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Yusheng Luo

On geometrically finite degenerations I: boundaries of main hyperbolic components

Abstract. We develop a theory of *quasi post-critically finite* degenerations of Blaschke products. This gives us tools to study the boundaries of hyperbolic components of rational maps in higher-dimensional moduli spaces. We use it to obtain a combinatorial classification of geometrically finite polynomials on the boundary of the *main hyperbolic component* $\mathcal{H}_d$, i.e., the hyperbolic component in the space of monic and centered polynomials that contains z^d. We also show that the closure $\overline{\mathcal{H}_d}$ is not a topological manifold with boundary for $d \geq 4$ by constructing *self-bumps* on its boundary.

Keywords. Hyperbolic component, geometrically finite polynomials, Blaschke product, self-bumps

Contents

1. Introduction

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. It is said to be *hyperbolic* if all the critical points converge to attracting periodic cycles under iteration. The hyperbolic maps form an open set in suitable moduli spaces, and a connected component U of this set is called a *hyperbolic component*. The topology of hyperbolic components has been studied extensively and is well understood in various settings [4, 38, 53]. However, the boundaries of hyperbolic components and the interactions between hyperbolic components remain

Yusheng Luo: Institute for Mathematical Sciences, Stony Brook University,
Stony Brook, NY 11794, USA; yusheng.s.luo@gmail.com

mysterious. In a series of two papers, we develop a theory of 'geometrically finite' degenerations to investigate these questions. In this paper:

- We study *quasi post-critically finite* degenerations of Blaschke products $f_n \in \mathcal{B}_d$, where $\mathcal{B}_d$ is the space of normalized and marked Blaschke products of degree d (see (1.1)). For such degenerations, we construct quasi-invariant trees modeled by a limiting simplicial tree map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ with rescaling limits. These quasi-invariant trees are the analogs of the Hubbard trees for post-critically finite polynomials. We prove a realization theorem for quasi-invariant trees and thus classify quasi post-critically finite degenerations in $\mathcal{B}_d$.

- A rational map is said to be *geometrically finite* if the critical points in the Julia set have finite orbits. Using the realization theorem, we obtain a classification of geometrically finite polynomials on the boundary of the main hyperbolic component of polynomials, $\mathcal{H}_d$, containing z^d.

- The study of quasi-invariant tree maps reveals many different accesses to a boundary point from $\mathcal{H}_d$ and thus 'self-bumps' occur on $\partial \mathcal{H}_d$, showing the closure $\overline{\mathcal{H}_d}$ is not a topological manifold with boundary for $d \geq 4$ (see Figure 1.3).

In the sequel [24], we study the convergence and divergence of quasi post-critically finite degenerations for rational maps. We prove the boundedness of quasi post-critically finite degenerations for hyperbolic rational maps with Sierpiński carpet Julia set. We also prove a 'double limit theorem' for 'quasi-Blaschke products' by giving a criterion for the convergence of simultaneous quasi post-critically finite degenerations on the two Fatou components of z^d. Together with the realization result proved in this paper, the convergence results can be applied to show the existence of polynomial mating (cf. [13, 49]).

Our theory runs parallel with the developments in Kleinian groups, and the results fit into Sullivan's well-known dictionary between the two fields. We summarize the comparisons in the following table.

Complex dynamics	Kleinian groups
Blaschke product	Fuchsian group
Quasi-Blaschke product	Quasi-Fuchsian group
Main hyperbolic component	Bers slice
Geometrically finite rational map	Geometrically finite Kleinian group
Sierpiński carpet rational map	Acylindrical Kleinian group
Geometrically finite polynomial on $\partial \mathcal{H}_d$	Cusp on the Bers boundary
Self-bump on $\partial \mathcal{H}_d$	Self-bump on the Bers boundary
Double limit theorem for quasi-Blaschke products	Double limit theorem for quasi-Fuchsian groups
Boundedness for Sierpiński carpet rational maps	Thurston's compactness theorem for acylindrical 3-manifolds

We now turn to a detailed statement of results.

Main hyperbolic component

A polynomial $P(z) = a_d z^d + \cdots + a_0$ is said to be *monic* if $a_d = 1$ and *centered* if $a_{d-1} = 0$. Let $\mathcal{P}_d$ be the space of all monic and centered polynomials. A degree d polynomial P with connected Julia set has $d - 1$ invariant external rays. A *marking* of P is a particular choice of an invariant external ray. A monic and centered polynomial P with connected Julia set has a unique choice of the Böttcher map normalized so that the derivative at infinity is 1. The angle 0 external ray under this normalization thus gives a marking of P. Therefore, P is regarded as a *marked polynomial*. If P has a Jordan curve Julia set, then a marking is equivalent to a choice of a repelling fixed point on its Julia set $J(P)$.

Let $\mathcal{H}_d \subseteq \mathcal{P}_d$ be the hyperbolic component that contains z^d. We call $\mathcal{H}_d$ the *main hyperbolic component* of degree d.

In the quadratic polynomial case, the Landing Theorem of Douady and Hubbard can be used to gain a complete understanding of geometrically finite polynomials on the boundary of a hyperbolic component and to determine which hyperbolic components of the Mandelbrot set have intersecting closures [37]. In particular, geometrically finite polynomials $\partial \mathcal{H}_2$ are in correspondence with rational rotation numbers $\mathbb{Q}/\mathbb{Z}$ (see [37, Theorem 6.5]).

To describe the dynamics of a geometrically finite polynomial $\widehat{P} \in \partial \mathcal{H}_d$ in higher degrees, we introduce the notion of *pointed Hubbard tree*. A polynomial P is said to be *post-critically finite* if its critical points have finite orbits. Given any geometrically finite polynomial $\widehat{P} \in \mathcal{P}_d$ with connected Julia set, there exists a post-critically finite polynomial $P \in \mathcal{P}_d$ with topologically conjugate dynamics on the Julia sets compatible with the markings [17]. The dynamics of P is described combinatorially by its (angled) Hubbard tree H.

If $\widehat{P} \in \partial \mathcal{H}_d$, there exists a special non-repelling fixed point $\widehat{p} = \lim p_n$ of $\widehat{P}$, where p_n is the attracting fixed point of $P_n \in \mathcal{H}_d$ and $P_n \to \widehat{P}$. This gives a fixed point $p \in H$ of P, and we call (H, p) the *pointed Hubbard tree* for $\widehat{P}$.

The pointed Hubbard tree (H, p) is said to be *simplicial* if there exists a finite simplicial structure on H for which P is a simplicial map, i.e., P sends an edge of H to an edge of H. We say (H', p') is a *pointed simplicial tuning* of (H, p) if (H', p') is constructed from (H, p) by 'replacing' the center p of local degree $\delta(p)$ by a simplicial pointed Hubbard tree of degree $\delta(p)$ and modifying the backward orbits of p accordingly (see §5 for the precise definition). We say a degree d pointed Hubbard tree (H, p) is *iterated-simplicial* if it can be inductively constructed from the trivial degree d pointed Hubbard tree $(H = \{p\}, p)$ by a sequence of pointed simplicial tunings. We first show

Theorem 1.1. *If $\widehat{P} \in \partial \mathcal{H}_d$ is geometrically finite, then the associated pointed Hubbard tree (H, p) is iterated-simplicial. Conversely, if a degree d pointed Hubbard tree (H, p) is iterated-simplicial, then there exists a corresponding geometrically finite polynomial $\widehat{P}$ on the boundary $\partial \mathcal{H}_d$.*

We remark that the classification naturally gives a level structure for geometrically finite polynomials on $\partial \mathcal{H}_d$. This level structure is a manifestation of the Schwarz lemma and the incompatible escaping rates for the critical points for the dynamics in the corresponding Fatou component (see Figure 6.4 and the discussion below Theorem 1.3).

Theorem 1.1 gives concrete combinatorial models for geometrically finite polynomials on $\partial \mathcal{H}_d$ and classifies all models that arise. Let us denote by $U_{\widehat{P}}$ the space of all geometrically finite polynomials corresponding to the same pointed Hubbard tree as $\widehat{P} \in \partial \mathcal{H}_d$. We do not know a description of $U_{\widehat{P}} \cap \partial \mathcal{H}_d$, and it is expected this space can be quite complicated in general [28] (see §6 for some partial answers). To discuss some of the subtleties of describing this space, it is well-known that the closure of a hyperbolic component $\mathcal{H}$ may not be *quasiconformally closed*: if $P \in \bar{\mathcal{H}}$, a quasiconformal deformation of P may not be in $\bar{\mathcal{H}}$ (see [20, 51]).

A hyperbolic component $\mathcal{H} \neq \mathcal{H}_d$ is said to be a *satellite component* of $\mathcal{H}_d$ if there exists a parabolic polynomial $\widehat{P} \in \partial \mathcal{H} \cap \partial \mathcal{H}_d$ that has conjugate dynamics on the Julia sets with any $P \in \mathcal{H}$. As any parabolic polynomial can be perturbed to a hyperbolic polynomial with conjugate dynamics on the Julia sets [17], we immediately have the following corollary:

Corollary 1.2. *Let $\mathcal{H} \neq \mathcal{H}_d$ be a hyperbolic component with connected Julia set. Let H be the Hubbard tree of the post-critically finite center $P \in \mathcal{H}$. Then $\mathcal{H}$ is a satellite component of $\mathcal{H}_d$ if and only if there exists a fixed point $p \in H$ such that (H, p) is iterated-simplicial.*

The direction that the pointed Hubbard tree for $\widehat{P} \in \partial \mathcal{H}_d$ is iterated-simplicial in Theorem 1.1 follows from an analysis of cut points in the Julia set (see §5). The other implication is proved by studying the degenerations of Blaschke products sketched in the following.

Blaschke products

For $d \geq 2$, we let $\mathcal{B}_d$ denote the space of *normalized and marked Blaschke products* $f : \mathbb{D} \to \mathbb{D}$ of the form

$$f(z) = z \prod_{i=1}^{d-1} \frac{z - a_i}{1 - \overline{a_i} z}, \quad \text{where } |a_i| < 1. \tag{1.1}$$

Note that $f(0) = 0$, and any proper holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ of degree d with a fixed point in $\mathbb{D}$ is holomorphically conjugate to a map in $\mathcal{B}_d$.

Any map $f \in \mathcal{B}_d$ can be extended to a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Viewed as a rational map, the Julia set $J(f)$ is the circle $\mathbb{S}^1$, and there is a unique homeomorphism $\eta_f : \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$, called the *marking*, that varies continuously with f, conjugates z^d to f, and is such that η_f is the identity map if $f(z) = z^d$.

The polynomials $P \in \mathcal{H}_d$ are in correspondence with $f \in \mathcal{B}_d$ by gluing $f \in \mathcal{B}_d$ with z^d using their markings on $\mathbb{S}^1$ [27, §5]; the resulting polynomial is denoted by

$P = f \sqcup z^d$. Thus, the study of $\partial \mathcal{H}_d$ is naturally related to the study of degenerations in $\mathcal{B}_d$. We study 'geometrically finite' degenerations in $\mathcal{B}_d$ which give us uniform control on the rescaling dynamics at the critical orbits.

A sequence $f_n \in \mathcal{B}_d$ is said to be (K-)*quasi post-critically finite* if we can label the critical points of f_n by $c_{i,n} \in \mathbb{D}$, $i = 1, \ldots, d-1$, in such a way that for each i, there exist a quasi pre-period l_i and a quasi period q_i with

$$d_{\mathbb{D}}(f_n^{l_i}(c_{i,n}), f_n^{l_i + q_i}(c_{i,n})) \leq K.$$

The uniform bounds allow us to construct a sequence of quasi-invariant trees $\mathcal{T}_n \subseteq \mathbb{D}$ for the hyperbolic metric $d_{\mathbb{D}}$ on $\mathbb{D}$, capturing all the interesting dynamics (see Theorem 2.2, which is interesting in its own right). The dynamics on $\mathcal{T}_n$ is described by a simplicial tree map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$, with *rescaling limits* $F_v : \mathbb{D}_v \to \mathbb{D}_{f(v)}$ between vertices (cf. [22]). This simplicial tree map plays a similar role to the Hubbard tree for a post-critically finite polynomial.

Let $P_n = f_n \sqcup z^d \in \mathcal{H}_d$ be the corresponding quasi post-critically finite sequence of polynomials. We show that the limit $P = \lim_{n \to \infty} P_n$ is geometrically finite, and the sequence of the quasi-invariant trees converges to a pointed Hubbard tree with 'decorations' (see Figures 1.1 and 1.2). This allows us to describe the pointed Hubbard tree of the limit using quasi-invariant trees, building a bridge between geometrically finite polynomials on $\partial \mathcal{H}_d$ and quasi post-critically finite degenerations in $\mathcal{B}_d$.
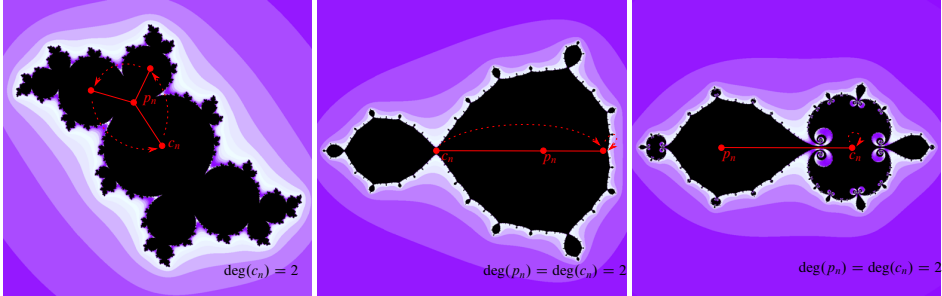

Fig. 1.1. Illustrations of quasi-invariant trees for quasi post-critically finite degenerations of $(f_n)_n \in \mathcal{B}_d$ in the bounded Fatou component of the corresponding polynomials $P_n = f_n \sqcup z^d$.

In §3, we define an abstract *angled tree map*

$$(f : (\mathcal{T}, p) \to (\mathcal{T}, p), \delta, \alpha = \{\alpha_v\})$$

with local degree function δ and angle functions α_v satisfying certain compatibility conditions. This data combinatorially classifies the simplicial tree map with the rescaling limits (cf. angled Hubbard tree in [45]). Every simplicial pointed Hubbard tree $P : (H, p) \to (H, p)$ gives an angled tree map. In §4, we define *admissible* angled tree maps and prove a realization theorem. In particular, we show

Fig. 1.2. Another example of quasi-invariant trees, where all vertices are quasi-fixed and have local degree 2. The trees converge to a 'decorated pointed Hubbard tree'. The corresponding pointed Hubbard tree is a 'cross', and 'splits' at the Julia branch point to get approximating quasi-invariant trees (see §6 for the definition of 'splitting').

Theorem 1.3. *Every simplicial pointed Hubbard tree is realizable by $f_n \in \mathcal{B}_d$ after an admissible splitting on its Julia branch points.*

The precise statement of the realization theorem can be found in Theorem 4.1 and Proposition 4.3. The definition of admissible splitting is given in §6 (see also Figure 1.2). The non-uniqueness of splittings is the source for self-bumps on $\partial \mathcal{H}_d$ (see Figure 1.3 and §7).

Theorem 1.3 allows us to construct geometrically finite polynomials $\widehat{P}_1 \in \partial \mathcal{H}_d$ for any simplicial pointed Hubbard tree (H_1, p_1).

The limit map $\widehat{P}_1$ has at most one attracting Fatou component which is necessarily fixed. If it has one, we show we can further degenerate its dynamics on this attracting Fatou component (and its backward orbits under $\widehat{P}_1$) by a quasi post-critically finite sequence, while staying on the boundary $\partial \mathcal{H}_d$. The limit of this sequence gives a geometrically finite polynomial $\widehat{P}_2$. The pointed Hubbard tree (H_2, p_2) of $\widehat{P}_2$ is a pointed simplicial tuning of (H_1, p_1). By induction, we thus construct geometrically finite polynomials on $\partial \mathcal{H}_d$ for any iterated-simplicial Hubbard tree and conclude the proof of Theorem 1.1.

Self-bumps on $\partial \mathcal{H}_d$

It is expected that the boundary of $\mathcal{H}_d$ is quite complicated for $d \geq 3$ [39]. We construct polynomials $\widehat{P} \in \partial \mathcal{H}_d$ with different accesses from the main hyperbolic component $\mathcal{H}_d$ (see Figure 1.3). Thus, *self-bumps* occur on $\partial \mathcal{H}_d$. More precisely, we prove

Theorem 1.4. *For any degree $d \geq 4$, there exists a geometrically finite polynomial $\widehat{P} \in \partial \mathcal{H}_d$ such that for any sufficiently small neighborhood U of $\widehat{P}$, the intersection $U \cap \mathcal{H}_d$ is disconnected.*

As an immediate corollary (cf. [30, Theorem A.1]), we have

Corollary 1.5. *The closure $\overline{\mathcal{H}_d}$ is not a topological manifold with boundary for $d \geq 4$.*
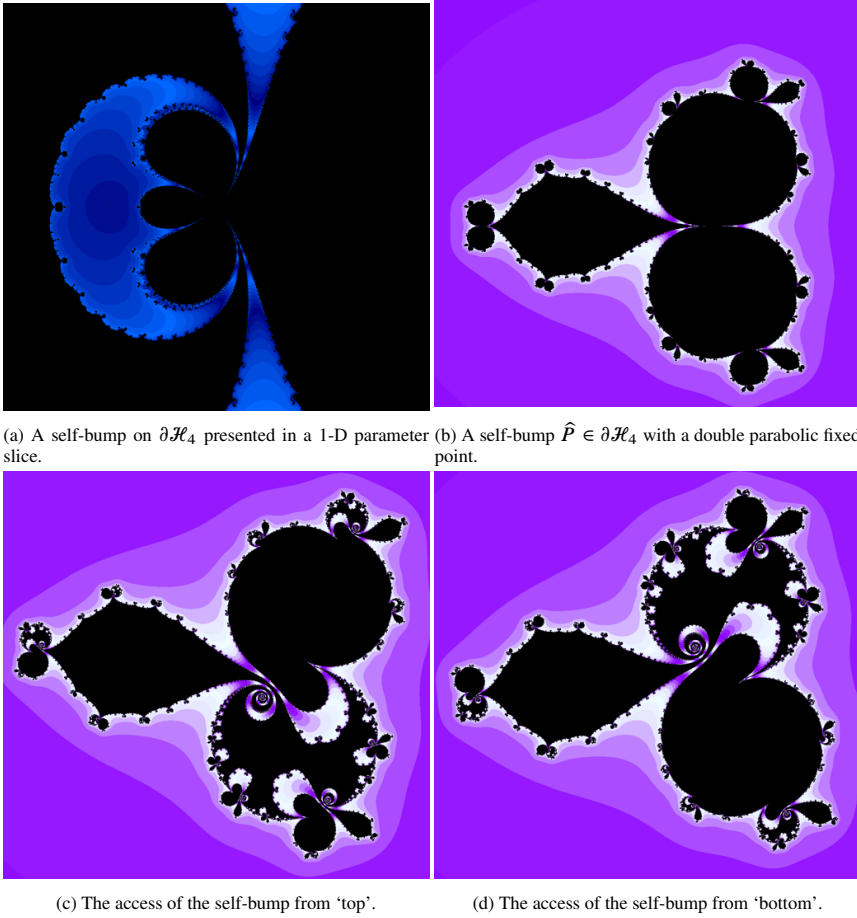
(a) A self-bump on $\partial \mathcal{H}_4$ presented in a 1-D parameter slice.

(b) A self-bump $\widehat{P} \in \partial \mathcal{H}_4$ with a double parabolic fixed point.

(c) The access of the self-bump from 'top'.

(d) The access of the self-bump from 'bottom'.

Fig. 1.3. A self-bump on $\partial \mathcal{H}_4$ with two different accesses.

Proof. By the characterization of J-stable rational maps (see [29, Theorem 4.2]), we have int $\overline{\mathcal{H}_d} = \mathcal{H}_d$. If $\overline{\mathcal{H}_d}$ were a topological manifold with boundary, then there would be a small neighborhood U of $\widehat{P}$ meeting the manifold's interior in a connected set, contrary to Theorem 1.4. Thus $\overline{\mathcal{H}_d}$ is not a topological manifold with boundary. ∎

Comparisons with Kleinian groups

We now discuss some similarities and differences between the theory of rational maps and Kleinian groups.

- *Dimension 1 vs. higher dimensions:* The Bers slice for a once punctured torus group has complex dimension 1. It is known [40] that the boundary is a Jordan curve, and hence is locally connected. When the deformation space has complex dimension ≥ 2,

it is known that its boundary spaces can be non-locally-connected [9]. For rational maps, $\partial\mathcal{H}_2$ is a Jordan curve. Many other hyperbolic components in a one-dimensional slice are also proved to be Jordan disks [48, 52]. For $d \geq 3$, it is conjectured that $\partial\mathcal{H}_d$ is not locally connected [39].

- *Geometric convergence:* A geometrically finite group G on the Bers boundary can be constructed by pinching a system of disjoint simple closed curves. The pinching deformations G_n converge both algebraically and geometrically to G. In particular, the limit sets converge in the Hausdorff topology to the limit set of G. This is in contrast with rational maps. For most (in a suitable sense) iterated-simplicial pointed Hubbard trees, the convergence to any corresponding polynomial $\widehat{P} \in \partial\mathcal{H}_d$ is *never geometric*: the Julia sets do not converge in the Hausdorff topology to $J(\widehat{P})$ for any $P_n \to \widehat{P}$ (see Figure 1.2). See [11, 31, 32] for related discussions.

- *Self-bump:* Self-bumps on the Bers boundary were first constructed by McMullen using projective structures [30]. This was generalized to other deformation spaces in [10]. The self-bumps on $\partial\mathcal{H}_d$ are direct analogues in the setting of rational maps. We use the sign of the imaginary part of the multipliers to distinguish different accesses (see §7), which are the analogues of the traces of the holonomy representations for Kleinian groups. Our method does not provide any self-bumps in degree 3.

Notes and references

A large portion of the boundary $\partial\mathcal{H}_3$ has been studied in [43] and a combinatorial model of $\partial\mathcal{H}_3$ is studied in [5]. The problem of how hyperbolic components are positioned for quadratic rational maps is studied in detail in [46, 47]. A related problem of self-bumps on $\partial\mathcal{H}_3$ is studied in [6, 7].

Degenerations of Fatou components have been studied extensively in terms of elementary moves like pinching and spinning [11, 18, 26, 44, 50]. The quasi post-critically finite degenerations in $\mathcal{B}_d$ generalize these operations, and thus provide a unified framework.

Other comparisons of $\mathcal{B}_d$ with Teichmüller theory can be found in [33–36]. The quasi-invariant trees are closely related to the ribbon $\mathbb{R}$-trees of [35] and the analogous constructions of isometric group actions on $\mathbb{R}$-trees for Kleinian groups [3, 41, 42] (see discussions in §2).

Related bumping problems for deformation spaces of Kleinian groups are studied in [1]. Rescaling limits of Blaschke products have also been studied in [21], and other applications of rescaling limits in complex dynamics can be found in [2, 12, 15, 22, 23, 25].

Outline of the paper

We define and prove some basic properties of quasi post-critical finite sequences of Blaschke products in §2. The abstract angled tree map is introduced in §3 and the realization theorem is proved in §4. One direction of Theorem 1.1 is proved in §5, and using the realization theorem for degenerations in $\mathcal{B}_d$, we finish the proof of Theorem 1.1 in §6. Finally, Theorem 1.4 is proved in §7.
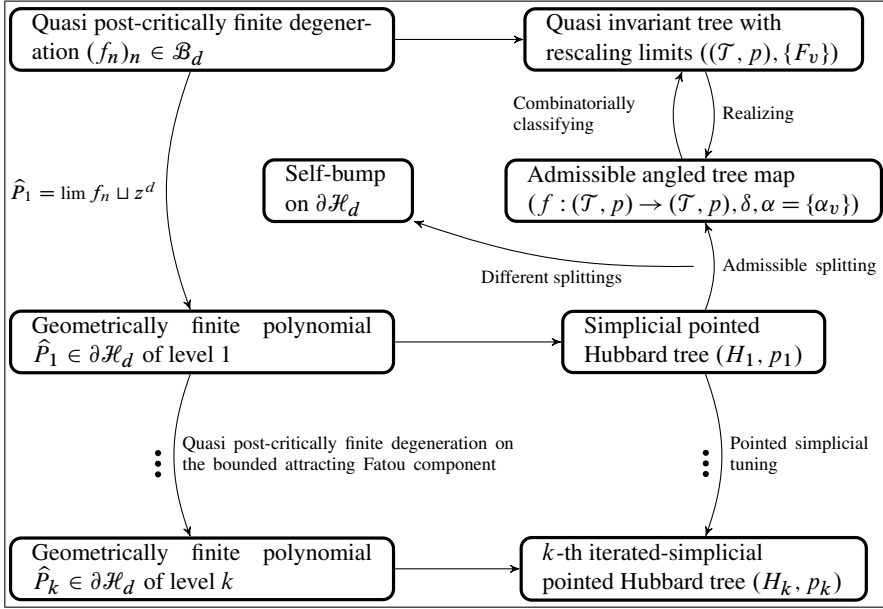
Fig. 1.4. The logical flow for different concepts in this paper.

2. Quasi post-critically finite sequences of Blaschke products

A proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ of degree $d \geq 2$ can be written as a *Blaschke product*

$$f(z) = e^{i\theta} \prod_{i=0}^{d} \frac{z - a_i}{1 - \overline{a_i} z},$$

where $|a_i| < 1$. By Schwarz reflection, any such map extends to a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Both perspectives are useful. In particular, throughout the paper we shall freely use the fact that f is defined on $\overline{\mathbb{D}}$.

By the Denjoy–Wolff theorem, there is a unique non-repelling fixed point of f on $\overline{\mathbb{D}}$, which puts a Blaschke product f into exactly three categories:

- f is *interior-hyperbolic* or simply *hyperbolic* if f has an attracting fixed point in $\mathbb{D}$;
- f is *parabolic* if f has a parabolic fixed point on $\mathbb{S}^1$;
- f is *boundary-hyperbolic* if f has an attracting fixed point on $\mathbb{S}^1$.

The parabolic Blaschke products can be further divided into *singly parabolic* or *doubly parabolic* depending on the multiplicities of the parabolic fixed points. The Julia set of a hyperbolic or a doubly parabolic Blaschke product is the circle $\mathbb{S}^1$, while the Julia set of a singly parabolic or a boundary-hyperbolic Blaschke product is a Cantor set on $\mathbb{S}^1$.

In this paper, we shall mainly focus on hyperbolic Blaschke products, although the parabolic and boundary-hyperbolic ones will appear as *rescaling limits* when we degenerate hyperbolic Blaschke products.

For $d \geq 2$, let

$$\mathcal{B}_d := \left\{ f(z) = z \prod_{i=1}^{d-1} \frac{z - a_i}{1 - \overline{a_i} z} : |a_i| < 1 \right\} \cong \mathbb{D}^{d-1}$$

be the space of (normalized) marked hyperbolic Blaschke products. Each map in $\mathcal{B}_d$ has an attracting fixed point at the origin. Any hyperbolic Blaschke product of degree d is conformally conjugate to a map in $\mathcal{B}_d$. We shall identify $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$. Under this identification, we set $m_d(t) = d \cdot t$. Note that for each $f \in \mathcal{B}_d$, there exists a unique quasisymmetric homeomorphism $\eta_f : \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ such that

(1) $\eta_f \circ m_d = f \circ \eta_f$,

(2) $f \mapsto \eta_f$ is continuous on $\mathcal{B}_d$ with $\eta_{z^d} = \mathrm{id}$.

In particular, the conjugacy η_f gives a way to label the periodic points of f on $\mathbb{S}^1$ by periodic points of m_d on $\mathbb{R}/\mathbb{Z}$ that varies continuously on $\mathcal{B}_d$.

Note that the space $\mathcal{B}_d \cong \mathbb{D}^{d-1}$ is not compact. We study a particular type of degenerations in $\mathcal{B}_d$:

Definition 2.1. Let $f_n \in \mathcal{B}_d$. We say that the sequence is *K-quasi post-critically finite* if we can label the critical points of f_n by $c_{i,n} \in \mathbb{D}$, $i = 1, \ldots, d-1$, in such a way that for each i, there exist l_i and q_i called *quasi pre-periods* and *quasi periods* respectively with

$$d_{\mathbb{D}}(f_n^{l_i}(c_{i,n}), f_n^{l_i+q_i}(c_{i,n})) \leq K.$$

We say f_n is *quasi post-critically finite* if it is K-quasi post-critically finite for some K.

A *ribbon structure* on a finite tree is the choice of a planar embedding up to isotopy. The ribbon structure can be specified by a cyclic ordering of the edges incident to each vertex. A finite tree with a ribbon structure is called a *finite ribbon tree*. Recall that a map $f : \mathcal{T} \to \mathcal{T}$ is said to be *simplicial* if f sends an edge to an edge.

