we will calculate its first homology. Let us use the symplectic basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$ for $H_1(\Sigma_4; \mathbb{Z})$ as in Figure 12. The homology classes of the vanishing cycles in this basis are $a = a' = c = c' = \alpha_1 + \alpha_3, x = z = \alpha_1 + \alpha_3 - \beta_1 + \beta_2 - \beta_3, b = b' = d = d' = \beta_1 - \beta_2 + \beta_3$ and $y = w = \alpha_1 + \alpha_3 + \beta_1 - \beta_2 + \beta_3$. In turn, the relations in $H_1(Y; \mathbb{Z})$ we obtain by setting the vanishing cycles equal to zero are $\alpha_3 = -\alpha_1$ and $\beta_3 = -\beta_1 + \beta_2$. Therefore $H_1(Y; \mathbb{Z})$ is freely generated by $\alpha_1, \beta_1, \alpha_2, \beta_2$, and $H_1(Y; \mathbb{Z}) = \mathbb{Z}^4$.

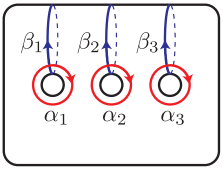


Fig. 12. The basis for $H_1(\Sigma_4; \mathbb{Z})$.

Since $\sigma(Y) = \mathrm{e}(Y)$ and $b_1(Y) = 4$, Y cannot be a rational or ruled surface. Since this genus-3 pencil has four base points, by [13, Theorem 1.2], Y is a symplectic Calabi–Yau 4-manifold, rational homology equivalent to T^4. (In fact one can show that $\pi_1(Y) \cong \mathbb{Z}^4$ and thus Y is homeomorphic to T^4.) In particular, we see that Y is spin. Alternatively, one can directly construct a quadratic form on Σ_3^4 that satisfies the conditions in Theorem 1 to show Y is spin.

Remark 13. There are other explicit monodromy factorizations for genus-3 pencils with four base points on symplectic Calabi–Yau 4-manifolds homeomorphic to T^4, obtained by the first author [10] and the second author and Hayano [33]. It is once again natural to ask whether the one we discovered here is Hurwitz equivalent to any of those, and we will also confirm that this is the case in Appendix A, which in particular will imply that this 4-manifold Y is *diffeomorphic* to the standard T^4.

A lift of the genus-3 relation: Here we pause our construction to capture a lift of relation (8) in $\mathrm{Mod}(\Sigma_3^4)$ to $\mathrm{Mod}(\Sigma_{3,2}^4)$. This lift will involve non-trivial point-pushing maps, and thus it will not yield an honest section, but we will plug this information in later to describe a *pseudosection* for the penultimate signature zero genus-9 Lefschetz fibration.

We first note the following simple yet useful observation (due to the anonymous referee). Suppose that $t_{c_1} \cdots t_{c_n} = t_{\delta_1} \cdots t_{\delta_b}$ is a positive factorization in $\mathrm{Mod}(\Sigma_g^b)$ with $b \geq 1$. As illustrated in Figure 13, we take an annulus with a marked point p in the inte-

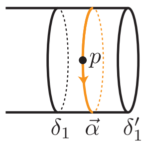


Fig. 13. Extension of Σ_g^b to $\Sigma_{g,1}^b$.

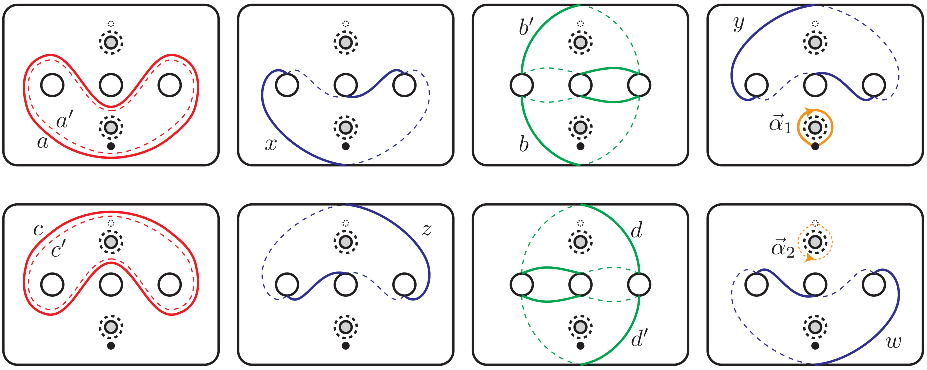


Fig. 14. A lift of the genus-3 relation to $\mathrm{Mod}(\Sigma_{3,2}^4)$.

rior and glue it with the surface Σ_g^b along the boundary component δ_1 to obtain a new surface $\Sigma_{g,1}^b$, where δ_1 is replaced by the new boundary component δ_1'. Then

$$t_{c_1} t_{c_2} \cdots t_{c_n} = t_{\delta_1} t_{\delta_2} \cdots t_{\delta_b} = t_{\delta_1} t_{\delta_1'}^{-1} \cdot t_{\delta_1'} t_{\delta_2} \cdots t_{\delta_b} = \mathcal{P}_{\vec{\alpha}} \cdot t_{\delta_1'} t_{\delta_2} \cdots t_{\delta_b},$$

where $\vec{\alpha}$ is an oriented loop based at p on the annulus as in Figure 13. This is a lift of the original relation to $\mathrm{Mod}(\Sigma_{g,1}^b)$, which only depends on the choice of a boundary component. Obviously, we can also apply this operation to multiple boundary components to get a lift with multiple marked points.

Going back to relation (8) in $\mathrm{Mod}(\Sigma_3^4)$, we put the first marked point near δ_1 on the front of the surface and the second near δ_4 on the back as in Figure 14. Then relation (8) lifts to $\mathrm{Mod}(\Sigma_{3,2}^4)$ as

$$t_a t_{a'} t_x t_b t_{b'} t_y t_c t_{c'} t_z t_d t_{d'} t_w \cdot \mathcal{P}_{\vec{\alpha}_1} \mathcal{P}_{\vec{\alpha}_2} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}, \tag{9}$$

where $\vec{\alpha}_1$ and $\vec{\alpha}_2$ are based loops parallel to δ_1 and δ_4, respectively. Note that we placed the point-pushing maps on the other side of t_{δ_1} and t_{δ_4} in the equation and accordingly the orientations of the loops were chosen so that the point-pushing maps become positive on that side.

3.3. A signature zero genus-9 Lefschetz fibration over the 2-sphere

We will now describe our genus-9 fibration (X, f), the total space of which has signature zero. Further properties of X, such as it being spin, will be explored in Section 4.1. Expectedly, its monodromy factorization will not contain any Dehn twists about separating curves (which would destroy the spin property), but it contains Dehn twists about a quadruple of bounding curves which split the Σ_9 into two copies of Σ_3^4. This extra property will be essential for our arguments in Section 5.

We will first explain how we obtain our signature zero Lefschetz fibration, without specifying all the embeddings involved in this construction, and thus without an explicit description of all the curves in the final monodromy factorization. We will also omit the marked point in this exposition. This first round of information suffices to justify the existence of a signature zero genus-9 Lefschetz fibration. Afterwards, we will describe the embeddings explicitly, and also include the marked point for the pseudosection calculation. The latter will take the chunk of this subsection.

Schematic construction: For simplicity, we omit all the marked points in any of the figures we will refer to here. We will breed six copies of our genus-3 pencil with monodromy factorization (8). As seen on the right in Figure 15, the curves $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ on Σ_9 bound two copies of Σ_3^4, which constitute the top and the bottom halves of Σ_9. We embed two copies of relation (8) in $\mathrm{Mod}(\Sigma_3^4)$ into $\mathrm{Mod}(\Sigma_9)$ via the embeddings Φ_1 and Φ_2 as explained in Figure 15 (and its caption), such that the first one is supported on the top and the second one on the bottom half:

$$t_{a_1}t_{a_1'}t_{x_1}t_{b_1}t_{b_1'}t_{y_1}t_{c_1}t_{c_1'}t_{z_1}t_{d_1}t_{d_1'}t_{w_1} = t_{\Delta_1}t_{\Delta_2}t_{\Delta_3}t_{\Delta_4},$$
$$t_{a_2}t_{a_2'}t_{x_2}t_{b_2}t_{b_2'}t_{y_2}t_{c_2}t_{c_2'}t_{z_2}t_{d_2}t_{d_2'}t_{w_2} = t_{\Delta_1}t_{\Delta_2}t_{\Delta_3}t_{\Delta_4}.$$

These curves are explicitly given in Figures 16 and 23. As usual, the Dehn twist curves in the two relations above are the images, under the respective embeddings Φ_1 and Φ_2, of the curves of relation (8) in the same order—and the same goes for our four other embeddings to follow.

Since these two relations have disjoint supports, we can combine them to get

$$t_{a_1}t_{a_1'}t_{a_2}t_{a_2'}t_{x_1}t_{x_2}t_{b_1}t_{b_1'}t_{b_2}t_{b_2'}t_{y_1}t_{y_2}t_{c_1}t_{c_1'}t_{c_2}t_{c_2'}t_{z_1}t_{z_2}t_{d_1}t_{d_1'}t_{d_2}t_{d_2'}t_{w_1}t_{w_2}$$
$$= t_{\Delta_1}^2 t_{\Delta_2}^2 t_{\Delta_3}^2 t_{\Delta_4}^2,$$

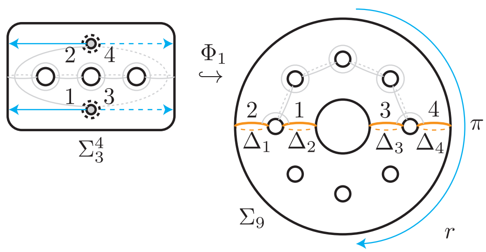


Fig. 15. The embedding Φ_1 of Σ_3^4 into Σ_9. First, we slide the four boundary components as indicated by the blue arrows. Then embed the surface into the top half of Σ_9. The labels $1, 2, 3, 4$ indicate how the boundary components are matched with the quadruple $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. Also, the gray curves, which fill Σ_3^4, and their embedded images are drawn so that the embedding Φ_1 is uniquely defined. The second embedding Φ_2 is given by Φ_1 followed by the π-rotation r, that is, $\Phi_2 := r \circ \Phi_1$.

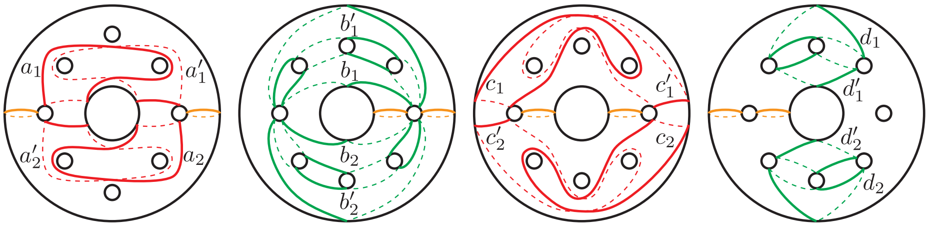


Fig. 16. The four bounding quadruples.

in $\mathrm{Mod}(\Sigma_9)$, which we can rewrite as

$$\underline{t_{a_1}t_{a_1'}t_{a_2}t_{a_2'}t_{\Delta_1}^{-1}t_{\Delta_4}^{-1}} \cdot t_{x_1}t_{x_2} \cdot \underline{t_{b_1}t_{b_1'}t_{b_2}t_{b_2'}t_{\Delta_3}^{-1}t_{\Delta_4}^{-1}} \cdot t_{y_1}t_{y_2}$$
$$\cdot \underline{t_{c_1}t_{c_1'}t_{c_2}t_{c_2'}t_{\Delta_2}^{-1}t_{\Delta_3}^{-1}} \cdot t_{z_1}t_{z_2} \cdot \underline{t_{d_1}t_{d_1'}t_{d_2}t_{d_2'}t_{\Delta_1}^{-1}t_{\Delta_2}^{-1}} \cdot t_{w_1}t_{w_2} = 1. \qquad (10)$$

The most essential feature of this relation is that it contains Dehn twists about four bounding quadruples, as singled out in Figure 16, such that the subsurfaces they cobound are diffeomorphic to Σ_3^4 with pairs of Δ_i as genus-1 bounding pairs in each Σ_3^4.

Now, let us examine the configuration of the curves $a_1, a_1', a_2, a_2', \Delta_1, \Delta_4$. Let S_3 be the subsurface bounded by the quadruple a_1, a_1', a_2, a_2' that contains Δ_1 and Δ_4. Then it is easy to observe that S_3 is diffeomorphic to Σ_3^4 and the pair Δ_1, Δ_4 splits S_3 into two genus-1 subsurfaces with boundary components $a_1, a_1', \Delta_1, \Delta_4$ and $a_2, a_2', \Delta_1, \Delta_4$, respectively. Turning to Figure 11 we note that the pair d, d' also splits Σ_3^4 into two genus-1 subsurfaces with boundary components $\delta_1, \delta_2, d, d'$ and $\delta_3, \delta_4, d, d'$, respectively. This means that the tuple of curves $(a_1, a_1', a_2, a_2', \Delta_1, \Delta_4)$ in S_3 is topologically equivalent to $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ in Σ_3^4. Therefore there exists an embedding Φ_3 of Σ_3^4 into Σ_9 such that

- Φ_3 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(a_1, a_1', a_2, a_2', \Delta_1, \Delta_4)$.

Identical arguments guarantee that there exist embeddings Φ_4, Φ_5, Φ_6 of Σ_3^4 into Σ_9 such that

- Φ_4 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(b_1, b_1', b_2, b_2', \Delta_3, \Delta_4)$,
- Φ_5 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(c_1, c_1', c_2, c_2', \Delta_2, \Delta_3)$,
- Φ_6 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(d_1, d_1', d_2, d_2', \Delta_1, \Delta_2)$.

Now the genus-3 relation (8) can be rearranged as

$$t_w t_a t_{a'} t_x t_b t_{b'} t_y t_c t_{c'} t_z = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} t_d^{-1} t_{d'}^{-1}.$$

So if we set $a_i := \Phi_i(a), a_i' := \Phi_i(a'), x_i := \Phi_i(x)$, and so on, then copies of this relation are embedded into Γ_9 as

$$t_{w_3}t_{a_3}t_{a_3'}t_{x_3}t_{b_3}t_{b_3'}t_{y_3}t_{c_3}t_{c_3'}t_{z_3} = t_{a_1}t_{a_1'}t_{a_2}t_{a_2'}t_{\Delta_1}^{-1}t_{\Delta_4}^{-1},$$
$$t_{w_4}t_{a_4}t_{a_4'}t_{x_4}t_{b_4}t_{b_4'}t_{y_4}t_{c_4}t_{c_4'}t_{z_4} = t_{b_1}t_{b_1'}t_{b_2}t_{b_2'}t_{\Delta_3}^{-1}t_{\Delta_4}^{-1},$$

$$t_{w_5} t_{a_5} t_{a'_5} t_{x_5} t_{b_5} t_{b'_5} t_{y_5} t_{c_5} t_{c'_5} t_{z_5} = t_{c_1} t_{c'_1} t_{c_2} t_{c'_2} t_{\Delta_2}^{-1} t_{\Delta_3}^{-1},$$
$$t_{w_6} t_{a_6} t_{a'_6} t_{x_6} t_{b_6} t_{b'_6} t_{y_6} t_{c_6} t_{c'_6} t_{z_6} = t_{d_1} t_{d'_1} t_{d_2} t_{d'_2} t_{\Delta_1}^{-1} t_{\Delta_2}^{-1}.$$

Then we can substitute those four relations into the underlined parts of relation (10) to obtain

$$t_{w_3} t_{a_3} t_{a'_3} t_{x_3} t_{b_3} t_{b'_3} t_{y_3} t_{c_3} t_{c'_3} t_{z_3} t_{x_1} t_{x_2} \cdot t_{w_4} t_{a_4} t_{a'_4} t_{x_4} t_{b_4} t_{b'_4} t_{y_4} t_{c_4} t_{c'_4} t_{z_4} t_{y_1} t_{y_2}$$
$$\cdot t_{w_5} t_{a_5} t_{a'_5} t_{x_5} t_{b_5} t_{b'_5} t_{y_5} t_{c_5} t_{c'_5} t_{z_5} t_{z_1} t_{z_2} \cdot t_{w_6} t_{a_6} t_{a'_6} t_{x_6} t_{b_6} t_{b'_6} t_{y_6} t_{c_6} t_{c'_6} t_{z_6} t_{w_1} t_{w_2} = 1. \qquad (11)$$

This is a positive factorization of the identity in $\mathrm{Mod}(\Sigma_9)$, so it provides a genus-9 Lefschetz fibration $f\colon X \to S^2$. Since we have only used copies of relation (8), which has signature zero, the total space X has signature zero.

