J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1335

Published by EMS Press

Wiesław Pawłucki

Strict $\mathcal{C}^p$-triangulations – a new approach to desingularization

To Jolanta

Received January 28, 2021; revised May 9, 2022

Abstract. Let R be any real closed field expanded by some o-minimal structure. Let $f : A \to R^d$ be a definable and continuous mapping defined on a definable, closed, bounded subset A of R^n. Let $\mathcal{E}$ be a finite family of definable subsets of R^n contained in A. Let p be any positive integer. We prove that then there exists a finite simplicial complex $\mathcal{T}$ in R^n and a definable homeomorphism $h : |\mathcal{T}| \to A$, where $|\mathcal{T}| := \bigcup \mathcal{T}$, such that for each simplex $\Delta \in \mathcal{T}$, the restriction of h to its relative interior $\mathring{\Delta}$ is a $\mathcal{C}^p$-embedding of $\mathring{\Delta}$ into R^n and moreover both h and $f \circ h$ are of class $\mathcal{C}^p$ in the sense that they have definable $\mathcal{C}^p$-extensions defined on an open definable neighborhood of $|\mathcal{T}|$ in R^n. We then call a pair $(\mathcal{T}, h)$ a *strict* $\mathcal{C}^p$*-triangulation* of A. In addition, this triangulation can be made compatible with $\mathcal{E}$ in the sense that for each $E \in \mathcal{E}$, $h^{-1}(E)$ is a union of some $\mathring{\Delta}$, where $\Delta \in \mathcal{T}$. We also give an application to approximation theory.

Keywords. O-minimal structure, semialgebraic set, $\mathcal{C}^p$-triangulation, strict $\mathcal{C}^p$-triangulation, capsule, detector

1. Introduction and Main Theorem

We will work with an arbitrary fixed o-minimal expansion of any real closed field R, e.g. the field $\mathbb{R}$ of real numbers with semialgebraic subsets of $\mathbb{R}^n$ spaces, where $n \in \mathbb{N}$. O-minimal geometry (see [4,21] for fundamental notions and results) is a far-reaching generalization of semialgebraic and subanalytic geometries (presented in [2, 3, 8, 11, 15, 19]). We will deal only with subsets of R^n and mappings $f : A \to R^m$, where $A \subset R^n$, which are *definable* in this structure (the mapping f is called definable if the graph of f is a definable subset of R^{n+m}). Therefore, as a rule we will skip the adjective "definable".

We adopt the following general definition. If $\mathcal{K}$ is any family of subsets of a set X, then by a *refinement* of $\mathcal{K}$ we understand any family $\mathcal{L}$ of subsets of X such that each

Wiesław Pawłucki: Institute of Mathematics, Jagiellonian University, 30-348 Kraków, Poland; wieslaw.pawlucki@im.uj.edu.pl

Mathematics Subject Classification (2020): Primary 32B25; Secondary 03C64, 14P10, 32B20, 32S45, 57R05

$L \in \mathcal{L}$ is contained in some $K \in \mathcal{K}$ and each $K \in \mathcal{K}$ is the union $\bigcup \mathcal{L}'$ of some subfamily $\mathcal{L}' \subset \mathcal{L}$. The term "refinement" will also be used in another sense: if $\mathcal{F}$ is a family of functions defined on a set X we will say that a family $\mathcal{G}$ of functions defined on X is a *refinement* of $\mathcal{F}$ if simply $\mathcal{F} \subset \mathcal{G}$.

If $\mathcal{K}$ is any family of subsets of a set X, then we will denote by $|\mathcal{K}|$ the union of all those subsets.

The interior of a subset A of a topological space will in general be denoted int A, but often we find the Bourbaki notation $\mathring{A}$ more handy, while for the closure of A we will use either $\overline{A}$ or cl A.

We adopt the standard definition of a *simplex* of dimension k in R^n as the convex hull of $k+1$ points $a_0, \dots, a_k$ affinely independent in R^n; i.e.

$$\Delta = [a_0, \dots, a_k] := \Big\{ \sum_{i=0}^{k} \alpha_i a_i : \alpha_i \geq 0 \ (i \in \{0, \dots, k\}),\ \sum_{i=0}^{k} \alpha_i = 1 \Big\}.$$

If $0 \leq i_0 < i_1 < \dots < i_l \leq k$, then the simplex $[a_{i_0}, \dots, a_{i_l}]$ is called a *face* of Δ of dimension l. The points $a_0, \dots, a_k$ are called *vertices* of Δ. The *boundary* $\partial\Delta$ of Δ is the union of all faces of Δ of dimension $< k$. Its *relative interior* is by definition

$$\mathring{\Delta} := \Delta \setminus \partial\Delta = (a_0, \dots, a_k) := \Big\{ \sum_{i=0}^{k} \alpha_i a_i : \alpha_i > 0 \ (i \in \{0, \dots, k\}),\ \sum_{i=0}^{k} \alpha_i = 1 \Big\}.$$

It will be convenient for us to use a more general notion of a *convex polyhedron* in R^n, which is defined as the convex hull of any finite subset of R^n. All polyhedra considered in this paper are assumed to be convex. It is clear that the notions of dimension, faces, boundary, vertices and relative interior generalize to all polyhedra and that polyhedra are definable in PL-geometry. For a polyhedron P in R^n and $l \in \{0, \dots, n\}$ we will denote by $P^{(l)}$ its *l-dimensional skeleton*, i.e. the union of all its faces of dimension $\leq l$.

By a *polyhedral complex* in R^n we will always understand a finite family $\mathcal{P}$ of (convex) polyhedra in R^n such that for each $P \in \mathcal{P}$ all faces of P belong to $\mathcal{P}$ and for each pair $P_1, P_2 \in \mathcal{P}$, $P_1 \cap P_2$ is empty or a common face of both P_1 and P_2. A polyhedral complex consisting of simplexes is called a *simplicial complex*. In fact, we will restrict our considerations to polyhedral complexes $\mathcal{P}$ such that $|\mathcal{P}|$ is a polyhedron of constant dimension n. Then a polyhedral complex can be defined as a finite family of polyhedra of dimension n such that the intersection of any two of them is their common face, if not empty. We will use this identification for simplicial complexes as well.

Throughout the paper, p denotes a positive integer.

Definition 1. Let A be any definable, bounded, closed subset of R^n. A *$\mathcal{C}^p$-triangulation* of A is a pair $(\mathcal{T}, h)$, where $\mathcal{T}$ is a simplicial complex in R^n and h is a definable homeomorphism of $|\mathcal{T}|$ onto A such that for each simplex $\Delta \in \mathcal{T}$ the restriction $h|\mathring{\Delta}$ is a $\mathcal{C}^p$-embedding of $\mathring{\Delta}$ into R^n. If $\mathcal{E}$ is any finite family of definable subsets of A we say that the triangulation $(\mathcal{T}, h)$ is compatible with $\mathcal{E}$ if for each $E \in \mathcal{E}$ the inverse image $h^{-1}(E)$ is a union of some $\mathring{\Delta}$, where $\Delta \in \mathcal{T}$. A $\mathcal{C}^p$-triangulation of A will be called a *strict $\mathcal{C}^p$-triangulation* of A if the mapping $h : |\mathcal{T}| \to R^n$ is of class $\mathcal{C}^p$ in the sense of the following definition.

Definition 2. If $f : B \to R^d$ is any definable mapping defined on a definable subset $B \subset R^n$ we say that f is of class $\mathcal{C}^p$ if it admits an extension $\tilde{f} : U \to R^d$ of class $\mathcal{C}^p$ defined on an open definable neighborhood U of B in R^n.

Main Theorem. *Let R be any real closed field expanded by some o-minimal structure. Let $f : A \to R^d$ be a definable and continuous mapping defined on a definable, closed, bounded subset A of R^n. Let $\mathcal{E}$ be a finite family of definable subsets of R^n contained in A. Let p be any positive integer.*

Then there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of A compatible with $\mathcal{E}$ and such that $f \circ h$ is of class $\mathcal{C}^p$.

In fact, we prove a more precise theorem that an arbitrary definable triangulation of the set A can be refined to a strict $\mathcal{C}^p$-triangulation at the same time smoothing the mapping f to the class $\mathcal{C}^p$. Namely, we have the following (compare Corollary 8.7).