In this section, we first construct a sequence of quasi-invariant trees $\mathcal{T}_n \subseteq \mathbb{D}$ for a quasi post-critically finite sequence $f_n \in \mathcal{B}_d$ capturing all interesting dynamical features.

Theorem 2.2. *Let $f_n \in \mathcal{B}_d$ be quasi post-critically finite. After passing to a subsequence, there exists a constant $K > 0$, a simplicial map*

$$f : (\mathcal{T}, p) \to (\mathcal{T}, p)$$

on a pointed finite ribbon tree with vertex set $\mathcal{V}$, a sequence of pointed finite ribbon trees $(\mathcal{T}_n, p_n)$ with $p_n \in \mathcal{T}_n \subseteq \mathbb{D}$, and a sequence of isomorphisms

$$\phi_n : (\mathcal{T}, p) \to (\mathcal{T}_n, p_n)$$

such that

- (Degenerating tree) *If $v_1 \neq v_2 \in \mathcal{V}$, then*

$$d_{\mathbb{D}}(\phi_n(v_1), \phi_n(v_2)) \to \infty.$$

- (Geodesic edge) *If $E = [v_1, v_2] \subseteq \mathcal{T}$ is an edge, then the corresponding edge $\phi_n(E) \subseteq \mathcal{T}_n$ is a hyperbolic geodesic segment connecting $\phi_n(v_1)$ and $\phi_n(v_2)$.*

- (Critically approximating) *Any critical point of f_n is within hyperbolic distance K of the vertex set $\mathcal{V}_n := \phi_n(\mathcal{V})$ of $\mathcal{T}_n$.*

- (Quasi-invariance on vertices) *If $v \in \mathcal{V}$, then*

$$d_{\mathbb{D}}(f_n(\phi_n(v)), \phi_n(f(v))) \leq K \quad \text{for all } n.$$

- (Quasi-invariance on edges) *If $E \subseteq \mathcal{T}$ is an edge and $x_n \in \phi_n(E)$, then there exists $y_n \in \phi_n(f(E))$ such that*

$$d_{\mathbb{D}}(f_n(x_n), y_n) \leq K \quad \text{for all } n.$$

If E is a periodic edge of period q, then

$$d_{\mathbb{D}}(f_n^q(x_n), x_n) \leq K \quad \text{for all } n.$$

To prepare ourselves for the construction, we will first review some results on degenerations of rational maps and geometric bounds for Blaschke products.

Degeneration of rational maps

Let Rat_d denote the space of rational maps of degree d. By fixing a coordinate system of $\widehat{\mathbb{C}} \cong \mathbb{P}^1(\mathbb{C})$, a rational map can be expressed as a ratio of two homogeneous polynomials, $f(z : w) = (P(z, w) : Q(z, w))$, where P and Q have degree $d = \deg(f)$ with no common divisors. Thus, using the coefficients of P and Q as parameters, we have

$$\mathrm{Rat}_d = \mathbb{P}^{2d+1} \setminus V(\mathrm{Res}),$$

where Res is the resultant of the polynomials P and Q, and $V(\mathrm{Res})$ is the hypersurface for which the resultant vanishes. This embedding gives a natural compactification $\overline{\mathrm{Rat}_d} = \mathbb{P}^{2d+1}$, which will be called the *algebraic compactification*. A map $f \in \overline{\mathrm{Rat}_d}$ can be written as

$$f = (P : Q) = (Hp : Hq),$$

where $H = \gcd(P, Q)$. We set

$$\varphi_f := (p : q),$$

which is a rational map of degree at most d. The zeros of H in $\mathbb{P}^1$ are called the *holes* of f, and the set of holes of f is denoted by $\mathcal{H}(f)$.

If the coefficients of $f_n \in \mathrm{Rat}_d$ converge to those of $f \in \overline{\mathrm{Rat}_d}$, we say that f is the *algebraic limit* of the sequence f_n. We also say f_n *converges algebraically* to f. The limit f is said to have *degree k* if φ_f has degree k. Abusing notation, sometimes we shall refer to φ_f as the algebraic limit of f_n. The following is useful in analyzing the limiting dynamics.

Lemma 2.3 ([12, Lemma 4.2]). *If $f_n \in \text{Rat}_d$ converges to f algebraically, then f_n converges to φ_f uniformly on compact subsets of $\widehat{\mathbb{C}} - \mathcal{H}(f)$.*

The following statement is also useful in many situations:

Lemma 2.4. *Let $f_n \in \text{Rat}_d$ converge to f algebraically with $\deg(f) \geq 1$. If $x \in \mathcal{H}(f)$, then there exists a sequence of critical points c_n for f_n with $c_n \to x$.*

Proof. Suppose for contradiction that this is not true. Let $x \in U$ be a small open neighborhood containing no critical points of f_n for all large n and let $C = \partial U$. Modifying U and changing the role of 0 and ∞ if necessary, we may assume that f is univalent on U, $f(C)$ is a simple closed curve and the image U' does not contain ∞. Since f_n converges uniformly to f on a neighborhood of C by Lemma 2.3, there exists a component $C_n \subseteq f_n^{-1}(f(C))$ that converges in the Hausdorff topology to C. Let U_n be the component of $\widehat{\mathbb{C}} - C_n$ that contains x. So for large n, U_n contains no critical points of f_n. Note that $f_n(\partial U_n) = f_n(C_n) = f(C) = \partial U'$, so $f_n : U_n \to U'$ is univalent. This is a contradiction as there exists a sequence of poles of f_n converging to the hole x, but $\infty \notin U'$. ∎

Geometric bounds for Blaschke products

The normalization $f(0) = 0$ imposed on maps $f \in \mathcal{B}_d$ gives the following compactness result:

Proposition 2.5. *Let $K \geq 0$, and let $f_n : \mathbb{D} \to \mathbb{D}$ be a sequence of proper holomorphic maps of degree d with $d_\mathbb{D}(0, f_n(0)) \leq K$. Then after passing to a subsequence, f_n converges compactly on $\mathbb{D}$ to a proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ (of potentially lower degree).*

By Schwarz reflection, f, f_n are rational maps. As rational maps, f_n converges algebraically to the map f of degree ≥ 1 with holes contained in $\mathbb{S}^1$.

Proof. If $f_n \in \mathcal{B}_d$, then the statement follows from [35, Proposition 10.2]. The statement for general sequences can be derived from the above by post-composing with a bounded sequence $M_n \in \text{Isom}(\mathbb{D})$ such that $M_n \circ f_n \in \mathcal{B}_d$. ∎

We remark that the rational map point of view in Proposition 2.5 gives more information. By Lemma 2.3, we know where the convergence of $f_n \to f$ fails to be uniform on $\mathbb{S}^1$.

By the Schwarz lemma, any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is distance non-increasing with respect to the hyperbolic metric. We will frequently use the following which gives a bound in the other direction:

Theorem 2.6 ([35, Theorem 10.11]). *There is a constant $R > 0$ such that for any holomorphic map $f : \mathbb{D} \to \mathbb{D}$ of degree d,*

(1) *if $d_\mathbb{D}([a, b], C(f)) > R$, then $d_\mathbb{D}(f(a), f(b)) = d_\mathbb{D}(a, b) + O(1)$;*

(2) *if $d_\mathbb{D}([a, b], f(C(f))) > R$, then $d_\mathbb{D}(f^{-1}(a), f^{-1}(b)) = d_\mathbb{D}(a, b) + O(1)$.*

Here $C(f)$ denotes the critical set of f, $[a, b]$ is the hyperbolic geodesic segment that connects a and b, and f^{-1} is any branch of the inverse map that is continuous along $[a, b]$.

Quasi-invariant tree

Let $f_n \in \mathcal{B}_d$ be a quasi post-critically finite sequence. We now explain how a pointed quasi-invariant tree can be constructed after possibly passing to a subsequence. This tree plays a similar role to the Hubbard tree for post-critically finite polynomials.

The construction is in the same spirit as that of the ribbon $\mathbb{R}$-tree in [35]; see also [23, 25]. One of the key differences is that two sequences $x_n, y_n \in \mathbb{D}$ are identified in the ribbon $\mathbb{R}$-tree if $\lim d_{\mathbb{D}}(x_n, y_n)/R_n = 0$, where R_n is a given rescaling factor; while they are identified in the quasi-invariant tree if they are a uniformly bounded distance apart. The more restrictive identification allows us to better control the dynamics and construct rescaling limits. Another problem with using the ribbon $\mathbb{R}$-tree is that it only sees the dynamics on the scale of R_n. To deal with incompatible escaping rates of the critical points, we therefore abandon the rescaling construction of [35].

In the following, we shall use (a_n) to denote a sequence, and a_n its n-th term. If there are multiple subindices, we shall use $(-)_n$ to emphasize that the index for the sequence is n.

Let $p_n = 0$ be the unique attracting fixed point of f_n in $\mathbb{D}$. Let

$$\widetilde{\mathcal{P}} := \{(f_n^j(c_{i,n}))_n : i = 1, \ldots, d - 1, \ j = 0, \ldots, l_i + q_i - 1\} \cup \{(p_n)\}.$$

Note that an element of $\widetilde{\mathcal{P}}$ is a sequence of points in $\mathbb{D}$ indexed by n, and the cardinality of $\widetilde{\mathcal{P}}$ is $1 + \sum_{i=1}^{d-1} l_i + q_i$.

Let $(v_n) \in \widetilde{\mathcal{P}}$. After passing to a subsequence of (f_n), we assume that the local degree $\deg_{v_n} f_n$ of f_n at v_n is constant in n and the limit

$$\lim_{n \to \infty} d_{\mathbb{D}}(v_n, w_n)$$

exists for any $(v_n), (w_n) \in \widetilde{\mathcal{P}}$ (which can possibly be ∞). This defines an equivalence relation:

$$(v_n) \sim (w_n) \quad \text{if} \quad \lim_{n \to \infty} d_{\mathbb{D}}(v_n, w_n) < \infty.$$

An equivalence class $\mathcal{C} \in \widetilde{\mathcal{P}}/\sim$ will be called a *cluster set*. We define the *degree* of $\mathcal{C}$ by

$$\deg(\mathcal{C}) = 1 + \sum_{(v_n) \in \mathcal{C}} (\deg_{v_n} f_n - 1).$$

Let $\mathcal{C} \in \widetilde{\mathcal{P}}/\sim$ be an equivalence class. Note that $\mathcal{C}$ is a finite set of sequences of points in $\mathbb{D}$. It is convenient to choose a representative $(v_n(\mathcal{C})) \in \mathcal{C}$ with the convention that $(v_n(\mathcal{C})) = (p_n)$ if $\mathcal{C} = [(p_n)]$. We denote

$$\mathcal{P} := \{(v_n(\mathcal{C})) : \mathcal{C} \in \widetilde{\mathcal{P}}/\sim\}.$$

Note that an element in $\mathcal{P}$ is a sequence of points in $\mathbb{D}$ indexed by n. Since $\tilde{\mathcal{P}}$ is a finite set, $\mathcal{P}$ is a finite set as well.

We also set

$$\mathcal{P}_n = \{v_n(\mathcal{C}) : \mathcal{C} \in \tilde{\mathcal{P}}/\sim\},$$

i.e., $\mathcal{P}_n$ consists of the n-th terms of the elements of $\mathcal{P}$. Note that $\mathcal{P}_n$ is a finite collection of points of $\mathbb{D}$. If $(v_n) \neq (w_n) \in \mathcal{P}$, then $d_{\mathbb{D}}(v_n, w_n) \to \infty$. Thus, after passing to a subsequence, we can assume that for any pair of distinct elements $(v_n), (w_n) \in \mathcal{P}$, we have $v_n \neq w_n$ for all n. Thus, each point $v_n \in \mathcal{P}_n$ uniquely determines an element $(v_n) \in \mathcal{P}$, which gives a canonical identification of $\mathcal{P}_n$ with $\mathcal{P}$. We define the degree of a point $v_n \in \mathcal{P}_n$ by

$$\deg(v_n) = \deg(\mathcal{C})$$

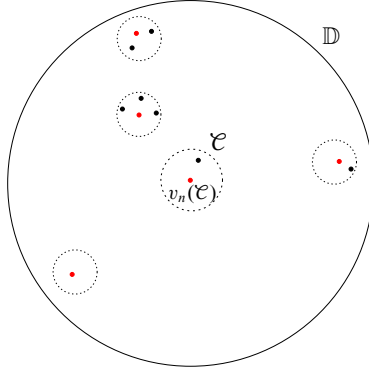if $(v_n) \in \mathcal{P}$ represents $\mathcal{C} \in \tilde{\mathcal{P}}/\sim$.

Fig. 2.1. A schematic illustration of cluster sets. We choose a representative (red) for each cluster. As $n \to \infty$, the distance is bounded between points in the same cluster, while goes to infinity between points in different clusters.

Index the elements of $\mathcal{P} = \{(b_{0,n})_n, (b_{1,n})_n, \ldots, (b_{m,n})_n\}$ with $(b_{0,n}) = (p_n)$. The sequence of quasi-invariant trees $\mathcal{T}_n$ is the 'spine' of the degenerating hyperbolic polygon $\mathrm{Cvx\,Hull}(\mathcal{P}_n)$ and is constructed inductively as follows:

As the base case, we define $\mathcal{T}_n^0 = \{b_{0,n}\}$ with vertex set $\mathcal{V}_n^0 = \{b_{0,n}\}$.

Assume that $\mathcal{T}_n^i$ is constructed with vertex set $\mathcal{V}_n^i := \{v_{1,n}, \ldots, v_{m_i,n}\}$ containing $\{b_{0,n}, \ldots, b_{i,n}\}$. Assume as induction hypotheses that

(1) $\lim_{n\to\infty} d_{\mathbb{D}}(v_{k,n}, v_{k',n}) = \infty$ for all $k \neq k' \le m_i$;

(2) $\min_{k=i+1,\ldots,m} d_{\mathbb{D}}(b_{k,n}, \mathrm{Cvx\,Hull}(\mathcal{V}_n^i)) \to \infty$ as $n \to \infty$;

(3) each edge of $\mathcal{T}_n^i$ is a hyperbolic geodesic segment.

After passing to a subsequence and changing indices, we assume that for all n,

$$d_{\mathbb{D}}(b_{i+1,n}, \mathrm{Cvx\,Hull}(\mathcal{V}_n^i)) = \min_{k=i+1,\ldots,m} d_{\mathbb{D}}(b_{k,n}, \mathrm{Cvx\,Hull}(\mathcal{V}_n^i)).$$

Let $a_{i+1,n}$ be the projection of $b_{i+1,n}$ to the convex hull Cvx Hull$(\mathcal{V}_n^i)$. After passing to a subsequence, we assume $\lim_{n\to\infty} d_{\mathbb{D}}(a_{i+1,n}, v_{k,n})$ exists (which can possibly be ∞) for all k (see Figure 2.2).

Since $a_{i+1,n}$ is on the boundary of Cvx Hull$(\mathcal{V}_n^i)$, there exist $u_n, w_n \in \mathcal{V}_n^i$ such that $a_{i+1,n}$ is on the hyperbolic geodesic segment $[u_n, w_n]$. Since $\mathcal{T}_n^i$ is a tree and each edge is a hyperbolic geodesic segment, there is a unique piecewise geodesic path in $\mathcal{T}_n^i$ that connects u_n and w_n. We denote it by

$$[v_{j_1,n} = u_n, v_{j_2,n}] \cup [v_{j_2,n}, v_{j_3,n}] \cup \cdots \cup [v_{j_{l-1},n}, v_{j_l,n} = w_n],$$

where $[v_{j_i,n}, v_{j_{i+1},n}]$ is an edge of $\mathcal{T}_n^i$. After passing to a subsequence, we assume that the subindices j_i do not depend on n.
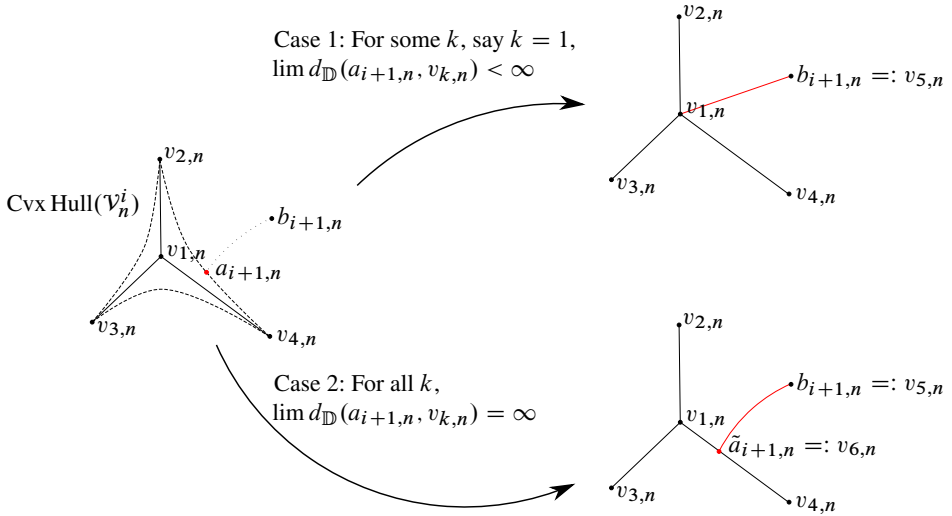

Fig. 2.2. An illustration of the inductive construction of $\mathcal{T}_n^i$.

We have two cases. If $a_{i+1,n}$ stay at a bounded distance from some vertex $v_{k,n} \in \mathcal{V}_n^i$, then we define

$$\mathcal{T}_n^{i+1} := \mathcal{T}_n^i \cup [v_{k,n}, b_{i+1,n}],$$

and set $\mathcal{V}_n^{i+1} := \mathcal{V}_n^i \cup \{b_{i+1,n}\}$ (see Figure 2.2).

Otherwise, since the hyperbolic triangles are thin, subdividing the polygon $v_{j_1,n}, \ldots, v_{j_k,n}$ into finitely many triangles, there exist

$$\tilde{a}_{i+1,n} \in [v_{j_1,n}, v_{j_2,n}] \cup \cdots \cup [v_{j_{l-1},n}, v_{j_l,n}] \subseteq \mathcal{T}_n^i$$

which stay at a bounded distance from $a_{i+1,n}$. Note that the bound depends only on the number of vertices. We define

$$\mathcal{T}_n^{i+1} := \mathcal{T}_n^i \cup [\tilde{a}_{i+1,n}, b_{i+1,n}],$$

and set the vertex set $\mathcal{V}_n^{i+1} := \mathcal{V}_n^i \cup \{\tilde{a}_{i+1,n}, b_{i+1,n}\}$ (see Figure 2.2).

By our construction, it is easy to verify that the three induction hypotheses are satisfied. Thus, by induction, let $\mathcal{T}_n = \mathcal{T}_n^m$ be the finite tree when all m points in $\mathcal{P}_n$ are added.

By construction, $\mathcal{P}_n \subseteq \mathcal{V}_n$, and any point in $\mathcal{V}_n - \mathcal{P}_n$ is a branch point for $\mathcal{T}_n$. Moreover, $\lim_{n \to \infty} d_{\mathbb{D}}(v_{k,n}, v_{k',n}) = \infty$ for all $k \neq k'$ and the edges are hyperbolic geodesic segments.

After passing to a further subsequence, we may assume that $\mathcal{T}_n$ are all canonically isomorphic with the same ribbon structure. Thus, we denote by $\mathcal{T}$ the underlying ribbon finite tree, with the isomorphisms

$$\phi_n : \mathcal{T} \to \mathcal{T}_n.$$

We denote the vertex set as $\mathcal{V}$. By our construction, $\phi_n^{-1}(\mathcal{P}_n) \subseteq \mathcal{V}$ is canonically identified with $\mathcal{P}$ and the identification does not depend on n. In this way, we shall abuse notation and regard $\mathcal{P} \subseteq \mathcal{V}$.

The local degree at a vertex is defined by $\delta(v) := \deg(\phi_n(v))$ if $v \in \mathcal{P}$, and $\delta(v) := 1$ if $v \notin \mathcal{P}$. There exists a unique vertex corresponding to the attracting fixed point for f_n, and we shall denote it by $p \in \mathcal{V}$. Note that by construction, $\delta(v) - 1$ equals the number of critical points of f_n counted with multiplicity that stay at a bounded distance from $\phi_n(v)$.

We first show that the vertex set is quasi forward invariant.

Lemma 2.7. *There exists an induced map $f : \mathcal{V} \to \mathcal{V}$ and a constant K such that for all $v \in \mathcal{V}$ and for all n,*

$$d_{\mathbb{D}}(f_n(\phi_n(v)), \phi_n(f(v))) \leq K.$$

Proof. First consider the case $v \in \mathcal{P}$. Note that there exists an induced map $\tilde{f} : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}}$ defined as follows. Let $(v_n) \in \widetilde{\mathcal{P}}$.

- If $(v_n) = (p_n)$, then $\tilde{f}((v_n)) = (v_n)$.
- If $(v_n) = (f_n^j(c_{i,n}))$ with $j \leq l_i + q_i - 2$, then $\tilde{f}((v_n)) = (f_n^{j+1}(c_{i,n}))$.
- If $(v_n) = (f_n^j(c_{i,n}))$ with $j = l_i + q_i - 1$, then $\tilde{f}((v_n)) = (f_n^{l_i}(c_{i,n}))$.

Note that in the first two cases, $\tilde{f}((v_n)) = (f_n(v_n))$. In the third case, the sequence $\tilde{f}((v_n)) = (f_n^{l_i}(c_{i,n}))$ stays at a bounded distance from $(f_n(v_n)) = (f_n^{l_i+q_i}(c_{i,n}))$ as f_n is quasi post-critically finite.

If $(v_n) \sim (w_n)$, then by the Schwarz lemma, $f_n(v_n)$ stay at a bounded distance from $f_n(w_n)$, so $\tilde{f}((v_n)) \sim \tilde{f}((w_n))$. Thus the map $\tilde{f} : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}}$ descends to a map $f : \mathcal{P} \to \mathcal{P}$. By our construction, there exists a constant K such that

$$d_{\mathbb{D}}(f_n(\phi_n(v)), \phi_n(f(v))) \leq K.$$

Now suppose $v \in \mathcal{V} - \mathcal{P}$. Then v is a branch point. Let b_1, b_2, b_3 be the closest points to v on $\mathcal{P}$ so that the convex hull of them in $\mathcal{T}$ forms a 'tripod' with v as the center. Then the angles $\angle \phi_n(b_i)\phi_n(v)\phi_n(b_j)$ are uniformly bounded away from 0 for $i \neq j$ by construction. Thus $\phi_n(v)$ is at a uniformly bounded distance from the hyperbolic geodesic segment $[\phi_n(b_i), \phi_n(b_j)]$ that connects $\phi_n(b_i)$ and $\phi_n(b_j)$. By construction, the geodesic

$[\phi_n(b_i), \phi_n(b_j)]$ is close to critical points (in the sense of the bound in Theorem 2.6) only possibly near its end points. Thus by Theorem 2.6, $d_{\mathbb{D}}(f_n(\phi_n(b_i)), f_n(\phi_n(b_j))) = d(\phi_n(b_i), \phi_n(b_j)) + O(1)$. Therefore, the angles $\angle f_n(\phi_n(b_i)) f_n(\phi_n(v)) f_n(\phi_n(b_j))$ are all uniformly bounded away from 0 for $i \neq j$. Thus, $f_n(\phi_n(v))$ is at a uniformly bounded distance from some branch point $\phi_n(v')$ of $\mathcal{T}_n$. Therefore, we define $f(v) = v'$ in this case, and the lemma follows. ∎

The above lemma allows us to define an induced map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ by extending f continuously on any edge $[v, w]$ to the arc $[f(v), f(w)]$. Consider the finite subtree $\mathcal{T}^P \subseteq \mathcal{T}$ as the convex hull of the periodic vertices of f. We first show

Lemma 2.8. *Any vertex* $v \in \mathcal{T}^P$ *is periodic and* $f : \mathcal{T}^P \to \mathcal{T}^P$ *is a simplicial map, i.e., if* $[v, w]$ *is an edge of* $\mathcal{T}^P$, *then* $[f(v), f(w)]$ *is also an edge of* $\mathcal{T}^P$.

Proof. If v is a vertex in $\mathcal{T}^P$, then there are two periodic vertices w_1, w_2 such that $v \in [w_1, w_2]$. Let q be the least common multiple of the periods of w_1 and w_2. By Lemma 2.7, $d_{\mathbb{D}}(\phi_n(w_i), f_n^q(\phi_n(w_i))) = O(1)$. By the Schwarz lemma, we have $d_{\mathbb{D}}(f_n^q(\phi_n(v)), f_n^q(\phi_n(w_i))) \leq d_{\mathbb{D}}(\phi_n(v), \phi_n(w_i))$, thus

$$d_{\mathbb{D}}(f_n^q(\phi_n(v)), \phi_n(w_i)) \leq d_{\mathbb{D}}(\phi_n(v), \phi_n(w_i)) + O(1).$$

Since the angle $\angle \phi_n(w_1)\phi_n(v)\phi_n(w_2)$ is uniformly bounded away from 0,

$$d_{\mathbb{D}}(\phi_n(w_1), \phi_n(w_2)) = d_{\mathbb{D}}(\phi_n(v), \phi_n(w_1)) + d_{\mathbb{D}}(\phi_n(v), \phi_n(w_2)) + O(1).$$

A standard hyperbolic geometry estimate using horocycles gives

$$d_{\mathbb{D}}(f_n^q(\phi_n(v)), \phi_n(v)) = O(1).$$

This means that v is periodic. Thus $\mathcal{T}^P$ is invariant under f.

A similar proof also shows that f is simplicial. Indeed, suppose it is not. Then there exists an edge $[w_1, w_2] \subseteq \mathcal{T}^P$ such that $[f(w_1), f(w_2)]$ is not an edge. Let $v \in (f(w_1), f(w_2))$ be a vertex. Then v is also periodic by the previous paragraph. Let q be the least common multiple of the periods of w_1, w_2, v. Note that

$$d_{\mathbb{D}}(\phi_n(f(w_1)), \phi_n(f(w_2))) = d_{\mathbb{D}}(\phi_n(f(w_1)), \phi_n(v)) + d_{\mathbb{D}}(\phi_n(v), \phi_n(f(w_2))) + O(1).$$

Thus applying Lemma 2.7 to f^{q-1}, we get

$$d_{\mathbb{D}}(\phi_n(w_1), \phi_n(w_2)) = d_{\mathbb{D}}(\phi_n(w_1), \phi_n(f^{q-1}(v))) \\ + d_{\mathbb{D}}(\phi_n(f^{q-1}(v))), \phi_n(w_2)) + O(1).$$

Therefore, $f^{q-1}(v) \in (w_1, w_2)$, a contradiction. ∎

Since the vertices of $\mathcal{T}_n$ are uniformly quasi-invariant, we show f_n is quasi-invariant on $\mathcal{T}_n$ with dynamics modeled by $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$.

Proposition 2.9. *There exists a constant K such that*

- *if $[v_1, v_2] \subseteq \mathcal{T}$ is an edge, then for any point $x_n \in [\phi_n(v_1), \phi_n(v_2)]$, there exists $y_n \in \phi_n(f([v_1, v_2]))$ such that*

$$d_{\mathbb{D}}(f_n(x_n), y_n) \leq K \quad \text{for all } n;$$

- *if $[v_1, v_2] \subseteq \mathcal{T}$ is a periodic edge of period q, then for any point $x_n \in [\phi_n(v_1), \phi_n(v_2)]$,*

$$d_{\mathbb{D}}(f_n^q(x_n), x_n) \leq K \quad \text{for all } n.$$

Proof. The proof uses the same estimates as in the proof of Lemma 2.8. By the Schwarz lemma,
$$d_{\mathbb{D}}(f_n(x_n), f_n(\phi_n(v_i))) \leq d_{\mathbb{D}}(x_n, \phi_n(v_i)).$$

Thus, by Lemma 2.7,

$$d_{\mathbb{D}}(f_n(x_n), \phi_n(f(v_i))) \leq d_{\mathbb{D}}(x_n, \phi_n(v_i)) + O(1).$$

By Theorem 2.6 and Lemma 2.7,

$$\begin{aligned} d_{\mathbb{D}}(\phi_n(f(v_1)), \phi_n(f(v_2))) &= d_{\mathbb{D}}(\phi_n(v_1), \phi_n(v_2)) + O(1) \\ &= d_{\mathbb{D}}(x_n, \phi_n(v_1)) + d_{\mathbb{D}}(x_n, \phi_n(v_2)) + O(1). \end{aligned}$$

Therefore, there exists a constant K such that

$$d_{\mathbb{D}}(f_n(x_n), \phi_n(f([v, w]))) \leq K.$$

A similar argument proves the part on periodic edges. ∎

Corollary 2.10. *The map f is non-expanding. More precisely, let E be an open edge of $\mathcal{T}$. If there exists k with $f^k(E) \cap E \neq \emptyset$, then $f^k : E \to E$ is a homeomorphism.*

Proof. By the Schwarz lemma, f_n is distance decreasing. By Proposition 2.9, the dynamics of f^k on E is approximated by f_n^k on $\phi_n(E)$. Thus f is non-expanding. ∎

Corollary 2.11. *We can add finitely many vertices to $\mathcal{T}$ so that $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is simplicial.*

Proof. By Corollary 2.10, each recurring edge is periodic. Thus, for each edge E, there exists n such that $f^n(E) \subseteq \mathcal{T}^P$. We call the smallest such integer the *generation* of E. By Lemma 2.8, f is simplicial on generation 0 edges. If E is generation 1, then $f(E)$ is a finite union of generation 0 edges. By adding finitely many vertices on E, we may assume f is simplicial on E. The corollary now follows by induction. ∎

Proof of Theorem 2.2. Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ with isomorphisms $\phi_n(\mathcal{T}, p) \to (\mathcal{T}_n, p_n)$ constructed as above. Then Corollary 2.11 shows f is simplicial. By construction of $\mathcal{T}_n$, the first three conditions are satisfied. By Lemma 2.7 and Proposition 2.9, the map is quasi-invariant. ∎

Rescaling limits

Let $v \in \mathcal{V}$. We define a *normalization at v* or a *coordinate at v* as a sequence $M_{v,n} \in$ Isom$(\mathbb{D})$ such that $M_{v,n}(0) = \phi_n(v)$. Note that there are many choices for $M_{v,n}$, and they differ by pre-composing with rotations that fix 0.

