



Laurent Manivel · Mateusz Michałek · Leonid Monin · Tim Seynnaeve ·
Martin Vodička

Complete quadrics: Schubert calculus for Gaussian models and semidefinite programming

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Abstract. We establish connections between the maximum likelihood degree (ML-degree) for linear concentration models, the algebraic degree of semidefinite programming (SDP), and Schubert calculus for complete quadrics. We prove a conjecture by Sturmfels and Uhler on the polynomiality of the ML-degree. We also prove a conjecture by Nie, Ranestad and Sturmfels providing an explicit formula for the degree of SDP. The interactions between the three fields shed new light on the asymptotic behaviour of enumerative invariants for the variety of complete quadrics. We also extend these results to spaces of general matrices and of skew-symmetric matrices.

Keywords. Complete quadrics, Lascoux coefficients, ML-degree, SDP-degree, Lascoux polynomials

1. Introduction

Maximum likelihood degree and quadrics

Although this paper is mainly about enumerative geometry and symmetric functions, the main motivations come from algebraic statistics and multivariate Gaussian models. These are generalizations of the well-known Gaussian distribution to higher dimensions. In the one-dimensional case, in order to determine a Gaussian distribution on \mathbb{R} , one needs to

Laurent Manivel: Institut de Mathématiques de Toulouse UMR 5219, Université de Toulouse, CNRS, 31062 Toulouse, France; manivel@math.cnrs.fr

Mateusz Michałek: Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany; mateusz.michalek@uni-konstanz.de

Leonid Monin: School of Mathematics, University of Bristol, Bristol BS8 1TW, UK; leonid.monin@bristol.ac.uk

Tim Seynnaeve: Mathematisches Institut, Universität Bern, 3012 Bern, Switzerland; tim.seynnaeve@math.unibe.ch

Martin Vodička: Faculty of Science, Šafárik University, 04180 Košice, Slovakia; martin.vodicka@upjs.sk

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specify its mean $\mu \in \mathbb{R}$ and its variance $\sigma \in \mathbb{R}_{>0}$. In the n -dimensional case, the mean is a vector $\mu \in \mathbb{R}^n$, and the second parameter is a positive definite $n \times n$ covariance matrix Σ . The corresponding Gaussian distribution on \mathbb{R}^n is given by

$$f_{\mu, \Sigma}(x) := \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)},$$

where T denotes the transpose. Equivalently to determining it by μ and Σ , one may represent the distribution by μ and the *concentration matrix* $K := \Sigma^{-1}$, which is also positive definite. Our primary interest lies in *linear concentration models*, i.e. statistical models which assume that K belongs to a fixed d -dimensional space \mathcal{L} of $n \times n$ symmetric matrices. These were introduced by Anderson half a century ago [1]. In particular, this means that Σ should belong to the set \mathcal{L}^{-1} of inverses of matrices from \mathcal{L} .

In statistics, typically one gathers data as sample vectors $x_1, \dots, x_s \in \mathbb{R}^n$. This allows one to estimate the mean μ as the mean of the x_i 's. Furthermore, each x_i provides a matrix $\Sigma_i := (x_i - \mu)(x_i - \mu)^T$. Next, one considers the *sample covariance matrix* S , that is, the mean of the Σ_i 's. Note that in most situations, it is not true that $S \in \mathcal{L}^{-1}$. The aim is then to find Σ that best explains the observations. From the point of view of statistics, it is natural to maximize the likelihood function

$$f_{\mu, \Sigma}(x_1) \cdots f_{\mu, \Sigma}(x_s),$$

that is, to find a positive definite matrix $\Sigma \in \mathcal{L}^{-1}$ for which the above value is maximal. Classical theorems in statistics assert that the solution to this optimization problem is essentially geometric [4, Theorems 3.6, 5.5], [23, Theorem 4.4]. Namely, under mild genericity assumptions, the optimal Σ is the unique positive definite matrix in \mathcal{L}^{-1} that maps to the same point as S under projection from \mathcal{L}^\perp .

This is one of the main reasons why the complex variety that is the Zariski closure of \mathcal{L}^{-1} (which abusing notation we also denote by \mathcal{L}^{-1}) and the rational map π defined as the projection from \mathcal{L}^\perp are intensively studied in algebraic statistics. Note that for generic \mathcal{L} , and after projectivization, π becomes a finite map. The following is the central definition of the article.

Definition 1.1 (ML-degree). The *ML-degree* of a linear concentration model represented by a space \mathcal{L} is the degree of the projection from the space \mathcal{L}^\perp restricted to the variety \mathcal{L}^{-1} .

The ML-degree is the basic measure of the complexity of the model. When \mathcal{L} is a generic space, the ML-degree only depends on the size n of the symmetric matrices and on the (affine) dimension d of \mathcal{L} . By a theorem of Teissier [39, 40] (cf. [22, Corollary 2.6]) or Sturmfels and Uhler [38, Theorem 1], the ML-degree equals the degree of the variety \mathcal{L}^{-1} . Following Sturmfels and Uhler [38] we denote it by $\phi(n, d)$. We refer algebraists interested in statistics to [10] for more information.

Definition 1.2. For $n \in \mathbb{Z}_{>0}$ and $1 \leq d \leq \binom{n+1}{2}$, we define $\phi(n, d)$ to be the degree of the variety \mathcal{L}^{-1} , where \mathcal{L} is a general d -dimensional linear subspace of $S^2\mathbb{C}^n$.

Thus, our main result concerns a very basic algebro-geometric object: the degree of the variety obtained by inverting all symmetric matrices in a general linear space. In Section 4, we confirm the following conjecture of Sturmfels and Uhler [38, p. 611], [22, Conjecture 2.8]. This theorem is actually a corollary of a similar polynomiality result for the *algebraic degree of semidefinite programming*, which will be discussed in the next section.

Theorem 1.3. *For any fixed positive integer d , the ML-degree $\phi(n, d)$ is polynomial in n .*

Astonishingly, it appears that the numbers $\phi(n, d)$ were studied for the last 150 years! In 1879 Schubert presented his fundamental results on quadrics satisfying various tangency conditions [31]. His contributions shaped the field of enumerative geometry, inspiring many mathematicians for centuries to come. A nondegenerate quadric being given, the set of its tangent hyperplanes (its projective dual, in modern language) is nothing other than the inverse quadric. This implies that $\phi(n, d)$ is also the solution to the following enumerative problem:

What is the number of nondegenerate quadrics in n variables, passing through $\binom{n+1}{2} - d$ general points and tangent to $d - 1$ general hyperplanes?

In modern language, such problems can be solved by performing computations in the cohomology ring of the *variety of complete quadrics*. This is now a classical topic with many beautiful results [7–9, 16, 20, 32, 34, 41–43]. In particular, the cohomology ring has been described by generators and relations, and algorithms have been devised that allow one to compute any given intersection number. But this only applies for n fixed. Algebraic statistics suggested changing the perspective and fixing d instead of n . This explains, in a way, why the polynomiality property of $\phi(n, d)$ is only proved now.

Semidefinite programming and projective duality

The second domain of mathematics that inspired our research is semidefinite programming (SDP), an important subject in optimization theory. The goal is to study linear optimization problems over spectrahedra. This subject is a direct generalization of linear programming, that is, optimization of linear functions over polyhedra. For a short introduction we refer to [24, Chapter 12].

The coordinates of the optimal solution for an SDP problem, defined over rational numbers, are algebraic numbers. Their algebraic degree is governed by *the algebraic degree of semidefinite programming*. For more information we refer to the fundamental article [25]. To stress the importance of this degree let us just quote that paper:

The algebraic degree of semidefinite programming addresses the computational complexity at a fundamental level. To solve the semidefinite programming exactly essentially reduces to solve a class of univariate polynomial equations whose degrees are the algebraic degree.

Let us provide a precise definition of the algebraic degree of SDP, in the language of algebraic geometry, without referring to optimization. (However, the fact that this definition is correct is actually a nontrivial result [25, Theorem 13].)

Definition 1.4. For $0 < m < \binom{n+1}{2}$ and $0 < r < n$, let $\mathcal{L} \subset S^2\mathbb{C}^n$ be a general linear space of symmetric matrices, of (affine) dimension $m + 1$, and let $SD_m^{r,n} \subset \mathbb{P}(\mathcal{L})$ denote the projectivization of the cone of matrices of rank at most r in \mathcal{L} . The *algebraic degree of semidefinite programming* $\delta(m, n, r)$ is the degree of the projective dual $(SD_m^{r,n})^*$ of $SD_m^{r,n}$ if this dual is a hypersurface, and zero otherwise.

Projective duality is a classical topic, to which a huge literature has been devoted. Computing the degree of a dual variety is well-known to be hard, especially when the variety in question is singular, which is often the case for our $SD_m^{r,n}$. Nevertheless, Ranestad and Graf von Bothmer [14] suggested the use of conormal varieties, and managed to obtain an algebraic expression for $\delta(m, n, r)$ in terms of what we call the *Lascoux coefficients*. These are integer coefficients that govern the Segre classes of the symmetric square of a given vector bundle; algebraically, they are defined by the formal identity

$$\prod_{1 \leq i \leq j \leq s} \frac{1}{1 - (x_i + x_j)} = \sum_I \psi_I s_{\lambda(I)}(x_1, \dots, x_s),$$

where the sum is over the increasing sets $I = (i_1 < \dots < i_s)$ of nonnegative integers, $\lambda(I) = (i_s - s + 1, \dots, i_2 - 1, i_1)$ is the associated partition, and $s_{\lambda(I)}(x_1, \dots, x_s)$ the corresponding Schur function in the variables x_1, \dots, x_s . These coefficients were introduced and studied in [16, 26], whose influence on our work cannot be underestimated. Graf von Bothmer and Ranestad found a formula for $\delta(m, n, r)$ in terms of the Lascoux coefficients (see Theorem 3.7). Delving into the combinatorics of those coefficients, in Section 4 we prove the following polynomiality result:

Theorem 1.5. *For any fixed $m, s > 0$, the function $\delta(m, n, n - s)$ is a polynomial in n .*

Moreover, in Section 5 we confirm [25, Conjecture 21], providing another explicit formula for $\delta(m, n, r)$, and another proof of the above theorem.

Theorem 1.6 ([25, Conjecture 21]). *Let m, n, s be positive integers. Then*

$$\delta(m, n, n - s) = \sum_{\sum I \leq m - s} (-1)^{m - s - \sum I} \psi_I b_I(n) \binom{m - 1}{m - s - \sum I}$$

where the sum is over sets of nonnegative integers of cardinality s .

In this formula, $\sum I = i_1 + \dots + i_s$, and $b_I(n)$ is a polynomial function of n defined in Section 5. Actually, $b_I(n)$ is obtained by evaluating a Schur Q -polynomial on n identical variables; by the work of Stembridge [35], it counts certain shifted tableaux of shape determined by I , numbered by integers not greater than n . Theorem 1.5 also confirms the polynomiality of the ML-degree, since elementary relations in the cohomology ring of the variety of complete quadrics imply the fundamental identity (see Corollary 3.6)

$$\phi(n, d) = \frac{1}{n} \sum_{\binom{s+1}{2} \leq d} s \delta(d, n, n - s).$$

Our approach also applies to linear spaces of general square matrices or of skew-symmetric matrices, and allows us to obtain closed formulas for the dual degrees of their determinantal loci. In particular, the analogues of the Nie–Ranestad–Sturmfels conjecture hold true. This means in particular that these dual degrees can be computed by the class formula, essentially as if those determinantal loci were always smooth (which is certainly not the case in general!). Since the dual degree is well-known to be extremely sensitive to singularities, it would be very interesting to have a conceptual explanation of this phenomenon.

Computations

The proofs of Theorems 1.3, 1.5, 1.6 give rise to explicit polynomial formulas for $\delta(m, n, n - s)$ and $\phi(n, d)$, which can be evaluated using software. We illustrate this on a Sage worksheet, available at <https://mathrepo.mis.mpg.de/>. So far the exact formula for $\phi(n, d)$ has only been known for $d \leq 5$ [6, 22, 36, 38]. We compute it explicitly for $d \leq 50$, confirming in particular [22, Conjecture 5.1].

2. Notation and preliminaries

2.A. Partitions, Schur polynomials

Notation 2.1. For a set $I = \{i_1, \dots, i_r\}$ of nonnegative integers, we assume $i_1 < \dots < i_r$ and we define the corresponding partition

$$\lambda(I) := (i_r - (r - 1), i_{r-1} - (r - 2), \dots, i_2 - 1, i_1).$$

Analogously, for a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, which means $\lambda_1 \geq \dots \geq \lambda_r \geq 0$ (zeroes are allowed), we define the corresponding set

$$I(\lambda) := \{\lambda_r, \lambda_{r-1} + 1, \dots, \lambda_2 + r - 2, \lambda_1 + r - 1\}.$$

The *length* of a partition λ (i.e. the number of nonzero entries) will be denoted by $\text{length}(\lambda)$. We denote by $|\lambda|$ the size of the partition, $\sum_{i=1}^r \lambda_i$, and by $\tilde{\lambda}$ the partition conjugate to λ , e.g. $\widetilde{(3, 1)} = (2, 1, 1)$.

We will abbreviate $\{0, \dots, n - 1\}$ to $[n]$. Let $\sum I := i_1 + \dots + i_r$ denote the sum of elements of I and $\#I = r$ its cardinality. For two sets $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_r\}$ we say that $I \leq J$ if $i_k \leq j_k$ for all $1 \leq k \leq r$.

Notation 2.2. For a partition λ we denote by s_λ the corresponding Schur polynomial.

Definition 2.3. Let I, J be sets of nonnegative integers of cardinality r . We define $s_{I,J}$ to be the unique integers which satisfy the polynomial equation

$$s_{\lambda(I)}(x_1 + 1, \dots, x_r + 1) = \sum_{J \leq I} s_{I,J} s_{\lambda(J)}(x_1, \dots, x_r).$$

Note that since $s_{\lambda(I)}$ is a homogeneous polynomial, we also have the identity

$$s_{\lambda(I)}(x_1 - 1, \dots, x_r - 1) = \sum_{J \leq I} (-1)^{\sum I - \sum J} s_{I,J} s_{\lambda(J)}(x_1, \dots, x_r).$$

As a consequence, the triangular matrices $(s_{I,J})_{I,J}$ and $((-1)^{\sum I - \sum J} s_{I,J})_{I,J}$ are inverses of each other.

Lemma 2.4. *Let $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_r\}$ be two sets of nonnegative integers. Let $M_{I,J} = (m_{kl})$ be the $r \times r$ matrix with $m_{kl} = \binom{i_k}{j_l}$. Then*

(a) $s_{I,J} = \det(M_{I,J})$,

(b) $s_{(d)}(x_1 + 1, \dots, x_r + 1) = \sum_{i=0}^d \binom{d+r-1}{d-i} s_{(i)}(x_1, \dots, x_r)$.

Proof. Part (a) is proved in [19, Section I.3, Example 10]. In particular, it implies

$$s_{[r+d],[r+i]} = \binom{d+r-1}{r+i-1} = \binom{d+r-1}{d-i}.$$

From this, the equation in part (b) becomes the defining equation for $s_{I,J}$. ■

2.B. Lascoux coefficients

Definition 2.5. We define the *Lascoux coefficients* ψ_I by the following formula:

$$s_{(d)}(\{x_i + x_j \mid 1 \leq i \leq j \leq r\}) = \sum_{\substack{\lambda(I) \vdash d \\ \#I=r}} \psi_I s_{\lambda(I)}(x_1, \dots, x_r).$$

Here $s_{(d)}$ is the complete symmetric polynomial of degree d in the $\binom{r+1}{2}$ variables $x_i + x_j$. Hence, the coefficients ψ_I appear in the expansion, in the Schur basis, of the complete symmetric polynomial evaluated at sums of variables.

Equivalently, the Lascoux coefficients appear in the expansion of the d -th Segre class of the second symmetric power of any rank r vector bundle in terms of its Schur classes. In particular, for the universal bundle \mathcal{U} over the Grassmannian $G(r, n)$ for $n \geq r + d$,

$$\text{Seg}_d(S^2\mathcal{U}) = \sum_{\substack{\lambda(I) \vdash d \\ \#I=r}} \psi_I \sigma_{\lambda(I)},$$

where σ_{λ} denote the Schubert classes in the Chow ring of the Grassmannian. (For $r \leq n < r + d$ the identity is still true, but some of the Schubert classes $\sigma_{\lambda(I)}$ will be zero.)

Example 2.6. Let us consider $r = 2$ and $n = 4$, i.e. the Grassmannian $G(2, 4)$. The rank 2 universal vector bundle \mathcal{U} has two Chern roots x_1, x_2 . Recall that the cohomology ring of $G(2, 4)$ is six-dimensional with basis corresponding to Young diagrams contained in the 2×2 square. We have formal equalities

$$x_1 + x_2 = -\square, \quad x_1 \cdot x_2 = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}.$$

The Chern roots of $S^2\mathcal{U}$ are $2x_1, x_1 + x_2, 2x_2$. Computing the elementary symmetric polynomials in those we obtain the respective three Chern classes:

$$-3 \begin{array}{|c|} \hline \square \\ \hline \end{array}, \quad 2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + 6 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad -4 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}.$$

By inverting the Chern polynomial we obtain the Segre classes:

$$3 \begin{array}{|c|} \hline \square \\ \hline \end{array}, \quad \mathbf{7} \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + 3 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad 10 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \quad 10 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}.$$

Their coefficients are the Lascoux coefficients, namely:

$$\psi_{0,2} = 3, \quad \psi_{0,3} = \mathbf{7}, \quad \psi_{1,2} = 3, \quad \psi_{1,3} = 10, \quad \psi_{2,3} = 10.$$

We use boldface and emphasis above and below to indicate the corresponding numbers. We may also compute them by expanding complete symmetric polynomials, where now x_1, x_2 are simply formal variables:

$$\begin{aligned} s_{(2)}(2x_1, x_1 + x_2, 2x_2) &= 7x_1^2 + 7x_2^2 + 10x_1x_2 \\ &= 7(x_1^2 + x_1x_2 + x_2^2) + 3x_1x_2 = \mathbf{7}s_{(2,0)}(x_1, x_2) + 3s_{(1,1)}(x_1, x_2). \end{aligned}$$

We note that Lascoux coefficients appear in many publications with different notations. In particular one needs to be careful with the shift: $\psi_{\{j_1, \dots, j_r\}}$ as defined above equals $\psi_{\{j_1+1, \dots, j_r+1\}}$ in [14]. On the other hand, our notation is consistent with [17, 25].