Strict $\mathcal{C}^p$-Refinement Theorem. *Under the assumptions of the Main Theorem, let $\mathcal{P}$ be a polyhedral complex in R^n and let $g : |\mathcal{P}| \to A$ be any definable homeomorphism. Then there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $|\mathcal{P}|$ such that $\mathcal{T}$ is a refinement of $\mathcal{P}$, $h(\Gamma) = \Gamma$ for any face Γ of any polyhedron $P \in \mathcal{P}$ and $g \circ h$ is a strict $\mathcal{C}^p$-triangulation of A compatible with the family $\mathcal{E}$ and such that $f \circ g \circ h$ is of class $\mathcal{C}^p$.*

The proof of both theorems is an interplay between PL- and o-minimal geometries. The general idea comes from our earlier paper about $\mathcal{C}^p$-parametrizations of sets definable in o-minimal structures [12]. In that paper we parametrized definable sets by $\mathcal{C}^p$-mappings defined on cubes (similarly to the classical analytic rectilinearization theorem for subanalytic sets [2, 11]), which inevitably spoils injectivity of the parametrization. Similarly, blowing-up operations evidently spoil injectivity. Instead of cubes or blowings-up we propose to use simplexes as in the classical triangulation theorem [21, Chapter 8], which can be adapted to give $\mathcal{C}^p$-triangulations (cf. [10]). The problem is to make a triangulating homeomorphism (extendable to) a $\mathcal{C}^p$-mapping. Our procedure of smoothing is based on the case of dimension 1, that is, on the Main Theorem for $n = 1$, the proof of which we will briefly explain now, assuming for simplicity that $d = 1$.

Without any loss of generality we can assume that $f : [a, b] \to R$ is a continuous definable function defined on a bounded, closed interval. There exists a finite sequence $c_0 = a < c_1 < \cdots < c_{s+1} = b$ such that for each $i \in \{0, \ldots, s\}$, the restriction $f|(c_i, c_{i+1})$ is of class $\mathcal{C}^{p+1}$ and either $|f'| \le 1$ on (c_i, c_{i+1}) or $|f'| > 1$ on (c_i, c_{i+1}). Now we use a simple but beautiful trick of Coste–Reguiat [5] reducing the problem to that where $|f'| \le 1$ on $[a, b] \setminus \{c_0, \ldots, c_{s+1}\}$. Namely, we define $g : [a, b] \to R$ by an inductive formula. First, we put $g(a) = g(c_0) = f(a)$. Then we define g on $[c_i, c_{i+1}]$ depending on the following two cases:

Case I: If $|f'| \le 1$ on $[c_i, c_{i+1}]$, then we put $g(x) := g(c_i) + x - c_i$ for $x \in [c_i, c_{i+1}]$.

Case II: If $|f'| > 1$ on $[c_i, c_{i+1}]$, then we put $g(x) := g(c_i) + |f(x) - f(c_i)|$ for $x \in [c_i, c_{i+1}]$.

Put $d_i = g(c_i)$ for $i \in \{0, \dots, s+1\}$. Observe that $g : [c_0, c_{s+1}] \to [d_0, d_{s+1}]$ is a strictly increasing homeomorphism such that $g'(x) \geq 1$ for $x \in [c_0, c_{s+1}] \setminus \{c_0, \dots, c_{s+1}\}$. Take now the inverse $h := g^{-1} : [d_0, d_{s+1}] \to [c_0, c_{s+1}]$. Then $0 < h'(y) \leq 1$ and $|(f \circ h)'(y)| \leq 1$ for each $y \in (d_i, d_{i+1})$, where $i \in \{0, \dots, s\}$. Now we use a trick of Yomdin–Gromov (see Lemma 4.1 and Corollary 4.2 below and compare with [9,22,23]). Passing perhaps to a finer subdivision one can assume that on each of the intervals (d_i, d_{i+1}) each of the derivatives $h^{(\nu)}$ and $(f \circ h)^{(\nu)}$, where $\nu \in \{2, \dots, p+1\}$, exists and has a constant sign. It follows that there exists a piecewise polynomial strictly increasing function $\omega : [\gamma_0, \gamma_{2s}] \to [d_0, d_s]$ of class $\mathcal{C}^p$, where $\gamma_0 < \gamma_1 < \dots < \gamma_{2s}$, such that

$$\omega(\gamma_{2i}) = d_i \quad (i \in \{0, \dots, s\}), \quad \omega(\gamma_{2i-1}) = \frac{d_{i-1} + d_i}{2} \quad (i \in \{1, \dots, s\}),$$

ω is p-flat at each γ_j and such that $h \circ \omega$ and $f \circ (h \circ \omega)$ are $\mathcal{C}^p$-functions flat at $\gamma_0, \dots, \gamma_{2s}$.

For $n > 1$ we use the same smoothing procedure but with parameters. In order to make it possible we introduce two devices: *capsules* which are cells without vertical line segments in the boundary (see Section 2) and *detectors* which are special differentiable functions of choice (see Section 3). A capsule in R^n can be treated as a family parametrized by an open subset D of R^{n-1} of vertical line segments shrinking to points when approaching the boundary of D. To these line segments we apply the above described smoothing procedure of our function f (cf. Lemma 5.1). This gives us $\mathcal{C}^p$-smoothing, but only in one (vertical) direction, say in the direction of the x_n-axis. More precisely, the partial derivative $\frac{\partial^p f}{\partial x_n^p}$ extends continuously by zero to the boundary of the capsule. It is important that we obtain this by substituting in f a homeomorphism which is of the form $\Phi(x', x_n) = (h(x'), \varphi(x', x_n))$, where $x' = (x_1, \dots, x_{n-1})$ and h is a homeomorphism of $\overline{D}$. Now we want to control the other partial derivatives, which may a priori be unbounded at the boundary. Consider first $\frac{\partial^p f}{\partial x_{n-1} \partial^{p-1} x_n}$. We want to control it from the level of the space R^{n-1}. For this purpose, we find a function $\omega : D \to R$ which detects in every vertical fiber a point at which the maximum of $|\frac{\partial^p f}{\partial x_{n-1} \partial^{p-1} x_n}|$ over the fiber is attained (up to a factor $1/2$). Such a detector can be found as smooth as we want and since it is contained in a capsule, it automatically extends continuously to the boundary. We apply the previous step to the function

$$\frac{\partial^{p-1} f}{\partial x_n^{p-1}}(x', \omega(x')) = \frac{\partial^{p-1} f}{\partial x_n^{p-1}}(x'', x_{n-1}, \omega(x'', x_{n-1})),$$

where $x'' = (x_1, \dots, x_{n-2})$ and x_{n-1} now plays the role of a "vertical variable". Hence, there exists a homeomorphism of the form

$$\Psi(x'', x_{n-1}) = (g(x''), \psi(x'', x_{n-1}))$$

such that

$$\frac{\partial}{\partial x_{n-1}}\left[\frac{\partial^{p-1} f}{\partial x_n^{p-1}}\big(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))\big)\right]$$

extends continuously by zero to the boundary of D (in fact, D has to be represented as a union of capsules beforehand). But

$$\begin{aligned}
&\frac{\partial}{\partial x_{n-1}}\left[\frac{\partial^{p-1} f}{\partial x_n^{p-1}}\big(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))\big)\right]\\
&= \frac{\partial^p f}{\partial x_{n-1}\partial x_n^{p-1}}\big(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))\big)\frac{\partial \psi}{\partial x_{n-1}}\\
&\quad + \frac{\partial^p f}{\partial x_n^p}\big(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))\big)\frac{\partial}{\partial x_{n-1}}\omega(g(x''), \psi(x'', x_{n-1})),
\end{aligned}$$

and the last line represents a function which is already known to have continuous extension by zero to the boundary. Hence the previous one has this property as well. It follows that

$$\begin{aligned}
&\left|\frac{\partial^p}{\partial x_{n-1}\partial x_n^{p-1}}\big[f\big(g(x''), \psi(x'', x_{n-1}), x_n\big)\big]\right|\\
&\qquad = \left|\frac{\partial^p f}{\partial x_{n-1}\partial x_n^{p-1}}\big(g(x''), \psi(x'', x_{n-1}), x_n\big)\frac{\partial \psi}{\partial x_{n-1}}\right|\\
&\qquad \le 2\left|\frac{\partial^p f}{\partial x_{n-1}\partial x_n^{p-1}}\big(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))\big)\frac{\partial \psi}{\partial x_{n-1}}\right|
\end{aligned}$$

has continuous extension by zero to the boundary. Repeating a similar reasoning we are able to achieve that $\frac{\partial^p f}{\partial^2 x_{n-1}\partial^{p-2} x_n}$ has continuous extension by zero to the boundary, and so on (cf. Propositions 8.2 and 8.3). We stop this procedure when we are able to apply the following *Basic $\mathcal{C}^p$-Extension Lemma*, which we here state in a slightly simplified form (compare Lemma 5.4):

Let P be a convex polyhedron in R^n of dimension n, let Σ be its face of dimension k such that $\Sigma \subset \{(x_1, \ldots, x_n) \in R^n : x_{k+1} = \cdots = x_n = 0\}$ and let $f : P \setminus \Sigma \to R$ be a $\mathcal{C}^p$-function such that all the partial derivatives

$$\frac{\partial^p f}{\partial x_{k+1}^{\alpha_{k+1}} \ldots \partial x_n^{\alpha_n}} \qquad (|\alpha| = \alpha_{k+1} + \cdots + \alpha_n = p)$$

have continuous extensions to Σ.