We shall think of the sequence $M_{v,n}$ as giving 'coordinates' at v, which gives an associated limiting disk $\mathbb{D}_v$ and a limiting circle $\mathbb{S}_v^1$ for the vertex v. More precisely, we say a sequence $z_n \in \overline{\mathbb{D}}$ *converges* to $z \in \overline{\mathbb{D}}_v$ *in v-coordinate*, denoted by $z_n \to_v z$ or $z = \lim_v z_n$, if

$$\lim_{n \to \infty} M_{v,n}^{-1}(z_n) = z.$$

Here the subindices in $\mathbb{D}_v$ and $\mathbb{S}_v^1$ are used to distinguish different coordinates at different vertices.

By Proposition 2.9, there exists a constant K such that

$$d_{\mathbb{D}}(0, M_{f(v),n}^{-1} \circ f_n \circ M_{v,n}(0)) = d_{\mathbb{D}}(M_{f(v),n}(0), f_n \circ M_{v,n}(0)) \leq K.$$

Thus, by Proposition 2.5, after passing to subsequences, the sequence

$$M_{f(v),n}^{-1} \circ f_n \circ M_{v,n}$$

converges compactly on $\mathbb{D}$ to a proper holomorphic map F_v on the unit disk. We call F_v the *rescaling limit* between v and $f(v)$.

We remark that it is important to keep track of the changing coordinates, thus one should really think of F_v as a map

$$F_v : \mathbb{D}_v \to \mathbb{D}_{f(v)}.$$

To emphasize this, we sometimes also write $F_v = F_{v \to f(v)}$.

We also remark that $M_{f(v),n}^{-1} \circ f_n \circ M_{v,n}$ and F_v extend to rational maps on $\widehat{\mathbb{C}}$ of degree ≥ 1, and $M_{f(v),n}^{-1} \circ f_n \circ M_{v,n}$ converges algebraically to F_v. This perspective is useful as it gives more information on where the convergence $M_{f(v),n}^{-1} \circ f_n \circ M_{v,n}$ to F_v fails to be uniform on $\mathbb{S}^1$.

More generally, $M_{f^k(v),n}^{-1} \circ f_n^k \circ M_{v,n}$ converges compactly on $\mathbb{D}$ to $F_{f^{k-1}(v)} \circ F_{f^{k-2}(v)} \circ \cdots \circ F_v$. We shall denote this composition by

$$F_v^k := F_{f^{k-1}(v)} \circ F_{f^{k-2}(v)} \circ \cdots \circ F_v.$$

In particular, if v is a periodic point of period q, then

$$M_{v,n}^{-1} \circ f_n^q \circ M_{v,n}$$

converges compactly on $\mathbb{D}$ to F_v^q. We shall call the map F_v^q the *first return rescaling limit* at v. As before, all these maps extend to rational maps on $\widehat{\mathbb{C}}$, and the convergences are algebraic.

Recall that $\delta(v)$ is the local degree of a vertex $v \in \mathcal{V}$, and $\delta(v) - 1$ equals the number of critical points of f_n counted with multiplicity that stay at a bounded distance from $\phi_n(v)$.

Thus, the rescaling limit F_v at v has degree $\delta(v)$. Also recall that $\eta_n : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ is the marking for $f_n \in \mathcal{B}_d$. So $\eta_n(0)$ is the marked repelling fixed point of f_n on $\mathbb{S}^1$.

By precomposing the normalizations with rotations, we make the following *anchored* convention on the normalizations. Throughout this paper, the normalizations will always be assumed to be anchored (see Figure 2.3).

Definition 2.12. Let $f_n \in \mathcal{B}_d$ be quasi post-critically finite, with simplicial tree map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$. The normalizations $M_{v,n}$ at vertices $v \in \mathcal{V}$ are said to be *anchored* if they satisfy the following conditions:

- If $v = p$ and $\delta(p) = 1$, then $M_{p,n}$ is chosen so that $\lim_p \eta_n(0) = 1 \in \mathbb{S}_p^1$.
- If $v = p$ and $\delta(p) \geq 2$, then $M_{p,n}$ is chosen so that $1 \in \mathbb{S}_p^1$ is the fixed point of F_p nearest to $\lim_p \eta_n(0)$ counterclockwise.
- If $v \neq p$, then $M_{v,n}$ is chosen so that $\lim_v \phi_n(p) = 1 \in \mathbb{S}_v^1$.

To illustrate the second case in Definition 2.12, consider the sequence

$$f_n(z) = z^3 \frac{z - (1 - 1/n)}{1 - (1 - 1/n)z}.$$

One can verify that it is quasi post-critically finite. Note that $L_{p,n}(z) = z$ is a normalization at p. Under this normalization, $\eta_n(0) \to_p 1 \in \mathbb{S}_p^1$ as $\eta_n(0) = 1$ is the marked fixed point for f_n for all n. This normalization $L_{p,n}$ does not satisfy the anchored convention as the rescaling limit

$$F_p(z) = \lim L_{p,n}^{-1} \circ f_n \circ L_{p,n}(z) = \lim f_n(z) = -z^3$$

has fixed points at $\pm i$. Since the fixed point i is closer to $1 = \lim_n \eta_n(0)$ counterclockwise, one can verify that the sequence $M_{p,n}(z) = iz$ is a normalization at p, and it satisfies the anchored convention.
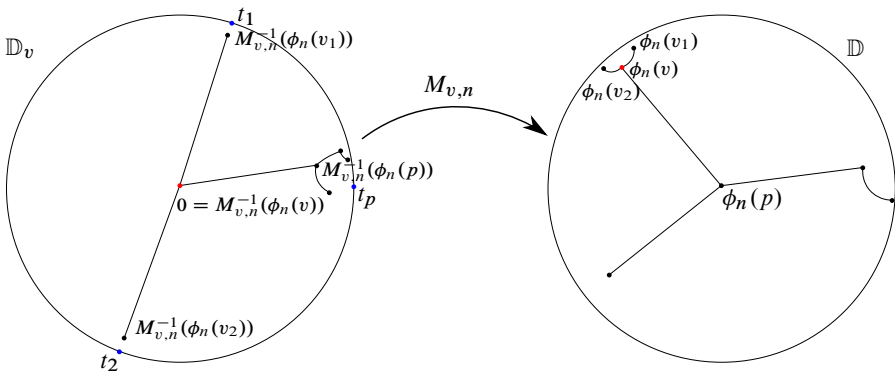

Fig. 2.3. An illustration of $M_{v,n}$ for $v \neq p$, and the marked points on $\mathbb{S}_v^1$. This normalization satisfies the anchored convention as $\lim_v \phi_n(p) = t_p = 1 \in \mathbb{S}_v^1$.

A marking of $\mathbb{S}^1$ at a vertex

Let $v_1, \ldots, v_l$ be the list of vertices adjacent to v in $\mathcal{T}$. Then $\{v_1, \ldots, v_l\}$ can be identified with the tangent space $T_v \mathcal{T}$ of $\mathcal{T}$ at v. By our construction, after passing to a subsequence, there exist l distinct points $t_1, \ldots, t_l \in \mathbb{S}_v^1$ with $t_i = \lim_v \phi_n(v_i)$ (see Figure 2.3). We denote this correspondence by the map

$$\xi_v : T_v \mathcal{T} \to \mathbb{S}_v^1.$$

Since the angles between different edges at a vertex in $\mathcal{T}_n$ are uniformly bounded from below by construction, in v-coordinate, $\mathcal{T}_n$ converges to the union of the geodesic rays connecting 0 and points in $\xi_v(T_v \mathcal{T})$. In particular, if $w \neq v$ is in the same component of $\mathcal{T} - \{v\}$ as v_i, then $\phi_n(w) \to_v t_i$.

Since f is injective on each edge, we have a well-defined tangent map $Df : T_v \mathcal{T} \to T_{f(v)} \mathcal{T}$. On the other hand, the rescaling limit F_v gives a map from $\mathbb{S}_v^1$ to $\mathbb{S}_{f(v)}^1$. It is easy to check that this marking of $\mathbb{S}^1$ at a vertex is compatible with the dynamics of the tangent map:

Lemma 2.13. *Let $v \in \mathcal{V}$. Then the holes of the rescaling limit F_v are contained in $\xi_v(T_v \mathcal{T}) \subseteq \mathbb{S}_v^1$. Moreover,*
$$F_v \circ \xi_v = \xi_{f(v)} \circ Df.$$

Proof. The first statement follows immediately from Lemma 2.4.

To prove the 'moreover' part, we let w be adjacent to v with $\phi_n(w) \to_v t$. Thus, $[\phi_n(v), \phi_n(w)] \to_v [0, t]$. Let $t' := \xi_{f(v)} \circ Df(w) \in \mathbb{S}_{f(v)}^1$. Then $[\phi_n(f(v)), \phi_n(f(w))] \to_{f(v)} [0, t']$. Suppose for contradiction that $F_v(t) \neq t'$. Note that $M_{f(v),n}^{-1} \circ f_n \circ M_{v,n}$ converges to F_v uniformly away from finitely many holes. Let K be the constant in Proposition 2.9. Then we conclude by uniform convergence that there exists $x \in [0, t]$ such that

$$d_{\mathbb{D}}([0, t'], M_{f(v),n}^{-1} \circ f_n \circ M_{v,n}(x)) > 2K.$$

This is a contradiction. ∎

Corollary 2.14. *Let $v \neq p \in \mathcal{V}$ be periodic of period q. Then F_v^q has a non-repelling fixed point at 1.*

Proof. Let $a \in T_v \mathcal{T}$ be the direction associated to p. Since f is simplicial on $\mathcal{T}$ and $f(p) = p$, we have $Df^q(a) = a$. Since the normalizations are anchored, the direction a corresponds to $1 \in \mathbb{S}_v^1$. By Lemma 2.13, $F_v(1) = 1$. It is non-repelling by the Schwarz lemma and the fact that the fixed point $\phi_n(p)$ converges to 1 in v-coordinate. ∎

Definition 2.15. We introduce an *anchored marking* to anchored normalizations $M_{v,n}$ by choosing a point $t_v \in \mathbb{S}_v^1$ with the following rules:

- if v is periodic, t_v is the periodic point $1 \in \mathbb{S}_v^1$;
- inductively, for v strictly pre-periodic we choose t_v as the nearest t to 1 in counter-clockwise orientation with $F(t) = t_{f(v)}$.

Landing of pre-periodic points

The pre-periodic points for f_n are marked by pre-periodic points of m_d on $\mathbb{R}/\mathbb{Z}$ using $\eta_n := \eta_{f_n}$. Let $x \in \mathbb{Q}/\mathbb{Z}$ be a rational angle. Then $\eta_n(x)$ is a pre-periodic point for f_n.

Denote the projection of $\eta_n(x)$ to the convex hull $\operatorname{Cvx Hull}(\mathcal{T}_n)$ by $\operatorname{proj}_{\mathcal{T}_n}(\eta_n(x))$. Since there are only countably many pre-periodic points, by a standard diagonal argument and passing to subsequences we may assume that the limit

$$\lim_{n \to \infty} d_{\mathbb{D}}(\operatorname{proj}_{\mathcal{T}_n}(\eta_n(x)), \phi_n(v))$$

exists (it can possibly be ∞) for all $v \in \mathcal{V}$ and for all pre-periodic points x. We say x *lands* at a vertex $v \in \mathcal{V}$ if

$$\lim_{n \to \infty} d_{\mathbb{D}}(\operatorname{proj}_{\mathcal{T}_n}(\eta_n(x)), \phi_n(v)) < \infty.$$

We say x *lands* on $\mathcal{V}$ if x lands at v for some $v \in \mathcal{V}$. The collection of $x \in \mathbb{Q}/\mathbb{Z}$ that land at v is called the *landing angles* at v.

After passing to a subsequence, we may assume the limits $\lim_v \eta_n(x)$ exist in $\mathbb{S}^1_v$ for all v and all pre-periodic points x. Another way to define the landing angles at v is to look at the rescaling limits. The following can be checked easily from the definition:

Lemma 2.16. *A pre-periodic point x lands at $v \in \mathcal{V}$ if and only if*

$$\lim_v \eta_n(x) \notin \xi_v(T_v\mathcal{T}).$$

If x does not land on $\mathcal{V}$, then there exists an edge $E = [v, w] \in \mathcal{T}$ such that the projection $\operatorname{proj}_{\mathcal{T}_n}(\eta_n(x))$ is within bounded distance of $[\phi_n(v), \phi_n(w)]$. We say x *lands on the edge E* in this case.

The *degree* of a cycle $C \subseteq \mathbb{S}^1$ for m_d is the least $e \geq 1$ such that $m_d|_C$ extends to a covering of the circle of degree e. A cycle C is said to be *simple* if its degree is 1 (see [36] for more details). We now show that almost all pre-periodic points land on $\mathcal{V}$:

Proposition 2.17. *All but finitely many pre-periodic points land on $\mathcal{V}$. Moreover, let $\mathcal{C} = \bigcup C_i$ be the set of periodic points that do not land on $\mathcal{V}$. Then*

- *every cycle C_i is simple with the same rotation number;*

- *$m_d|_{\mathcal{C}}$ preserves the cyclic ordering of $\mathcal{C}$.*

Any strictly pre-periodic point that does not land on $\mathcal{V}$ is eventually mapped to $\mathcal{C}$.

Proof. Let x be a periodic point of period q for m_d. Suppose that x does not land on $\mathcal{V}$. We first show that the multiplier at $\eta_n(x)$ converges to 1. Let $a_n \in \mathcal{T}_n$ be at a uniformly bounded distance from $\operatorname{proj}_{\mathcal{T}_n}(\eta_n(x)) \in \operatorname{Cvx Hull}(\mathcal{T}_n)$. Note that $d_{\mathbb{D}}(a_n, \phi_n(v)) \to \infty$ for any vertex $v \in \mathcal{V}$.

Let $M_{a,n} \in \operatorname{Isom}(\mathbb{D})$ with $M_{a,n}(0) = a_n$. Similar to the case of vertices of $\mathcal{T}_n$, we can define a-limit in this setting. In this a-coordinate, after passing to a subsequence, $\mathcal{T}_n$ converges to a geodesic passing through 0. Thus, there exist distinct points $t_1, t_2 = -t_1 \in \mathbb{S}^1_a$

corresponding to the a-limits of vertices. After passing to a subsequence, we assume $\eta_n(x) \to_a t \in \mathbb{S}^1_a$. Since a_n and $\mathrm{proj}_{\mathcal{T}_n}(\eta_n(x))$ are a uniformly bounded distance apart, we have $t \neq t_i$.

After passing to a subsequence, by Propositions 2.9 and 2.5, $M_{a,n}^{-1} \circ f_n^q \circ M_{a,n}$ converges algebraically to a degree 1 map F. An argument as for Lemma 2.13 gives $F(t_i) = t_i$. Since $\eta_n(x)$ is fixed by f_n^q and t is not a hole for F, $F(t) = t$. Thus, $F = \mathrm{id}$ as it fixes three points. Therefore, $(f_n^q)'(\eta_n(x)) \to 1$, in particular $\log |(f_n^q)'(\eta_n(x))| \leq \log 2$ for all sufficiently large n. By [36, Theorem 1.1], these cycles C_i are all simple with the same rotation number, and $m_d|_{\mathcal{C}}$ preserves the cyclic ordering of $\mathcal{C}$.

Since there are only finitely many simple cycles with the same rotation number (see [36, §2]), all but finitely many periodic points land on $\mathcal{V}$.

There are only finitely many edges in $\mathcal{T}$ that are mapped to the edges landed by $\mathcal{C}$. Thus by pulling back, there are only finitely many strictly pre-periodic points landing at edges, and they all come from backward orbits of $\mathcal{C}$. The proposition now follows. ∎

A vertex $v \in \mathcal{V}$ is said to be *simple* if $\delta(v) = 1$, and *critical* if $\delta(v) \geq 2$. The same proof also gives

Proposition 2.18. *Let $v \in \mathcal{V}$. If $f^k(v)$ is simple for all $k \geq 0$, then there are only finitely many pre-periodic points landing at v.*

We immediately have the following, which will be used to construct the dual lamination later.

Corollary 2.19. *There are at most two pre-periodic points landing on an edge E; more precisely, at most one from each side of the edge.*

Proof. Suppose for contradiction that there are two pre-periodic points landing on the same side of the edge E. Then the ribbon structure will give infinitely many periodic points landing on E, contradicting Proposition 2.17. ∎

Pullback of quasi-invariant tree

Given the quasi-invariant tree $\mathcal{T}_n$ for f_n modeled by $f : \mathcal{T} \to \mathcal{T}$, we can construct the pullback quasi-invariant tree $\mathcal{T}_n^1$ as follows.

Let $v \in \mathcal{V}$. Choose an ordering of the d preimages of $\phi_n(v)$ under f_n, and denote them by $w_{1,n}(v), \ldots, w_{d,n}(v)$. Define

$$\widetilde{\mathcal{P}}^1 := \{(w_{i,n}(v)) : i = 1, \ldots, d, \ v \in \mathcal{V}\}.$$

Note that an element of $\widetilde{\mathcal{P}}^1$ is a sequence of points in $\mathbb{D}$ indexed by n and it is a finite set. After passing to a subsequence, we assume that the limits

$$\lim_{n \to \infty} d_{\mathbb{D}}(w_{i,n}(v), w_{j,n}(u)) \quad \text{and} \quad \lim_{n \to \infty} d_{\mathbb{D}}(w_{i,n}(v), \phi_n(u))$$

exist for all $i, j \in \{1, \ldots, d\}$ and all $v, u \in \mathcal{V}$. This defines an equivalence relation on $\widetilde{\mathcal{P}}^1$, and an equivalence class is called a *cluster set*. A cluster set $[(w_n)]$ is said to be *new* if $\lim_{n\to\infty} d_{\mathbb{D}}(w_n, \phi_n(\mathcal{V})) = \infty$. We choose a representative (w_n) for each new cluster set.

Note that by construction, $(f_n(w_n)) = (\phi_n(v))$ for some $v \in \mathcal{V}$. If $a_n \in \mathcal{T}_n$ are such that $d_{\mathbb{D}}(a_n, \phi_n(\mathcal{V})) \to \infty$, then $d_{\mathbb{D}}(f_n(a_n), \phi_n(\mathcal{V})) \to \infty$ by Proposition 2.9. Thus,

$$\lim_{n\to\infty} d_{\mathbb{D}}(w_n, \mathcal{T}_n) = \infty \quad \text{for any new cluster } [(w_n)]. \tag{2.1}$$

So we can apply the same inductive method in the construction of $\mathcal{T}_n$ to add these new cluster sets, and get a new quasi-invariant tree $\mathcal{T}_n^1 \supseteq \mathcal{T}_n$.

After passing to a subsequence, we may assume the trees $\mathcal{T}_n^1$ are all isomorphic with the same ribbon structure. We denote by $\mathcal{T}^1$ the underlying ribbon finite tree, with the isomorphisms

$$\phi_n : \mathcal{T}^1 \to \mathcal{T}_n^1.$$

A simplicial model of the dynamics can be constructed for the pullback and is denoted by

$$f : (\mathcal{T}^1, p) \to (\mathcal{T}^0, p) := (\mathcal{T}, p) \subseteq (\mathcal{T}^1, p).$$

The rescaling limits are defined similarly and the same proof of Lemma 2.13 shows the compatibility of the local dynamics with the tangent map. We remark that the new vertices are all simple, so the rescaling limits are defined by degree 1 maps. Each vertex $v \in \mathcal{V}^0$ of $\mathcal{T}^0$ has d preimages in $\mathcal{T}^1$ counted with multiplicity. By (2.1), if E is an edge of $\mathcal{T}^0$, then it is also an edge of $\mathcal{T}^1$, i.e., the inclusion map from $\mathcal{T}^0$ to $\mathcal{T}^1$ is simplicial.

We also remark that a priori, there are many choices in this construction. They all give the same simplicial model $f : (\mathcal{T}^1, p) \to (\mathcal{T}^0, p)$, as they are equal to the unique one constructed combinatorially in §3.

The pullback can be iterated, and we denote the k-th pullback simplicial model by $f : (\mathcal{T}^k, p) \to (\mathcal{T}^{k-1}, p) \subseteq (\mathcal{T}^k, p)$. We also denote by $f : (\mathcal{T}^\infty, p) \to (\mathcal{T}^\infty, p)$ the union of these maps.

Dual lamination of $\mathcal{T}$

Let $A_v \subseteq \mathbb{Q}/\mathbb{Z}$ be the set of rational angles landing at v. Since $\eta_n : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ is a homeomorphism, and $\mathcal{T}_n$ is a ribbon tree, for any $v \neq w \in \mathcal{V}$, A_v and A_w are *unlinked*, i.e., there exists an interval $I \subseteq \mathbb{R}/\mathbb{Z}$ such that $A_v \subseteq I$ and $A_w \subseteq \mathbb{R}/\mathbb{Z} - I$.

Two intervals $J_1, J_2 \subseteq \mathbb{R}/\mathbb{Z}$ are said to be *essentially the same* if $\mathrm{Int}(J_1) = \mathrm{Int}(J_2)$ and $\overline{J_1} = \overline{J_2}$. Let $E = [v, w]$ be an edge. We claim that there exists an essentially unique interval I with $A_v \subseteq I$ and $A_w \subseteq \mathbb{R}/\mathbb{Z} - I$. Otherwise, there are infinitely many rational angles that are separated by A_v and A_w; since $[v, w]$ is an edge, these angles must land on E, which is a contradiction to Corollary 2.19. We denote $\partial I = \{t_E^\pm\}$, and call $t_E^\pm$ the *dual angles* for E. We remark that it is possible that I is a degenerate interval consisting of a single point. In this case, ∂I consists of a single point.

An equivalent way to compute $t_E^\pm$ is as follows. Let $x_E \in T_v \mathcal{T}$ be the tangent vector at v associated to E. We consider the set

$$I_E(v) := \left\{ t \in \mathbb{R}/\mathbb{Z} : \lim_v \eta_n(t) = \xi_v(x_E) \in \mathbb{S}_v^1 \right\} \subseteq \mathbb{R}/\mathbb{Z}.$$

Since η_n is a homeomorphism, $I_E(v)$ is an interval. By Lemma 2.16, one can verify that $\partial I_E(v) = t_E^\pm$. Note that if the other boundary point $w \in \partial E$ is chosen, then $I_E(w)$ is essentially the complement of $I_E(v)$ in $\mathbb{R}/\mathbb{Z}$, so the boundary points are the same.
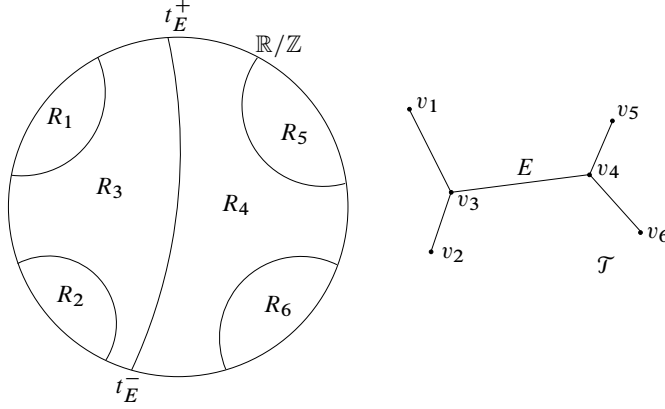

Fig. 2.4. An illustration of the dual lamination. The rational angles in $\partial R_i \cap \mathbb{R}/\mathbb{Z}$ land at v_i. The dual angles $t_E^\pm$ for the edge E are labeled on the figure.

It can be verified that these angles are compatible with the dynamics:

Proposition 2.20. *Let E be an edge of $\mathcal{T}$ with dual angles $t_E^\pm$. Then $m_d(t_E^\pm)$ are the dual angles for $f(E)$.*

A *lamination* $\mathcal{L}$ is a family of disjoint hyperbolic geodesics in $\mathbb{D}$ together with the two end points in $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, whose union $|\mathcal{L}| := \bigcup \mathcal{L}$ is closed. An element of the lamination is called a *leaf* of the lamination.

We define the leaf associated with the edge E as the hyperbolic geodesics in $\mathbb{D}$ connecting $t_E^\pm \in \mathbb{R}/\mathbb{Z} \cong \partial \mathbb{D}$. It is easy to check that leaves for different edges have disjoint interiors.

The *dual finite lamination* for $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is defined as the union of all leaves for edges of $\mathcal{T}$, and is denoted by $\mathcal{L}_{\mathcal{T}}^F$. The leaves can be constructed for edges of any pullbacks of $\mathcal{T}$. We call the closure of the union of leaves for edges in $\mathcal{T}^\infty$ the *dual lamination* of $\mathcal{T}$, and denote it by $\mathcal{L}_{\mathcal{T}}$.

We remark that the lamination $\mathcal{L}_{\mathcal{T}}$ is defined abstractly using dual angles in $\mathbb{R}/\mathbb{Z}$. The leaves of $\mathcal{L}_{\mathcal{T}}$ do not lie in the same disk $\mathbb{D}$ where $f_n \in \mathcal{B}_d$ or $\mathcal{T}_n$ is defined.

The lamination $\mathcal{L}_{\mathcal{T}}$ gives an equivalence relation $\sim_{\mathcal{T}}$ on $\mathbb{R}/\mathbb{Z}$: $a \sim_{\mathcal{T}} b$ if there exists a finite chain of leaves connecting a and b. Note that different laminations may generate the same equivalence relation.

Periodic Fatou and Julia points

For reasons that will appear apparent later in §6, we introduce

Definition 2.21. Let $v \in \mathcal{V}$. It is called a *Fatou point* if it is eventually mapped to a critical periodic orbit, and a *Julia point* otherwise.

Let $a, b \in \mathcal{V}$. We use $[a, b], (a, b), [a, b)$ and $(a, b]$ to denote the paths in $\mathcal{T}$ that connect a, b, with appropriate boundary points removed.

Lemma 2.22. *If $v \neq p$ is a periodic point, and $[p, v)$ contains a periodic Fatou point, then the first return rescaling limit F at v has an attracting fixed point on $\mathbb{S}^1_v$.*

Proof. After passing to an iterate, we may assume that f fixes $[p, v]$. Let $t^v_p := \lim_v \phi_n(p)$. By Lemma 2.13, t^v_p is a fixed point of the rescaling limit F_v. Since f_n fixes $\phi_n(p)$, by the Schwarz lemma, $M^{-1}_{v,n} \circ f_n \circ M_{v,n}$ moves points towards $M^{-1}_{v,n}(\phi_n(p))$. Since $M^{-1}_{v,n} \circ f_n \circ M_{v,n}$ converges to F_v and $M^{-1}_{v,n}(\phi_n(p)) \to t^v_p$, the fixed point t^v_p is non-repelling.

Suppose for contradiction that t^v_p is parabolic. Then $d_{\mathbb{D}}(x, F(x))$ can be made arbitrarily small[1] by making x close to t^v_p. Therefore, for any $\epsilon > 0$, there exists x_n close to t^v_p in v-coordinate so that $d_{\mathbb{D}}(x_n, f_n(x_n)) < \epsilon$ for all sufficiently large n.

On the other hand, let c be the fixed Fatou point in $[p, v)$. Let F_c be the rescaling limit at c. Note that $\deg F_c = \delta(c) \geq 2$, as c is a Fatou point.

