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# Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase

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**Abstract.** We prove sharp spectral transition in the arithmetics of phase between localization and singular continuous spectrum for Diophantine almost Mathieu operators. We also determine exact exponential asymptotics of eigenfunctions and of corresponding transfer matrices throughout the localization region. This uncovers a universal structure in their behavior governed by the exponential phase resonances. The structure features a new type of hierarchy, where self-similarity holds upon alternating reflections.

**Keywords.** Quasiperiodic operator, almost Mathieu operator, Anderson localization, hierarchical structure

## 1. Introduction

Since this paper continues the program started in [36], we advise the readers less familiar with the subject to start with the first section of [36] or recent reviews [31, 32] for general background and useful introductory remarks. To avoid repetitions, we focus here only on the additional introductory considerations more directly relevant to the subject at hand.

Unlike random, one-dimensional quasiperiodic operators (3) feature spectral transitions with changes of parameters. The transitions between absolutely continuous and singular spectrum are governed by vanishing/non-vanishing of the Lyapunov exponent [19, 39, 45]. In the regime of positive Lyapunov exponents (also called supercritical in the analytic case, with the name inspired by the almost Mathieu operator) there are also more delicate transitions: between localization (point spectrum with exponentially decaying

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eigenfunctions) and singular continuous spectrum. They are governed by the resonances: eigenvalues of box restrictions that are too close to each other in relation to the distance between the boxes, leading to small denominators in various expansions. All known proofs of localization are based, in one way or another, on avoiding resonances and removing resonance-producing parameters, while all known proofs of singular continuous spectrum and even some of the absolutely continuous spectrum [2] are based on showing their abundance.

For quasiperiodic operators, one category of resonances are the ones determined entirely by the frequency. Indeed, for smooth potentials, large coefficients in the continued fraction expansion of the frequency lead to almost repetitions and thus resonances, regardless of the values of other parameters. Such resonances were first understood and exploited to show singular continuous spectrum for the Liouville frequencies in [9, 10], based on [24].[1] The strength of frequency resonances is measured by the arithmetic parameter

$$\beta(\alpha) = \limsup_{k\to\infty} -\frac{\ln \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{1}$$

where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \inf_{\ell\in\mathbb{Z}} |x-\ell|$. Another class of resonances, appearing for all *even* potentials, was discovered in [38], where it was shown for the first time that the arithmetic properties of the phase also play a role and may lead to singular continuous spectrum even for the Diophantine frequencies. Indeed, for even potentials, phases with almost symmetries lead to resonances, regardless of the values of other parameters. The strength of phase resonances is measured by the arithmetic parameter

$$\delta(\alpha,\theta) = \limsup_{k\to\pm\infty} -\frac{\ln \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|}. \tag{2}$$

In both these cases, the strength of the resonances is in competition with the exponential growth controlled by the Lyapunov exponent. It was conjectured in 1994 [28] that for the almost Mathieu family – the prototypical quasiperiodic operator – the above two types of resonances are the only ones that appear and the competition between the Lyapunov growth and resonance strength resolves, in both cases, in a sharp way.

Namely, separating frequency and phase resonances, the *frequency conjecture* was that for the $\alpha$-Diophantine phases, there is a transition from singular continuous to pure point spectrum precisely at $\beta(\alpha) = L$, where $L$ is the Lyapunov exponent. The *phase conjecture* was that for Diophantine frequencies, there is a transition from singular continuous to pure point spectrum precisely at $\delta(\alpha,\theta) = L$.

The frequency conjecture was recently proved [8,36]. In this paper, we prove the phase conjecture (Theorem 1.1). Moreover, our proof of the pure point part of the conjecture uncovers a universal structure of the eigenfunctions throughout the entire pure point spectrum regime (Theorem 1.2), which, in the presence of exponentially strong resonances,

[1] According to [46], the fact that the Diophantine properties of the frequencies should play a role was first observed in [44].

demonstrates a new phenomenon that we call a *reflective hierarchy*, when the eigenfunctions feature self-similarity upon proper reflections (Theorem 1.3). While the existence of the continued fraction based hierarchy in the behavior of eigenfunctions was predicted already in [11], the phenomenon of reflective hierarchy was not even previously described in the (vast) physics literature, presumably because the dependence on the phase is still underappreciated by the physics community, and in particular numerical experiments tend to be performed for phase zero only.

This paper is both dual and complementary to the recent work [36]. There we found a sharp way to deal with frequency resonances, leading to both the discovery of a universal hierarchical structure of eigenfunctions driven by the continued fraction expansion of $\alpha$ and the sharp spectral transition in frequency. We note that the sharp transition in frequency for a.e. phase was first proved in [8], with pure point part being based on dual reducibility, without the analysis of the eigenfunctions. In contrast, sharp transition in phase that we prove here is not currently approachable by any other means. While several results approaching the transition in frequency have been obtained in the last 15 years, e.g. [6], there have been no results on the transition in phase for $0 < \delta < \infty$. Moreover, the palindromic nature of phase resonances is fundamentally different from the repetition nature of the frequency ones, requiring very different technical and conceptual solutions in order to go sharp, and leading to a completely different, *reflective*, hierarchy.

The universality of the hierarchical structure described in Theorem 1.3 is twofold: not only the same universal function governs the behavior around each exponential phase resonance upon reflection and renormalization, it is the same structure for all the parameters involved: Diophantine frequency $\alpha$, phase $\theta$ with $\delta(\alpha, \theta) < L$ and eigenvalue $E$. Moreover, we expect this picture to be universal for a large class of potentials with symmetry-based resonances. While the hierarchical structure governed by the frequency resonances in [36] is conjectured to hold, for a.e. phase, throughout the entire class of analytic potentials, the structure discovered here requires evenness of the function defining the potential, and moreover, in general, resonances of other types may also be present. However, we conjecture that for general even analytic potentials for a.e. frequency only finitely many other exponentially strong resonances will appear, thus the structure described in this paper will hold for the corresponding class, with $\ln \lambda$ replaced by the Lyapunov exponent $L(E)$ throughout.

The *almost Mathieu operator* (AMO) is the (discrete) quasiperiodic Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda v(\theta + n\alpha)u(n), \tag{3}$$

with $v(\theta) = \cos 2\pi\theta$, where $\lambda$ is the *coupling*, $\alpha$ is the *frequency*, and $\theta$ is the *phase*.

It is the central quasiperiodic model for a multitude of reasons (see, e.g., [31]). In particular, it comes from physics and attracts continued interest there. It first appeared in Peierls [43], and arises as related, in two different ways, to a two-dimensional electron subject to a perpendicular magnetic field. It plays the central role in the Thouless et al. theory of the integer quantum Hall effect. For further background, history, and surveys of results see [16, 18, 30–32, 40, 42] and references therein.

The frequency $\alpha$ is called *Diophantine* if there exist $\kappa, \tau > 0$ such that for $k \neq 0$,

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\tau}{|k|^{\kappa}}. \tag{4}$$

From now on, unless otherwise noted, we will always assume $\alpha$ is Diophantine. When we need to refer to (4) in a non-quantitative way, we will sometimes call it the *Diophantine condition* (DC) on $\alpha$.[2]

The operator $H = H_{\lambda,\alpha,\theta}$ is said to have *Anderson localization* if it has pure point spectrum with exponentially decaying eigenfunctions.

We have

**Theorem 1.1.** (1) *$H_{\lambda,\alpha,\theta}$ has Anderson localization if $|\lambda| > e^{\delta(\alpha,\theta)}$.*

(2) *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $1 < |\lambda| < e^{\delta(\alpha,\theta)}$.*

(3) *$H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum if $|\lambda| < 1$.*

**Remark.** (1) We will prove part (2) for all irrational $\alpha$, and general even Lipschitz $v$ in (3); see Theorem 4.2.

(2) Part (3) is known for all $\alpha, \theta$ [3] and is included here for completeness.

(3) Parts (1) and (2) of Theorem 1.1 verify the phase part of the conjecture in [28], as stated there. The frequency part was recently proved in [8, 36].

Singular continuous spectrum was first established for $1 < |\lambda| < e^{c\delta(\alpha,\theta)}$, for sufficiently small $c$ [38]. One can see that even with tight upper semicontinuity bounds the argument of [38] does not work for $c > 1/2$. Here we introduce new ideas to remove the factor of 2 and approach the actual threshold.

Anderson localization for Diophantine $\alpha$ and $\delta(\alpha, \theta) = 0$ was proved in [29]. The argument was theoretically extendable to $|\lambda| > e^{C\delta(\alpha,\theta)}$ for a large $C$ but not beyond. Therefore, the case of $\delta(\alpha, \theta) > 0$ was completely open before. In fact, the localization method of [29] could not deal with exponentially strong resonances. The first way to handle exponentially strong *frequency* resonances was developed in [6]. That method however could not approach the threshold. An important technical achievement of [36] was to develop a way to handle frequency resonances that works up to the very transition and leads to sharp bounds. In this paper we develop the first, and at the same time the sharp, way to treat exponential *phase* resonances.

Recently, it became possible to prove pure point spectrum in a non-constructive way, avoiding the localization method, using instead reducibility for the dual model [8] (see also [33]), as was first done, in the perturbative regime, in [12]. Coupled with recent arguments [4, 5, 27, 49] that allow one to conjugate the global transfer-matrix cocycle into the local almost reducibility regime[3] and proceed by almost reducibility, this offers a

[2] It is rather straightforward to extend all the results to the case $\beta(\alpha) = 0$, without any changes in formulations. We present the proof under condition (4) just for a slight simplification of some arguments.

[3] For the Diophantine case this is Eliasson's regime [20].

powerful technique that led to a solution of the measure-theoretic version of the frequency part of the conjecture of [28] by Avila–You–Zhou [8] and a corresponding sharp result for the supercritical regime in the extended Harper model [25]. However, those methods lose control on phases, thus are not applicable to study transitions in phase. More recently, after the appearance of an earlier version of this paper, there has been an important achievement in the work of Ge–You [22] who developed arithmetic Aubry duality, allowing one to obtain localization statements for arithmetically specific sets of phases based on structural reducibility estimates for dual operators, a method that works also in the multidimensional case. It seems to have the potential for an alternative proof of localization in the regime $|\lambda| > e^{\delta(\alpha,\theta)}$, which would be quite interesting. Conversely, the current analysis has the potential for making certain deeper reducibility-related conclusions based on duality in the opposite direction, as in e.g. [7].

Our proof of localization is based on determining the *exact asymptotics* of the generalized eigenfunctions for $|\lambda| > e^{\delta(\alpha,\theta)}$. Namely, exponential phase resonances lead to a certain reflection in the exponential shape of the eigenfunction, of magnitude and at scales determined by the strength of the resonance. Clearly, the potential $\cos 2\pi(\theta + n\alpha)$ is symmetric with respect to $x_0 \in \mathbb{Z}$ such that $\sin \pi(2\theta + x_0\alpha) = 0$ and "almost" symmetric if $\sin \pi(2\theta + x_0\alpha)$ is small. It turns out that the asymptotics can be determined by following such local minima.

Namely, for any $\ell$, let $x_0 \in \mathbb{Z}$ (we can choose any one if $x_0$ is not unique) be such that

$$|\sin \pi(2\theta + x_0\alpha)| = \min_{|x|\le 2|\ell|} |\sin \pi(2\theta + x\alpha)|.$$

Let $\eta = 0$ if $2\theta + x_0\alpha \in \mathbb{Z}$, otherwise let $\eta \in (0,\infty)$ be given by

$$|\sin \pi(2\theta + x_0\alpha)| = e^{-\eta|\ell|}. \tag{5}$$

Define $f : \mathbb{Z} \to \mathbb{R}^+$ as follows:

$$f(\ell) = \begin{cases} e^{-|\ell|\ln|\lambda|} & \text{if } x_0 \cdot \ell \le 0, \\ e^{-(|x_0|+|\ell-x_0|)\ln|\lambda|}e^{\eta|\ell|} + e^{-|\ell|\ln|\lambda|} & \text{if } x_0 \cdot \ell > 0. \end{cases}$$

We say that $\phi \ne 0$ is a *generalized eigenfunction* of $H$ with *generalized eigenvalue* $E$ if for some $C < \infty$ and all $k \in \mathbb{Z}$,

$$H\phi = E\phi \quad \text{and} \quad |\phi(k)| \le \hat{C}(1 + |k|). \tag{6}$$

For a fixed generalized eigenvalue $E$ and corresponding generalized eigenfunction $\phi$ of $H_{\lambda,\alpha,\theta}$, let $U(\ell) = \binom{\phi(\ell)}{\phi(\ell-1)}$. We have

**Theorem 1.2.** *Assume* $\ln|\lambda| > \delta(\alpha,\theta)$. *Then for any* $\varepsilon > 0$, *there exists* $K$ *such that for any* $|\ell| \ge K$, $U(\ell)$ *satisfies*

$$f(\ell)e^{-\varepsilon|\ell|} \le \|U(\ell)\| \le f(\ell)e^{\varepsilon|\ell|}. \tag{7}$$

*In particular, the eigenfunctions decay at the rate* $\ln|\lambda| - \delta(\alpha,\theta)$.

**Remark.** • For $\delta = 0$ we have, for any $\varepsilon > 0$,

$$e^{-(\ln|\lambda|+\varepsilon)|\ell|} \le f(\ell) \le e^{-(\ln|\lambda|-\varepsilon)|\ell|}.$$

This implies that the eigenfunctions decay precisely at the rate of the Lyapunov exponent $\ln|\lambda|$.

• For $\delta > 0$, by the definition of $\delta$ and $f$ we have, for any $\varepsilon > 0$,

$$f(\ell) \le e^{-(\ln|\lambda|-\delta-\varepsilon)|\ell|}. \tag{8}$$

• By the definition of $\delta$ again, there exists a subsequence $\{\ell_i\}$ such that

$$|\sin \pi(2\theta + \ell_i\alpha)| \le e^{-(\delta-\varepsilon)|\ell_i|}.$$

By the DC on $\alpha$, one has

$$|\sin \pi(2\theta + \ell_i\alpha)| = \min_{|x|\le 2|\ell_i|} |\sin \pi(2\theta + x\alpha)|.$$

Then

$$f(\ell_i) \ge e^{-(\ln|\lambda|-\delta+\varepsilon)|\ell_i|}. \tag{9}$$

This implies the eigenfunctions decay precisely at the rate $\ln|\lambda| - \delta(\alpha,\theta)$.

• If $x_0$ is not unique, then by the DC on $\alpha$, $\eta$ in (5) is necessarily smaller than any $\epsilon$ for $\ell$ large. Then

$$e^{-(\ln|\lambda|+\varepsilon)|\ell|} \le \|U(\ell)\| \le e^{-(\ln|\lambda|-\varepsilon)|\ell|}.$$

The behavior described in Theorem 1.2 happens around every point.[4] This, coupled with effective control of parameters at the local maxima, allows us to uncover the self-similar nature of the eigenfunctions. Hierarchical behavior of solutions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [1, 48], has remained an important open challenge even at the physics level. In [36] we obtained universal hierarchical structure of the eigenfunctions for all frequencies $\alpha$ and phases with $\delta(\alpha,\theta) = 0$. In studying the eigenfunctions of $H_{\lambda,\alpha,\theta}$ for $\delta(\alpha,\theta) > 0$ we obtain a different kind of universality throughout the pure point spectrum regime, which features a self-similar hierarchical structure upon proper *reflections*.

Assume the phase $\theta$ satisfies $0 < \delta(\alpha,\theta) < \ln|\lambda|$. Fix $0 < \varsigma < \delta(\alpha,\theta)$.

Let $k_0$ be the the position of a global maximum point of $|\phi|$.[5] Let $K_i$ be the positions of exponential resonances of the phase $\theta' = \theta + k_0\alpha$ defined by

$$\|2\theta + (2k_0 + K_i)\alpha\|_{\mathbb{R}/\mathbb{Z}} \le e^{-\varsigma|K_i|}. \tag{10}$$

This means that $|v(\theta' + \ell\alpha) - v(\theta' + (K_i - \ell)\alpha)| \le Ce^{-\varsigma|K_i|}$, uniformly in $\ell$, or, in other words, the potential $v_n = v(\theta + n\alpha)$ is $e^{-\varsigma|K_i|}$-almost symmetric with respect to $(k_0 + K_i)/2$.