Then there exists a closed subset E of Σ of dimension $< k$ such that f extends to a $\mathcal{C}^p$-function defined on $P \setminus E$.

At the beginning of the proof we can assume without any loss of generality that in the Main Theorem, instead of any definable A, we have a big convex polyhedron P in R^n containing A, because of the definable version of the Tietze Theorem (see [1, Lemma 6.6], [21, Chapter 8, (3.10)]). The next initial reduction is that by the classical $\mathcal{C}^p$-triangulation theorem (see [10], [21, Chapter 8]) there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}, g)$ of P compatible with A such that $\mathcal{T}$ is a simplicial complex in R^n, $|\mathcal{T}| = P$ and $f \circ g|\mathring{\Delta}$ is of class $\mathcal{C}^p$ for any $\Delta \in \mathcal{T}$. We will be working under the inductive hypothesis that our theorems are true in dimensions $< n$ and use descending induction on the dimension k of a face of a polyhedron as described above.

We should stress that the above sketch of proof is oversimplified. For example, in general we have to use a number of detectors rather than one.

The advantage of our method of desingularization is that it works for an arbitrary o-minimal structure, including in particular the following two examples:

(1) the o-minimal structure of $\mathbb{R}$-subanalytic sets and mappings, i.e. the structure generated on the ordered field $\mathbb{R}$ of real numbers by real analytic bounded subsets of $\mathbb{R}^n$ ($n \in \mathbb{N}$) and all power functions $(0, \infty) \ni t \mapsto t^\alpha \in (0, \infty)$ with real irrational α (for a $\mathcal{C}^p$-rectilinearization and uniformization theorems in this structure see [18]),

(2) an o-minimal structure of Le Gal and Rolin [14] which does not admit $\mathcal{C}^\infty$ cell decompositions.

These examples explain why in our Main Theorem we deal with finite differentiability classes rather than $\mathcal{C}^\infty$. Besides, the $\mathcal{C}^\infty$-analogue of the theorem, if taken literally, is not true even in the semialgebraic case, as shown by the example of the function $f(t) = |t|$. Indeed, if there existed a semialgebraic $\mathcal{C}^\infty$-homeomorphism g of a neighborhood of 0 onto a neighborhood of 0 such that $g(0) = 0$ and $f \circ g$ were $\mathcal{C}^\infty$, then g would be analytic, so $g \sim t^k$ for some positive odd integer k; hence $h \circ g \sim |t|^k$, a contradiction.

The case $p = 1$ has already been proved in a slightly weaker form for semialgebraic category by Ohmoto and Shiota [16], who used strict $\mathcal{C}^1$-triangulations to develop integration on sets with singularities. Our Main Theorem for $p = 1$ in full extent has been proved by Czapla and Pawłucki [6].

If $R = \mathbb{R}$ is the field of real numbers, versions of our Main Theorem for locally definable sets and mappings are possible. They will be a subject of a separate article.

Throughout the paper we use the linear projections

$$\pi^n_m : R^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m) \in R^m$$

where $m \leq n$.

2. Capsules

We define two special notions which will play an essential role in the proof of the Main Theorem. These are *capsules* studied in the present section and *detectors* to which the next section is devoted.

A *capsule* in R^{n+1} is a subset K of R^{n+1} of the form

$$K = \{(x, t) \in D \times R : \alpha(x) \leq t \leq \beta(x)\},$$

where D is a subset of R^n such that $D = \overline{\operatorname{int} D}$, $\operatorname{int} D$ is bounded, connected and $\alpha, \beta : D \to R$ are continuous functions such that $\alpha < \beta$ on $\operatorname{int} D$ and $\alpha = \beta$ on ∂D.

Proposition 2.1. *For any subset E of R^{n+1} the following conditions are equivalent:*

(2.1.1) *E is a finite union of capsules in R^{n+1}.*

(2.1.2) *$E = \overline{\operatorname{int} E}$ is bounded and ∂E does not contain any nontrivial line segment parallel to the t-axis.*

(2.1.3) *E is a finite union of capsules in R^{n+1} whose interiors are pairwise disjoint.*

Proof. Obviously (2.1.1) implies (2.1.2). Assume now (2.1.2) satisfied. Let $\pi : R^{n+1} \ni (x,t) \mapsto x \in R^n$. Since $\operatorname{int} E$ is bounded and $\pi(E)$ is closed,

$$\pi(E) = \pi(\overline{\operatorname{int} E}) = \overline{\pi(\operatorname{int} E)} \subset \overline{\operatorname{int} \pi(E)} \subset \pi(E),$$

hence $\pi(E) = \overline{\operatorname{int} \pi(E)}$. Take a cell decomposition of R^{n+1} compatible with $\operatorname{int} E$ and with ∂E (cf. [21, Chapter 3, (2.11)]). This allows us to represent $\operatorname{int} E$ as a finite union of pairwise disjoint cells of the form

$$(\varphi, \psi) := \{(x,t) : x \in S,\ \varphi(x) < t < \psi(x)\},$$

where $S \subset \pi(\operatorname{int} E)$, $\varphi, \psi : S \to R$ are continuous, $\varphi < \psi$ on S and the graphs[1] of φ and ψ are contained in ∂E. Applying classical triangulation to $\pi(\operatorname{int} E)$ and all S (see [21, Chapter 8, (1.7)]) we can additionally assume that $S = \pi(\varphi, \psi)$ satisfies *Łojasiewicz's (s)-condition* (see [15, Section 25]): each point $a \in \overline{S} \setminus S$ admits a neighborhood basis $\mathcal{U}$ in R^n such that the trace $U \cap S$ of each $U \in \mathcal{U}$ on S is connected. Then the set of all limit values of φ at each point $a \in \overline{S} \setminus S$ can be identified with

$$\overline{\varphi} \cap (\{a\} \times R) = \{a\} \times \bigcap \{\overline{\varphi(U \cap S)} : U \in \mathcal{U}\},$$

which is a nonempty, connected subset of the vertical line $\{a\} \times R$ and of ∂E at the same time, hence a singleton. Consequently, both φ and ψ have continuous extensions $\overline{\varphi}, \overline{\psi} : \overline{S} \to R$ to $\overline{S}$ and next, by the Tietze Theorem, to the whole $\pi(E)$. Using all these extensions and min and max functions we can find a sequence of continuous functions

$$\alpha_1 \le \cdots \le \alpha_p : \pi(E) \to R$$

such that

(2.1.4) for each $x \in \pi(\operatorname{int} E)$ the fiber $(\operatorname{int} E)_x$ is a union of some intervals $(\alpha_i(x), \alpha_j(x))$, where $i < j$,

(2.1.5) $\pi^{-1}(\pi(\operatorname{int} E)) \cap \partial E \subset \bigcup_i \alpha_i$.

Refining the sequence $\alpha_1, \ldots, \alpha_p$ by some extra functions we can assume that all the sets

$$(\alpha_i, \alpha_{i+1}) := \{(x,t) : x \in \pi(E),\ \alpha_i(x) < t < \alpha_{i+1}(x)\}$$

[1] We identify mappings with their graphs, denoting both by the same letter.

are connected and nonempty. It follows from (2.1.5) that if $(\alpha_i, \alpha_{i+1}) \cap \operatorname{int} E \neq \emptyset$, then $(\alpha_i, \alpha_{i+1}) \subset \operatorname{int} E$. Let $\{i_1 < \cdots < i_s\} = \{i : (\alpha_i, \alpha_{i+1}) \subset \operatorname{int} E\}$. Then by (2.1.4),

$$(\alpha_{i_1}, \alpha_{i_1+1}) \cup \cdots \cup (\alpha_{i_s}, \alpha_{i_s+1})$$

is dense in int E, hence in E. Let $P_\nu := \pi(\alpha_{i_\nu}, \alpha_{i_\nu+1})$. Now if $x \in \overline{P_\nu} \setminus P_\nu$ and $x \in \pi(\operatorname{int} E)$, then of course $\alpha_{i_\nu}(x) = \alpha_{i_\nu+1}(x)$ and if $x \in \overline{P_\nu} \setminus P_\nu$ and $x \notin \pi(\operatorname{int} E)$, then $\{x\} \times [\alpha_{i_\nu}(x), \alpha_{i_\nu+1}(x)] \subset \partial E$, hence again $\alpha_{i_\nu}(x) = \alpha_{i_\nu+1}(x)$. However, $\overline{(\alpha_{i_\nu}, \alpha_{i_\nu+1})}$ may not be a capsule yet because the condition $\operatorname{int} \overline{P_\nu} = P_\nu$ may not be satisfied. To solve this problem we prove the following lemma.