We claim that $t^c_v := \lim_c \phi_n(v)$ is a repelling fixed point of F_c. There are two cases. If $c = p$, then $F_c = F_p$ has a fixed point at 0, which is necessarily attracting as $\deg F_c \geq 2$. Thus, t^c_v is repelling. If $c \neq p$, then by the same argument as for t^v_p, the point $t^c_p := \lim_c \phi_n(p) \in \mathbb{S}^1_c$ is a non-repelling fixed point of F_c. Since t^c_v is different from t^c_p, it is repelling. This proves the claim.

Therefore for $y \in \mathbb{D}_c$ near t^c_v, $d_{\mathbb{D}}(y, F_c(y)) \geq K$ for some K depending on the multiplier of the repelling fixed point t^c_v under F_c, and the angle $\angle F_c(y) y t^c_v$ is greater than or equal to $\pi/2$. Thus, for all sufficiently large n, there exists y_n close to t^c_v in c-coordinate such that $d_{\mathbb{D}}(y_n, f_n(y_n)) \geq K$, and the angle $\angle f(y_n) y_n x_n$ is greater than or equal to $\pi/2$. By choosing ϵ small enough, we have $d_{\mathbb{D}}(f_n(x_n), f_n(y_n)) > d_{\mathbb{D}}(x_n, y_n)$, which contradicts the Schwarz lemma. ∎

Corollary 2.23. *If $v \neq p$ is a periodic Julia point, and $[p, v)$ contains a periodic Fatou point, then v is not a branch point of $\mathcal{T}$.*

Proof. Suppose for contradiction that v is a branch point; then the first return rescaling limit F at v is the identity map as it has degree 1 and fixes three points on the circle. This is a contradiction as F has an attracting fixed point on the circle by Lemma 2.22. ∎

We call a periodic Fatou point v *parabolic* or *boundary-hyperbolic* if the first return rescaling limit is parabolic or boundary-hyperbolic. If p is critical, then $[p, v)$ contains a periodic Fatou point for any $v \neq p$, so we have

---

[1] Note that if t_p is attracting, the hyperbolic distance $d_{\mathbb{D}}(x, F(x))$ is uniformly bounded away from 0.

Corollary 2.24. *If $\delta(p) \geq 2$, then*

- *every periodic branch point is a Fatou point;*
- *every periodic Fatou point other than p is boundary-hyperbolic.*

3. Angled tree map

In this section, we introduce abstract angled tree maps that give combinatorial descriptions of the pointed quasi-invariant trees for $f_n \in \mathcal{B}_d$. The construction is similar to the angled Hubbard tree introduced in [45] with two major differences:

- to work with quasi post-critically finite degenerations in $\mathcal{B}_d$, the angled tree maps in our setting are simplicial and marked;
- to capture the dynamics of parabolic or boundary-hyperbolic rescaling limits, the angles are allowed to be $0^{\pm}$.

We remark that unlike the angled Hubbard trees, the angle 0 plays a special role in our setting.

Let $(\mathcal{T}, p)$ be a pointed finite tree with a ribbon structure and vertex set $\mathcal{V}$. Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be a simplicial map. The tangent space at a vertex v is identified with the set of incident edges to v and is denoted by $T_v\mathcal{T}$. We define the local degree function $\delta : \mathcal{V} \to \mathbb{Z}_{\geq 1}$ which assigns an integer $\delta(v) \geq 1$ to each vertex $v \in \mathcal{V}$. A vertex v is said to be *critical* if $\delta(v) \geq 2$ and *simple* otherwise. The *degree* of the map f is defined by

$$\deg(f) := 1 + \sum_{v \in \mathcal{V}} (\delta(v) - 1),$$

and we always assume $\deg(f) \geq 2$. We also assume that $\mathcal{T}$ is *non-trivial*, i.e. it contains more than one point. We say $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is *minimal* if $\mathcal{T}$ is the convex hull of all critical orbits and $\mathcal{V}$ is the smallest set containing all critical orbits such that f is simplicial.

Angle structure on $\mathcal{T}$

We identify $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and $m_d : \mathbb{S}^1 \to \mathbb{S}^1$ is the multiplication by d map. By our convention, m_1 is the identity map. For $d \geq 2$, m_d gives a topological model of the dynamics on the Julia set of a degree d hyperbolic and a doubly parabolic Blaschke product.

To set up a framework that also works for singly parabolic or boundary-hyperbolic Blaschke products uniformly, we consider an extended circle $\mathbb{S}_d^1$, which is naturally regarded as a *cyclically ordered set* (see [35, §2]). As a set, $\mathbb{S}_d^1$ is constructed from $\mathbb{S}^1$ by adding (formal symbols) x^-, x^+ for any point x in the backward orbit of 0 under m_d for $d \geq 2$. The cyclic ordering on $\mathbb{S}_d^1$ is defined so that x^- (or x^+) is regarded as a point infinitesimally smaller than x (or bigger than x respectively) in the standard identification of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. We use the convention that $\mathbb{S}_1^1 = \mathbb{S}^1$.

Given any integer $k \geq 1$, the map m_k naturally extends to $m_k : \mathbb{S}_d^1 \to \mathbb{S}_d^1$ by setting $m_k(x^\pm) = m_k(x)^\pm$. This is well-defined because if x is in the backward orbit of 0 under m_d, then $m_k(x)$ is also in that orbit.

If f is a degree d boundary-hyperbolic Blaschke product, i.e., f has an attracting fixed point a on the circle, then the Julia set J of f is a Cantor set on $\mathbb{S}^1$. The complement $\mathbb{S}^1 - J$ consists of countably many intervals, which are all eventually mapped to the unique interval $I \subseteq \mathbb{S}^1$ that contains the attracting fixed point a. The boundary ∂I consists of two repelling fixed points of f. Let $\mathcal{O}(a)$ be the backward orbit of the attracting fixed point a. Then there exists a bijective map $\eta_f : \mathbb{S}_d^1 \to J(f) \cup \mathcal{O}(a)$ which preserves the cyclic ordering and such that

$$f \circ \eta_f = \eta_f \circ m_d.$$

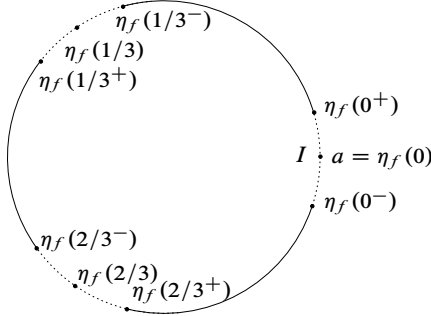Note that $\eta_f(0) = a$ and $\eta_f(0^\pm) = \partial I$.

Fig. 3.1. An illustration of the conjugacy η_f for a degree 3 boundary-hyperbolic Blaschke product. The Julia set is a Cantor set, constructed by removing the backward orbits of the interval I.

To model the dynamics of the pullbacks, we construct $\mathbb{S}_{d,D}^1$ by adding x^-, x^+ to $\mathbb{S}^1$ if $m_D(x)$ is in the backward orbit of 0 under m_d, and the cyclic ordering is constructed in the same way. Note that by this construction, $m_D : \mathbb{S}_{d,DD'}^1 \to \mathbb{S}_{d,D'}^1$ is a degree D covering between cyclically ordered sets (see [35, §2] for detailed definitions). Note that $\mathbb{S}_{d,1}^1 = \mathbb{S}_d^1$. We remark that the intervals can be defined naturally for $\mathbb{S}_{d,D}^1$, and we denote them by $[a,b], (a,b], [a,b), (a,b)$ with appropriate boundary points removed.

If $v \in \mathcal{V}$ has pre-period l and period q, we define the *cumulative degree* of v to be

$$\Delta(v) = \delta(f^l(v))\delta(f^{l+1}(v))\ldots\delta(f^{l+q-1}(v)),$$

and its *cumulative pre-periodic degree* to be

$$\Delta_{\mathrm{pre}}(v) = \delta(v)\delta(f(v))\ldots\delta(f^{l-1}(v)).$$

We use the convention that $\Delta_{\mathrm{pre}}(v) = 1$ for all periodic vertices.

We attach an extended circle $\mathbb{S}_{\Delta(v),\Delta_{\mathrm{pre}}(v)}^1$ to a vertex $v \in \mathcal{V}$. We define an *angle function* α at v as an injective map

$$\alpha_v : T_v \mathcal{T} \hookrightarrow \mathbb{S}_{\Delta(v),\Delta_{\mathrm{pre}}(v)}^1.$$

We say α is *regular* at v if $\alpha_v(T_v\mathcal{T}) \subseteq \mathbb{S}^1 \subseteq \mathbb{S}^1_{\Delta(v),\Delta_{\mathrm{pre}}(v)}$.

Definition 3.1. We say an angle function α is *compatible* if for any $v \in \mathcal{V}$,

(1) α_v is *cyclically compatible*: if $x_1, x_2, x_3 \in T_v\mathcal{T}$ are clockwise oriented, then so are $\alpha_v(x_1), \alpha_v(x_2), \alpha_v(x_3)$;

(2) α_v is *dynamically compatible*:

- if $v = p$ and $\delta(p) = 1$, then there exists a rigid rotation R, which is necessarily a rational rotation, such that $R \circ \alpha_p = \alpha_p \circ Df|_{T_p\mathcal{T}}$;

- otherwise, $m_{\delta(v)} \circ \alpha_v = \alpha_{f(v)} \circ Df|_{T_v\mathcal{T}}$;

(3) α_v is *p-compatible*: if $v \neq p$ is periodic and $x \in T_v\mathcal{T}$ is the tangent vector in the direction of p, then $\alpha_v(x) = 0$.

We remark that if $v \neq p$ is a periodic point of period q, and $x \in T_v\mathcal{T}$ is in the direction of p, then $D_v f^q(x) = x$ as f is simplicial. Thus condition (3) is compatible with the dynamics.

Definition 3.2. An *angled tree map* is a triple

$$(f : (\mathcal{T}, p) \to (\mathcal{T}, p), \delta, \alpha = \{\alpha_v\})$$

of a simplicial map on a pointed finite ribbon tree together with a local degree function δ and a compatible angle function α which is regular at p.

We shall use $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ or simply $\mathcal{T}$ to denote an angled tree map if the dynamics, local degree function and angle function are not ambiguous.

Pullback of an angled tree map

Given an angled tree map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$, one can naturally construct a new angled tree map by pulling back by the dynamics. More precisely, take $v \in \mathcal{V}$ and a tangent direction $a \in T_{f(v)}\mathcal{T}$. Let S_a be the component of $\mathcal{T} - \{f(v)\}$ corresponding to the direction $a \in T_{f(v)}\mathcal{T}$. Let $B \subseteq \mathbb{S}^1_{\Delta(v),\Delta_{\mathrm{pre}}(v)}$ be the preimage of $\alpha(a) \in \mathbb{S}^1_{\Delta(f(v)),\Delta_{\mathrm{pre}}(f(v))}$ under the corresponding map from $\mathbb{S}^1_{\Delta(v),\Delta_{\mathrm{pre}}(v)}$ to $\mathbb{S}^1_{\Delta(f(v)),\Delta_{\mathrm{pre}}(f(v))}$. We attach a copy of S_a at every point in $B - \alpha(T_v\mathcal{T})$. The dynamics extend naturally to the new copies by identifications, and the angular structures are defined by pulling back with the identity map m_1.

Let $\mathcal{T}^1$ be the angled tree constructed from $\mathcal{T}$ by running the above algorithm for all vertices v and all tangent directions $a \in T_w\mathcal{T}$. We call

$$f : (\mathcal{T}^1, p) \to (\mathcal{T}^0, p) := (\mathcal{T}, p) \subseteq (\mathcal{T}^1, p)$$

the *(first) pullback* of $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$.

Note that each vertex $v \in \mathcal{T}^0$ has exactly d preimages in $\mathcal{T}^1$ counted with multiplicity, and the inclusion map $i : \mathcal{T}^0 \to \mathcal{T}^1$ is simplicial. Also note that the map $f : (\mathcal{T}^1, p) \to (\mathcal{T}^1, p)$ is no longer minimal.

This construction can be iterated. We denote the k-th pullback by $f : (\mathcal{T}^k, p) \to (\mathcal{T}^k, p)$, and the union of the pullbacks by $f : (\mathcal{T}^\infty, p) \to (\mathcal{T}^\infty, p)$.

External angles, markings and anchored conditions

Similar to the abstract Hubbard trees (see [45, Chapter III, §4]), external angles can be defined using the dynamics on $\mathcal{T}$. They can be described as follows.

Let $d = \deg(f)$ be the degree of $f : \mathcal{T} \to \mathcal{T}$. Let $\mathcal{T}^\infty$ be the union of the pullbacks of $\mathcal{T}$. Let $\epsilon(\mathcal{T}^\infty)$ be the set of ends of $\mathcal{T}^\infty$, viewed as a topological space. Note that the ribbon structure of $\mathcal{T}^\infty$ makes $\epsilon(\mathcal{T}^\infty)$ a cyclically ordered set. Since $\mathcal{T}$ is non-trivial, $\epsilon(\mathcal{T}^\infty)$ is infinite. The simplicial map $f : \mathcal{T}^\infty \to \mathcal{T}^\infty$ induces a map $f_* : \epsilon(\mathcal{T}^\infty) \to \epsilon(\mathcal{T}^\infty)$ (cf. [35, §3]). There is a natural cyclical order-preserving map

$$\phi : \epsilon(\mathcal{T}^\infty) \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$$

that transports the dynamics of f_* on $\epsilon(\mathcal{T}^\infty)$ to the dynamics of m_d on $\mathbb{R}/\mathbb{Z}$ (cf. [35, §6]). We call ϕ an *external angle marking*, or simply a *marking*. Since the image $\phi(\epsilon(\mathcal{T}^\infty))$ is invariant under m_d^{-1}, it is necessarily dense. We remark that there are $d - 1$ choices of markings, and any two markings are related by post-composition with an element of the automorphism group $\mathbb{Z}/(d-1)$ of m_d. An angled tree map together with a marking is called a *marked angled tree map*.

Note that the p-compatibility condition in Definition 3.1 gives the unique normalization of angle functions at a periodic vertex $v \neq p$. Given a marking ϕ on an angled tree map $\mathcal{T}$, we can normalize the angle function at all other vertices with the following anchored convention, which is an analogue of Definitions 2.12 and 2.15.

We remark that 0 may not be in the image $\phi(\epsilon(\mathcal{T}^\infty))$. But since $\phi(\epsilon(\mathcal{T}^\infty))$ is dense in $\mathbb{R}/\mathbb{Z}$, there exists a sequence $\gamma_n \in \epsilon(\mathcal{T}^\infty)$ with $\phi(\gamma_n) \to 0^-$, i.e., $\phi(\gamma_n)$ approaches 0 from below. We represent each end γ_n by a path in $\mathcal{T}^\infty$ that starts at p. Since α is regular at p, α_p extends to

$$\alpha_p : T_p \mathcal{T}^\infty \to \mathbb{S}^1 \subseteq \mathbb{S}^1_{\Delta(p)}.$$

Since each path γ_n gives a tangent vector $x_n \in T_p \mathcal{T}^\infty$, we can associate an internal angle $t_n := \alpha_p(x_n)$ to γ_n. With the standard topology on $\mathbb{S}^1$, one can verify that the limit of t_n exists and is independent of the sequence γ_n that we choose. We call $t_p := \lim t_n \in \mathbb{S}^1$ the *internal angle* at p *with respect to the marking* (see Figure 3.2).

Definition 3.3. Let $(f : (\mathcal{T}, p) \to (\mathcal{T}, p), \delta, \alpha)$ be a marked angled tree map. Let t_p be the internal angle at p with respect to the marking. It is said to be *anchored* if the following hold:

- if $\delta(p) = 1$, then $t_p = 0$;
- if $\delta(p) \geq 2$, then $t_p \in (-\frac{1}{\Delta(p)-1}, 0]$;
- if $v \neq p$ is strictly pre-periodic and $x \in T_v \mathcal{T}$ is the tangent vector in the direction of p, then $\alpha_v(x) \in (-\frac{1}{\delta(v)}, 0]$.

We remark that by post-composing the angled functions with rotations that keep the compatibility condition, one can always make the angled tree map anchored. Throughout this paper, we shall always assume that an angled tree map is marked and the angle functions are anchored.
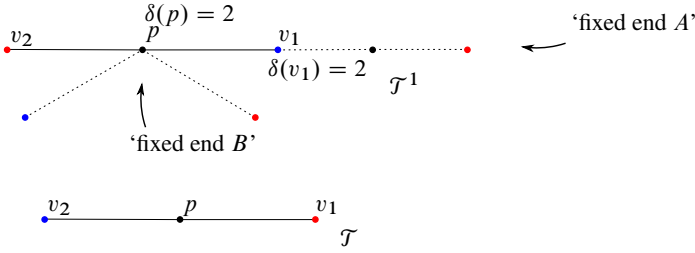
Fig. 3.2. Vertices in $\mathcal{T}^1$ are sent to vertices in $\mathcal{T} = \mathcal{T}^0$ with the same color. This degree 3 angled tree map has two markings ϕ_A, ϕ_B. The corresponding ends of zero external angle are illustrated by the two arrows. Note that $0 \in \mathbb{R}/\mathbb{Z}$ is in the image of ϕ_A but not in the image of ϕ_B. The corresponding internal angles at p with respect to ϕ_A and ϕ_B are $t_{p,A} = 1/3$ and $t_{p,B} = 0$.

Realizing angled tree maps

Let $f_n \in \mathcal{B}_d$ be a quasi post-critically finite sequence. By the discussion in §2, we have an induced (marked) simplicial map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ with a local degree function δ modeling the dynamics of the critical orbits. By construction, f is minimal. The cumulative degree and cumulative pre-periodic degree for v are defined similarly.

To define the angle functions, we assign the angle $0 \in \mathbb{S}^1_{\Delta(v), \Delta_{\mathrm{pre}}(v)}$ to the marked point $t_v \in \mathbb{S}^1_v$ (see Definition 2.15). The angles are then determined by the conjugacy of the tangent map with the rescaling limit (see Lemma 2.13). We remark that the angles assigned at periodic Julia points v are rather artificial. We use the convention to assign the angles $i/\nu, i = 0, \ldots, \nu - 1$, according to their cyclic order where ν is the valence at v.

The above construction gives an angled tree map. An angled tree map $(f : (\mathcal{T}, p) \to (\mathcal{T}, p), \delta, \alpha)$ that arises in this way is said to be *realized* by $f_n \in \mathcal{B}_d$. Let $\mathcal{T}_n$ be the sequence of quasi-invariant trees for f_n. Since we will be working with pullbacks, we shall say $\mathcal{T}_n$ *realizes* the angled tree map $\mathcal{T}$ or the angled tree map $\mathcal{T}$ is *realized* by $\mathcal{T}_n$ when the underlying maps are not ambiguous. We can associate an angled tree map to the k-th pullback $\mathcal{T}_n^k$ of the quasi-invariant tree $\mathcal{T}_n$. One can verify that this angled tree map is the k-th pullback of the angled tree map $\mathcal{T}$. We shall say $\mathcal{T}_n^k$ realizes $\mathcal{T}^k$.

Admissible angled tree maps

Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be an angled tree map. We now describe a sufficient condition for realization. A periodic vertex v is said to be *attached to p* if $[p, v)$ contains no Fatou point. Here $[p, v)$ is the path in $\mathcal{T}$ that connects p and v with the boundary point v removed. The *core $\mathcal{T}^C$* of an angled tree map is defined as the convex hull of all periodic vertices attached to p. Since f is simplicial, any vertex in $\mathcal{T}^C$ is periodic. Note that if $\delta(p) \geq 2$, then $\mathcal{T}^C = \{p\}$.

A vertex is said to be an *end point* of the finite tree $\mathcal{T}$ if the valence at v is 1. We say a tree $\mathcal{T}$ is *star-shaped* if there exists a unique vertex in $\mathcal{T}$ that is not an end point. The unique vertex is called the *center* for the star-shaped tree. For $k \geq 2$, we call a star-

shaped tree with k end points a *k-star*. By our definition, a tree with a single vertex is also star-shaped with no end point.

Definition 3.4. The core $\mathcal{T}^C$ is said to be *critically star-shaped* if

- $\mathcal{T}^C$ is star-shaped with center p;
- every end point of $\mathcal{T}^C$ is a periodic Fatou point;
- for any vertex $v \in \mathcal{T}^C$, the angle function α_v is regular at v.

If $\delta(p) \geq 2$, then $\mathcal{T}^C = \{p\}$ and the conditions are trivially satisfied. If $\delta(p) = 1$, these conditions give a way to 'normalize' the dynamics at p.

Definition 3.5. An angled tree map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is said to be *admissible* if

- the core $\mathcal{T}^C$ is critically star-shaped;
- every periodic branch point other than p is a Fatou point.

Assume $\mathcal{T}^C$ is critically star-shaped. By Corollary 2.23, the last condition above is necessary for realization. Thus for $\delta(p) \geq 2$, admissibility is a necessary condition for realization. In §6, we introduce a notion of admissible splitting of a simplicial pointed Hubbard tree. We shall see that this produces an admissible angled tree map (see Proposition 6.2).

4. Realizing admissible angled tree map

In this section, we shall prove the following theorem:

Theorem 4.1. *If a minimal angled tree map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is admissible, then it is realizable.*

The proof is by induction. We start by showing that any degree 2 admissible angled tree map is realizable. We then show we can build a degree $d + 1$ admissible angled tree map from a degree d one. By the induction hypothesis, the degree d admissible angled tree map is realizable by a sequence of $g_n \in \mathcal{B}_d$. The induction step is proved by carefully adding a zero of the Blaschke product. More precisely, we construct a sequence $f_n = e^{i\theta_n} \frac{z - a_n}{1 - \overline{a}_n z} \cdot g_n$ where θ_n and a_n are chosen carefully, and we show that the corresponding angled tree map is as desired.

For our purposes, it is convenient to define the distance $d_{\mathcal{T}}(v, w)$ between two vertices in $\mathcal{T}$ as the number of edges in the shortest path between v and w. Since f is simplicial, f is distance non-increasing with respect to this metric $d_{\mathcal{T}}$.

More precise statement

A degree d proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is said to be

- *1-anchored* if $f(1) = 1$;
- *fixed point centered* if $f(0) = 0$;

- *zeros centered* if the sum

$$a_1 + \cdots + a_d$$

  of the points of $f^{-1}(0)$ (counted with multiplicity) is equal to 0;

- *M-uni-critical* if the critical points are all within hyperbolic distance M of 0.

Take $f \in \mathcal{B}_d$. By conjugating with a rotation fixing 0 that sends $\eta_f(0)$ to 1, we can canonically identify f with a 1-anchored, fixed point centered Blaschke product. The following definition can be found in [38, §4]:

Definition 4.2. Let $\mathcal{S} \subseteq \mathcal{V}$ be an invariant subset. To each $v \in \mathcal{S}$ we associate a copy $\mathbb{D}_v$ of the disk, and we define the *normalized mapping scheme* $\mathcal{F}$ for $\mathcal{S}$ as a collection of 1-anchored proper holomorphic maps

$$\mathcal{F}_v : \mathbb{D}_v \to \mathbb{D}_{f(v)}$$

of degree $\delta(v)$, which are either fixed point centered or zeros centered according as v is periodic or strictly pre-periodic under $f : \mathcal{S} \to \mathcal{S}$.

Denote by $\mathcal{S}_p \subseteq \mathcal{V}$ the backward orbit of p. If $\delta(p) \geq 2$, the rescaling limits give a normalized mapping scheme on $\mathcal{S}_p$. Since the rescaling limits are marked by the anchored convention (see Definition 2.15), we say that conjugacies ϕ_v between the rescaling limits and mapping schemes have *compatible markings* if $\phi_v(t_v) = 1$. We prove the following more precise and technical statement which immediately implies Theorem 4.1:

Proposition 4.3. *Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be a minimal admissible angled tree map of degree d. When $\delta(p) \geq 2$, let $\mathcal{F}$ be a normalized mapping scheme on $\mathcal{S}_p$. There exists a K-quasi post-critically finite sequence $f_n \in \mathcal{B}_d$ realizing the angled tree map such that*

(1) *the rescaling limits on $\mathcal{S}_p$ are conjugate to $\mathcal{F}$ with compatible markings;*

(2) *there exists a constant M depending only on the tree map f (and thus independent of $\mathcal{F}$) such that*

  (a) *for any periodic cycle $\mathcal{C}$ other than p, there exists a periodic point $v \in \mathcal{C}$ such that the first return rescaling limit*

$$F_v^q : \mathbb{D}_v \to \mathbb{D}_v$$

  is M-uni-critical, where q is the period of v;

  (b) *for any strictly pre-periodic vertex $w \notin \mathcal{S}_p$, the rescaling limit $F_w : \mathbb{D}_w \to \mathbb{D}_{f(w)}$ is M-uni-critical and the critical values are within hyperbolic distance M of $0 \in \mathbb{D}_{f(w)}$.*

Note that condition (1) is vacuously satisfied when $\delta(p) = 1$. We remark that as we degenerate $\mathcal{F}$, K may go to infinity. Since M is independent of $\mathcal{F}$, the last condition says that nevertheless, the critical points for each vertex v stay at a uniformly bounded distance from each other. Thus, the first return rescaling limit at v is almost uni-critical. This provides compactness that will be used in the successive degenerations (see Lemma 4.9 and Proposition 6.15).

We also remark that the 'M-uni-critical' condition may not hold for all periodic points in the periodic cycle: if there are two or more critical vertices in the periodic cycle, the fact that the first return map is M-uni-critical at one periodic vertex does not mean that the map is M-uni-critical at other periodic vertices in this cycle.

Estimates in hyperbolic geometry

We start with some useful estimates that will be used frequently in the proof. The following lemma follows from the Schwarz lemma and Koebe distortion theorem.

Lemma 4.4. *Let U be a simply connected domain in $\mathbb{C}$ with hyperbolic metric $\rho_U |dz|$. Then for $z \in U$,*

$$\frac{1}{2d_{\mathbb{R}^2}(z, \partial U)} \leq \rho_U(z) \leq \frac{2}{d_{\mathbb{R}^2}(z, \partial U)}.$$

The following fact in hyperbolic geometry is very useful.

Lemma 4.5. *Let $z, w \in \mathbb{D}$ with $|z| = 1 - \delta$ and $|w| = 1 - \delta^s$. If $0 \leq s < 1$, then*

$$sd_{\mathbb{D}}(0, z) \leq d_{\mathbb{D}}(0, w) \leq sd_{\mathbb{D}}(0, z) + \log 2.$$

If $s > 1$, then

$$s(d_{\mathbb{D}}(0, z) - \log 2) \leq d_{\mathbb{D}}(0, w) \leq sd_{\mathbb{D}}(0, z).$$

Proof. Note that $d_{\mathbb{D}}(0, z) = \log \frac{2-\delta}{\delta}$ and $d_{\mathbb{D}}(0, w) = \log \frac{2-\delta^s}{\delta^s}$. So

$$d_{\mathbb{D}}(0, w) - sd_{\mathbb{D}}(0, z) = \log(2 - \delta^s) - \log(2 - \delta)^s.$$

If $s < 1$, then $(2 - \delta)^s \leq 2 - \delta^s$, so $d_{\mathbb{D}}(0, w) - sd_{\mathbb{D}}(0, z) \in [0, \log 2]$.
If $s > 1$, then $(2 - \delta)^s \geq 2 - \delta^s$, so $d_{\mathbb{D}}(0, w) - sd_{\mathbb{D}}(0, z) \in [-s \log 2, 0]$. ∎

Given $a \in \mathbb{D}$, we denote

$$\hat{a} = a/|a| \in \mathbb{S}^1, \quad \delta_a = 1 - |a| \quad \text{and} \quad \rho_a = d_{\mathbb{D}}(0, a).$$

Lemma 4.6. *Let $M_a(z) = \frac{z-a}{1-\bar{a}z}$. Then for $0 < s < 1$,*

$$|M_a(z) + \hat{a}| \leq 2\delta_a^{1-s} \quad \text{for all } z \in B(0, 1 - \delta_a^s) \supseteq B_{\mathbb{H}^2}(0, s\rho_a),$$

where $B_{\mathbb{H}^2}(0, s\rho_a)$ is the hyperbolic ball centered at 0 with radius $s\rho_a$.