The lemma below gives a closed formula for the Lascoux coefficients, in terms of Pfaffians.

Lemma 2.7 ([26, Prop. 7.12]). *Let $I = \{i_1, \dots, i_r\}$ be a set of nonnegative integers. If $r = 1, 2$ then ψ_I is given by $\psi_{\{i\}} = 2^i$ and $\psi_{\{i,j\}} = \sum_{k=i+1}^j \binom{i+j}{k}$ respectively. For $r > 2$, ψ_I can be computed as*

$$\psi_I = \begin{cases} \text{Pf}(\psi_{\{i_k, i_l\}})_{0 < k < l \leq n} & \text{for } r \text{ even,} \\ \text{Pf}(\psi_{\{i_k, i_l\}})_{0 \leq k < l \leq n} & \text{for } r \text{ odd,} \end{cases}$$

where $\psi_{\{i_0, i_k\}} := \psi_{\{i_k\}}$.

2.C. SDP-degree and ML-degree

Recall Definitions 1.2 and 1.4 of the ML-degree $\phi(n, d)$ and the SDP-degree $\delta(m, n, r)$.

Remark 2.8. For our polynomiality results in Section 4, it will be useful to extend the definitions of ϕ and δ as follows:

- For $d > \binom{n+1}{2}$, we put $\phi(n, d) = 0$.
- For $m = 0$ and $r < n$, we define $\delta(0, n, r) = 0$.
- For $m \geq \binom{n+1}{2}$ or $s \geq n$, we put $\delta(m, n, n-s) = 0$, with one exception: in the case $m = \binom{n+1}{2}$ and $s = n$, we define $\delta(\binom{n+1}{2}, n, 0) = 1$. See also Remark 3.9.

Now $\phi(n, d)$ is defined for all $n, d > 0$, and $\delta(m, n, n-s)$ is defined for all $m, n, s > 0$.

For later reference, we recall the description of $\delta(m, n, r)$ in terms of the bidegrees of a conormal variety:

Theorem 2.9 ([25, Theorem 10]). *Let $Z_r \subseteq \mathbb{P}(S^2V) \times \mathbb{P}(S^2V^*)$ be the conormal variety to the variety $SD^r \subseteq \mathbb{P}(S^2V)$ of matrices of rank at most r . Explicitly, Z_r consists of pairs (X, Y) of symmetric matrices, up to scalars, with $\text{rk } X \leq r$, $\text{rk } Y \leq n - r$, and $X \cdot Y = 0$. Then the multidegree of Z_r is given by*

$$[Z_r] = \sum_{m=0}^{\binom{n+1}{2}} \delta(m, n, r) H_1^m H_{n-1}^{\binom{n+1}{2}-m}.$$

Here H_1 and H_{n-1} denote the pull-backs of the hyperplane classes from $\mathbb{P}(S^2V)$ and $\mathbb{P}(S^2V^*)$.

Remark 2.10. From this description, we immediately get the following duality relation (see also [25, Proposition 9]):

$$\delta(m, n, n-s) = \delta\left(\binom{n+1}{2} - m, n, s\right).$$

3. ML-degrees via complete quadrics

3.A. Spaces of complete quadrics

Let V be an n -dimensional vector space over \mathbb{C} . The space $CQ(V)$ of complete quadrics is a particular compactification of the space of smooth quadrics in $\mathbb{P}(V)$, or equivalently, of the space of invertible symmetric matrices (up to scalars) $\mathbb{P}(S^2V)^\circ \subset \mathbb{P}(S^2V)$. The space $CQ(V)$ has several equivalent descriptions; below we will describe some of them. For more information we refer the reader to [16, 20, 41].

For $A \in S^2(V)$ let $\bigwedge^k A \in S^2(\bigwedge^k V)$ be the corresponding form on $\bigwedge^k V$. If we view A as a symmetric matrix, then $\bigwedge^k A$ is the matrix of $k \times k$ minors of A . In particular, $\bigwedge^{n-1} A$ is the inverse of A up to scaling (by the determinant).

Definition 3.1. The *space of complete quadrics*, $CQ(V)$, is the closure of $\phi(\mathbb{P}(S^2V)^\circ)$, where

$$\phi : \mathbb{P}(S^2V)^\circ \rightarrow \mathbb{P}(S^2V) \times \mathbb{P}(S^2(V \wedge V)) \times \cdots \times \mathbb{P}(S^2(\bigwedge^{n-1} V))$$

is given by

$$[A] \mapsto ([A], [\bigwedge^2 A], \dots, [\bigwedge^{n-1} A]).$$

To simplify the notation we will also denote $CQ(V)$ by CQ_n .

The natural projection to the j -th factor induces a regular map

$$\pi_j : CQ(V) \rightarrow \mathbb{P}(S^2(\bigwedge^j V)).$$

In particular, the map $\pi_1 : CQ(V) \rightarrow \mathbb{P}(S^2V)$, which is an isomorphism on $\phi(\mathbb{P}(S^2V)^\circ)$, can be described as a sequence of blow-downs. This provides the second description of the space of complete quadrics.

Definition 3.2. The *space of complete quadrics*, $CQ(V)$, is the successive blow-up of $\mathbb{P}(S^2V)$:

$$CQ(V) = Bl_{\tilde{D}^{n-1}} Bl_{\tilde{D}^{n-2}} \dots Bl_{D^1} \mathbb{P}(S^2V),$$

where \tilde{D}^i is the proper transform of the space of symmetric matrices of rank at most i under the previous blow-ups.

Theorem 3.3 ([43, Theorem 6.3]). *Definitions 3.1 and 3.2 of the space of complete quadrics are equivalent.*

The space of complete quadrics admits other descriptions; we would like to mention two of them. The first one describes the space of complete quadrics as an equivariant compactification of the space of invertible symmetric matrices, which is a spherical homogeneous space:

$$\mathbb{P}(S^2V)^\circ \simeq \mathrm{SL}_n / N(\mathrm{SO}_n),$$

where $N(\mathrm{SO}_n)$ is the normalizer of SO_n . The second description realises the space of complete quadrics as a subvariety of the Kontsevich moduli space of stable maps to the Lagrangian Grassmannian; for details see [22, 41].

The space of complete quadrics has two natural series of special classes of divisors. The first series S_1, \dots, S_{n-1} is given by the classes of the (strict transforms) of the exceptional divisors E_1, \dots, E_{n-1} of the successive blow-ups in Definition 3.2 (which are precisely the SL_n -invariant prime divisors on $CQ(V)$). The second series L_1, \dots, L_{n-1} is obtained by pulling back the hyperplane classes by the projections π_1, \dots, π_{n-1} .

Proposition 3.4. *The classes L_1, \dots, L_{n-1} form a basis of $\mathrm{Pic}(CQ(V))$, in which the classes S_1, \dots, S_{n-1} are given by the relations*

$$S_i = -L_{i-1} + 2L_i - L_{i+1}$$

with $L_0 = L_n := 0$.

Proof. These relations were already known to Schubert [30]. For a modern treatment, see for example [20, Proposition 3.6 and Theorem 3.13]. ■

The inverse relations are given by the $(n-1) \times (n-1)$ matrix M given by

$$M_{i,j} = \min(i, j) - ij/n.$$

In particular, we have

$$\begin{aligned} nL_1 &= (n-1)S_1 + (n-2)S_2 + \dots + S_{n-1}, \\ nL_{n-1} &= S_1 + 2S_2 + \dots + (n-1)S_{n-1}. \end{aligned} \tag{3.1}$$

3.B. Intersection theory

We are ready to relate the computation of the ML-degree and of the algebraic degree of semidefinite programming to the intersection theory of $CQ(V)$.

Proposition 3.5. *The ML-degree and the algebraic degree of semidefinite programming can be computed as the following intersection numbers on $CQ(V)$:*

$$\begin{aligned}\phi(n, d) &= \int_{CQ_n} L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1}, \\ \delta(m, n, r) &= \int_{E_r} L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1} = \int_{CQ_n} S_r L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1} \\ &= \int_{\mathbb{P}(S^2\mathcal{U}) \times_{G(r,n)} \mathbb{P}(S^2\mathcal{Q}^*)} \tilde{L}_1^{\binom{n+1}{2}-m-1} \tilde{L}_{n-1}^{m-1},\end{aligned}$$

where E_r and $\mathbb{P}(S^2\mathcal{U}) \times_{G(r,n)} \mathbb{P}(S^2\mathcal{Q}^*)$ are birational and \tilde{L}_i is the transform of L_i .

Proof. As the morphisms π_1 and π_{n-1} resolve the inversion map $\mathbb{P}(S^2V) \dashrightarrow \mathbb{P}(S^2V^*)$, we can compute the degree of \mathcal{L}^{-1} , for $\mathcal{L} \subseteq \mathbb{P}(S^2V^*)$ a general $d-1$ -dimensional linear subspace, as $\pi_1^*(H_1^{\binom{n+1}{2}-d})\pi_{n-1}^*(H_{n-1}^{d-1})$, where H_1 and H_{n-1} are hyperplane classes in $\mathbb{P}(S^2V)$ and $\mathbb{P}(S^2V^*)$ respectively. Indeed, the factor $H_1^{\binom{n+1}{2}-d}$ imposes $\binom{n+1}{2}-d$ linear conditions on matrices in $\mathbb{P}(S^2V)$, defining a linear subspace $\mathbb{P}(\mathcal{L})$ of projective dimension $d-1$, while the factor H_{n-1}^{d-1} imposes $d-1$ linear conditions on $\mathbb{P}(\mathcal{L}^{-1})$, hence computing the degree $\phi(n, d)$.

For $\delta(m, n, r)$, the main observation is that E_r is birationally equivalent to the conormal variety Z_r . Indeed, the inductive properties of the spaces of complete quadrics imply that E_r can be described as a space of relative complete quadrics; more precisely,

$$E_r = CQ_r(\mathcal{U}) \times_{G(r,n)} CQ_{n-r}(\mathcal{Q}^*).$$

The equality above may be derived from the quotient construction of the variety of complete quadrics [22, 41]. In particular, the variety E_r is birationally equivalent to $Y_r = \mathbb{P}(S^2\mathcal{U}) \times_{G(r,n)} \mathbb{P}(S^2\mathcal{Q}^*)$. As observed for example in [14], this is also a smooth model for the conormal variety Z_r . By Theorem 2.9 we have

$$\delta(m, n, r) = \int_{Z_r} L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1}.$$

As the divisors L_i are base point free, the above intersection on Z_r is realized by a zero-dimensional scheme, which may be assumed not to be supported on any closed, proper subvariety. In fact, by Kleiman's transversality theorem this zero-dimensional scheme consists precisely of $\delta(m, n, r)$ reduced points. As Y_r and Z_r are birational, we may consider classes of divisors \tilde{L}_i on Y_r which are transforms of L_i on Z_r . We conclude as before that the intersection of respective \tilde{L}_i 's also consists of $\delta(m, n, r)$ reduced points:

$$\delta(m, n, r) = \int_{Z_r} L_1^{\binom{n+1}{2}-m-1} L_{n-1}^{m-1} = \int_{Y_r} \tilde{L}_1^{\binom{n+1}{2}-m-1} \tilde{L}_{n-1}^{m-1}.$$

This implies our claim. ■

Since S_r projects in $\mathbb{P}(S^2V)$ to the locus of matrices of rank at most r , we must have $S_r L_1^{\binom{n+1}{2}-m-1} = 0$ when m is smaller than the codimension of this locus. Similarly $S_r L_{n-1}^{m-1} = 0$ when m is large enough. One can deduce that the following *Pataki inequalities* are necessary and sufficient conditions for $\delta(m, n, r)$ to be nonzero [25, Proposition 5 and Theorem 7]:

$$\binom{n-r+1}{2} \leq m \leq \binom{n+1}{2} - \binom{r+1}{2}. \quad (3.2)$$

We can then use (3.1) to write the ML-degree in terms of the SDP-degree:

Corollary 3.6. *For any $n, d > 0$, the following fundamental relation holds:*

$$\phi(n, d) = \frac{1}{n} \sum_{1 \leq \binom{s+1}{2} \leq d} s \delta(d, n, n-s).$$

Proof. First, for $d > \binom{n+1}{2}$, both sides are 0, and for $d = \binom{n+1}{2}$ both sides equal 1 (see Remark 2.8), so the relation holds. For $1 \leq d < \binom{n+1}{2}$, we can write

$$\begin{aligned} \phi(n, d) &= \int_{CQ_n} L_1^{\binom{n+1}{2}-d} L_{n-1}^{d-1} \\ &= \frac{1}{n} \int_{CQ_n} L_1^{\binom{n+1}{2}-d-1} L_{n-1}^{d-1} \sum_{s=1}^{n-1} s S_{n-s} \\ &= \frac{1}{n} \sum_{s=1}^{n-1} s \delta(d, n, n-s) = \frac{1}{n} \sum_{1 \leq \binom{s+1}{2} \leq d} s \delta(d, n, n-s). \end{aligned} \quad (3.3)$$

The last equality follows from the Pataki inequalities, since $\delta(d, n, n-s) = 0$ whenever $\binom{s+1}{2} > d$. ■

All our computations can now be reduced to the intersection theory of Grassmannians.

Theorem 3.7 ([14, Theorem 1.1]). *For $0 < m < \binom{n+1}{2}$ and $0 < r < n$,*

$$\delta(m, n, r) = \sum_{\substack{I \subset [n] \\ \#I = n-r \\ \sum I = m-n+r}} \psi_I \psi_{[n] \setminus I}.$$

Idea of proof. As already mentioned, $\delta(m, n, r)$ can be computed as an intersection number on Y_r , which is a fiber bundle over the Grassmannian $G(r, n)$. By push-forward, one obtains

$$\delta(m, n, r) = \int_{G(r, n)} \text{Seg}_{\binom{n+1}{2}-m-\binom{r+1}{2}}(S^2\mathcal{U}) \text{Seg}_{m-\binom{n-r+1}{2}}(S^2\mathcal{Q}^*).$$

We then obtain the theorem by expanding these Segre classes, and using the fundamental duality properties of Schubert classes. ■

Remark 3.8. Recall that our definition of ψ_I is shifted with respect to [14], which explains why our formula looks slightly different.

Remark 3.9. One can easily verify that the above formula is still true for the extended definition of δ from Remark 2.8. The only nontrivial case is $\delta\left(\binom{n+1}{2}, n, 0\right) = \psi_{[n]}\psi_{[n]\setminus[n]} = 1$.

4. Polynomiality of the ML-degree

In this section and the next one, we present three proofs of the following polynomiality result for the algebraic degree of semidefinite programming:

Theorem 4.1. *For any fixed $m, s > 0$, the function $\delta(m, n, n - s)$ is a polynomial in n . Moreover, this polynomial vanishes at $n = 0$.*

As an immediate corollary, we obtain one of the main results of this paper: the polynomiality of the ML-degree for linear concentration models. This property was first conjectured by Sturmfels and Uhler [38] and confirmed in small, special cases in [6, 22, 36].

Theorem 4.2. *For any fixed $d > 0$, the function $\phi(n, d)$ is a polynomial for $n > 0$.*

Proof. For all $n, d > 0$, by Corollary 3.6, we have

$$\phi(n, d) = \frac{1}{n} \sum_{1 \leq \binom{s+1}{2} \leq d} s\delta(d, n, n - s). \quad (4.1)$$

By Theorem 4.1 every term on the right hand side of (4.1) is a polynomial divisible by n , hence the theorem follows. ■

Each of our proofs of Theorem 4.1 has its advantages. The first one is quite elementary, being based on algebraic recursive formulas, which also have a geometric meaning. It provides very efficient methods for explicit computations. The second proof is more technical, but it allows one to derive the leading coefficients of the polynomials we study. The last one is simply a corollary of the conjecture of Nie, Ranestad and Sturmfels that we prove in Section 5.

Our first two proofs of Theorem 4.1 are based on the following theorem.

Theorem 4.3. *Let $I = \{i_1, \dots, i_r\}$ be a set of strictly increasing nonnegative integers. For $n \geq 0$ the function*

$$LP_I(n) := \begin{cases} \psi_{[n]\setminus I} & \text{if } I \subseteq [n], \\ 0 & \text{otherwise,} \end{cases}$$

is a polynomial.

Before we prove Theorem 4.3 let us note that it immediately implies Theorem 4.1. Indeed, by Theorem 3.7, we have

$$\delta(m, n, n - s) = \sum_{\substack{I \subseteq [n] \\ \#I = s \\ \sum I = m - s}} \psi_I \psi_{[n]\setminus I} = \sum_{\substack{\#I = s \\ \sum I = m - s}} \psi_I LP_I(n).$$

By Theorem 4.3, each of the summands is a polynomial in n that vanishes for $n = 0$. Moreover, the number of summands in the last sum depends only on m and s , and not on n . Thus $\delta(m, n, n - s)$ is also a polynomial in n , which proves Theorem 4.1, and hence Theorem 4.2.

In the remainder of this section, we will present two proofs of Theorem 4.3. But let us first give a few examples.

Example 4.4. By induction, one can check the following formulas for LP_I , when I has cardinality 1 or 2:

$$LP_{(i)}(n) = \binom{n}{j+1}, \quad LP_{(0,j)}(n) = j \binom{n+1}{j+2},$$

and more generally, for $i < j$,

$$LP_{(i,j)}(n) = \frac{(j-i)[n+1]_{j+2}}{(i+1)!(j+1)!(i+j+2)!} \sum_{d=0}^i (-1)^d a_{i,d} (i+j+1-d)! [n]_{i-d},$$

where $a_{i,d} = \prod_{k=0}^{d-1} (i-k)(i-k+1)$ and $[n]_d = n(n-1)\dots(n-d+1)$.

4.A. First proof

The following recursive relations are central to our first proof.

Lemma 4.5. (1) For $j_1 > 0$ we have

$$\psi_{\{j_1, \dots, j_r\}} = (r+1)\psi_{\{0, j_1, \dots, j_r\}} - 2 \sum_{\ell=1}^r \psi_{\{0, j_1, \dots, j_{\ell-1}, \dots, j_r\}}, \quad (4.2)$$

where the summation is over ℓ for which $j_{\ell} - 1 > j_{\ell-1}$ and we set $j_0 := 0$.