Lemma. *Let P be a bounded open subset of R^n and let $\alpha, \beta : \overline{P} \to R$ be two continuous functions such that $\alpha < \beta$ on P and $\alpha = \beta$ on ∂P. Then $\overline{(\alpha, \beta)}$ can be represented as a finite union of capsules with pairwise disjoint interiors.*

Proof of Lemma. Without any loss of generality we can assume that $\alpha \equiv 0$. Next, using classical triangulation we reduce the problem to PL-geometry. Then the subset $A := (\operatorname{int} \overline{P}) \setminus P$ is contained in a finite number $H_1, \ldots, H_q$ of affine hyperplanes, with q minimal possible. We argue by induction on q. By an affine change of coordinates in R^n, we can assume that $H_q = \{(x_1, \ldots, x_n) : x_n = 0\}$. Then the function $\gamma(x) := M x_n$, with $|M|$ large enough, cuts the cell $\overline{(0, \beta)}$ into two $\overline{(0, \max(0, \min(\gamma, \beta)))}$ and $\overline{(\max(0, \min(\gamma, \beta)), \beta)}$, for each of which $q' < q$. ∎

This ends the proof of Proposition 2.1.

Remark 2.2. If E fulfills the conditions of Proposition 2.1 and $\lambda_j : \pi(E) \to R$ ($j \in \{1, \ldots, r\}$) is a given finite family of continuous functions, then there exists a finite family of continuous functions $\alpha_1 \leq \cdots \leq \alpha_s : R^n \to R$ such that E is a union of some capsules of the form $\overline{(\alpha_i, \alpha_{i+1})}$ which are compatible with every λ_j in the sense that either $\lambda_j(x) \leq t$ for all $(x, t) \in \overline{(\alpha_i, \alpha_{i+1})}$, or $\lambda_j(x) \geq t$ for all $(x, t) \in \overline{(\alpha_i, \alpha_{i+1})}$.

Remark 2.3. If $K_0, K_1, \ldots, K_p$ are capsules in R^{n+1} and $K_\nu \subset K_0$ when $1 \leq \nu \leq p$, then there exists a finite family of continuous functions $\alpha_1 \leq \cdots \leq \alpha_s : R^n \to R$ such that $\overline{(\alpha_i, \alpha_{i+1})}$ ($i \in \{0, \ldots, s-1\}$) is a family of capsules which is a refinement of $K_0, \ldots, K_p$.

Corollary 2.4. *For any finite family $\mathcal{K}$ of capsules in R^{n+1} there exists a finite family $\mathcal{L}$ of capsules in R^{n+1} which is a refinement of $\mathcal{K}$ and the interiors of capsules from $\mathcal{L}$ are pairwise disjoint.*

Proposition 2.5. *Let K be any capsule in R^{n+1} and let $\mathcal{V}$ be a finite family of open subsets of* int K *covering the whole* int K. *Then there exists a finite family $\mathcal{L}$ of capsules in R^{n+1} whose interiors are pairwise disjoint, $\bigcup \mathcal{L} = K$ and for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ such that* $\operatorname{int} L \subset V$.

Proof. Put $K = \{(x, t) \in D \times R : \alpha(x) \leq t \leq \beta(x)\}$. There are two parts of the proof.

Part I. We first prove by induction on k that if A is any subset of int D of dimension k, then there exists a finite family $\mathcal{L}$ of capsules in R^{n+1} such that for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ containing $\operatorname{int} L$ and for each $a \in A$ there exist $L \in \mathcal{L}$ and $\varepsilon > 0$ such that $\{a\} \times (\alpha(a), \alpha(a) + \varepsilon) \subset \operatorname{int} L$.

Applying a triangulation of D compatible with A, we can assume that A is an open subset of $R^k = \{(x_1, \ldots, x_n) : x_{k+1} = \cdots = x_n = 0\}$. Partitioning A, using the induction hypothesis and cell decomposition, we can assume that A is connected, and there exists one $V \in \mathcal{V}$ and a function

$$\eta : A \to (0, \infty)$$

such that $\{a\} \times (\alpha(a), \alpha(a) + \eta(a)] \subset V$ for each $a \in A$. Replacing η by $\tilde{\eta}(a) := \min\{\eta(a), d(a, \overline{A} \setminus A)\}$, we can assume that $\eta(a) \to 0$ when $d(a, \overline{A} \setminus A) \to 0$. For each $t \in [\alpha(a), \alpha(a) + \eta(a)]$ put $\rho(a, t) := \frac{1}{2} d((a, t), K \setminus V)$. Since for each $a \in A$, $\rho(a, \alpha(a)) = 0$ and $\rho(a, t) > 0$ when $t > \alpha(a)$, we can modify η in such a way that

$$(\alpha(a), \alpha(a) + \eta(a)] \ni t \mapsto \rho(a, t) \in (0, \infty)$$

is strictly increasing. Again by partitioning A and using the induction hypothesis we can assume that η is continuous, and replacing η by $\tilde{\eta}(a) := \min\{\eta(a), d(a, \overline{A} \setminus A)\}$, we can assume that $\eta(a) \to 0$ when $d(a, \overline{A} \setminus A) \to 0$. It follows from the definition of ρ that for each $a \in A$ and $t \in (\alpha(a), \alpha(a) + \eta(a)]$,

$$\{(x_1, \ldots, x_n, t) : a = (x_1, \ldots, x_k),\ (x_{k+1}^2 + \cdots + x_n^2)^{1/2} \le \rho(a, t)\} \subset V.$$

Now we define the desired capsule. Put

$$E := \{(x_1, \ldots, x_n) : a = (x_1, \ldots, x_k) \in \overline{A},\ (x_{k+1}^2 + \cdots + x_n^2)^{1/2} \le \rho(a, \alpha(a) + \eta(a))\}$$

and

$$L := \big\{(x_1, \ldots, x_n, t) : (x_1, \ldots, x_n) \in E,$$
$$\rho^{-1}\big(x_1, \ldots, x_k, (x_{k+1}^2 + \cdots + x_n^2)^{1/2}\big) \le t \le \alpha(x_1, \ldots, x_k) + \eta(x_1, \ldots, x_k)\big\},$$

where ρ^{-1} denotes the inverse of ρ with respect to the last variable.

Part II. According to Part I, there exists a finite family $\mathcal{L}$ of capsules in R^{n+1} such that for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ containing $\operatorname{int} L$ and for each $a \in D$ there exist $L \in \mathcal{L}$ and $\varepsilon > 0$ such that $\{a\} \times (\alpha(a), \alpha(a) + \varepsilon) \subset \operatorname{int} L$ and there exist $M \in \mathcal{L}$ and $\theta > 0$ such that $\{a\} \times (\beta(a), \beta(a) - \theta) \subset \operatorname{int} M$.

By Corollary 2.4 there exists a finite family $\mathcal{L}'$ of capsules in R^{n+1} which is a refinement of $\mathcal{L} \cup \{K\}$ and consists of capsules with pairwise disjoint interiors. It follows that if $L' \in \mathcal{L}'$ and L' is not contained in any of the capsules from $\mathcal{L}$, then L' is of the form

$$L' = \{(x, t) : x \in Q,\ \gamma(x) \le t \le \delta(x)\},$$

where $\mathcal{V}$ is an open covering of $L'|\operatorname{int} Q = \{(x, t) : x \in \operatorname{int} Q,\ \gamma(x) \le t \le \delta(x)\}$. Thus to finish the proof it suffices to prove the following.

If $K = \{(x, t) \in D \times R : \alpha(x) \le t \le \beta(x)\}$ is a capsule in R^{n+1}, $K^* := K \cap (\partial D \times R)$, $\mathcal{V}$ is a finite family of open subsets of R^{n+1} such that $K \setminus K^* \subset \bigcup \mathcal{V}$ and A is a subset *of* int D *of dimension* k*, then there exists a finite family* $\mathcal{L}$ *of capsules in* R^{n+1} *contained in* K *such that* $\bigcup \mathcal{L} \setminus K^*$ *is a neighborhood of* $K|A$ *in* $K \setminus K^*$ *and for each* $L \in \mathcal{L}$ *there exists* $V \in \mathcal{V}$ *such that* $L \setminus K^* \subset V$.