Proof. Note that

$$|M_a(z) + \hat{a}| = \left| \frac{z - a + \hat{a} - \hat{a}\bar{a}z}{1 - \bar{a}z} \right| = \left| \frac{\delta_a z + \hat{a}\delta_a}{1 - \bar{a}z} \right|.$$

Since $|z| < 1$ and $|\hat{a}| = 1$, it follows that $|\delta_a z + \hat{a}\delta_a| \leq 2\delta_a$. Since $|z| \leq 1 - \delta_a^s$, we have $|1 - \bar{a}z| \geq \delta_a^s$. Thus, $|M_a(z) + \hat{a}| \leq 2\delta_a/\delta_a^s = 2\delta_a^{1-s}$. ∎

More generally, we have the following estimate in terms of hyperbolic geometry:

Lemma 4.7. *Let $\theta > 0$. There exists a constant $C = C(\theta)$ such that the following holds. Let $M_a(z) = \frac{z-a}{1-\bar{a}z}$. Let $z \in \mathbb{D}$ be such that the angle $\angle 0za$ between the hyperbolic geodesic segments $[0, z]$ and $[z, a]$ satisfies $\angle 0za \geq \theta$. Then*

$$|M_a(z) + \hat{a}| \leq C\delta,$$

where δ is the positive number such that the hyperbolic ball $B_{\mathbb{H}^2}(0, \rho)$ with $\rho := d_{\mathbb{D}}(a, z)$ has Euclidean radius $1 - \delta$.

Proof. Since the angle between $[a, z]$ and $[z, 0]$ is bounded below by θ, there exists a constant $C_1 = C_1(\theta)$ such that $\rho_a \geq \rho_z + \rho - C_1$, where $\rho_a = d_{\mathbb{D}}(0, a)$ and $\rho_z = d_{\mathbb{D}}(0, z)$. Thus, there exists a constant $C = C(\theta)$ such that $\delta_a \leq \frac{C}{2}\delta_z\delta$, where $\delta_a = 1 - |a|$ and $\delta_z = 1 - |z|$. By the same computation as in Lemma 4.6, we have $|M_a(z) + \hat{a}| \leq 2\delta_a/\delta_z$ $\leq C\delta$. ∎

We shall also use the following estimate for critical points:

Lemma 4.8. *Let $C, \theta > 0$. Let $x_1, \ldots, x_k, z \in \mathbb{D}$ be $k + 1$ points with angles $\angle x_i z x_j \geq \theta$ for any pairs i, j. Let $f : \mathbb{D} \to \mathbb{D}$ be a proper map of degree d such that $f(z) = a$ and $f(x_i) = b$ for all i. Suppose that*

$$d_{\mathbb{D}}(z, x_i) \leq d_{\mathbb{D}}(a, b) + C.$$

Then there exists a constant $R = R(C, \theta, d)$ such that there are $k - 1$ critical points (counted with multiplicity) of f in $B_{\mathbb{H}^2}(z, R)$.

Proof. Let $L = d_{\mathbb{D}}(a, b)$, and let $z_{i,t}$ and a_t be the points on the geodesic segments $[z, x_i]$ and $[a, b]$ with $d_{\mathbb{D}}(z, z_{i,t}) = d_{\mathbb{D}}(a, a_t) = t$. Since $d_{\mathbb{D}}(z_{i,t}, x_i) \leq L + C - t$, by the Schwarz lemma we have

$$f(z_{i,t}) \in B_{\mathbb{H}^2}(a, t) \cap B_{\mathbb{H}^2}(b, L + C - t).$$

Thus, there exists a constant $R_1 = R_1(C)$ such that $d_{\mathbb{D}}(f(z_{i,t}), a_t) \leq R_1$. Hence there exists a constant $R_2 = R_2(R_1, d)$ such that $d_{\mathbb{D}}(z_{i,t}, f^{-1}(a_t)) \leq R_2$ (see [35, Corollary 10.3]). Since the angles $\angle x_i z x_j$ are bounded from below, there exists a constant $R_3 = R_3(\theta)$ such that the balls $B_{\mathbb{H}^2}(z_{i,t}, R_2)$ are disjoint for each $t \geq R_3$. Thus there are k different preimages of a_{R_3} in the ball $B_{\mathbb{H}^2}(z, R_2 + R_3)$. We have $f(B_{\mathbb{H}^2}(z, R_2 + R_3)) \subseteq B_{\mathbb{H}^2}(a, R_2 + R_3)$ by the Schwarz lemma. Let U be the component of $f^{-1}(B_{\mathbb{H}^2}(a, R_2 + R_3))$ that contains z. Since $f : \mathbb{D} \to \mathbb{D}$ is proper, U is simply connected. By Theorem 2.6, there exists a constant $R = R(R_2, R_3, d) = R(C, \theta, d)$ such that $U \subseteq B_{\mathbb{H}^2}(z, R)$. The degree of $f : U \to B_{\mathbb{H}^2}(a, R_2 + R_3)$ is at least k, so there are at least $k - 1$ critical points in $U \subseteq B_{\mathbb{H}^2}(z, R)$, proving the lemma. ∎

The following uniform separation property will also be used:

Lemma 4.9. *Let $f : \mathbb{D} \to \mathbb{D}$ be a proper map of degree d that is M-uni-critical. Then there exists $\theta = \theta(M) > 0$ such that for any $t \in \mathbb{S}^1$, any two points in $f^{-1}(t)$ are at least a distance of θ apart in the standard Euclidean metric on $\mathbb{S}^1$.*

Proof. Since post-composing by $\text{Isom}(\mathbb{D})$ does not change the statement of the lemma, we may assume $f \in \mathcal{B}_d$. Since the critical set is contained in $B_{\mathbb{H}^2}(0, M)$, f lives in a compact subset of $\mathcal{B}_d$, and the conclusion follows. ∎

Realization for degree 2 angled tree maps

Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be a minimal admissible angled tree map. If $\delta(p) = 2$, then $\mathcal{T} = \{p\}$ and we can simply take the constant sequence $f_n(z) = \mathcal{F}_p(z)$. Therefore we may assume $\delta(p) = 1$. By the compatibility condition, there exists a rigid rotation R such that $R \circ \alpha_p = \alpha_p \circ Df|_{T_p \mathcal{T}}$. Note that here R is necessarily a rational rotation. By the admissibility condition, the angled tree map is uniquely determined by this rotation number:

$$\mathcal{T} = \bigcup_{k=0}^{q-1} f^k([p, c]),$$

where c is the unique critical vertex of $\mathcal{T}$. The realization for such angled tree maps has been established in [35, §14] using 'strong convergence'. We give a different proof here, as the induction step uses the same idea.

Lemma 4.10. *Proposition* 4.3 *holds in degree* 2.

Proof. By the discussion in the previous paragraph, we may assume that $\delta(p) = 1$ with rotation number r/q. Thus conditions (1) and (2) in Proposition 4.3 are trivially satisfied, and it suffices to construct a quasi post-critically finite $f_n \in \mathcal{B}_2$ realizing $\mathcal{T}$ of rotation number r/q.

Let $R(z) = e^{i2\pi r/q} z$ be the rigid rotation with rotation number r/q. Let $M_n(z) = \frac{z - a_n}{1 - \bar{a}_n z}$ where $a_n \in \mathbb{R}^+$ and $d_{\mathbb{D}}(0, a_n) = n$. Note $M_n(z) \to -z$ algebraically.

We consider the sequence $f_n(z) = -R(z)M_n(z)$ which is conjugate to a map in $\mathcal{B}_2$ by some rotation. Abusing notation, we shall ignore this difference and show that f_n is quasi post-critically finite and realizes the angled tree map with rotation number r/q.

Let $b_n \in \mathbb{R}^+$ with $d_{\mathbb{D}}(0, b_n) = n/2$ be the hyperbolic midpoint of $[0, a_n]$. Since the zeros of f_n are 0 and a_n, by Lemma 4.8 there exists some constant K_1 such that

$$d_{\mathbb{D}}(b_n, c_n) \le K_1 \tag{4.1}$$

where c_n is the critical point of f_n. Since $d_{\mathbb{D}}(0, b_n) = \frac{1}{2} d_{\mathbb{D}}(0, a_n)$, we have $1 - |b_n| = 1 - |R(b_n)| \ge \delta_{a_n}^{1/2}$ by Lemma 4.5. Thus, by Lemma 4.6, the error term $|f_n(b_n) - R(b_n)|$ satisfies

$$|f_n(b_n) - R(b_n)| = |R(b_n)| \, |M_n(b_n) + 1| \le 2\delta_{a_n}^{1/2} \tag{4.2}$$

where $\delta_{a_n} = 1 - |a_n|$.

By the Schwarz lemma, $d_{\mathbb{D}}(0, f_n(b_n)) \le n/2$, so $1 - |f_n(b_n)| \ge \delta_{a_n}^{1/2}$ as well. Therefore, by Lemma 4.4, the hyperbolic metric along the Euclidean segment

$[f_n(b_n), R(b_n)]_{\mathbb{R}^2}$ is bounded above by

$$\rho_{\mathbb{H}^2}(z)|dz| \leq \frac{2}{\delta_{a_n}^{1/2}}|dz|. \tag{4.3}$$

Thus, by (4.2) and (4.3),

$$d_{\mathbb{D}}(f_n(b_n), R(b_n)) \leq 2\delta_{a_n}^{1/2} \cdot \frac{2}{\delta_{a_n}^{1/2}} = 4.$$

A similar proof and the Schwarz lemma show that for $k = 1, \ldots, q$,

$$d_{\mathbb{D}}(f_n^k(b_n), R^k(b_n)) \leq K_2$$

for some constant K_2. By the Schwarz lemma and (4.1), for $k = 1, \ldots, q$,

$$d_{\mathbb{D}}(f_n^k(c_n), R^k(b_n)) \leq K_3$$

for some constant K_3. Therefore, f_n is a quasi post-critically finite sequence and realizes the angled tree map with rotation number r/q. $\blacksquare$

Construction of the reduction

Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be a minimal admissible angled tree map of degree $d + 1$. We shall construct a new admissible angled tree map of degree d by removing a 'furthest' critical point.

Let $\mathcal{T}^1$ be the first pullback of $\mathcal{T}$. Then p has exactly $d + 1$ preimages in $\mathcal{T}^1$ counted with multiplicity. Since f is simplicial, $f : \mathcal{T}^1 \to \mathcal{T}^1$ is distance non-increasing with respect to the edge metric $d_{\mathcal{T}^1}$. Since the inclusion map $i : \mathcal{T} \to \mathcal{T}^1$ is simplicial, $d_{\mathcal{T}^1}$ agrees with $d_{\mathcal{T}}$ on $\mathcal{T}$.

It is convenient to introduce a radius function $r : \mathcal{V}^1 \to \mathbb{N}$ by setting $r(v) = d_{\mathcal{T}^1}(v, p)$. Denote $r_f = \max_{v \in f^{-1}(p)} r(v)$. We say $w \in f^{-1}(p) \subseteq \mathcal{T}^1$ is a *furthest preimage* of p if $r(w) = r_f$.

Lemma 4.11. *Let $v \in \mathcal{V} \subseteq \mathcal{V}^1$. Then*

$$r(v) + r(f(v)) \leq r_f.$$

Proof. First assume v is a critical vertex. Suppose for contradiction that $r(v) + r(f(v)) > r_f$. Note that $f(v) \neq p$, as otherwise, v gives a further preimage of p, contrary to the definition of r_f.

Let $a \in T_{f(v)}\mathcal{T} \subseteq T_{f(v)}\mathcal{T}^1$ be the tangent vector associated to the component containing p. By construction of $\mathcal{T}^1$, the tangent vector a has $\delta(v) \geq 2$ preimages in $T_v\mathcal{T}^1$ under the tangent map $Df : T_v\mathcal{T}^1 \to T_{f(v)}\mathcal{T} \subseteq T_{f(v)}\mathcal{T}^1$. At least one direction will increase the radius function. Thus, by considering the pullback of $[p, f(v)]$ in the increasing direction, we can find a preimage x of p with radius $r(x) = r(v) + r(f(v)) > r_f$, which is a contradiction.

If v is post-critical, let $v = f^l(c)$ where c is critical. Since f is distance non-increasing, $r(f(v)) \leq r(v) \leq r(f(c)) \leq r(c)$, so $r(v) + r(f(v)) \leq r_f$.

If v is neither critical nor post-critical, since f is assumed to be minimal on $\mathcal{T}$, v is not an end point. Let w_1, w_2 be two critical or post-critical vertices such that $v \in [w_1, w_2]$ and there are no other critical or post-critical points in (w_1, w_2). Thus, f restricts to an isomorphism from $[w_1, w_2]$ to $[f(w_1), f(w_2)]$. Note that at least one of the two points, say w_1, has larger radius than v. Write $r(v) = r(w_1) - l$ for some positive integer l. Since f is an isomorphism from $[w_1, w_2]$ to $[f(w_1), f(w_2)]$, we have $r(f(v)) \leq r(f(w_1)) + l$. Thus, $r(v) + r(f(v)) \leq r(w_1) + r(f(w_1)) \leq r_f$. ∎

Let w be a preimage of p in $\mathcal{T}^1$. Then there exists at least one critical point on the path $[p, w] \subseteq \mathcal{T}^1$. A critical point $c_w \in \mathcal{V}$ on $[p, w]$ is said to be *furthest* if $r(c_w) \geq r(c)$ for any critical point $c \in \mathcal{V}$ on $[p, w]$. We call c_w the critical point *associated to w*.

Lemma 4.12. *If $w \in \mathcal{V}^1$ is a furthest preimage of p, then*

(1) $r_f = r(c_w) + r(f(c_w))$;

(2) $\mathcal{T} \cap [p, w] = [p, c_w]$.

Proof. Since there are no critical points in $(c_w, w]$, we have

$$d_{\mathcal{T}^1}(c_w, w) = d_{\mathcal{T}^1}(f(c_w), p) = r(f(c_w))$$

and the first statement follows.

Suppose for contradiction that the second statement does not hold. Since f is minimal, there exists a post-critical vertex v' on $(c_w, w]$. Note that v' has a preimage v in $\mathcal{T}$ as v' is post-critical. Since f is distance non-increasing, we have $r(f(c_w)) \leq r(c_w) < r(v') \leq r(v)$, so $r(v) + r(v') = r(v) + r(f(v)) > r_f$, which contradicts Lemma 4.11. ∎

Definition 4.13. A preimage $w \in \mathcal{V}^1$ of p is said to be *critically furthest* if

- $r(w) = r_f$;
- $r(c_w) = \max\{r(c_v) : f(v) = p \text{ and } r(v) = r_f\}$.

We remark that c_w may not be a furthest critical point. It is only furthest among the critical points associated to furthest preimages of p.

Let w be a critically furthest preimage of p. Note that $[p, w]$ is a path in the pullback $\mathcal{T}^1$. The following lemma says that the associated critical point $c_w \in [p, w]$ is an end point of the original tree $\mathcal{T}$.

Lemma 4.14. *If $w \in \mathcal{V}^1$ is a critically furthest preimage of p, then c_w is an end point of $\mathcal{T}$.*

Proof. Suppose for contradiction that c_w is not an end point of $\mathcal{T}$. There is at least one direction $a \in T_{c_w}\mathcal{T}$ for which the radius function is increasing. By Lemma 4.11, $Df(a) \in T_{f(c_w)}\mathcal{T}$ corresponds to the component containing p. Thus, by considering the pullback of $[p, f(c_w)]$ in this direction, we find another furthest preimage v of p. Since $[p, c_w]$ is

strictly contained in $[p, v] \cap \mathcal{T}$, by Lemma 4.12 (2), c_w is not the furthest critical point on the path $[p, v]$. Let c_v be the associated critical point for v. Then $r(c_v) > r(c_w)$. Thus, w is not critically furthest, which is a contradiction. ∎

Let w be a critically furthest preimage of p. We first define

$$\tilde{f} = f : (\mathcal{T}, p) \to (\mathcal{T}, p)$$

with the same degree function and angle function except at $c := c_w$, where $\tilde{\delta}(c) = \delta(c) - 1$. Since c is an end point by Lemma 4.14, it is easy to check the above definition gives an admissible angled tree map of degree d.

Let $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$ be the angled subtree where the dynamics $\tilde{f}$ is minimal. We call

$$(\tilde{f} : (\widetilde{\mathcal{T}}, p) \to (\widetilde{\mathcal{T}}, p), \tilde{\delta}, \tilde{\alpha})$$

the *reduction* of f.

Note that if $\delta(c) \geq 3$, then the dynamics $\tilde{f}$ on $\mathcal{T}$ is minimal, so $\widetilde{\mathcal{T}} = \mathcal{T}$. If $\delta(c) = 2$, then the dynamics $\tilde{f}$ may or may not be minimal on $\mathcal{T}$ (depending on whether c is in the post-critical set of other critical points or not). In any case, $\mathcal{T}$ is the convex hull of $\widetilde{\mathcal{T}}$ and the orbit of c.

Induction step for realization

Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be a minimal admissible angled tree map of degree $d + 1$. If $\delta(p) \geq 2$, let $\mathcal{F}$ be a marked and normalized mapping scheme on $\mathcal{S}_p$.

If $\delta(p) = d + 1$, then Proposition 4.3 is vacuously satisfied by taking the constant sequence $f_n(z) = \mathcal{F}_p(z)$. Thus, we assume $\delta(p) \leq d$. Note that in this case, $\tilde{\delta}(p) = \delta(p)$ and $\widetilde{\mathcal{F}}_p = \mathcal{F}_p$.

We will break up the proof of Proposition 4.3 into three steps:

- Using induction, we first show that the quasi-invariant tree $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is realized (Lemma 4.15), which already implies Theorem 4.1.

- We then show that the conditions for periodic rescaling limits in Proposition 4.3 are satisfied (see IH4 below and Lemma 4.16).

- Finally, we perform surgery so that the conditions for strictly pre-periodic rescaling limits in Proposition 4.3 are satisfied (Lemma 4.17).

Let $(\tilde{f} : (\widetilde{\mathcal{T}}, p) \to (\widetilde{\mathcal{T}}, p), \tilde{\delta}, \tilde{\alpha})$ be the reduction of f. We assume the following technical and auxiliary induction hypotheses for the reduction. It is easy to verify that all these induction hypotheses are satisfied for the degree 2 base case.

IH1. *There exists a quasi post-critically finite sequence $\tilde{f}_n$ of degree d realizing $\tilde{f}$: $(\widetilde{\mathcal{T}}, \tilde{p}) \to (\widetilde{\mathcal{T}}, \tilde{p})$, with isomorphisms $\tilde{\phi}_n : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}_n$, such that the rescaling limit $\widetilde{F}_p$ at p is conjugate to $\widetilde{\mathcal{F}}_p$ with compatible marking.*

IH2. *There exists a sequence $R_n \to \infty$ such that*

$$d_{\mathbb{D}}(\tilde{\phi}_n(v), \tilde{\phi}_n(w)) = d_{\tilde{\mathcal{T}}}(v, w)R_n + O(1),$$

where $O(1)$ depends on $\tilde{f}$ and $\tilde{\mathcal{F}}$.

IH3. *Let $v \in \tilde{\mathcal{V}}$ with $f(v) \neq p$. Let $s_0 = \lim_{f(v)} \tilde{\phi}_n(p) \in \mathbb{S}^1_{f(v)}$ and $\tilde{F}^{-1}_v(s_0) = \{t_1, \ldots, t_{\tilde{\delta}(\tilde{v})}\} \subseteq \mathbb{S}^1_v$. Then there exists $\tilde{\theta} = \tilde{\theta}(\tilde{f})$ such that*

$$|t_i - t_j| \geq \tilde{\theta} \quad \text{for all } i \neq j.$$

Let $w_{i,n}$ be the nearest zero of $\tilde{f}_n$ with $\lim_v w_{i,n} = t_i$. Then

$$d_{\mathbb{D}}(\tilde{\phi}_n(v), w_{i,n}) = d_{\mathbb{D}}(\tilde{\phi}_n(v), w_{j,n}) \quad \text{for all } i \neq j.$$

IH4. *There exists a constant $\tilde{L} = \tilde{L}(\tilde{f})$ such that for any periodic cycle $\mathcal{C}$ other than p, we can order the cycle as $\mathcal{C} = \{v_1, \ldots, v_q\}$ so that*

$$\lim_{n \to \infty} d_{\mathbb{D}}(\tilde{\phi}_n(v_{i+1}), \tilde{f}_n(\tilde{\phi}_n(v_i))) \leq \tilde{L} \quad \text{for all } i = 1, \ldots, q-1.$$

We remark that IH3 is vacuously satisfied if $\delta(\tilde{v}) = 1$. The nearest zero in IH3 exists as $\tilde{F}_v(t_i) = s_0$. By Lemma 4.8, IH3 gives a constant $\tilde{R}$, depending only $\tilde{f}$, such that $B_{\mathbb{H}^2}(\phi_n(v), \tilde{R})$ contains $\delta(v) - 1$ critical points of $\tilde{f}_n$. Thus IH4 implies that the rescaling limit

$$\tilde{F}^q_{v_1} : \mathbb{D}_{v_1} \to \mathbb{D}_{v_1}$$

is $\tilde{M}$-uni-critical for some constant $\tilde{M} = \tilde{M}(\tilde{f})$ depending only on $\tilde{f}$.

We also remark that the estimate in IH4 may not hold for $i = q$, as the hyperbolic distance between a critical point and its image under the first return map $\tilde{F}^q_{v_1}$ may depend on $\mathcal{F}$.

Lemma 4.15. *There exists a quasi post-critically finite sequence $f_n \in \mathcal{B}_d$ which realizes $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$, and satisfies IH1–IH3.*

Proof. We consider two cases.

Case (1): $\tilde{\mathcal{T}} = \mathcal{T}$. We set $\phi_n(v) = \tilde{\phi}_n(v)$ for $v \in \mathcal{V}$. Let $M_{c,n}, M_{\phi(c),n} \in \text{Isom}(\mathbb{D})$ be the coordinate at c and $\phi(c)$ respectively. Let $\tilde{F}_c$ be the rescaling limit at c. Since $\delta(p) \leq d$, we have $c \neq p$.

If $f(c) = p$, set $w_n = \phi_n(c)$.

Otherwise, let $s_0 = \lim_{f(c)} \phi_n(p) \in \mathbb{S}^1_{f(c)}$. Let $t_1, \ldots, t_{\tilde{\delta}(c)} \in \mathbb{S}^1_c$ be the preimages of s_0 under $\tilde{F}_c$. Let $w_{i,n}$ be the nearest zero of $\tilde{f}_n$ to c_n with $\lim_c w_{i,n} = t_i$.

By IH3, we choose a point $t \neq t_i \in \mathbb{S}^1_c$ so that the $t, t_1, \ldots, t_{\tilde{\delta}(c)}$ are at least θ apart, where θ depends only on f. Let $w_n \in \mathbb{D}$ be such that

- $M_{c,n}(w_n)$ lies on the geodesic ray $[0, t)$;
- $d_{\mathbb{D}}(\phi_n(c), w_n) = d_{\mathbb{D}}(\phi_n(c), w_{i,n})$ for any i.

Let $\hat{w}_n = \frac{w_n}{|w_n|} \in \mathbb{S}^1$ and $A_n(z) = \frac{z - w_n}{1 - \overline{w}_n z}$.

Consider the sequence

$$f_n(z) = \frac{-1}{\widehat{w}_n} A_n(z) \tilde{f}_n(z),$$

which is conjugate to a map in $\mathcal{B}_d$ by some rotation. Again, abusing notation, we shall not distinguish f_n and its conjugate in $\mathcal{B}_d$ in this proof.

Throughout this proof, if not specified, the constants and $O(1)$ depend on the tree map f and the mapping scheme $\mathcal{F}$. Let $v \in \mathcal{V}$. We first claim that there exists a constant K_1 with

$$d_{\mathbb{D}}(f_n(\phi_n(v)), \tilde{f}_n(\phi_n(v))) \le K_1.$$

Denote $\delta_n := 1 - |w_n|$ and $\rho_n := d_{\mathbb{D}}(0, w_n)$. Then $\rho_n = r(w)R_n + O(1)$. Let $s = r(v)/r(w)$. By Lemma 4.7, there exists some constant K_2 such that the error term satisfies

$$|f_n(\phi_n(v)) - \tilde{f}_n(\phi_n(v))| = |\tilde{f}_n(\phi_n(v))| \, |A_n(\phi_n(v)) + \widehat{w}_n| \le K_2 \delta_n^{1-s}. \qquad (4.4)$$

By Lemma 4.11, we have $r(v) + r(f(v)) \le r_f = r(w)$. Thus

$$d_{\mathbb{D}}(0, \tilde{f}_n(\phi_n(v))) = r(f(v))R_n + O(1) \le (1-s)\rho_n + O(1).$$

Since $\left| \frac{-1}{\widehat{w}_n} A_n(z) \right| < 1$ for $z \in \mathbb{D}$,

$$d_{\mathbb{D}}(0, f_n(\phi_n(v))) \le d_{\mathbb{D}}(0, \tilde{f}_n(\phi_n(v))) \le (1-s)\rho_n + O(1).$$

Thus, by Lemma 4.4, the hyperbolic metric at z along the Euclidean segment $[f_n(\phi_n(v)), \tilde{f}_n(\phi_n(v))]_{\mathbb{R}^2}$ satisfies

$$\rho_{\mathbb{H}^2}(z)|dz| \le K_3 \frac{1}{\delta_n^{1-s}} |dz|.$$

Together with (4.4), we conclude that

$$d_{\mathbb{D}}(f_n(\phi_n(v)), \tilde{f}_n(\phi_n(v))) \le K_1$$

for some constant K_1. Thus, the vertices for $\mathcal{T}_n$ are K_1-quasi-invariant. The same proof as in Proposition 2.9 shows that $\mathcal{T}_n$ is quasi-invariant under f_n.

Note that f_n and $\tilde{f}_n$ have the same set of zeros except for w_n. If $f(c) = p$, i.e., $c = w$, then $w_n = \phi_n(c)$, so $\delta(c) = \tilde{\delta}(c) + 1$.

Otherwise, by IH3 and Lemma 4.8, there are $\delta(c)$ critical points within a bounded distance of $\phi_n(c)$, so $\delta(c) = \tilde{\delta}(c) + 1$. The same argument also shows $\delta(v) = \tilde{\delta}(v)$ for all $v \ne c$. A similar estimate on the first pullback tree $\mathcal{T}^1$ allows us to verify that we have the correct marking for rescaling limits. So f_n realizes the angled tree map $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$.

By construction, IH1–IH3 are satisfied. This proves Case (1).

Case (2): $\tilde{\mathcal{T}} \subsetneq \mathcal{T}$**.** We show we can reduce to the first case. During the reduction, we will encounter angled tree maps that are no longer minimal. The notion of realization naturally generalizes to such maps.

If $c \in \mathcal{T}$ is eventually mapped into $\widetilde{\mathcal{T}}$ by $\tilde{f}$, then there exists $k \geq 1$ with $\mathcal{T} \subseteq \widetilde{\mathcal{T}}^k$ for some pullback of $\widetilde{\mathcal{T}}$, which is realized by the quasi-invariant tree $\widetilde{\mathcal{T}}_n^k$ for $\tilde{f}_n$. It is easy to see that $\widetilde{\mathcal{T}}^k$ satisfies all the induction hypotheses as all the additional vertices have degree 1. Applying the argument in Case (1) to $\widetilde{\mathcal{T}}^k$, we get the conclusion of Lemma 4.15.

Thus, we assume that the orbit of c does not intersect $\widetilde{\mathcal{T}}$.

Suppose c is periodic with period q. Let $a = \text{proj}_{\widetilde{\mathcal{T}}}(c)$ be the projection of c to $\widetilde{\mathcal{T}}$. Since c is periodic, and f is distance non-increasing, a is periodic as well. Since there are no periodic Julia branch points other than p, c is adjacent to a and the period of a divides q.

If a is a periodic Fatou point, then the first return rescaling limit $\widetilde{F}$ at a has degree ≥ 2. Let $t \in \mathbb{S}^1_{\Delta(a), \Delta_{\text{pre}}(a)}$ be the angle associated to the direction of c. Then the corresponding repelling periodic point $s \in \mathbb{S}^1_a$ for $\widetilde{F}$ is not a hole for $\widetilde{F}$ by Lemma 2.13.

Let $M_{a,n} \in \text{Isom}(\mathbb{D})$ be the local coordinate at a. We define $\tilde{\phi}_n(c)$ so that

- $M_{a,n}(\tilde{\phi}_n(c))$ lies in the geodesic ray $[0, s)$;
- $d_{\mathbb{D}}(0, M_n(\tilde{\phi}_n(c))) = R_n = d_{\mathcal{T}}(a, c) R_n$.

We define $\tilde{\phi}_n(\tilde{f}^k(c)) = \tilde{f}_n^k(\tilde{\phi}_n(c))$ for $k = 1, \ldots, p-1$. We construct

$$\mathcal{T}_n = \widetilde{\mathcal{T}}_n \cup \bigcup_{k=0}^{q-1} [\tilde{\phi}_n(\tilde{f}^k(a)), \tilde{\phi}_n(\tilde{f}^k(c))].$$

It is easy to verify that the map $\tilde{f}_n$ on $\mathcal{T}_n$ realizes the (non-minimal) angled tree map $\tilde{f} : (\mathcal{T}, p) \to (\mathcal{T}, p)$ and satisfies all the induction hypotheses as all the additional vertices have degree 1. Thus the lemma follows from the argument in Case (1).