(2) For $j_1 = 0$ we have

$$\psi_{\{j_1, j_2, \dots, j_r\}} = \sum_{j_{\ell} \leq j'_{\ell} < j_{\ell+1}} \psi_{\{j'_1, \dots, j'_{r-1}\}}. \quad (4.3)$$

Proof. For the first formula see [26, p. 446], [17, (A.15.7)] or [28, pp. 163–166].

To prove the second formula, recall that $s_{(d)}$ is the complete homogeneous symmetric polynomial of degree d , and we have

$$s_{(d)}(\{x_i + x_j \mid 1 \leq i \leq j \leq r\}) = \sum_{\substack{\lambda(I) \vdash d \\ \#I=r}} \psi_I s_{\lambda(I)}(x_1, \dots, x_r).$$

Substituting $x_r = 0$ we obtain

$$\begin{aligned} \sum_{i=0}^d s_{(i)}(\{x_i + x_j \mid 1 \leq i \leq j \leq r-1\}) s_{(d-i)}(x_1, \dots, x_{r-1}) \\ = s_{(d)}(\{x_i + x_j \mid 1 \leq i \leq j \leq r-1\}, x_1, \dots, x_{r-1}) \\ = \sum_{\substack{\lambda(I) \vdash d \\ \text{length}(\lambda(I)) \leq r-1}} \psi_I s_{\lambda(I)}(x_1, \dots, x_{r-1}). \end{aligned}$$

We note that $\text{length}(\lambda(I)) \leq r - 1$ if and only if $0 \in I$. On the other hand, we may apply Pieri's rule to

$$\begin{aligned} \sum_{i=0}^d s_{(i)}(\{x_i + x_j \mid 1 \leq i \leq j \leq r-1\}) s_{(d-i)}(x_1, \dots, x_{r-1}) \\ = \sum_{i=0}^d \left(\sum_{\substack{\lambda(I) \vdash i \\ \#I=r-1}} \psi_I s_{\lambda(I)}(x_1, \dots, x_{r-1}) \right) s_{(d-i)}(x_1, \dots, x_{r-1}). \end{aligned}$$

Comparing the coefficients of Schur polynomials in both expressions gives the desired formula. \blacksquare

First proof of Theorem 4.3. We proceed by induction first on $\#I$, then on $\sum I := \sum_{i_j \in I} i_j$. The base case is $I = \emptyset$, when $\psi_{\{0, \dots, n-1\}} = 1$.

For the induction step, fix I , and assume the theorem has been proven for all I' with $\#I' < \#I$, and for all I' with $\#I' = \#I$ and $\sum I' < \sum I$. We consider two cases:

Case 1: $i_1 = 0$. We claim that for every $n \geq 0$,

$$LP_I(n) = (n - r + 1)LP_{I \setminus \{0\}}(n) - 2 \sum_{\ell: i_{\ell+1} > i_{\ell} + 1} LP_{I \setminus \{0, i_{\ell}\} \sqcup \{i_{\ell} + 1\}}(n),$$

where for summation we formally assume $i_{r+1} = +\infty$. Indeed, if $n \leq i_r$ then both sides are 0, and if $n > i_r$ then the equation is precisely Lemma 4.5 (1).

Case 2: $i_1 > 0$. We claim that for every $n \geq 0$,

$$LP_I(n) - LP_I(n-1) = \sum_J LP_J(n-1),$$

where the sum is over all $J \neq I$ of the form $\{i_1 - \epsilon_1, \dots, i_r - \epsilon_r\}$ with $\epsilon_{\ell} \in \{0, 1\}$. Again, if $n \leq i_r$ then both sides are 0, and if $n > i_r$ then the equation is precisely Lemma 4.5 (2).

In both cases, it follows that LP_I is a polynomial. \blacksquare

4.B. Second proof

Our second proof is based on an explicit interpretation of ψ_I as a sum of minors in the Pascal triangle. We denote by E the Pascal triangle matrix, i.e. $E_{ij} = \binom{i}{j}$. We will always consider only finite submatrices of E so despite the fact that it is an infinite matrix there will be no computations with infinite matrices.

Notation 4.6. For an $n \times n$ matrix A and sets $I, J \subset [n]$ we denote by $A_{I,J}$ the $\#I \times \#J$ matrix which is obtained from A by taking rows indexed by I and columns indexed by J . Here we index rows and columns from 0. In the case $I = J$ we write simply $A_{I,I} = A_I$.

For sets $K, C \subset \mathbb{N}$ with $\#K = \#C$ we denote by $V(K, C)$ the Vandermonde matrix with entries $V(K, C)_{ij} = k_{i+1}^{c_j}$. We also set $V(K) := V(K, [\#K])$, i.e. $V(K)_{ij} = k_{i+1}^j$.

For two sets $A, B \subset \mathbb{N}$ we denote by $\varepsilon^{A,B}$ the sign of the permutation of $A \cup B$ determined by A, B if they are disjoint. If they are not, we define $\varepsilon^{A,B} = 0$.

We begin with a characterization of ψ_I as a sum of minors of the matrix E , which follows from [17, Proposition 2.8].

Proposition 4.7. *The following equality holds:*

$$\psi_I = \sum_{J \leq I} \det(E_{I,J}).$$

In what follows we will need the following lemma that may be easily proved by induction.

Lemma 4.8. *Let a, b be nonnegative integers.*

- (a) *If $a > b$ then $\sum_{i=0}^a (-1)^i \binom{a}{i} i^b = 0$.*
- (b) *If $a = b$ then $\sum_{i=0}^a (-1)^{a-i} \binom{a}{i} i^b = a!$.*

To compute special minors of the matrix E we use the following lemma.

Lemma 4.9. *Let $I = \{i_1, \dots, i_r\} \subset [n]$. Then*

$$\det(E_{[n] \setminus [r], [n] \setminus I}) = \frac{\prod_{1 \leq j < k \leq n-r} (i_k - i_j)}{(r-1)!(r-2)! \dots 2!1!} = \frac{\det(V(I))}{(r-1)!(r-2)! \dots 2!1!}$$

Proof. We fix r and proceed by induction on n . The case $i_r < n-1$ is trivial. If $i_r = n-1$ we express the determinant via Laplace expansion along the n -th row, and use the induction hypothesis and Lemma 4.8 to conclude. ■

Now we are able to present our second proof of Theorem 4.3.

Second proof of Theorem 4.3. Let $\#I = r$ and $m := i_r + 1$. First, assume $n \geq m$. We use the formula from Proposition 4.7. We express the determinants $E_{[n] \setminus I, J}$ using the Laplace expansion along the first $m-r$ rows, choosing the columns indexed by L . For the rest we use Lemma 4.9. To simplify notation we let $K := [n] \setminus J$. We obtain

$$\begin{aligned} \psi_{[n] \setminus I} &= \sum_{J \leq [n] \setminus I} \det(E_{[n] \setminus I, J}) \\ &= \sum_{J \leq [n] \setminus I} \sum_{\substack{L \subseteq J \\ \#L = m-r}} \varepsilon^{L, J \setminus L} \det(E_{[m] \setminus I, L}) \det(E_{[n] \setminus [m], J \setminus L}) \\ &= \sum_{\substack{\#L = m-r \\ L \subseteq [m] \setminus I}} \det(E_{[m] \setminus I, L}) \sum_{\substack{\#K=r \\ K \cap L = \emptyset \\ K \subset [n]}} \varepsilon^{L, [n] \setminus (K \cup L)} \det(E_{[n] \setminus [m], [n] \setminus (K \cup L)}) \\ &= \sum_{\substack{\#L = m-r \\ L \subseteq [m] \setminus I}} \det(E_{[m] \setminus I, L}) \sum_{\substack{\#K=r \\ K \subset [n]}} \varepsilon^{L, K} \varepsilon^{L, [n] \setminus L} \frac{\det(V(L \cup K))}{(m-1)!(m-2)! \dots 2!1!} \\ &= \sum_{\substack{\#L = m-r \\ L \subseteq [m] \setminus I}} \frac{\varepsilon^{L, [n] \setminus L} \det(E_{[m] \setminus I, L})}{(m-1)!(m-2)! \dots 2!1!} \sum_{\substack{\#K=r \\ K \subset [n]}} \det(V^*(L \cup K)) \end{aligned}$$

where $V^*(L \cup K)$ is the matrix $V(L \cup K)$ with the rows indexed by L put first. Note that we may drop the assumption $L \leq [m] \setminus I$, since otherwise $\det(E_{[m] \setminus I, L}) = 0$. Similarly, we can extend our sum and drop the condition $L \cap K = \emptyset$ since we add only zero terms. If we fix L and denote the elements of K by $k_1 < \dots < k_r$, then $\det(V^*(L \cup K))$ is clearly a polynomial in k_1, \dots, k_r . Then

$$\sum_{\substack{\#K=r \\ K \subset [n]}} \det(V^*(L \cup K)) = \sum_{0 \leq k_1 < \dots < k_r < n} \det(V^*(L \cup K))$$

is a polynomial in n for L fixed. Moreover, the sum over L does not depend on n and therefore also $\psi_{[m] \setminus I}$ is a polynomial in n . Our computations are correct only for $n \geq m$. However, the last expression makes sense and is a polynomial for all $n \geq 0$. Clearly, it is 0 for $n < m$. This proves the theorem. \blacksquare

With this approach we can even compute the leading coefficient of LP_I . For this we will need two technical lemmas. The proof of the first one is straightforward, e.g. by induction.

Lemma 4.10. *Let a_1, \dots, a_r be nonnegative integers. Then*

$$\sum_{0 \leq k_1 < k_2 < \dots < k_r < n} k_1^{a_1} k_2^{a_2} \dots k_r^{a_r}$$

is a polynomial in n of degree $\sum_{i=1}^r a_i + r$, with leading coefficient

$$\frac{1}{(a_1 + 1)(a_1 + a_2 + 2) \dots (a_1 + \dots + a_r + r)}.$$

Lemma 4.11. *The following identity of rational functions in r variables holds:*

$$\sum_{\sigma \in S_r} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)} + x_{\sigma(2)}) \dots (x_{\sigma(1)} + \dots + x_{\sigma(r)})} = \frac{\prod_{i>j} (x_i - x_j)}{\prod_i x_i \prod_{i>j} (x_i + x_j)}.$$

Proof. We proceed by induction on r . It is easy to check that for $r = 1, 2$ the statement holds. For $r > 2$, we split the sum depending on $\sigma(r)$ and apply the induction hypothesis to the partial sums:

$$\begin{aligned} & \sum_{\sigma \in S_r} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)} + x_{\sigma(2)}) \dots (x_{\sigma(1)} + \dots + x_{\sigma(r)})} \\ &= \frac{1}{x_1 + \dots + x_r} \sum_{k=1}^r \sum_{\substack{\sigma \in S_r \\ \sigma(r)=k}} (-1)^\sigma \frac{1}{(x_{\sigma(1)})(x_{\sigma(1)} + x_{\sigma(2)}) \dots (x_{\sigma(1)} + \dots + x_{\sigma(r-1)})} \\ &= \frac{1}{x_1 + \dots + x_r} \sum_{k=1}^r (-1)^{r-k} \frac{\prod_{i>j; i, j \neq k} (x_i - x_j)}{\prod_{i \neq k} x_i \prod_{i>j; i, j \neq k} (x_i + x_j)} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(x_1 + \cdots + x_r) \prod_i x_i \prod_{i>j} (x_i + x_j)} \sum_{k=1}^r (-1)^{r-k} x_k \prod_{i>j; i, j \neq k} (x_i - x_j) \prod_{i \neq k} (x_i + x_k) \\
&= \frac{1}{(x_1 + \cdots + x_r) \prod_i x_i \prod_{i>j} (x_i + x_j)} Q(x_1, \dots, x_r),
\end{aligned}$$

where Q is a homogeneous polynomial of degree $\binom{r}{2} + 1$. Moreover, Q is skew-symmetric: if we exchange x_i and x_j we just change the sign. Therefore

$$Q(x_1, \dots, x_r) = \prod_{i>j} (x_i - x_j) R(x_1, \dots, x_r)$$

for R a symmetric polynomial of degree 1. This implies that R is a multiple of $x_1 + \cdots + x_r$. Finally, it is easy to check that the coefficient of $x_r^r x_{r-1}^{r-2} x_{r-2}^{r-3} \cdots x_2$ in Q is 1. Therefore $R = x_1 + \cdots + x_r$ and the proof is complete. ■

Theorem 4.12. *The polynomial LP_I has degree $\sum I + \#I$. Its leading coefficient is*

$$\frac{\prod_{j>k} (i_j - i_k)}{(i_1 + 1)! \cdots (i_r + 1)! \prod_{j>k} (i_j + i_k + 2)}.$$

Proof. We continue with the calculation from the second proof of Theorem 4.3. We take the Laplace expansion of Vandermonde along the first $m - r$ rows. We get

$$\begin{aligned}
&\sum_{\substack{\#L=m-r \\ L \subseteq [m] \setminus I}} \varepsilon^{L, [m] \setminus L} \det(E_{[m] \setminus I, L}) \sum_{\substack{\#K=r \\ K \subseteq [n]}} \det(V^*(L \cup K)) \\
&= \sum_{\substack{\#L=m-r \\ L \subseteq [m] \setminus I}} \varepsilon^{L, [m] \setminus L} \det(E_{[m] \setminus I, L}) \sum_{\substack{\#C=m-r \\ C \subseteq [m]}} \varepsilon^{C, [m] \setminus C} \det(V(L, C)) \sum_{\substack{\#K=r \\ K \subseteq [n]}} \det(V(K, [m] \setminus C)) \\
&= \sum_{\substack{\#C=m-r \\ C \subseteq [m]}} \varepsilon^{C, [m] \setminus C} \sum_{\substack{\#L=m-r \\ L \subseteq [m] \setminus I}} \varepsilon^{L, [m] \setminus L} \det(E_{[m] \setminus I, L}) \det(V(L, C)) \sum_{\substack{\#K=r \\ K \subseteq [n]}} \det(V(K, [m] \setminus C)) \\
&= \sum_{\substack{\#C=m-r \\ C \subseteq [m]}} \varepsilon^{C, [m] \setminus C} \det(A) \sum_{\substack{\#K=r \\ K \subseteq [n]}} \det(V(K, [m] \setminus C)),
\end{aligned}$$

where $A := (\text{diag}(1, -1, \dots, -1^{m-1}) E_{[m] \setminus I, [m]} V([m], C))$. Let $[m] \setminus I = \{b_1, \dots, b_{m-r}\}$, $C = \{c_1, \dots, c_{m-r}\}$, where, as always, we assume that the elements of these sets are ordered increasingly. Notice that $c_{m-r} < b_{m-r}$ implies that the last row of the matrix A is 0 by Lemma 4.8 and so is $\det A$. In general, if $c_i < b_i$, then $A_{[m-r] \setminus [i-1], [i]} = 0$ and we also get $\det A = 0$.

The necessary condition for $\det A \neq 0$ is $c_i \geq b_i$ for all $1 \leq i \leq m - r$. Therefore, we will sum only over such sets C . In the border case when $C = [m] \setminus I$ we find that the matrix A is upper triangular and by Lemma 4.8 we have $\varepsilon^{C, [m] \setminus C} \det A = (b_1)! \cdots (b_{m-r})!$.

Note that $\sum_{\#K=r, K \subseteq [n]} \det(V(K, [m] \setminus C))$ is clearly a polynomial in n of degree at most $\sum([m] \setminus C) + r = \binom{m}{2} + r - \sum C$. Since we are summing only over C with

$\sum C \geq \sum([m] \setminus I)$, we immediately see that the degree of this polynomial is at most $\sum I + r$. Moreover, the only summand which contributes to the term of degree $\sum I + r$ is the one with $C = [m] \setminus I$. We finish the proof of the theorem by computing this summand. In this case we get the polynomial

$$\begin{aligned} \widetilde{LP}_I(n) &:= \sum_{\substack{\#K=r \\ K \subset [n]}} \det(V(K, [m] \setminus C)) \\ &= \sum_{\sigma \in S_r} \sum_{0 \leq k_1 < \dots < k_r < n} (-1)^\sigma k_1^{i_{\sigma(1)}} \dots k_r^{i_{\sigma(r)}}. \end{aligned}$$

By Lemma 4.10 the leading coefficient of \widetilde{LP}_I is

$$\sum_{\sigma \in S_r} (-1)^\sigma \frac{1}{(i_{\sigma(1)} + 1)(i_{\sigma(1)} + i_{\sigma(2)} + 2) \dots (i_{\sigma(1)} + \dots + i_{\sigma(r)} + r)}$$

Now we apply Lemma 4.11 for $x_j = i_j + 1$ to conclude that the leading coefficient is

$$\frac{\prod_{j>k} (i_j - i_k)}{\prod_j (i_j + 1) \prod_{j>k} (i_j + i_k + 2)},$$

which is obviously non-zero. This shows that the degree of LP_I is $\sum I + r$ and its leading coefficient is

$$\begin{aligned} &\frac{1}{(m-1)!(m-2)! \dots 1!} \cdot (b_1!) \dots (b_{m-r})! \cdot \frac{\prod_{j>k} (i_j - i_k)}{\prod_j (i_j + 1) \prod_{j>k} (i_j + i_k + 2)} \\ &= \frac{\prod_{j>k} (i_j - i_k)}{(i_1)! \dots (i_r)! \prod_j (i_j + 1) \prod_{j>k} (i_j + i_k + 2)}. \quad \blacksquare \end{aligned}$$

Corollary 4.13. *The polynomial $\delta(m, n, n - s)$ from Theorem 4.1 has degree m , and the polynomial $\phi(n, d)$ from Theorem 4.2 has degree $d - 1$.*

5. The Nie–Ranestad–Sturmfels conjecture

In this section we present a proof of the formula for the degree of semidefinite programming which was conjectured by Nie, Ranestad and Sturmfels [25]. The formula was proved so far only for special values of the parameters. To state it we introduce the following coefficients.

Definition 5.1 (Coefficients b_I). Let I be a set of s nonnegative integers. We define

$$b_I(n) = Q_{I+\mathbf{1}_s}(\underbrace{1/2, \dots, 1/2}_{n \text{ times}}),$$

where $I + \mathbf{1}_s$ is the set obtained from I by adding 1 to each of its elements. The function $Q_{I+\mathbf{1}_s}$ is the Schur Q -function [19, Section III.8] and its argument $1/2$ appears n times.