[4] While the required largeness $K$ in Theorem 1.2 depends on $E$, the more technically relevant local version, Theorem 5.1, only requires largeness bigger than a *constant* that depends only on $\alpha, \lambda$.

[5] One can take any one if there are several.

Since $\alpha$ is Diophantine, we have

$$|K_i| \geq c e^{c|K_{i-1}|}, \tag{11}$$

where $c$ depends on $\varsigma$ and $\alpha$ through the Diophantine constants $\kappa, \tau$. On the other hand, $K_i$ is necessarily an infinite sequence.

Let $\phi$ be an eigenfunction, and $U(k) = \binom{\phi(k)}{\phi(k-1)}$. We say $k$ is a *local $K$-maximum* if $\|U(k)\| \geq \|U(k+s)\|$ for all $s - k \in [-K, K]$. We emphasize that by a "local $K$-maximum" we mean, from now on, its position in $\mathbb{Z}$, not the value.

We first describe the hierarchical structure of local maxima informally. There exists a *constant* $\hat{K}$ such that there is a local $cK_j$-maximum $b_j$ within distance $\hat{K}$ of each resonance $K_j$. The exponential behavior of the eigenfunction in the local $cK_j$-neighborhood of each such local maximum, normalized by the value at the local maximum, is given by the *reflection* of $f$. Moreover, this describes the entire collection of local maxima of depth 1, that is, $K$ such that $K$ is a local $cK$-maximum. Then we have a similar picture in the vicinity of $b_j$: there are local $cK_i$-maxima $b_{j,i}$, $i < j$, within distance $\hat{K}^2$ of each $K_j - K_i$. The exponential (on the $K_i$ scale) behavior of the eigenfunction in the local $cK_i$-neighborhood of each such local maximum, normalized by the value at the local maximum, is given by $f$. Then we get the next level maxima $b_{j,i,s}$, $s < i$, in the $\hat{K}^3$-neighborhood of $K_j - K_i + K_s$ and reflected behavior around each, and so on, with reflections alternating with steps. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{j_0,j_1,\ldots,j_s}$, with each "depth $s+1$" local maximum $b_{j_0,j_1,\ldots,j_s}$ being in the corresponding vicinity of the "depth $s$" local maximum $b_{j_0,j_1,\ldots,j_{s-1}} \approx k_0 + \sum_{i=0}^{s-1}(-1)^i K_{j_i}$ and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, with $b_{j_0,j_1,\ldots,j_{s-1}}$ determined with $\hat{K}^s$ precision, thus it presents an accurate picture as long as $K_{j_s} \gg \hat{K}^s$.

We now describe the hierarchical structure precisely.

**Theorem 1.3.** *Assume the sequence $K_i$ satisfies (10) for some $\varsigma > 0$. Then there exists $\hat{K}(\alpha, \lambda, \theta, \varsigma) < \infty$[6] such that for any $j_0 > j_1 > \cdots > j_k \geq 0$ with $K_{j_k} \geq \hat{K}^{k+1}$, for each $0 \leq s \leq k$ there exists a local $\frac{\varsigma}{2\ln\lambda} K_{j_s}$-maximum[7] $b_{j_0,j_1,\ldots,j_s}$ such that the following holds:*

(I) $|b_{j_0,j_1,\ldots,j_s} - k_0 - \sum_{i=0}^{s}(-1)^i K_{j_i}| \leq \hat{K}^{s+1}$.

(II) *For any $\varepsilon > 0$, if $C\hat{K}^{k+1} \leq |x - b_{j_0,j_1,\ldots,j_k}| \leq \frac{\varsigma}{4\ln\lambda}|K_{j_k}|$, where $C$ is a large constant depending on $\alpha, \lambda, \theta, \varsigma$ and $\varepsilon$, then for each $s = 0, 1, \ldots, k$,*

$$f((-1)^{s+1}x_s)e^{-\varepsilon|x_s|} \leq \frac{\|U(x)\|}{\|U(b_{j_0,j_1,\ldots,j_s})\|} \leq f((-1)^{s+1}x_s)e^{\varepsilon|x_s|}, \tag{12}$$

*where $x_s = x - b_{j_0,j_1,\ldots,j_s}$.*

[6] $\hat{K}$ depends on $\theta$ through $2\theta + k\alpha$; see (2).

[7] Actually, it can be a local $(\frac{\varsigma}{\ln\lambda} - \varepsilon)K_{j_s}$-maximum for any $\varepsilon > 0$.

**Remark 1.4.** Actually (12) holds for $x$ with $C\hat{K}^{k+1} \le |x - b_{j_0,j_1,\ldots,j_k}| \le (\frac{\varsigma}{2\ln\lambda} - \varepsilon)|K_{j_k}|$ for any $\varepsilon > 0$.

Thus the behavior of $\phi(x)$ is described by the same universal $f$ in each $\frac{\varsigma}{2\ln\lambda}K_{j_s}$ window around the corresponding local maximum $b_{j_0,j_1,\ldots,j_s}$ after alternating reflections. The positions of the local maxima in the hierarchy are determined up to errors that at all but possibly the last step are superlogarithmically small in $K_{j_s}$. We call such a structure *reflective hierarchy*.

We are not aware of previous results describing the structure of eigenfunctions for Diophantine $\alpha$ (the structure in the regime $\beta > 0$ is described in [36]). Certain results indicating the hierarchical structure in the corresponding semiclassical/perturbative regimes were previously obtained in the works of Sinai, Helffer–Sjöstrand, and Buslaev–Fedotov (see [21,26,47], and also [51] for another model).

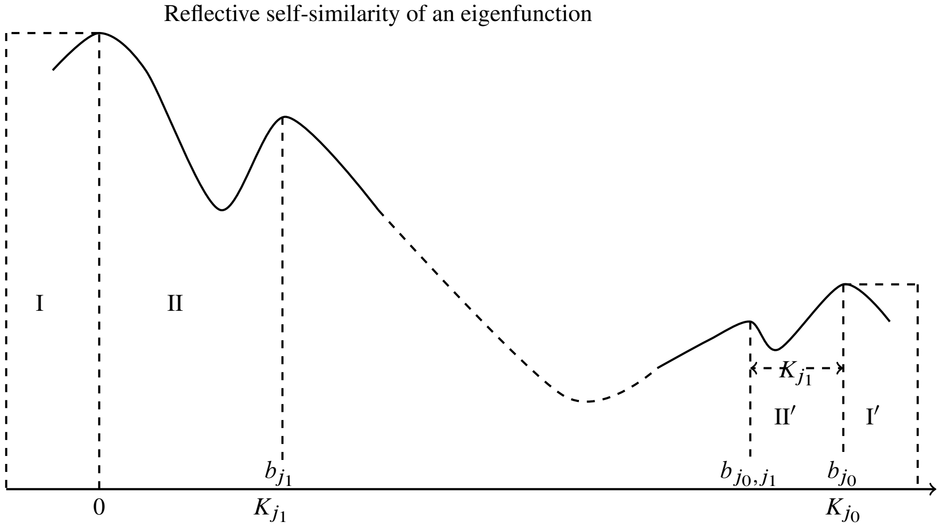


**Fig. 1.** This depicts reflective self-similarity of an eigenfunction with global maximum at 0. The self-similarity: I′ is obtained from I by scaling the $x$-axis proportional to the ratio of the heights of the maxima in I and I′. II′ is obtained from II by scaling the $x$-axis proportional to the ratio of the heights of the maxima in II and II′. The behavior in the regions I′, II′ mirrors the behavior in I, II upon reflection and corresponding dilation.

Our final main result is the asymptotics of the transfer matrices. Let $A_0 = I$ and for $k \ge 1$,

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha)\cdots A(\theta),$$

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha),$$

where

$$A(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}.$$

$A_k$ is called the ($k$-step) *transfer matrix*. As is clear from the definition, it also depends on $\theta$, $\lambda$ and $E$, but since those parameters will usually be fixed, we omit them from the notation.

We define a new function $g : \mathbb{Z} \to \mathbb{R}^+$ as follows:

$$g(\ell) = \begin{cases} e^{|\ell| \ln |\lambda|} & \text{if } x_0 \cdot \ell \le 0 \text{ or } |x_0| > |\ell|, \\ e^{(\ln |\lambda| - \eta)|\ell|} + e^{|2x_0 - \ell| \ln |\lambda|} & \text{if } x_0 \cdot \ell \ge 0 \text{ and } |x_0| \le |\ell| \le 2|x_0|, \\ e^{(\ln |\lambda| - \eta)|\ell|} & \text{if } x_0 \cdot \ell \ge 0 \text{ and } |\ell| > 2|x_0|. \end{cases}$$

We have

**Theorem 1.5.** *Under the conditions of Theorem 1.2, we have*

$$g(\ell)e^{-\varepsilon|\ell|} \le \|A_\ell\| \le g(\ell)e^{\varepsilon|\ell|}. \tag{13}$$

Let $\psi(\ell)$ denote any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ that is linearly independent of $\phi(\ell)$. Let $\tilde{U}(\ell) = \binom{\psi(\ell)}{\psi(\ell-1)}$. An immediate counterpart of (13) is the following

**Corollary 1.6.** *Under the conditions of Theorem 1.2, the vectors $\tilde{U}(\ell)$ satisfy*

$$g(\ell)e^{-\varepsilon|\ell|} \le \|\tilde{U}(\ell)\| \le g(\ell)e^{\varepsilon|\ell|}. \tag{14}$$

Our analysis also gives

**Corollary 1.7.** *Under the conditions of Theorem 1.2, we have*

(i)
$$\limsup_{k\to\infty} \frac{\ln \|A_k\|}{k} = \limsup_{k\to\infty} \frac{-\ln \|U(k)\|}{k} = \ln |\lambda|,$$

(ii)
$$\liminf_{k\to\infty} \frac{\ln \|A_k\|}{k} = \liminf_{k\to\infty} \frac{-\ln \|U(k)\|}{k} = \ln |\lambda| - \delta,$$

(iii) *outside a sequence of lower density* $1/2$,

$$\lim_{k\to\infty} \frac{-\ln \|U(k)\|}{|k|} = \ln |\lambda|, \tag{15}$$

(iv) *outside a sequence of lower density* $0$,

$$\lim_{k\to\infty} \frac{\ln \|A_k\|}{|k|} = \ln |\lambda|. \tag{16}$$

Thus our analysis presents the second, after [36], study of the dynamics of Lyapunov–Perron non-regular points, in a natural setting. It is interesting to remark that (16) also holds throughout the pure point regime of [36]. As in [36], the fact that $g$ is not always the reciprocal of $f$ leads to exponential tangencies between contracted and expanded directions with the rate approaching $-\delta$ along a subsequence. Tangencies are an attribute of non-uniform hyperbolicity and are usually viewed as a difficulty to avoid through e.g.

the parameter exclusion (e.g. [13, 15, 50]). Our analysis allows studying them in detail and uncovers the hierarchical structure of exponential tangencies positioned precisely at resonances. This will be explored in future work.

While many of our statements are on the surface parallel (in fact, dual) to those of [36], the shape of the universal structure is completely different for phase vs frequency resonances, and the treatment of symmetry-based resonances required development of completely new techniques both for sharp small denominator analysis around the phase resonances and sharp palindromic arguments for the lower bounds. In fact, the only two technical ingredients that are similar to the arguments used in [36] to prove localization in the presence of exponential frequency resonance are those presented, in universal versions, in Theorem 3.2 (a uniformity statement for any Diophantine $\alpha$) and Theorem 3.3 (a resonant block expansion theorem for any one-dimensional operator), proved in Appendices A and B respectively. Those statements can be of use for proving localization for other models. The rest of the argument is based on new ideas specific to the phase resonance situation.

Moreover, we mention that the methods developed in this paper have made it possible to determine the *exact* exponent of the exponential decay rate in expectation for the two-point function [35], the first result of this kind for any model. In particular, even despite a remarkable progress in arguments based on quantitative dual reducibility, the lower bound is not currently accessible by other means [23]. Methods of this paper have also recently led to the first ever example of dynamical localization in absence of SULE [37].

The rest of this paper is organized as follows. We list the definitions and standard preliminaries in Section 2. Section 3 is devoted to the upper bound on the generalized eigenfunction in Theorem 1.2, establishing sharp upper bounds for any eigensolution in the resonant case.

In Section 4 we prove the sharp transition in Theorem 1.1, and a lower bound on the generalized eigenfunctions in Theorem 1.2. The part on the singular continuous spectrum, in particular, requires a new approach to the palindromic argument in order to remove a factor of 4 inherent in the previous proofs, and the sharp lower bound in the localization regime requires an even more delicate approach. In Section 5, we use the local version of Theorem 1.2 and establish reflective hierarchical structure of *resonances* to prove the reflective hierarchical structure in Theorem 1.3. In Section 6, we study the growth of transfer matrices and prove Theorem 1.5, and Corollaries 1.6 and 1.7. Except for the (mostly standard) statements listed in the preliminaries and Lemma A.1, this paper is entirely self-contained.

## 2. Preliminaries

Without loss of generality, we assume $\lambda > 1$ and $\ell > 0$. If $2\theta \in \alpha\mathbb{Z} + \mathbb{Z}$, then $\delta(\alpha, \theta) = 0$, in which case Theorem 1.2 follows from [34] and Theorem 1.3 by vacuousness. Thus in what follows we always assume $2\theta \notin \alpha\mathbb{Z} + \mathbb{Z}$.

For any solution of $H_{\lambda,\alpha,\theta}\varphi = E\varphi$ we have, for any $k, m$,

$$\begin{pmatrix} \varphi(k+m) \\ \varphi(k+m-1) \end{pmatrix} = A_k(\theta + m\alpha) \begin{pmatrix} \varphi(m) \\ \varphi(m-1) \end{pmatrix}. \tag{17}$$

The Lyapunov exponent is given by

$$L(E) = \lim_{k\to\infty} \frac{1}{k} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_k(\theta)\| \, d\theta. \tag{18}$$

The Lyapunov exponent can be computed precisely for $E$ in the spectrum of $H_{\lambda,\alpha,\theta}$. We denote the spectrum by $\Sigma_{\lambda,\alpha}$ (it does not depend on $\theta$).

**Lemma 2.1** ([17]). *For $E \in \Sigma_{\lambda,\alpha}$ and $\lambda > 1$, we have $L(E) = \ln \lambda$.*

Recall that we always assume $E \in \Sigma_{\lambda,\alpha}$, so by upper semicontinuity and unique ergodicity, one has

$$\ln \lambda = \lim_{k\to\infty} \sup_{\theta\in\mathbb{R}/\mathbb{Z}} \frac{1}{k} \ln \|A_k(\theta)\|, \tag{19}$$

that is, the convergence in (19) is uniform with respect to $\theta \in \mathbb{R}$. More precisely, for all $\varepsilon > 0$,

$$\|A_k(\theta)\| \le e^{(\ln\lambda+\varepsilon)k} \quad \text{for } k \text{ large enough.} \tag{20}$$

We start with the basic setup going back to [29]. Let us denote

$$P_k(\theta) = \det(R_{[0,k-1]}(H_{\lambda,\alpha,\theta} - E)R_{[0,k-1]}),$$

where $R_{[a,b]}$ is the coordinate restriction to $[a,b] \subset \mathbb{Z}$.

It is easy to check that

$$A_k(\theta) = \begin{pmatrix} P_k(\theta) & -P_{k-1}(\theta+\alpha) \\ P_{k-1}(\theta) & -P_{k-2}(\theta+\alpha) \end{pmatrix}. \tag{21}$$

For any interval $I \subset \mathbb{Z}$, define the Green's function as

$$G_I = \big(R_I(H_{\lambda,\alpha,\theta} - E)R_I\big)^{-1}$$

if $R_I(H-E)R_I$ is invertible. We remark that $G_I$ depends on $E, \lambda, \alpha$ and $\theta$.