We proceed again by induction on k. Take a cell decomposition $\mathcal{C}$ of $\bigcup \mathcal{V}$ compatible with each $V \in \mathcal{V}$ and with $K|A$. Let

$$\{B_1, \dots, B_s\} = \{\pi(C) : C \in \mathcal{C},\ C \subset K|A,\ \dim \pi(C) = k\}.$$

Now we apply the induction hypothesis to

$$E := [A \setminus (B_1 \cup \cdots \cup B_s)] \cup (\overline{B}_1 \cap \operatorname{int} D \setminus B_1) \cup \cdots \cup (\overline{B}_s \cap \operatorname{int} D \setminus B_s).$$

There exists a finite family $\mathcal{L}$ of capsules in R^{n+1} contained in K such that $\bigcup \mathcal{L} \setminus K^*$ is a neighborhood of $K|E$ in $K \setminus K^*$ and for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ such that $L \setminus K^* \subset V$. Fix one $B_\mu = B$. Then

$$K|B = [\gamma_0, \gamma_1] \cup \cdots \cup [\gamma_{m-1}, \gamma_m],$$

where $\gamma_\nu : B \to R$ ($\nu \in \{0, \dots, m\}$) are continuous, $\gamma_0 < \cdots < \gamma_m$, $\gamma_0 = \alpha|B$, $\gamma_m = \beta|B$ and each $[\gamma_\nu, \gamma_{\nu+1}]$ is contained in some $V \in \mathcal{V}$. There is an open subset T_0 of B such that $\overline{T_0} \cap \operatorname{int} D \subset B$ and $\bigcup \mathcal{L} \setminus K^*$ is a neighborhood of $K|(B \setminus T_0)$. Take also open subsets T_1, T_2 of B such that $\overline{T_i} \cap \operatorname{int} D \subset T_j \subset \overline{T_j} \cap \operatorname{int} D \subset B$ if $0 \le i < j \le 2$. By the Tietze Theorem for each $\nu \in \{1, \dots, m\}$ there exists a continuous function

$$\tilde{\gamma}_\nu : \overline{T_2} \to R$$

such that $\tilde{\gamma}_\nu|\overline{T_1} = \gamma_\nu|\overline{T_1}$, $\tilde{\gamma}_\nu|\partial T_2 = \gamma_{\nu-1}|\partial T_2$ and $\gamma_{\nu-1} \le \tilde{\gamma}_\nu \le \gamma_\nu$ on $\overline{T_2}$. Then

$$\bigcup_{\nu=1}^{m} [\gamma_{\nu-1}|\overline{T_2}, \tilde{\gamma}_\nu] \setminus K^*$$

is a neighborhood of $K|\overline{T_0} \cap \operatorname{int} D$ in $K \setminus K^*$. We build a similar neighborhood over every B_μ. Applying Proposition 2.1 we finish the proof. ■

Remark 2.6. The reader will easily check that Propositions 2.1 and 2.5 as well as Remarks 2.2 and 2.3 hold true in the PL-structure.

In Section 8 we will need the following lemma.

Lemma 2.7. *Every* PL*-capsule in* R^{n+1} *is a finite union of convex* PL*-capsules whose interiors are pairwise disjoint.*

Proof. The boundary ∂S of any PL-capsule S is contained in a finite number of graphs of affine functions,

$$\partial S \subset \varphi_1 \cup \cdots \cup \varphi_s,$$

where s is the smallest possible. We argue by induction on the number q of φ_ν such that S is not contained in just one closed half-space cut by φ_ν. If $q = 0$, then clearly S is convex.

Otherwise there is ν such that

$$T_1 := \operatorname{cl}\{(x, y) \in \operatorname{int} S : y < \varphi_\nu(x)\} \quad \text{and} \quad T_2 := \operatorname{cl}\{(x, y) \in \operatorname{int} S : y > \varphi_\nu(x)\}$$

are finite unions of PL-capsules, for which the number q is smaller. The lemma follows. ∎

3. Detectors

In this section we will need $\mathcal{C}^p$-partitions of unity. Although it is well-known that $\mathcal{C}^p$-partitions of unity exist in any o-minimal structure, for the reader's convenience and to make the paper self-contained, we give a short proof in the first two lemmas.

Lemma 3.1. *Let Ω be an open subset of R^n and let A and B be two closed, disjoint subsets of Ω. Then there exists a $\mathcal{C}^p$-function $\varphi : \Omega \to [0, 1]$ such that $\varphi = 1$ on A and $\varphi = 0$ on B.*

Proof. By the Whitney extension theorem in the version from [13], there exists a $\mathcal{C}^p$-function $\psi : \Omega \to R$ such that $\psi = 1$ on A and $\psi = 0$ on B. Now it suffices to put $\varphi := \lambda \circ \psi$, where $\lambda : R \to [0, 1]$ is a $\mathcal{C}^p$-function such that $\lambda(0) = 0$ and $\lambda(1) = 1$. ∎

Lemma 3.2. *Let Ω be an open subset of R^n and let $A_1, \ldots, A_m$ be a finite family of closed and pairwise disjoint subsets of Ω. Then there exist $\mathcal{C}^p$-functions $\varphi_j : \Omega \to [0, 1]$ ($j \in \{1, \ldots, m\}$) such that*

$$\sum_{j=1}^{m} \varphi_j(x) = 1 \quad \textit{for all } x \in \Omega,$$

and for each $j \in \{1, \ldots, m\}$, $\varphi_j = 1$ on A_j.

Proof. Induction on m. Let $m > 1$. By the induction hypothesis there are $\psi_1, \ldots, \psi_{m-1} : \Omega \to [0, 1]$ of class $\mathcal{C}^p$ such that

$$\sum_{i=1}^{m-1} \psi_i(x) = 1 \quad \text{for all } x \in \Omega,$$

and $\psi_i = 1$ on A_i. By Lemma 3.1 there exists a $\mathcal{C}^p$-function $\sigma_1 : \Omega \to [0, 1]$ such that $\sigma_1 = 1$ on A_m and $\sigma_1 = 0$ on $A_1 \cup \cdots \cup A_{m-1}$. There exists an open neighborhood U of A_m in Ω such that $\sigma_1 > 0$ on U and $U \subset \Omega \setminus (A_1 \cup \cdots \cup A_{m-1})$. By Lemma 3.1 there exists a $\mathcal{C}^p$-function $\sigma_2 : \Omega \to [0, 1]$ such that $\sigma_2 = 1$ on $\Omega \setminus U$ and $\sigma_2 = 0$ on A_m. Then the $\mathcal{C}^p$-function

$$\sigma_1 + \sigma_2 : \Omega \to (0, 2]$$

is positive on Ω, so we can build the following $\mathcal{C}^p$-functions on Ω:

$$\rho_1(x) := \frac{\sigma_1(x)}{\sigma_1(x) + \sigma_2(x)} \quad \text{and} \quad \rho_2(x) := \frac{\sigma_2(x)}{\sigma_1(x) + \sigma_2(x)}.$$

Of course, $\rho_1(x) + \rho_2(x) \equiv 1$, $\rho_1 = 0$ on $A_1 \cup \cdots \cup A_{m-1}$, while $\rho_2 = 0$ on A_m; hence $\rho_1 = 1$ on A_m and $\rho_2 = 1$ on $A_1 \cup \cdots \cup A_{m-1}$. Finally, we put $\varphi_1 := \psi_1\rho_2, \ldots, \varphi_{m-1} := \psi_{m-1}\rho_2$ and $\varphi_m := \rho_1$. ■

Proposition 3.3. *Let Ω be an open subset of R^n, E a closed subset of Ω of dimension k, and C a convex, closed bounded subset of R^m such that* $\operatorname{int} C \neq \emptyset$. *Let $f : E \times C \to [0, \infty)$ be a continuous function and define*

$$g(x) := \sup_{y \in C} f(x, y) \quad \textit{for } x \in E.$$

Assume that $g(x) > 0$ for all $x \in E$. Let $p \in \mathbb{N}$.

Then there exists a family $\omega_j : \Omega \to \operatorname{int} C$ $(j \in \{0, \ldots, k\})$ of $\mathcal{C}^p$-mappings such that

$$\tfrac{1}{2} g(x) < \sup_j f(x, \omega_j(x)) \quad \textit{for all } x \in E.$$

The mappings ω_j will be called detectors of class $\mathcal{C}^p$ for f over E.