Since the Schur Q -function $Q_{I+\mathbf{1}_s}$ is homogeneous of degree $\sum I + s$, there is an equivalent formula:

$$b_I(n) = \left(\frac{1}{2}\right)^{\sum I+s} Q_{I+\mathbf{1}_s}(\underbrace{1, \dots, 1}_{n \text{ times}}),$$

These coefficients may be computed recursively as described in [25, Section 6]. We note that in this reference the authors use the convention that $I \subset \{1, \dots, n\}$ while in this article $I \subset [n] = \{0, \dots, n-1\}$. This results in the difference in notation for the coefficient b_I exchanging I and $I + \mathbf{1}_s$.

The main theorem of this section, confirming the Nie–Ranestad–Sturmfels conjecture, is the following.

Theorem 5.2 ([25, Conjecture 21]). *Let m, n, s be positive integers. Then*

$$\delta(m, n, n-s) = \sum_{\sum I \leq m-s} (-1)^{m-s-\sum I} \psi_I b_I(n) \binom{m-1}{m-s-\sum I},$$

where the sum is over sets of nonnegative integers of cardinality s .

As we already mentioned, Theorem 4.1 is an immediate corollary of Theorem 5.2, since the coefficients $b_I(n)$ are known to be polynomials. Hence, as soon as we have proven Theorem 5.2, we have a third proof of Theorem 4.1.

Remark 5.3. We note that if the Pataki inequality (3.2), $m \geq \binom{s+1}{2}$, is not satisfied, then both sides of the equality above are trivially zero.

For the rest of the section we fix the numbers m, n, s as in the statement of the theorem. Theorem 5.2 presents a relation between the numbers $b_I(n)$ and ψ_I , our proof of which will be algebraic, with the coefficients $s_{I,J}$ from Definition 2.3 playing a prominent role. The following lemma describes the relations between $b_I(n)$ and $s_{I,J}$:

Lemma 5.4. *Let I be a set of s nonnegative integers. Then*

$$b_I(n) = \sum_{J \leq I} \left(\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} LP_J(n), \quad LP_I(n) = \sum_{J \leq I} \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} b_J(n).$$

These two identities are equivalent, by the discussion following Definition 2.3. We present two proofs: one based on simple algebra, and one on more sophisticated methods from algebraic geometry.

For the first proof, let us recall two statements from linear algebra which will allow us to prove Pfaffian formulas also for set complements.

Lemma 5.5 (Jacobi's Theorem). *Let A be an $n \times n$ matrix, and A^C its cofactor matrix. Then*

$$\det(A_{[n] \setminus I, [n] \setminus J}) = \det(A_{I,J}^C) \det(A)^{\#I-1}$$

for all sets $I, J \subset [n]$ with $\#I = \#J$.

Corollary 5.6. *The cofactor matrix A^C of an $n \times n$ skew-symmetric matrix A is given by*

$$A_{ij}^C = \text{Pf}(A_{[n] \setminus \{i,j\}}) \text{Pf}(A).$$

Lemma 5.7. *Let $I = \{i_1, \dots, i_r\}$ be a set of nonnegative integers. Then*

$$\psi_{[n] \setminus I} = \begin{cases} \text{Pf}(\psi_{[n] \setminus \{i_k, i_l\}})_{0 < k < l \leq r} & \text{for } \#I \text{ even,} \\ \text{Pf}(\psi_{[n] \setminus \{i_k, i_l\}})_{0 \leq k < l \leq r} & \text{for } \#I \text{ odd,} \end{cases}$$

where $\psi_{[n] \setminus \{i_0, i_k\}} := \psi_{[n] \setminus \{i_k\}}$.

Proof. Let us consider the case where both n and $\#I$ are even. Consider the skew-symmetric matrix A such that $A_{k,l} = \psi_{\{k,l\}}$ for $0 \leq k < l < n$. Then using Lemmas 5.5 and 2.7 we get

$$\psi_{[n] \setminus I} = \text{Pf}(A_{[n] \setminus I}) = \text{Pf}(A_I^C) \text{Pf}(A)^{\#I-1} = \text{Pf}(A_I^C),$$

since $\det(A) = \psi_{\{0,1,\dots,n-1\}} = 1$. Moreover, by Corollary 5.6, the entries of the cofactor matrix A^C are $\text{Pf}(A_{[n] \setminus \{k,l\}}) \text{Pf}(A) = \psi_{[n] \setminus \{k,l\}}$, which proves the lemma in this case.

The proof in the other cases is similar. The only difference is that we consider a different matrix A . If n is odd we take $A = (\psi_{\{k,l\}})_{-1 \leq k < l < n}$ and if n is even and $\#I$ is odd we take $A = (\psi_{\{k,l\}})_{-2 \leq k < l < n}$. We interpret $\psi_{\{-1,k\}}$ and $\psi_{\{-2,k\}}$ as $\psi_{\{k\}}$ and we put $\psi_{\{-1,-2\}} = 1$. Then we conclude in the same way. ■

Corollary 5.8.

$$\#I \psi_{[n] \setminus I} = \begin{cases} 2 \sum_{1 \leq k < l \leq r} (-1)^{k+l+1} \psi_{[n] \setminus \{i_k, i_l\}} \psi_{[n] \setminus (I \setminus \{i_k, i_l\})} & \text{if } \#I \text{ is even,} \\ 2 \sum_{0 \leq k < l \leq r} (-1)^{k+l+1} \psi_{[n] \setminus \{i_k, i_l\}} \psi_{[n] \setminus (I \setminus \{i_k, i_l\})} & \text{if } \#I \text{ is odd,} \end{cases}$$

where $\psi_{[n] \setminus \{i_0, i_k\}} := \psi_{[n] \setminus \{i_k\}}$.

Proof. For every skew-symmetric $r \times r$ matrix A (with r even) and every $k = 1, \dots, r$, we have the following recursive formula for the Pfaffian:

$$\text{Pf}(A) = \sum_{l=1}^{k-1} (-1)^{k+l} a_{k,l} \text{Pf}(A_{\hat{k}\hat{l}}) - \sum_{l=k+1}^r (-1)^{k+l} a_{k,l} \text{Pf}(A_{\hat{k}\hat{l}}),$$

where $A_{\hat{k}\hat{l}}$ is the submatrix obtained by removing the k -th and l -th rows and columns. Summing over all k gives the desired equality. ■

Remark 5.9. If we define $\psi_{[n] \setminus I} = 0$ for $I = \{i_1, \dots, i_r\}$ a multiset/partition with at least one repeated entry, the recursion from Corollary 5.8 still holds.

Remark 5.10. Corollary 5.8 can be seen as a recursive relation between the polynomials $LP_I(n)$ from Theorem 4.3. In particular, we can obtain in this way one more proof of Theorem 4.3.

First proof of Lemma 5.4. We will use induction on the length of I , which we will denote by s . The base cases $s = 1, 2$ are left to the reader.

We proceed with the general case $s > 2$. We will assume that s is even; the odd case is analogous. Since $b_I = \text{Pf}(b_{i_p, i_q})_{1 \leq p < q \leq s}$, we have (as in Corollary 5.8) the following recursive relations between the b_I 's:

$$s b_I = 2 \sum_{1 \leq p < q \leq n} (-1)^{p+q+1} b_{\{i_p, i_q\}} b_{I \setminus \{i_p, i_q\}}.$$

In order to use induction, we need to show that

$$\begin{aligned} s \sum_{J \leq I} 2^{\sum J} s_{I, J} \psi_{[n] \setminus J} &= 2 \sum_{1 \leq p < q \leq n} (-1)^{p+q+1} \\ &\times \left(\sum_{J \leq \{i_p, i_q\}} 2^{\sum J} s_{\{i_p, i_q\}, J} \psi_{[n] \setminus J} \right) \left(\sum_{J \leq I \setminus \{i_p, i_q\}} 2^{\sum J} s_{I \setminus \{i_p, i_q\}, J} \psi_{[n] \setminus J} \right). \end{aligned}$$

This follows immediately from the following claim:

Claim 5.11. *For every $J \leq I$, where J may have repeated elements,*

$$\begin{aligned} s_{I, J} \psi_{[n] \setminus J} &= \frac{2}{s} \sum_{1 \leq p < q \leq n} (-1)^{p+q+1} \\ &\times \left(\sum_{1 \leq s < t < n} s_{\{i_p, i_q\}, \{j_s, j_t\}} \psi_{[n] \setminus \{j_s, j_t\}} s_{I \setminus \{i_p, i_q\}, J \setminus \{j_s, j_t\}} \psi_{[n] \setminus (J \setminus \{j_s, j_t\})} \right). \end{aligned}$$

Indeed, using Laplace expansion, for any t, u we can write

$$s_{I, J} = \sum_{p < q} (-1)^{p+q+t+u} s_{\{i_p, i_q\}, \{j_t, j_u\}} s_{I \setminus \{i_p, i_q\}, J \setminus \{j_t, j_u\}}.$$

Hence, the right hand side can be rewritten as

$$2 s_{I, J} \sum_{1 \leq t < u < n} (-1)^{t+u+1} \psi_{[n] \setminus \{j_t, j_u\}} \psi_{[n] \setminus (J \setminus \{j_t, j_u\})}.$$

It remains to show that

$$s \psi_{[n] \setminus J} = 2 \sum_{1 \leq t < u < n} (-1)^{t+u+1} \psi_{[n] \setminus \{j_t, j_u\}} \psi_{[n] \setminus (J \setminus \{j_t, j_u\})}.$$

But this is precisely Corollary 5.8, which concludes the first proof of the formula. \blacksquare

The ideas of the second proof were suggested to us by Andrzej Weber.

Second proof of Lemma 5.4. We start with a projection formula, which is a special case of [13, (4.7)]. The proofs of the formula were first provided by Pragacz [28, 29]. Note that this formula is stated in terms of Schur P -polynomials, while we work with Schur Q -polynomials, which accounts for an additional factor of a power of 2.

For a vector bundle \mathcal{E} of rank n over some base X , we consider the relative Grassmannian $G^s(\mathcal{E})$ of rank s quotients of \mathcal{E} , with its projection π to X . We denote by \mathcal{K}

and \mathcal{Q} the relative tautological subbundle and quotient bundle of $\pi^*\mathcal{E}$, of respective ranks $r = n - s$ and s . Then

$$Q_{I+1_s}(\mathcal{E}) = \pi_*(c_{\text{top}}(\mathcal{K} \otimes \mathcal{Q})Q_{I+1_s}(\mathcal{Q})), \quad (5.1)$$

where $+1_s$ indicates adding 1 to all s elements of I (cf. [13, Example 2, p. 50]). Moreover, the formula of [13, (4.5)] and [27, Proposition 2.2] can be written as

$$Q_{I+1_s}(\mathcal{Q}) = 2^s c_{\text{top}}(\bigwedge^2 \mathcal{Q})s_{\lambda(I)+1_s}(\mathcal{Q}) = c_{\text{top}}(S^2\mathcal{Q})s_{\lambda(I)}(\mathcal{Q}).$$

Since $\pi^*\mathcal{E}$ is an extension of \mathcal{Q} by \mathcal{K} , the bundle $\pi^*S^2\mathcal{E}$ admits a filtration whose successive quotients are $S^2\mathcal{Q}$, $\mathcal{K} \otimes \mathcal{Q}$ and $S^2\mathcal{K}$. Hence the identity

$$c(\mathcal{K} \otimes \mathcal{Q})c(S^2\mathcal{Q}) = s(S^2\mathcal{K})\pi^*c(S^2\mathcal{E}).$$

Equation (5.1) can thus be rewritten as

$$Q_{I+1_s}(\mathcal{E}) = c(S^2\mathcal{E})\pi_*(s(S^2\mathcal{K})s_{\lambda(I)}(\mathcal{Q}))|_{\text{deg}=\sum I+s},$$

where the last symbols mean we only keep the component of degree $\sum I + s$.

Now suppose that $\mathcal{E} = \mathcal{E}_0 \otimes L$ for some line bundle L and a trivial vector bundle \mathcal{E}_0 . Then $G^s(\mathcal{E})$ is a trivial bundle over X , while $\mathcal{K} = \mathcal{K}_0 \otimes L$ and $\mathcal{Q} = \mathcal{Q}_0 \otimes L$ are obtained by pull-back of the tautological and quotient bundles \mathcal{K}_0 , \mathcal{Q}_0 over a fixed Grassmannian $G^s(\mathbf{C}^n)$ (we omit the pull-backs in notation for simplicity). By Definition 2.3 (where formally the x_i 's are the Chern roots of \mathcal{Q} and we need to homogenize by using $c_1(L)$), we have

$$s_{\lambda(I)}(\mathcal{Q}) = \sum_{J \leq I} s_{I,J} s_{\lambda(J)}(\mathcal{Q}_0) \delta^{\sum I - \sum J},$$

where $\delta = c_1(L)$. Moreover, the Segre classes of $S^2\mathcal{K}_0^*$ and $S^2\mathcal{K}$ are related by

$$s(S^2\mathcal{K}) = \sum_{\ell \geq 0} (1 + 2\delta)^{-(\frac{r+1}{2})-\ell} s_{(\ell)}(S^2\mathcal{K}_0^*).$$

Plugging these two formulas into the previous one, and expanding each $s_{(\ell)}(S^2\mathcal{K}_0^*)$ in terms of $s_{\lambda(H)}(\mathcal{K}_0^*)$ using Definition 2.5 (where H is simply another index for a subset of nonnegative integers, like I and J), we see that $Q_{I+1_s}(\mathcal{E})$ equals

$$\sum_{J \leq I} \sum_H (1 + 2\delta)^{\binom{n+1}{2} - \binom{r+1}{2} - |\lambda(H)|} \delta^{\sum I - \sum J} s_{I,J} \psi_H \pi_*(s_{\lambda(H)}(\mathcal{K}_0^*)s_{\lambda(J)}(\mathcal{Q}_0))|_{\text{deg}=\sum I+s}.$$

Now recall that the Schur classes $s_\alpha(\mathcal{K}_0^*)$ and $s_\beta(\mathcal{Q}_0)$, for partitions $\alpha \subset (s^r)$ and $\beta \subset (s^k)$, that are nonzero, give dual bases of Schubert cycles on the Grassmannian $G^s(\mathbf{C}^n)$. This can be expressed as

$$\pi_*(s_{\lambda(H)}(\mathcal{K}_0^*)s_{\lambda(J)}(\mathcal{Q}_0)) = \delta_{H,[n]/J},$$

where $\delta_{H,[n]/J}$ is the Kronecker delta. Note that $H = [n]/J$ implies that $|\lambda(H)| + |\lambda(J)| = sr$. We thus get the formula

$$Q_{I+1_s}(\mathcal{E}) = \left(\sum_{J \leq I} (1 + 2\delta)^{s+\sum J} \delta^{\sum I - \sum J} s_{I,J} \psi_{[n]/J} \right) |_{\text{deg}=\sum I+s}.$$

But since the degree of the polynomial in brackets is exactly $\sum I + s$, we just need to keep its top degree component, that is,

$$Q_{I+1_s}(\mathcal{E}) = \sum_{J \leq I} 2^{\sum J + s} s_{I,J} \psi_{[m]/J} \delta^{\sum I + s}.$$

We conclude by applying this formula to the bundle $\mathcal{E} = \mathcal{O}(1)^{\oplus n}$ over the projective space. \blacksquare

Remark 5.12. We could also deduce the lemma by applying the last formula formally to the bundle $\mathcal{E} = \mathcal{O}(1/2)^{\oplus n}$, obtaining directly the defining formula for b_I from Definition 5.1. Even though \mathcal{E} is not a line bundle, by the squaring principle [15, pp. 76–78] we can formally compute with \mathcal{E} as if it were a line bundle.

Lemma 5.13. *Let J be a set of nonnegative integers of length s with $\sum J \leq m - s$. Then*

$$\sum_{\substack{I \geq J \\ \sum I \leq m-s}} \psi_I \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} \binom{m-1}{m-s-\sum I} = \begin{cases} 0 & \text{if } \sum J < m-s, \\ \psi_J & \text{if } \sum J = m-s. \end{cases}$$

Proof. We prove the lemma at the same time for all the J 's by multiplying the above equation by the Schur polynomial $s_{\lambda(J)}(x_1, \dots, x_s)$ and summing up. Since the Schur polynomials form a basis of the space of symmetric polynomials, the statement of the lemma is equivalent to the following polynomial identity:

$$\begin{aligned} \sum_{\sum J \leq m-s} \sum_{\substack{I \geq J \\ \sum I \leq m-s}} \psi_I \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} \binom{m-1}{m-s-\sum I} s_{\lambda(J)}(x_1, \dots, x_s) \\ = \sum_{\sum J = m-s} \psi_J s_{\lambda(J)}(x_1, \dots, x_s). \end{aligned}$$

By Definition 2.5, the right hand side is equal to $s_{(m-s-\binom{s}{2})}(x_i + x_j \mid 1 \leq i < j \leq s)$. For the left hand side we can use Definition 2.3 of the coefficients $s_{I,J}$:

$$\begin{aligned} \sum_{\sum J \leq m-s} \sum_{\substack{I \geq J \\ \sum I \leq m-s}} \psi_I \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} \binom{m-1}{m-s-\sum I} s_{\lambda(J)}(x_1, \dots, x_s) \\ = \sum_{\sum I \leq m-s} \psi_I \binom{m-1}{m-s-\sum I} \sum_{J \leq I} \left(-\frac{1}{2}\right)^{\sum I - \sum J} s_{I,J} s_{\lambda(J)}(x_1, \dots, x_s) \\ = \sum_{\sum I \leq m-s} \psi_I \binom{m-1}{m-s-\sum I} s_{\lambda(I)}(x_1 - 1/2, \dots, x_s - 1/2) \\ = \sum_{i=\binom{s}{2}}^{m-s} \sum_{\sum I = i} \binom{m-1}{m-s-i} \psi_I s_{\lambda(I)}(x_1 - 1/2, \dots, x_s - 1/2) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=\binom{s}{2}}^{m-s} \binom{m-1}{m-s-i} \sum_{\Sigma I=i} \psi_I s_{\lambda(I)}(x_1 - 1/2, \dots, x_s - 1/2) \\
&= \sum_{i=\binom{s}{2}}^{m-s} \binom{m-1}{m-s-i} s_{(i-\binom{s}{2})}(x_i + x_j - 1 \mid 1 \leq i \leq j \leq s) \\
&= s_{(m-s-\binom{s}{2})}(x_i + x_j \mid 1 \leq i \leq j \leq s).
\end{aligned}$$

In the last equality we have applied Lemma 2.4 to the variables $x_i + x_j - 1$. ■

Now we are able to present the proof of Theorem 5.2:

Proof of Theorem 5.2. We replace $b_I(n)$ by the expression from Lemma 5.4, change the order of summation and use Lemma 5.13 in the last step:

$$\begin{aligned}
&\sum_{\Sigma I \leq m-s} (-1)^{m-s-\Sigma I} \psi_I b_I(n) \binom{m-1}{m-s-\Sigma I} \\
&= \sum_{\Sigma I \leq m-s} \sum_{J \leq I} s_{I,J} \psi_{[n] \setminus J} \left(\frac{1}{2}\right)^{\Sigma I - \Sigma J} (-1)^{m-s-\Sigma I} \psi_I \binom{m-1}{m-s-\Sigma I} \\
&= \sum_{\Sigma J \leq m-s} (-1)^{m-s-\Sigma J} \psi_{[n] \setminus J} \sum_{\substack{I \geq J \\ \Sigma I \leq m-s}} s_{I,J} \left(-\frac{1}{2}\right)^{\Sigma I - \Sigma J} \psi_I \binom{m-1}{m-s-\Sigma I} \\
&= \sum_{\Sigma J = m-s} (-1)^{m-s-\Sigma J} \psi_{[n] \setminus J} \psi_J = \delta(m, n, n-s). \quad \blacksquare
\end{aligned}$$

6. General square matrices

The results from the previous sections have natural analogues if we replace the space of symmetric matrices (“type C ”) with the space of skew-symmetric matrices (“type D ”), or with the space of general matrices (“type A ”). This section will be devoted to the latter case, and the next section to the former one.