Explicit construction: We will now describe explicit embeddings $\Phi_3, \Phi_4, \Phi_5, \Phi_6$ of Σ_3^4 (with or without a marked point) into $\Sigma_{9,1}$ to obtain an explicit monodromy factorization of the genus-9 fibration, as well as to pin-point a pseudosection.

On Σ_9 we take two parallel copies of Δ_2, $\hat{\Delta}_2$ and $\check{\Delta}_2$, and add a marked point between them as in Figure 17. The first two embeddings Φ_1 and Φ_2 are now regarded as embeddings of Σ_3^4 into $\Sigma_{9,1}$ in a straightforward fashion, which provide the two relations

$$t_{a_1} t_{a'_1} t_{x_1} t_{b_1} t_{b'_1} t_{y_1} t_{c_1} t_{c'_1} t_{z_1} t_{d_1} t_{d'_1} t_{w_1} = t_{\Delta_1} t_{\hat{\Delta}_2} t_{\Delta_3} t_{\Delta_4},$$
$$t_{a_2} t_{a'_2} t_{x_2} t_{b_2} t_{b'_2} t_{y_2} t_{c_2} t_{c'_2} t_{z_2} t_{d_2} t_{d'_2} t_{w_2} = t_{\Delta_1} t_{\check{\Delta}_2} t_{\Delta_3} t_{\Delta_4}$$

in $\mathrm{Mod}(\Sigma_{9,1})$. Combining them as before, we get a factorization (with no point-pushing map yet)

$$\underline{t_{a_1} t_{a'_1} t_{a_2} t_{a'_2} t_{\Delta_1}^{-1} t_{\Delta_4}^{-1}} \cdot t_{x_1} t_{x_2} \cdot \underline{t_{b_1} t_{b'_1} t_{b_2} t_{b'_2} t_{\Delta_3}^{-1} t_{\Delta_4}^{-1}} \cdot t_{y_1} t_{y_2}$$
$$\cdot \underline{t_{c_1} t_{c'_1} t_{c_2} t_{c'_2} t_{\check{\Delta}_2}^{-1} t_{\Delta_3}^{-1}} \cdot t_{z_1} t_{z_2} \cdot \underline{t_{d_1} t_{d'_1} t_{d_2} t_{d'_2} t_{\Delta_1}^{-1} t_{\hat{\Delta}_2}^{-1}} \cdot t_{w_1} t_{w_2} = 1. \qquad (12)$$

We can now describe our embeddings $\Phi_3, \Phi_4, \Phi_5, \Phi_6$. We first modify the presentation of the genus-3 relation (9) so that one of the bounding pair sits in a "standard position" as illustrated in Figure 19. Take the surface $\Sigma_{3,2}^4$ in Figure 14 and slightly slide the boundary components as indicated in Figure 18. Then we conjugate relation (9) by the series of Dehn twists

$$t_\beta t_\gamma t_\alpha t_\beta t_\gamma t_\alpha t_\eta^{-1}$$

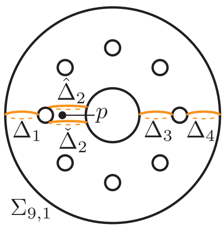


Fig. 17. The curves $\Delta_1, \hat{\Delta}_2, \check{\Delta}_2, \Delta_3, \Delta_4$ on $\Sigma_{9,1}$.

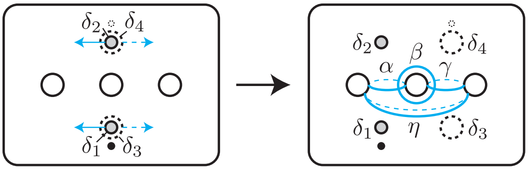


Fig. 18. Sliding the boundary components and the curves for the conjugation.

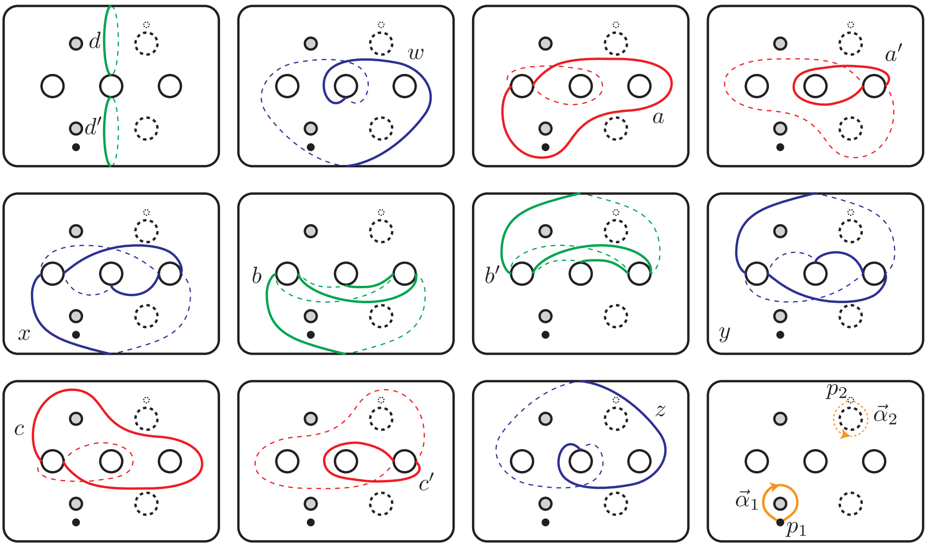


Fig. 19. Monodromy curves of the genus-3 pencil with simpler d and d'.

with the curves given on the right of Figure 18. This puts the bounding pair d, d' in the standard position as shown in Figure 19, where the images of the other curves are also given. Here we keep using the same symbols for the curves as those before the conjugation.

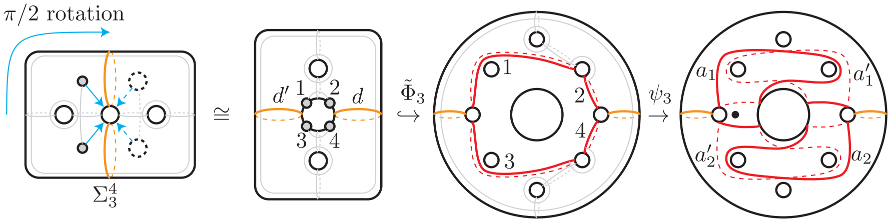
By cyclic permutation and commutativity, relation (9) becomes

$$\mathcal{P}_{\vec{\alpha}_2} \cdot t_w t_a t_{a'} t_x t_b t_{b'} t_y t_c t_{c'} t_z \cdot \mathcal{P}_{\vec{\alpha}_1} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} t_d^{-1} t_{d'}^{-1}. \tag{13}$$

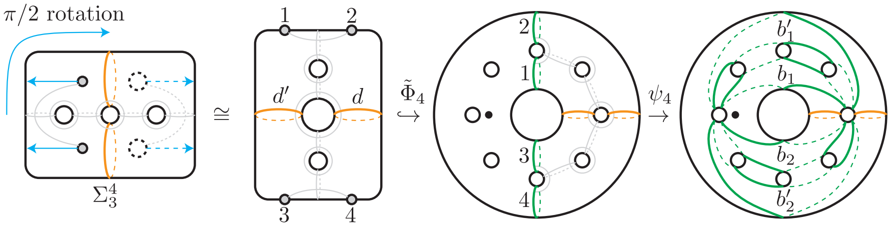
We will embed relation (13) with the curves in Figure 19 into $\mathrm{Mod}(\Sigma_{9,1})$. For Φ_3 and Φ_4 we do not need the marked points. For Φ_5 we include the marked point p_1 and forget p_2, whereas for Φ_6 we keep p_2 and forget p_1. In Figures 20 (a)–(d), we describe the embeddings

$$\Phi_3 : \Sigma_3^4 \to \Sigma_{9,1}, \quad \Phi_4 : \Sigma_3^4 \to \Sigma_{9,1}, \quad \Phi_5 : \Sigma_{3,1}^4 \to \Sigma_{9,1}, \quad \Phi_6 : \Sigma_{3,1}^4 \to \Sigma_{9,1}$$

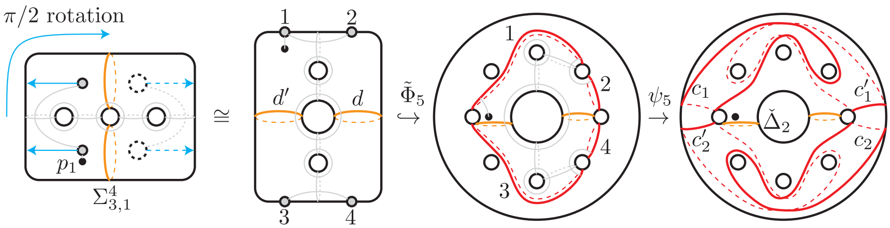
as the compositions of embeddings $\widetilde{\Phi}_i$ (which are easier to visualize) and diffeomorphisms ψ_i of $\Sigma_{9,1}$. The embeddings $\widetilde{\Phi}_i$ are uniquely specified by describing how the gray curves that fill the genus-3 surfaces are embedded. The diffeomorphisms ψ_3, ψ_4, ψ_5



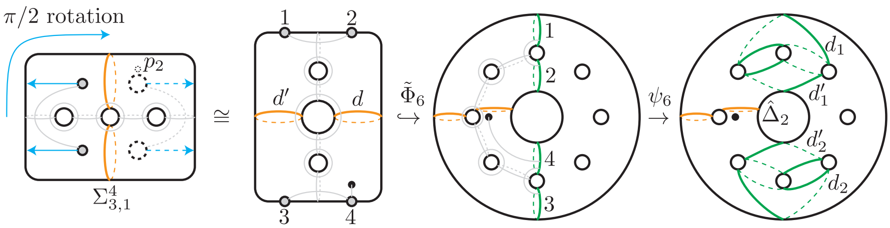
(a) The embedding $\Phi_3 = \psi_3 \circ \tilde{\Phi}_3$.



(b) The embedding $\Phi_4 = \psi_4 \circ \tilde{\Phi}_4$.

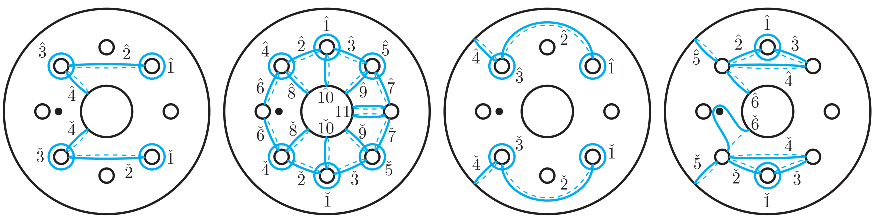


(c) The embedding $\Phi_5 = \psi_5 \circ \tilde{\Phi}_5$.



(d) The embedding $\Phi_6 = \psi_6 \circ \tilde{\Phi}_6$.

Fig. 20. The embeddings.



(a) Curves for ψ_3 (b) for ψ_4 (c) for ψ_5 (d) for ψ_6

Fig. 21. Dehn twist curves for the diffeomorphisms $\psi_3, \psi_4, \psi_5, \psi_6$.

and ψ_6, are given by the products of Dehn twists below. Here t_i means the right-handed Dehn twist about the curve labeled i in the respective Figures 21 (a)–(d):

$$\begin{aligned}
\psi_3 &:= t_{\check{4}}^{-1} t_{\hat{4}}^{-1} t_{\check{3}}^{-1} t_{\hat{3}}^{-1} t_{\check{2}}^{-1} t_{\hat{2}}^{-1} t_{\check{1}}^{-1} t_{\hat{1}}^{-1}, \\
\psi_4 &:= t_{11}^{2} t_{\check{10}} t_{\hat{10}} t_{\check{9}}^{-1} t_{\hat{9}}^{-1} t_{\check{8}}^{-1} t_{\hat{8}}^{-1} t_{\check{5}}^{-1} t_{\hat{5}}^{-1} t_{\check{4}}^{-1} t_{\hat{4}}^{-1} t_{\check{7}}^{-1} t_{\hat{7}}^{-1} t_{\check{6}}^{-1} t_{\hat{6}}^{-1} \\
&\quad \cdot t_{\check{3}}^{-1} t_{\hat{3}}^{-1} t_{\check{2}}^{-1} t_{\hat{2}}^{-1} t_{\check{1}}^{-1} t_{\hat{1}}^{-1} t_{\check{5}}^{-1} t_{\hat{5}}^{-1} t_{\check{4}}^{-1} t_{\hat{4}}^{-1} t_{\check{3}}^{-1} t_{\hat{3}}^{-1} t_{\check{2}}^{-1} t_{\hat{2}}^{-1} t_{\check{1}}^{-1} t_{\hat{1}}^{-1}, \\
\psi_5 &:= t_{\check{4}} t_{\hat{4}} t_{\check{3}} t_{\hat{3}} t_{\check{2}} t_{\hat{2}} t_{\check{1}} t_{\hat{1}}, \\
\psi_6 &:= t_{\check{6}}^{-1} t_{\hat{6}}^{-1} t_{\check{5}} t_{\hat{5}} t_{\check{4}} t_{\hat{4}} t_{\check{3}}^{-1} t_{\hat{3}}^{-1} t_{\check{2}}^{-1} t_{\hat{2}}^{-1} t_{\check{1}}^{-1} t_{\hat{1}}^{-1} t_{\check{3}}^{-1} t_{\hat{3}}^{-1} t_{\check{2}}^{-1} t_{\hat{2}}^{-1} t_{\check{1}}^{-1} t_{\hat{1}}^{-1}.
\end{aligned}$$

So

- Φ_3 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(a_1, a_1', a_2', a_2, \Delta_4, \Delta_1)$,
- Φ_4 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(b_1, b_1', b_2, b_2', \Delta_4, \Delta_3)$,
- Φ_5 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(c_1, c_1', c_2', c_2, \Delta_3, \check{\Delta}_2)$,
- Φ_6 maps $(\delta_1, \delta_2, \delta_3, \delta_4, d, d')$ to $(d_1, d_1', d_2, d_2', \hat{\Delta}_2, \Delta_1)$.

Under these embeddings, the modified genus-3 relation (13) yields

$$\begin{aligned}
t_{w_3} t_{a_3} t_{a_3'} t_{x_3} t_{b_3} t_{b_3'} t_{y_3} t_{c_3} t_{c_3'} t_{z_3} &= t_{a_1} t_{a_1'} t_{a_2} t_{a_2'} t_{\Delta_1}^{-1} t_{\Delta_4}^{-1}, \\
t_{w_4} t_{a_4} t_{a_4'} t_{x_4} t_{b_4} t_{b_4'} t_{y_4} t_{c_4} t_{c_4'} t_{z_4} &= t_{b_1} t_{b_1'} t_{b_2} t_{b_2'} t_{\Delta_3}^{-1} t_{\Delta_4}^{-1}, \\
t_{w_5} t_{a_5} t_{a_5'} t_{x_5} t_{b_5} t_{b_5'} t_{y_5} t_{c_5} t_{c_5'} t_{z_5} \cdot \mathcal{P}_{\vec{\alpha}_1} &= t_{c_1} t_{c_1'} t_{c_2} t_{c_2'} t_{\check{\Delta}_2}^{-1} t_{\Delta_3}^{-1}, \\
\mathcal{P}_{\vec{\alpha}_2} \cdot t_{w_6} t_{a_6} t_{a_6'} t_{x_6} t_{b_6} t_{b_6'} t_{y_6} t_{c_6} t_{c_6'} t_{z_6} &= t_{d_1} t_{d_1'} t_{d_2} t_{d_2'} t_{\Delta_1}^{-1} t_{\hat{\Delta}_2}^{-1}.
\end{aligned}$$

The images of the curves under these embeddings are given in Figures 23. In addition, the oriented loops $\vec{\alpha}_1, \vec{\alpha}_2$ for the point-pushing maps are given in Figure 22. By substituting all into relation (12), we get

$$\begin{aligned}
&t_{w_3} t_{a_3} t_{a_3'} t_{x_3} t_{b_3} t_{b_3'} t_{y_3} t_{c_3} t_{c_3'} t_{z_3} t_{x_1} t_{x_2} \cdot t_{w_4} t_{a_4} t_{a_4'} t_{x_4} t_{b_4} t_{b_4'} t_{y_4} t_{c_4} t_{c_4'} t_{z_4} t_{y_1} t_{y_2} \\
&\qquad \cdot t_{w_5} t_{a_5} t_{a_5'} t_{x_5} t_{b_5} t_{b_5'} t_{y_5} t_{c_5} t_{c_5'} t_{z_5} \cdot \underline{\mathcal{P}_{\vec{\alpha}_1} \cdot t_{z_1} t_{z_2} \cdot \mathcal{P}_{\vec{\alpha}_2}} \\
&\qquad\qquad \cdot t_{w_6} t_{a_6} t_{a_6'} t_{x_6} t_{b_6} t_{b_6'} t_{y_6} t_{c_6} t_{c_6'} t_{z_6} t_{w_1} t_{w_2} = 1.
\end{aligned}$$

Here z_1 and z_2 are disjoint, $\vec{\alpha}_1$ and z_2 are disjoint, and z_1 and $\vec{\alpha}_2$ are disjoint, so we get

$$\mathcal{P}_{\vec{\alpha}_1} \cdot t_{z_1} t_{z_2} \cdot \mathcal{P}_{\vec{\alpha}_2} = \mathcal{P}_{\vec{\alpha}_1} \cdot t_{z_2} t_{z_1} \cdot \mathcal{P}_{\vec{\alpha}_2} = t_{z_2} \cdot \mathcal{P}_{\vec{\alpha}_1} \mathcal{P}_{\vec{\alpha}_2} \cdot t_{z_1}.$$

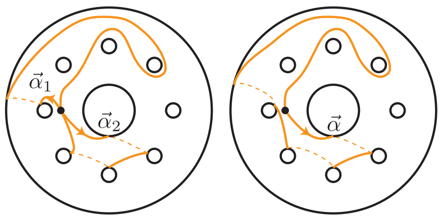


Fig. 22. Curves of the point-pushing maps.