Proof. Induction on k. If $k = 0$ it suffices to know that there exists a $\mathcal{C}^p$-mapping $\omega : \Omega \to C$ which has prescribed values at a finite number of points, which is an immediate consequence of existence of definable $\mathcal{C}^p$-partitions of unity (Lemma 3.2).

Suppose now that $k > 0$. By definable choice there exists a mapping $\omega_k : E \to \operatorname{int} C$ such that

$$\tfrac{1}{2} g(x) < f(x, \omega_k(x)) \quad \text{for all } x \in E. \tag{3.3.1}$$

There exists a closed subset E_1 of E of dimension $l < k$ such that $E \setminus E_1$ is a $\mathcal{C}^p$-submanifold of R^n of dimension k and $\omega_k | E \setminus E_1$ is a $\mathcal{C}^p$-mapping. Moreover, by [13] we can assume that $E \setminus E_1$ can be represented as a finite union

$$E \setminus E_1 = \bigcup_\nu \Gamma_\nu \tag{3.3.2}$$

of pairwise disjoint k-dimensional $\mathcal{C}^p$-submanifolds each of which, in some linear coordinate system, is the graph

$$\Gamma_\nu = \big\{\big(x_1, \ldots, x_k, \gamma^\nu_{k+1}(x_1, \ldots, x_k), \ldots, \gamma^\nu_n(x_1, \ldots, x_k)\big) : (x_1, \ldots, x_k) \in D_\nu\big\},$$

of a $\mathcal{C}^p$-mapping $\gamma^\nu = (\gamma^\nu_{k+1}, \ldots, \gamma^\nu_n) : D_\nu \to R^{n-k}$ defined on some open subset $D_\nu \subset R^k$.

Via the natural projection

$$D_\nu \times R^{n-k} \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k, \gamma^\nu(x_1, \ldots, x_k)) \in \Gamma_\nu,$$

$\omega_k | \Gamma_\nu$ can be extended to a $\mathcal{C}^p$-mapping on a neighborhood of Γ_ν; hence $\omega_k | E \setminus E_1$ can be extended to a $\mathcal{C}^p$-mapping defined on a neighborhood of $E \setminus E_1$. Consequently, $\omega_k | E \setminus E_1$ extends to a $\mathcal{C}^p$-Whitney field defined on $E \setminus E_1$. By the induction hypothesis, there exist $\mathcal{C}^p$-mappings $\omega_j : \Omega \to \operatorname{int} C$ $(j \in \{0, \dots, k-1\})$ such that

$$\tfrac{1}{2} g(x) < \sup_j f(x, \omega_j(x)) \quad \text{for all } x \in E_1 \tag{3.3.3}$$

There exists an open neighborhood W of E_1 in Ω such that (3.3.3) holds true for all $x \in W \cap E$. Then $E \setminus W$ is a closed subset of Ω contained in $E \setminus E_1$. By the Whitney Extension Theorem, there exists a $\mathcal{C}^p$-mapping $F : \Omega \to R^m$ which extends $\omega_k|(E \setminus W)$. Then $U := F^{-1}(\operatorname{int} C)$ is an open neighborhood of $E \setminus W$ in Ω. By Lemma 3.2, there exist $\mathcal{C}^p$-functions $\varphi_1, \varphi_2 : \Omega \to [0,1]$ such that $\varphi_1 + \varphi_2 \equiv 1$, $\varphi_1 = 1$ on $E \setminus W$ and $\varphi_2 = 1$ on $\Omega \setminus U$. Choose any $c_0 \in \operatorname{int} C$ and put $\tilde{\omega}_k := \varphi_1 F + \varphi_2 c_0$. Then $\omega_0, \dots, \omega_{k-1}, \tilde{\omega}_k$ is the desired sequence for E. ∎

Example 3.2. The following example shows that the assumption $g(x) > 0$ for all $x \in E$ in Proposition 3.3 cannot be omitted. Put

$$E := \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 \le 1/4\} \quad \text{and} \quad C = [0,1].$$

Consider $f : E \times C \to [0, \infty)$ defined in the following way:

$$f(x_1, x_2, y) = 0 \quad \text{when } x_1^2 + x_2^2 > 0 \text{ and } y \le \frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)};$$

$$f(x_1, x_2, y) = y - \frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)} \quad \text{when } x_1^2 + x_2^2 > 0 \text{ and}$$
$$\frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)} \le y \le \frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)} + x_1^2 + x_2^2;$$

$$f(x_1, x_2, y) = 2(x_1^2 + x_2^2) - \left(y - \frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)}\right) \quad \text{when } x_1^2 + x_2^2 > 0 \text{ and}$$
$$\frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)} + x_1^2 + x_2^2 \le y \le \frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)} + 2(x_1^2 + x_2^2);$$

$$f(x_1, x_2, y) = 0 \quad \text{when } x_1^2 + x_2^2 > 0 \text{ and } \frac{|x_1|\,|x_2|}{2(x_1^2 + x_2^2)} + 2(x_1^2 + x_2^2) \le y \le 1;$$

$$f(x_1, x_2, y) = 0 \quad \text{when } x_1^2 + x_2^2 = 0.$$

Clearly, $g(x_1, x_2) = x_1^2 + x_2^2$ and f does not admit continuous detectors over E.

4. Yomdin–Gromov trick and a smoothing homeomorphism ω

The aim of this section is to present a method of smoothing functions of one variable (Corollary 4.5), mimicking Yomdin and Gromov (cf. [9, Section 4.1], [22, 23]), which appeared useful to get smooth parametrizations of subsets definable in o-minimal structures (see [12]). It is crucial in the proof of our basic Lemma 5.1 in the next section. We will come to Corollary 4.5, starting from a more elementary lemma.

Lemma 4.1. *Let $\lambda : (a, b) \to R$ be a definable $\mathcal{C}^{p+1}$-function, where $p \in \mathbb{N}$, $p \geq 1$, defined on an open interval $(a, b) \subset R$ such that, for each $\nu \in \{2, \dots, p+1\}$, $\lambda^{(\nu)} \geq 0$ on (a, b) or $\lambda^{(\nu)} \leq 0$ on (a, b). Then, for any closed interval $[t - r, t + r] \subset (a, b)$, where $r \in R$ and $r > 0$,*

$$|\lambda^{(p)}(t)| \leq 2^{\binom{p+2}{2}-2} \sup_{[t-r,t+r]} |\lambda| \frac{1}{r^p}.$$

Proof. The same as the proof of [12, Lemma 2.1]. ∎

Applying Lemma 4.1 to λ' in place of λ and $\mu - 1$ in place of p, we have

Corollary 4.2. *Under the assumptions of Lemma 4.1, where $p \geq 2$,*

$$|\lambda^{(\mu)}(t)| \leq C_p \sup_{(a,b)} |\lambda'| \frac{1}{|t - a|^{\mu-1}}$$

for all $t \in (a, \frac{a+b}{2}]$ and $\mu \in \{2, \dots, p\}$, where $C_p := 2^{\binom{p+1}{2}-2}$. In particular, if λ' is bounded, i.e. $|\lambda'| \leq M$, where $M \in R$ and $M > 0$, then

$$(4.2.1) \qquad |\lambda^{(\mu)}(t)| \leq C_p M \frac{1}{|t - a|^{\mu-1}} \quad \textit{for all } t \in \left(a, \frac{a+b}{2}\right], \ \mu \in \{2, \dots, p\}.$$

Lemma 4.3. *Let $\lambda : (a, c] \to R$ be a definable $\mathcal{C}^p$-function, where $a, c \in R$, $a < c$, such that*

$$(4.3.1) \qquad |\lambda^{(\mu)}(t)| \leq L \frac{1}{|t - a|^{\mu-1}} \quad \textit{for all } t \in (a, c], \ \mu \in \{1, \dots, p\},$$

where $L \in R$ is a positive constant. Fix $m \in \mathbb{N}$, $m \geq p + 1$. Fix any $\alpha \in R$. Put $\varphi(\tau) := \lambda(a + (\tau - \alpha)^m)$ for $\tau \in (\alpha, \beta]$, where $\beta = \alpha + \sqrt[m]{c - a}$.

Then there exists a positive constant M depending only on L and m such that $|\varphi^{(\mu)}(\tau)| \leq M|\tau - \alpha|^{m-\mu}$ for all $\tau \in (\alpha, \beta]$ and $\mu \in \{1, \dots, p\}$. Consequently, φ has a unique extension to a $\mathcal{C}^p$-function $\varphi : [\alpha, \beta] \to R$, p-flat at α.