6.A. Codegrees of smooth determinantal loci

Let M_n denote the space of complex matrices of size n , and $D^{n-r,n} \subset \mathbb{P}(M_n)$ the locus of matrices of rank at most $n-r$. Denote by $D_m^{n-r,n}$ its intersection with a general m -dimensional projective space. Its dimension is $d = m - r^2$ when this is nonnegative, otherwise it is empty. The analogues of Pataki’s inequalities are given by

Proposition 6.1. *The dual variety of $D_m^{n-r,n}$ is a hypersurface if and only if*

$$r^2 \leq m \leq n^2 - (n-r)^2.$$

As was done in [25] for symmetric matrices, the degree of this dual variety can be computed by classical means when $D_m^{n-r,n}$ is smooth, which is equivalent to $r^2 \leq m \leq r^2 + 2r$. The class formula gives, in terms of topological Euler characteristics,

$$\deg (D_m^{n-r,n})^* = (-1)^d (\chi(D_m^{n-r,n}) - 2\chi(D_{m-1}^{n-r,n}) + \chi(D_{m-2}^{n-r,n})).$$

The Euler characteristics of smooth degeneracy loci have been computed by Pragacz [26]. For $\varphi : F \rightarrow E$ a morphism of vector bundles of ranks f, e over a variety X , the formula given in [13, p. 57] is

$$\chi(D_r(\varphi)) = \int_X P_r(E, F)c(X),$$

where $c(X)$ denotes the total Chern class of X , while $P_r(E, F)$ is a universal polynomial in the Chern classes of E and F . Explicitly,

$$P_r(E, F) = \sum_{\lambda, \mu} (-1)^{|\lambda|+|\mu|} D_{\lambda, \mu}^{n-r, m-r} s_{(m-r)^{n-r} + \lambda, \tilde{\mu}}(E - F),$$

where the sum is over partitions λ and μ of lengths $n - r$ and $m - r$ respectively, and $\tilde{\mu}$ is the dual partition of μ . Moreover, the coefficients $D_{\lambda, \mu}^{n-r, m-r}$ defined in [13] encode the Segre classes of a tensor product of vector bundles. (In what follows, we will rather use the notations of [17]; see Definition 6.7.)

We want to apply this formula to $D_m^{n-r,n}$, which we consider as the degeneracy locus $D_{n-r}(\varphi)$ of the tautological morphism $\varphi : F = \mathcal{O}(-1)^{\oplus n} \rightarrow \mathcal{O}^{\oplus n}$ over $X = \mathbb{P}^m$. Since $c(\mathbb{P}^m) - 2hc(\mathbb{P}^{m-1}) + h^2c(\mathbb{P}^{m-2}) = (1 + h)^{m-1}$, where h denotes the hyperplane class, we get the formula

$$\deg (D_m^{n-r,n})^* = \sum_{\lambda, \mu} (-1)^{|\lambda|+|\mu|} \binom{m-1}{r^2 + |\lambda| + |\mu|} D_{\lambda, \mu}^{r,r} s_{(r)^r + \lambda, \tilde{\mu}}(\underbrace{1, \dots, 1}_{n \text{ times}}),$$

the sum being taken over partitions λ and μ of length r . Note that the dependence on n for r and m fixed is only in the last term, more precisely in the number of 1's on which the Schur functions are evaluated. This dependence is well known to be polynomial in n ; very explicitly, for any partition ν ,

$$s_\nu(\underbrace{1, \dots, 1}_{n \text{ times}}) = \dim S_\nu \mathbb{C}^n = c_\nu(n)/h(\nu),$$

where c_ν is the content polynomial and $h(\nu)$ is the product of the hook lengths of ν [19]. A priori this formula is only valid in the range $r^2 \leq m \leq r^2 + 2r$, when $D_m^{n-r,n}$ is smooth. That it should be true in general would be an analogue of the NRS conjecture in type A. We will prove below that this statement is correct.

We introduce the following notations, similar to those we used for symmetric matrices.

Definition 6.2. We define $\delta_A(m, n, r)$ to be the degree of the variety $(D_m^{r,n})^*$ if it is a hypersurface, and zero otherwise. Here $D_m^{r,n}$ is the variety of $n \times n$ matrices of rank at most r , intersected with a general (projective) m -dimensional subspace. Equivalently, if

we let $Z_r \subset \mathbb{P}(V^* \otimes V) \times \mathbb{P}(V^* \otimes V)$ be the variety of pairs of matrices (X, Y) , up to scalars, with $X \cdot Y = Y \cdot X = 0$, $\text{rk } X \leq r$, $\text{rk } Y \leq n - r$, then the multidegree of Z_r is equal to

$$[Z_r] = \sum_m \delta_A(m, n, r) H_1^{n^2-m} H_{n-1}^m,$$

where H_1 and H_{n-1} denote the pull-backs of the hyperplane classes from $\mathbb{P}(V^* \otimes V)$ and $\mathbb{P}(V^* \otimes V)$, respectively.

Definition 6.3. The number $\phi_A(n, d)$ is the degree of the variety \mathcal{L}^{-1} , where $\mathcal{L} \subseteq \mathbb{P}(M_n)$ is a general linear subspace of dimension $d - 1$.

6.B. Complete collineations

The correct space to work with is the *space of complete collineations* [16,20,33,41,42,44]. It can actually be defined for rectangular matrices, but for the sake of simplicity we will restrict ourselves to square matrices.

Definition 6.4. Let V and W be two vector spaces of equal dimension n . The space $\mathbb{P}(V^* \otimes W)$ represents linear maps from V to W ; the open subset of rank n linear maps is denoted by $\mathbb{P}(V^* \otimes W)^\circ$. Then the space of *complete collineations*, $CC(V, W)$, is defined as the closure of the image of the map

$$\phi : \mathbb{P}(V^* \otimes W)^\circ \rightarrow \mathbb{P}(V^* \otimes W) \times \mathbb{P}(\wedge^2 V^* \otimes \wedge^2 W) \times \cdots \times \mathbb{P}(\wedge^{n-1} V^* \otimes \wedge^{n-1} W)$$

given by

$$[A] \mapsto ([A], [\wedge^2 A], \dots, [\wedge^{n-1} A]).$$

As before, in coordinates this map sends a matrix to its minors of various sizes.

As in the symmetric case, the space of complete collineations can be constructed by blowing up $\mathbb{P}(M_n)$ along the subvariety of rank 1 matrices, then the strict transform of the subvariety of matrices of rank at most 2, and so on. As such, it admits a first series S_1, \dots, S_{n-1} of special classes of divisors: the classes of (the strict transforms of) the exceptional divisors E_1, \dots, E_{n-1} of these successive blow-ups. A second natural series L_1, \dots, L_{n-1} of classes of divisors can be obtained by pulling back the hyperplane classes under the projections $\pi_i : CC(V, W) \rightarrow \mathbb{P}(\wedge^i V^* \otimes \wedge^i W)$.

The analogue of Proposition 3.4 holds:

Proposition 6.5. L_1, \dots, L_{n-1} form a basis of $\text{Pic}(CC(V, W))$, in which the S_i 's are given by the formulas

$$S_i = -L_{i-1} + 2L_i - L_{i+1},$$

with the convention that $L_0 = L_n := 0$.

Proof. Follows from [20, Proposition 3.6, Theorem 3.13]. ■

Proposition 6.6. *The numbers ϕ_A and δ_A can be computed as intersection products of the variety of complete collineations:*

$$\begin{aligned}\phi_A(n, d) &= \int_{CC_n} L_1^{n^2-d} L_{n-1}^{d-1}, \\ \delta_A(m, n, r) &= \int_{CC_n} S_r L_1^{n^2-m-1} L_{n-1}^{m-1} = \int_{E_r} L_1^{n^2-m-1} L_{n-1}^{m-1}.\end{aligned}$$

This implies the analogue of (3.3):

$$\phi_A(n, d) = \frac{1}{n} \sum_{r=1}^{n-1} r \delta_A(d, n, n-r). \quad (6.1)$$

Definition 6.7. We define *type A Lascoux coefficients* $d_{I,J}$ as follows. For $X = (x_1, \dots, x_k)$ and $Y = (y_1, \dots, y_l)$ two sets of indeterminates, we denote by $X + Y$ the set of indeterminates $x_i + y_j$, $1 \leq i \leq k$, $1 \leq j \leq l$. Then the $d_{I,J}$'s are defined by the formal identity

$$s_{(d)}(X + Y) = \sum_{\substack{\#I=k, \#J=l \\ |\lambda(I)| + |\lambda(J)| = d}} d_{I,J} s_{\lambda(I)}(X) s_{\lambda(J)}(Y).$$

Equivalently, for the product of the universal bundles $\mathcal{U}_1 \otimes \mathcal{U}_2$ over a product of Grassmannians $G(k, m) \times G(l, n)$,

$$\text{Seg}_d(\mathcal{U}_1 \otimes \mathcal{U}_2) = \sum_{\substack{\#I=k, \#J=l \\ |\lambda(I)| + |\lambda(J)| = d}} d_{I,J} \sigma_{\lambda(I)}^1 \sigma_{\lambda(J)}^2.$$

Analogously to Theorem 3.7, we have the following formula for δ_A :

Theorem 6.8.

$$\delta_A(m, n, r) = \sum_{\substack{I, J \subset [n] \\ \#I = \#J = n-r \\ \sum I + \sum J = m-n+r}} d_{I,J} d_{[n] \setminus I, [n] \setminus J}.$$

6.C. Induction relations and polynomiality

We denote by $D(t)$ the infinite matrix with entries $D(t)_{ij} = \binom{t+i+j}{i}$. This matrix gives us a formula for $d_{I,J}$ [17, Proposition 2.8].

Proposition 6.9. *Let $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_s\}$ be two sets of nonnegative integers with $r \leq s$. Then*

$$d_{I,J} = \begin{cases} \det D(s-r)_{I, \{j_{s-r+1}-(s-r), \dots, j_s-(s-r)\}} & \text{if } j_i = i-1 \text{ for all } 1 \leq i \leq s-r, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\#I = \#J$ then $d_{I,J} = \det D(0)_{I,J}$.

Lemma 6.10. (1) Let $I = \{i_1, \dots, i_s\}$ and $J = \{j_1, \dots, j_s\}$ with $i_1, j_1 > 1$. Write $I_0 = \{0\} \cup I$ and $J_0 = \{0\} \cup J$. Then

$$d_{I,J} = (s+1)d_{I_0,J_0} - \sum_{p=1}^s d_{(I_0 \setminus \{i_p\}) \cup \{i_p-1\}, J_0} - \sum_{q=1}^s d_{I_0, (J_0 \setminus \{j_q\}) \cup \{j_q-1\}}.$$

(Here, if $(I_0 \setminus \{i_p\}) \cup \{i_p-1\}$ is a multiset, then $d_{(I_0 \setminus \{i_p\}) \cup \{i_p-1\}, J_0} = 0$.)

(2) For $i_1 = 0$ or $j_1 = 0$ we have

$$d_{\{i_1, \dots, i_s\}, \{j_1, j_2, \dots, j_s\}} = \sum_{\substack{i_\ell \leq i'_\ell < i_{\ell+1} \\ j_\ell \leq j'_\ell < j_{\ell+1}}} d_{\{i'_1, \dots, i'_{s-1}\}, \{j'_1, \dots, j'_{s-1}\}}.$$

Proof. (1) We expand the determinant $\det D(0)_{I_0, J_0}$ along each row, and sum up:

$$\begin{aligned} (s+1)d_{I_0, J_0} &= \sum_{p,q=0}^s (-1)^{p+q} \binom{i_p + j_q}{i_p} d_{I_0 \setminus \{i_p\}, J_0 \setminus \{j_q\}} \\ &= d_{I,J} + \sum_{p=1}^s (-1)^p d_{I_0 \setminus \{i_p\}, J} + \sum_{q=1}^s (-1)^q d_{I, J_0 \setminus \{j_q\}} \\ &\quad + \sum_{p,q=1}^s (-1)^{p+q} \left(\binom{i_p + j_q - 1}{i_p} + \binom{i_p + j_q - 1}{i_p - 1} \right) d_{I_0 \setminus \{i_p\}, J_0 \setminus \{j_q\}} \\ &= d_{I,J} + \sum_{p=1}^s \sum_{q=0}^s (-1)^{p+q} \binom{i_p + j_q - 1}{i_p - 1} d_{I_0 \setminus \{i_p\}, J_0 \setminus \{j_q\}} \\ &\quad + \sum_{q=1}^s \sum_{p=0}^s (-1)^{p+q} \binom{i_p + j_q - 1}{i_p} d_{I_0 \setminus \{i_p\}, J_0 \setminus \{j_q\}} \\ &= d_{I,J} + \sum_{p=1}^s d_{(I_0 \setminus \{i_p\}) \cup \{i_p-1\}, J_0} + \sum_{q=1}^s d_{I_0, (J_0 \setminus \{j_q\}) \cup \{j_q-1\}}. \end{aligned}$$

(2) The proof of the second formula is similar to the proof of (4.3) in Lemma 4.5. We only consider the case $i_1 = 0$ and in $s_{(d)}(\{x_i + y_j \mid 1 \leq i, j \leq s\})$ we substitute $x_s = 0$. This yields

$$d_{\{i_1, \dots, i_s\}, \{j_1, \dots, j_s\}} = \sum_{j_{\ell-1} < j'_\ell \leq j_\ell} d_{\{i_{2-1}, \dots, i_{s-1}\}, \{j'_1, \dots, j'_s\}}.$$

Then by Proposition 6.9 all summands with $j'_1 > 0$ are zero. This allows us to substitute $y_s = 0$ in $s_{(d)}(\{x_i + y_j \mid 1 \leq i \leq s-1, 1 \leq j \leq s\})$ and deduce the lemma analogously to formula (4.3). \blacksquare

Theorem 6.11. Let $I = \{i_1, \dots, i_r\}$ and $J = \{j_1, \dots, j_r\}$ be two sets of strictly increasing nonnegative integers. The function defined for $n \geq 0$ by

$$LP_{I,J}^A(n) := \begin{cases} d_{[n] \setminus I, [n] \setminus J} & \text{if } I, J \subset [n], \\ 0 & \text{otherwise,} \end{cases}$$

is a polynomial in n .

Proof. From Lemma 6.10 it follows that

$$LP_{I',J}^A(n) = (n-r+1)LP_{I'\setminus\{0\},J\setminus\{0\}}^A(n) \\ - \sum_{\ell: i_{\ell+1} > i_\ell + 1} LP_{(I'\setminus\{0, i_\ell\}) \sqcup \{i_\ell + 1\}, J \setminus \{0\}}^A(n) - \sum_{\ell: j_{\ell+1} > j_\ell + 1} LP_{I' \setminus \{0\}, (J \setminus \{0, j_\ell\}) \sqcup \{j_\ell + 1\}}^A(n)$$

if $i_0 = j_0 = 0$, and otherwise

$$LP_{I',J}^A(n) = \sum_{I'',J''} LP_{I'',J''}^A(n-1),$$

where the sum is over all pairs $(I'', J'') = (\{i_1 - \epsilon_1, \dots, i_r - \epsilon_r\}, \{j_1 - \mu_1, \dots, j_r - \mu_r\})$ with $\epsilon_\ell, \mu_\ell \in \{0, 1\}$. As in the first proof of Theorem 4.3, it follows by induction that $LP_{I',J}^A$ is a polynomial. ■

Theorem 6.12. *For any fixed m, s , the function $\delta_A(m, n, n-s)$ is a polynomial in n .*

Proof. Follows from Theorems 6.8 and 6.11. ■

Theorem 6.13. *For any fixed d , the function $\phi_A(n, d)$ is a polynomial for $n > 0$.*

Proof. Follows from (6.1) and Theorem 6.12. ■

6.D. Proof of the NRS conjecture for type A

We start with the following analogue of Lemma 5.13.

Lemma 6.14. *Let K, L be sets of r nonnegative integers with $\sum K + \sum L \leq m-r$. Then*

$$\sum_{I \geq K} d_{I,L} (-1)^{\sum I - \sum K} s_{I,K} \binom{m-1}{m-r-\sum I - \sum L} = d_{K,L}$$

if $\sum K + \sum L = m-r$, while this sum vanishes if $\sum K + \sum L < m-r$.