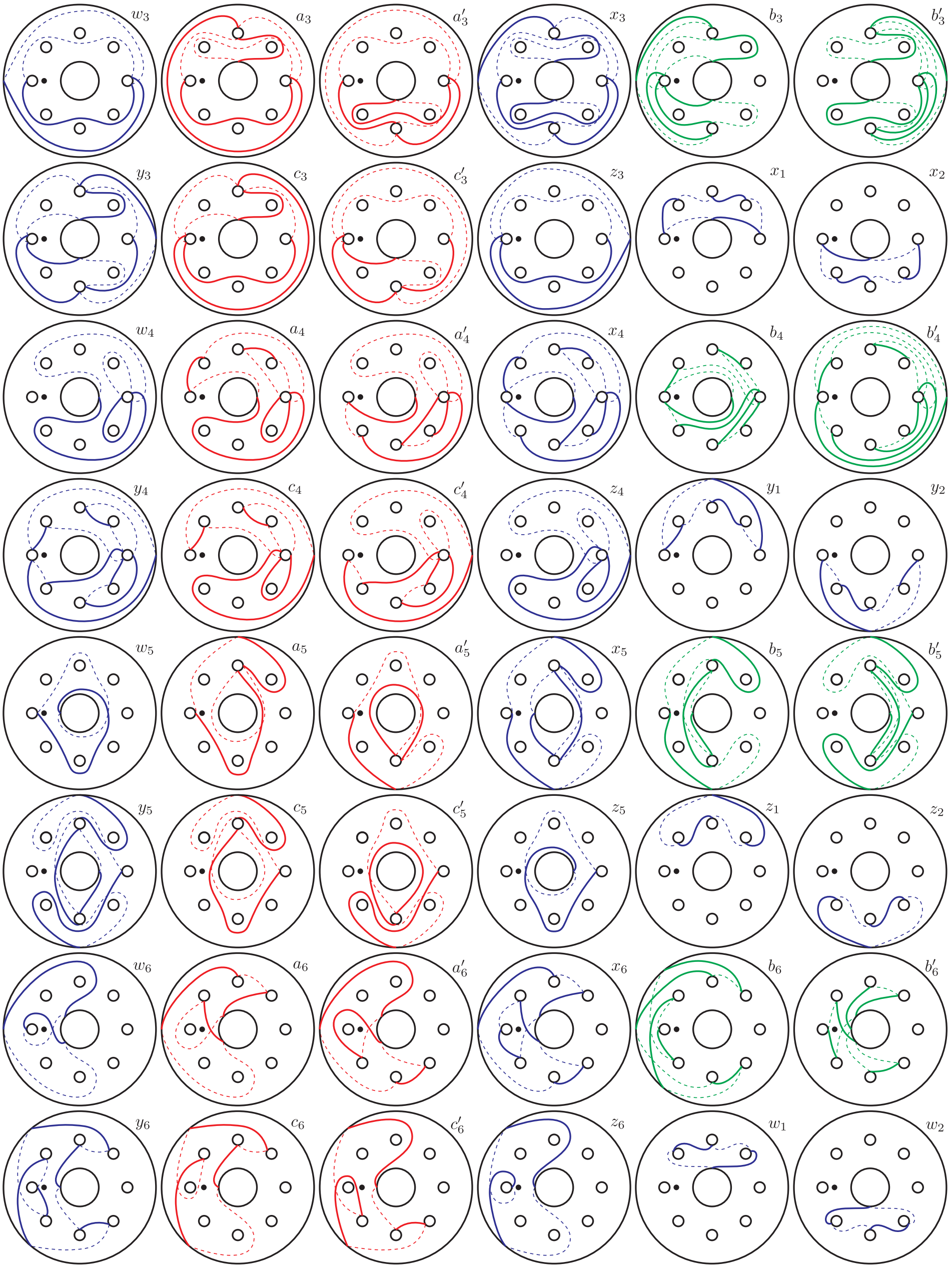


Fig. 23. Vanishing cycles of the genus-9 Lefschetz fibration with signature zero.

Then the product $\mathcal{P}_{\vec{\alpha}_1}\mathcal{P}_{\vec{\alpha}_2}$ is equal to the single point-pushing map along $\vec{\alpha}$, which is homotopic to the concatenation $\vec{\alpha}_2 \cdot \vec{\alpha}_1$.

Hence, we obtain the factorization in $\mathrm{Mod}(\Sigma_{9,1})$:

$$t_{w_3}t_{a_3}t_{a'_3}t_{x_3}t_{b_3}t_{b'_3}t_{y_3}t_{c_3}t_{c'_3}t_{z_3}t_{x_1}t_{x_2}\cdot t_{w_4}t_{a_4}t_{a'_4}t_{x_4}t_{b_4}t_{b'_4}t_{y_4}t_{c_4}t_{c'_4}t_{z_4}t_{y_1}t_{y_2} \\ \cdot t_{w_5}t_{a_5}t_{a'_5}t_{x_5}t_{b_5}t_{b'_5}t_{y_5}t_{c_5}t_{c'_5}t_{z_5}\cdot t_{z_2}\cdot \mathcal{P}_{\vec{\alpha}}\cdot t_{z_1} \\ \cdot t_{w_6}t_{a_6}t_{a'_6}t_{x_6}t_{b_6}t_{b'_6}t_{y_6}t_{c_6}t_{c'_6}t_{z_6}t_{w_1}t_{w_2}=1, \quad (14)$$

where all the Dehn twist curves are as in Figure 23. Forgetting the marked point (and thus the point-pushing map in the factorization), we get an explicit monodromy factorization of our signature zero Lefschetz fibration.

4. Proof of Theorem A

Here is an outline for our proof of Theorem A: Since the novel genus-9 Lefschetz fibration (X, f) we built in the previous section will serve as one of the main building blocks of our constructions, we will first analyze the algebraic and differential topology of X in some detail. We will then first prove all the statements for genus-9 fibrations, deferring the construction of higher genus examples till the end. We will first construct Lefschetz fibrations with prescribed signatures, and then the spin ones. All will be done by taking products of conjugated positive factorizations of the identity in $\mathrm{Mod}(\Sigma_g)$ (corresponding to twisted fiber sums of the fibrations) and breedings (one of which corresponds to lantern substitution, thus the rational blow-down), which will require extra care in the spin case.

4.1. *The topology of the signature zero genus-9 Lefschetz fibration*

The essential information we need for later arguments is summed up as follows:

Theorem 14. *There is a symplectic genus-9 Lefschetz fibration (X, f) with monodromy factorization*

$$t_{w_3}t_{a_3}t_{a'_3}t_{x_3}t_{b_3}t_{b'_3}t_{y_3}t_{c_3}t_{c'_3}t_{z_3}t_{x_1}t_{x_2}t_{w_4}t_{a_4}t_{a'_4}t_{x_4}t_{b_4}t_{b'_4}t_{y_4}t_{c_4}t_{c'_4}t_{z_4}t_{y_1}t_{y_2} \\ \cdot t_{w_5}t_{a_5}t_{a'_5}t_{x_5}t_{b_5}t_{b'_5}t_{y_5}t_{c_5}t_{c'_5}t_{z_5}t_{z_2}t_{z_1}t_{w_6}t_{a_6}t_{a'_6}t_{x_6}t_{b_6}t_{b'_6}t_{y_6}t_{c_6}t_{c'_6}t_{z_6}t_{w_1}t_{w_2}=1, \quad (15)$$

in $\mathrm{Mod}(\Sigma_9)$, where the Dehn twist curves are as in Figure 23. The total space X is a spin symplectic 4-manifold of general type, which is not deformation equivalent to any compact complex surface, where $e(X)=16$, $\sigma(X)=0$, and $H_1(X)=\mathbb{Z}^7\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2$.

Proof. The Lefschetz fibration (X, f) prescribed by the positive factorization (15) we have constructed in the previous section is a symplectic fibration with respect to a Gompf–Thurston symplectic form ω we can equip it with. While it will take some effort to prove that X is spin, all other claims on the differential topology of X follow from our claims on its algebraic topology: Since X is spin, it is obviously minimal, and $c_1^2(X)=2\mathrm{e}(X)+3\sigma(X)=32>0$, so the Kodaira dimension of the symplectic 4-manifold (X,ω) is $\kappa(X)=2$, in other words, it is of *general type*. Because $b_1(X)$ is odd, X is not even homotopy equivalent to a compact complex surface of general type, which are all known to be Kähler.

We calculate the algebraic invariants of X next.

Euler characteristic and signature: The Euler characteristic of X is easiest to calculate by $\mathrm{e}(X) = 4 - 4g + n$, where the genus of the Lefschetz fibration is $g = 9$, and the number of nodes, corresponding to the Dehn twists in the monodromy factorization, is $n = 48$.

Although the calculation of the signature of a Lefschetz fibration usually requires computer assistance to run an algorithm, we leveraged the fact that we have built the monodromy factorization (15) for (X, f) from scratch, using only basic relations in the mapping class group. Thanks to the work of Endo and Nagami [21], we easily calculate the signature of X by an algebraic count of the relations we have employed to derive the final positive factorization of the identity in $\mathrm{Mod}(\Sigma_9)$. Embeddings of relations into higher genus surfaces, cancellations of positive and negative Dehn twists, and Hurwitz moves do not affect the signature calculation. Since we built our genus-9 factorization through embeddings of the signature zero genus-3 relation, cancellations and Hurwitz moves, we conclude that $\sigma(X) = 0$.

Note that we calculated the signature of our genus-2 and genus-3 pencils in the same way, recalling that every use of the 2-chain and the lantern relation contributes -7 and $+1$ to the signature count [21]. Algebraically, we used one 2-chain and seven lantern relations to derive the monodromy factorization of the genus-2 pencil, whereas these numbers are doubled for the genus-3 pencil. In turn, a total of twelve 2-chain and 84 lantern relations yield the positive factorization for (X, f). (These large numbers might demonstrate why it is more feasible to build such a relation in multiple steps via breedings.)

First homology: The total space of a genus-g Lefschetz fibration over the 2-sphere has a handle decomposition $(D^2 \times \Sigma_g) \cup \sum_i h_i \cup (D^2 \times \Sigma_g)$, where h_i are 2-handles attached along the loops on Σ_g, which are the Dehn twist curves c_i in the monodromy factorization of the fibration [31]. Therefore, the first homology of the total space can be calculated by taking a quotient $H_1(\Sigma_g)$, as the first $D^2 \times \Sigma_g$ contains all the 1-handles, by the abelianized relations induced by the attaching circles of all the 2-handles, which are the vanishing cycles c_i, and the attaching circle of the last 2-handle coming from the second $D^2 \times \Sigma_g$ in the above decomposition.

Accordingly, we will calculate $H_1(X)$ using the monodromy factorization of our genus-9 fibration. Let $\{\alpha_i, \beta_i\}$ be the homology generators of $H_1(\Sigma_9)$, represented by the loops in Figure 24. After picking an auxiliary orientation on each Dehn twist curve in Figure 23, and calculating the number of its algebraic intersections with $\{\alpha_i, \beta_i\}$, we can easily read off the relation induced by the corresponding vanishing cycle. We tabulate this data below in a way the coefficients are easily visible, as it will be vital to our arguments for determining the spin type of X as well.

Homology classes of the vanishing cycles:

$$
\begin{aligned}
w_3 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+2\alpha_7+2\alpha_8+2\alpha_9 \quad +2\beta_3 \quad -\beta_7,\\
a_3 &= \alpha_1+3\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+2\alpha_7+3\alpha_8+2\alpha_9 \quad +2\beta_3 \quad -\beta_7 \quad -\beta_9,
\end{aligned}
$$

$$
\begin{aligned}
a_3' &= \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+\alpha_5+3\alpha_6+2\alpha_7+2\alpha_8+2\alpha_9 +2\beta_3 -\beta_7 -\beta_9,\\
x_3 &= \alpha_1+3\alpha_2+2\alpha_3+3\alpha_4+\alpha_5+3\alpha_6+2\alpha_7+3\alpha_8+2\alpha_9 +2\beta_3 -\beta_7 -2\beta_9,\\
b_3 &= \alpha_2 +\alpha_4 +\alpha_6 +\alpha_8 +\beta_7 -2\beta_9,\\
b_3' &= \alpha_2 +\alpha_4 +\alpha_6 +\alpha_8 +\beta_3 -2\beta_9,\\
y_3 &= \alpha_1 +\alpha_2+2\alpha_3 +\alpha_4+\alpha_5 +\alpha_6+2\alpha_7 +\alpha_8+2\alpha_9 +\beta_3 -2\beta_7 +2\beta_9,\\
c_3 &= \alpha_1 +\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+2\alpha_7 +\alpha_8+2\alpha_9 +\beta_3 -2\beta_7 +\beta_9,\\
c_3' &= \alpha_1+2\alpha_2+2\alpha_3 +\alpha_4+\alpha_5 +\alpha_6+2\alpha_7+2\alpha_8+2\alpha_9 +\beta_3 -2\beta_7 +\beta_9,\\
z_3 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+2\alpha_6+2\alpha_7+2\alpha_8+2\alpha_9 +\beta_3 -2\beta_7,\\
x_1 &= \alpha_2 +\alpha_8 +\beta_1-\beta_2 +\beta_3 +\beta_7-\beta_8 -\beta_9,\\
x_2 &= \alpha_4 +\alpha_6 +\beta_3-\beta_4+\beta_5-\beta_6 +\beta_7 -\beta_9,\\
w_4 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 +\beta_3 -\beta_9,\\
a_4 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 +\beta_1-\beta_2 +\beta_3 +\beta_7-\beta_8 -\beta_9,\\
a_4' &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 +\beta_3-\beta_4+\beta_5-\beta_6 +\beta_7 -\beta_9,\\
x_4 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 +\beta_1-\beta_2 +\beta_3-\beta_4+\beta_5-\beta_6 +2\beta_7-\beta_8 -\beta_9,\\
b_4 &= \beta_1-\beta_2 +\beta_3-\beta_4+\beta_5-\beta_6 +2\beta_7-\beta_8 -\beta_9,\\
b_4' &= \beta_1-\beta_2 +\beta_3-\beta_4+\beta_5-\beta_6 +2\beta_7-\beta_8,\\
y_4 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 -\beta_1+\beta_2 -\beta_3+\beta_4-\beta_5+\beta_6 -2\beta_7+\beta_8,\\
c_4 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 -\beta_1+\beta_2 -\beta_3 -\beta_7+\beta_8,\\
c_4' &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 -\beta_3+\beta_4-\beta_5+\beta_6 -\beta_7,\\
z_4 &= \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5 +\alpha_6 +\alpha_8 -\beta_3,\\
y_1 &= \alpha_2 +\alpha_8 -\beta_1+\beta_2 -\beta_3 -\beta_7+\beta_8,\\
y_2 &= \alpha_4 +\alpha_6 -\beta_3+\beta_4-\beta_5+\beta_6 -\beta_7,\\
w_5 &= \alpha_1 +\alpha_5 +2\alpha_9 -\beta_7 +\beta_9,\\
a_5 &= \alpha_1 +\alpha_2 +\alpha_5 +\alpha_8+2\alpha_9 -\beta_7,\\
a_5' &= \alpha_1 +\alpha_4+\alpha_5 +\alpha_6 +2\alpha_9 -\beta_7,\\
x_5 &= \alpha_1 +\alpha_2 +\alpha_4+\alpha_5 +\alpha_6 +\alpha_8+2\alpha_9 -\beta_7 -\beta_9,\\
b_5 &= \alpha_2 +\alpha_4 +\alpha_6 +\alpha_8 -\beta_7 -\beta_9,\\
b_5' &= \alpha_2 +\alpha_4 +\alpha_6 +\alpha_8 -\beta_3 -\beta_9,\\
y_5 &= \alpha_1 -\alpha_2 -\alpha_4+\alpha_5 -\alpha_6 -\alpha_8+2\alpha_9 +\beta_3 +\beta_9,\\
c_5 &= \alpha_1 -\alpha_2 +\alpha_5 -\alpha_8+2\alpha_9 +\beta_3,\\
c_5' &= \alpha_1 -\alpha_4+\alpha_5 -\alpha_6 +2\alpha_9 +\beta_3,\\
z_5 &= \alpha_1 +\alpha_5 +2\alpha_9 +\beta_3 -\beta_9,
\end{aligned}
$$