By Cramer's rule for given $x_1$ and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$, one has

$$|G_I(x_1, y)| = \left| \frac{P_{x_2-y}(\theta + (y+1)\alpha)}{P_k(\theta + x_1\alpha)} \right|, \tag{22}$$

$$|G_I(y, x_2)| = \left| \frac{P_{y-x_1}(\theta + x_1\alpha)}{P_k(\theta + x_1\alpha)} \right|. \tag{23}$$

By (20) and (21), the numerators in (22) and (23) can be bounded uniformly with respect to $\theta$. Namely, for any $\varepsilon > 0$,

$$|P_k(\theta)| \le e^{(\ln\lambda+\varepsilon)k} \tag{24}$$

for $k$ large enough.

**Definition 2.2.** Fix $\tau > 0$. A point $y \in \mathbb{Z}$ will be called $(\tau, k)$ *regular* if there exists an interval $[x_1, x_2]$ containing $y$, where $x_2 = x_1 + k - 1$, such that

$$|G_{[x_1,x_2]}(y, x_i)| < e^{-\tau|y-x_i|} \quad \text{and} \quad |y - x_i| \geq \frac{1}{40}k \quad \text{for } i = 1, 2.$$

It is easy to check that for any solution of $H_{\lambda,\alpha,\theta}\varphi = E\varphi$,

$$\varphi(x) = -G_{[x_1,x_2]}(x_1, x)\varphi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\varphi(x_2 + 1), \tag{25}$$

where $x \in I = [x_1, x_2] \subset \mathbb{Z}$.

**Definition 2.3.** We say that the set $\{\theta_1, \ldots, \theta_{k+1}\}$ is $\epsilon$-*uniform* if

$$\max_{x\in[-1,1]} \max_{i=1,\ldots,k+1} \prod_{j=1,\, j\neq i}^{k+1} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{k\epsilon}. \tag{26}$$

Let $A_{k,r} = \{\theta \in \mathbb{R} \mid P_k(\theta + \frac{1}{2}(k-1)\alpha)| \leq e^{(k+1)r}\}$ with $k \in \mathbb{N}$ and $r > 0$. We have the following lemma.

**Lemma 2.4** ([6, Lemma 9.3]). *Suppose* $\{\theta_1, \ldots, \theta_{k+1}\}$ *is* $\epsilon_1$*-uniform. Then there exists some* $\theta_i$ *in* $\{\theta_1, \ldots, \theta_{k+1}\}$ *such that* $\theta_i \notin A_{k,\ln\lambda-\epsilon}$ *if* $\epsilon > \epsilon_1$ *and* $k$ *is sufficiently large.*

## 3. Localization

Let $\alpha$ be Diophantine and $\delta(\alpha, \theta)$ be given by (2).

Recall that for $\phi$ a generalized eigenfunction and $E$ the corresponding generalized eigenvalue of $H_{\lambda,\alpha,\theta}$, we denote $U(\ell) = \binom{\phi(\ell)}{\phi(\ell-1)}$. In this part we will prove the localization part of Theorem 1.1 and the upper bound of Theorem 1.2.

**Theorem 3.1.** *Suppose* $\ln\lambda > \delta(\alpha, \theta)$. *For any* $\varepsilon > 0$ *and any generalized eigenfunction* $\phi$ *there exists* $K$ *such that for any* $|\ell| \geq K$, $U(\ell)$ *satisfies*

$$\|U(\ell)\| \leq f(\ell)e^{\varepsilon|\ell|}. \tag{27}$$

*In particular,* $H_{\lambda,\alpha,\theta}$ *satisfies Anderson localization, and the following upper bound holds for the generalized eigenfunction:*

$$\|U(\ell)\| \leq e^{-(\ln\lambda-\delta-\varepsilon)|\ell|}. \tag{28}$$

By Schnol's Theorem [14] if every generalized eigenfunction of $H$ decays exponentially, then $H$ satisfies Anderson localization. Therefore, in order to prove Theorem 3.1, it suffices to prove only its first part.

Without loss of generality assume $|\phi(0)|^2 + |\phi(-1)|^2 = 1$. Let $\psi$ be any solution of $H_{\lambda,\alpha,\theta}\psi = E\psi$ linearly independent of $\phi$, i.e., $|\psi(0)|^2 + |\psi(-1)|^2 = 1$ and

$$\phi(-1)\psi(0) - \phi(0)\psi(-1) = c,$$

where $c \neq 0$.

Then by the constancy of the Wronskian, one has

$$\phi(y-1)\psi(y)-\phi(y)\psi(y-1)=c. \tag{29}$$

We will also denote by $\varphi$ an *arbitrary* solution, so either $\psi$ or $\phi$, and denote $U^{\varphi}(y)=\binom{\varphi(y)}{\varphi(y-1)}$. Let $U(y)=\binom{\phi(y)}{\phi(y-1)}$ and $\tilde{U}(y)=\binom{\psi(y)}{\psi(y-1)}$.

Below, $\varepsilon>0$ is always sufficiently small and $\frac{p_n}{q_n}$ is the $n$th convergent of the continued fraction expansion of $\alpha$.

We will make repeated use of the following two theorems that can be useful also in other situations. The first theorem is an arithmetic statement that holds for any Diophantine $\alpha$.

**Theorem 3.2** (Uniformity Theorem). *Let $I_1, I_2$ be two disjoint intervals in $\mathbb{Z}$ such that $\# I_1=s_1q_n$ and $\# I_2=s_2q_n$, where $s_1,s_2\in\mathbb{Z}^+$. Suppose $|j|\le Csq_n$ for any $j\in I_1\cup I_2$ and $s\le q_n^C$, where $s=s_1+s_2$. Let $\gamma>0$ be such that*

$$e^{-\gamma s q_n}=\min_{i,j\in I_1\cup I_2}|\sin\pi(2\theta+(i+j)\alpha)|. \tag{30}$$

*Then for any $\varepsilon>0$, $\{\theta_j=\theta+j\alpha\}_{j\in I_1\cup I_2}$ is $\gamma+\varepsilon$-uniform if $n$ is large enough (not depending on $\gamma$).*

The second theorem holds for any one-dimensional (not necessarily quasiperiodic or even ergodic) Schrödinger operator. It is the technique to establish exponential decay with respect to the distance to the resonances. Let $H:\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})$ be given by

$$(Hu)(n)=u(n+1)+u(n-1)+v_nu(n), \tag{31}$$

Fix $\gamma>0$. For a generalized eigenfunction $\varphi$ of $H$ we set $r_y^{\varphi}:=\max_{|\sigma|\le 10\gamma}|\varphi(y+\sigma k)|$. We have

**Theorem 3.3** (Block Expansion Theorem). *Suppose $y_1,y_2\in\mathbb{Z}$ are such that $y_2-y_1=k$. Suppose there exists some $\tau>0$ such that for any $y\in[y_1+\gamma k, y_2-\gamma k]$, $y$ is $(\tau,k_1)$ regular for some $\frac{\gamma}{20}k<k_1\le\frac{1}{2}\min\{|y-y_1|,|y-y_2|\}$. Then for large enough $k$ (depending on $\tau$ and $\gamma$),*

$$r_y^{\varphi}\le\max\left\{r_{y_1}^{\varphi}\exp\{-\tau(|y-y_1|-3\gamma k)\}, r_{y_2}^{\varphi}\exp\{-\tau(|y-y_2|-3\gamma k)\}\right\} \tag{32}$$

*for all $y\in[y_1+10\gamma k, y_2-10\gamma k]$.*

These two theorems are similar in spirit to the statements in [36], with the ones related to Theorem 3.2 being in turn modifications of the ones appearing in [6]. While these techniques were developed specifically to treat the non-Diophantine case, these ideas turn out to be relevant for the case of phase resonances as well. Theorem 3.3 is essentially the block-expansion technique of multiscale analysis, coupled with certain extremality arguments, an idea used also in [36]. We expect Theorem 3.2 to be useful for various one-frequency quasiperiodic problems, and Theorem 3.3 for general one-dimensional models. We present the proofs in Appendices A and B respectively.

Now we return to $H_{\lambda,\alpha,\theta}$. The following lemma establishes the non-resonant decay for any Diophantine $\alpha$ and any $\theta$.

**Lemma 3.4.** *Suppose $k_0 \in [-2Ck, 2Ck]$ is such that*

$$|\sin \pi(2\theta + \alpha k_0)| = \min_{|x|\leq 2Ck} |\sin \pi(2\theta + \alpha x)|,$$

*where $C \geq 1$ is a constant. Let $\gamma, \varepsilon$ be small positive constants. Let $y_1 = 0, y_2 = k_0, y_3 \in [-2Ck, 2Ck]$. Assume $y \in [y_i, y_j]$ with $|y_i - y_j| \geq k$ and $y_s \notin [y_i, y_j]$ for $s \neq i, j$. Suppose $|y_i|, |y_j| \leq Ck$ and $|y - y_i| \geq 10\gamma k$ and $|y - y_j| \geq 10\gamma k$. Then for large enough $k$,*

$$r_y^{\varphi} \leq \max\left\{r_{y_i}^{\varphi} \exp\{-(L-\varepsilon)(|y - y_i| - 3\gamma k)\}, r_{y_j}^{\varphi} \exp\{-(L-\varepsilon)(|y - y_j| - 3\gamma k)\}\right\}. \tag{33}$$

**Remark 3.5.** We note that this lemma establishes, in particular, almost localization in the sense of [7] with decay rate $\ln \lambda - \varepsilon$ for any $\varepsilon > 0, \theta \in \mathbb{R}, \lambda > 1$, and $\alpha$ in DC.

*Proof of Lemma 3.4.* By the DC on $\alpha$, there exist $\tau', \kappa' > 0$ such that for any $x \neq k_0$ and $|x| \leq 2Ck$,

$$|\sin \pi(2\theta + x\alpha)| \geq \tau'/k^{\kappa'}. \tag{34}$$

Fix $y'$. For any $p$ satisfying $|p - y'| \geq \gamma k$, $|p| \geq \gamma k$ and $|p - k_0| \geq \gamma k$, let

$$d_p = \frac{1}{10} \min\{|p|, |p - k_0|, |p - y'|\}.$$

Let $\frac{p_n}{q_n}$ be the $n$th convergent of the continued fraction expansion of $\alpha$. Let $n$ be the largest integer such that

$$2q_n \leq d_p,$$

and let $s$ be the largest positive integer such that $2sq_n \leq d_p$. Notice that $2q_{n+1} > d_p$ and by the Diophantine condition on $\alpha$, we have $s \leq q_n^C$.

*Case 1:* $0 \leq k_0 < p$. We define intervals

$$I_1 = [-2sq_n, -1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].$$

*Case 2:* $0 \leq p < k_0$. If $p \leq k_0/2$, we set

$$I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p - 1].$$

If $p > k_0/2$, we set

$$I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p, p + 2sq_n - 1].$$

*Case 3:* $p < k_0 \leq 0$. We set

$$I_1 = [0, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].$$

*Case 4:* $k_0 < p < 0$. If $p \leq k_0/2$, we set

$$I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p - 1].$$

If $p > k_0/2$, we set

$$I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [p, p + 2sq_n - 1].$$

*Case 5:* $k_0 \le 0 < p$. We set

$$I_1 = [0, 2sq_n - 1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].$$

*Case 6:* $p < 0 \le k_0$. We set

$$I_1 = [-2sq_n, -1], \quad I_2 = [p - 2sq_n, p + 2sq_n - 1].$$

Using the small divisor condition (34) and the construction of $I_1, I_2$, we have in any case

$$\min_{i,j \in I_1 \cup I_2} |\sin \pi(2\theta + (i + j)\alpha)| \ge \tau'/k^{\kappa'}.$$

By Theorem 3.2, for any $\varepsilon > 0$, we see that in each case $\{\theta_j = \theta + j\alpha\}_{j \in I_1 \cup I_2}$ is $\varepsilon$-uniform. Hence by Lemma 2.4, there exists some $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{6sq_n-1, \ln\lambda-\varepsilon}$.

We have the following simple lemma, to be used repeatedly in the rest of the paper.

**Lemma 3.6.** *Let $a_n \to \infty$ and $0 < t < 1$. Then for sufficiently large $n$ and $|j| < ta_n$ we have $\theta_j = \theta + j\alpha \in A_{2a_n-1, \ln\lambda-\varepsilon}$.*

*Proof.* Assume $\theta_j \notin A_{2a_n-1, \ln\lambda-\varepsilon}$ for some $|j| < ta_n$.

Let $I = [j - a_n + 1, j + a_n - 1] = [x_1, x_2]$. We have $x_1 < 0 < x_2$ and

$$|x_i| > (1 - t)a_n. \tag{35}$$

By (22)–(24), one has

$$|G_I(0, x_i)| \le e^{(\ln\lambda+\varepsilon)(2a_n-1-|x_i|)-(2a_n-1)(\ln\lambda-\varepsilon)}.$$

Using (25), we obtain

$$|\phi(-1)|, |\phi(0)| \le \sum_{i=1,2} e^{\varepsilon a_n} |\varphi(x_i')| e^{-|x_i| \ln\lambda}, \tag{36}$$

where $x_1' = x_1 - 1$ and $x_2' = x_2 + 1$. Because of (35), the inequality (36) implies $|\phi(-1)|, |\phi(0)| \le e^{-(1-t-\varepsilon)\ln\lambda a_n}$. This contradicts $|\phi(-1)|^2 + |\phi(0)|^2 = 1$. ■

Lemma 3.6 implies that $j_0$ must belong to $I_2$.

Set $I = [j_0 - 3sq_n + 1, j_0 + 3sq_n - 1] = [x_1, x_2]$. By (22)–(24) again, one has

$$|G_I(p, x_i)| \le e^{(\ln\lambda+\varepsilon)(6sq_n-1-|k-x_i|)-(6sq_n-1)(\ln\lambda-\varepsilon)} \le e^{\varepsilon sq_n} e^{-|p-x_i|\ln\lambda}.$$

Notice that $|p - x_1|, |p - x_2| \ge sq_n - 1$. Thus for any $p \in [y_i + \gamma k, y_j - \gamma k]$, $p$ is $(6sq_n - 1, \ln\lambda - \varepsilon)$ regular. Block expansion (Theorem 3.3) now implies the lemma. ■

**Remark 3.7.** Recall that $U^\varphi(y) = \binom{\varphi(y)}{\varphi(y-1)}$. By (17) and (20), we have

$$Ce^{-(\ln\lambda+\varepsilon)|k_1-k_2|}\|U^\varphi(k_2)\| \le \|U^\varphi(k_1)\| \le Ce^{(\ln\lambda+\varepsilon)|k_1-k_2|}\|U^\varphi(k_2)\|. \tag{37}$$

Thus (33) implies

$$\|U^\varphi(y)\| \le \max\big\{\|U^\varphi(y_i)\|\exp\{-(\ln\lambda-\varepsilon)(|y-y_i|-14\gamma k)\}, \\ \|U^\varphi(y_j)\|\exp\{-(\ln\lambda-\varepsilon)(|y-y_j|-14\gamma k)\}\big\}. \tag{38}$$

**Lemma 3.8.** *Fix* $0 < t < \ln\lambda$. *Suppose*

$$|\sin\pi(2\theta+\alpha\mathcal{K})| = e^{-t|\mathcal{K}|}. \tag{39}$$

*Then for large* $|\mathcal{K}|$ *(depending on* $\kappa$, $\tau$ *and* $t$*),*

$$\|U^\varphi(\mathcal{K})\| \le \max\{\|U^\varphi(0)\|, \|U^\varphi(2\mathcal{K})\|\}\, e^{-(\ln\lambda-t-\varepsilon)|\mathcal{K}|}. \tag{40}$$

*Proof.* Without loss of generality assume $\mathcal{K} > 0$. By the DC on $\alpha$, we have

$$|\sin\pi(2\theta+\alpha\mathcal{K})| = \min_{|x|\le 8\mathcal{K}} |\sin\pi(2\theta+\alpha x)|.$$

Furthermore, there exist $\tau', \kappa' > 0$ such that for any $x \ne \mathcal{K}$ and $|x| \le 8\mathcal{K}$,

$$|\sin\pi(2\theta+x\alpha)| \ge \tau'/\mathcal{K}^{\kappa'}.$$

Let $\gamma$ be any small positive constant and define $r_y^\varphi = \max_{|\sigma|\le 10\gamma}|\varphi(y+\sigma\mathcal{K})|$. Let $\frac{p_n}{q_n}$ be the $n$th convergent of the continued fraction expansion of $\alpha$. Let $n$ be the largest integer such that

$$\left(\frac{t+C\varepsilon}{\ln\lambda-t-C\varepsilon}+1\right)q_n \le \frac{\mathcal{K}}{2},$$

where $C$ is a large constant depending on $\lambda, t$. Let $s$ be the largest positive integer such that $sq_n \le \frac{1}{2}\mathcal{K}$. Then $s > \frac{t+C\varepsilon}{\ln\lambda-t-C\varepsilon}$. Since also $(s+1)q_n \ge \frac{1}{2}\mathcal{K}$, we obtain

$$2s\frac{q_n}{\mathcal{K}} > \frac{t}{\ln\lambda} + C\varepsilon. \tag{41}$$

We define intervals

$$I_1 = [-sq_n, sq_n-1], \quad I_2 = [\mathcal{K}-sq_n, \mathcal{K}+sq_n-1].$$

Let $\theta_j = \theta + j\alpha$ for $j \in I_1 \cup I_2$. The set $\{\theta_j\}_{j\in I_1\cup I_2}$ consists of $4sq_n$ elements.