Proof. Using Lemma 2.4 we compute $s_{(m-r^2)}(X+Y)$ to be

$$\sum_{k=r^2}^m \binom{m-1}{m-k} s_{(k-r^2)}(X+Y-1) \\ = \sum_{k=r^2}^m \binom{m-1}{m-k} \sum_{\sum I + \sum L = k-r} d_{I,L} s_{\lambda(I)}(X-1) s_{\lambda(L)}(Y) \\ = \sum_{k=r^2}^m \binom{m-1}{m-k} \sum_{\sum I + \sum L = k-r} d_{I,L} \sum_{K \leq I} (-1)^{\sum I - \sum K} s_{I,K} s_{\lambda(K)}(X) s_{\lambda(L)}(Y).$$

Comparing this expansion with that of Definition 6.7 yields the claim. ■

Definition 6.15. For two partitions $\lambda = \lambda(I)$, $\mu = \lambda(J)$ of length r , we define the polynomial

$$a_{I,J}(n) := s_{(r)^r + \lambda, \tilde{\mu}}(\underbrace{1, \dots, 1}_{n \text{ times}}).$$

Lemma 6.16. *Let I, J be two sets of r nonnegative integers. Then*

$$\begin{aligned} a_{I,J}(n) &= \sum_{L \leq I} s_{I,L} d_{[n] \setminus L, [n] \setminus J}, \\ d_{[n] \setminus I, [n] \setminus J} &= \sum_{L \leq I} (-1)^{\sum I - \sum L} s_{I,L} a_{L,J}(n). \end{aligned}$$

Proof. These two formulas being equivalent, we will prove the first one. We use the projection formula given in [26, Lemma 3.1], [13, Proposition 1, p. 51], where one considers two vector bundles E, F of respective ranks n, m over some variety X . The Grassmann bundles $G_s E$ and $G^s F$ (parametrizing rank s subspaces and rank s quotients respectively) over X admit tautological and quotient bundles S_E, Q_E and S_F, Q_F , where S_E and Q_F have rank s . Let $\tau : G := G_s E \times_X G^s F \rightarrow X$ denote the total projection. The variety G is endowed with the vector bundle $H := \tau^* \text{Hom}(F, E) / \text{Hom}(Q_F, S_E)$. In our situation we will suppose that $m = n$, and let $r = n - s$, which is the rank of both S_F and Q_E . Then the projection formula asserts that for any two partitions λ, μ of length at most r ,

$$s_{(r)^r + \lambda, \tilde{\mu}}(E - F) = \tau_*(s_\lambda(Q_E) s_\mu(S_F) c_{\text{top}}(H)).$$

(In the original formula $s_\mu(S_F)$ is replaced by $s_{\tilde{\mu}}(-S_F)$, but they are equal.) We can replace $c_{\text{top}}(H)$ by $c(H)$ and keep only the term of the correct degree, which yields

$$c(\text{Hom}(F, E)) \tau_*(s_\lambda(Q_E) s_\mu(S_F) s(\text{Hom}(Q_F, S_E)))_{|\text{deg}=r^2+|\lambda|+|\mu|}.$$

Now we specialize to the case where $E = E_0 \otimes L$ and $F = F_0$, where E_0, F_0 are trivial vector bundles of rank n , and L is a line bundle on X with $c_1(L) = \delta$. In this case

$$c(\text{Hom}(F, E)) = (1 + \delta)^{s^2}.$$

The Grassmann bundles $G_s E$ and $G^s F$ are then the trivial bundles $G_s E_0 \times X$ and $G^s F_0 \times X$ respectively, while the tautological and quotient bundles are $S_E = S_{E_0} \otimes L$, $Q_E = Q_{E_0} \otimes L$, $S_F = S_{F_0}$ and $Q_F = Q_{F_0}$, where we omit the obvious pull-backs. In this situation,

$$s(\text{Hom}(Q_F, S_E)) = \sum_{\ell \geq 0} (1 + \delta)^{-s^2 - \ell} s_\ell(\text{Hom}(Q_{F_0}, S_{E_0})).$$

Let $\lambda = \lambda(I)$ and $\mu = \lambda(J)$. Using Definition 6.7, Definition 2.3 and the duality properties of Schubert classes, we deduce that

$$s_{(r)^r + \lambda, \tilde{\mu}}(E - F) = a_{I,J}(n) \delta^{r^2 + |\lambda| + |\mu|}$$

can be computed by picking the term of the correct degree in

$$\sum_{\ell \geq 0} (1 + \delta)^{n^2 - s^2 - \ell} \sum_L \delta^{\sum I - \sum L} s_{I,L} d_{[n]/J, [n]/L},$$

where the size of L is constrained by the relation $\ell + \sum L = n(n-1) - s(s-1) - \sum J$. This implies that $n^2 - s^2 - \ell + \sum I - \sum L = r + \sum I + \sum J = r^2 + |\lambda| + |\mu|$. So the term of the correct degree is actually the term of maximal degree in δ , and the claim follows. \blacksquare

We are now ready to prove the NRS conjecture in type A .

Theorem 6.17.

$$\delta_A(m, n, n-r) = \sum_{\substack{\#I=\#L=r, \\ \sum I + \sum L \leq m-r}} d_{I,L} (-1)^{m-r-\sum I - \sum L} \binom{m-1}{m-r-\sum I - \sum L} a_{I,L}(n).$$

Proof. Using Theorem 6.8, Lemma 6.14 and Lemma 6.16, we get

$$\begin{aligned} \delta_A(m, n, n-r) &= \sum_{\substack{K,L \\ \sum K + \sum L = m-r}} d_{K,L} d_{[n]/K, [n]/L} \\ &= \sum_{\substack{K,L \\ \sum K + \sum L \leq m-r}} (-1)^{m-r-\sum K - \sum L} \sum_{I \geq K} \sum_L d_{I,L} \\ &\quad \times (-1)^{\sum I - \sum K} s_{I,K} \binom{m-1}{m-r-\sum I - \sum L} d_{[n]/K, [n]/L} \\ &= \sum_{\substack{I,L \\ \sum I + \sum L \leq m-r}} (-1)^{m-r-\sum I - \sum L} d_{I,L} \\ &\quad \times \binom{m-1}{m-r-\sum I - \sum L} \sum_{K \leq I} (-1)^{\sum I - \sum K} s_{I,K} d_{[n]/K, [n]/L} \\ &= \sum_{\substack{I,L \\ \sum I + \sum L \leq m-r}} (-1)^{m-r-\sum I - \sum L} d_{I,L} \binom{m-1}{m-r-\sum I - \sum L} a_{I,L}(n). \quad \blacksquare \end{aligned}$$

7. Skew-symmetric matrices

7.A. Codegrees of smooth skew-symmetric determinantal loci

Let A_n denote the space of skew-symmetric complex matrices of size n , and $AD^{n-r, n} \subset \mathbb{P}(A_n)$ the locus of matrices of rank at most $n-r$, where $n-r$ is always supposed to be even. Denote by $AD_m^{n-r, n}$ its intersection with a general m -dimensional projective space. Its dimension is $d = m - \binom{r}{2}$ when this is nonnegative, otherwise it is empty. The analogs of Pataki's inequalities are given by

Proposition 7.1. *The dual variety of $AD_m^{n-r,n}$ is a hypersurface if and only if*

$$\binom{r}{2} \leq m \leq \binom{n}{2} - \binom{n-r}{2}.$$

As in the previous case, the degree of this dual variety can be computed by classical means when $AD_m^{n-r,n}$ is smooth, which is equivalent to $\binom{r}{2} \leq m \leq \binom{r}{2} + 2r$. The class formula gives, in terms of topological Euler characteristics,

$$\deg(AD_m^{n-r,n})^* = (-1)^d (\chi(AD_m^{n-r,n}) - 2\chi(AD_{m-1}^{n-r,n}) + \chi(AD_{m-2}^{n-r,n})).$$

The Euler characteristics of smooth skew-symmetric degeneracy loci have also been computed by Pragacz [26]. For E a vector bundle of rank e over a variety X , and $\varphi : E^* \rightarrow E$ a skew-symmetric morphism, the formula given in [13, p. 64] is

$$\chi(D_s(\varphi)) = \int_X P_s(E)c(X),$$

where $c(X)$ denotes the total Chern class of X , while $P_s(E)$ is a universal polynomial in the Chern classes of E . Explicitly,

$$P_s(E) = \sum_{\ell(\lambda) \leq n-s} (-1)^{|\lambda|} [\lambda + \rho(n-s-1)] P_{\lambda + \rho(n-s-1)}(E),$$

where the coefficients $[\lambda + \rho(n-s-1)]$ are those appearing in the Segre class of the skew-symmetric square of a vector bundle of rank $n-s$. These coefficients were denoted α_I in [17], which we will rather follow, where I is a set of $r = n-s$ nonnegative integers.

Let us apply this formula to $AD_m^{n-r,n}$, which we consider formally as the degeneracy locus $D_s(\varphi)$ of the tautological skew-symmetric morphism $\phi : F = \mathcal{O}(-\frac{1}{2})^{\oplus n} \rightarrow \mathcal{O}(\frac{1}{2})^{\oplus n}$ over $X = \mathbb{P}^m$. Since $c(\mathbb{P}^m) - 2hc(\mathbb{P}^{m-1}) + h^2c(\mathbb{P}^{m-2}) = (1+h)^{m-1}$, where h denotes the hyperplane class, we get the formula

$$\deg(AD_m^{n-r,n})^* = \sum_I \binom{m-1}{m-\sum I} \alpha_I P_I(\underbrace{1, \dots, 1}_{n \text{ times}}),$$

where the sum is over all sets I of r nonnegative integers. Once again the dependence on n for r fixed is only in the last term, more precisely in the number of 1's on which the Schur P -functions are evaluated. We have already seen that this dependence is well known to be polynomial in n .

A priori this formula is only valid in the range $\binom{r}{2} \leq m \leq \binom{r}{2} + 2r$, when $AD_m^{n-r,n}$ is smooth. That it should be true in general would be an analogue of the NRS conjecture in type D . We will prove below that this statement is correct. Our notations for the dual degrees will be as follows:

Definition 7.2. Define $\delta_D(m, n, r)$ to be the degree of the variety $(AD_m^{2r, 2n})^*$ if it is a hypersurface, and zero otherwise. Here $AD_m^{2r, 2n}$ is the variety of rank at most $2r$ skew-symmetric $2n \times 2n$ matrices, intersected with a general (projective) m -dimensional subspace. Equivalently, if we let $Z_r \subset \mathbb{P}(\wedge^2 V^*) \times \mathbb{P}(\wedge^2 V)$ be the variety of pairs (X, Y)

of matrices, up to scalars, with $X \cdot Y = 0$, $\text{rk } X \leq 2r$, $\text{rk } Y \leq n - 2r$, then the multidegree of Z_r is equal to

$$[Z_r] = \sum_m \delta_D(m, n, r) H_1^{\binom{n}{2}-m} H_{n-1}^m,$$

where H_1 and H_{n-1} are the pull-backs of the hyperplane classes from $\mathbb{P}(\wedge^2 V^*)$ and $\mathbb{P}(\wedge^2 V)$.

7.B. Complete skew-symmetric forms

A well-known feature of skew-symmetric forms is that the cases of odd and even sizes are quite different. In particular, the following definition only makes sense in the even case.

Definition 7.3. The number $\phi_D(n, d)$ is the degree of the variety \mathcal{L}^{-1} , where $\mathcal{L} \subseteq \mathbb{P}(A_{2n})$ is a general linear subspace of dimension $d - 1$.

In this section, we will only be working with skew-symmetric matrices of even size $2n \times 2n$. The relevant space to deal with is then the *space of complete skew-forms*. Just as with complete quadrics, there are many ways of constructing this space. Here we give just two, referring the reader to the literature [2, 21, 41] for other equivalent definitions.

Definition 7.4. Let V be a $2n$ -dimensional vector space. The *space of complete skew-forms*, $CS(V)$, is defined as the closure of $\phi(\mathbb{P}(\wedge^2 V)^\circ)$, where

$$\phi : \mathbb{P}(\wedge^2 V)^\circ \rightarrow \mathbb{P}(\wedge^2 V) \times \mathbb{P}(\wedge^4 V) \times \cdots \times \mathbb{P}(\wedge^{2n-2} V)$$

is given by

$$[A] \mapsto ([A], [\wedge^2 A], \dots, [\wedge^{n-1} A]).$$

We note that here $\wedge^i A$ is viewed as an element of $\wedge^{2i} V$; see also [2, Section 3]. In coordinates, the map $\wedge^2 V \rightarrow \wedge^{2i} V$ sends the entries of a skew-symmetric matrix to the Pfaffians of its principal $2i \times 2i$ submatrices.

For simplicity we will also use the notation $CS_{2n} = CS(\mathbb{C}^{2n})$.

As in the symmetric case, the space of complete skew-forms can be constructed by blowing up $\mathbb{P}(A_{2n})$ along the subvariety of rank 2 matrices, then the strict transform of the subvariety of matrices of rank at most 4, and so on. As such, it admits a first series S_1, \dots, S_{n-1} of special classes of divisors: the classes of (the strict transforms of) the exceptional divisors E_1, \dots, E_{n-1} of these successive blow-ups. A second natural series L_1, \dots, L_{n-1} of classes of divisors can be obtained by pulling back the hyperplane classes under the projections $\pi_i : CS(V) \rightarrow \mathbb{P}(\wedge^{2i} V)$.

The analogue of Proposition 3.4 holds:

Proposition 7.5. *The classes L_1, \dots, L_{n-1} form a basis of $\text{Pic}(CS(V))$, in which the S_i 's are given by*

$$S_i = -L_{i-1} + 2L_i - L_{i+1}$$

with $L_0 = L_n := 0$.

Proof. Follows from [21, Proposition 3.6, Theorem 3.9]. ■

As with symmetric matrices, the numbers ϕ_D and δ_D can be expressed as intersection products in the Chow ring of CS_{2n} :

Proposition 7.6.

$$\begin{aligned}\phi_D(n, d) &= \int_{CS_{2n}} L_1^{\binom{2n}{2}-d} L_{n-1}^{d-1}, \\ \delta_D(m, n, r) &= \int_{CS_{2n}} S_r L_1^{\binom{2n}{2}-m-1} L_{n-1}^{m-1} = \int_{E_r} L_1^{\binom{2n}{2}-m-1} L_{n-1}^{m-1}.\end{aligned}$$

Proof. Analogous to the proof of Proposition 3.5. ■

From the two propositions above and the Pataki inequalities, we deduce that

$$\phi_D(n, d) = \frac{1}{n} \sum_{\binom{2n}{2} \leq d} r \delta_D(d, n, n-r), \quad (7.1)$$

an analogue of (3.3).

Definition 7.7. We define type D Lascoux coefficients α_I as follows. For $X = (x_1, \dots, x_k)$ a set of indeterminates, we denote by $\lambda(X)$ the set of indeterminates $x_i + x_j$, $1 \leq i < j \leq k$. Then the α_I 's are defined by the formal identity

$$s_{(d)}(\lambda(X)) = \sum_{\substack{\#I=k \\ |\lambda(I)|=d}} \alpha_I s_{\lambda(I)}(X).$$

Equivalently, for the universal bundle \mathcal{U} over a Grassmannian $G(k, m)$,

$$\text{Seg}_d(\wedge^2 \mathcal{U}) = \sum_{\substack{\#I=k \\ |\lambda(I)|=d}} \alpha_I \sigma_{\lambda(I)}.$$

For more about these coefficients, see [17, Proposition A.16].

Theorem 7.8.

$$\delta_D(m, n, r) = \sum_{\substack{I \subset [2n] \\ \#I=2n-2r \\ \sum I=m}} \alpha_I \alpha_{[2n] \setminus I}.$$

Proof. Analogous to the proof of Theorem 3.7. ■

7.C. Induction relations and polynomiality

We will now prove the polynomiality (or more precisely, quasipolynomiality) of $\alpha_{[k] \setminus I}$. The following recursive relations will be central to our proof:

Lemma 7.9. (1) For $j_1 > 0$ we have

$$\alpha_{\{j_1, \dots, j_s\}} = \begin{cases} \alpha_{\{0, j_1, \dots, j_s\}} & \text{if } s \text{ is even,} \\ 0 & \text{if } s \text{ is odd.} \end{cases} \quad (7.2)$$

(2) For $j_1 = 0$ we have

$$\alpha_{\{j_1, \dots, j_s\}} = \sum_{j_\ell \leq j'_\ell < j_{\ell+1}} \alpha_{\{j'_1, \dots, j'_{s-1}\}}. \quad (7.3)$$

Proof. The first formula is [26, p. 446], [17, (A.16.3)] and [28, pp. 163–66]. The proof of the second formula is analogous to the proof of (4.3) in Lemma 4.5. ■

Theorem 7.10. Let $I = \{i_1, \dots, i_s\}$ be a set of strictly increasing nonnegative integers. For $k \geq 0$ the function

$$LP_I^D(k) := \begin{cases} \alpha_{[k] \setminus I} & \text{if } I \subset [k], \\ 0 & \text{otherwise,} \end{cases}$$

is a quasipolynomial in k with period 2, i.e. for both even k and odd k it is a polynomial.

Proof. We proceed as in the first proof of Theorem 4.3 by induction on $\#I$ and then on $\sum I$ using the relations from Lemma 7.9. The difference is that in the case $i_0 = 0$ we have

$$LP_I^D(n) = \begin{cases} LP_{I \setminus 0}^D(n) & \text{if } n - \#I \text{ is even,} \\ 0 & \text{if } n - \#I \text{ is odd,} \end{cases}$$

which is clearly by induction hypothesis a quasipolynomial in n with period 2. The rest is analogous to the proof of Theorem 4.3. ■

From Theorems 7.8 and 7.10 we deduce the polynomiality of δ_D :

Theorem 7.11. For any fixed m, s , the function $\delta_D(m, n, n - s)$ is a polynomial in n .

Using (7.1), we also get the polynomiality of ϕ_D :

Theorem 7.12. For any fixed d , the function $\phi_D(n, d)$ is a polynomial for $n > 0$.

7.D. Proof of the NRS conjecture in type D

The proof of Theorem 7.16 will be extremely similar to that of the original NRS conjecture.