Proof. Without any loss of generality we can assume that $a = 0 = \alpha$. Then $\varphi(\tau) = \lambda(\tau^m)$. For each $\mu \in \{1, \dots, p\}$,

$$\begin{aligned}\varphi^{(\mu)}(\tau) = a_{1\mu}\tau^{m-\mu}\lambda'(\tau^m) + a_{2\mu}\tau^{2m-\mu}\lambda''(\tau^m) + a_{3\mu}\tau^{3m-\mu}\lambda^{(3)}(\tau^m) \\ + \dots + a_{\mu\mu}\tau^{\mu m-\mu}\lambda^{(\mu)}(\tau^m),\end{aligned}$$

where $a_{i\mu}$ are positive integers defined inductively by

$$a_{1\mu} = \frac{m!}{(m - \mu)!}, \quad a_{i\mu} = m a_{i-1,\mu-1} + (im - \mu + 1)a_{i,\mu-1}, \quad a_{\mu\mu} = m^\mu.$$

By (4.3.1), it follows that

$$\begin{aligned}|\varphi^{(\mu)}(\tau)| &\leq a_{1\mu}\tau^{m-\mu}L + a_{2\mu}\tau^{2m-\mu}\frac{L}{\tau^m} + a_{3\mu}\tau^{3m-\mu}\frac{L}{\tau^{2m}} + \dots + a_{\mu\mu}\tau^{\mu m-\mu}\frac{L}{\tau^{(\mu-1)m}} \\ &= L(a_{1\mu} + \dots + a_{\mu\mu})\tau^{m-\mu}.\end{aligned}$$

∎

It will be convenient to have the p-flatness of a parametrization of the interval $[a, c]$ at the right end as well. That is why we use the following increasing parametrization of $[\alpha, \beta]$, p-flat at the right end:

$$\tau := \alpha + \sqrt[m]{c-a} - (\delta - s)^m,$$

where $\gamma \in R$ is arbitrary, $s \in [\gamma, \delta]$ and $\delta = \gamma + \sqrt[2m]{c-a}$. This leads us to the following.

Corollary 4.4. *Let $\lambda : (a, b) \to R$ be a $\mathcal{C}^{p+1}$-function, where $p \in \mathbb{N}$, $p \geq 1$, defined on $(a, b) \subset R$ such that λ' is bounded and, for each $\nu \in \{2, \ldots, p+1\}$, $\lambda^{(\nu)} \geq 0$ on (a, b) or $\lambda^{(\nu)} \leq 0$ on (a, b). Let $m \in \mathbb{N}$, $m \geq p+1$. Fix any $\gamma_0 \in R$ and let $\gamma_1 := \gamma_0 + \sqrt[2m]{(b-a)/2}$, $\gamma_2 := \gamma_1 + \sqrt[2m]{(b-a)/2} = \gamma_0 + 2\sqrt[2m]{(b-a)/2}$. Put*

$$\omega(a, b; s) := \begin{cases} a + \left[\sqrt[m]{(b-a)/2} - (\gamma_1 - s)^m\right]^m & \text{if } s \in [\gamma_0, \gamma_1], \\ b - \left[\sqrt[m]{(b-a)/2} - (s - \gamma_1)^m\right]^m & \text{if } s \in [\gamma_1, \gamma_2]. \end{cases}$$

Then $\omega : [\gamma_0, \gamma_2] \to [a, b]$ is an increasing homeomorphism such that $\omega(\gamma_0) = a$, $\omega(\gamma_1) = \frac{a+b}{2}$, $\omega(\gamma_2) = b$ and $\lambda \circ \omega$ extends uniquely to a $\mathcal{C}^p$-function $\lambda \circ \omega : [\gamma_0, \gamma_2] \to R$ p-flat at γ_0, γ_1 and γ_2.

Corollary 4.5. *Let $y_0 \leq y_1 \leq \cdots \leq y_r$ be (at most) $r+1$ points in R. Let $\lambda : [y_0, y_r] \to R$ be a continuous function such that, for each $i \in \{0, \ldots, r-1\}$, if $y_i < y_{i+1}$, then $\lambda|(y_i, y_{i+1})$ satisfies the assumptions of Corollary 4.4. Let $m \in \mathbb{N}$, $m \geq p+1$. Define a sequence*

$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{2r}$$

of points in R inductively: $\gamma_0 \in R$ arbitrary, $\gamma_{2i+1} := \gamma_{2i} + \sqrt[2m]{(y_{i+1} - y_i)/2}$, $\gamma_{2i+2} := \gamma_{2i+1} + \sqrt[2m]{(y_{i+1} - y_i)/2}$ ($i \in \{0, \ldots, r-1\}$). Put

$$\omega(y_0, \ldots, y_r; s) := \begin{cases} y_i + \left[\sqrt[m]{(y_{i+1} - y_i)/2} - (\gamma_{2i+1} - s)^m\right]^m & \text{if } s \in [\gamma_{2i}, \gamma_{2i+1}], \\ y_{i+1} - \left[\sqrt[m]{(y_{i+1} - y_i)/2} - (s - \gamma_{2i+1})^m\right]^m & \text{if } s \in [\gamma_{2i+1}, \gamma_{2i+2}], \end{cases}$$

for $i \in \{0, \ldots, r-1\}$ and

$$\omega(y_0, \ldots, y_r; s) := \begin{cases} y_0 - (\gamma_0 - s)^m & \text{if } s \in (-\infty, \gamma_0], \\ y_r + (s - \gamma_{2r})^m & \text{if } s \in [\gamma_{2r}, \infty). \end{cases}$$

Then $\omega : R \to R$ is an increasing homeomorphism of class $\mathcal{C}^p$ such that $\omega(\gamma_{2i}) = y_i$ and $\omega(\gamma_{2i+1}) = \frac{y_i + y_{i+1}}{2}$ ($i \in \{0, \ldots, r-1\}$), and $\lambda \circ \omega : [\gamma_0, \gamma_{2r}] \to R$ is of class $\mathcal{C}^p$, p-flat at $\gamma_0, \ldots, \gamma_{2r}$.

5. Basic lemmata

In this section we prove three lemmata of technical importance, each of which is of a different nature.

5.1. Smoothing with a parameter

Lemma 5.1 below is a first version with parameter of the one-dimensional smoothing described in the Introduction, which will be enhanced later in Proposition 8.1.

Lemma 5.1. *Let D be a bounded subset of R^{n-1} such that $D = \overline{\operatorname{int} D}$, and let m, p be positive integers such that $m \geq p+1$. Let*

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r : D \to R$$

be a finite sequence of continuous functions such that $\mathcal{K} := \{\overline{(\alpha_i, \alpha_{i+1})} : i \in \{0, \ldots, r-1\}\}$ is a family of capsules in R^n. Let $\mathcal{K}_1 \subset \mathcal{K}$ and put $A := |\mathcal{K}|$ and $A_1 := |\mathcal{K}_1|$. Let $f = (f_1, \ldots, f_d) : A_1 \to R^d$ be a continuous mapping such that for each $K \in \mathcal{K}_1$ there exist continuous partial derivatives

$$\frac{\partial^\sigma (f|\mathring{K})}{\partial x_n^\sigma} \quad \text{for } \sigma \in \{1, \ldots, p+1\}.$$

Then there exists a finite sequence of continuous functions

$$\delta_0 \leq \delta_1 \leq \cdots \leq \delta_k : D \to R$$

and a homeomorphism $\Phi : [\delta_0, \delta_k] \to [\alpha_0, \alpha_r]$ such that:

(5.1.1) *Φ is of the form $\Phi(x', \xi_n) = (x', \varphi(x', \xi_n))$, where $x' = (x_1, \ldots, x_{n-1})$.*

(5.1.2) *For each $j \in \{0, \ldots, k-1\}$ the derivatives*

$$\frac{\partial^\sigma \varphi}{\partial \xi_n^\sigma} \quad \text{for } \sigma \in \{1, \ldots, p+1\}$$

exist and are continuous in (δ_j, δ_{j+1}) and have continuous extensions by zero to $\overline{(\delta_j, \delta_{j+1})}$; moreover,

$$\frac{\partial \varphi}{\partial \xi_n} > 0 \quad \text{on } (\delta_j, \delta_{j+1}).$$

(5.1.3) *The sequence $\theta_j(x') := \varphi(x', \delta_j(x'))$, where $x' \in D$ and $j \in \{0, \ldots, k\}$, is a refinement of $\alpha_0, \ldots, \alpha_r$; in particular, $\alpha_0 = \theta_0$ and $\alpha_r = \theta_k$.*

(5.1.4) *$\mathcal{L} := \{\overline{(\delta_j, \delta_{j+1})} : j \in \{0, \ldots, k-1\}\}$ is a family of capsules in R^n such that $\{\Phi(L) : L \in \mathcal{L}\}$ is a family of capsules which is a refinement of $\mathcal{K}$.*

(5.1.5) *Put $\mathcal{L}_1 := \{L \in \mathcal{L} : \Phi(L) \subset K \text{ for some } K \in \mathcal{K}_1\}$. For each $L \in \mathcal{L}_1$, there exist continuous partial derivatives*

$$\frac{\partial^\sigma (f \circ \Phi|\mathring{L})}{\partial \xi_n^\sigma} \quad (\sigma \in \{1, \ldots, p+1\})$$

and those for $\sigma \in \{1, \ldots, p\}$ extend continuously by zero to L.