By Theorem 3.2 and (39), the set $\{\theta_j\}_{j\in I_1\cup I_2}$ is $(\frac{t\mathcal{K}}{4sq_n}+\varepsilon)$-uniform. In view of Lemma 2.4, there exists some $j_0 \in I_1 \cup I_2$ such that $\theta_{j_0} \notin A_{4sq_n-1,\ln\lambda-\frac{t\mathcal{K}}{4sq_n}-\varepsilon}$.

First assume $j_0 \in I_2$.

Set $I = [j_0-2sq_n+1, j_0+2sq_n-1] = [x_1, x_2]$. By (22)–(24), one has

$$|G_I(\mathcal{K}, x_i)| \le e^{(\ln\lambda+\varepsilon)(4sq_n-1-|\mathcal{K}-x_i|)-(4sq_n-1)(\ln\lambda-\frac{t\mathcal{K}}{4sq_n}-\varepsilon)}.$$

Using (25), we obtain

$$|\varphi(\mathcal{K}-1)|, |\varphi(\mathcal{K})| \leq \sum_{i=1,2} e^{(t+\varepsilon)\mathcal{K}} |\varphi(x_i')| e^{-|\mathcal{K}-x_i|\ln\lambda}, \tag{42}$$

where $x_1' = x_1 - 1$ and $x_2' = x_2 + 1$.

Fix small $\gamma = \varepsilon/C$, where $C$ is a large constant depending on $\lambda, t$.

If $x_i' \in [-10\gamma\mathcal{K}, 10\gamma\mathcal{K}]$, $x_i' \in [\mathcal{K} - 10\gamma\mathcal{K}, \mathcal{K} + 10\gamma\mathcal{K}]$ or $x_i' \in [2\mathcal{K} - 10\gamma\mathcal{K}, 2\mathcal{K} + 10\gamma\mathcal{K}]$, we bound $\varphi(x_i')$ in (42) by $r_0^\varphi$, $r_{\mathcal{K}}^\varphi$ or $r_{2\mathcal{K}}^\varphi$ respectively. In other cases, we bound $\varphi(x_i')$ in (42) with (33) using $k_0 = \mathcal{K}$, $y = x_i'$ and $y' = -\mathcal{K}, 2\mathcal{K}$ or $3\mathcal{K}$. Then we have

$$\begin{aligned}
&|\varphi(\mathcal{K}-1)|, |\varphi(\mathcal{K})| \\
&\leq \max\big\{r^\varphi_{\mathcal{K}\pm 2\mathcal{K}} \exp\{-(2\ln\lambda - t - C\gamma - \varepsilon)\mathcal{K}\}, r^\varphi_{\mathcal{K}\pm\mathcal{K}} \exp\{-(\ln\lambda - t - C\gamma - \varepsilon)\mathcal{K}\}, \\
&\qquad\qquad r^\varphi_{\mathcal{K}} \exp\{-(\ln\lambda - \varepsilon)2sq_n + (t + C\gamma)\mathcal{K}\}\big\}.
\end{aligned}$$

However, by (41), the inequality

$$|\varphi(\mathcal{K}-1)|, |\varphi(\mathcal{K})| \leq r^\varphi_{\mathcal{K}} \exp\{-(\ln\lambda - \varepsilon)2sq_n + (t + C\gamma)\mathcal{K}\} \leq e^{-\varepsilon\mathcal{K}} r^\varphi_{\mathcal{K}}$$

cannot happen, so we must have

$$|\varphi(\mathcal{K}-1)|, |\varphi(\mathcal{K})| \leq \exp\{-(\ln\lambda - t - C\gamma - \varepsilon)\mathcal{K}\} \max\{r^\varphi_{\mathcal{K}\pm\mathcal{K}}, e^{-\mathcal{K}\ln\lambda} r^\varphi_{\mathcal{K}\pm 2\mathcal{K}}\}. \tag{43}$$

Notice that by (37), one has

$$r^\varphi_{\mathcal{K}\pm 2\mathcal{K}} \leq e^{(\ln\lambda + C\gamma)\mathcal{K}} r^\varphi_{\mathcal{K}\pm\mathcal{K}}.$$

Then (43) becomes

$$\|U^\varphi(\mathcal{K})\| \leq \exp\{-(\ln\lambda - t - C\gamma - \varepsilon)\mathcal{K}\} \max\{r_0^\varphi, r_{2\mathcal{K}}^\varphi\}.$$

By (37) again, one has

$$r_y^\varphi e^{-(\ln\lambda + \varepsilon)10\gamma\mathcal{K}} \leq \|U^\varphi(y)\| \leq r_y^\varphi e^{(\ln\lambda + \varepsilon)10\gamma\mathcal{K}}.$$

Thus

$$\begin{aligned}
\|U^\varphi(\mathcal{K})\| &\leq \max\{\|U^\varphi(0)\|, \|U^\varphi(2\mathcal{K})\|\} e^{-(\ln\lambda - t - C\gamma - \varepsilon)|\mathcal{K}|} \\
&\leq \max\{\|U^\varphi(0)\|, \|U^\varphi(2\mathcal{K})\|\} e^{-(\ln\lambda - t - \varepsilon)|\mathcal{K}|}.
\end{aligned} \tag{44}$$

This implies (40). Thus in order to prove the lemma, it suffices to exclude the case $j_0 \in I_1$.

Suppose $j_0 \in I_1$. Notice that $I_1 + \mathcal{K} = I_2$ (i.e., $I_2$ can be obtained from $I_1$ by moving by $\mathcal{K}$ units). Following the proof of (44), we get (move $-\mathcal{K}$ units in (44))

$$\|U^\varphi(0)\| \leq \max\{\|U^\varphi(-\mathcal{K})\|, \|U^\varphi(\mathcal{K})\|\} e^{-(\ln\lambda - t - \varepsilon)|\mathcal{K}|}.$$

This contradicts $\|U^\phi(0)\| = 1$. ∎

*Proof of Theorem* 3.1. Without loss of generality, assume $\ell > 0$.

For any $\varepsilon > 0$, let $\gamma = \varepsilon/C > 0$, where $C$ is a large constant that may depend on $\lambda$ and $\delta$. Let $x_0'$ (we can choose any one if $x_0'$ is not unique) be such that

$$|\sin \pi(2\theta + x_0'\alpha)| = \min_{|x| \leq 4|\ell|} |\sin \pi(2\theta + x\alpha)|.$$

Let $\eta' \in (0, \infty)$ be given by the equation

$$|\sin \pi(2\theta + x_0'\alpha)| = e^{-\eta'|\ell|}. \tag{45}$$

*Case 1:* $|\sin \pi(2\theta + x_0'\alpha)| \neq |\sin \pi(2\theta + x_0\alpha)|$. This implies $|x_0'| > 2\ell$. In this case for any $\varepsilon > 0$, we have $\eta \leq \varepsilon$ if $\ell$ is large enough by the Diophantine condition. Let $y = \ell$, $C = 2$, $k = 2\ell$, and $y' = 2\ell$ in Lemma 3.4. Then $k_0 = x_0'$ and we obtain

$$|\phi(\ell)|, |\phi(\ell - 1)| \leq e^{-(\ln \lambda - C\gamma)\ell}.$$

This implies the right inequality of (7) in this case.

*Case 2:* $|\sin \pi(2\theta + x_0'\alpha)| = |\sin \pi(2\theta + x_0\alpha)|$, so $\eta = \eta'$. If $x_0 \leq 0$, let $y = \ell$, $C = 2$, $k = 2\ell$ and $y' = 2\ell$ in Lemma 3.4. Then Theorem 3.1 holds by (33).

Now we consider the case $x_0 > 0$. We split the proof into two subcases.

*Subcase (i):* $\eta \leq \gamma$. Fix some $y \in [\gamma\ell, 2\ell - \gamma\ell]$. Let $n$ be such that $q_n \leq \frac{1}{20} \min\{y, 2\ell - y\} < q_{n+1}$, and let $s$ be the largest positive integer such that $sq_n \leq \frac{1}{20} \min\{y, 2\ell - y\}$. We set

$$I_1 = [-2sq_n, 2sq_n - 1], \quad I_2 = [y - 2sq_n, y - 1].$$

By the definition of $\eta'$, $\eta$ and of $I_1$, $I_2$, we have

$$\min_{i,j \in I_1 \cup I_2} |\sin \pi(2\theta + (j + i)\alpha)| \geq e^{-\eta'\ell} = e^{-\eta\ell}.$$

By Theorem 3.2, the set $\{\theta_j = \theta + j\alpha\}_{j \in I_1 \cup I_2}$ is $2\gamma$-uniform. As in the proof of Lemma 3.4, there exists some $j_0 \in I_2$ such that $\theta_{j_0} \notin A_{6sq_n - 1, \ln \lambda - 3\gamma}$. Thus $y$ is $(\ln \lambda - 3\gamma, 6sq_n)$-regular. By block expansion (Theorem 3.3 with $y_1 = 0$, $y_2 = 2\ell$, $\tau = \ln \lambda - 3\gamma$), we get

$$|\phi(\ell)|, |\phi(\ell - 1)| \leq e^{-(\ln \lambda - C\gamma)\ell},$$

This implies the right inequality of (7).

*Subcase (ii):* $\eta \geq \gamma$. By the definition of $\delta(\alpha, \theta)$ and the fact that $\delta(\alpha, \theta) < \ln \lambda$, we must have

$$\frac{\gamma}{\ln \lambda}\ell \leq |x_0| \leq 2\ell. \tag{46}$$

Applying Lemma 3.8 with $\mathcal{K} = x_0$ to the generalized eigenfunction $\phi(k)$, we have

$$\|U(x_0)\| = \|U^\phi(x_0)\| \leq e^{-(\ln \lambda - \varepsilon)|x_0|} e^{\eta\ell}. \tag{47}$$

Applying Lemma 3.4 with $y = \ell$, $k = 2\ell$, $C = 2$, $y' = 2\ell$, $k_0 = x_0$, $\varphi = \phi$, considering $\ell > x_0$ and $\ell \leq x_0$ separately, and using (47), we obtain Theorem 3.1.

**Remark 3.9.** By (20), we have

$$\|U(\ell)\| \geq \|A_\ell\|^{-1}\|U(0)\| \geq e^{-(\ln\lambda+\varepsilon)\ell}.$$

This already implies the left inequality of (7), except for Subcase (ii).

## 4. Palindromic arguments

### *4.1. Singular continuous spectrum*

We first show that if $0 < \ln|\lambda| < \delta(\alpha,\theta)$, then $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum, which is the second part of Theorem 1.1:

**Theorem 4.1.** *Let $H_{\lambda,\alpha,\theta}$ be an almost Mathieu operator with $|\lambda| > 1$. For any irrational number $\alpha$ and $\theta \in \mathbb{R}$, define $\delta(\alpha,\theta) \in [0,\infty]$ by (2). Then $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if* $\ln|\lambda| < \delta(\alpha,\theta)$.

Actually, we can prove a more general result.

**Theorem 4.2.** *Let $H_{v,\alpha,\theta}$ be a discrete Schrödinger operator,*

$$(H_{v,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + v(\theta+n\alpha)u(n),$$

*where $v : \mathbb{T} \to \mathbb{R}$ is an even Lipschitz continuous function. For any irrational number $\alpha$ and $\theta \in \mathbb{R}$, define $\delta(\alpha,\theta) \in [0,\infty]$ by (2). Then $H_{v,\alpha,\theta}$ has no eigenvalues in the regime $\{E \in \mathbb{R} : L(E) < \delta(\alpha,\theta)\}$, where $L(E)$ is the Lyapunov exponent.*

Theorem 4.1 follows directly from Theorem 4.2, Lemma 2.1 and Kotani theory [39,41].

By the definition of $\delta(\alpha,\theta)$, for any $\varepsilon > 0$ there exists a sequence $\{k_i\}_{i=1}^{\infty}$ such that

$$\|2\theta + k_i\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-(\delta-\varepsilon)|k_i|}. \tag{48}$$

Without loss of generality assume $k_i > 0$.

*Proof of Theorem 4.2.* Suppose not and let $u$ be an $\ell^2(\mathbb{Z})$ solution, i.e., $H_{v,\alpha,\theta}u = Eu$, with $L(E) < \delta(\alpha,\theta)$. Without loss of generality assume

$$\|u\|_{\ell^2}^2 = \sum_n |u(n)|^2 = 1.$$

We let $u_i(n) = u(k_i - n)$, $V(n) = v(\theta+n\alpha)$ and $V_i(n) = v(\theta+(k_i-n)\alpha)$. Then by (48), evenness and Lipschitz continuity of $v$ one has, for all $n \in \mathbb{Z}$,

$$|V(n) - V_i(n)| \leq Ce^{-(\delta-\varepsilon)|k_i|}. \tag{49}$$

We also have

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n), \tag{50}$$

$$u_i(n+1) + u_i(n-1) + V_i(n)u_i(n) = Eu_i(n). \tag{51}$$

Let $W(n) = W(f,g) = f(n+1)g(n) - f(n)g(n+1)$ be the Wronskian, as usual, and let

$$\Phi(n) = \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}, \quad \Phi_i(n) = \begin{pmatrix} u_i(n) \\ u_i(n-1) \end{pmatrix}.$$

By a standard calculation using (49)–(51), we have

$$\begin{aligned} |W(u,u_i)(n) - W(u,u_i)(n-1)| &\leq |V(n) - V_i(n)|\,|u(n)u_i(n)| \\ &\leq Ce^{-(\delta-\varepsilon)|k_i|}|u(n)u_i(n)|. \end{aligned}$$

This implies, for any $m > 0$ and $n$,

$$\begin{aligned} |W(u,u_i)(n+m) - W(u,u_i)(n-1)| &\leq Ce^{-(\delta-\varepsilon)|k_i|}\sum_{j=0}^{m-1}|u(n+j)u_i(n+j)| \\ &\leq Ce^{-(\delta-\varepsilon)|k_i|}, \end{aligned} \tag{52}$$

where the second inequality holds because $\|u\|_{\ell^2} = \|u_i\|_{\ell^2} = 1$.

Notice that $\sum_n |W(u,u_i)(n)| \leq 2$. Thus for some $n$,

$$|W(u,u_i)(n)| \leq Ce^{-(\delta-\varepsilon)|k_i|}.$$

By (52), we must have

$$|W(u,u_i)(n)| \leq Ce^{-(\delta-\varepsilon)|k_i|} \tag{53}$$

for all $n$.