If a is a periodic Julia point and $a \neq p$, then since $\mathcal{T}$ is admissible, a is not a branch point for $\mathcal{T}$. Thus a is an end point of $\widetilde{\mathcal{T}}$. Therefore, if $t \in \mathbb{S}^1_{\Delta(a), \Delta_{\text{pre}}(a)}$ is the angle at a associated to the direction of c, it corresponds to the unique repelling fixed point $s \in \mathbb{S}^1_a$ for the degree 1 first return rescaling limit $\widetilde{F}$ at a. The proof is similar to the previous case.

If a is a periodic Julia point and $a = p$, then $\mathcal{T}$ is a star-shaped tree. The lemma then follows directly from a similar argument to that for Lemma 4.10 (see also [35, §14]).

Finally, suppose c is strictly pre-periodic. Let $b = \tilde{f}^l(c)$ where l is the pre-period, and $a = \text{proj}_{\widetilde{\mathcal{T}}}(b)$. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the convex hull of $\widetilde{\mathcal{T}}$ and the orbit of b. The same argument as for c periodic shows that $\mathcal{T}'$ is realized by $\tilde{f}_n$. Then a similar argument to the case when c is mapped into $\mathcal{T}$ shows that by pulling back, $\tilde{f} : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is realized by $\tilde{f}_n$ and satisfies all the induction hypotheses. Thus, the argument in Case (1) shows that $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is also realizable. This proves Case (2). ∎

Rescaling limits for periodic orbits

We now prove that the bound for periodic orbits is independent of $\mathcal{F}$.

Lemma 4.16. *The realization f_n for $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ satisfies* IH4.

Proof. Recall that we have two cases after the reduction. We consider Case (1) for $\tilde{\mathcal{T}} = \mathcal{T}$ here. The same modification as in Lemma 4.15 can be used to prove Case (2) for $\tilde{\mathcal{T}} \subsetneq \mathcal{T}$.

After passing to a subsequence, we may assume that all limits exist in the following discussion. Let $\{v_1, \ldots, v_q\}$ be a periodic cycle. Changing the ordering if necessary, by IH4, there exists $M_1 = M_1(\tilde{f})$ such that for $i = 1, \ldots, q - 1$,

$$\lim_{n \to \infty} d_{\mathbb{D}}(\phi_n(v_{i+1}), \tilde{f}_n(\phi_n(v_i))) \leq M_1. \tag{4.5}$$

We claim that after passing to a subsequence, there exists a constant $M_2 = M_2(f)$ such that for all $i = 1, \ldots, q$,

$$\lim_{n \to \infty} d_{\mathbb{D}}(f_n(\phi_n(v_i)), \tilde{f}_n(\phi_n(v_i))) \leq M_2. \tag{4.6}$$

Let w be the new preimage of p. Then $r(w) = r_f$. Since $v_i \in \mathcal{V}$, by Lemma 4.11 we have $r(v_i) + r(f(v_i)) \leq r_f$. Since v_i is periodic, $r(f(v_i)) = r(v_i)$. Thus $r(v_i) \leq r_f/2$.

Suppose we have strict inequality $r(v_i) < r_f/2$. Then the error term is

$$|f_n(\phi_n(v_i)) - \tilde{f}_n(\phi_n(v_i))| \leq K_1 \delta_n^{1 - r(v_i)/r_f},$$

where K_1 is a constant depending on f and $\mathcal{F}$. The hyperbolic metric at $\tilde{f}_n(\phi_n(v_i))$ and $f_n(\phi_n(v_i))$ is

$$\rho_{\mathbb{H}^2}(f_n(\phi_n(v_i)))|dz| \leq \rho_{\mathbb{H}^2}(\tilde{f}_n(\phi_n(v_i)))|dz| \leq K_2 \delta_n^{-r(f(v_i))/r_f}|dz|,$$

where K_2 is a constant depending on f and $\mathcal{F}$. Since $\delta_n \to 0$, and

$$1 - (r(v_i) + r(f(v_i)))/r_f = 1 - 2r(v_i)/r_f > 0,$$

we have

$$\lim_{n \to \infty} d_{\mathbb{D}}(f_n(\phi_n(v_i)), \tilde{f}_n(\phi_n(v_i))) \leq \lim_{n \to \infty} K_1 K_2 \delta_n^{1 - \frac{r(v_i) + r(f(v_i))}{r_f}} = 0.$$

Thus, after passing to a subsequence, the claim follows by simply taking the constant $M_2 = 1$.

Therefore, we only need to consider the case $r(v_i) = r_f/2$. Suppose v_i is not the midpoint of $[p, w]$. Then a similar estimate to the one above would also give

$$\lim_{n \to \infty} d_{\mathbb{D}}(f_n(\phi_n(v_i)), \tilde{f}_n(\phi_n(v_i))) = 0,$$

so the claim follows in this case as well.

Finally, suppose v_i is the midpoint of $[p, w]$. By IH3 and Lemma 4.9, there exists a constant $\theta = \theta(f)$ such that for any n, we have

$$\angle \phi_n(w)\phi_n(v_i)\phi_n(p) \geq \theta.$$

By the Schwarz lemma,

$$d_{\mathbb{D}}(0, \tilde{f}_n(\phi_n(v_i))) \leq d_{\mathbb{D}}(\phi_n(w), \phi_n(v_i)).$$

Let $\epsilon_n > 0$ be such that $|\tilde{f}_n(\phi_n(v_i))| = 1 - \epsilon_n$. Recall that $f_n(z) = \frac{-1}{\widehat{w}_n} A_n(z) \tilde{f}_n(z)$, where $A_n(z) = \frac{z - \phi_n(w)}{1 - \overline{\phi_n(w)} z}$, and $\widehat{w}_n = \frac{\phi_n(w)}{|\phi_n(w)|} \in \mathbb{S}^1$. By Lemma 4.7, there exists a constant $K_3 = K_3(\theta)$ such that for all n, $|A_n(\phi_n(v_i)) + \widehat{w}_n| \leq K_3 \epsilon_n$. Therefore,

$$|f_n(\phi_n(v_i)) - \tilde{f}_n(\phi_n(v_i))| = |\tilde{f}_n(\phi_n(v_i))| \left| 1 + \frac{1}{\widehat{w}_n} A_n(\phi_n(v_i)) \right| \leq K_3 \epsilon_n.$$

By Lemma 4.5,

$$\rho_{\mathbb{H}^2}(f_n(\phi_n(v_i)))|dz| \leq \rho_{\mathbb{H}^2}(\tilde{f}_n(\phi_n(v_i)))|dz| \leq \frac{4}{\epsilon_n}|dz|.$$

Therefore,

$$d_{\mathbb{D}}(f_n(\phi_n(v_i)), \tilde{f}_n(\phi_n(v_i))) \leq 4K_3.$$

Since K_3 depends only on f, the claim follows.

Combining (4.5) and (4.6), there exists $M_3 = M_3(f)$ such that

$$\lim_{n \to \infty} d_{\mathbb{D}}(\phi_n(v_{i+1}), f_n(\phi_n(v_i))) \leq M_3 \quad \text{for all } i = 1, \ldots, q - 1. \qquad \blacksquare$$

Rescaling limits for strictly pre-periodic orbits

Let $f_n \in \mathcal{B}_d$ be the quasi post-critically finite sequence realizing $\mathcal{T}$, constructed as above. Using a standard quasi-conformal surgery argument (cf. [38, Theorem 5.7] or [8]), we show

Lemma 4.17. *We can modify f_n so that it satisfies the conditions of Proposition 4.3.*

Proof. Let $f_n \in \mathcal{B}_d$ be constructed as above. Then f_n has the correct rescaling limit at the periodic orbits. We need to modify the dynamics on the strictly pre-periodic vertices to get the desired rescaling limits while keeping the rescaling limits on periodic points unchanged. Assume $\delta(p) \geq 2$, and let v be a strictly pre-periodic point with $f(v) = p$. The other strictly pre-periodic points can be treated using the same argument.

If $\delta(v) = 1$ then we can modify the marking ϕ_n so that $F_v(0) = 0$, and thus v is marked and normalized.

If $\delta(v) \geq 2$, we choose a large ball $B(0, r) \subseteq \mathbb{D}_p$ containing all critical values of the mapping scheme $\mathcal{F}_v$ and F_v. We choose a larger ball $B(0, s) \subseteq \mathbb{D}_p$ so that $\mathcal{F}_v^{-1}(B(0, r)) \subseteq F_v^{-1}(B(0, s))$. Let $U = \mathcal{F}_v^{-1}(B(0, r)) \subseteq \mathbb{D}_v$ and $V = F_v^{-1}(B(0, s)) \subseteq \mathbb{D}_v$. Since $f_n \circ M_{v,n}$ converges compactly on $\mathbb{D}$ to F_v, there exists a component V_n of $(f_n \circ M_{v,n})^{-1}(B(0, s))$ approximating V. For sufficiently large n, we define

$$\tilde{g}_n(z) = \begin{cases} f_n(z), & z \notin M_{v,n}(V_n), \\ \mathcal{F}_v \circ M_{v,n}^{-1}(z), & z \in M_{v,n}(\overline{U}), \\ H_n(z), & z \in M_{v,n}(V_n - \overline{U}), \end{cases}$$

where $H_n(z)$ is a K-quasi-regular degree $\delta(v)$ covering between the annuli $M_{v,n}(V_n - \overline{U})$ and $B(0, s) - \overline{B(0, r)}$ interpolating boundary values. Note K can be chosen to be independent of n. Let μ'_n be the Beltrami differential on $\mathbb{D}$ associated to $\tilde{g}_n$. Then $\mu'_n = 0$

away from the annulus $M_{v,n}(V_n - \bar{U})$. For sufficiently large n, orbits of $\tilde{g}_n$ can pass through $M_{v,n}(V_n - \bar{U})$ at most once. We extend μ'_n to a Beltrami differential on $\hat{\mathbb{C}}$ by reflecting along $\mathbb{S}^1$. Thus, we can construct a $\tilde{g}_n$-invariant Beltrami differential μ_n which has bounded dilatation. By the measurable Riemann mapping theorem, there exists a K-quasiconformal map ψ_n and a Blaschke product g_n such that

$$\tilde{g}_n = \psi_n^{-1} \circ g_n \circ \psi_n.$$

Let $L_{v,n} \in \text{Isom}(\mathbb{D})$ with $L_{v,n}(0) = \psi_n(v_n)$, and set $\psi_{v,n} = L_{v,n}^{-1} \circ \psi_n \circ M_{v,n}$. We have the following commutative diagram:

$$
\begin{array}{ccccc}
(\mathbb{D}, 0) & \xrightarrow{M_{v,n}} & (\mathbb{D}, v_n) & \xrightarrow{\tilde{g}_n} & (\mathbb{D}, \tilde{g}_n(v_n)) \\
\downarrow{\psi_{v,n}} & & \downarrow{\psi_n} & & \downarrow{\psi_n} \\
(\mathbb{D}, 0) & \xrightarrow{L_{v,n}} & (\mathbb{D}, \psi_n(v_n)) & \xrightarrow{g_n} & (\mathbb{D}, g_n(\psi_n(v_n)))
\end{array}
$$

After passing to a subsequence, $M_{v,n} \circ \tilde{g}_n$ converges compactly on $\mathbb{D}$ to a proper map of degree $\delta(v)$, and $\psi_n, \psi_{v,n}$ converge to K-quasiconformal maps that preserve the circle. Thus, $L_{v,n} \circ g_n$ converges to a proper map of degree $\delta(v)$. We denote this rescaling limit by G_v. Since for sufficiently large n, $\psi_{v,n}$ is conformal on U where $\tilde{g}_n \circ M_{v,n} = \mathcal{F}_v$ and U contains all critical points of $\mathcal{F}_v$, G_v is conformally conjugate to $\mathcal{F}_v$ (see [38, Lemma 5.10]). Adjusting the coordinate $L_{v,n}$ and interpolating the function H_n if necessary, we can assume that the rescaling limit G_v is normalized and has compatible marking.

We now show the surgery does not change the rescaling limits on periodic points. Let $w \in \mathcal{V}$ be a periodic point. Without loss of generality, we assume w is fixed. Let $\Omega \subseteq \mathbb{D}_w$ be a compact set.

We claim that for all sufficiently large n, the orbit of $z \in M_{w,n}(\Omega)$ under f_n does not pass through $M_{v,n}(V_n)$. Indeed, since v is strictly pre-periodic, there exists k_0 such that for all $k \geq k_0$ and all sufficiently large n,

$$d_{\mathbb{D}}(p_n, f_n^{-k}(M_{v,n}(V_n))) \geq 2 d_{\mathcal{T}}(p, w) R_n.$$

Therefore by the Schwarz lemma, for all sufficiently large n and $z \in M_{w,n}(\Omega)$, $f_n^k(z) \notin M_{v,n}(V_n)$ for all $k \geq k_0$. On the other hand, since w is fixed, for sufficiently large n and $z \in M_{w,n}(\Omega)$, $f_n^k(z) \notin M_{v,n}(V_n)$ for all $k \leq k_0$. Therefore, $(M_{w,n}^{-1})^* \mu_n$ converges to 0 in L^1 norm. Let $L_{w,n} \in \text{Isom}(\mathbb{D})$ with $L_{w,n}(0) = \psi_n(w_n)$ so that

$$\psi_{w,n} := L_{w,n}^{-1} \circ \psi_n \circ M_{w,n}$$

fixes $0, 1, \infty$. Thus, $\psi_{w,n}$ converges uniformly to the identity map (see [19, Proposition 4.7.2]). Denote $g_{w,n} = L_{w,n}^{-1} \circ g_n \circ L_{w,n}$ and $\tilde{g}_{w,n} = M_{w,n}^{-1} \circ \tilde{g}_n \circ M_{w,n}$. We have the commutative diagram

$$
\begin{array}{ccccccc}
(\mathbb{D}, 0) & \xrightarrow{M_{w,n}} & (\mathbb{D}, w_n) & \xrightarrow{\tilde{g}_n} & (\mathbb{D}, \tilde{g}_n(w_n)) & \xleftarrow{M_{w,n}} & (\mathbb{D}, \tilde{g}_{w,n}(0)) \\
\downarrow{\psi_{w,n}} & & \downarrow{\psi_n} & & \downarrow{\psi_n} & & \downarrow{\psi_{w,n}} \\
(\mathbb{D}, 0) & \xrightarrow{L_{w,n}} & (\mathbb{D}, \psi_n(w_n)) & \xrightarrow{g_n} & (\mathbb{D}, g_n(\psi_n(w_n))) & \xleftarrow{L_{w,n}} & (\mathbb{D}, g_{w,n}(0))
\end{array}
$$

Therefore, the rescaling limit $G_w := \lim_{n\to\infty} L_{w,n}^{-1} \circ g_n \circ L_{w,n}$ equals F_w, i.e., we have not changed the rescaling limits at the periodic points.

By choosing the region of modification carefully (see [38, Lemma 5.9]), the same argument allows us to modify all strictly pre-periodic points in $\mathcal{V}$ simultaneously. It is easy to verify that the modified sequence realizes $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ and satisfies the two conditions in Proposition 4.3.   ∎

5. Pointed Hubbard trees

In this section, we will prove the following necessary condition for a geometrically finite polynomial to be on the boundary of $\mathcal{H}_d$:

Proposition 5.1. *Let (H, p) be the pointed Hubbard tree for a geometrically finite polynomial $\widehat{P} \in \overline{\mathcal{H}_d}$. Then (H, p) is iterated-simplicial.*

Pointed Hubbard trees

Recall that given a monic and centered polynomial P with connected Julia set, there exists a unique Böttcher map normalized with derivative 1 at infinity. This gives a unique choice of the angle 0 external ray, and thus a marking on P. In this section, all polynomials considered are monic and centered.

Given a geometrically finite polynomial $\widehat{P}$ with connected Julia set, it can be perturbed into a subhyperbolic polynomial with topologically conjugate dynamics on the Julia set [17]. A quasi-conformal surgery then gives a post-critically finite polynomial P associated to $\widehat{P}$ [18].

For a post-critically finite polynomial P, the Hubbard tree H, introduced in [14], is defined as the 'regulated hull' of the critical and post-critical points in the filled Julia set K_P. More precisely, an arc $I \subseteq K_P$ is called *regulated* if its intersection with any bounded Fatou component consists of (at most two) segments of internal rays. The Hubbard tree is the minimal closed regulated connected subset of $K(P)$ containing the critical and post-critical points (see [45, §1]).

The polynomial restricts to a map $P : H \to H$. In our setting, the Hubbard tree H is marked by the Böttcher map. We say that P is *simplicial* on H if there exists a finite simplicial structure on H for which P is a simplicial map, i.e., P sends an edge of H to an edge of H. Abusing notation, we call H a *simplicial Hubbard tree* if the map $P : H \to H$ is simplicial.

The realization of Hubbard trees has been studied in [45]. It is proved that an abstract angled Hubbard tree is realizable by a post-critically finite polynomial if and only if the tree is expanding. Thus, we shall not distinguish the Hubbard trees of monic and centered post-critically finite polynomials from the abstract expanding angled Hubbard trees with markings.

Definition 5.2. Let H be a Hubbard tree and $p \in H$ be a fixed point of P. The pair (H, p) is called a *pointed Hubbard tree*.

A point $z \in J$ on the Julia set is said to be a *cut point* if $J - \{z\}$ is disconnected. The *valence* at z is defined as the number of components of $J - \{z\}$.

Let $\widehat{P} \in \partial \mathcal{H}_d$ be a geometrically finite polynomial. There exists a special non-repelling fixed point $\hat{p} = \lim p_n$ of $\widehat{P}$, where p_n is the attracting fixed point of $P_n \in \mathcal{H}_d$ and $P_n \to \widehat{P}$. We remark that $\hat{p}$ can still be an attracting fixed point for $\widehat{P}$ (see the middle or the right example in Figure 1.1). This happens when some critical points stay at a bounded hyperbolic distance from the attracting fixed point, while some other critical points escape.

Let P be the corresponding post-critically finite polynomial with Hubbard tree H. The special non-repelling fixed point $\hat{p}$ gives a fixed point H constructed as follows. We remark that the reason why we need to do some modification is that if the non-repelling fixed point $\hat{p}$ is on the Julia set, the corresponding fixed point for P may or may not be on the Hubbard tree H.

- If $\hat{p}$ is attracting, then it is contained in a fixed critical Fatou component. We set p to be the corresponding Fatou fixed point in H.

- If $\hat{p}$ is a parabolic end point, then $\hat{p}$ is on the boundary of a unique fixed critical Fatou component. If the corresponding Julia fixed point is in H, we set p to be that point; otherwise, we set p to be the Fatou fixed point in H corresponding to the fixed critical Fatou component.

- If $\hat{p}$ is a parabolic cut point, then the corresponding Julia fixed point is contained in H. We set p to be the corresponding Julia fixed point.

We call (H, p) the pointed Hubbard tree *corresponding* to $\widehat{P}$.

For $f \in \mathcal{B}_d$, any point in $\mathbb{D}$ is mapped towards the attracting fixed point 0 by the Schwarz lemma. The following key proposition is a manifestation of the Schwarz lemma for maps on $\partial \mathcal{H}_d$.

Proposition 5.3. *Let $\widehat{P} \in \overline{\mathcal{H}_d}$ be geometrically finite with the special non-repelling fixed point $\hat{p}$. If $\hat{v}$ is a periodic cut point in $J = J(\widehat{P})$ with valence v, then $\hat{v}$ is parabolic. Moreover, let $K = K(\widehat{P})$ be the filled Julia set of $\widehat{P}$.*

- *If $\hat{v} = \hat{p}$, then $\hat{v}$ has exactly v attracting basins which are in bijective correspondence with components of $K - \{\hat{v}\}$.*

- *If $\hat{v} \neq \hat{p}$, then $\hat{v}$ has exactly $v - 1$ attracting basins which are in bijective correspondence with components of $K - \{\hat{v}\}$ that do not contain $\hat{p}$.*

Proof. The fact that $\hat{v}$ is parabolic follows immediately from the stability of landing rays (see [16, Lemma B.1]). Note that there are exactly v external rays landing at $\hat{v}$.

If $\hat{v} = \hat{p}$, then for any approximating polynomial $P_n \in \mathcal{H}_d$ of $\widehat{P}$, there are $v + 1$ periodic points (v repelling and one attracting) of P_n that converge to $\hat{v}$. Thus the parabolic multiplicity of $\hat{v}$ is $v + 1$, so there are $v + 1$ attracting basins. Since each component of $K - \{\hat{v}\}$ can correspond to at most one attracting basin of $\hat{v}$, and different components of $K - \{\hat{v}\}$ correspond to different attracting basins, the first case follows.

If $\hat{v} \neq \hat{p}$, then the parabolic multiplicity of $\hat{v}$ is v.

We claim the component of $K - \{\hat{v}\}$ containing $\hat{p}$ does not give an attracting basin. Note that the proposition follows from the claim as there are $v - 1$ attracting basins and $v - 1$ components of $K - \{\hat{v}\}$ that do not contain $\hat{p}$. We shall prove the claim using the corresponding pointed Hubbard tree (H, p). The periodic cut points for $\hat{P}$ are in correspondence with the periodic Julia cut points in H and there are only finitely many periodic cut points as they are all parabolic.

Suppose $v \in H$ is a closest periodic Julia cut point to p that does not satisfy the claim, i.e., all periodic Julia cut points on $[p, v)$ satisfy the claim. If there exist Julia periodic points in $[p, v)$, let w be the one furthest from p. Let u_w be the vertex adjacent to $\hat{w}$ on (w, v). Since $\hat{w}$ satisfies the claim, u_w is a Fatou periodic point. If there are no Julia periodic points in $[p, v)$, then p is a Fatou fixed point and let $u_w = p$. Since $\hat{v}$ does not satisfy the claim, the adjacent vertex $u_v \in [u_w, v)$ is a Fatou fixed point. If $u_w \neq u_v$, there exists a Julia periodic point in (u_w, u_v) which is necessarily a cut point, contradicting the assumption that w is the furthest periodic Julia point. Thus $u_w = u_v$. Let Ω be the corresponding Fatou component for $\hat{P}$. Then Ω converges to two distinct boundary points under iteration, which is a contradiction. The claim follows and we conclude the proof of the proposition. ∎

As an illustration of the second case, consider the geometrically finite polynomial $\hat{P} \in \partial \mathcal{H}_4$ in Figure 1.3 (b). There is a parabolic fixed point $\hat{v} \neq \hat{p}$ with valence 3. This fixed point has parabolic multiplicity 3, and is on the common boundary of three Fatou components U_1, U_2, U_3. The Fatou component U_1 that contains $\hat{p}$ gives a repelling direction at $\hat{v}$. The attracting basins at $\hat{v}$ are thus in bijective correspondence with the other two Fatou components.

Pointed simplicial tuning

If $\delta(p) \geq 2$, we define a combinatorial operation called *pointed simplicial tuning* on a pointed Hubbard tree (H, p).

Let (H_Q, p') be a marked pointed simplicial Hubbard tree of degree $\delta(p)$ associated to a monic and centered polynomial Q. The marking gives an identification of the incident edges with external rays for H_Q. Let H_p be the regulated hull of H_Q with the landing points of those external rays (viewed as an abstract angled tree). As the first step, we remove p and glue the H_p to the incident edges at p. We remark that to be more precise, we need to remove $T_p = H \cap U_p$, where U_p is the Fatou component of p for the polynomial P associated to the Hubbard tree H, and glue back H_p.

Let $w \in H$ be a preimage of p. Suppose that $\delta(w) = 1$. Note that the incident edges of w correspond to vertices in H_p; we can thus remove w and glue the regulated hull of those corresponding vertices in H_p (viewed as an abstract angled tree) to the incident edges at w.

If $\delta(w) \geq 2$, we need to specify the pullback map as follows. Let

$$P_w : \mathbb{C} \to \mathbb{C}$$

be a polynomial of degree $\delta(w)$ with critical values $z_1, \ldots, z_k$ such that under Q, each z_i is eventually mapped to a periodic critical cycle or a repelling periodic cycle on the boundary of a bounded Fatou component for Q.

Let $\tilde{H}_p$ be the regulated hull of H_p and the orbits of z_i (viewed as an abstract angled tree). The condition on z_i guarantees that Q is simplicial on $\tilde{H}_p$. Let H_w be the pullback $P_w^{-1}(\tilde{H}_p)$. As before, we remove w and glue back in a homeomorphic copy of H_w, and also replace H_p with $\tilde{H}_p$. The dynamics and angle structures are defined naturally.

Inductively, we replace the backward orbits of p in the vertex set of H by pullbacks with the above algorithm. The algorithm terminates as the number of branch points and critical points in H is finite.

This algorithm may produce some end points that are not critical or post-critical. After deleting these end points and the corresponding edges if necessary, we obtain a new expanding pointed angled Hubbard tree (H', p'), which we call a *pointed simplicial tuning* of (H, p) (see Figure 5.1). We remark that depending on how the pointed simplicial tuning is performed, (H', p') may or may not be simplicial.

Unlike general tuning, the pointed simplicial tuning can only be performed finitely many times, as the degree of the marked fixed point p' is strictly smaller than the degree of p after the operation. We say a pointed Hubbard tree (H, p) is *iterated-simplicial* if it can be constructed from the trivial pointed Hubbard tree by a sequence of pointed simplicial tunings.
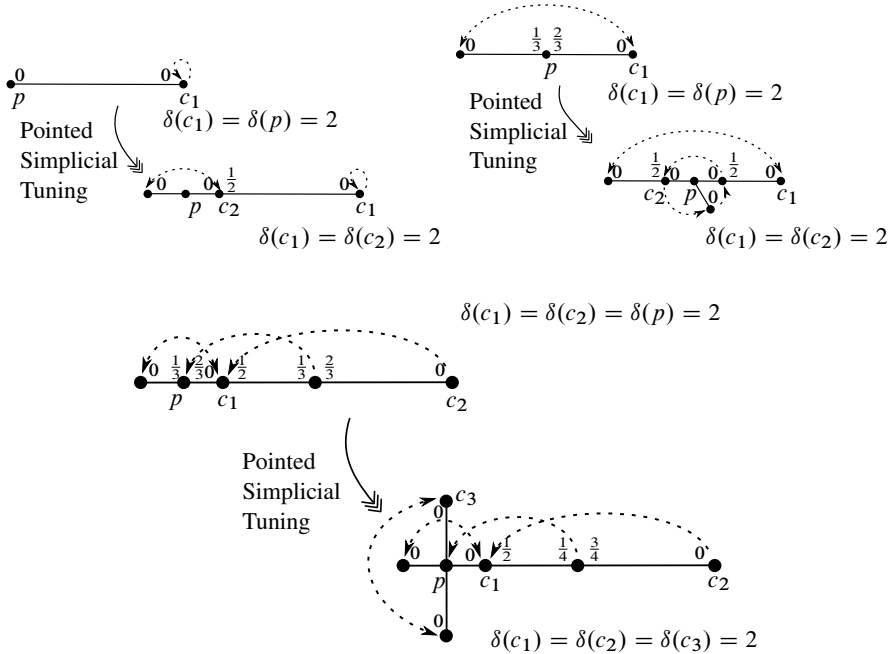

Fig. 5.1. Illustrations of pointed simplicial tunings. The angles for edges are labeled on the graph. The resulting pointed Hubbard trees for the first two are not simplicial.

Pointed simplicial quotient

An inverse operation for pointed simplicial tuning is *pointed simplicial quotient*. Let us fix a simplicial structure, i.e., a vertex set V for H. We assume that V is forward invariant, and contains all critical points. A subtree $p \in S \subseteq H$ is said to be *invariant* if $P(S) \subseteq S$. An invariant subtree is said to be *simplicial* if P is a simplicial map on S. The union of any two simplicial invariant subtrees is again a simplicial invariant subtree. Thus the maximal simplicial invariant subtree is well-defined.

Lemma 5.4. *Let (H, p) be the pointed Hubbard tree corresponding to a geometrically finite polynomial $\widehat{P} \in \overline{\mathcal{H}_d}$. Let $a \in T_p H$ be a periodic tangent direction at p of period q, and let $E = [p, w]$ be the corresponding incident edge. Then P^q maps E homeomorphically to E.*

Proof. Suppose p is a Julia fixed point. Then the adjacent vertices are periodic Fatou points by Proposition 5.3, and P is simplicial on the union of these edges.