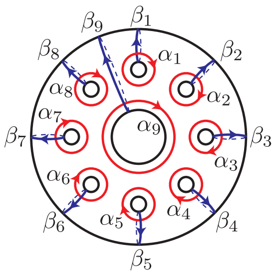


Fig. 24. The symplectic basis for $H_1(\Sigma_9;\mathbb{Z})$.

$$
\begin{array}{rllllllllll}
z_1 = & & \alpha_2 & & & & +\alpha_8 & +\beta_1-\beta_2 & & -\beta_8, & \\
z_2 = & & & \alpha_4 & & +\alpha_6 & & & -\beta_4+\beta_5-\beta_6, & & \\
w_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & & & +\beta_7 & -2\beta_9, \\
a_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & -\beta_1+\beta_2 & & +\beta_8 & -2\beta_9, \\
a'_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & & +\beta_4-\beta_5+\beta_6 & & -2\beta_9, \\
x_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & -\beta_1+\beta_2 & +\beta_4-\beta_5+\beta_6 & -\beta_7+\beta_8 & -2\beta_9, \\
b_6 = & & & & & & & \beta_1-\beta_2 & -\beta_4+\beta_5-\beta_6 & +\beta_7-\beta_8, & \\
b'_6 = & & & & & & & \beta_1-\beta_2 & -\beta_4+\beta_5-\beta_6 & +\beta_7-\beta_8 & +\beta_9, \\
y_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & +\beta_1-\beta_2 & -\beta_4+\beta_5-\beta_6 & +\beta_7-\beta_8 & -\beta_9, \\
c_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & +\beta_1-\beta_2 & & -\beta_8 & -\beta_9, \\
c'_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & & -\beta_4+\beta_5-\beta_6 & & -\beta_9, \\
z_6 = & \alpha_1 & & & +\alpha_5 & +\alpha_6+2\alpha_7 & +\alpha_8 & & & -\beta_7 & -\beta_9, \\
w_1 = & & \alpha_2 & & & & +\alpha_8 & -\beta_1+\beta_2 & & +\beta_8 & -\beta_9, \\
w_2 = & & & \alpha_4 & & +\alpha_6 & & & +\beta_4-\beta_5+\beta_6 & & -\beta_9.
\end{array}
$$

Recall that there is one more 2-handle we have to consider, and as we explained in the proof of Proposition 8 in Section 2, its attaching circle is homologous to the sum of the oriented curves of the point-pushing maps for *any* chosen marked point inducing a lift of the monodromy factorization from $\mathrm{Mod}(\Sigma_9)$ to $\mathrm{Mod}(\Sigma_{9,1})$. (If we could find a section of (X, f), the lift for the corresponding marked point would have no point-pushing maps, so this homology class would be trivial.) Looking at the point-pushing map in relation (14) in $\mathrm{Mod}(\Sigma_{9,1})$ we get the following additional relation in homology:

$$s = \alpha_2 + \alpha_8 + \beta_4 - \beta_5 + \beta_6 - \beta_7 - \beta_9 = 0.$$

With this explicit presentation in hand, determining the finitely generated abelian group $H_1(X)$ is now a straightforward calculation. We will show that $H_1(X) = \mathbb{Z}^7 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

From $b'_4 - b_4 = \beta_9$ we see $\beta_9 = 0$. Then, $a_3 - w_3 = \alpha_2 + \alpha_8 - \beta_9$, $a'_3 - w_3 = \alpha_4 + \alpha_6 - \beta_9$ give $\alpha_2 + \alpha_8 = 0$, $\alpha_4 + \alpha_6 = 0$. Since $b_3 = \beta_7$ modulo $\beta_9 = \alpha_2 + \alpha_8 = \alpha_4 + \alpha_6 = 0$, we have $\beta_7 = 0$. Similarly, $b'_3 = 0$ gives $\beta_3 = 0$. Now $x_1 = \beta_1 - \beta_2 - \beta_8$ and $x_2 = -\beta_4 + \beta_5 - \beta_6$ modulo $\beta_9 = \alpha_2 + \alpha_8 = \alpha_4 + \alpha_6 = \beta_3 = \beta_7 = 0$, thus $\beta_1 - \beta_2 - \beta_8 = -\beta_4 + \beta_5 - \beta_6 = 0$, or $\beta_1 = \beta_2 + \beta_8$ and $\beta_5 = \beta_4 + \beta_6$. From $w_5 = 0$ with $\beta_7 = \beta_9 = 0$ we see $\alpha_1 + \alpha_5 + 2\alpha_9 = 0$. Looking at $w_3 = 0$ together with $\alpha_1 + \alpha_5 + 2\alpha_9 = \alpha_2 + \alpha_8 = \alpha_4 + \alpha_6 = \beta_3 = \beta_7 = 0$ we get $2\alpha_3 + 2\alpha_7 = 0$. Combining $0 = w_4 + w_6 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + 2\alpha_8 + \beta_3 + \beta_7 - 3\beta_9$ and $\alpha_2 + \alpha_8 = \alpha_4 + \alpha_6 = 2\alpha_3 + 2\alpha_7 = \beta_3 = \beta_7 = \beta_9 = 0$ we obtain $2\alpha_1 + 2\alpha_5 = 0$. Since we also know $\alpha_1 + \alpha_5 + 2\alpha_9 = 0$ we deduce $4\alpha_9 = 0$. We turn to $w_6 = 0$ and substitute $\alpha_1 + \alpha_5 = -2\alpha_9$, $\beta_7 = \beta_9 = 0$ to get $-2\alpha_9 + \alpha_6 + 2\alpha_7 + \alpha_8 = 0$, or $\alpha_8 = -\alpha_6 - 2\alpha_7 + 2\alpha_9$. In summary, we have

$$
\begin{aligned}
&2(\alpha_3 + \alpha_7) = 0,\\
&4\alpha_9 = 0,\\
&\alpha_2 = -\alpha_8 = \alpha_6 + 2\alpha_7 - 2\alpha_9,\\
&\alpha_4 = -\alpha_6,
\end{aligned}
$$

$$\begin{aligned}
\alpha_5 &= -\alpha_1 - 2\alpha_9,\\
\alpha_8 &= -\alpha_6 - 2\alpha_7 + 2\alpha_9,\\
\beta_1 &= \beta_2 + \beta_8,\\
\beta_3 &= 0,\\
\beta_5 &= \beta_4 + \beta_6,\\
\beta_7 &= 0,\\
\beta_9 &= 0.
\end{aligned}$$

It is easy to check that any other relations that come from the vanishing cycles can be deduced from the above relations. The extra relation coming from the point-pushing map is

$$s = \alpha_2 + \alpha_8 + \beta_4 - \beta_5 + \beta_6 - \beta_7 - \beta_9 = 0.$$

However, this relation can be deduced already from the relations coming from vanishing cycles since $\alpha_2 + \alpha_8 = \beta_4 - \beta_5 + \beta_6 = \beta_7 = \beta_9 = 0$. This implies that we have a pseudosection. So we can take $g_1 = \alpha_1, g_2 = \alpha_3, g_3 = \alpha_6, g_4 = \beta_2, g_5 = \beta_4, g_6 = \beta_6, g_7 = \beta_8, g_8 = \alpha_9, g_9 = \alpha_3 + \alpha_7$ as generators and the only relations among them are $4g_8 = 0$ and $2g_9 = 0$. We conclude that

$$H_1(X) = \mathbb{Z}^7 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2.$$

Other algebraic invariants: While we can derive an explicit presentation of $\pi_1(X)$ in the same fashion we did for $H_1(X)$ earlier, we will not actually need all of this more massive presentation for any of our arguments to follow, even the ones that involve killing the fundamental group. Instead, it will suffice to observe that in the monodromy factorization of (X, f), we have three disjoint non-separating curves, coming from the bounding quadruple of curves x_1, z_1, x_2, z_2; see Figure 25. These three curves simultaneously kill three generators of $\pi_1(\Sigma_9)$.

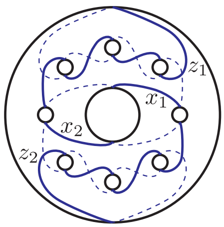


Fig. 25. The bounding quadruple.

As is the case for any closed oriented 4-manifold, the remaining homology groups of X, as well as $b_2^+(X)$ and $b_2^-(X)$, are already determined by $\mathrm{e}(X)$, $\sigma(X)$ and $H_1(X)$ by Poincaré duality and the universal coefficient theorem. In particular, $H_2(X) = \mathbb{Z}^{28} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and $b_2^+(X) = b_2^-(X) = 14$. Moreover, X can be seen to have an even intersection form, which will follow from its spin type.

Spin type: To build a spin Lefschetz fibration, our heuristic has been to breed exclusively Lefschetz pencils on spin 4-manifolds, as the monodromy factorizations of such pencils consist of Dehn twists that preserve a spin structure on a compact surface with boundary [14]. The groundwork for describing spin structures on X was laid out in Section 2, and particularly in Theorem 9.

From our calculation of the $\mathbb{Z}$-homology classes of the vanishing cycles on Σ_9 in the symplectic basis $\{\alpha_i, \beta_i\}$ represented by the oriented curves with the same labels in Figure 24, we easily deduce the $\mathbb{Z}_2$-homology classes of them. We will single out two spin structures on X, restrictions of which to the regular fiber will have different Arf invariants. Let the quadratic form q_0 on $H_1(\Sigma_9; \mathbb{Z}_2)$ be defined by

$$q_0(\alpha_i) = q_0(\beta_j) = 1 \quad \text{for all } i, j \text{ except for } q_0(\beta_9) = 0,$$

and the quadratic form q_1 by

$$q_1(\alpha_i) = q_1(\beta_j) = 1 \quad \text{for all } i, j \text{ except for } q_1(\alpha_7) = q_1(\beta_9) = 0.$$

Both are easily seen to evaluate as 1 on all the vanishing cycles. Let $s_0, s_1 \in \mathrm{Spin}(\Sigma_g)$ be the spin structures corresponding to these two forms q_0, q_1, which have Arf invariants $\mathrm{Arf}(q_0) = 0$ and $\mathrm{Arf}(q_1) = 1$, respectively.

Next, recall that the homology relation $\alpha_2 + \alpha_8 + \beta_4 - \beta_5 + \beta_6 - \beta_7 - \beta_9 = 0$ induced by the pseudosection s is already generated by the relations induced by the vanishing cycles we listed during our calculation of $H_1(X)$. So by Proposition 8, the homology class of the regular fiber $[F]$ is primitive in $H_2(X)$. As we have (X, f) with a primitive fiber class and signature $\sigma(X) \equiv 0 \pmod{16}$, by Theorem 9, the existence of the quadratic form q_1 with $\mathrm{Arf}(q_1) = 1$ now implies that we have spin structures $\mathfrak{s}_i$ on X coming from the spin structures s_i, for each $i = 0, 1$. Note that $\mathfrak{s}_0, \mathfrak{s}_1$ are only two of the 512 spin structures on X, for $H^1(X; \mathbb{Z}_2) = \mathbb{Z}_2^9$ acts freely and transitively on $\mathrm{Spin}(X)$. ■

Remark 15. It might be interesting for symplectic construction enthusiasts to observe how the topology of the pieces we built evolved: Our most elementary building blocks are a genus-0 pencil (corresponding to the lantern relation) and genus-1 pencils (corresponding to 6-holed torus and 2-chain relations) on rational surfaces, breeding of which gives us a genus-2 pencil on the ruled surface $T^2 \times S^2$. All have symplectic Kodaira dimension $\kappa = -\infty$. Using copies of the genus-2 pencil on the ruled surface, we obtained a symplectic Calabi–Yau 4-torus, which has $\kappa = 0$. We can also re-present our construction of the genus-9 fibration so that it is made out of two copies of a genus-5 pencil (each one of which is obtained by breeding a pair of our genus-3 pencils), along with two more copies of the genus-3 pencil. These genus-5 pencils have $\kappa = 1$. Lastly, our resulting genus-9 fibration (X, f) yields a symplectic 4-manifold with $\kappa = 2$.

Remark 16. Although we have used the breeding technique to effectively produce symplectic 4-manifolds out of smaller symplectic 4-manifolds, it is not inherently a symplectic operation. One first produces achiral pencils/fibrations on 4-manifolds that are often not symplectic, and only after pairing the vanishing cycles of all the negative nodes with matching vanishing cycles for positive nodes can we eliminate them to get a symplectic pencil/fibration. In general, the matching pairs would be visible only after a sequence of Hurwitz moves, or equivalently, a sequence of handle slides between the Lefschetz 2-handles. To approximate a surgery description, one can interpret the breeding construction as a two-step process, where one first takes a fiber sum of achiral fibrations, whose monodromies are now supported on a larger surface, and then attaches a 5-dimensional 3-handle. (The latter can be seen by looking at the handle diagram of the neighborhood of the two matching nodes; see [10].) As observed in [31], when this matching vanishing cycle is trivial in the fundamental group of the complement, eliminating the two Lefschetz 2-handles is equivalent to removing an $S^2 \times S^2$ or $S^2 \widetilde{\times} S^2$ summand from a connected sum (and capping off with D^4).

4.2. Lefschetz fibrations with prescribed signatures

We are now ready to prove our main theorem.

Proof of Theorem A. To keep our presentation simple, here we will not attempt to keep the topology of the arbitrary signature examples small, but instead, we will refine our arguments in specific cases, as we will also do in Section 5. In fact, the following bit of information will be enough to run all our arguments: *There is a signature zero genus-9 Lefschetz fibration (X, f), where X is spin and $\pi_1(X)$ has subgroups of arbitrarily large index.* The existence of such a fibration is guaranteed by Theorem 14. We can assume, after a global conjugation, that the Lefschetz fibration (X, f) has a monodromy factorization in $\mathrm{Mod}(\Sigma_9)$:

$$t_{B_1}\, P_1 = 1, \tag{16}$$

where P_1 is a product of positive Dehn twists, and B_1 is the non-separating curve in Figure 26 (a).