Now we split the discussion into the cases of $k_i$ odd or even.

*Case 1:* $k_i$ is even. Let $m_i = k_i/2$. Then

$$\Phi(m_i) = \begin{pmatrix} u(m_i) \\ u(m_i-1) \end{pmatrix}, \quad \Phi_i(m_i) = \begin{pmatrix} u(m_i) \\ u(m_i+1) \end{pmatrix}.$$

Applying (53) with $n = m_i - 1$, we have

$$|u(m_i)|\,|u(m_i+1) - u(m_i-1)| \leq Ce^{-(\delta-\varepsilon)|k_i|}.$$

This implies

$$|u(m_i)| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|} \tag{54}$$

or

$$|u(m_i+1) - u(m_i-1)| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|}. \tag{55}$$

If (54) holds, by (50) we also have

$$|u(m_i+1) + u(m_i-1)| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|}. \tag{56}$$

Putting (54) and (56) together, we get

$$\|\Phi(m_i) + \Phi_i(m_i)\| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|}. \tag{57}$$

If (55) holds, we have

$$\|\Phi(m_i) - \Phi_i(m_i)\| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|}. \tag{58}$$

Thus in Case 1 there exists $\iota \in \{-1, 1\}$ such that

$$\|\Phi(m_i) + \iota\Phi_i(m_i)\| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|}.$$

Let $T_i^1$ and $T_i^2$ be the transfer matrices with potentials $V$ and $V_i$ respectively, taking $\Phi(m_i), \Phi_i(m_i)$ to $\Phi(0), \Phi_i(0)$.

By (20), (49), the usual uniform upper semicontinuity and telescoping, one has

$$\|T_i^1\|, \|T_i^2\| \leq Ce^{(L(E)+\varepsilon)m_i}, \quad \|T_i^1 - T_i^2\| \leq Ce^{(L(E)-2\delta+\varepsilon)m_i}.$$

Then

$$\begin{aligned}
\|\Phi(0) + \iota\Phi_i(0)\| &= \|T_i^1\Phi(m_i) + \iota T_i^2\Phi_i(m_i)\| \\
&= \|T_i^1\Phi(m_i) + \iota T_i^1\Phi_i(m_i) - \iota T_i^1\Phi_i(m_i) + \iota T_i^2\Phi_i(m_i)\| \\
&\leq \|T_i^1\|\|\Phi(m_i) + \iota\Phi_i(m_i)\| + \|T_i^1 - T_i^2\|\,\|\Phi_i(m_i)\| \\
&\leq e^{-(\delta-L(E)-\varepsilon)m_i} + e^{(L(E)-2\delta+\varepsilon)m_i} \\
&\leq 2e^{-(\delta-L(E)-\varepsilon)m_i}.
\end{aligned} \tag{59}$$

This implies $\|\Phi(0)\| - \|\Phi(2m_i + 1)\| \to 0$, which is impossible because $u \in \ell^2(\mathbb{Z})$.

*Case 2:* $k_i$ is odd. Let $\tilde{m}_i = \frac{k_i-1}{2}$. Then

$$\Phi(\tilde{m}_i + 1) = \begin{pmatrix} u(\tilde{m}_i + 1) \\ u(\tilde{m}_i) \end{pmatrix}, \quad \Phi_i(\tilde{m}_i + 1) = \begin{pmatrix} u(\tilde{m}_i) \\ u(\tilde{m}_i + 1) \end{pmatrix}.$$

Applying (53) with $n = \tilde{m}_i$, we have

$$|u(\tilde{m}_i) + u(\tilde{m}_i + 1)|\,|u(\tilde{m}_i) - u(\tilde{m}_i + 1)| \leq Ce^{-(\delta-\varepsilon)|k_i|}.$$

This implies

$$|u(\tilde{m}_i) + u(\tilde{m}_i + 1)| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|}$$

or

$$|u(\tilde{m}_i + 1) - u(\tilde{m}_i)| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|}.$$

Thus in Case 2, there also exists $\iota \in \{-1, 1\}$ such that

$$\|\Phi(\tilde{m}_i + 1) + \iota\Phi_i(\tilde{m}_i + 1)\| \leq Ce^{-\frac{1}{2}(\delta-\varepsilon)|k_i|},$$

and by the arguments of Case 1, we can also get a contradiction. ■

### 4.2. *Lower bound on the eigenfunctions*

Now we turn to the proof of the left inequality in (7). Our key argument for the lower bound is

**Lemma 4.3.** *Suppose for some* $\mathcal{K} > 0$ *and* $0 < t < \ln\lambda$,

$$\|2\theta + \mathcal{K}\alpha\|_{\mathbb{R}/\mathbb{Z}} = e^{-t\mathcal{K}}.$$

*Then for any* $\varepsilon > 0$ *we must have, for large* $\mathcal{K}$,

$$\|U(\mathcal{K})\| \geq e^{-(\ln\lambda - t + \varepsilon)\mathcal{K}}. \tag{60}$$

*Proof.* We define $\hat{\phi}(n) = \phi(\mathcal{K} - n)$, $V(n) = 2\lambda\cos 2\pi(\theta + n\alpha)$ and $\hat{V}(n) = 2\lambda \cdot \cos 2\pi(\theta + (\mathcal{K} - n)\alpha)$. Then by the assumption one has, for all $n \in \mathbb{Z}$,

$$|V(n) - \hat{V}(n)| \leq Ce^{-t\mathcal{K}}. \tag{61}$$

We also have

$$\phi(n+1) + \phi(n-1) + V(n)u(n) = E\phi(n), \tag{62}$$

$$\hat{\phi}(n+1) + \hat{\phi}(n-1) + \hat{V}(n)\hat{\phi}(n) = E\hat{\phi}(n). \tag{63}$$

Let

$$\hat{U}(n) = \begin{pmatrix} \hat{\phi}(n) \\ \hat{\phi}(n-1) \end{pmatrix}.$$

Suppose for some small $\sigma > 0$,

$$\|U(\mathcal{K})\| \leq e^{-(\ln\lambda - t + \sigma)\mathcal{K}}.$$

By Lemma 3.4 and (37) ($k_0 = \mathcal{K}$, $y = n$, $y' = 2n$), for any $\mathcal{K} \leq |n| \leq C\mathcal{K}$ we have

$$\begin{aligned}\|U(n)\| &\leq e^{-|n-\mathcal{K}|\ln\lambda} e^{\varepsilon|n|}\|U(\mathcal{K})\| + e^{-(\ln\lambda - \varepsilon)|n|} \\ &\leq e^{-(\ln\lambda-\varepsilon)|n|} e^{(t-\sigma)\mathcal{K}}.\end{aligned}$$

By Lemma 3.4 again, for $|n| \leq \mathcal{K}$ we have

$$\begin{aligned}\|U(n)\| &\leq \max\{e^{-|n|\ln\lambda}, e^{-|n-\mathcal{K}|\ln\lambda}\|U(\mathcal{K})|\|\} e^{\varepsilon\mathcal{K}} + e^{-(\ln\lambda-\varepsilon)|n|} \\ &\leq e^{-|n|\ln\lambda} e^{\varepsilon\mathcal{K}} + e^{-(2\mathcal{K}-|n|)\ln\lambda} e^{(t-\sigma+\varepsilon)\mathcal{K}}.\end{aligned}$$

This implies, for $|n| \leq C\mathcal{K}$,

$$|\hat{\phi}(n)|\,|\phi(n)| = |\phi(\mathcal{K}-n)|\,|\phi(n)| \leq e^{-(\ln\lambda - t + \sigma - \varepsilon)\mathcal{K}} + e^{-(\ln\lambda-\varepsilon)\mathcal{K}}.$$

By a standard calculation using (61)–(63), for any $|n| \leq C|\mathcal{K}|$ we have

$$\begin{aligned}|W(\phi,\hat{\phi})(n) - W(\phi,\hat{\phi})(n-1)| &\leq |V(n) - \hat{V}(n)\|\phi(n)\hat{\phi}(n)| \\ &\leq e^{-t\mathcal{K}}|\phi(n)\hat{\phi}(n)| \leq e^{-(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}},\end{aligned}$$

where $\sigma' = \min\{\sigma, t\}$. This implies, for any $0 < m \leq C\mathcal{K}$ and $|n| \leq C\mathcal{K}$,

$$|W(\phi,\hat{\phi})(n+m) - W(\phi,\hat{\phi})(n-1)| \leq \sum_{j=0}^{m-1} e^{-(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}} \leq e^{-(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}. \tag{64}$$

By (28), for some $n_0 = C\mathcal{K}$ we must have

$$|\phi(n_0)|, |\phi(n_0-1)| \le e^{-(\ln\lambda-\delta-\varepsilon)n_0} \le e^{-(\ln\lambda+\sigma')\mathcal{K}}.$$

This implies

$$|W(\phi,\hat\phi)(n_0)| \le e^{-(\ln\lambda+\sigma')\mathcal{K}}.$$

Combining this with (64), we must have

$$|W(\phi,\hat\phi)(n)| \le e^{-(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}} \tag{65}$$

for all $|n| \le C\mathcal{K}$.

Now we split the discussion into the cases of odd or even $\mathcal{K}$.

*Case 1:* $\mathcal{K}$ is even. Let $m = \mathcal{K}/2$. Then

$$U(m) = \begin{pmatrix} \phi(m) \\ \phi(m-1) \end{pmatrix}, \quad \hat U(m) = \begin{pmatrix} \phi(m) \\ \phi(m+1) \end{pmatrix}.$$

Applying (65) with $n = m-1$, we have

$$|\phi(m)|\,|\phi(m+1)-\phi(m-1)| \le e^{-(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}.$$

This implies

$$|\phi(m)| \le e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}} \tag{66}$$

or

$$|\phi(m+1)-\phi(m-1)| \le e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}. \tag{67}$$

If (66) holds, by (62) we also have

$$|\phi(m+1)+\phi(m-1)| \le e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}. \tag{68}$$

Putting (66) and (68) together, we get

$$\|U(m)+\hat U(m)\| \le e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}. \tag{69}$$

If (67) holds, we have

$$\|U(m)-\hat U(m)\| \le e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}. \tag{70}$$

Thus in Case 1 there exists $\iota \in \{-1,1\}$ such that

$$\|U(m)+\iota\hat U(m)\| \le e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}.$$

Let $T$ and $\hat T$ be the transfer matrices associated to potentials $V$ and $\hat V$, taking $U(m), \hat U(m)$ to $U(0), \hat U(0)$ respectively.

By (20), (61), the usual uniform upper semicontinuity and telescoping, one has

$$\|T\|, \|\hat T\| \le e^{(\ln\lambda+\varepsilon)m}. \quad \|T-\hat T\| \le e^{(\ln\lambda-2t+\varepsilon)m}.$$

By the right inequality of (7) ($\ell = m, x_0 = \mathcal{K}$), it is easy to see that

$$\|\hat{U}(m)\| \leq e^{-(\ln\lambda-\varepsilon)m}. \tag{71}$$

Then, as in (59), we have

$$\begin{aligned}\|U(0) + \iota\hat{U}(0)\| &\leq \|T\|\,\|U(m) + \iota\hat{U}(m)\| + \|T - \hat{T}\|\,\|\hat{U}(m)\| \\ &\leq e^{(\ln\lambda+\varepsilon)m} e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}} + e^{(\ln\lambda-2t+\varepsilon)m} e^{-m\ln\lambda}.\end{aligned}$$

This implies $\|U(0)\| - \|U(2m+1)\| \to 0$, which is impossible because $\phi \in \ell^2(\mathbb{Z})$.

*Case 2:* $\mathcal{K}$ is odd. Let $\tilde{m} = \frac{\mathcal{K}-1}{2}$. Then

$$U(\tilde{m}+1) = \begin{pmatrix}\phi(\tilde{m}+1)\\ \phi(\tilde{m})\end{pmatrix}, \quad \hat{U}(\tilde{m}+1) = \begin{pmatrix}\phi(\tilde{m})\\ \phi(\tilde{m}+1)\end{pmatrix}.$$

Combining this with (65), we have

$$|\phi(\tilde{m}) + \phi(\tilde{m}+1)|\,|\phi(\tilde{m}) - \phi(\tilde{m}+1)| \leq e^{-(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}.$$

This implies

$$|\phi(\tilde{m}) + \phi(\tilde{m}+1)| \leq e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}},$$

or

$$|\phi(\tilde{m}+1) - \phi(\tilde{m})| \leq e^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}.$$

Thus in Case 2, there also exists $\iota \in \{-1, 1\}$ such that

$$\|U(\tilde{m}+1) + \iota\hat{U}(\tilde{m}+1)\| \leq Ce^{-\frac{1}{2}(\ln\lambda+\sigma'-\varepsilon)\mathcal{K}}.$$

As before, we also get a contradiction. ∎

*Proof of the left inequality of* (7). The left inequality of (7) already follows except for Subcase (ii) in the proof of Theorem 3.1, by Remark 3.9.

Thus we only need to consider the case when $\eta \geq \gamma = \varepsilon/C$. Letting $t = \eta|\ell|/|x_0|$ and $\mathcal{K} = x_0$ in Lemma 4.3, we obtain

$$\|U(x_0)\| \geq e^{-(\ln\lambda+\varepsilon)|x_0|} e^{\eta|\ell|}.$$

Together with (37), this completes the proof. ∎

## 5. Universal reflective hierarchical structure

We first present the local version of Theorem 1.2. The definition of $f(\ell)$ in Theorem 1.2 depends on $\theta$ and $\alpha$. Thus sometimes we will write $f_{\alpha,\theta}(\ell)$ to make clear what $\theta$ is used.

**Theorem 5.1.** *Fix $\delta$ with $0 < \delta < \ln \lambda$. Suppose $\alpha$ is Diophantine. Let $\varepsilon > 0$ be small enough. Then there exists $L_0 = L_0(\lambda, \alpha, \delta, \hat{C})$* [8] *such that if for all $k$ with $L_1 \le |k| \le C|\ell|$ we have*

$$\|2\theta + 2s_0\alpha + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge e^{-(\delta+\varepsilon)|k|}, \tag{72}$$

*and the solution of $H\phi = E\phi$ satisfies (6) for all $k$ with $|k - s_0| \le C|\ell|$ and $\|U(s_0)\| = 1$, where $C = C(\alpha, \delta, \lambda)$ is a large constant and $L_0 \le L_1 \le |\ell|/C$, then the following statement holds:*

*Let $x_0$ (we can choose any one if $x_0$ is not unique) be such that*

$$|\sin \pi(2\theta + 2s_0\alpha + x_0\alpha)| = \min_{|x| \le 2|\ell|} |\sin \pi(2\theta + 2s_0\alpha + x\alpha)|.$$

*Then if $|x_0| \ge L_1$, we have*

$$f_{\alpha,\theta+s_0\alpha}(\ell)e^{-\varepsilon|\ell|} \le \|U(\ell)\| \le f_{\alpha,\theta+s_0\alpha}(\ell)e^{\varepsilon|\ell|}. \tag{73}$$

*If $|x_0| \le L_1$, we have*

$$e^{-\ln \lambda|\ell|}e^{-\varepsilon|\ell|} \le \|U(\ell)\| \le e^{-\ln \lambda|\ell|}e^{\varepsilon|\ell|}. \tag{74}$$

*Proof.* *Case 1: $|x_0| \ge L_1$.* In Sections 3 and 4, we completed the proof of Theorem 1.2. It is immediate that if we shift the operator by $s_0$ units[9] and replace the definition of the generalized eigenfunctions $\phi$ with the assumption of (6) only on the scale $C|\ell|$, our arguments will hold for (73) directly. In order to avoid repetition, we omit the proof.