Lemma 7.13. Let J be a set of r nonnegative integers with $\sum J \leq m$. Then

$$\sum_{\substack{I \supseteq J \\ \sum I \leq m}} \alpha_I \left(-\frac{1}{2} \right)^{\sum I - \sum J} s_{I, J} \binom{m-1}{m - \sum I} = \begin{cases} 0 & \text{if } \sum J < m, \\ \alpha_J & \text{if } \sum J = m. \end{cases}$$

Proof. Given a set $X = (x_1, \dots, x_r)$ of r variables, we denote by $\lambda(X)$ the set of variables $(x_i + x_j, i < j)$. Using Lemma 2.4 we compute $s_{(m-\binom{r}{2})}(\lambda(X))$ to be

$$\begin{aligned} \sum_{k=\binom{r}{2}}^m \binom{m-1}{m-k} s_{(k-\binom{r}{2})}(\lambda(X) - 1) &= \sum_{k=\binom{r}{2}}^m \binom{m-1}{m-k} s_{(k-\binom{r}{2})}(\lambda(X - 1/2)) \\ &= \sum_{k=\binom{r}{2}}^m \binom{m-1}{m-k} \sum_{\Sigma I=k} \alpha_I s_{\lambda(I)}(X - 1/2) \\ &= \sum_{k=\binom{r}{2}}^m \binom{m-1}{m-k} \sum_{\Sigma I=k} \alpha_I \sum_{J \leq I} s_{I,J}(-1/2)^{\Sigma I - \Sigma J} s_{\lambda(J)}(X). \end{aligned}$$

Comparing this expression with the expansion in Definition 6.7 yields the claim. \blacksquare

Definition 7.14. For I a set of nonnegative integers, we define

$$d_I(n) := P_I(\underbrace{1/2, \dots, 1/2}_{n \text{ times}}).$$

Analogously to Definition 5.1, we have an equivalent formula:

$$d_I(n) = \left(\frac{1}{2}\right)^{\Sigma I} P_I(\underbrace{1, \dots, 1}_{n \text{ times}}).$$

Like the Schur P -polynomials themselves [19, Section III.8], these polynomials may be computed recursively. The following lemma describes the relation between $d_I(n)$ and the Lascoux coefficients α_J .

Lemma 7.15. *Let I be a set of r nonnegative integers. Then*

$$\begin{aligned} d_I(n) &= \sum_{J \leq I} \left(\frac{1}{2}\right)^{\Sigma I - \Sigma J} s_{I,J} \alpha_{[n] \setminus J}, \\ \alpha_{[n] \setminus I} &= \sum_{J \leq I} \left(-\frac{1}{2}\right)^{\Sigma I - \Sigma J} s_{I,J} d_J(n). \end{aligned}$$

Proof. The two formulas are equivalent; we shall prove the first one. For a vector bundle \mathcal{E} of rank n over some base X , we consider the relative Grassmannian $G^r(\mathcal{E})$ of rank r quotients of \mathcal{E} , with its projection π to X . We denote by \mathcal{S} and \mathcal{Q} the relative tautological subbundle and quotient bundle of $\pi^* \mathcal{E}$, of respective ranks $s = n - r$ and r . Then for $I = (i_1 < \dots < i_r)$,

$$P_I(\mathcal{E}) = \pi_*(c_{\text{top}}(\mathcal{S} \otimes \mathcal{Q}) P_I(\mathcal{Q}))$$

(cf. [13, Example 2, p. 50]). Moreover, the result of [13, (4.5)] and [27, Proposition 2.2] can be written as

$$P_I(\mathcal{Q}) = c_{\text{top}}(\bigwedge^2 \mathcal{Q}) s_{\lambda(I)}(\mathcal{Q}).$$

Since $\pi^*\mathcal{E}$ is an extension of \mathcal{Q} by \mathcal{S} , the bundle $\pi^*(\wedge^2 \mathcal{E})$ admits a filtration whose successive quotients are $\wedge^2 \mathcal{Q}$, $\mathcal{S} \otimes \mathcal{Q}$ and $\wedge^2 \mathcal{S}$. Hence the identity

$$c(\mathcal{S} \otimes \mathcal{Q})c(\wedge^2 \mathcal{Q}) = s(\wedge^2 \mathcal{S}^*)\pi^*c(\wedge^2 \mathcal{E}).$$

Equation (5.1) can thus be rewritten as

$$P_I(\mathcal{E}) = c(\wedge^2 \mathcal{E})\pi_*(s(\wedge^2 \mathcal{S}^*)s_{\lambda(I)}(\mathcal{Q}))|_{\deg=\sum I},$$

where the last symbols mean we only keep the component of degree $\sum I$.

Now suppose $\mathcal{E} = \mathcal{E}_0 \otimes L$ for a line bundle L and a trivial vector bundle \mathcal{E}_0 . Then

$$c(\wedge^2 \mathcal{E}) = (1 + 2\delta) \binom{s}{2}.$$

Moreover, $G^r(\mathcal{E})$ is a trivial bundle over X , while $\mathcal{S} = \mathcal{S}_0 \otimes L$ and $\mathcal{Q} = \mathcal{Q}_0 \otimes L$ are obtained by pulling back the tautological and quotient bundles $\mathcal{S}_0, \mathcal{Q}_0$ over a fixed Grassmannian $G^r(\mathbf{C}^n)$ (we omit the pull-backs for simplicity). By Definition 2.3 we have

$$s_{\lambda(I)}(\mathcal{Q}) = \sum_{J \leq I} s_{I,J} s_{\lambda(J)}(\mathcal{Q}_0) \delta^{\sum I - \sum J},$$

where $\delta = c_1(L)$. Moreover, the Segre classes of $\wedge^2 \mathcal{S}_0^*$ and $\wedge^2 \mathcal{S}^*$ are related by the formula

$$s(\wedge^2 \mathcal{S}^*) = \sum_{\ell \geq 0} (1 + 2\delta)^{-\binom{r}{2} - \ell} s_\ell(\wedge^2 \mathcal{S}_0^*).$$

Plugging these two formulas into the previous one, we get

$$P_I(\mathcal{E}) = \sum_{J \leq I} \sum_L (1 + 2\delta)^{\binom{n}{2} - \binom{r}{2} - |\lambda(L)|} \delta^{\sum I - \sum J} s_{I,J} \alpha_L \pi_*(s_{\lambda(L)}(\mathcal{S}_0^*) s_{\lambda(J)}(\mathcal{Q}_0))|_{\deg=\sum I}.$$

Now recall that the Schur classes $s_\alpha(\mathcal{S}_0^*)$ and $s_\beta(\mathcal{Q}_0)$, for partitions $\alpha \subset (r^s)$ and $\beta \subset (s^r)$, which are nonzero, give dual bases of Schubert cycles on the Grassmannian $G^r(\mathbf{C}^n)$. This can be expressed as

$$\pi_*(s_{\lambda(L)}(\mathcal{S}_0^*) s_{\lambda(J)}(\mathcal{Q}_0)) = \delta_{L, [n]/J},$$

where $\delta_{S,T}$ is the Kronecker delta. Note that $L = [n]/J$ implies that $|\lambda(L)| + |\lambda(J)| = rs$. We thus get the formula

$$P_I(\mathcal{E}) = \left(\sum_{J \leq I} (1 + 2\delta)^{\sum J} \delta^{\sum I - \sum J} s_{I,J} \alpha_{[n]/J} \right) |_{\deg=\sum I}.$$

But since the degree of the polynomial in brackets is exactly $\sum I$, we just need to keep its top degree component, that is,

$$P_I(\mathcal{E}) = \sum_{J \leq I} 2^{\sum J} s_{I,J} \alpha_{[n]/J} \delta^{\sum I}.$$

We conclude by applying formally this formula to the bundle $\mathcal{E} = \mathcal{O}(1)^{\oplus n}$ over the projective space. ■

Theorem 7.16. *Let m, n, r be positive integers. Then*

$$\delta_D(m, n, n - r) = \sum_{\Sigma I \leq m} (-1)^{m - \Sigma I} \alpha_I d_I(n) \binom{m - 1}{m - \Sigma I},$$

where the sum is over sets of nonnegative integers of cardinality r .

Proof. We replace $d_I(n)$ by the expression from Lemma 5.4, change the order of summation and use Lemma 7.13 in the last step:

$$\begin{aligned} & \sum_{\Sigma I \leq m} (-1)^{m - \Sigma I} \alpha_I d_I(n) \binom{m - 1}{m - \Sigma I} \\ &= \sum_{\Sigma I \leq m} \sum_{J \leq I} s_{I, J} \alpha_{[2n] \setminus J} \left(\frac{1}{2}\right)^{\Sigma I - \Sigma J} (-1)^{m - \Sigma I} \alpha_I \binom{m - 1}{m - \Sigma I} \\ &= \sum_{\Sigma J \leq m} (-1)^{m - \Sigma J} \alpha_{[2n] \setminus J} \sum_{\substack{I \geq J \\ \Sigma I \leq m}} s_{I, J} \left(-\frac{1}{2}\right)^{\Sigma I - \Sigma J} \alpha_I \binom{m - 1}{m - \Sigma I} \\ &= \sum_{\Sigma J = m} \alpha_{[2n] \setminus J} \alpha_J = \delta_D(m, n, n - s). \quad \blacksquare \end{aligned}$$

8. Representation theory

In this section we will establish a formula which expresses the ML-degree as a linear combination of dimensions of irreducible representations of SL_n . Our construction is based on the following folklore lemma.

Lemma 8.1. *Let X be a smooth complete N -dimensional algebraic variety, and D_1, D_2 two divisors on X . Then the following identity holds:*

$$\int_X D_1^i D_2^{N-i} = \chi((1 - \mathcal{O}(-D_1))^i (1 - \mathcal{O}(-D_2))^{N-i}),$$

where χ denotes the holomorphic Euler characteristic.

Proof. By the additivity and multiplicativity properties of the Chern character we have

$$\mathrm{ch}(1 - \mathcal{O}(-D_i)) = \sum_{k \geq 1} (-1)^{k+1} \frac{D_i^k}{k!},$$

and therefore

$$\begin{aligned} & \mathrm{ch}((1 - \mathcal{O}(-D_1))^i (1 - \mathcal{O}(-D_2))^{N-i}) \\ &= \left(\sum_{k \geq 1} (-1)^{k+1} \frac{D_1^k}{k!} \right)^i \left(\sum_{k \geq 1} (-1)^{k+1} \frac{D_2^k}{k!} \right)^{N-i} = D_1^i D_2^{N-i}. \end{aligned}$$

Finally, by the Riemann–Roch theorem we get

$$\chi((1 - \mathcal{O}(-D_1))^i (1 - \mathcal{O}(-D_2))^{N-i}) = \int_X D_1^i D_2^{N-i} \text{td}(X) = \int_X D_1^i D_2^{N-i}. \quad \blacksquare$$

We are going to apply Lemma 8.1 to the computation of the ML-degree

$$\phi(n, d) = \int_{CQ_n} L_1^{N+1-d} L_{n-1}^{d-1},$$

where $N = \binom{n+1}{2} - 1$ denotes the dimension of the variety of complete quadrics, CQ_n . We will denote by Λ the character lattice of SL_n , and for a fundamental $\lambda \in \Lambda$, we will denote by V_λ the corresponding irreducible representation of SL_n . We will also denote by $\alpha_1, \dots, \alpha_{n-1}$ and $\omega_1, \dots, \omega_{n-1}$ the simple roots and fundamental weights of SL_{n-1} , respectively.

For a character $\lambda \in \Lambda$, let us define an SL_n -representation W_λ by

$$W_\lambda := \bigoplus_{\nu \in \Lambda} V_{2\nu}^*,$$

where the sum is taken over dominant weights of the form $\nu = \lambda - \sum_{i=1}^{n-1} k_i \alpha_i$ with $k_i \in \mathbb{Z}_{\geq 0}$. In particular, $W_\lambda = 0$ if λ cannot be represented as a sum $\nu + \sum_{i=1}^{n-1} k_i \alpha_i$ with ν a dominant weight and $k_i \in \mathbb{Z}_{\geq 0}$. For $\lambda = (i-1)\omega_1 + (j-1)\omega_{n-1} - \sum_{l=1}^{n-1} \omega_l$, we will denote the representation W_λ by $W_{i,j}^n$.

Theorem 8.2. *With the notation as above, the following identity holds:*

$$\phi(n, d) = 1 + \sum_{\substack{0 \leq i \leq N-d+1 \\ 0 \leq j \leq d-1 \\ i+j > 0}} (-1)^{i+j+N} \binom{N+1-d}{i} \binom{d-1}{j} \dim(W_{i,j}^n).$$

Proof. By Lemma 8.1 we have

$$\begin{aligned} \phi(n, d) &= \chi((1 - \mathcal{L}_1^{-1})^{N+1-d} (1 - \mathcal{L}_{n-1}^{-1})^{d-1}) \\ &= \sum_{\substack{0 \leq i \leq N+1-d \\ 0 \leq j \leq d-1}} (-1)^{i+j} \binom{N+1-d}{i} \binom{d-1}{j} \chi(\mathcal{L}_1^{-i} \otimes \mathcal{L}_{n-1}^{-j}). \end{aligned}$$

Since both \mathcal{L}_1 and \mathcal{L}_{n-1} are globally generated, and both are pull-backs of ample line bundles by birational morphisms, their Iitaka dimensions satisfy $\kappa(\mathcal{L}_1) = \kappa(\mathcal{L}_{n-1}) = N$. By [3, Theorem 2.2], the cohomology of their negative powers must therefore vanish in degree lower than the dimension, so that for $i \geq 0, j \geq 0$ and $i+j > 0$,

$$\begin{aligned} \chi(\mathcal{L}_1^{-i} \otimes \mathcal{L}_{n-1}^{-j}) &= (-1)^N h^N(CQ_n, \mathcal{L}_1^{-i} \otimes \mathcal{L}_{n-1}^{-j}) \\ &= (-1)^N h^0(CQ_n, K_{CQ_n} \otimes \mathcal{L}_1^i \otimes \mathcal{L}_{n-1}^j). \end{aligned}$$

The canonical divisor K_{CQ_n} of the space of complete quadrics is given by [18, Corollary 3]:

$$K_{CQ_n} = -L_1 - L_{n-1} - \sum_{l=1}^{n-1} L_l.$$

In particular, CQ_n is Fano [20] and $\chi(CQ_n, \mathcal{O}) = 1$. Finally, the spaces of global sections of line bundles on complete quadrics have been computed, as SL_n -representations, by De Concini and Procesi. It is a direct corollary of [8, Theorem 8.3] that

$$H^0(CQ_n, \mathcal{L}_1^{a_1} \otimes \cdots \otimes \mathcal{L}_{n-1}^{a_{n-1}}) = W_{a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}}.$$

In particular, $H^0(CQ_n, K_{CQ_n} \otimes \mathcal{L}_1^i \otimes \mathcal{L}_{n-1}^j) = W_{ij}^n$, so our result follows. \blacksquare

Example 8.3. Let us compute $\phi(3, 1)$, $\phi(3, 2)$ and $\phi(3, 3)$ with the help of Theorem 8.2. First we notice that

$$W_{i,0}^3 = W_{0,i}^3 = 0 \text{ for } i \leq 5, \quad W_{i,1}^3 = W_{1,i}^3 = 0 \text{ for } i \leq 3.$$

Therefore we get

$$\phi(3, 1) = 1 + \sum_{1 \leq i \leq 5} (-1)^{i+1} \binom{5}{i} \dim(W_{i,0}^3) = 1,$$

$$\phi(3, 2) = 1 + \sum_{\substack{0 \leq i \leq 4 \\ 0 \leq j \leq 1 \\ i+j > 0}} (-1)^{i+j+1} \binom{4}{i} \dim(W_{i,j}^3) = 1 + \dim(W_{4,1}^3) = 1 + \dim(V_0) = 2,$$

$$\phi(3, 3) = 1 + \sum_{\substack{0 \leq i \leq 3 \\ 0 \leq j \leq 2 \\ i+j > 0}} (-1)^{i+j+1} \binom{3}{i} \binom{2}{j} \dim(W_{i,j}^3) = 1 - 3 \cdot \dim(W_{2,2}^3) + \dim(W_{3,2}^3),$$

where $W_{2,2}^3 = V_0$ and $W_{3,2}^3 = V_{2\omega_1}^*$. By Weyl's dimension formula,

$$\dim(V_{i\omega_1 + j\omega_2}) = \frac{(i+1)(j+1)(i+j+2)}{2},$$

we get $\phi(3, 3) = 1 - 3 \cdot 1 + 6 = 4$.

Remark 8.4. Our representation-theoretic approach gives a closed formula for the ML-degree. However, already for $n = 4$, the computation analogous to Example 8.3 is quite involved. One reason why this computation is more complicated than other formulas is that we obtain the answer to our intersection problem as a virtual representation, not only its dimension. One could say that this gives more information than we ask for.

9. Future directions and conjectures

9.A. Dual degrees of defective determinantal loci

For symmetric matrices, Theorem 4.1 asserts that when $SD_m^{n-s,n}$ is not dual defective, that is, when Pataki's inequalities (3.2) are satisfied, its codegree depends polynomially on n when s and m are fixed. By the fundamental duality relation (Remark 2.10), this also means that the codegree of $SD_{\binom{n+1}{2}-m}^{s,n}$ depends polynomially on n for s and m fixed.

What happens in the defective cases? For example, by [15], one has the formula

$$\begin{aligned} \text{codeg}(SD_{\binom{n+1}{2}-1}^{s,n}) &= \text{deg}(SD_{\binom{n+1}{2}-1}^{n-s,n}) \\ &= \prod_{j=1}^s \frac{\binom{n+j-1}{s-j+1}}{\binom{2j-1}{j-1}}, \end{aligned}$$

which is clearly a polynomial in n . Note that in this case the dual defect is $\binom{s+1}{2} - 1$, independently of n . Could it happen that the following holds?

Conjecture 9.1. *For any fixed $m > 0$,*

- (1) *the dual defect of $SD_{\binom{n+1}{2}-m}^{s,n}$ is equal to $\max(0, \binom{s+1}{2} - m)$, independently of n ,*
- (2) *the codegree of $SD_{\binom{n+1}{2}-m}^{s,n}$ depends polynomially on n .*

Similar statements should hold for general and skew-symmetric matrices. For general matrices, the degrees of the determinantal loci have been computed in [15]. In particular, this implies that

$$\begin{aligned} \text{codeg}(D_{n^2-1}^{s,n}) &= \text{deg}(D_{n^2-1}^{n-s,n}) \\ &= \prod_{j=1}^s \frac{\binom{n+j-1}{2j-1}}{\binom{s+j-1}{2j-1}} \end{aligned}$$

is a polynomial in n . In this case the dual defect is $s^2 - 1$, independently of n . Could it happen that the following holds?