Any signature: There are $\phi_1, \phi_2 \in \mathrm{Mod}(\Sigma_9)$ such that $\phi_1(B_1) = B_2$ and $\phi_2(B_1) = B_2'$, where B_1, B_2 and B_2' are the curves shown in Figure 26 (a). Conjugating the monodromy factorization (16) with ϕ_i for each $i = 1, 2$, we get $t_{B_2} P_2 = 1$ and $t_{B_2'} P_3 = 1$ in $\mathrm{Mod}(\Sigma_9)$, where $P_i = P_1^{\phi_i}$ is the product of positive Dehn twists about the images of the Dehn twist curves in P_1 under ϕ_i, in the same word order. After cyclic permutations, we get product

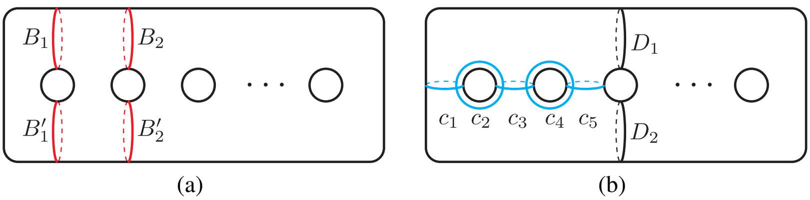


Fig. 26. Embeddings of lantern and 5-chain relation curves.

factorizations $t_{B_1} P_1 \cdot P_1 t_{B_1} = 1$ and $t_{B_2} P_2 \cdot P_3 t_{B_2'} = 1$, and in turn we get

$$t_{B_1}^2 t_{B_2} t_{B_2'} P_1^2 P_2 P_3 = 1 \tag{17}$$

in $\mathrm{Mod}(\Sigma_9)$, where t_{B_1}, t_{B_2} and $t_{B_2'}$ all commute with each other. The signature of the corresponding Lefschetz fibration, which is nothing but a twisted fiber sum of four copies of the signature zero fibration (X, f), is zero. Since B_1, B_1', B_2 and B_2' (where B_1' is isotopic to B_1) cobound a subsurface Σ_0^4, we can make a lantern substitution (or equivalently breed with the genus-0 pencil on $\mathbb{CP}^2$) in the monodromy factorization (17), and get

$$t_x t_y t_z P_1^2 P_2 P_3 = 1 \tag{18}$$

in $\mathrm{Mod}(\Sigma_9)$, where x, y, z are the images of the standard lantern curves under the embedding of the above Σ_2^2 into Σ_9. Since the signature of the lantern relation is 1, the factorization (18) prescribes a new Lefschetz fibration (X_1, f_1) with $\sigma(X_1) = 1$.

By taking fiber sum of copies of (X_1, f_1) with copies of any genus-9 Lefschetz fibration with a negative signature, we get Lefschetz fibrations (X_k, f_k) with $\sigma(X_k) = k$, for any prescribed $k \in \mathbb{Z}$. For instance, by taking the fiber sum of k copies of (X_1, f_1), we obtain a Lefschetz fibration (X_k, f_k) with $\sigma(X_k) = k$, for any $k \in \mathbb{Z}^+$. If we employ in this procedure any of the Lefschetz fibrations discussed in Example 6, since their vanishing cycles already kill $\pi_1(\Sigma_g)$, we get simply-connected examples with prescribed signature. For example, we can take the fiber sum of (X_{41}, f_{41}) with a genus-9 Lefschetz fibration on $\mathbb{CP}^2 \# 41\overline{\mathbb{CP}^2}$ which has the monodromy factorization (3), to get a simply-connected Lefschetz fibration with signature 1. For later reference, let (X_k, f_k) denote a fixed family of *simply-connected* genus-9 Lefschetz fibrations with signature $k \in \mathbb{Z}$ we obtain through these arguments.

Spin with any signature divisible by 16: The spin structure $\mathfrak{s}$ on X is induced by some spin structure on Σ_9, which corresponds to a quadratic form on $H_1(\Sigma_9; \mathbb{Z}_2)$ mapping all the Dehn twist curves in the monodromy factorization, and in particular B_1, to 1.

Let $\{\alpha_i, \beta_i\}$ be the curves in Figure 27 (a), which constitute a symplectic basis for $H_1(\Sigma_9; \mathbb{Z}_2)$. We claim that there is a spin structure $s \in \mathrm{Spin}(\Sigma_9)$ where the corresponding quadratic form q is such that

$$q(\beta_1) = q(\beta_3) = q(\alpha_1) = q(\alpha_2) = 1 \quad \text{and} \quad q(\beta_2) = 0,$$

and q sends every vanishing cycle (which we will get after a conjugation) to 1.

While the values of the remaining basis elements under q will not matter for our arguments, we will rely on the fiber genus g being at least 3. We will obtain s as the image, under a diffeomorphism φ of Σ_9, of the initial spin structure on Σ_9 yielding the spin structure $\mathfrak{s}$ on (X, f). (The promised quadratic form q will then be obtained by pulling back under φ^{-1} the quadratic form for the initial spin structure.) We will require φ to fix B_1. We will obtain this diffeomorphism φ as a composition of several diffeomorphisms of Σ_9, each realizing a well-known symplectic basis change for $H_1(\Sigma_9; \mathbb{Z}_2)$; cf. [16,35]. At each step, we take the new spin structure we get under the diffeomorphism of Σ_9. Note that

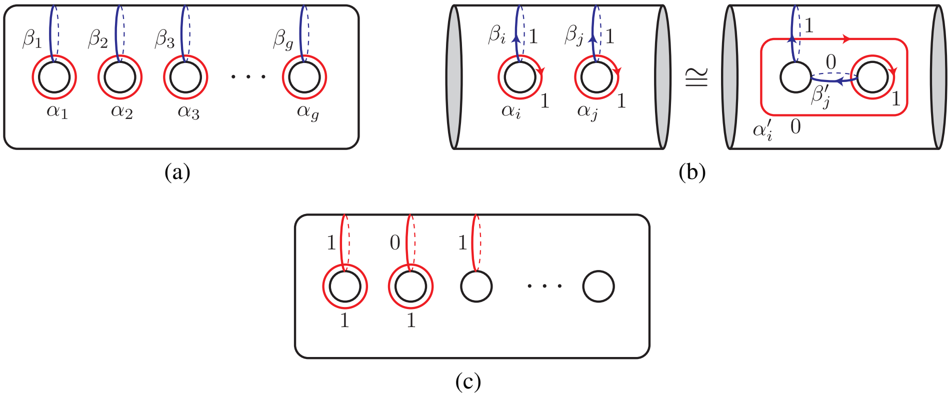


Fig. 27. Basis changes.

every Dehn twist curve in the monodromy factorization will be conjugated by the diffeomorphisms, and will be mapped to 1 again under the new quadratic form corresponding to the new spin structure.

Below, we will focus on the values of the ordered pair of elements α_i, β_i in $\mathbb{Z}_2 \times \mathbb{Z}_2$ under the quadratic form q, and will simply call them a $(q(\alpha_i), q(\beta_i))$ *pair*.

First of all, note that if $q(\alpha_i) = q(\beta_i) = 0$ for some i, we can replace α_i with $\alpha_i' := t_{\beta_i}(\alpha_i)$ so that $q(\alpha_i') = 1$, or β_i with $\beta_i' := t_{\alpha_i}(\beta_i)$ so that $q(\beta_i') = 1$. Therefore, any time $q(\alpha_i)\,q(\beta_i) = 0$, we can find a pair of geometrically dual curves in a small neighborhood of $\alpha_i \cup \beta_i$, where q maps our pick of one of the two curves to 1 and the other one to 0. So there are diffeomorphisms (which obviously change the spin structure) that trade between a $(0,0)$, a $(1,0)$ or a $(0,1)$ pair locally.

Secondly, suppose $q(\alpha_i)\,q(\beta_i) = 1 = q(\alpha_j)\,q(\beta_j)$ for some $i \neq j$. It is easy to see that there is a diffeomorphism supported in a Σ_2^2 neighborhood of $(\alpha_i \cup \beta_i) \sqcup (\alpha_j \cup \beta_j)$ which replaces α_i and β_j with α_i' and β_j' shown in Figure 27 (b), while fixing β_i and α_j. Here α_i', β_i and α_j, β_j' are disjoint geometrically dual pairs contained in Σ_2^2 as well, but now $q(\alpha_i') = q(\beta_j') = 0$. Thus, we see that any time we have two disjoint $(1,1)$ pairs, there is a diffeomorphism which replaces them with disjoint $(0,1)$ and $(1,0)$ pairs, and vice versa.

We can now complete the proof of our claim. All we have initially is that we have a quadratic form which maps the monodromy curves to 1, and in particular $q(\beta_1) = 1$, since $\beta_1 = B_1$. Suppose $q(\alpha_1) = 0$. Since the genus of the surface $g \geq 3$, it follows from the above observations that, after a diffeomorphism supported away from $\alpha_1 \cup \beta_1$, we can get a $(1,0)$ pair α_j, β_j. Under the inverse of the diffeomorphism we described in the previous paragraph, we can switch these two disjoint $(0,1)$ and $(1,0)$ pairs with two disjoint $(1,1)$ pairs, all the while fixing β_1. Relabeling the quadratic form corresponding to the new spin structure as q, we now have $q(\alpha_1) = q(\beta_1) = 1$. Finally, after another diffeomorphism supported away from the new $\alpha_1 \cup \beta_1$, and relabeling again the quadratic form for the new spin structure, we can assume that α_2, β_2 is a $(1,0)$ pair and $q(\beta_3) = 1$ as desired; see Figure 27 (c). This completes the proof of our claim.

Let c_1, c_2, c_3, c_4, c_5 be as in Figure 26 (b). For $s \in \mathrm{Spin}(\Sigma_9)$ we obtained above, and q the quadratic form corresponding to s, we easily see that now $q(c_i) = 1$, so $t_{c_i} \in \mathrm{Mod}(\Sigma_9, s)$ for all i. In particular, $\phi = t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5} \in \mathrm{Mod}(\Sigma_9, s)$, where one can easily verify that $\phi(c_i) = c_{i+1}$ for each $i = 1, \ldots, 4$. (Alternatively, for each $i = 1, \ldots, 4$, we can take $\psi_i = t_{c_i} t_{c_{i+1}} \in \mathrm{Mod}(\Sigma_9, s)$ so that $\psi_i(c_i) = c_{i+1}$.) After conjugating the positive factorization (16) with the diffeomorphism φ we described above, we can further conjugate the resulting positive factorization with powers of ϕ, and get five positive factorizations $Q_i := t_{c_i} P_1^{\phi^{i-1}\varphi} = 1$ for $i = 1, \ldots, 5$. (Recall that $B_1 = c_1$.) Next, take the product factorization $(Q_1 Q_2 Q_3 Q_4 Q_5)^6 = 1$, which, after conjugating away the products of positive Dehn twists $P_1^{\phi_{i-1}\varphi}$ in Q_i (i.e. after the corresponding Hurwitz moves), yields a positive factorization

$$(t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^6 \, Q = 1 \tag{19}$$

in $\mathrm{Mod}(\Sigma_9)$, where Q is the product of all the remaining Dehn twists conjugated away from t_{c_i}. This is in fact a factorization in $\mathrm{Mod}(\Sigma_9, s)$. Because it is a product of signature zero relations, the signature of this last relation is also zero.

Now, by substituting the *inverse* of the 5-chain relation in the factorization (19), or equivalently breeding with the achiral counter-part of the genus-2 pencil on the K3 surface given in Example 4 (which we do by embedding the relation into the subsurface Σ_2^2 that is a neighborhood of the chain $c_1, \ldots, c_5$), we get a new positive factorization

$$R := t_{D_1} t_{D_2} \, Q = 1 \tag{20}$$

in $\mathrm{Mod}(\Sigma_9)$. It is evident from Figure 27 (c) that $D_1 = \beta_3$ and D_2 is homologous to it, so $q(D_1) = q(D_2) = 1$. Thus, this too is in fact a factorization in $\mathrm{Mod}(\Sigma_9, s)$, i.e. the quadratic form q maps all the Dehn twist curves in the factorization (20) to 1. Following our heuristic, we have once again bred with an achiral pencil on a spin 4-manifold. This new relation has signature 16, since we had a signature zero factorization, and inserted the inverse 5-chain relation, which has signature 16 [21] (it is the signature of the K3 surface with the opposite orientation after all).

To be able to apply Theorem 9 and equip the new fibration with a spin structure, we still need to confirm that it also has a primitive fiber class. While this is indeed the case for the fibration corresponding to the factorization (19), which is nothing but a twisted fiber sum of several copies of (X, f), the substitution we have made above requires a new calculation for the pseudosection. There is an easier way to get what we want, which we will discuss next, in two distinct cases:

Suppose $\mathrm{Arf}(s) = 1$. Let $H = 1$ be the positive factorization (3) in Example 6 for a genus-9 fibration on $\mathbb{CP}^2 \# 41\overline{\mathbb{CP}}^2$. Recall that the unique quadratic form on $H_1(\Sigma_9; \mathbb{Z}_2)$ that maps all the Dehn twist curves in H to 1 also has Arf invariant 1. So, there is some diffeomorphism ψ of Σ_9 sending the spin structure corresponding to that quadratic form to the spin structure s we had. Now take the product positive factorization

$$R^6 \, (H^2)^{\psi} = 1 \tag{21}$$

in $\mathrm{Mod}(\Sigma_9)$, which is in fact a factorization in $\mathrm{Mod}(\Sigma_9, s)$. It also has signature $6 \cdot 16 + 2(-40) = 16$. Because the Dehn twist curves in H already kill $\pi_1(\Sigma_9)$, by the particular case of Proposition 8, the fiber class of the Lefschetz fibration (Z_1, h_1), prescribed by the positive factorization (21), is primitive. So Z_1 is spin by Theorem 9.

Suppose $\mathrm{Arf}(s) = 0$. Consider the positive factorization (4) in Example 7. For $n = 8$ and $m = 1$, this prescribes a genus-9 Lefschetz fibration on $(T^2 \times S^2) \# 32\overline{\mathbb{CP}}^2$. While there are four quadratic forms on $H_1(\Sigma_9; \mathbb{Z}_2)$ which map all the Dehn twist curves in the factorization (4) to 1, they all have Arf invariant 0. As explained in Example 7, there is a twisted fiber sum of two copies of this fibration, which gives a spin genus-9 Lefschetz fibration with a section on the knot surgered $E(8)_K$, where K is any genus-1 fibered knot, say a trefoil. Let $Y = 1$ be a positive factorization corresponding to this latter fibration. So it has signature -64, and since the fibration has a section, $\pi_1(\Sigma_9)/N = \pi_1(E(8)_K) = 1$, where N is the subgroup of $\pi_1(\Sigma_9)$ normally generated by the Dehn twist curves in Y. So the Dehn twist curves in the positive factorization Y already kill $\pi_1(\Sigma_g)$. (In fact, here we chose $m = 1$ so one can also see directly how to twist the fiber sum to get a spin fibration with trivial π_1 as desired.) The unique quadratic form on $H_1(\Sigma_9; \mathbb{Z}_2)$ which maps all the monodromy curves to 1 necessarily has Arf invariant 0. It follows that there is again some diffeomorphism ψ of Σ_9 sending the spin structure corresponding to that quadratic form to the spin structure s we had. Now take the product positive factorization

$$R^5 Y^{\psi} = 1 \tag{22}$$

in $\mathrm{Mod}(\Sigma_9)$, which is in fact a factorization in $\mathrm{Mod}(\Sigma_9, s)$. It too has signature $5 \cdot 16 + (-64) = 16$. Because the Dehn twist curves in Y already kill $\pi_1(\Sigma_9)$, once again, by the particular case of Proposition 8 and Theorem 9, for (Z_1, h_1) denoting the genus-9 Lefschetz fibration prescribed by the positive factorization (22), we conclude that Z_1 is spin.

Note that either of the two cases, with $\mathrm{Arf}(s) = 1$ or 0, may occur depending on the initial choice of $s \in \mathrm{Spin}(\Sigma_9)$ which yields the spin structure on (X, f). (Indeed, the signature zero genus-9 Lefschetz fibration (X, f) of Theorem 14 admits spin structures of both types.) Also note that, in either case, the vanishing cycles of (Z_1, h_1) we constructed kill $\pi_1(\Sigma_9)$.

To finish our proof, take a fiber sum of k copies of (Z_1, h_1) to produce a genus-9 Lefschetz fibration (Z_k, h_k) with $\sigma(Z_k) = 16k$, for any prescribed $k \in \mathbb{Z}^+$. By Proposition 8 and Theorem 9, all Z_k are spin, and since the vanishing cycles of even one copy already kill $\pi_1(\Sigma_9)$, all Z_k are simply-connected. Negative signature examples are already realized by fiber sums of Lefschetz fibrations on knot surgered K3 surfaces discussed in Example 7, for any genus-4 fibered knot. Finally, to get *simply-connected* examples of signature zero, one can simply lower the power of R by 1 in the monodromy factorizations (21) and (22). For later reference, let (Z_k, h_k) denote a fixed family of *simply-connected* genus-9 Lefschetz fibrations with Z_k spin and $\sigma(Z_k) = 16k$, for $k \in \mathbb{Z}$.