*Case 2: $|x_0| \le L_1$.* In this case (74) follows directly from Lemma 3.4 by shifting the operator by $s_0$ units. ■

**Remark 5.2.** In order to obtain (73), we only need condition (72) for $|\ell|/C \le |k| \le C|\ell|$ and condition (6) for $|k| \le C|\ell|$. Moreover, if we assume that condition (72) also holds for $|k| \le L_1$, then (73) holds in both cases.

We will now prove Theorem 1.3.

**Theorem 5.3.** *Fix $\varsigma_1 > 0$, $0 < \delta < \ln \lambda$ and $s_0 \in \mathbb{Z}$. Then there exists a constant $L_0 = L_0(\alpha, \lambda, \delta, \varsigma_1)$ such that the following statement holds. Let $L_1 \ge L_0$. Suppose $K$ satisfies $|K| \ge CL_1$ and*

$$\|2\theta + 2s_0\alpha + K\alpha\|_{\mathbb{R}/\mathbb{Z}} \le e^{-\varsigma_1|K|}, \tag{75}$$

*and for all $k$ with $L_1 \le |k| \le C|K|$,*

$$\|2\theta + 2s_0\alpha + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge e^{-(\delta+\varepsilon)|k|}, \tag{76}$$

[8] We omit the dependence on $\varepsilon$ whenever $\varepsilon$ is (implicitly) present in the statement.

[9] Given $s_0 \in \mathbb{Z}$ and an operator $H_1$ on $\ell^2(\mathbb{Z})$, we call $H_2 = U^{-1}H_1U$ the $s_0$ shift of $H_1$, where $U$ is the unitary operator on $\ell^2(\mathbb{Z})$ given by $(Uf)(n) = f(n - s_0)$, $f \in \ell^2(\mathbb{Z})$.

*and $s_0$ is a $CK$-local maximum, where $C = C(\alpha, \lambda, \delta, \varsigma_1)$ is a large constant. Then there exists a $\frac{3\varsigma_1}{4\ln\lambda}K$-local maximum*[10] *$b_K$ such that*

$$|b_K - K - s_0| \le 2L_1. \tag{77}$$

*Proof.* By shifting the operator, we can assume $s_0 = 0$. Let $\epsilon$ be such that

$$\|2\theta + 2s_0\alpha + K\alpha\|_{\mathbb{R}/\mathbb{Z}} = e^{-\epsilon|K|}.$$

Then $\varsigma_1 \le \epsilon \le \delta + \varepsilon$.

By Theorem 5.1[11] with $\ell = x_0 = K$, one has

$$e^{-(\ln\lambda - \epsilon + \varepsilon)|K|} \le \frac{\|U(s_0 + K)\|}{\|U(s_0)\|} \le e^{-(\ln\lambda - \epsilon - \varepsilon)|K|}. \tag{78}$$

By Theorem 5.1 again, one has

$$\sup_{|k|\le\varepsilon|K|} \|U(K+k)\| = \sup_{|k|\le\frac{3\varsigma_1}{4\ln\lambda}|K|} \|U(K+k)\|. \tag{79}$$

Thus there exists a $\frac{3\varsigma_1}{4\ln\lambda}|K|$-local maximum $b_K$ such that

$$|b_K - K| \le \varepsilon|K|. \tag{80}$$

Suppose (77) does not hold. Then there exists $k_0$ with $2L_1 \le |k_0| \le \varepsilon K$ such that

$$\|U(K+k_0)\| = \sup_{|k|\le\varepsilon|K|} \|U(K+k)\| = \sup_{|k|\le\frac{3\varsigma_1}{4\ln\lambda}|K|} \|U(K+k)\|. \tag{81}$$

where $L_1$ is such that (75) and (76) hold.

*Case 1:* $\min_{|k|\le 2|k_0|} \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge e^{-\varepsilon|k_0|}$. Let $\frac{p_n}{q_n}$ be the $n$th convergent of the continued fraction expansion of $\alpha$. For $\gamma > 0$ (we will let $\gamma = \varepsilon/C$), let $n$ be the largest integer such that

$$2q_n \le \gamma|k_0|,$$

and let $s$ be the largest positive integer such that $2sq_n \le \gamma|k_0|$.

We define intervals $I_1 = [sq_n, sq_n - 1]$ and $I_2 = [K + k_0 - sq_n, K + k_0 + sq_n - 1]$.

**Claim 1.** *We have*

$$\min_{i,i'\in I_1\cup I_2} \|2\theta + (i + i')\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge e^{-\varepsilon|k_0|} \tag{82}$$

*and for any distinct $i, i' \in I_1 \cup I_2$,*

$$\|(i - i')\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge e^{-\varepsilon|k_0|}. \tag{83}$$

[10] $3/4$ can be replaced with $1 - \varepsilon$ for any $\varepsilon > 0$.

[11] $s_0$ is a local maximum so that $\hat{C}$ in (6) is 1, thus the largeness in Theorem 5.1 does not depend on $\hat{C}$.

By Theorem 3.2 and the DC condition on $\alpha$, $\{\theta_i\}_{i\in I_1\cup I_2}$ is $\varepsilon$-uniform. In view of Lemma 2.4, there exists some $i_0$ with $i_0 \in I_1 \cup I_2$ such that $\theta_{i_0} \notin A_{4sq_n-1,\ln\lambda-\varepsilon}$. By Lemma 3.6, $i_0$ cannot be in $I_1$ so must be in $I_2$. Set $I = [i_0 - 2sq_n + 1, i_0 + 2sq_n - 1] = [x_1, x_2]$. By (22)–(24) again, one has

$$|G_I(K + k_0, x_i)| \le e^{(\ln\lambda+\varepsilon)(4sq_n-1-|K_j+k_0-x_i|)-(4sq_n-1)(\ln\lambda-\varepsilon)} \le e^{\varepsilon sq_n} e^{-|K+k_0-x_i|\ln\lambda}.$$

Notice that $|K + k_0 - x_1|, |K + k_0 - x_2| \ge sq_n - 1$. By (25) and (81),

$$|\phi(K + k_0)| \le e^{-(\ln\lambda-\varepsilon)sq_n}(|\phi(x_1)| + |\phi(x_0)|) \le e^{-(\ln\lambda-\varepsilon)sq_n}\|U(K + k_0)\|.$$

Similarly,

$$|\phi(K + k_0 - 1)| \le e^{-(\ln\lambda-\varepsilon)sq_n}\|U(K + k_0)\|.$$

The last two inequalities imply that

$$\|U(K + k_0)\| \le e^{-(\ln\lambda-\varepsilon)sq_n}\|U(K + k_0)\|. \tag{84}$$

Since $2(s + 1)q_n \ge \gamma|k_0|$ and $|k_0| \ge 2L_1$, (84) is impossible.

*Case 2:* $\min_{|k|\le 2|k_0|}\|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \le e^{-\varepsilon|k_0|}$ for some $\varepsilon > 0$. In this case we define, as before, intervals $I_1$ around 0 and $I_2$ around $K + k_0$.

Suppose $i \in I_1$. For $i' \in I_2$, we have

$$\begin{aligned}\|2\theta + (i + i')\alpha\|_{\mathbb{R}/\mathbb{Z}} &\ge \|(i + i' - K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta + K\alpha\|_{\mathbb{R}/\mathbb{Z}}\\ &\ge \|(i + i' - K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - e^{-\epsilon|K|}\end{aligned} \tag{85}$$

and

$$\begin{aligned}\|(i - i')\alpha\|_{\mathbb{R}/\mathbb{Z}} &\ge \|2\theta + (i - i' + K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|-2\theta - K\alpha\|_{\mathbb{R}/\mathbb{Z}}\\ &\ge \|2\theta + (i - i' + K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - e^{-\epsilon|K|}.\end{aligned} \tag{86}$$

Suppose $i \in I_2$. For $i' \in I_1$, we have

$$\begin{aligned}\|2\theta + (i + i')\alpha\|_{\mathbb{R}/\mathbb{Z}} &\ge \|(i - K + i')\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta + K\alpha\|_{\mathbb{R}/\mathbb{Z}}\\ &\ge \|(i - K + i')\alpha\|_{\mathbb{R}/\mathbb{Z}} - e^{-\epsilon|K|}\end{aligned} \tag{87}$$

and

$$\begin{aligned}\|(i - i')\alpha\|_{\mathbb{R}/\mathbb{Z}} &\ge \|2\theta - (i - K - i')\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|2\theta + K\alpha\|_{\mathbb{R}/\mathbb{Z}}\\ &\ge \|2\theta - (i - K - i')\alpha\|_{\mathbb{R}/\mathbb{Z}} - e^{-\epsilon|K|}.\end{aligned} \tag{88}$$

For $i' \in I_2$, we have

$$\begin{aligned}\|2\theta + (i + i')\alpha\|_{\mathbb{R}/\mathbb{Z}} &\ge \|-2\theta + (i - K + i' - K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|4\theta + 2K\alpha\|_{\mathbb{R}/\mathbb{Z}}\\ &\ge \|2\theta - (i - K + i' - K)\alpha\|_{\mathbb{R}/\mathbb{Z}} - 2e^{-\epsilon|K|}\end{aligned} \tag{89}$$

and

$$\|(i-i')\alpha\|_{\mathbb{R}/\mathbb{Z}} = \|(i-K-(i'-K))\alpha\|_{\mathbb{R}/\mathbb{Z}}. \tag{90}$$

Conditions (85)–(90) imply that the small divisor conditions on $\theta_i+\theta_{i'}$ and $\theta_i-\theta_{i'}$ get swapped upon shifting the elements in $I_2$ by $K$ units.

Let $|x_0|\leq 2|k_0|$ be such that $\|2\theta+x_0\alpha\|_{\mathbb{R}/\mathbb{Z}}\leq e^{-\varepsilon|k_0|}$.

*Case 2.1:* $|k_0+x_0|\geq\varepsilon|k_0|$. In this case let $[x_1,x_2]=[K+k_0-\varepsilon|k_0|,K+k_0+\varepsilon|k_0|]$. By the small divisor conditions (85)–(90) and following the proof of (33), and (38), we get

$$\begin{aligned}\|U(K+k_0)\| &\leq e^{-(\ln\lambda-\varepsilon)|x_1-K-k_0|}\|U(x_1)\|+e^{-(\ln\lambda-\varepsilon)|x_2-K-k_0|}\|U(x_2)\|\\ &\leq e^{-(\ln\lambda-\varepsilon)\varepsilon|k_0|}\|U(x_1)\|+e^{-(\ln\lambda-\varepsilon)\varepsilon|k_0|}\|U(x_2)\|\\ &\leq e^{-(\ln\lambda-\varepsilon)\varepsilon|k_0|}\|U(K+k_0)\|,\end{aligned} \tag{91}$$

where the third inequality holds because $K+k_0$ is the local maximum. (91) is also impossible for $|k_0|\geq 2L_1$.

*Case 2.2:* $|k_0+x_0|\leq\varepsilon|k_0|$. In this case, $|x_0|\geq\frac{1}{2}|k_0|\geq L_1$ so that condition (76) holds for all $|k|\geq|x_0|$. By the small divisor conditions (85)–(90) again, and following the proof of (40), we get (using (81))

$$\|U(K+k_0)\|\leq\|U(K+k_0)\|e^{-(\ln\lambda-\delta-\varepsilon)|k_0|}.$$

This is also impossible. ■

*Proof of Claim 1.* Without loss of generality assume $i\in I_1$. For $i'\in I_2$, by the DC condition on $\alpha$ we have

$$\|2\theta+(i+i')\alpha\|_{\mathbb{R}/\mathbb{Z}}\geq\|(i+i'-K)\alpha\|_{\mathbb{R}/\mathbb{Z}}-\|2\theta+K\alpha\|_{\mathbb{R}/\mathbb{Z}}\geq e^{-\varepsilon|k_0|}$$

and

$$\|(i-i')\alpha\|_{\mathbb{R}/\mathbb{Z}}\geq\|2\theta+(i-i'+K)\alpha\|_{\mathbb{R}/\mathbb{Z}}-\|-2\theta-K\alpha\|_{\mathbb{R}/\mathbb{Z}}\geq e^{-\varepsilon|k_0|}.$$

For $i'\in I_1$, the proof is trivial. ■

*Proof of Theorem 1.3.* Without loss of generality, assume $k_0=0$. Let $\hat{K}=L_0(\alpha,\lambda,\delta,\varsigma)$ in Theorem 5.3.

By Theorem 5.3 with $s_0=0$, $K=K_{j_0}$, $\varsigma_1=\varsigma$ and $L_1=\hat{K}$, there exists a local $\frac{3\varsigma}{4\ln\lambda}K_{j_0}$-maximum $b_{j_0}$ such that $|b_{j_0}-K_{j_0}|\leq 2\hat{K}$. Let $b_{j_0}-K_{j_0}=b'_{j_0}$ with $|b'_{j_0}|\leq 2\hat{K}$.

Shifting the operator $H_{\lambda,\alpha,\theta}$ by $b_{j_0}$ units, we get $H_{\lambda,\alpha,\theta+b_{j_0}\alpha}$. By the conditions of Theorem 1.3, $\zeta<\delta+\varepsilon<\ln\lambda$, we have

$$\begin{aligned}\|2(\theta+b_{j_0}\alpha)+k\alpha\|_{\mathbb{R}/\mathbb{Z}} &\geq\|2\theta-(2b'_{j_0}\alpha+k\alpha)\|_{\mathbb{R}/\mathbb{Z}}-\|4\theta+2K_{j_0}\alpha\|_{\mathbb{R}/\mathbb{Z}}\\ &\geq\|2\theta-(2b'_{j_0}\alpha+k\alpha)\|_{\mathbb{R}/\mathbb{Z}}-2e^{-(\varsigma+\varepsilon)|K_{j_0}|}\\ &\geq e^{-(\delta+\varepsilon)(|k|+4\hat{K})}\geq e^{-(\delta+\varepsilon)|k|}\end{aligned} \tag{92}$$

for all $\frac{1}{2}\hat{K}^2 \le |k| \le \frac{\varsigma}{\ln\lambda}|K_{j_0}|$. Similarly,

$$\begin{aligned}\|2(\theta + b_{j_0}\alpha) + (-K_{j_1} - 2b'_{j_0})\alpha\|_{\mathbb{R}/\mathbb{Z}} &\le \|2\theta + K_{j_1}\alpha\|_{\mathbb{R}/\mathbb{Z}} + \|4\theta + 2K_{j_0}\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\le \|2\theta + K_{j_1}\alpha\|_{\mathbb{R}/\mathbb{Z}} - 2e^{-(\varsigma+\varepsilon)|K_{j_0}|} \\ &\le e^{-\frac{3}{4}\varsigma|-K_{j_1}-2b'_{j_0}|}. \end{aligned} \tag{93}$$

By Theorem 5.3 with $s_0 = b_{j_0}$, $K = -K_{j_1} - 2b'_{j_0}$, $\varsigma_1 = \frac{3}{4}\varsigma$ and $L_1 = \frac{1}{2}\hat{K}^2$, there exists a local $\frac{9\varsigma}{16\ln\lambda}K_{j-1}$-maximum $b_{K_{j_0},K_{j_1}}$ such that

$$|b_{j_0,j_1} - b_{j_0} - (-K_{j_1} - 2b'_{j_0})| \le \hat{K}^2.$$

This implies $b_{j_0,j_1} = K_{j_0} - b'_{j_0} - K_{j_1} + b'_{j_1}$ with $|b'_{j_1}| \le \hat{K}^2$.