Suppose p is a Fatou fixed point. Let Ω be the corresponding fixed Fatou component for $\widehat{P}$. Let $x \in \partial\Omega$ be the corresponding periodic point in the direction a. If $w = x$, then P^q is a homeomorphism on $[p, w]$. Otherwise, by Proposition 5.3, w is a periodic Fatou point and the corresponding Fatou component is attracted to Ω at x, and the conclusion follows. ∎

Corollary 5.5. *Let (H, p) be the pointed Hubbard tree corresponding to a geometrically finite polynomial $\widehat{P} \in \overline{\mathcal{H}_d}$. If H is non-trivial, i.e., $H \neq \{p\}$, then the maximal simplicial invariant subtree S is non-trivial.*

We say S' is a *subtree preimage* of S if S' is a subtree (with the simplicial structure given by V) and $P(S') \subseteq S$. The maximal one exists by taking unions. Let S be the maximal simplicial invariant subtree of (H, p). Note that S itself is a maximal subtree preimage of S. We construct a new tree H' from H by collapsing S and each inductive maximal subtree-preimages to a point and let $p' \in H'$ be the point associated with S. Similarly, the new vertex set V' is constructed from V by collapsing, and the map P induces a map P' on V' by $P'([x]) = [P(x)]$ where $[x]$ represents the vertex in V' that $x \in V$ collapses to. The local degrees can also be recovered by counting the multiplicities of all collapsed critical vertices.

Lemma 5.6. *The map $P' : V' \to V'$ satisfies $P'(a) \neq P'(b)$ whenever $[a, b]$ is an edge in H'.*

Proof. Let $[a, b]$ be an edge. Let x, y be vertices for V that lie above a, b respectively. We may assume that $[x, y]$ is an edge of H. Suppose for contradiction that $P'(x) = P'(y)$; then $[P(x)] = [P(y)]$. Thus, $P(x)$ and $P(y)$ are contained in some inductive maximal subtree preimage S' of S, so $[P(x), P(y)] \subseteq S'$. Therefore $[x] = [y]$ by maximality of subtree preimage, which is a contradiction. ∎

To construct the angles at the new vertices, we first show that the conclusion of Lemma 5.4 also holds for (H', p').

Lemma 5.7. *Let $a' \in T_{p'}H'$ be a periodic tangent direction of period q, and let $E' = [p', w']$ be the corresponding incident edge. Then $(P')^q$ maps E' homeomorphically to E'.*

Proof. The tangent direction a' corresponds to an edge $E = [x, y] \subseteq H$ with $x \in S$. Since a' has period q, $P^q([x, y])$ contains $[x, y]$. Let $I = [x', y'] \subseteq [x, y]$ be the interval closest to x such that P^q maps I homeomorphically to $[x, y]$. If $x = x'$, then x is a periodic cut point. By Proposition 5.3, $[x, y]$ corresponds to an attracting petal for the corresponding parabolic point $\hat{x} \in J(\hat{P})$. Thus, we can extend S and still get a simplicial action, which contradicts the maximality of S.

Therefore, $x \neq x'$. Suppose for contradiction that $y \neq y'$. By pulling back, there exists a periodic point on (x', y'), which is not on the boundary of any periodic Fatou component. This point gives a repelling periodic cut point for $\hat{P}$, contradicting Proposition 5.3. Thus $y = y'$, and hence $(P')^q$ maps E' homeomorphically to E'. ∎

Let $a' \in T_{p'}H'$ be a periodic tangent direction of period q corresponding to the edge $[x, y] \subseteq H$. As in the proof of Lemma 5.7, we have a map $P^q : [x, y] \supseteq [x', y] \to [x, y]$. Thus, there exists a point $w \in [x', y]$ with period q, corresponding to a periodic end for the polynomial Q associated with S. We define the angle at the direction a' by the external angle that lands at t. The angles for pre-periodic tangent directions and inductive preimages of p' are defined by pullback. Since H is expanding, it is not hard to verify that H' is expanding as well. We call (H', p') the *pointed simplicial quotient* for (H, p). It can be verified that this operation is an inverse for pointed simplicial tuning. More precisely, if (H', p') is a pointed simplicial quotient of (H, p), then (H, p) can be constructed from (H', p') using pointed simplicial tuning.

This process can be iterated. The same proof of Lemma 5.7 using Proposition 5.3 shows that unless the Hubbard tree is trivial, the maximal simplicial invariant subtree is non-trivial. Since pointed simplicial quotient reduces the number of vertices, the process eventually terminates at the trivial Hubbard tree. This proves Proposition 5.1.

6. Boundary of $\mathcal{H}_d$

In this section, we prove the other direction of Theorem 1.1. Let $\hat{P} \in \overline{\mathcal{H}_d}$ be a marked geometrically finite polynomial associated with the marked pointed Hubbard tree (H, p). Let P be the corresponding marked post-critically finite polynomial. For each critical and post-critical Fatou component Ω of P, we choose a point $t(\Omega) \in \partial\Omega$ as a marking which satisfies $P(t(\Omega)) = t(P(\Omega))$ (see [38, §5]). The boundary marking is chosen using an anchored convention as in Definitions 2.12 and 2.15 with respect to the external angle 0. The topological conjugacy carries this boundary marking to $\hat{P}$. Recall that a Blaschke

product is *M*-uni-critical if the critical set is contained in $B_{\mathbb{H}^2}(0, M)$. A proper holomorphic map $f : U \to V$ between two connected and simply connected proper subsets of $\mathbb{C}$ is said to be *M*-uni-critical if there exist uniformizing maps ϕ_U and ϕ_V such that $\phi_V^{-1} \circ f \circ \phi_U : \mathbb{D} \to \mathbb{D}$ is *M*-uni-critical.

Let $\mathcal{S}_p \subseteq V$ be the set of backward orbits of p and let $\mathcal{F}$ be a normalized mapping scheme on $\mathcal{S}_p$ (see Definition 4.2). We first prove the following (cf. Proposition 4.3):

Proposition 6.1. *Let (H, p) be a marked simplicial pointed Hubbard tree of degree d. When $\delta(p) \geq 2$, let $\mathcal{F}$ be a normalized mapping scheme on $\mathcal{S}_p$. There exists a sequence $P_n \in \mathcal{H}_d$ converging to a geometrically finite polynomial $\widehat{P} \in \overline{\mathcal{H}_d}$ associated to (H, p). Moreover,*

(1) *the dynamics of $\widehat{P}$ on the Fatou components corresponding to $\mathcal{S}_p$ is conjugate to $\mathcal{F}$ (with compatible markings);*

(2) *there exists a constant M depending only on (H, p) (and independent of $\mathcal{F}$) such that*

    (a) *for any periodic Fatou point $v \neq p$ of period q, $\widehat{P}^q$ on the corresponding Fatou component Ω_v is M-uni-critical;*

    (b) *for any strictly pre-periodic Fatou point $w \notin \mathcal{S}_p$, if k is the smallest positive integer such that $f^k(w)$ is critical, then $\widehat{P}^k : \Omega_w \to \Omega_{f^k(w)}$ is M-uni-critical and the critical values are within hyperbolic distance M of the critical points in $\Omega_{f^k(w)}$.*

Simplicial pointed Hubbard tree and admissible angled tree map

Let (H, p) be a marked simplicial pointed Hubbard tree. We first associate to it an admissible angled tree map. Indeed, the pointed Hubbard tree comes with a local degree function δ. The dynamics of the polynomial P gives angles between any pair of edges incident to a vertex (see [45]). For a periodic Julia vertex, we follow the same convention as in §3 for realizing angled tree maps: any two consecutive edges (in cyclic ordering at the vertex) have angle $\frac{1}{\nu}$ where ν is the valence at the vertex. To specify the angle 0 of the marking, we use the anchored convention in Definitions 3.1 and 3.3. Therefore, $P : (H, p) \to (H, p)$ is naturally an angled tree map.

The core of H is critically star-shaped. Indeed, if $\delta(p) \geq 2$, then this is vacuously true. Otherwise, each vertex v adjacent to p is in a periodic Fatou point.

On the other hand, the Hubbard tree H may contain many periodic Julia branch points. In the following, we introduce an operation on these branch points, which we call *split modification*, to get an admissible angled tree map.

Let $v \neq p$ be a periodic Julia branch point. After passing to an iterate, we may assume that v is fixed. Let S be the star-shaped subtree consisting of all vertices adjacent to v. Let a_0 be the vertex in S corresponding to the direction associated to p, and label the other vertices by $a_1, \ldots, a_m$ in counterclockwise order. Since P is simplicial on H and fixes p,

a_0 is fixed, and thus all a_i are fixed. By Proposition 5.3, each a_i is a fixed Fatou point for $i = 1, \ldots, m$.

We modify H locally in S by first removing v and its incident edges. On the first level, we choose $k_1 \in \{1, \ldots, m\}$ and connect a_0 to a_{k_1}. On the second level, we choose $k_{2,1} \in \{1, \ldots, k_1 - 1\}$ and $k_{2,2} \in \{k_1 + 1, \ldots, m\}$ and connect a_{k_1} to $a_{k_{2,1}}$ and $a_{k_{2,2}}$. Inductively, $k_1, k_{2,1}, k_{2,2}$ divide the set $\{1, \ldots, m\}$ into four subsets (some subset may be empty), and we proceed as above for each of the subintervals. The trees $\tilde{S}$ that can be constructed in this way will be called *admissible splittings* (see Figure 6.1). For an admissible splitting, each vertex a_i, $i = 0, 1, \ldots, m$, can be assigned a level, which is the edge distance between a_i and a_0. An edge connects a level k to a level $k + 1$ vertex.

(a) A star-shaped neighborhood of a periodic Julia branch point v with two different admissible splittings with angles specified.

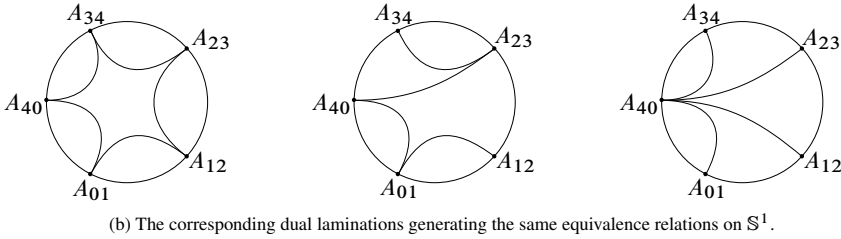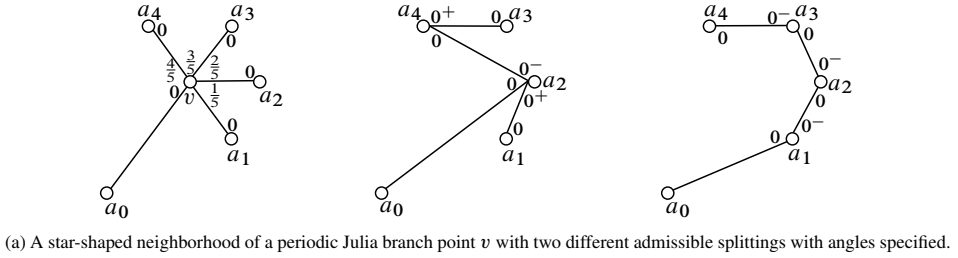(b) The corresponding dual laminations generating the same equivalence relations on $\mathbb{S}^1$.

Fig. 6.1. The split modification and dual laminations.

The dynamics are modified in S so that each edge of $\tilde{S}$ is fixed. The local degree function δ is defined to be the same as before the modification. The angle function at a_i is modified with the following rule (see Figure 6.1):

- if $a_i a_j$ is an edge where a_j is closer to a_0 than a_i is, we set the angle of the tangent direction corresponding to a_j to be 0;
- if $a_i a_j$ is an edge where a_j is further from a_0 than a_i is, we set the angle of the tangent direction corresponding to a_j to be 0^+ if $j < i$ and 0^- if $j > i$;
- the other angles remain the same.

We also modify H on the backward orbits of vertices in S by pullback. We will assume that $w \neq v \in \mathcal{V}$ is a preimage of v. The general construction on backward orbits can be done by induction.

If $\delta(w) = 1$, then we remove the 1-neighborhood of w and glue back a copy of $\tilde{S}$. Note that there is a unique way of gluing $\tilde{S}$ back by the dynamics.

If $\delta(w) \geq 2$, then the pullback is not unique. First, we modify the simplicial structure of $\tilde{S}$ by adding a vertex at the 'midpoint' of each edge. We call a degree $\delta(w)$ branched covering $f : \tilde{S}' \to \tilde{S}$ *admissible* if the branch locus is contained in the midpoints of the edges of $\tilde{S}$. We call $\tilde{S}'$ an *admissible* $\delta(w)$ *branched cover* of $\tilde{S}$. The pullback is constructed by removing the 1-neighborhood of w and then gluing back a copy of the admissible $\delta(w)$ branched cover $\tilde{S}'$. Similar to the degree 1 case, the gluing is determined by the dynamics.

Note that in both cases, we may introduce some new vertices because the preimage of the full 1-neighborhood of a periodic simple Julia branch point may not be in the Hubbard tree. Those new vertices are defined to have local degree 1, and the angle functions are defined by pullback.

Let $\mathcal{T}$ be the tree after the split modification over all periodic Julia branch vertices. Removing vertices of $\mathcal{T}$ if necessary, we may assume that f is minimal. We then have the following

Proposition 6.2. *Let (H, p) be a simplicial pointed Hubbard tree. Let $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be the minimal angled tree map after performing admissible split modification over all periodic Julia branch vertices. Then $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ is admissible.*

We remark that, conjecturally, these different splittings and pullbacks will all result in different accesses of points on $\partial \mathcal{H}$ and create complicated topology of $\overline{\mathcal{H}_d}$. To prove Theorem 1.1, we only need the existence of one admissible splitting. To prove Theorem 1.4, we only need two different admissible splittings (see §7). It would be interesting to know what such combinatorial operation can tell us about the 'complexity' of $\partial \mathcal{H}_d$.

Proposition 6.3. *For any admissible split modification $(\mathcal{T}, p)$ of (H, p), the dual laminations $\mathscr{L}_{\mathcal{T}}$ and $\mathscr{L}_H$ generate the same equivalence relations on $\mathbb{S}^1$.*

Proof. By induction and pullback, it suffices to consider the splitting at a periodic Julia branch point $v \in H$. Without loss of generality, we may assume v is fixed. Let $a_0, \ldots, a_k$ be the vertices adjacent to v. Then there are $k + 1$ angles $A_0, \ldots, A_k$ landing at v, where A_i corresponds to the access between the Fatou components of a_i and a_{i+1}.

Let S be the subtree containing $a_0, \ldots, a_k$ after the modification. By the dynamics, A_i is the only angle landing at the right side of the union of directed edges from a_i to a_{i+1} (see Figure 6.1). Since the angle at the vertex a_i between an incident edge in S and an incident edge outside of S is strictly positive, A_i does not land at any other edges. Therefore, $A_0, \ldots, A_k$ form an equivalence class of the equivalence relation generated by the lamination for $\mathcal{T}$. The claim now follows. ∎

Carathéodory convergence

A *disk* is a simply connected and connected open subset in $\mathbb{C}$. It is said to be *hyperbolic* if it is not $\mathbb{C}$. For a sequence of pointed disks (U_n, u_n), we say (U_n, u_n) converges to (U, u) in the *Carathéodory topology* if

- $u_n \to u$;
- for any compact set $K \subseteq U$, $K \subseteq U_n$ for all sufficiently large n;
- for any open connected set N containing u, if $N \subseteq U_n$ for all sufficiently large n, then $N \subseteq U$.

Equivalently, the convergence means that $u_n \to u$ and for any subsequence such that $\widehat{\mathbb{C}} - U_n \to K$ in the Hausdorff topology on compact subsets of the sphere, U is equal to the component of $\widehat{\mathbb{C}} - K$ containing u (see [29, §5.1]).

Similarly, we say a sequence of proper holomorphic maps between pointed disks $f_n : (U_n, u_n) \to (V_n, v_n)$ converges to $f : (U, u) \to (V, v)$ if

- $(U_n, u_n), (V_n, v_n)$ converge to $(U, u), (V, v)$ in the Carathéodory topology;
- for all sufficiently large n, f_n converges to f uniformly on compact subsets of U.

We have the following compactness result:

Theorem 6.4 ([29, Theorem 5.2]). *The set of disks $(U_n, 0)$ containing $B(0, r)$ for some $r > 0$ is compact in the Carathéodory topology.*

The definition of Carathéodory convergence naturally generalizes to simply connected and connected open subsets of $\widehat{\mathbb{C}}$. Recall that by Lemma 2.3, a sequence of degree d rational maps R_n converges uniformly on $\widehat{\mathbb{C}}$ to a rational map R if and only if R_n converges algebraically to R and R has degree d. The following lemma is very useful in studying degenerations of quasi post-critically finite Blaschke products.

Lemma 6.5. *Let $R_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a sequence of degree d rational maps converging uniformly on $\widehat{\mathbb{C}}$ to $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Let U_n be a sequence of invariant hyperbolic disks for R_n, and $x_n \in U_n$. If there exists K such that $d_{U_n}(x_n, R_n(x_n)) \leq K$ for all n, then after passing to a subsequence, either*

- $\lim_{n \to \infty} x_n = x$ *and x is fixed by R; or*
- *(U_n, x_n) converges in the Carathéodory topology to (U, x), and consequently $R_n : (U_n, x_n) \to (U_n, R_n(x_n))$ converges to $R : (U, x) \to (U, R(x))$.*

Proof. After passing to a subsequence, we may assume $x_n \to x$. We may assume $U_n \subseteq \mathbb{C}$ for all sufficiently large n and $x \in \mathbb{C}$. Suppose x is not fixed by R. We claim there exists a Euclidean ball of radius r such that $B(x_n, r) \subseteq U_n$ for all sufficiently large n. Suppose not; then the hyperbolic metric $\rho_{U_n}(x_n)|dz|$ at x_n is going to infinity by Lemma 4.4. Since R_n converges to R uniformly and x is not fixed, for sufficiently large n the Euclidean distance between x_n and $R_n(x_n)$ is bounded away from 0. Thus, $d_{U_n}(x_n, R_n(x_n))$ is unbounded, which is a contradiction. Therefore, after passing to a subsequence, the pointed disk (U_n, x_n) converges in the Carathéodory topology to (U, x) by Theorem 6.4. ∎

We remark that in the first case, the pointed disks (U_n, x_n) usually diverge in the space of pointed disks with the Carathéodory topology.

A typical example of divergent pointed disks can be constructed as follows. Let $x_n = 0$ and let $U_n = B(-1 + 1/n, 1) \cup B(1 - 1/n, 1)$ be the union of two unit balls centered

at $-1 + 1/n$ and $1 - 1/n$. As $n \to \infty$, $\widehat{\mathbb{C}} - U_n$ converges in the Hausdorff topology to $K = \widehat{\mathbb{C}} - (B(-1, 1) \cup B(1, 1))$ which contains the limit point $0 = \lim x_n$, so (U_n, x_n) diverges.

As an application of the Carathéodory limits, we have

Proposition 6.6. *If $f_n \in \mathcal{B}_d$ is a quasi post-critically finite sequence, then after passing to a subsequence, the corresponding polynomials P_n converge to a geometrically finite polynomial $\widehat{P}$.*

Proof. Let $\tilde{c}_n$ be a critical point of f_n and let $\tilde{x}_n$ be an iterate of $\tilde{c}_n$ with

$$d_{\mathbb{D}}(\tilde{x}_n, f_n^q(\tilde{x}_n)) \leq K.$$

Let x_n and c_n be the corresponding points for P_n. After passing to a subsequence, we assume $P_n \to P$, $x_n \to x_\infty \in \mathbb{C}$ and $c_n \to c_\infty \in \mathbb{C}$.

Let U_n be the bounded Fatou component of P_n. If x_∞ is fixed by P^q, then c is pre-periodic. If x_∞ is not fixed by P^q, then by Lemma, 6.5, after passing to a subsequence, the pointed disk $P_n : (U_n, x_n) \to (U_n, P_n^q(x_n))$ converges to $P : (U_\infty, x_\infty) \to (U_\infty, P^q(x_\infty))$. So U_∞ is contained in the Fatou set. Hence c_∞ is in the Fatou set.

Since any critical point of P is approximated by critical points of P_n, we conclude that P is geometrically finite. ∎

Construction of geometrically finite polynomials

Let (H, p) be a marked simplicial pointed Hubbard tree and $f : (\mathcal{T}, p) \to (\mathcal{T}, p)$ be the angled tree map after the admissible split modification. Let $\mathcal{F}$ be a normalized mapping scheme on $\mathcal{S}_p$ if $\delta(p) \geq 2$. Let $f_n \in \mathcal{B}_d$ be as in Proposition 4.3 and $P_n = f_n \sqcup z^d$ be the corresponding polynomials. By Proposition 6.6, after passing to a subsequence, P_n converges to a geometrically finite polynomial $\widehat{P}$. Denote by U_n the bounded Fatou component of P_n. Using the conjugacy between $\mathbb{D}$ and U_n, the quasi-invariant tree for f_n corresponds to a quasi-invariant tree for P_n in U_n. Abusing notation, we denote this quasi-invariant tree for P_n as $\mathcal{T}_n \subseteq U_n$.

We shall now prove that $\widehat{P}$ has the property desired in Proposition 6.1. Recall a vertex $v \in \mathcal{V}$ is said to be a *Fatou point* if v is eventually mapped to a critical periodic orbit; and it is called a *Julia point* otherwise.

Lemma 6.7. *Let $v \in \mathcal{V}$ be a periodic Fatou point of period q and let v_n be the corresponding point for P_n. After passing to a subsequence, (U_n, v_n) converges in the Carathéodory topology to $(U_{v,\infty}, v_\infty)$, and the map*

$$\widehat{P}^q : (U_{v,\infty}, v_\infty) \to (U_{v,\infty}, \widehat{P}^q(v_\infty))$$

is conformally conjugate to the first return rescaling limit

$$F_v^q : (\mathbb{D}_v, 0) \to (\mathbb{D}_v, F_v^q(0)).$$

Proof. The proof is similar to that of Proposition 6.6. Let $\tilde{c}_n$ be a critical point of f_n^q in the cluster associated to v. Let c_n be the corresponding critical point for P_n. After passing to a subsequence, we assume $c_n \to c_\infty$ and $P_n \to \widehat{P}$.

If $\widehat{P}^q(c_\infty) \neq c_\infty$, since $d_{U_n}(c_n, v_n)$ is bounded, the claim on the Carathéodory limit follows by Lemma 6.5. Since each P_n^q is conformally conjugate on U_n to f_n^q, the limit $\widehat{P}^q$ on $U_{v,\infty}$ is conformally conjugate to F_v^q.

If $\widehat{P}^q(c_\infty) = c_\infty$, then c_∞ is a superattracting fixed point for $\widehat{P}^q$. Thus there exists an open set $U \ni c_\infty$ such that $\widehat{P}^q(U)$ is compactly contained in U. Since P_n converges to $\widehat{P}$, for sufficiently large n the iterates of P_n are normal on U. Thus $U \subset U_n$ for sufficiently large n. The rest of the argument is the same as in the previous case. ∎

By pulling back, we immediately get

Corollary 6.8. *Let $v \in \mathcal{V}$ be a Fatou point. Then after passing to a subsequence, (U_n, v_n) converges in the Carathéodory topology to $(U_{v,\infty}, v_\infty)$ and*

$$\widehat{P} : (U_{v,\infty}, v_\infty) \to (U_{f(v),\infty}, \widehat{P}(v_\infty))$$

is conformally conjugate to the rescaling limit

$$F_v : (\mathbb{D}_v, 0) \to (\mathbb{D}_{f(v)}, F_v(0)).$$

By Lemma 6.7 and Corollary 6.8, if $v \in \mathcal{V}$ is a Fatou point, then $U_{v,\infty}$ is contained in a Fatou component Ω_v of $\widehat{P}$.

Corollary 6.9. *Suppose $\delta(p) \geq 2$. Then $U_{p,\infty} = \Omega_p$ and the Fatou component Ω_p is attracting. If $v \neq p \in \mathcal{V}$ is a periodic Fatou point, then the Fatou component Ω_v is parabolic.*

Proof. Since $\delta(p) \geq 2$, by Lemma 6.7, Ω_p is attracting. Since $U_{p,\infty}$ is invariant under $\widehat{P}$ and contains the attracting fixed point, we have $U_{p,\infty} = \Omega_p$. Since $\widehat{P} \in \overline{\mathcal{H}_d}$, there exists at most one attracting Fatou component, so Ω_v is not attracting as $v \neq p$. So Ω_v is parabolic by Proposition 6.6. ∎

Recall T is a v *star-shaped tree* if T is a union of the v arcs $[c, x_i]$, $i = 1, \ldots, v$, glued at c. We say T is an *open v star-shaped tree* if T is a union of the v arcs $[c, x_i)$, $i = 1, \ldots, v$, glued at c.

After passing to a subsequence, we let $\mathcal{T}_\infty$ be the Hausdorff limit of $\mathcal{T}_n$. We show the following (see Figure 1.2):

Lemma 6.10. *Let $v \in \mathcal{V}$ be a periodic Fatou point of period q. Then $\mathcal{T}_\infty \cap U_{v,\infty}$ is an open star-shaped tree where each limit point on $\partial U_{v,\infty}$ is pre-periodic. Each tangent direction $a \in T_v\mathcal{T}$ gives one limit point $s_a \in \partial U_{v,\infty}$. The limiting graph $\mathcal{T}_\infty$ gives an attracting direction for s_a if a corresponds to the direction of p and a repelling direction otherwise.*

Proof. We first note that the quasi-invariant trees converge to an open star-shaped tree in $\mathbb{D}_v$. For any compact subset $K \subseteq U_{v,\infty}$, $\mathcal{T}_\infty \cap K$ is a star-shaped tree by Lemma 6.7.

Let x be a limit point of $\mathcal{T}_\infty \cap U_{v,\infty}$ in $\partial U_{v,\infty}$. Then there exists a sequence $x_n \to x$ and $x_n \in \mathcal{T}_n \cap U_{v,\infty}$.

After passing to a subsequence, we may assume x_n is either quasi-fixed or quasi pre-fixed by $\widehat{P}^k$ for some large k dividing q. If x_n is quasi-fixed, then $d_{U_n}(x_n, P_n^k(x_n)) \le K$. Since $x_n \to \partial U_{v,\infty}$, the hyperbolic metric $\rho_{U_n}(x_n)|dz|$ goes to infinity by Lemma 4.4. Thus the Euclidean distance $d_{\mathbb{R}^2}(x_n, P_n^k(x_n))$ goes to 0 and we conclude that x is fixed by $\widehat{P}^k$. If x_n is quasi pre-fixed, then the same argument shows x is pre-fixed. Since the fixed points of $\widehat{P}^k$ are discrete, each tangent gives one limit point.

The last statement follows from comparing with the dynamics of F_v^q on $\mathbb{D}_v$.   ■

We remark that different tangent directions of $\mathcal{T}$ at v may give the same limit point, thus the closure $\overline{\mathcal{T}_\infty \cap U_{v,\infty}}$ is a graph that is not necessarily a tree (see Figure 1.2). The same proof also gives the following more general statement, where x_n are allowed to be on the edges of $\mathcal{T}_n$:

Lemma 6.11. *Let $x_n \in \mathcal{T}_n \cap U_n$ be quasi periodic with period q that converges to x. If x is not fixed by $\widehat{P}^q$, then (U_n, x_n) converges to (U, x) in the Carathéodory topology and $\mathcal{T}_\infty \cap U$ is an open star-shaped tree whose limit points on ∂U are pre-periodic. Each tangent direction $a \in T_x \mathcal{T}$ gives one limit point s_a. The limiting graph $\mathcal{T}_\infty$ gives an attracting direction for s_a if a corresponds to the direction of p and a repelling direction otherwise.*

Proposition 6.12. *Let $v \ne w \in \mathcal{V}$ be Fatou points. Then the Fatou components Ω_v and Ω_w are not equal.*

Proof. Interchanging v and w if necessary, we may assume v is closer to p than w is. Assume v, w are periodic; the case when v, w are strictly pre-periodic is proved similarly. After passing to an iterate, we assume v and w are fixed. If $v = p$, then the statement follows from Corollary 6.9. Thus we assume $v, w \ne p$.

Let $s_{v,\infty} \in \partial U_{v,\infty}$ and $s_{w,\infty} \in \partial U_{v,\infty}$ be the corresponding parabolic fixed points. Let E_v, E_w be the incident edges at v and w in the direction of p. Let $s_{v,n} \in E_{v,n} \subseteq \mathcal{T}_n$ (and $s_{w,n}$) be a sequence that converges to $s_{v,\infty}$ (and $s_{w,\infty}$ respectively). Suppose for contradiction that $\Omega_v = \Omega_w$. Then $s_{v,\infty} = s_{w,\infty}$ and U_v, U_w are in the same attracting petal. We consider two cases.

Case (1): v, w are in the same component of $\mathcal{T} - \{p\}$. Let b be the point furthest from p on $[p, v] \cap [p, w]$.