Higher genera: For any index d subgroup of $\pi_1(X)$, there exists a d-fold cover $\pi_d : \widetilde{X}(d) \to X$. The composition $\widetilde{f}(d) := f \circ \pi_d : \widetilde{X}(d) \to S^2$ yields a Lefschetz fibration on $\widetilde{X}(d)$ of genus $g = 8d + 1$, and since the signature is multiplicative under unbranched coverings, $(\widetilde{X}(d), \widetilde{f}(d))$ too is a signature zero Lefschetz fibration. We can thus follow the same construction scheme as above, now for genus $g = 8d + 1$ fibrations.

For arbitrary signatures, take a twisted fiber sum of four copies of $(\widetilde{X}(d), \widetilde{f}(d))$ so that we get Dehn twist curves cobounding a subsurface Σ_0^4, and then make a lantern substitution to get a genus $g = 8d + 1$ Lefschetz fibration $(\widetilde{X}_1(d), \widetilde{f}_1(d))$ with signature 1. Then, by taking fiber sums of copies of $(\widetilde{X}_1(d), \widetilde{f}_1(d))$ and copies of any genus-g Lefschetz fibration with a negative signature, we can build $(\widetilde{X}_k(d), \widetilde{f}_k(d))$ with $\sigma(\widetilde{X}_k(d)) = k$, for any given $k \in \mathbb{Z}$. Once again, picking the negative signature summands as simply-connected ones, like the ones in Example 6, we can assume that $\widetilde{X}_k(d)$ are all simply-connected.

For spin examples, first note that if $\pi : \widetilde{Y} \to Y$ is a finite covering of a spin 4-manifold Y, then $\widetilde{Y}$ is also spin: The tangent bundle $T\widetilde{Y}$ is isomorphic to the pull-back bundle $\pi^*(TY)$, and by functoriality, the second Stiefel–Whitney class of $\widetilde{Y}$ is $w_2(T\widetilde{Y}) = \pi^* w_2(TY)$. So for $w_2(Y) = w_2(TY) = 0$ in $H^2(Y; \mathbb{Z}_2)$, we have $w_2(\widetilde{Y}) = w_2(T\widetilde{Y}) = 0$ in $H^2(\widetilde{Y}; \mathbb{Z}_2)$ as well. Since w_2 is the only obstruction to the existence of a spin structure on an orientable manifold, the signature zero Lefschetz fibrations $(\widetilde{X}(d), \widetilde{f}(d))$ of genus $g = 8d + 1$ we described above are all spin. It follows that there is a spin structure on Σ_g with a quadratic form that evaluates as 1 on all the Dehn twist curves in a monodromy factorization of $(\widetilde{X}(d), \widetilde{f}(d))$. As the argument we gave in the genus $g = 9$ case applies just the same to any other genus $g \geq 3$, we can then build a twisted fiber sum of copies of $(\widetilde{X}(d), \widetilde{f}(d))$, with a monodromy factorization of the form $(t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^6 Q = 1$ in $\mathrm{Mod}(\Sigma_g, s)$, for some spin structure s. Once again, by an inverse 5-chain substitution, we get a new genus-g Lefschetz fibration, with a monodromy factorization in $\mathrm{Mod}(\Sigma_g, s)$. By fiber summing this fibration with simply-connected, spin, genus-g fibrations, which come with a quadratic form on $H_1(\Sigma_g; \mathbb{Z}_2)$ with the same Arf invariant as q, we obtain a simply-connected, spin, genus-g Lefschetz fibration $(\widetilde{Z}_1(d), \widetilde{h}_1(d))$, with $\sigma(\widetilde{Z}_1(d)) = 16$. For this, we observe that for any $g = 8d + 1$, there exists a spin genus-g Lefschetz fibration among the ones in Examples 7, where the quadratic form that evaluates as 1 on all the monodromy curves has the desired Arf invariant. Finally, by taking further sums as explained in the genus-9 case, we can get simply-connected, spin, genus-g Lefschetz fibrations $(\widetilde{Z}_k(d), \widetilde{h}_k(d))$ with $\sigma(\widetilde{Z}_k(d)) = 16k$, for any prescribed $k \in \mathbb{Z}^+$, and in turn examples with any signature $16k$, for $k \in \mathbb{Z}$.

By the initial assumption on $\pi_1(X)$, we can take the covering degree d to be arbitrarily large, and obtain the above examples with arbitrarily high genera.

Other properties: We equip all our examples with a Gompf–Thurston symplectic form so they become symplectic Lefschetz fibrations. Set $(\widetilde{X}_0(1), \widetilde{f}_0(1)) = (X_0, f_0)$ and $(\widetilde{Z}_0(1), \widetilde{h}_0(1)) = (Z_0, h_0)$, which are the simply-connected and signature zero examples for genus $g = 9$ where the latter is spin. Taking further fiber sums with copies of the simply-connected, signature zero genus $g = 8d + 1$ Lefschetz fibration $(\widetilde{X}_0(d), \widetilde{f}_0(d))$, or in the spin case, with copies of $(\widetilde{Z}_0(d), \widetilde{h}_0(d))$, we get an infinite family of simply-connected examples. Since fiber sums of Lefschetz fibrations of genus $g \geq 1$ are always minimal [61] (see also [11]), these symplectic 4-manifolds are minimal.

This completes the proof of Theorem A. ■

Remark 17. While all the examples of arbitrary signature Lefschetz fibrations over the 2-sphere we have presented in this article are of fiber genus $g \equiv 1 \pmod 8$, we do not have any reason to believe that the gaps in values of g are essential. On the other hand, it is interesting to determine the smallest g for a genus-g Lefschetz fibration with positive signature, and even more so, with zero signature (as should be evident from our proof above, one can then generate examples with any other signatures). For the restricted class of hyperelliptic Lefschetz fibrations, one can derive upper bounds using Endo's signature formula [20], and in particular any genus $g = 1$ or 2 fibration is known to have negative signature; see also [45]. Therefore, the question really is:

Are there genus-g Lefschetz fibrations over the 2*-sphere with arbitrary signatures for* $3 \leq g \leq 8$*?*

Note that if one asks the analogous question for Lefschetz fibrations over the 2-disk instead, *with no restriction on their global monodromy*, then it is easier to generate positive signature examples, and there is a satisfactory answer that follows from the work of Çengel and Karakurt [17], where we see that $g = 1$ suffices in this case.

Remark 18. It was conjectured by Stipsicz that all symplectic Lefschetz fibrations over the 2-sphere had negative signatures [44, 56], and this constituted an open problem for over 20 years; see e.g. [44, 45, 56], [57, Problem 6.3], and [39, Problems 7.4]. Besides the lack of examples with positive signatures in the literature, as far as we know, not all negative values were known to be realized as the signature of Lefschetz fibrations over the 2-sphere either—especially for fixed fiber genus. Since the signature is additive under fiber sums, the gaps in the latter case were essentially due to lack of examples with signatures close to zero. Prior to our work, the largest known signatures in the literature we know of had signature $\sigma = -4$ (realized by the examples of Matsumoto, Cadavid and Korkmaz for every even $g \geq 2$, and by the examples of the first author [10] for every odd $g \geq 3$), with the exception of $\sigma = -3$ in the $g = 2$ case (realized by the examples of Xiao [62] and Baykur–Korkmaz [15]).

Remark 19. We expect most of our examples to be non-holomorphic. The main building blocks, and many non-simply-connected examples we can produce, have odd first Betti numbers, and thus their total spaces are not even homotopy equivalent to compact complex surfaces. It would be interesting to know to what extent an analogue of Theorem A holds in the holomorphic category:

Are there holomorphic Lefschetz fibrations over $\mathbb{CP}^1$ *with arbitrary signatures?*

Many examples with various negative signatures appear in the works of Persson, Peter, Xiao among many others. It is claimed in [47] that some of the holomorphic fibrations on positive signature compact complex surfaces described in [49] are Lefschetz, but the authors' construction in the positive index case appears to always involve multiple fibers. (Existence of multiple fibers have no bearing on the rest of the arguments of [47].) In fact the authors successfully build their *negative* signature examples as Lefschetz fibrations, and use this to deduce the simple-connectivity of their complex surfaces, whereas for their positive signature examples, they take a detour and use different arguments to kill the fundamental group [49].

5. Proofs of Corollary C and Theorem D

In this final section we focus on the signature zero case and apply our techniques to present novel constructions of symplectic 4-manifolds homeomorphic but not diffeomorphic to $\#_m(S^2 \times S^2)$, the connected sum of m copies of $S^2 \times S^2$. Here m is necessarily odd since the holomorphic Euler characteristic of any almost complex 4-manifold is an integer, implying that the Betti numbers of a symplectic 4-manifold satisfies the equality $1 - b_1 + b_2^+ \equiv 0 \pmod 2$. We will first produce such examples as Lefschetz fibrations over the 2-sphere and prove Corollary C. We will then produce examples with smaller topology by applying symplectic surgeries to certain Lefschetz fibrations over the 2-torus and prove Theorem D.

5.1. Exotic $\#_m(S^2 \times S^2)$ as symplectic Lefschetz fibrations

Any example of simply-connected, spin, signature zero symplectic Lefschetz fibration granted by Theorem A is necessarily homeomorphic to $\#_m(S^2 \times S^2)$ (for some large m), by Freedman [28]. Whereas by the works of Taubes [59, 60], no symplectic 4-manifold with $b_2^+ > 1$ can be diffeomorphic to such a connected sum, thus such a symplectic 4-manifold would be an exotic $\#_m(S^2 \times S^2)$. Corollary C to our main theorem promises two refinements: we can get explicit examples for $m = 127$ and also for *every* odd $m \geq 415$. Although we are going to produce all our examples with fiber genus $g = 9$, one can adopt the construction scheme below to generate higher genus examples as well (for different values of m), as we did in the proof of Theorem A.

Proof of Corollary C. Let (X, f) be the signature zero genus-9 Lefschetz fibration of Theorem 14, equipped with the spin structure $\mathfrak{s}_0$. Recall that $\mathfrak{s}_0$ is induced by $s_0 \in \mathrm{Spin}(\Sigma_9)$ with a quadratic form q_0 that evaluates as 1 on all the basis elements α_i, β_i but β_9 given in Figure 24.

For an easier presentation of our arguments, let us first note that there is a diffeomorphism of Σ_9 which fixes the spin structure s_0 and maps the bounding quadruple (z_1, x_1, z_2, x_2) (see Figure 25) in the monodromy factorization (15) of (X, f), to the quadruple $(\beta_1, \beta_1', \beta_5, \beta_5')$ shown in Figure 29 (a). For instance, we can take the diffeomorphism

$$\phi_1 = t_6^{-1} t_8^{-1} t_7^{-1} t_6^{-1} t_{\check{5}} t_{\hat{5}} t_{\check{4}} t_{\hat{4}} t_{\check{1}} t_{\hat{1}} t_{\check{5}} t_{\hat{5}} t_{\check{4}} t_{\hat{4}} t_{\check{3}} t_{\hat{3}} t_{\check{2}} t_{\hat{2}} t_{\check{1}} t_{\hat{1}}$$

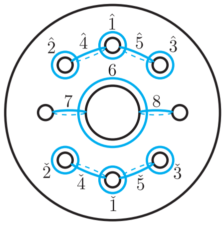


Fig. 28. The Dehn twist curves for the spin diffeomorphism ϕ_1.

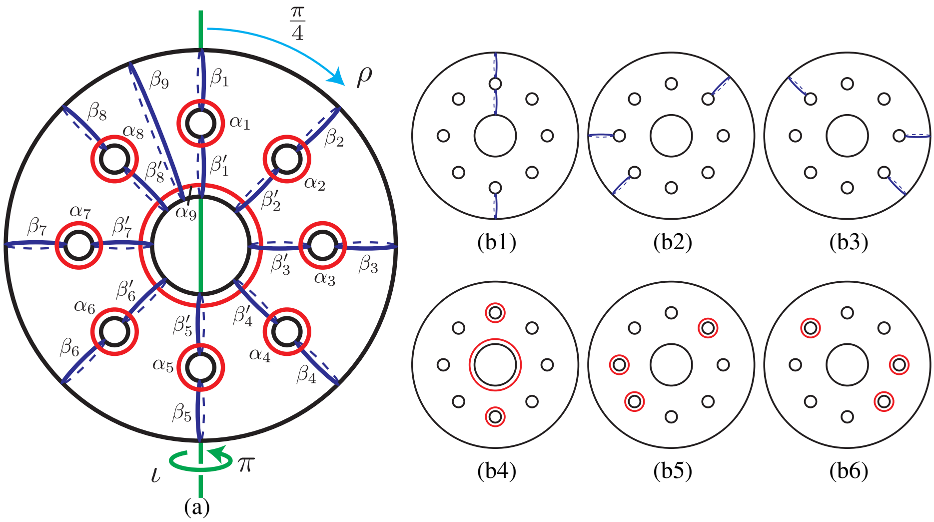


Fig. 29. Left: The curves $\alpha_i, \beta_i, \beta_i'$ and the diffeomorphisms ρ and ι of Σ_9. Right: $\beta_1, \beta_1', \beta_5$ and their images under the diffeomorphisms ϕ_k.

where the Dehn twist curves are as shown in Figure 28. One can easily verify that $\phi_1 \in \mathrm{Mod}(\Sigma_9, s_0)$ by looking at its action on the symplectic basis elements in Figure 24. (Here not every Dehn twist we employed is in $\mathrm{Mod}(\Sigma_g, s_0)$, but their product ϕ_1 is.) After a global conjugation with ϕ_1 and Hurwitz moves, we get a monodromy factorization for (X, f) of the form

$$t_{\beta_1} t_{\beta_1'} t_{\beta_5} t_{\beta_5'}\, P_1 = 1 \quad \text{in } \mathrm{Mod}(\Sigma_9) \tag{23}$$

where P_1 is the product of the remaining Dehn twists. This is in fact a factorization in $\mathrm{Mod}(\Sigma_9, s_0)$.

Simple-connectivity: We will first show how to get simply-connected fibrations. For $\alpha_i, \beta_j, \beta_j'$ as in Figure 29 (a), we have $q_0(\alpha_i) = q_0(\beta_j) = q_0(\beta_j') = 1$ for all $i = 1, \ldots, 9$ and $j = 1, \ldots, 8$. It follows that the Dehn twists $t_{\alpha_i}, t_{\beta_j}, t_{\beta_j'}$ are in $\mathrm{Mod}(\Sigma_9, s_0)$ for all i and j as above. Also note that the clockwise $\pi/4$ rotation ρ about the center of the figure and the involution ι about the y-axis as illustrated in Figure 29 (a) both preserve s_0. It is now easy to see that there are diffeomorphisms $\phi_k \in \mathrm{Mod}(\Sigma_9, s_0)$ for each

$k = 2, \ldots, 6$ which takes the curves $\beta_1, \beta_1', \beta_5$ to the basis elements α_i, β_i as shown in Figure 29 (b1)–(b6). For instance we can take

$$\begin{aligned}
\phi_2 &= t_{\alpha_7} t_{\beta_7} t_{\beta_7'} t_{\alpha_7} t_{\alpha_9} t_{\beta_7'} t_{\beta_2'} t_{\alpha_9} \rho, \\
\phi_3 &= \iota \phi_2, \\
\phi_4 &= t_{\beta_1} t_{\alpha_1} t_{\beta_1'} t_{\alpha_9} t_{\beta_5} t_{\alpha_5}, \\
\phi_5 &= t_{\beta_7} t_{\alpha_7} t_{\beta_6} t_{\alpha_6} t_{\beta_2} t_{\alpha_2} \phi_2, \\
\phi_6 &= t_{\beta_8} t_{\alpha_8} t_{\beta_4} t_{\alpha_4} t_{\beta_3} t_{\alpha_3} \phi_3.
\end{aligned}$$

Homotoping the curves $\alpha_i, \beta_i, i = 1, \ldots, 9$, to share a common base point, and orienting them, we get a basis for $\pi_1(\Sigma_9)$. So the curves $\{\alpha_i, \beta_i\}_{i=1}^9$ normally generate $\pi_1(\Sigma_9)$. In turn, we see that $\beta_1, \beta_1', \beta_5$, and their images under ϕ_k, $k = 2, \ldots, 6$, together normally generate $\pi_1(\Sigma_9)$. (Here β_9 is normally generated by β_1 and β_1', i.e. a product of conjugates of β_1 and β_1'.)