Shifting the operator $H_{\lambda,\alpha,\theta}$ by $b_{j_0,j_1}$ units, we get $H_{\lambda,\alpha,\theta+b_{j_0,j_1}\alpha}$. Thus

$$\begin{aligned}&\|2(\theta + b_{j_0,j_1}\alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\quad\ge \|2\theta - 2b'_{j_0}\alpha + 2b'_{j_1}\alpha + k\alpha)\|_{\mathbb{R}/\mathbb{Z}} - 2\|2\theta + K_{j_0}\alpha\|_{\mathbb{R}/\mathbb{Z}} - 2\|2\theta + K_{j_1}\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\quad\ge \|2\theta + (-2b'_{K_{j_0}} + 2b'_{j_1} + k)\alpha)\|_{\mathbb{R}/\mathbb{Z}} - 4e^{-(\varsigma+\varepsilon)|K_{j_1}|} \\ &\quad\ge e^{-(\delta+\varepsilon)(|k|+2\hat{K}+2\hat{K}^2)} \ge e^{-(\delta+\varepsilon)|k|} \end{aligned} \tag{94}$$

for all $\frac{1}{2}(\hat{K} + \hat{K}^2)\hat{K} \le |k| \le \frac{\varsigma}{\ln\lambda}|K_{j-1}|$. Similarly,

$$\begin{aligned}\|2(\theta + b_{j_0,j_1}\alpha) + (K_{j-2} + 2b'_{K_j} - 2b'_{j_1})\alpha\|_{\mathbb{R}/\mathbb{Z}} &\le \|2\theta + K_{j_2}\alpha\|_{\mathbb{R}/\mathbb{Z}} + 2e^{-(\varsigma+\varepsilon)|K_{j_1}|} \\ &\le e^{-\frac{3}{4}\varsigma|K_{j_2}+2b'_{j_0}-2b'_{j_1}|}. \end{aligned} \tag{95}$$

By Theorem 5.3 with $s_0 = b_{j_0,j_1}$, $K = K_{j_2} + 2b'_{j_0} - 2b'_{j_1}$, $\varsigma_1 = \frac{3}{4}\varsigma$ and $L_1 = \frac{1}{2}(\hat{K}^2 + \hat{K}^3)$, there exists a local $\frac{9\varsigma}{16\ln\lambda}K_{j_2}$-maximum $b_{j_0,j_1,j_2}$ such that

$$b_{j_0,j_1,j_2} = K_{j_0} + b'_{j_0} - K_{j_1} - b'_{j_1} + K_{j_2} + b'_{j_2}$$

with $|b'_{j_2}| \le \hat{K}^2 + \hat{K}^3$.

Define $a_n = \hat{K}^2(\hat{K} + 1)^{n-2}$ for $n \ge 2$ and $a_1 = \hat{K}$. Then $a_n = \hat{K}\sum_{i=1}^{n-1} a_i$. Notice that by (11),

$$\sum_{i=0}^{s} \|2\theta + K_{j_i}\alpha\|_{\mathbb{R}/\mathbb{Z}} \le \sum_{i=0}^{s} e^{-(\varsigma+\varepsilon)|K_{j_i}|} \le 2e^{-(\varsigma+\varepsilon)|K_{j_s}|}. \tag{96}$$

We will prove that for any $1 \le s \le k$ there exists a local $\frac{9\varsigma}{16\ln\lambda}K_{j_s}$-maximum $b_{j_0,j_1,\ldots,j_s}$ such that

$$b_{j_0,j_1,\ldots,j_s} = \sum_{i=0}^{s} [(-1)^i K_{j_i} + (-1)^{i-s} b'_{j_i}] \tag{97}$$

with $|b'_{j_i}| \le a_{i+1}$ by induction on $s$.

Assume that (97) holds for $s$. We will prove that it holds for $s + 1$.

Shifting the operator $H_{\lambda,\alpha,\theta}$ by $b_{j_0,j_1,\ldots,j_s}$ units, we get $H_{\lambda,\alpha,\theta+b_{j_0,j_1,\ldots,j_s}\alpha}$. Arguing as in (94) we have

$$\begin{aligned}
&\|2(\theta + b_{j_0,j_1,\ldots,j_s}\alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \\
&\quad \geq \left\|2\theta + \Big(2\sum_{i=0}^{s}(-1)^{i+1}b'_{j_i}\Big)\alpha + (-1)^{s+1}k\alpha\right\|_{\mathbb{R}/\mathbb{Z}} - 2\sum_{i=0}^{s}\|2\theta + K_{j_i}\alpha\|_{\mathbb{R}/\mathbb{Z}} \\
&\quad \geq \left\|2\theta + \Big(2\sum_{i=0}^{s}(-1)^{i+1}b'_{j_i}\Big)\alpha + (-1)^{s+1}k\alpha\right\|_{\mathbb{R}/\mathbb{Z}} - 2\sum_{i=0}^{s}e^{-(\varsigma+\varepsilon)|K_{j_i}|} \qquad (98) \\
&\quad \geq e^{-(\delta+\varepsilon)(|k|+2\sum_{i=1}^{s+1}a_i)} \\
&\quad \geq e^{-(\delta+\varepsilon)|k|} \qquad (99)
\end{aligned}$$

for all $\frac{1}{2}a_{s+2} \leq |k| \leq \frac{\varsigma}{\ln\lambda}|K_{j_s}|$, since $\sum_{i=1}^{s+1}a_i = \frac{1}{\hat{K}}a_{s+2}$. Similarly to (93), we have

$$\begin{aligned}
&\left\|2(\theta + b_{j_0,j_1,\ldots,j_s}\alpha) + \Big((-1)^{s+1}K_{j_{s+1}} + 2\sum_{i=0}^{s}(-1)^{s+i+1}b'_{j_i}\Big)\alpha\right\|_{\mathbb{R}/\mathbb{Z}} \\
&\qquad \leq \|2\theta + K_{j_{s+1}}\alpha\|_{\mathbb{R}/\mathbb{Z}} + 4e^{-(\varsigma+\varepsilon)|K_{j_1}|} \\
&\qquad \leq e^{-\frac{3}{4}\varsigma|(-1)^{s+1}K_{j_{s+1}}+2\sum_{i=0}^{s}(-1)^{s+i+1}b'_{j_i}|}. \qquad (100)
\end{aligned}$$

By Theorem 5.3 with $s_0 = b_{j_0,j_1,\ldots,j_s}$, $K = (-1)^{s+1}K_{j_{s+1}} + 2\sum_{i=0}^{s}(-1)^{s+i+1}b'_{j_i}$, $\varsigma_1 = \frac{3}{4}\varsigma$ and $L_1 = \frac{1}{2}a_{s+2}$, there exists a local $\frac{9\varsigma}{16\ln\lambda}K_{j_{s+1}}$-maximum $b_{j_0,j_1,\ldots,j_{s+1}}$ such that

$$\begin{aligned}
b_{j_0,j_1,\ldots,j_{s+1}} &= b_{j_0,j_1,\ldots,j_s} + (-1)^{s+1}K_{j_{s+1}} + 2\sum_{i=0}^{s}(-1)^{s+i+1}b'_{j_i} + b'_{j_{s+1}} \\
&= \sum_{i=0}^{s-1}[(-1)^iK_{j_i} + (-1)^{i-s-1}b'_{j_i}]
\end{aligned}$$

with $|b'_{j_{s+1}}| \leq a_{s+2}$.

Since

$$\left|b_{j_0,j_1,\ldots,j_s} - \sum_{i=0}^{s}(-1)^iK_{j_i}\right| \leq \sum_{i=0}^{s}|b'_{j_i}| \leq \sum_{i=1}^{s+1}a_i \leq (\hat{K}+1)^{s+1},$$

the proof of item (I) of Theorem 1.3 is complete.

Now we start to prove (II). Fix some $0 \leq s \leq k$. Let us consider a local $\frac{9\varsigma}{16\ln\lambda}K_{j_s}$-maximum $b_{j_0,j_1,\ldots,j_s}$ and shift the operator by $b_{j_0,j_1,\ldots,j_s}$ units. We get the operator $H_{\lambda,\alpha,\theta+b_{j_0,j_1,\ldots,j_s}\alpha}$. As in (98), we also have

$$\begin{aligned}
&\|2(\theta + b_{j_0,j_1,\ldots,j_s}\alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} \\
&\quad \leq \left\|2\theta + \Big(2\sum_{i=0}^{s}(-1)^{i+1}b'_{j_i}\Big)\alpha + (-1)^{s+1}k\alpha\right\|_{\mathbb{R}/\mathbb{Z}} + 2\sum_{i=0}^{s}e^{-(\varsigma+\varepsilon)|K_{j_i}|} \qquad (101)
\end{aligned}$$

for all $a_{s+2} \leq |k| \leq \frac{\varsigma}{\ln\lambda}|K_{j_s}|$.

Actually, the definition of $f(\ell)$ in Theorems 1.2 and 5.1 depends on $\theta$ and $\alpha$. Thus we will use $f_{\alpha,\theta}(\ell)$ with $|\ell| \geq Ca_{s+2}$. Let $\ell_0$ be such that

$$|\sin \pi(2\theta + 2b_{j_0,j_1,\ldots,j_s}\alpha + \ell_0\alpha)| = \min_{|x|\leq 2|\ell|} |\sin \pi(2\theta + 2b_{j_0,j_1,\ldots,j_s}\alpha + x\alpha)|.$$

By (98) and (101), for $|\ell_0| \geq a_{s+2}$ we have

$$e^{-\varepsilon|\ell|} f_{\alpha,\theta}((-1)^{s+1}\ell) \leq f_{\alpha,\theta+b_{j_0,j_1,\ldots,j_s}\alpha}(\ell) \leq f_{\alpha,\theta}((-1)^{s+1}\ell)e^{\varepsilon|\ell|}. \tag{102}$$

If $|\ell_0| \leq a_{s+2}$, we have

$$e^{-\varepsilon|\ell|}e^{-\ln\lambda|\ell|} \leq f_{\alpha,\theta}(\ell) \leq e^{-\ln\lambda|\ell|}e^{\varepsilon|\ell|}, \tag{103}$$

since $|\ell| \geq Ca_{s+2}$.

Let $x_s = x - b_{j_0,j_1,\ldots,j_s}$. If $|x_s| \in [Ca_{s+2}, \frac{1}{C}\frac{\varsigma}{\ln\lambda}|K_{j_s}|]$, assertion (II) of Theorem 1.3 follows from Theorem 5.1 and (102) and (103).

If $|x_s| \in [\frac{1}{C}\frac{\varsigma}{\ln\lambda}|K_{j_s}|, \frac{\varsigma}{4\ln\lambda}|K_{j_s}|]$, (II) follows from Lemma 3.4 and the fact that $b_{j_0,j_1,\ldots,j_s}$ is a local $\frac{9\varsigma}{16\ln\lambda}K_{j_s}$-maximum. Notice that in this case

$$e^{-\varepsilon|x_s|}e^{-\ln\lambda|x_s|} \leq f_{\alpha,\theta}((-1)^{s+1}x_s) \leq e^{-\ln\lambda|x_s|}e^{\varepsilon|x_s|}. \qquad \blacksquare$$

## 6. Asymptotics of the transfer matrices

*Proof of Theorem* 1.5. Without loss of generality, we consider $\ell > 0$. First assume $x_0 < 0$ or $\eta \leq \gamma = \varepsilon/C$. By Theorem 1.2, in those cases, one has

$$\|U(\ell)\| \leq e^{-(\ln\lambda-\varepsilon)\ell}.$$

By (17), we have

$$\|A_\ell\| \geq \|U(\ell)\|^{-1} \geq e^{(\ln\lambda-\varepsilon)\ell}.$$

Combining this with (20), the conclusion follows.

Now we turn to the case when $x_0 > 0$ and $\eta > \gamma$. We will assume $\ell > 0$ is large enough. By (46), $x_0 > 0$ is large enough. Thus below we always assume $x_0$ is large.

**Theorem 6.1.** *Under the above assumptions, let $k$ be such that $jx_0 \leq k < (j+1)x_0$ with $k \geq x_0/8$, where $j = 0, 1$. Then*

$$\|A_k\| \leq \max\{e^{-|k-jx_0|\ln\lambda}\|A_{jx_0}\|, e^{-|k-(j+1)x_0|\ln\lambda}\|A_{(j+1)x_0}\|\}\, e^{\varepsilon k}, \tag{104}$$

$$\|A_k\| \geq \max\{e^{-|k-jx_0|\ln\lambda}\|A_{jx_0}\|, e^{-|k-(j+1)x_0|\ln\lambda}\|A_{(j+1)x_0}\|\}\, e^{-\varepsilon k}. \tag{105}$$

*Proof.* Apply (38) with $k_0 = x_0$, $y = k$ $y' = 2x_0$ and $\varphi = \psi$. For $jx_0 \leq k < (j+1)x_0$ with $k \geq x_0/8$, we have

$$\|\tilde{U}(k)\| \leq \max\{e^{-|k-jx_0|\ln\lambda}\|\tilde{U}(jx_0)\|, e^{-|k-(j+1)x_0|\ln\lambda}\|\tilde{U}((j+1)x_0)\|\}\, e^{\varepsilon k}. \tag{106}$$

By Last–Simon's arguments [41, (8.6)], one has

$$\|A_k\| \geq \|A_k \tilde{U}(0)\| \geq c\|A_k\|. \tag{107}$$

Then (104) holds by (107) and (106).

(105) holds directly by (20). ∎

**Lemma 6.2.** *For any* $2x_0 \leq k \leq Cx_0$,

$$e^{-\varepsilon x_0}\|A_{2x_0}\|e^{\ln\lambda|k-2x_0|} \leq \|A_k\| \leq e^{\varepsilon x_0}\|A_{2x_0}\|e^{\ln\lambda|k-2x_0|}.$$

*Proof.* The right inequality holds directly. It suffices to show the left inequality.

By (40) and noting $t \leq \delta + \varepsilon$, we have

$$\|\tilde{U}(x_0)\| \leq \max\{e^{-(\ln\lambda-\delta-\varepsilon)x_0}\|\tilde{U}(0)\|, e^{-(\ln\lambda-\delta-\varepsilon)x_0}\|\tilde{U}(2x_0)\|\}.$$

Clearly, $\|\tilde{U}(x_0)\| \leq e^{-(\ln\lambda-\delta-\varepsilon)x_0}\|\tilde{U}(0)\|$ cannot happen: otherwise, since $\|U(x_0)\| \leq e^{-(\ln\lambda-\delta-\varepsilon)x_0}\|U(0)\|$, we must have

$$|\phi(x_0)\psi(x_0-1) - \phi(x_0-1)\psi(x_0)| \leq e^{-(\ln\lambda-\delta-\varepsilon)x_0},$$

contrary to (29).