Conjecture 9.2. *For any fixed $m > 0$,*

- (1) *the dual defect of $D_{n^2-m}^{s,n}$ is equal to $\max(0, s^2 - m)$, independently of n ,*
- (2) *the codegree of $D_{n^2-m}^{s,n}$ depends polynomially on n .*

Finally, for skew-symmetric matrices, the degrees of the determinantal loci have also been computed in [15]. In particular, this implies that

$$\begin{aligned} \text{codeg}(AD_{\binom{2n}{2}-1}^{2s,2n}) &= \text{deg}(AD_{\binom{2n}{2}-1}^{2n-2s,2n}) \\ &= \frac{1}{2^{2s-1}} \prod_{j=1}^{2s-1} \frac{\binom{2n+j-1}{2s-j}}{\binom{2j-1}{j-1}} \end{aligned}$$

is a polynomial in n . In this case the dual defect is $\binom{2s}{2} - 1$, independently of n . Could it happen that the following holds, and a similar statement in odd dimensions?

Conjecture 9.3. *For any fixed $m > 0$,*

- (1) *the dual defect of $AD_{\binom{2n}{2}-m}^{2s,2n}$ is equal to $\max(0, \binom{2n}{2} - m)$, independently of n ,*
- (2) *the codegree of $AD_{\binom{2n}{2}-m}^{2s,2n}$ depends polynomially on n .*

9.B. Dual degrees of singular varieties

There is a general Plücker formula for the degree of the dual variety X^* of a possibly singular projective variety $X \subset \mathbb{P}^n$ [11, Theorem 1.1]:

$$\deg X^* = (-1)^n (\chi(\text{Eu}_X) - 2\chi(\text{Eu}_{X_1}) + \chi(\text{Eu}_{X_2})),$$

where X_1 (resp. X_2) is a general hyperplane section (resp. codimension 2 linear section) of X . Here Eu_X denotes the *Euler obstruction* of X [12, Examples 4.2.9 and 19.1.7]. Our Theorem 5.2 confirming the NRS conjecture, and their analogues for the other types of determinantal loci, would immediately follow if one could prove that Pragacz's formulas [26] for the topological Euler characteristics of determinantal loci, which a priori are only valid in the smooth case, can also be used in the singular case, and compute the Euler characteristics of the Euler obstructions. This is an approach we plan to investigate in the near future, and which could potentially also apply to the defective cases.

9.C. Polynomiality of intersection products

Theorem 4.2 shows that for fixed d , the product

$$\phi(n, d) = \int_{CQ_n} L_1^{d-1} L_{n-1}^{\binom{n+1}{2}-d}$$

is a polynomial in n . Can this result be generalized to other intersection products on CQ ? More precisely, we have the following question:

Question 9.4. *Let $d_1, \dots, d_k \in \mathbb{N}$ with $\sum d_i = d - 1$. Is*

$$\int_{CQ_n} L_1^{d_1} L_2^{d_2} \dots L_k^{d_k} L_{n-1}^{\binom{n+1}{2}-d}$$

a polynomial in $n > k + 1$?

More generally: by which cohomology classes can one replace $L_1^{d_1} L_2^{d_2} \dots L_k^{d_k}$ for the polynomiality property to hold?

9.D. Representation theory

The version of the NRS conjecture that we proved in type A (Theorem 6.17) expresses the codegree $\delta_A(m, n, n - s)$ as a linear combination of dimensions of Schur modules of $\text{GL}(n, \mathbb{C})$, with highest weights and multiplicities depending only on m and r . In other words, it is obtained as the dimensional evaluation of a fixed character, depending only on m and s . A natural question is: what is really this character? Is there a natural (combinatorial, or geometric) interpretation of the corresponding representations?

The same question can be raised both in types B and D , where the dependence in n of the codegrees $\delta(m, n, n - s)$ and $\delta_D(m, n, n - s)$ only appears through the number of 1's on which a certain combination of Schur Q -functions or Schur P -functions is evaluated. We mentioned in the introduction that these evaluations of Schur Q -functions or Schur P -functions count certain types of shifted tableaux. An alternative interpretation is that

they give the dimensions of certain representations of the queer Lie superalgebra $\mathfrak{q}(n)$ (see for example [5, Theorem 4.11]). So $\delta(m, n, n - s)$ and $\delta_D(m, n, n - s)$ can also be interpreted as dimensions of certain representations of $\mathfrak{q}(n)$, whose characters only depend on m and s . What are these representations? Do they admit natural constructions or interpretations?

9.E. Noncommutative matroids

We believe that our results may also be viewed in the context of noncommutative matroids. Indeed, suppose we restrict to diagonal matrices, instead of symmetric. One may still consider the rational map given by inverting matrices, which gives the classical Cremona transformation. The famous resolution of that graph given by the permutohedral variety is the analogue of the variety of complete quadrics. A representable matroid may be viewed as a subspace of the space of diagonal matrices. Many interesting invariants of this matroid may be read off from the cohomology class of its strict transform in the permutohedral variety. In this analogy, the representable symmetric noncommutative matroid, would be a subspace of symmetric matrices. Its crucial invariants should come from the cohomology class of the strict transform of that subspace to the variety of complete quadrics. For future work, it would be interesting to dare to define noncommutative matroids as special cohomology classes of the variety of complete quadrics.

9.F. Algebraic statistics

In terms of algebraic statistics, the number $\phi(n, d)$ is equal to the maximum likelihood degree of a general *linear concentration model*. A related quantity is the maximum likelihood degree of a general *linear covariance model*. For $\mathcal{L} \subseteq S^2(\mathbb{C}^n)$, the ML-degree of the associated linear concentration model is the number of pairs $(\Sigma, K) \in S^2(\mathbb{C}^n)^2$ satisfying

$$\Sigma \cdot K = \text{Id}_n, \quad K \in \mathcal{L}, \quad \Sigma - S \in \mathcal{L}^\perp,$$

where S is generic. In contrast, the ML-degree of the associated linear covariance model is the number of pairs $(\Sigma, K) \in S^2(\mathbb{C}^n)^2$ for which

$$\Sigma \cdot K = \text{Id}_n, \quad \Sigma \in \mathcal{L}, \quad KSK - K \in \mathcal{L}^\perp.$$

In [37], it was conjectured that for generic \mathcal{L} , these ML-degrees are also polynomial in n . In future work, we plan to apply our geometric methods to prove this conjecture.

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References

- [1] Anderson, T. W.: Estimation of covariance matrices which are linear combinations or whose inverses are linear combinations of given matrices. In: *Essays in Probability and Statistics*, Univ. North Carolina Press, Chapel Hill, NC, 1–24 (1970) Zbl 0265.62023 MR 0277057

- [2] Bertram, A.: An application of a log version of the Kodaira vanishing theorem to embedded projective varieties. arXiv:[alg-geom/9707001](https://arxiv.org/abs/alg-geom/9707001) (1997)
- [3] Brion, M.: Une extension du théorème de Borel–Weil. *Math. Ann.* **286**, 655–660 (1990) Zbl [0718.14011](https://zbmath.org/?q=ser/0718.14011) MR [1045394](https://mathscinet.org/mr/1045394)
- [4] Brown, L. D.: *Fundamentals of Statistical Exponential Families with Applications in Statistical Decision Theory*. Inst. Math. Statist. Lecture Notes Monogr. Ser. 9, Institute of Mathematical Statistics, Hayward, CA (1986) Zbl [0685.62002](https://zbmath.org/?q=ser/0685.62002) MR [882001](https://mathscinet.org/mr/882001)
- [5] Brundan, J.: Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra $q(n)$. *Adv. Math.* **182**, 28–77 (2004) Zbl [1048.17003](https://zbmath.org/?q=ser/1048.17003) MR [2028496](https://mathscinet.org/mr/2028496)
- [6] Chardin, M., Eisenbud, D., Ulrich, B.: Hilbert series of residual intersections. *Compos. Math.* **151**, 1663–1687 (2015) Zbl [1329.13023](https://zbmath.org/?q=ser/1329.13023) MR [3406441](https://mathscinet.org/mr/3406441)
- [7] De Concini, C., Goresky, M., MacPherson, R., Procesi, C.: On the geometry of quadrics and their degenerations. *Comment. Math. Helv.* **63**, 337–413 (1988) Zbl [0693.14023](https://zbmath.org/?q=ser/0693.14023) MR [960767](https://mathscinet.org/mr/960767)
- [8] De Concini, C., Procesi, C.: Complete symmetric varieties. In: *Invariant Theory (Montecatini, 1982)*, Lecture Notes in Math. 996, Springer, Berlin, 1–44 (1983) Zbl [0581.14041](https://zbmath.org/?q=ser/0581.14041) MR [718125](https://mathscinet.org/mr/718125)
- [9] De Concini, C., Procesi, C.: Complete symmetric varieties. II. Intersection theory. In: *Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983)*, Adv. Stud. Pure Math. 6, North-Holland, Amsterdam, 481–513 (1985) Zbl [0596.14041](https://zbmath.org/?q=ser/0596.14041) MR [803344](https://mathscinet.org/mr/803344)
- [10] Drton, M., Sturmfels, B., Sullivant, S.: *Lectures on Algebraic Statistics*. Oberwolfach Sem. 39, Birkhäuser, Basel (2009) Zbl [1166.13001](https://zbmath.org/?q=ser/1166.13001) MR [2723140](https://mathscinet.org/mr/2723140)
- [11] Ernström, L.: A Plücker formula for singular projective varieties. *Comm. Algebra* **25**, 2897–2901 (1997) Zbl [0891.14012](https://zbmath.org/?q=ser/0891.14012) MR [1458736](https://mathscinet.org/mr/1458736)
- [12] Fulton, W.: *Intersection Theory*. *Ergeb. Math. Grenzgeb. (3)* Springer, Berlin (1998) Zbl [0541.14005](https://zbmath.org/?q=ser/0541.14005) MR [732620](https://mathscinet.org/mr/732620)
- [13] Fulton, W., Pragacz, P.: *Schubert Varieties and Degeneracy Loci*. Lecture Notes in Math. 1689, Springer, Berlin (1998) Zbl [0913.14016](https://zbmath.org/?q=ser/0913.14016) MR [1639468](https://mathscinet.org/mr/1639468)
- [14] Graf von Bothmer, H.-C., Ranestad, K.: A general formula for the algebraic degree in semidefinite programming. *Bull. London Math. Soc.* **41**, 193–197 (2009) Zbl [1185.14047](https://zbmath.org/?q=ser/1185.14047) MR [2496496](https://mathscinet.org/mr/2496496)
- [15] Harris, J., Tu, L. W.: On symmetric and skew-symmetric determinantal varieties. *Topology* **23**, 71–84 (1984) Zbl [0534.55010](https://zbmath.org/?q=ser/0534.55010) MR [721453](https://mathscinet.org/mr/721453)
- [16] Laksov, D.: Completed quadrics and linear maps. In: *Algebraic Geometry, Bowdoin, 1985* (Brunswick, ME, 1985), Proc. Sympos. Pure Math. 46, Amer. Math. Soc., Providence, RI, 371–387 (1987) Zbl [0679.14030](https://zbmath.org/?q=ser/0679.14030) MR [927988](https://mathscinet.org/mr/927988)
- [17] Laksov, D., Lascoux, A., Thorup, A.: On Giambelli’s theorem on complete correlations. *Acta Math.* **162**, 143–199 (1989) Zbl [0695.14023](https://zbmath.org/?q=ser/0695.14023) MR [989395](https://mathscinet.org/mr/989395)
- [18] Lozano Huerta, C.: Birational geometry of the space of complete quadrics. *Int. Math. Res. Notices* **2015**, 12563–12589 Zbl [1329.14031](https://zbmath.org/?q=ser/1329.14031) MR [3431630](https://mathscinet.org/mr/3431630)
- [19] Macdonald, I. G.: *Symmetric Functions and Hall Polynomials*. Oxford Math. Monogr., Clarendon Press, Oxford Univ. Press, New York (1995) Zbl [0899.05068](https://zbmath.org/?q=ser/0899.05068) MR [1354144](https://mathscinet.org/mr/1354144)
- [20] Massarenti, A.: On the birational geometry of spaces of complete forms I: collineations and quadrics. *Proc. London Math. Soc. (3)* **121**, 1579–1618 (2020) Zbl [1473.14031](https://zbmath.org/?q=ser/1473.14031) MR [4144371](https://mathscinet.org/mr/4144371)
- [21] Massarenti, A.: On the birational geometry of spaces of complete forms II: Skew-forms. *J. Algebra* **546**, 178–200 (2020) Zbl [1442.14056](https://zbmath.org/?q=ser/1442.14056) MR [4032731](https://mathscinet.org/mr/4032731)
- [22] Michałek, M., Monin, L., Wiśniewski, J. A.: Maximum likelihood degree, complete quadrics, and \mathbb{C}^* -action. *SIAM J. Appl. Algebra Geom.* **5**, 60–85 (2021) Zbl [1461.62228](https://zbmath.org/?q=ser/1461.62228) MR [4219257](https://mathscinet.org/mr/4219257)

- [23] Michałek, M., Sturmfels, B., Uhler, C., Zwiernik, P.: Exponential varieties. *Proc. London Math. Soc.* (3) **112**, 27–56 (2016) Zbl [1345.14048](#) MR [3458144](#)
- [24] Michałek, M., Sturmfels, B.: *Invitation to Nonlinear Algebra*. *Grad. Stud. Math.* 211, Amer. Math. Soc. (2021) Zbl [1477.14001](#) MR [4423369](#)
- [25] Nie, J., Ranestad, K., Sturmfels, B.: The algebraic degree of semidefinite programming. *Math. Program.* **122**, 379–405 (2010) Zbl [1184.90119](#) MR [2546336](#)
- [26] Pragacz, P.: Enumerative geometry of degeneracy loci. *Ann. Sci. École Norm. Sup.* (4) **21**, 413–454 (1988) Zbl [0687.14043](#) MR [974411](#)
- [27] Pragacz, P.: Algebro-geometric applications of Schur S - and Q -polynomials. In: *Topics in Invariant Theory* (Paris, 1989/1990), *Lecture Notes in Math.* 1478, Springer, Berlin, 130–191 (1991) Zbl [0783.14031](#) MR [1180989](#)
- [28] Pragacz, P.: Symmetric polynomials and divided differences in formulas of intersection theory. In: *Parameter Spaces* (Warszawa, 1994), *Banach Center Publ.* 36, Inst. Math., Polish Acad. Sci. Warszawa, 125–177 (1996) Zbl [0851.05094](#) MR [1481485](#)
- [29] Pragacz, P.: A Gysin formula for Hall–Littlewood polynomials. *Proc. Amer. Math. Soc.* **143**, 4705–4711 (2015) Zbl [1327.14034](#) MR [3391029](#)
- [30] Schubert, H.: Allgemeine Anzahlfunctionen für Kegelschnitte, Flächen und Räume zweiten Grades in n Dimensionen. *Math. Ann.* **45**, 153–206 (1894) Zbl [25.1038.03](#) MR [1510859](#)
- [31] Schubert, H.: *Kalkül der abzählenden Geometrie*. Springer, Berlin (1979) Zbl [0417.51008](#) MR [555576](#)
- [32] Semple, J. G.: On complete quadrics. *J. London Math. Soc.* **23**, 258–267 (1948) Zbl [0047.39902](#) MR [28605](#)
- [33] Semple, J. G.: The variety whose points represent complete collineations of S_r on S'_r . *Rend. Mat. Appl.* (5) **10**, 201–208 (1951) Zbl [0044.16904](#) MR [48847](#)
- [34] Semple, J. G.: On complete quadrics. II. *J. London Math. Soc.* **27**, 280–287 (1952) Zbl [0047.39902](#) MR [48846](#)
- [35] Stembridge, J. R.: Shifted tableaux and the projective representations of symmetric groups. *Adv. Math.* **74**, 87–134 (1989) Zbl [0677.20012](#) MR [991411](#)
- [36] Stückrad, J.: On quasi-complete intersections. *Arch. Math. (Basel)* **58**, 529–538 (1992) Zbl [0724.14031](#) MR [1161918](#)
- [37] Sturmfels, B., Timme, S., Zwiernik, P.: Estimating linear covariance models with numerical nonlinear algebra. *Algebr. Statist.* **11**, 31–52 (2020) Zbl [1461.62230](#) MR [4196404](#)
- [38] Sturmfels, B., Uhler, C.: Multivariate Gaussian, semidefinite matrix completion, and convex algebraic geometry. *Ann. Inst. Statist. Math.* **62**, 603–638 (2010) Zbl [1440.62255](#) MR [2652308](#)
- [39] Teissier, B.: Cycles évanescents, sections planes et conditions de Whitney. In: *Singularités à Cargèse* (Cargèse, 1972), *Astérisque* **7–8**, 285–362 (1973) Zbl [0295.14003](#) MR [0374482](#)
- [40] Teissier, B.: Variétés polaires. II. Multiplicités polaires, sections planes, et conditions de Whitney. In: *Algebraic Geometry* (La Rábida, 1981), *Lecture Notes in Math.* 961, Springer, Berlin, 314–491 (1982) Zbl [0585.14008](#) MR [708342](#)
- [41] Thaddeus, M.: Complete collineations revisited. *Math. Ann.* **315**, 469–495 (1999) Zbl [0986.14030](#) MR [1725990](#)
- [42] Tyrrell, J. A.: Complete quadrics and collineations in S_n . *Mathematika* **3**, 69–79 (1956) Zbl [0074.15502](#) MR [80352](#)
- [43] Vainsencher, I.: Schubert calculus for complete quadrics. In: *Enumerative Geometry and Classical Algebraic Geometry* (Nice, 1981), *Progr. Math.* 24, Birkhäuser, Boston, MA, 199–235 (1982) Zbl [0501.14032](#) MR [685770](#)
- [44] Vainsencher, I.: Complete collineations and blowing up determinantal ideals. *Math. Ann.* **267**, 417–432 (1984) Zbl [0544.14033](#) MR [738261](#)