Thus the product monodromy factorization in $\mathrm{Mod}(\Sigma_9, s_0)$,

$$t_{\beta_1} t_{\beta_1'} t_{\beta_5}\, Q_1\, t_{\beta_2} t_{\beta_6} t_{\beta_7}\, Q_2\, t_{\beta_3} t_{\beta_4} t_{\beta_8}\, Q_3\, t_{\alpha_1} t_{\alpha_5} t_{\alpha_9}\, Q_4\, t_{\alpha_2} t_{\alpha_6} t_{\alpha_7}\, Q_5\, t_{\alpha_3} t_{\alpha_4} t_{\alpha_8}\, Q_6 = 1, \quad (24)$$

which prescribes a twisted fiber sum of six copies of (X, f), yields a simply-connected symplectic Lefschetz fibration (Z, h) admitting a spin structure coming from $s_0 \in \mathrm{Spin}(\Sigma_9)$ with quadratic form q_0. Here $Q_1 = t_{\beta_5'} P_1$ and $Q_i = Q_1^{\phi_k}$ for $k = 2, \ldots, 6$. A straightforward Euler characteristic calculation shows that the simply-connected, spin, signature zero 4-manifold Z has the same Euler characteristic as $\#_{127}(S^2 \times S^2)$. Since both are smoothable simply-connected 4-manifolds, and have even intersection forms of the same rank, by Freedman, they are homeomorphic.

Stable range: Next we will show that we can get symplectic Lefschetz fibrations homeomorphic to $\#_m(S^2 \times S^2)$ for any odd $m \geq 415$. Consider the factorization (23), which is $t_{\beta_1} t_{\beta_1'} t_{\beta_5} t_{\beta_5'} P_1 = 1$, and its global conjugation by ρ, which reads $(t_{\beta_1} t_{\beta_1'} t_{\beta_5} t_{\beta_5'} P_1)^\rho = t_{\beta_2} t_{\beta_2'} t_{\beta_6} t_{\beta_6'} (P_1)^\rho = 1$. By taking a product of six copies of each, and then applying Hurwitz moves, we obtain

$$(t_{\beta_1} t_{\beta_1'} t_{\beta_6} t_{\beta_6'})^6 (t_{\beta_2} t_{\beta_2'} t_{\beta_5} t_{\beta_5'})^6 R = 1 \quad (25)$$

where R is the product of the remaining Dehn twists. This is a monodromy in $\mathrm{Mod}(\Sigma_9, s_0)$ for a twisted fiber sum of 12 copies of (X, f) that we will denote by (Z', h').

In this new factorization, the quadruples $\beta_1, \beta_1', \beta_6, \beta_6'$ and $\beta_2, \beta_2', \beta_5, \beta_5'$ both bound copies of Σ_2^4 in Σ_9. We can thus embed two copies of the monodromy factorization (6) we had for our signature zero spin genus-2 pencil, and cancel out the boundary twists against the Dehn twists about these bounding quadruples. It is time to remember that the spin structure we described on this genus-2 pencil admits a quadratic form which evaluates as 1 on each of the homology basis elements we described in (7). We can thus embed this relation so that the symplectic pairs are mapped to $\alpha_7, \beta_7, \alpha_8, \beta_8$ and to $\alpha_3, \beta_3, \alpha_4, \beta_4$, which in turn guarantees that the new monodromy curves are all mapped to 1 under the quadratic form q_0 we had. Repeating this for the other five subfactorizations, we can embed the genus-2 pencil monodromy a total of 12 times into the product monodromy we had for (Z', h'). Each time we breed with the genus-2 pencil, we get four more vanishing cycles, and after 12 breedings, we get 48 more.

Now to get a Lefschetz fibration homeomorphic to $\#_m(S^2 \times S^2)$ for any given odd $m \geq 415$, write m as $m = 415 + 24n + 2k$, where n and k are unique non-negative integers for $k < 12$. We first take (Z', f') and apply k genus-2 breedings as described above, and then take the fiber sum of the resulting fibration with (Z, f) and n copies of (X, f), using any gluing that preserves $s_0 \in \mathrm{Spin}(\Sigma_9)$. (Untwisted fiber sums prescribed by a product of the monodromy factorizations we gave for these fibrations would do the job.) No matter how the fundamental group changes after the genus-2 breedings, the extra fiber sum summand (Z, f) ensures that the resulting fibration is simply-connected. Finally, calculating the Euler characteristic, we conclude that the simply-connected, spin, signature zero total space of the genus-9 Lefschetz fibration we have is homeomorphic to $\#_m(S^2 \times S^2)$.

This concludes the proof of Corollary C. ∎

Remark 20. It is plausible that, with a more detailed study of the fundamental group of (X, f) and that of twisted fiber sums, one can further improve Corollary C to get smaller values of m. Since we will produce examples of exotic $\#_m(S^2 \times S^2)$ with much smaller topology in Theorem D, here we content ourselves with examples we could get without getting bogged down in technical details. It should be easy to observe that following a similar construction scheme one can also get minimal symplectic Lefschetz fibrations homeomorphic but not diffeomorphic to $\#_m(S^2 \tilde{\times} S^2)$. This can be achieved essentially with less effort, since we are no longer concerned about matching spin structures, and a single breeding with a non-spin pencil yields an odd intersection form.

Remark 21. Involutions on genus-1 Lefschetz fibrations were used by Finashin, Kireck and Viro [24] to produce an infinite family of exotically knotted non-orientable surfaces in S^4; namely, a family of pairwise non-diffeomorphic surfaces, all ambiently homeomorphic to a standard embedding of $\#_{10}\mathbb{RP}^2$ with normal Euler number 16. These were obtained as the fixed point sets of involutions on exotic elliptic surfaces, descending to the quotient, the standard S^4. This result was later improved by Finashin [23], who produced exotically knotted $\#_6\mathbb{RP}^2$ with normal Euler number 8, and more recently by Havens [34], who, by using Finashin's techniques, obtained further irreducible examples. The starting point of the latter works is once again involutions on genus-1 Lefschetz fibrations. In stark contrast, no examples of exotically knotted *orientable* surfaces in S^4 are known, and there have been several attempts to show that there are no smooth knottings of the standardly embedded ones (that bound handlebodies), which is known as the *Smooth Unknotting Conjecture*. Since any such examples necessarily come from an involution on an exotic $\#_m(S^2 \times S^2)$ with fixed point set homeomorphic to Σ_m, explicit examples as in Corollary C, which one may attempt to build the desired involutions on, were sought for quite a while. Similarly, examples of exotic knottings of the standardly embedded non-orientable surface with Euler number zero would come from an involution on an exotic $\#_m(S^2 \tilde{\times} S^2)$. With the explicit fibration structure, the examples we built in this section come as prime candidates for this strategy.

5.2. Smaller, but not fibered, exotic $\#_m(S^2 \times S^2)$

To produce symplectic 4-manifolds homeomorphic but not diffeomorphic to connected sums of smaller numbers of $S^2 \times S^2$, we will adopt an approach similar to the one employed by the first author and Korkmaz to build an exotic $\mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2}$ in [15, §5.3]: First, we will extend our spin, signature zero Lefschetz fibration (X, f) over the 2-sphere to a Lefschetz fibration (X', f') over the 2-torus. While this will enlarge the fundamental group, now the symplectic 4-manifold (X', ω') will have many homologically essential Lagrangian tori carrying the generators of $\pi_1(X')$. We will then apply Luttinger surgeries to these tori in the same fashion as in, for example, [1–3, 9, 25], to derive a symplectic 4-manifold with trivial fundamental group which is homeomorphic to $\#_{23}(S^2 \times S^2)$. We will then show how to produce an infinite family of such examples and how to obtain similar examples in the homeomorphism class of $\#_m(S^2 \times S^2)$ for every odd $m \geq 23$.

Proof of Theorem D. We noted in the previous subsection that the signature zero genus-9 Lefschetz fibration (X, f) of Theorem 14 has a monodromy factorization of the form $t_{\beta_1} t_{\beta_1'} t_{\beta_5} Q_1 = 1$ in $\mathrm{Mod}(\Sigma_9)$, where $\beta_1, \beta_1', \beta_5$ are as in Figure 29 (a), and Q_1 is a product of the remaining positive Dehn twists. Given any disjoint, non-separating curves A_1, A_2, A_3 that are linearly independent in $H_1(\Sigma_9)$, there is a diffeomorphism of Σ_9 that takes $(\beta_1, \beta_1', \beta_5)$ to (A_1, A_2, A_3). Conjugating with this diffeomorphism, we thus get a monodromy factorization of the form $t_{A_1} t_{A_2} t_{A_3} R = 1$ in $\mathrm{Mod}(\Sigma_9, s)$ for (X, f), where R is a product of 45 positive Dehn twists and s is any spin structure on Σ_9 yielding a spin structure $\mathfrak{s}$ on X.

Symplectic 4-manifolds homeomorphic to $\#_{23}(S^2 \times S^2)$: For any $\mathcal{A}, \mathcal{B} \in \mathrm{Mod}(\Sigma_9)$ that commute with each other, we can extend the Lefschetz fibration (X, f) over the 2-sphere to a Lefschetz fibration (X', f') over the 2-torus with a monodromy factorization

$$[\mathcal{A}, \mathcal{B}]\, t_{A_1} t_{A_2} t_{A_3}\, R = 1 \quad \text{in } \mathrm{Mod}(\Sigma_9), \tag{26}$$

where the first term is the commutator of $\mathcal{A}$ and $\mathcal{B}$. Moreover, when $\mathcal{A}, \mathcal{B} \in \mathrm{Mod}(\Sigma_9, s)$, this becomes a factorization in $\mathrm{Mod}(\Sigma_9, s)$, and can be seen to prescribe a spin Lefschetz fibration over the 2-torus. For our construction to follow, it will suffice to take $\mathcal{A} = \mathcal{B} = 1$. In this case, the extension of the spin structure of X to that of X' can be easily seen by first viewing X' as $X' = (X \setminus \nu(F)) \cup (\Sigma_9 \times \Sigma_1^1)$, where $\nu(F)$ is a fibered neighborhood of a regular fiber $F \cong \Sigma_9$ of (X, f). Taking the product of $s \in \mathrm{Spin}(\Sigma_9)$ with any $s' \in \mathrm{Spin}(\Sigma_1^1)$ we get a spin structure on $\Sigma_9 \times \Sigma_1^1$ inducing the same spin structure on its boundary as the restriction of the spin structure of X to $\partial\nu(F) \cong \Sigma_9 \times S^1$. Gluing these spin structures we get a spin structure $\mathfrak{s}'$ on X'. In particular, X' has an even intersection form.

We have $\mathrm{e}(X') = 4(g-1)(h-1) + l = 4 \cdot 8 \cdot 0 + 48 = 48$ (where g, h are the fiber and base genera, l is the number of Lefschetz critical points), and $\sigma(X') = \sigma(X) = 0$ since the factorization (26) of (X', f') is obtained from the monodromy factorization of (X, f) by adding a trivial relation for the trivial commutator.

We now set up our notation for the Luttinger surgeries and the fundamental group calculation, following the conventions in [2,25]. Consider the surface Σ_9 with its standard cell decomposition prescribed by a regular 36-gon with vertex x, and with edges labeled as $\prod_{i=1}^{9} a_i b_i a_i^{-1} b_i^{-1}$ as we go around the perimeter. Similarly take $\Sigma_1^1 = T^2 \setminus D$, where D is an open disk, with its standard cell decomposition given by a rectangle with a hole, with vertex y and with edges labeled as $aba^{-1}b^{-1}$. So $\{a_i, b_i\}_{i=1}^{9}$ generate $\pi_1(\Sigma_9)$ and $\{a, b\}$ generate $\pi_1(\Sigma_1^1)$ at the base points x and y, respectively. Finally, for any $c \in \{a_i, b_i, a, b\}$ let c', c'' denote the parallel copies of the curve c on the same surface, as in [25, Figure 2]. By a slight abuse of notation, we will also denote any curve of the form $c \times y$ or $x \times c$ in $\Sigma_9 \times \Sigma_1^1$ by c.

Going forward, we take the Dehn twist curves A_i in the factorization (26) to be isotopic to a_i, for $i = 1, 2, 3$ and we assume D above is an open disk in the base T^2 of the fibration $f' \colon X' \to T^2$ containing all the critical values. As we added a trivial commutator, i.e. $\mathcal{A} = \mathcal{B} = 1$, we have $\pi_1(X') \cong (\langle a, b\rangle \times \pi_1(\Sigma_g))/N'$, where N' is the subgroup normally generated in $\langle a, b\rangle \times \pi_1(\Sigma_g)$ by the Dehn twist curves of the monodromy, together with an extra relation of the form $[a, b] = \mathcal{W}$ for some product of commutators $\mathcal{W} \in [\pi_1(\Sigma_9), \pi_1(\Sigma_9)]$. (The existence of $\mathcal{W}$ is implied by the existence of a pseudosection of (X, f), and we would have $\mathcal{W} = 1$ if (X, f) had an honest section.) This can be most easily seen by applying the Seifert–van Kampen theorem to the decomposition $X' = (X \setminus \nu(F)) \cup (\Sigma_9 \times \Sigma_1^1)$. Nonetheless, for our fundamental group calculation below, it will suffice to just know that N' contains relations induced by three Dehn twist curves a_1, a_2, a_3.

The parallel transport of any $\alpha \in \{a_i', b_i', b_i''\}$ over any $\gamma \in \{a', b', b''\}$ is a Lagrangian torus T fibered over γ. Through the trivialization $X' \setminus (f'^{-1}(D)) \cong \Sigma_9 \times \Sigma_1^1$, we can view T as a Lagrangian $\alpha \times \gamma$ with respect to a product symplectic form on $\Sigma_9 \times \Sigma_1^1$. Note that for the normal neighborhoods $\nu(\alpha), \nu(\gamma)$ of α, γ in Σ_9, Σ_1^1, the Weinstein neighborhood of the Lagrangian torus $\alpha \times \gamma$ is $\nu(\alpha) \times \nu(\gamma)$. Encoding the surgery information by the triple (T, λ, k), as in [2,25], we claim that performing the following disjoint Luttinger surgeries in X':

$$(a_1' \times b', b', 1),\ (b_2'' \times a', a', 1), (a_i' \times b', a_i', 1), (b_j' \times b'', b_j', 1) \qquad \text{for } i = 4, \dots, 9 \text{ and } j = 1, \dots, 9$$

results in a simply-connected symplectic 4-manifold X''. We take $x \times y$ as the base point for the fundamental group calculation. Per the choices we made here, we can invoke the work of Baldridge and Kirk [9] to deduce that $\pi_1(X'')$ has a presentation with generators a_i, b_i, a, b, for which the following relations hold (among several others we do not include here):

$$\begin{aligned} &a_1 = a_2 = a_3 = 1, \\ &\mu' \, b = \mu \, a \ = \mu_i \, a_i = \mu_j' \, b_j = 1 \quad \text{for } i = 4, \dots, 9 \text{ and } j = 1, \dots, 9, \end{aligned}$$

where the first three relations come from the vanishing cycles of our fibration, and μ', μ, μ_i, μ_j' in the second line are the meridians of the surgered Lagrangian tori, given by conjugates of commutators of the pairs $\{b_1, a\}$, $\{a_2, b\}$, $\{b_i, a\}$, $\{a_j, a\}$, respectively. (While we can write out the exact commutators following [9, 25], there will be no need for these details for our calculation to follow.) Since $a_2 = 1$, the second commutator is trivial, so $a = 1$ by the relation $\mu\, a = 1$. Since all others are commutators of a, they too are trivial, which implies that $b = a_i = b_j = 1$ for all $i = 4, \dots, 9$ and $j = 1, \dots 9$, by the remaining surgery relations above. Now, because $a_1 = a_2 = a_3 = 1$, we see that all the generators of $\pi_1(X'')$ are trivial, so X'' is simply-connected.

As we obtained X'' from X' via Luttinger surgeries, X'' admits a symplectic form ω'' [8]. Since these surgeries along tori do not change the Euler characteristic or the signature, we have $\mathrm{e}(X'') = \mathrm{e}(X') = 48$ and $\sigma(X'') = \sigma(X') = 0$. Moreover, each surgery is performed along a Lagrangian torus from a pair of geometrically dual Lagrangian tori, which describe a hyperbolic pair in $H_2(X)$ with respect to the intersection form. These pairs can all be seen to be disjoint. (For example we can take the Lagrangian tori $b_1'' \times a'$, $a_2' \times b'$, $b_i'' \times a'$, $a_j'' \times a''$ as the duals.) Since the intersection form $Q_{X'}$ is an extension of that of $Q_{X''}$ by such hyperbolic pairs, it follows that X'' is also even, and as $H_1(X'')$ has no 2-torsion, we conclude that X'' is spin. By Freedman's celebrated result, X'' is homeomorphic to $\#_{23}\,(S^2 \times S^2)$.