Thus we must have

$$\|\tilde{U}(x_0)\| \leq e^{-(\ln\lambda-\delta-\varepsilon)x_0}\|\tilde{U}(2x_0)\|. \tag{108}$$

The lemma holds directly if $k \leq 2x_0 + \frac{\varepsilon}{C}x_0$. If $k - 2x_0 \geq \frac{\varepsilon}{C}x_0$, by (38) again ($k_0 = x_0, y = 2x_0, y' = k, \gamma = \varepsilon/C$) one has

$$\|\tilde{U}(2x_0)\| \leq \max\{e^{-(\ln\lambda-\varepsilon)x_0}\|\tilde{U}(x_0)\|, e^{-(\ln\lambda-\varepsilon)|k-2x_0|}\|\tilde{U}(k)\|\}.$$

Combining this with (108), we must have

$$\|\tilde{U}(k)\| \geq e^{(\ln\lambda-\varepsilon)|k-2x_0|}\|\tilde{U}(2x_0)\|.$$

In view of (107), we get the left inequality. ∎

**Lemma 6.3.** *The following holds:*

$$e^{(\ln\lambda-\varepsilon)x_0} \leq \|A_{x_0}\| \leq e^{(\ln\lambda+\varepsilon)x_0}, \tag{109}$$

$$e^{(\ln\lambda-\varepsilon)2x_0}e^{-\eta\ell} \leq \|A_{2x_0}\| \leq e^{(\ln\lambda+\varepsilon)2x_0}e^{-\eta\ell}. \tag{110}$$

*Proof.* We first prove (109). The right inequality holds by (20) directly. Thus it suffices to show the left one. By (38), for any $x_0/8 \leq k < x_0$, one has

$$\|U(k)\| \leq \max\{e^{-k\ln\lambda}, e^{-|k-x_0|\ln\lambda}\|U(x_0)\|\}\, e^{\varepsilon k}.$$

Clearly

$$\|A_k\| \geq \|U(k)\|^{-1}, \tag{111}$$

so by (104) we must have, for any $x_0/8 \leq k < x_0$,

$$\max\{e^{-k\ln\lambda}, e^{-|k-x_0|\ln\lambda}\|A_{x_0}\|\}\, e^{\varepsilon k} \geq (\max\{e^{-k\ln\lambda}, e^{-|k-x_0|\ln\lambda}\|U(x_0)\|\})^{-1}\, e^{-\varepsilon k}. \tag{112}$$

Recall that by (47) and (60),

$$e^{-(\ln\lambda-\eta'+\varepsilon)x_0} \leq \|U(x_0)\| \leq e^{-(\ln\lambda-\eta'-\varepsilon)x_0}, \tag{113}$$

where $\eta' = \frac{\ell}{x_0}\eta$. Let

$$k_0 = x_0 - \frac{\eta'}{2\ln\lambda}x_0.$$

One has $k_0 \geq x_0/2$, so by (113),

$$\max\{e^{-k_0\ln\lambda}, e^{-|k_0-x_0|\ln\lambda}\|U(x_0)\|\} \leq e^{-(\ln\lambda-\eta'/2)x_0}e^{\varepsilon k_0}.$$

Combining this with (112), one has

$$\max\{e^{-k_0\ln\lambda}, e^{-|k_0-x_0|\ln\lambda}\|A_{x_0}\|\} \geq e^{(\ln\lambda-\eta'/2)x_0}e^{-\varepsilon k_0}.$$

This implies

$$\|A_{x_0}\| \geq e^{(\ln\lambda-\varepsilon)x_0}.$$

Now we prove (110). By (7) ($\ell = 2x_0$), one has

$$e^{-(\ln\lambda+\varepsilon)2x_0}e^{\eta' x_0} \leq \|U(2x_0)\| \leq e^{-(\ln\lambda-\varepsilon)2x_0}e^{\eta' x_0}. \tag{114}$$

Combining this with (111), one has

$$\|A_{2x_0}\| \geq e^{(\ln\lambda-\varepsilon)2x_0}e^{-\eta' x_0}.$$

Thus it remains to prove the right inequality of (110). By [41, (8.5) and (8.7)] we have

$$\|A_k U(0)\|^2 \leq \|A_k\|^2 m(k)^2 + \|A_k\|^{-2}, \tag{115}$$

where

$$m(k) \leq C\sum_{p=k}^{\infty}\frac{1}{\|A_p\|^2}. \tag{116}$$

If $k \geq Cx_0$ ($C$ may depend on $\ln\lambda, \delta$), by Theorem 1.2 we have

$$\|A_k\| \geq \|U(k)\|^{-1} \geq e^{(\ln\lambda-\delta-\varepsilon)k} \tag{117}$$

and by (20) we have

$$\|A_{2x_0}\| \leq e^{(\ln\lambda+\varepsilon)2x_0}.$$

Combining this with (117), we obtain

$$\|A_k\| \geq \|A_{2x_0}\|e^{\frac{\ln\lambda-\delta}{2}k} \tag{118}$$

for $k \geq Cx_0$, where $C$ is large enough.

If $2x_0 \leq k \leq Cx_0$, by Lemma 6.2 we have

$$\|A_k\| \geq \|A_{2x_0}\| e^{(\ln\lambda-\varepsilon)|k-2x_0|} e^{-\varepsilon x_0}. \tag{119}$$

Thus by (116), (118) and (119),

$$m(2x_0) \leq \|A_{2x_0}\|^{-2} e^{\varepsilon x_0}. \tag{120}$$

Let $k = 2x_0$ in (115). Then

$$\|U(2x_0)\| \leq \frac{e^{\varepsilon x_0}}{\|A_{2x_0}\|}.$$

Thus by (114), we obtain

$$\|A_{2x_0}\| \leq e^{(2\ln\lambda-\eta'-\varepsilon)x_0}.$$ ■

Theorem 1.5 for the remaining case ($\eta \geq \gamma = \varepsilon/C$ and $x_0 > 0$) now follows directly from Theorem 6.1 and Lemmas 6.2 and 6.3.

*Proof of Corollary* 1.6. The corollary follows from Theorem 1.5 and (107). ■

*Proof of Corollary* 1.7. (i) and (ii) follow from Theorem 1.5 and Corollary 1.6 directly.

Fix some small $\varepsilon_1, \varepsilon_2 > 0$. By the definition of $\delta$, there exists a sequence $n_j$ (assume $n_j > 0$ for simplicity) such that

$$e^{-(\delta+\varepsilon_1)n_j} \leq \|2\theta + n_j\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\frac{\delta}{2}n_j}.$$

By the Diophantine condition on $\alpha$, we have

$$n_{j+1} \geq e^{n_j/C}.$$

We prove (15) first. By Theorem 1.2, for any $|k| \in [\varepsilon_2 n_{j+1}, n_{j+1}/2]$ one has

$$\|U(k)\| \leq e^{-(\ln\lambda-\varepsilon_1)|k|}.$$

This implies (15) by the arbitrariness of $\varepsilon_1, \varepsilon_2$.

Now we turn to the proof of (16). By Theorem 1.5, for any $|k| \in [\varepsilon_2 n_{j+1}, n_{j+1}]$ one has

$$\|A_k\| \geq e^{(\ln\lambda-\varepsilon_1)|k|}.$$

This implies (16). ■

## Appendix A. Uniformity

The following lemma is critical when we prove Theorem 3.2.

**Lemma A.1** ([6, Lemma 9.7]). *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ and $0 \leq k_0 \leq q_n - 1$ be such that $|\sin\pi(x+k_0\alpha)| = \inf_{0\leq k\leq q_n-1}|\sin\pi(x+k\alpha)|$. Then for some absolute constant $C > 0$,*

$$-C\ln q_n \leq \sum_{k=0,k\neq k_0}^{q_n-1} \ln|\sin\pi(x+k\alpha)| + (q_n-1)\ln 2 \leq C\ln q_n. \tag{121}$$

*Proof of Theorem* 3.2. Let $i_0, j_0 \in I_1 \cup I_2$ be such that $|\sin\pi(2\theta + (i_0 + j_0)\alpha)| = \min_{i,j\in I_1\cup I_2} |\sin\pi(2\theta + (i + j)\alpha)|$. By the Diophantine condition on $\alpha$, there exist $\tau', \kappa' > 0$ such that for any $i + j \neq i_0 + j_0$ and $i, j \in I_1 \cup I_2$,

$$|\sin\pi(2\theta + (i + j)\alpha)| \geq \frac{\tau'}{(sq_n)^{\kappa'}}. \tag{122}$$

Also for all distinct $i, j \in I_1 \cup I_2$, we have

$$|\sin\pi(j - i)\alpha| \geq \frac{\tau'}{(sq_n)^{\kappa'}}. \tag{123}$$

In (26), let $x = \cos 2\pi a$, $k = sq_n - 1$ and take the logarithm. Then

$$\ln \prod_{j\in I_1\cup I_2,\, j\neq i} \frac{|\cos 2\pi a - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|}$$
$$= \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| - \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|.$$

First, we estimate $\sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j|$. Obviously,

$$\sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j|$$
$$= \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\sin\pi(a + \theta_j)| + \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\sin\pi(a - \theta_j)| + (sq_n - 1)\ln 2$$
$$= \Sigma_+ + \Sigma_- + (sq_n - 1)\ln 2.$$

Both $\Sigma_+$ and $\Sigma_-$ consist of $s$ terms of the form of (121), plus $s$ terms of the form

$$\ln \min_{j=0,1,\ldots,q_n} |\sin\pi(x + j\alpha)|,$$

minus $\ln|\sin\pi(a \pm \theta_i)|$. Thus, using (121) $s$ times for $\Sigma_+$ and $\Sigma_-$ respectively, one has

$$\sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \leq -sq_n \ln 2 + Cs\ln q_n. \tag{124}$$

If $a = \theta_i$, we obtain

$$\sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j|$$
$$= \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\sin\pi(\theta_i + \theta_j)| + \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\sin\pi(\theta_i - \theta_j)| + (sq_n - 1)\ln 2$$
$$= \Sigma_+ + \Sigma_- + (sq_n - 1)\ln 2, \tag{125}$$

where

$$\Sigma_+ = \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\sin\pi(2\theta + (i + j)\alpha)|, \quad \Sigma_- = \sum_{j\in I_1\cup I_2,\, j\neq i} \ln|\sin\pi(i - j)\alpha|.$$

We will estimate $\Sigma_+$. Set $J_1 = [1, s_1]$ and $J_2 = [s_1 + 1, s]$, which are two adjacent disjoint intervals of length $s_1, s_2$ respectively. Then $I_1 \cup I_2$ can be represented as a disjoint union of segments $B_j,\ j \in J_1 \cup J_2$, each of length $q_n$. Applying (121) to each $B_j$, we obtain

$$\Sigma_+ \geq -sq_n \ln 2 + \sum_{j \in J_1 \cup J_2} \ln |\sin \pi \hat{\theta}_j| - Cs \ln q_n - \ln |\sin 2\pi(\theta + i\alpha)|, \tag{126}$$

where

$$|\sin \pi \hat{\theta}_j| = \min_{\ell \in B_j} |\sin \pi(2\theta + (\ell + i)\alpha)|. \tag{127}$$

By (30) and (122), we have

$$\sum_{j \in J_1 \cup J_2} \ln |\sin \pi \hat{\theta}_j| \geq -\gamma sq_n - Cs \ln sq_n. \tag{128}$$

Putting (128) in (126), we get

$$\Sigma_+ \geq -sq_n \ln 2 - \gamma sq_n - Cs \ln sq_n. \tag{129}$$

Similarly, replacing (30), (122) with (123), and arguing as in the proof of (129), we obtain

$$\Sigma_- > -sq_n \ln 2 - Cs \ln sq_n. \tag{130}$$

From (125), (129) and (130), one has

$$\sum_{j \in I_1 \cup I_2,\, j \neq i} \ln |\cos 2\pi\theta_i - \cos 2\pi\theta_j| \geq -sq_n \ln 2 - \gamma sq_n - Cs \ln sq_n. \tag{131}$$

By (124) and (131), we have

$$\max_{i \in I_1 \cup I_2} \prod_{j \in I_1 \cup I_2,\, j \neq i} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{sq_n(\gamma + C\frac{\ln sq_n}{q_n})}.$$

By the assumption $s \leq q_n^C$ we get, for any $\varepsilon > 0$ and large $n$,

$$\max_{i \in I_1 \cup I_2} \prod_{j \in I_1 \cup I_2,\, j \neq i} \frac{|x - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} < e^{sq_n(\gamma + \varepsilon)}.$$

This completes the proof. ∎

## Appendix B. Block Expansion Theorem

*Proof of Theorem 3.3.* For any $\hat{y} \in [y_1 + \gamma k, y_2 - \gamma k]$, by the assumption there exists an interval $I(\hat{y}) = [x_1, x_2] \subset [y_1, y_2]$ such that $\hat{y} \in I(\hat{y})$ with $\frac{\gamma}{20}k \leq |I(\hat{y})| \leq \frac{1}{2}\operatorname{dist}(y, \{y_1, y_2\})$, and

$$\operatorname{dist}(\hat{y}, \partial I(\hat{y})) \geq \frac{1}{40}|I(\hat{y})| \geq \frac{\gamma}{800}k \tag{132}$$

and

$$|G_{I(\hat{y})}(\hat{y}, x_i)| \leq e^{-\tau|\hat{y}-x_i|}, \quad i = 1, 2, \tag{133}$$

where $\{x_1, x_2\} = \partial I(\hat{y})$ is the boundary of $I(\hat{y})$. For $z \in \partial I(\hat{y})$, let $z'$ be the neighbor of $z$ (i.e., $|z - z'| = 1$) not belonging to $I(\hat{y})$.

If $x_2 + 1 \leq y_2 - \gamma k$ or $x_1 - 1 \geq y_1 + \gamma k$, we can expand $\varphi(x_2 + 1)$ or $\varphi(x_1 - 1)$ using (25). We can continue this process until we arrive at $z$ such that $z + 1 > y_2 - \gamma k$ or $z - 1 < y_1 + \gamma k$, or the number of iterations reaches $\lfloor 1600/\gamma \rfloor$. Then, by (25),

$$\varphi(y) = \sum_{s; z_{i+1} \in \partial I(z_i')} G_{I(y)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}'), \tag{134}$$

where in each term of the summation one has $y_1 + \gamma k + 1 \leq z_i \leq y_2 - \gamma k - 1$, $i = 1, \ldots, s$, and either $z_{s+1} \notin [y_1 + \gamma k + 1, y_2 + \gamma k - 1]$ and $s + 1 < \lfloor 1600/\gamma \rfloor$, or $s + 1 = \lfloor 1600/\gamma \rfloor$. We should mention that $z_{s+1} \in [y_1, y_2]$.

If $z_{s+1} \in [y_1, y_1 + \gamma k]$ and $s + 1 < \lfloor 1600/\gamma \rfloor$, this implies

$$|\varphi(z_{s+1}')| \leq r_{y_1}^{\varphi}.$$

By (133), for such terms we have

$$\begin{aligned} &|G_{I(y)}(y, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}')| \\ &\qquad \leq r_{y_1}^{\varphi} e^{-\tau(|y-z_1| + \sum_{i=1}^{s} |z_i' - z_{i+1}|)} \leq r_{y_1}^{\varphi} e^{-\tau(|y-z_{s+1}| - (s+1))} \\ &\qquad \leq r_{y_1}^{\varphi} e^{-\tau(|y-y_1| - \gamma k - 1600/\gamma)}. \end{aligned} \tag{135}$$

If $z_{s+1} \in [y_2 - \gamma k, y_2]$ and $s + 1 < \lfloor 1600/\gamma \rfloor$, by the same arguments we have

$$|G_{I(y)}(y, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}')| \leq r_{y_2}^{\varphi} e^{-\tau(|y-y_2| - \gamma k - 1600/\gamma)}. \tag{136}$$

If $s + 1 = \lfloor 1600/\gamma \rfloor$, using (132) and (133) we obtain

$$|G_{I(y)}(y, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}')| \leq e^{-\tau \frac{\gamma}{800} k \lfloor 1600/\gamma \rfloor} |\varphi(z_{s+1}')|. \tag{137}$$

Notice that the total number of terms in (134) is at most $2^{\lfloor 1600/\gamma \rfloor}$ and $|y - y_1|, |y - y_2| \geq 10\gamma k$. By (135)–(137), we have

$$|\varphi(y)| \leq \max \left\{ r_{y_1}^{\varphi} e^{-\tau(|y-y_1| - 3\gamma k)}, r_{y_2}^{\varphi} e^{-\tau(|y-y_2| - 3\gamma k)}, \max_{p \in [y_1, y_2]} e^{-\tau k} |\varphi(p)| \right\}. \tag{138}$$

Now we will show that for any $p \in [y_1, y_2]$, one has $|\varphi(p)| \leq \max\{r_{y_1}^{\varphi}, r_{y_2}^{\varphi}\}$. Then (138) implies Theorem 3.3. Otherwise, by the definition of $r_{y_1}^{\varphi}$ and $r_{y_2}^{\varphi}$, if $|\varphi(p')|$ is the largest $|\varphi(z)|$ with $z \in [y_1 + 10\gamma k + 1, y_2 - 10\gamma k - 1]$, then $|\varphi(p')| > \max\{r_{y_1}^{\varphi}, r_{y_2}^{\varphi}\}$. Applying (138) to $\varphi(p')$ and noticing that $|p' - y_1|, |p' - y_2| \geq 10\gamma k$, we get

$$|\varphi(p')| \leq \max\{e^{-7\tau\gamma k} r_{y_1}^{\varphi}, e^{-7\tau\gamma k} r_{y_2}^{\varphi}, e^{-\tau k} |\varphi(p')|\}.$$

This is impossible because $|\varphi(p')| > \max\{r_{y_1}^{\varphi}, r_{y_2}^{\varphi}\}$. ∎

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