If $b = v$, i.e., $v \in [p, w]$, we consider the oriented arcs $[p_n, w_n] \subseteq \mathcal{T}_n$ (see Figure 6.2, left). Since $s_{v,\infty} = s_{w,\infty}$, the limit of the oriented arcs $[s_{v,n}, s_{w,n}] \subseteq [p_n, w_n]$ contains a simple oriented loop γ containing $s_{v,\infty}$. By Lemma 6.11, $\mathcal{T}_\infty$ is contained in the bounded Fatou sets of $\widehat{P}$ except at finitely many pre-periodic points. Since γ is simple, $\gamma = \bigcup_{j=1}^k \gamma_j$ is a finite union of closed smooth arcs γ_i, whose interiors are all contained in Ω_v. Since $[p, w]$ is fixed, $\partial \gamma_j$ are fixed points on $\partial \Omega_v$. Since P_n sends points on $[p_n, w_n]$ towards p_n, γ_j gives a repelling direction for one of the end points and an attracting direction for the other. Therefore $k = 1$ and γ is a loop in $\Omega_v \cup \{s_{v,\infty}\}$.
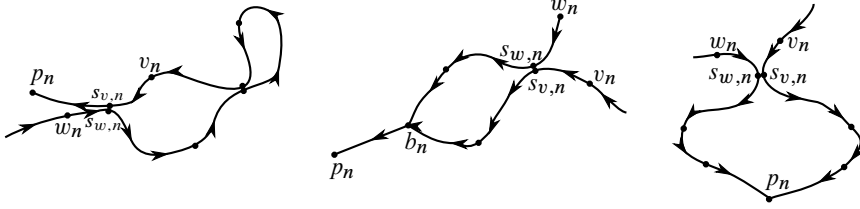
Fig. 6.2. A schematic diagram for the limit of quasi-invariant trees $\mathcal{T}_n$. The arrows represent the dynamics by P_n: the points on $\mathcal{T}_n$ are moving towards the fixed point p_n.

The loop γ does not enclose w_∞ as $\gamma \cap U_{w,\infty} = \emptyset$ and $\partial U_{w,\infty}$ intersects the Julia set. Similarly, γ does not enclose $p_\infty = \lim p_n$. Since $[p_n, w_n]$ is an oriented arc, its limit S containing γ does not have a transverse intersection. If $v \notin \mathcal{T}^C$, then $p_\infty \neq s_{v,\infty}$. Thus the limits of $[p_n, s_{v,n}]$ and $[s_{v,n}, s_{w,n}]$ are non-trivial and give two repelling directions separating $U_{v,\infty}$ and $U_{w,\infty}$, so they are not in the same attracting basin, which is a contradiction. Otherwise, $\alpha_v(T_v \mathcal{T}) \subseteq \mathbb{S}^1$. Thus, the limit of the arc for $[p, w]$ in $\mathbb{D}_v$ separates periodic points on $\mathbb{S}^1_v$, so γ encloses repelling periodic points, which is a contradiction.

If $b \neq v$, since $\mathcal{T}$ is admissible, b is a critical fixed point (see Figure 6.2, middle). The limit of $[b_n, s_{v,n}] \cup [b_n, s_{w,n}]$ contains a simple loop γ containing b_∞. An argument as above shows that γ is contained in Ω_b except at one point. Since the limit of the arc for $[v, w]$ in $\mathbb{D}_b$ separates periodic points on $\mathbb{S}^1_b$, γ encloses repelling periodic points, which is a contradiction.

Case (2): v, w are in different components of $\mathcal{T} - \{p\}$. If $v, w \notin \mathcal{T}^C$, then the proof is the same as above by considering the limit $[p_n, s_{v,n}] \cup [p_n, s_{w,n}]$ (see Figure 6.2, right). Otherwise, label the adjacent vertices to p by $v^1, \ldots, v^k$ counterclockwise. Let $t^i \in \mathbb{S}^1$ be the unique fixed point of m_d landing on the right side of the oriented arc $[v^i, p] \cup [p, v^{i+1}]$. If t^i does not land at v^i (or v^{i+1}), then the first return rescaling limit F_{v^i} (or $F_{v^{i+1}}$) has a repelling direction in the clockwise (or counterclockwise) direction from the fixed point $0 \in \mathbb{S}^1_{v^i}$ (or $\mathbb{S}^1_{v^{i+1}}$), giving a repelling direction that separates $U_{v_i,\infty}$ and $U_{v_{i+1},\infty}$. Therefore, Ω_{v^i} are all different and the statement follows. ∎

We prove the lamination of $\widehat{P}$ gives the desired equivalence relation.

Proposition 6.13. *Let $a \in H$ be a Julia point. Then there exists a corresponding pre-periodic point $\widehat{a} \in J(\widehat{P})$ such that an external angle lands at a if and only if it lands at $\widehat{a}$.*

Proof. We assume a is periodic, as the strictly pre-periodic case can be proved by pull-back. After passing to an iterate, we may assume a is fixed.

If $a \neq p$, denote the adjacent vertices by $a_0, \ldots, a_{m-1}$. We assume that the vertices are labeled counterclockwise and a_0 is the unique vertex that is closer to p than a is. Each a_i is a periodic Fatou point as H is simplicial. There are exactly m external rays landing at a. Let $A_0, \ldots, A_{m-1}$ be the angles landing at a, where A_i corresponds to the access between the Fatou components associated to a_i and a_{i+1}. Let $S \subseteq \mathcal{T}$ be the convex
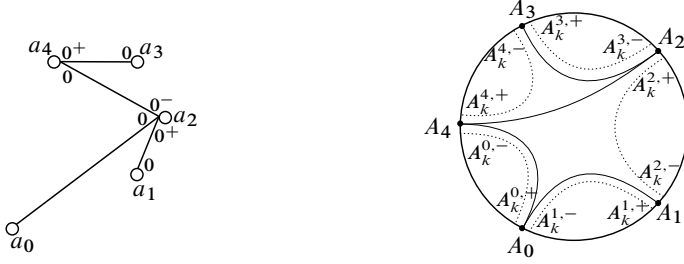
Fig. 6.3. The subtree S after the split modification on the left with its dual lamination on the right.

hull of $a_0, \ldots, a_m$ after the admissible splitting (see Figure 6.3). Let $\widehat{s}_i \in \partial U_{a_i, \infty}$ be the fixed point of $\widehat{P}$ associated to the direction towards a_0. Note $\widehat{s}_i$ is a parabolic fixed point for $\widehat{P}$.

By Proposition 6.3, we can find periodic points $s_k^{i,+} = \lim_{a_i} \eta_n(A_k^{i,+})$ and $s_k^{i,-} = \lim_{a_i} \eta_n(A_k^{i,-})$ in $\mathbb{S}_{a_i}^1$ that are not holes such that $A_k^{i,+}$ and $A_k^{i,-}$ converge to A_i from below and A_{i-1} from above in counterclockwise orientation (see Figure 6.3). Let $\widehat{s}_k^{i,\pm}$ be the corresponding periodic points of $\widehat{P}$. We let R_k be the domain bounded by geodesics of $[\widehat{s}_k^{i,-}, \widehat{s}_k^{i,+}]$ in the Fatou component and external rays of angles $A_k^{i,\pm}$ for $i = 0, \ldots, m-1$. Note that $R_{k+1} \subseteq R_k$ and the fixed points $\widehat{s}_i$ $i = 1, \ldots, m-1$, are all contained in R_n. For sufficiently large n, the external angles landing at $\{\widehat{s}_i : i = 1, \ldots, m-1\}$ are exactly A_j, $j = 0, \ldots, m-1$. Since $\Omega_{a_i} \neq \Omega_{a_j}$ by Proposition 6.12, there are at least $m-1$ attracting petals as the attracting fixed point of P_n does not converge to $\widehat{s}_i$. By counting the multiplicity of the parabolic fixed points, $\widehat{s}_i$ must all be the same. Let $\widehat{a}$ be this parabolic fixed point. Then the angles landing at it are exactly A_i, $i = 0, \ldots, m-1$.

If $a = p$ with valence k, then by Proposition 6.12 there are k attracting petals at the parabolic fixed point $\widehat{a}$. The same nested domain argument shows the angles landing at $\widehat{a}$ are exactly the k angles landing at a. ∎

Proposition 6.14. *The polynomials $\widehat{P}$ and P have topologically conjugate dynamics on their Julia sets.*

Proof. By Proposition 6.12, the critical and post-critical Fatou components of $\widehat{P}$ are in correspondence with those of P. By Proposition 6.13, the external angles landing at roots of critical, post-critical Fatou components and critical, post-critical Julia points are the same for $\widehat{P}$ and P. Since these landing angles uniquely determine the dynamics of $\widehat{P}$ and P on their Julia sets, the proposition follows. ∎

M-uni-critical doubly parabolic Blaschke product

Up to conjugation, there exists a unique uni-critical doubly parabolic Blaschke product for each degree:

$$f(z) = \frac{z^d + a}{1 + az^d} \quad \text{for } a = \frac{d-1}{d+1}.$$

Recall that a degree d proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is M-*uni-critical* if the critical points are contained in $B_{\mathbb{H}^2}(0, M)$. The following compactness result is useful and interesting.

Proposition 6.15. *The space of degree d M-uni-critical doubly parabolic Blaschke products is bounded in the space of degree d proper maps on $\mathbb{D}$.*

Proof. Suppose for contradiction that this space is not bounded. Then there exists a sequence of M-uni-critical doubly parabolic Blaschke products f_n that are degenerating when viewed as rational maps. We normalize by rotation so that 1 is the parabolic fixed point. Since f_n is degenerating and the critical points are all within hyperbolic distance M of 0, after passing to a subsequence, f_n converges algebraically to a constant function t by Lemma 2.4.

Suppose $t = 1$. By Lemma 4.9, the preimages $f_n^{-1}(1)$ are uniformly separated. Thus we can choose an arc $1 \in \gamma \subseteq \mathbb{S}^1$ so that f_n is injective on γ and its end points $\partial\gamma$ are not holes. Since $f_n(\partial\gamma) \to 1$ and $f_n(\gamma)$ contains 1, for sufficiently large n, 1 has at least one attracting direction on $\mathbb{S}^1$, which is a contradiction.

Suppose $t \neq 1$. Choose a disk U with $1 \in U$, $t \notin \bar{U}$ and such that ∂U contains no holes. Then for sufficiently large n, U contains no critical values of f_n and $f_n^{-1}(\partial U) \cap \partial U = \emptyset$ as $f_n(\partial U) \to t$. Thus the component V of $f_n^{-1}(U)$ containing 1 is contained in U, and f_n is univalent on V, which is a contradiction to 1 being a parabolic fixed point. $\blacksquare$

Proof of Proposition 6.1. By our construction, $P_n \to \widehat{P}$ and $\widehat{P}$ corresponds to the Hubbard tree (H, p) by Proposition 6.14. The dynamics on Ω_v for $v \in \mathcal{S}_p$ is conjugate to $\mathcal{F}$ by Lemma 6.7 and Corollary 6.9. Let $M_1 = M_1(f)$ be the constant in Proposition 4.3. If $\mathcal{C} \subseteq \mathcal{V}$ is a periodic Fatou cycle of period q, there exists $v \in \mathcal{C}$ for which F_v^q is M_1-uni-critical. Since $\widehat{P}^q : U_{v_1,\infty} \to U_{v_1,\infty}$ is conjugate to $F_{v_1}^q$ on $\mathbb{D}_{v_1}$ by Lemma 6.7, $\widehat{P}^q$ on $U_{v_1,\infty}$ is M_1-uni-critical. Since $U_{v_1,\infty} \subseteq \Omega_{v_1}$, and the inclusion is distance non-increasing (with respect to the hyperbolic metric) by the Schwarz lemma, $\widehat{P}^q$ on the Fatou component Ω_{v_1} is also M_1-uni-critical. Since $\widehat{P}^q$ conjugates to a doubly parabolic Blaschke product on Ω_{v_1}, by Proposition 6.15, $d_{\Omega_{v_1}}(c, \widehat{P}^q(c)) \leq M_2$ for all critical points c and some constant M_2 depending only on M_1. Therefore, $\widehat{P}^q$ on Ω_w is M-uni-critical for all $w \in \mathcal{C}$, with M depending only on the tree map.

The same argument works for strictly pre-periodic points, and the proposition follows. $\blacksquare$

Degenerations on $\partial \mathcal{H}_d$

Let $S \subseteq \overline{\mathcal{H}_d}$ be the space of all geometrically finite polynomials associated to a pointed simplicial Hubbard tree (H, p). Proposition 6.1 implies that

(1) S projects onto the space $\mathcal{B}^{\mathcal{S}_p}$ of all normalized mapping schemes on $\mathcal{S}_p$ (see Definition 4.2);

(2) each fiber contains at least one polynomial which is M-uni-critical on all other Fatou components.
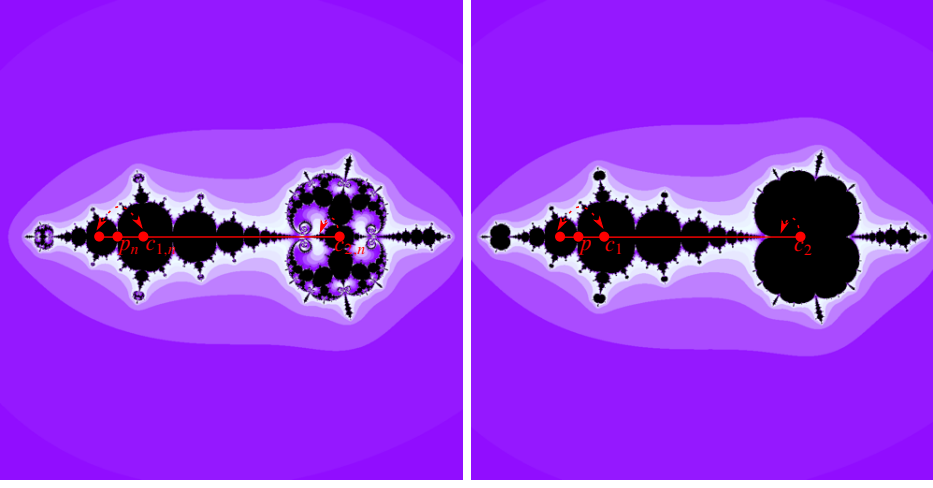
Fig. 6.4. The convergence to a degree 3 geometrically finite polynomial $\hat{P}$ (on the right) with iterated-simplicial pointed Hubbard tree that is not simplicial. The critical point $c_{2,n}$ is 'hidden' behind $c_{1,n}$. The escaping rates of the two critical points are incompatible, and $c_{2,n}$ is not quasi pre-periodic. $\hat{P}$ cannot be obtained directly by a quasi post-critically finite degeneration.

The space $\mathcal{B}^{\mathcal{S}_p}$ is the product of (normalized) spaces of Blaschke products (see [38, §4]). The degenerations on $\mathcal{B}^{\mathcal{S}_p}$ can be defined similarly:

Definition 6.16. Let $\mathcal{F}_n \in \mathcal{B}^{\mathcal{S}_p}$. We say that this sequence is *K-quasi post-critically finite* if we can label the critical points of $\mathcal{F}_n$ as $c_{1,n}, \ldots, c_{k,n}$ in such a way that for each i, there exists a quasi pre-period l_i and a quasi period q_i with

$$d_{\mathbb{D}}(\mathcal{F}_n^{l_i}(c_{i,n}), \mathcal{F}_n^{l_i+q_i}(c_{i,n})) \leq K.$$

Let $q_n \in \mathbb{D}_p$ be the unique attracting fixed point for $\mathcal{F}_n$. For $a \in \mathcal{S}_p$, define $Q_{a,n}$ inductively with $Q_{p,n} = \{q_n\}$ and $Q_{a,n} = \mathcal{F}_{a,n}^{-1}(Q_{f(a),n})$.

For each vertex $a \in \mathcal{S}_p$, the quasi-invariant trees $\mathcal{T}_{a,n}$ are constructed similarly to the case of $\mathcal{B}_d$. The dynamics are modeled by a collection of simplicial maps

$$\mathcal{F}_a : (\mathcal{T}_a, Q_a) \to (\mathcal{T}_{f(a)}, Q_{f(a)}),$$

with rescaling limits $F_v : \mathbb{D}_v \to \mathbb{D}_{\mathcal{F}(v)}$ where v is a vertex of $\mathcal{T}_a$, $a \in \mathcal{S}_p$. Here Q_a is a finite set corresponding to $Q_{a,n}$.

These simplicial maps with the rescaling limits are combinatorially modeled by an *angled tree mapping scheme*, a collection of simplicial maps $\mathcal{F}_a : (\mathcal{T}_a, Q_a) \to (\mathcal{T}_{f(a)}, Q_{f(a)})$ with local degree functions and anchored, compatible angle functions. The angled tree mapping schemes are said to be *admissible* if $\mathcal{F}_p : (\mathcal{T}_p, q) \to (\mathcal{T}_p, q)$ is admissible.

Minimality of the angled tree mapping schemes is defined similarly to angled tree maps. Note that for a minimal angled tree mapping scheme, $\mathcal{F}_p : (\mathcal{T}_p, q) \to (\mathcal{T}_p, q)$ may

not be minimal as it may contain vertices corresponding to orbits of critical points in $\mathbb{D}_v$ with $v \neq p$.

Let $\mathcal{S}_q \subseteq \bigcup_{a \in \mathcal{S}_p} \mathcal{T}_a$ be the backward orbits of q under the angled tree mapping scheme $\mathcal{F}$.

Since the induced dynamics $f : \mathcal{S}_p \to \mathcal{S}_p$ is particularly easy (it has a unique fixed point at p and all the other points are in the backward orbits of p), by pulling back the degeneration on $\mathbb{D}_p$, Proposition 4.3 can be easily generalized to degenerations in $\mathcal{B}^{\mathcal{S}_p}$:

Proposition 6.17. *Let $\mathcal{F} : \bigcup_{a \in \mathcal{S}_p} (\mathcal{T}_a, Q_a) \to \bigcup_{a \in \mathcal{S}_p} (\mathcal{T}_a, Q_a)$ be a minimal admissible angled tree mapping scheme for $\mathcal{S}_p$. When $\delta(q) \geq 2$, let $\mathcal{G}$ be a normalized mapping scheme on $\mathcal{S}_q$. There exists a K-quasi post-critically finite sequence $\mathcal{F}_n \in \mathcal{B}^{\mathcal{S}_p}$ realizing $\mathcal{F}$ such that*

(1) *the rescaling limits on $\mathcal{S}_q$ are conjugate to $\mathcal{G}$;*

(2) *there exists a constant M depending only on the angled tree mapping scheme $\mathcal{F}$ (and thus independent of $\mathcal{G}$) such that*

    (a) *for any periodic cycle $\mathcal{C}$ other than q, there exists a periodic point $v \in \mathcal{C}$ such that the first return rescaling limit*

$$F_v^k : \mathbb{D}_v \to \mathbb{D}_v$$

    is M-uni-critical, where k is the period of v;

    (b) *for any strictly pre-periodic vertex $w \notin \mathcal{S}_q$, the rescaling limit $F_w : \mathbb{D}_w \to \mathbb{D}_{f(w)}$ is M-uni-critical and the critical values are within hyperbolic distance M of $0 \in \mathbb{D}_{f(w)}$.*

Let (H', p') be a pointed simplicial tuning of (H, p). Then there exists an associated angled tree mapping scheme

$$\mathcal{F}_a : (\mathcal{T}_a, Q_a) \to (\mathcal{T}_{f(a)}, Q_{f(a)}).$$

By performing an admissible splitting on $\mathcal{T}_p$ and pull back the modifications accordingly to $\mathcal{T}_a$, $a \neq p$, we can define a quasi post-critically finite degeneration $\mathcal{F}_n \in \mathcal{B}^{\mathcal{S}_p}$ by Proposition 6.17. By Proposition 6.1, we can find a sequence $Q_n \in S \subseteq \overline{\mathcal{H}_d}$ whose dynamics on Fatou components of $\mathcal{S}_p$ are conjugate to $\mathcal{F}_n \in \mathcal{B}^{\mathcal{S}_p}$ while remaining M-uni-critical on all other Fatou components.

Since $\overline{\mathcal{H}_d}$ is compact, after passing to a subsequence, Q_n converges to $Q \in \overline{\mathcal{H}_d}$. The same argument as in Proposition 6.6, which uses Lemma 6.5, shows that Q is geometrically finite. By Proposition 6.15, the Fatou component $\Omega_{v,n}$ of Q_n for $v \in \mathcal{V} - \mathcal{S}_p$ converges to a corresponding Fatou component Ω_v. The same argument as for Proposition 6.12 shows that the corresponding Fatou components of Ω_v are different for different Fatou vertices of $\mathcal{T}_a$, $a \in \mathcal{S}_p$. Thus, a similar proof to those for Propositions 6.13 and 6.14 shows that the associated pointed Hubbard tree for Q is (H', p'). Moreover, by Proposition 6.17, the space $S' \subseteq \overline{\mathcal{H}_d}$ of all geometrically finite polynomials associated to (H', p') again satisfies the two properties listed at the beginning of this subsection. Therefore, by induction, we have

Proposition 6.18. *Let* (H, p) *be an iterated-simplicial Hubbard tree. Then there exists a geometrically finite polynomial* $\widehat{P} \in \overline{\mathcal{H}_d}$ *associated to it.*

Proof of Theorem 1.1. The theorem follows from Propositions 5.1 and 6.18. ∎

7. Self-bumps on $\partial \mathcal{H}_d$

In this section, we shall prove Theorem 1.4. We first explain the phenomenon with an example in degree 4, which can then be easily generalized to any higher degree.

Consider the geometrically finite polynomial

$$\widehat{P}(z) = z^4 - \tfrac{3}{8}z^2 + \tfrac{9}{8}z - \tfrac{3}{256}.$$

It has a superattracting fixed point at $-\tfrac{3}{4}$, and a double parabolic fixed point at $\tfrac{1}{4}$. The associated pointed Hubbard tree is a tripod

$$H = [v, a_0] \cup [v, a_1] \cup [v, a_2]$$

where $p = a_0$ and the vertices are ordered counterclockwise (see Figure 1.3 (b)).

The dynamics fixes all three edges. There are two distinct admissible split modifications, resulting in two different admissible angled tree maps:

$$^1\mathcal{T} := [a_0, v] \cup [v, a_1] \cup [a_1, a_2],$$
$$^2\mathcal{T} := [a_0, v] \cup [v, a_2] \cup [a_2, a_1].$$

Let $^1P_n, {}^2P_n \in \mathcal{H}_d$ be the two sequences of polynomials associated to $^1\mathcal{T}$ and $^2\mathcal{T}$, where every polynomial in the sequences is assumed to have a superattracting fixed point associated to a_0 (see Figure 1.3 (c, d)). There are three bounded critical Fatou components for $\widehat{P}$, and each Fatou component is fixed and contains one critical point. One is superattracting, and the other two are parabolic. Thus, the dynamics on the Fatou components are rigid (see [27, §6]). Hence both ^1P_n and ^2P_n converge to $\widehat{P}$.

Denote the repelling fixed points by $^1x_{1,n}, {}^1x_{2,n}, {}^1x_{3,n}$ and $^2x_{1,n}, {}^2x_{2,n}, {}^2x_{3,n}$ respectively. Then $^ix_{j,n}$ converges to the parabolic fixed point $\tfrac{1}{4}$ for $\widehat{P}$. We label them so that the three fixed points are ordered counterclockwise and $^ix_{2,n}$ corresponds to the fixed point that is accessible from the positive real axis in the limit. To prove that $\widehat{P}$ gives a self-bump on $\partial \mathcal{H}_d$, we study the multipliers of the three fixed points.

Residue computation

Let f be a holomorphic function defined in a neighborhood of $z_0 \in \mathbb{C}$, and suppose z_0 is an isolated fixed point of f. The *residue* of f at the fixed point z_0 is

$$\mathrm{res}(f, z_0) = \mathrm{Res}_{z=z_0} \frac{dz}{f(z) - z}$$

where the right-hand side is the residue of the 1-form at z_0. The residue is invariant under conformal changes of coordinate, thus it can also be defined for a fixed point at ∞.

If the multiplier at z_0 is $\lambda \neq 1$, then the residue is

$$\operatorname{res}(f, z_0) = \frac{1}{\lambda - 1}.$$

Thus, z_0 is a repelling fixed point if and only if

$$\Re e(\operatorname{res}(f, z_0)) > -1/2.$$

Let C be an oriented closed curve that bounds a domain D with no fixed point on C. Then the Residue Theorem shows that

$$\sum_{z_i \text{ fixed point in } D} \operatorname{res}(f, z_i) = \frac{1}{2\pi i} \int_C \frac{dz}{f(z) - z}.$$

For a global rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, we have

$$\sum_{z_i \text{ fixed point}} \operatorname{res}(f, z_i) = -1.$$

Back to our setting, an explicit computation shows $\operatorname{res}(\widehat{P}, 1) = 1$.

Lemma 7.1. *Let $U \subseteq \mathcal{P}_4$ be any sufficiently small neighborhood of $\widehat{P}$. Let $f \in U \cap \mathcal{H}_4$, and z be any repelling fixed point of f. Then $\Im\mathfrak{m}(f'(z)) \neq 0$.*

Proof. Let z_1, z_2, z_3 be the three repelling fixed points of f. If we choose U sufficiently small, then the multipliers λ_i are close to 1 and the sum

$$\sum_{i=1}^{3} \operatorname{res}(f, z_i) = \sum_{i=1}^{3} \frac{1}{\lambda_i - 1}$$

is close to $\operatorname{res}(\widehat{P}, 1) = 1$. If $\Im\mathfrak{m}(\lambda_1) = 0$, then $\lambda_1 > 1$, so

$$\Re e\left(\frac{1}{\lambda_2 - 1} + \frac{1}{\lambda_3 - 1}\right)$$

is very negative. Thus at least one of the multipliers is attracting, which is a contradiction. ∎

To prove that the intersection of any sufficiently small neighborhood U of $\widehat{P}$ with $\mathcal{H}_4$ is disconnected, it suffices to show that the *signatures*, i.e., signs of the imaginary parts of the multipliers of ${}^i x_{j,n}$, are different for the two sequences.

Signatures of a simple parabolic point

Let f be a holomorphic function in a neighborhood of $0 \in \mathbb{C}$. Assume that 0 is an isolated simple parabolic fixed point of f. After a conformal change of coordinate, we may assume that

$$f(z) = z + z^2 + O(z^3)$$

where $a > 0$. Note that near 0, the positive real axis is a repelling direction, while the negative real axis is an attracting direction. Assume that we perform a small perturbation so that the parabolic fixed point splits into two repelling fixed points. After conjugating with $z \mapsto z + c$ if necessary, we may assume that these two fixed points are $\pm \epsilon$, symmetric with respect to 0. Since we assume that the two fixed points are both repelling, ϵ is not a real number. By this normalization, an easy computation shows that the signatures of $\pm \epsilon$ equal the signs of the imaginary parts of $\pm \epsilon$.

Splitting the double parabolic point

To compute the signature, we degenerate in two steps, which allows us to consider perturbations of only simple parabolic points. Consider the degree 4 polynomial $Q \in \partial \mathcal{H}_4$ with a superattracting fixed point of local degree 3 and with a parabolic fixed point on the boundary of the immediate superattracting basin. We can degenerate the dynamics on the attracting Fatou component while staying on $\partial \mathcal{H}_4$ (see Proposition 6.1).

We choose the marking for the degree 3 Blaschke product so that $0 \in \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ corresponds to the parabolic fixed point of Q. We call this repelling fixed point the *marked fixed point*. We construct two geometrically finite sequences of Blaschke products ${}^1 f_n$ and ${}^2 f_n$, with quasi-invariant trees

$$ {}^i \mathcal{T} = [p, a], $$

where p and a are fixed points of local degree 2. The rescaling limits at a are different: For ${}^1 \mathcal{T}$, the marked fixed point has angle 0^+ at the vertex a, while it has angle 0^- at the vertex a for ${}^2 \mathcal{T}$. We also assume that both sequences have a superattracting fixed point. Then the corresponding sequences of polynomials ${}^1 Q_n$ and ${}^2 Q_n$ both converge to $\widehat{P}(z)$ (see Figure 1.3 (c, d)).

For sufficiently large n, we can perturb ${}^1 Q_n$ and ${}^2 Q_n$ slightly to get two polynomials ${}^1 P_n$ and ${}^2 P_n$ in $\mathcal{H}_4$, with ${}^i P_n \to \widehat{P}$. As before, we denote the three repelling fixed points of ${}^i P_n$ by ${}^i x_{1,n}, {}^i x_{2,n}, {}^i x_{3,n}$, where ${}^i x_{2,n}$ corresponds to the fixed point that is accessible from the positive real axis in the limit (see Figure 1.3).

Note that ${}^i P_n$ is constructed from ${}^i Q_n$ by splitting the simple parabolic fixed point into two repelling fixed points. Using the orientation of the dynamics near the simple parabolic fixed point of ${}^i Q_n$, we can compute that the signature at ${}^1 x_{2,n}$ is $+$, while the signature at ${}^2 x_{2,n}$ is $-$. Thus, by Lemma 7.1, Theorem 1.4 holds for degree 4.

If we replace the superattracting fixed point by a superattracting fixed point of degree $d - 2$, Theorem 1.4 holds for any degree $d \geq 4$.

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