To obtain an infinite family of examples, first note that the Lagrangian torus $a_3' \times b'$ and its dual (which we can take as $b_3'' \times a'$) in X' are disjoint from all the other tori (and their dual tori) we surgered. So it descends to a homologically essential Lagrangian torus T in X'', for ω'' agrees with ω' away from the surgery tori [8]. As shown by Gompf [30], we can perturb ω'' so that T becomes a self-intersection zero symplectic torus. Importantly, $\pi_1(X'' \setminus T) = \pi_1(X) = 1$, which follows from the fact that the meridian of T in X' was a conjugate of a commutator of the pair $\{b_3, a\}$ that became trivial in $\pi_1(X'')$. Hence, we can perform Fintushel–Stern knot surgery [26] to X'' along T, using an infinite family of fibered knots K_n with distinct Alexander polynomials, and produce an infinite family of symplectic 4-manifolds (X_n'', ω_n'') which are pairwise non-diffeomorphic but are all homeomorphic to X'', and thus to $\#_{23}\,(S^2 \times S^2)$.

Lastly, observe that we can run the same construction using spin, signature zero genus $g = 8d + 1$ Lefschetz fibrations $(\widetilde{X}(d), \widetilde{f}(d))$ we built in the proof of Theorem A. In this case, we will have a similar list of Luttinger surgeries along the Lagrangian tori in the extended fibration $(\widetilde{X}(d)', \widetilde{f}(d)')$ over the 2-torus, except now the indices i and j will run up to $8d + 1$. This way we see that there are symplectic Lefschetz fibrations over the 2-torus which are equivalent via Luttinger surgeries to symplectic 4-manifolds homeomorphic to $\#_{24d-1}(S^2 \times S^2)$, for any $d \in \mathbb{Z}^+$.

Stable range: As our observation above provides exotic $\#_m(S^2 \times S^2)$ only for $m \equiv 23 \pmod{24}$, we still need to show how to get examples for *every* odd $m \geq 23$. We will achieve this by taking symplectic fiber sums of (X'', ω'') above with copies of a small symplectic 4-manifold we will quickly derive from [25] as follows: Take $Y_0 = \Sigma_2 \times \Sigma_2$ with a product symplectic form ω_0. As before, let a_i, b_i and c_j, d_j denote the π_1 generators of the first and second copies of Σ_2 in Y_0, let x and y be the base points we take on them, and for any $c \in \{a_i, b_i, c_j, d_j\}$ let the parallel copies c', c'' of c be described in the same fashion. Performing the following Luttinger surgeries in Y_0:[2]

$$(b'_1 \times c''_1, b'_1, 1),\ (a'_2 \times c'_2, a'_2, 1), (b'_2 \times c''_2, b'_2, 1),$$
$$(a''_2 \times d'_1, d'_1, 1), (a'_1 \times c'_2, c'_2, 1), (a''_1 \times d'_2, d'_2, 1),$$

we obtain the desired symplectic 4-manifold Y. Clearly $\mathrm{e}(Y) = \mathrm{e}(Y_0) = 4$ and $\sigma(Y) = \sigma(Y_0) = 0$. The following relations hold in $\pi_1(Y)$, based at $x \times y$, between the generators a_i, b_i, c_j, d_j:

$$\mu_1\, b_1 = \mu_2\, a_2 = \mu_3\, b_2 = \mu_4\, d_1 = \mu_5\, c_2 = \mu_6\, d_2 = 1,$$

where μ_k, for $k = 1, \dots, 6$, are the meridians of the surgered Lagrangian tori, given by conjugates of commutators of the pairs $\{a_1, d_1\}$, $\{b_2, d_2\}$, $\{a_2, d_2\}$, $\{b_2, c_1\}$, $\{b_1, d_2\}$, $\{b_1, c_2\}$, respectively. We claim that a_1 and c_1 normally generate $\pi_1(Y)$. To see this, add extra relations $a_1 = c_1 = 1$. Then the first and fourth commutators are trivial, so $b_1 = d_1 = 1$ by the relations $\mu_1\, b_1 = \mu_4\, d_4 = 1$. But $b_1 = 1$ implies that the fifth and sixth commutators are trivial, and thus $c_2 = d_2 = 1$ by the relations $\mu_5\, c_2 = \mu_6\, d_2 = 1$. Since now $d_2 = 1$, the second and third commutators are trivial, so $a_2 = b_2 = 1$ as well, by the corresponding relations $\mu_2\, a_2 = \mu_3\, b_2 = 1$. Hence trivializing a_1 and c_1 kills all of $\pi_1(Y)$ as claimed. If we let T' and T'' denote the homologically essential Lagrangian tori in Y descending from $a'_1 \times c'_1$ and $a'_2 \times c'_1$ in Y_0 (which, along with their geometric duals $b''_1 \times d'_1$ and $b'_2 \times d''_1$, are disjoint from the other surgered tori), their meridians μ' and μ'' are conjugates of a commutator of $\{b_1, d_1\}$ and of $\{b_2, d_1\}$, respectively. It follows that $\pi_1(Y \setminus (T' \sqcup T''))$ is normally generated by a_1 and c_1 as well.

Recall that by perturbing the symplectic form on X'' we got a self-intersection zero symplectic torus T in X'' with $\pi_1(X'' \setminus T) = 1$. Let us continue denoting the perturbed symplectic form on X'' as ω''. Similarly, after perturbing the symplectic form, the Lagrangian tori T' and T'' (which are homologically essential and independent) become symplectic in (Y, ω_Y). We can thus take the symplectic fiber sum of (X'', ω'') and (Y, ω_Y) along T and T' to get (X''_1, ω''_1).

We have $\mathrm{e}(X''_1) = \mathrm{e}(X'') + \mathrm{e}(Y) - 2\mathrm{e}(T^2) = 48 + 4 = 52$ and $\sigma(X''_1) = \sigma(X'') + \sigma(Y) = 0 + 0 = 0$. Since the image of the generators of $\pi_1(\partial(\nu T'))$ under the boundary inclusion map are a_1, c_1 and μ, and since $\pi_1(X'' \setminus T) = 1$ and $\pi_1(Y \setminus T')$ is normally generated by a_1 and c_1, by applying the Seifert–van Kampen theorem to the decomposition $X''_1 = (X'' \setminus \nu T) \cup (Y \setminus \nu T')$, we conclude that $\pi_1(X''_1) = 1$. A quick way to see that X''_1 is spin is the following: Reversing the order of Luttinger surgeries and the symplectic fiber sum, which were performed along disjoint subsurfaces, we could obtain X''_1 by first taking a symplectic fiber sum of (X'', ω'') with (Y_0, ω_0) along T and $a'_1 \times c'_1$,

[2]These surgeries are essentially the same as the ones employed in the construction of the homology $S^2 \times S^2$ in [25], except here we do not perform surgeries along the Lagrangian tori $a'_1 \times c'_1$ and $a'_2 \times c'_1$ (as we have different plans for them) and we simply took all the surgery coefficients to be $+1$ (since the effect of surgery coefficient ± 1 on the π_1 calculation will be simply killing a generator or its inverse).

which we can assume to be spin by taking a spin structure on the product 4-manifold Y_0 whose restriction on the fiber sum region agrees with that of the restriction of the spin structure on X'' [30]. In particular, we get a 4-manifold with an even intersection form. But then, when we perform the Luttinger surgeries along the above tori contained in the Y_0 factor, the result is X''_1 and the intersection form only changes by removing the hyperbolic pairs corresponding to the surgery tori and their duals. Therefore X''_1 too has an even intersection form, and since $H_1(X''_1)$ has no 2-torsion, X''_1 is spin. Hence X''_1 is a simply-connected spin symplectic 4-manifold, which is homeomorphic to $\#_{25}(S^2 \times S^2)$ by Freedman. Knot surgery along the other symplectic torus T'' that descends from Y to X''_1 yields an infinite family of pairwise non-diffeomorphic symplectic 4-manifolds in the same homeomorphism class.

Now for $k > 1$, we can build a symplectic 4-manifold (X''_k, ω''_k) by taking a symplectic fiber sum of (X'', ω'') and k copies of (Y, ω_Y) by first fiber summing (X'', ω'') and (Y, ω_Y) along T and T' as above, then—without performing the knot surgery—fiber summing the resulting symplectic 4-manifold (X''_1, ω''_1) with the next copy of (Y, ω_Y) along T'' (which descends to X''_1) and T', and repeating the latter until we add all k copies of (Y, ω_Y). At the very end of this procedure, we can perform knot surgery along T'' coming from the very last copy of Y to produce infinitely many pairwise non-diffeomorphic symplectic 4-manifolds homeomorphic to X''_k as before. A straightforward calculation as above shows that the symplectic 4-manifold X''_k has $\pi_1(X''_k) = 1$, $\mathrm{e}(X''_k) = 48 + 4k$ and $\sigma(X''_k) = 0$. Thus X''_k is homeomorphic to $\#_{23+2k}(S^2 \times S^2)$, for $k \in \mathbb{Z}^+$. ■

Remark 22. Unlike the previous works [4,5,47], our construction of exotic $\#_m(S^2 \times S^2)$ does not build on a compact complex surface produced by algebraic geometers. The spin symplectic 4-manifold (X', ω'), to which we applied symplectic surgeries, has $b_1(X') = 9$, and thus it cannot even be homotopy equivalent to a compact complex surface. Moreover, since there can only be finitely many deformation classes of simply-connected complex surfaces with the same Chern numbers ($c_1^2 = 2\mathrm{e} + 3\sigma$ and $c_2 = \mathrm{e}$), all but finitely many of our exotic symplectic $\#_m(S^2 \times S^2)$ (for fixed m) are necessarily non-complex. Performing the knot surgeries we employed in our constructions, using *non-fibered* knots instead with distinct Alexander polynomials, we also get infinitely many exotic $\#_m(S^2 \times S^2)$ (for fixed m) which do not admit symplectic structures [26].

Appendix A. Hurwitz equivalence for the genus-2 and genus-3 pencils

Here we address the question of whether the signature zero, spin genus-2 and genus-3 pencils we constructed in Sections 3.1 and 3.2 are new additions to the literature. Many genus-2 pencils on ruled surfaces were obtained in [32], and genus-3 pencils on symplectic Calabi–Yau surfaces with $b_1 > 0$ in [10, 33]. While our constructions are new, we are able to observe that the monodromy factorizations of the genus-2 and genus-3 pencils we constructed here are in fact Hurwitz equivalent to the monodromy factorizations of pencils in [10, 32].

A.1. The genus-2 pencil on $T^2 \times S^2$

In [32], the second author described several lifts of the monodromy factorization in $\mathrm{Mod}(\Sigma_2)$ for Matsumoto's well-known genus-2 Lefschetz fibration [42] to monodromy factorizations in $\mathrm{Mod}(\Sigma_2^4)$ for genus-2 pencils on ruled surfaces. Here we will observe that the genus-2 pencil we constructed in Section 3.1 is isomorphic to one of these.

The pencil referred to as $W_{\mathrm{II}A}$ in [32] has the monodromy factorization

$$t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1} t_{B_{0,2}} t_{B_{1,2}} t_{B_{2,2}} t_{C_2} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}, \tag{27}$$

where the curves are as shown in Figure 30.

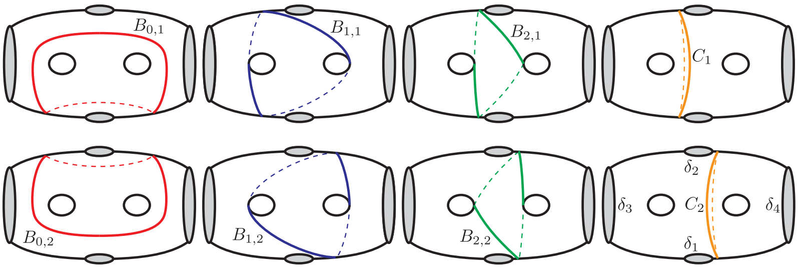


Fig. 30. The monodromy curves for the pencil $W_{\mathrm{II}A}$ of [32].

We perform the following Hurwitz moves to this factorization:

$$\begin{aligned} t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} &= t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1} \underline{t_{B_{0,2}} t_{B_{1,2}} t_{B_{2,2}} t_{C_2}} \\ &\sim t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1} \underline{t_{B_{1,2}} t_{B_{2,2}} t_{C_2}} t_{B'_{0,2}} \\ &\sim t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1} \underline{t_{B_{2,2}} t_{C_2}} t_{B'_{1,2}} t_{B'_{0,2}} \\ &\sim t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1} t_{C_2} t_{B'_{2,2}} t_{B'_{1,2}} t_{B'_{0,2}}, \end{aligned}$$

where $B'_{0,2} = t_{C_2}^{-1} t_{B_{2,2}}^{-1} t_{B_{1,2}}^{-1}(B_{0,2})$, $B'_{1,2} = t_{C_2}^{-1} t_{B_{2,2}}^{-1}(B_{1,2})$, $B'_{2,2} = t_{C_2}^{-1}(B_{2,2})$, and as it turns out that $B_{0,1}$ and $B'_{0,2}$ are disjoint,

$$\sim t_{B'_{0,2}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1} t_{C_2} t_{B'_{2,2}} t_{B'_{1,2}} t_{B_{0,1}}.$$

To compare with the monodromy factorization of our genus-2 pencil, push the boundary components δ_1 and δ_2 as shown in Figure 7 (c). Then we recognize that the last expression exactly coincides with

$$t_{B_0} t_{B_1} t_{B_2} t_C t_{C'} t_{B'_2} t_{B'_1} t_{B'_0}$$

with the curves in Figure 8, which is the monodromy factorization (6) of the genus-2 pencil we built in Section 3.1. (In fact, we can also show that the 6-holed torus relation (5) we obtained while building our genus-2 pencil is Hurwitz equivalent to the 6-holed torus relation of Korkmaz–Ozbagci [38].)

A.2. The genus-3 pencil on T^4

In [10], the first author constructed symplectic genus-3 Lefschetz pencils in every rational homology type of symplectic Calabi–Yau surfaces with $b_1 > 0$. We will show that our signature zero, spin, genus-3 pencil with monodromy factorization (8) in $\mathrm{Mod}(\Sigma_3^4)$ is Hurwitz equivalent to the genus-3 pencil on a symplectic Calabi–Yau 4-torus in [10]. After applying a cyclic permutation to the monodromy factorization (8), we get

$$\begin{aligned} t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} &= \underline{t_{d'} t_w t_a t_{a'} t_x t_b} \cdot \underline{t_{b'} t_y t_c t_{c'} t_z t_d} \\ &\sim \underline{t_w t_a t_{a'} t_x t_b} t_{A_2} \cdot \underline{t_y t_c t_{c'} t_z t_d} t_{A_2'} \\ &\sim t_a \underline{t_{a'} t_x t_b} t_{A_1} t_{A_2} \cdot t_c \underline{t_{c'} t_z t_d} t_{A_1'} t_{A_2'} \\ &\sim t_a t_x t_b t_{A_0} t_{A_1} t_{A_2} \cdot t_c t_z t_d t_{A_0'} t_{A_1'} t_{A_2'} \end{aligned}$$

where $A_2 = t_b^{-1} t_x^{-1} t_{a'}^{-1} t_a^{-1} t_w^{-1}(d')$, $A_2' = t_d^{-1} t_z^{-1} t_{c'}^{-1} t_c^{-1} t_y^{-1}(b')$, $A_1 = t_b^{-1} t_x^{-1} t_{a'}^{-1} t_a^{-1}(w)$, $A_1' = t_d^{-1} t_z^{-1} t_{c'}^{-1} t_c^{-1}(y)$, $A_0 = t_b^{-1} t_x^{-1}(a')$, $A_0' = t_d^{-1} t_z^{-1}(c')$. By relabeling $B_0 = a$, $B_1 = x$, $B_2 = b$, $B_0' = c$, $B_1' = z$, $B_2' = d$, we obtain

$$t_{B_0} t_{B_1} t_{B_2} t_{A_0} t_{A_1} t_{A_2} \cdot t_{B_0'} t_{B_1'} t_{B_2'} t_{A_0'} t_{A_1'} t_{A_2'} = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4} \tag{28}$$

which, after suitably sliding the boundary components, coincides with the positive factorization $W = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4}$ in [10]. As shown in [33], by further Hurwitz moves and a global conjugation, one can see that the monodromy factorization of the latter pencil is equivalent to that of the holomorphic Lefschetz pencil on the standard T^4 by Smith [52]. This array of arguments shows that the symplectic Calabi–Yau surface Y which is the total space of our genus-3 pencil in Section 3.2 is in fact diffeomorphic to T^4.

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