(5.1.6) *On each capsule $L \in \mathcal{L}$ the function φ is of the form either*

$$\xi_n^{2m} + a_1(x')\xi_n^{2m-1} + \cdots + a_{2m}(x'),$$

where $a_1, \ldots, a_{2m} : D \to R$ are continuous (in particular when $L \notin \mathcal{L}_1$), or

$$\pm f_\varkappa^{-1}(x', \pm\xi_n^{2m} + a_1(x')\xi_n^{2m-1} + \cdots + a_{2m}(x')),$$

where $a_1, \ldots, a_{2m} : D \to R$ are continuous and where $\varkappa \in \{1, \ldots, d\}$ and $f_\varkappa^{-1}$ denotes the inverse of $f_\varkappa$ with respect to the variable x_n on the capsule $\Phi(L)$ on which

$$\left|\frac{\partial f_\varkappa}{\partial x_n}\right| \geq c^{-1} \quad \textit{with some constant } c > 1.$$

Proof. Fix any $c > 1$. By Proposition 2.5, passing perhaps to a refinement of $\mathcal{K}$ one can assume that for each $K \in \mathcal{K}$ we have either

$$\left|\frac{\partial f_\varkappa}{\partial x_n}\right| \leq c \quad \text{in } \mathring{K} \text{ for all } \varkappa \in \{1, \ldots, d\}, \tag{5.1.7}$$

or

$$\left|\frac{\partial f_\varkappa}{\partial x_n}\right| \geq c^{-1} \quad \text{in } \mathring{K} \text{ for some } \varkappa \in \{1, \ldots, d\}, \tag{5.1.8}$$

and in the second case among $f_\varkappa$ satisfying (5.1.8) there is one, denote it by f_K, such that

$$\left|\frac{\partial f_\varkappa}{\partial x_n}\right| \Big/ \left|\frac{\partial f_K}{\partial x_n}\right| \leq c^d \quad \text{in } \mathring{K} \text{ for all } \varkappa \in \{1, \ldots, d\}. \tag{5.1.9}$$

Now we define a function $\lambda : [\alpha_0, \alpha_r] \to R$ inductively as follows. Put first

$$\lambda(x', \alpha_0(x')) := \alpha_0(x') \quad \text{for } x' \in D.$$

We define λ on $[\alpha_i, \alpha_{i+1}]$ according to the following two cases.

Case I. If $\overline{(\alpha_i, \alpha_{i+1})} \notin \mathcal{K}_1$ or if $\overline{(\alpha_i, \alpha_{i+1})} \in \mathcal{K}_1$ and (5.1.7) is satisfied on (α_i, α_{i+1}), then put

$$\lambda(x', x_n) := \lambda(x', \alpha_i(x')) + x_n - \alpha_i(x') \quad \text{for } (x', x_n) \in [\alpha_i, \alpha_{i+1}].$$

Case II. If $K = \overline{(\alpha_i, \alpha_{i+1})} \in \mathcal{K}_1$ and (5.1.8) is satisfied on (α_i, α_{i+1}), then put

$$\lambda(x', x_n) := \lambda(x', \alpha_i(x')) + |f_K(x', x_n) - f_K(x', \alpha_i(x'))| \quad \text{for } (x', x_n) \in [\alpha_i, \alpha_{i+1}].$$

(Compare the description of the one-dimensional case in our introduction.)

Put $\Lambda(x', x_n) := (x', \lambda(x', x_n)))$. Then Λ is a homeomorphism of $[\alpha_0, \alpha_r]$ onto $[\beta_0, \beta_r]$, where $\beta_i(x') := \lambda(x', \alpha_i(x'))$ $(x' \in D, i \in \{0, \ldots, r\})$ and $\overline{(\beta_i, \beta_{i+1})}$ $(i \in \{0, \ldots, r-1\})$ are capsules in R^n.

The partial derivatives

$$\frac{\partial^\sigma \lambda}{\partial x_n^\sigma} \quad (\sigma \in \{1, \dots, p+1\})$$

exist and are continuous in every (α_i, α_{i+1}) and $\frac{\partial \lambda}{\partial x_n} \equiv 1$ or $\frac{\partial \lambda}{\partial x_n} \geq c^{-1}$ on (α_i, α_{i+1}); hence $\lambda : [\alpha_0, \alpha_r] \to R$ is continuous, strictly increasing with respect to x_n. Let

$$\Psi : [\beta_0, \beta_r] \ni (x', \zeta_n) \mapsto (x', \psi(x', \zeta_n)) \in [\alpha_0, \alpha_r]$$

denote the inverse homeomorphism to Λ. Then

$$0 < \frac{\partial \psi}{\partial \zeta_n}(x', \zeta_n) = \frac{1}{\frac{\partial \lambda}{\partial x_n}(x', \psi(x', \zeta_n))} \leq \max\{1, c\} = c$$

on every (β_i, β_{i+1}). Fix now any $K = \overline{(\alpha_i, \alpha_{i+1})} \in \mathcal{K}$.

If K is as in Case I, then for each $(x', \zeta_n) \in (\beta_i, \beta_{i+1})$,

$$\beta_i(x') + \psi(x', \zeta_n) - \alpha_i(x') \equiv \zeta_n, \quad \text{hence} \quad \psi(x', \zeta_n) = \zeta_n - \beta_i(x') + \alpha_i(x');$$

consequently, if $K \in \mathcal{K}_1$, then for each $\varkappa \in \{1, \dots, d\}$,

$$\left| \frac{\partial (f_\varkappa \circ \Psi)}{\partial \zeta_n}(x', \zeta_n) \right| = \left| \frac{\partial f_\varkappa}{\partial x_n}(x', \psi(x', \zeta_n)) \right| \leq c.$$

If $K \in \mathcal{K}_1$ is as in Case II, then for each $(x', \zeta_n) \in (\beta_i, \beta_{i+1})$,

$$\beta_i(x') + |f_K(x', \psi(x', \zeta_n)) - f_K(x', \alpha_i(x'))| \equiv \zeta_n,$$

hence

$$\psi(x', \zeta_n) = f_K^{-1}(x', \pm(\zeta_n - \beta_i(x'))) + f_K(x', \alpha_i(x'));$$

consequently, for each $\varkappa \in \{1, \dots, d\}$,

$$\left| \frac{\partial (f_\varkappa \circ \Psi)}{\partial \zeta_n}(x', \zeta_n) \right| = \left| \frac{\partial f_\varkappa}{\partial x_n}(x', \psi(x', \zeta_n)) \right| \Big/ \left| \frac{\partial f_K}{\partial x_n}(x', \psi(x', \zeta_n)) \right| \leq c^d.$$

By Corollary 2.4, passing to a refinement $\overline{(\gamma_j, \gamma_{j+1})}$ $(j \in \{0, \dots, s-1\})$ of the capsules $\overline{(\beta_i, \beta_{i+1})}$, where $\gamma_0 \leq \gamma_1 \leq \dots \leq \gamma_s$ is a refinement of $\beta_0 \leq \dots \leq \beta_r$, we can additionally assume that for each $j \in \{0, \dots, s-1\}$ and each $\sigma \in \{2, \dots, p+1\}$ we have either

(5.1.10) $$\left| \frac{\partial^\sigma \psi}{\partial \zeta_n^\sigma}(x', \zeta_n) \right| \leq c \quad \text{on } (\gamma_j, \gamma_{j+1}),$$

or

(5.1.11) $$\left| \frac{\partial^\sigma \psi}{\partial \zeta_n^\sigma}(x', \zeta_n) \right| \geq c^{-1} \quad \text{on } (\gamma_j, \gamma_{j+1}),$$

and similarly, for each $\varkappa \in \{1, \dots, d\}$, either

$$\left|\frac{\partial^\sigma (f_\varkappa \circ \Psi)}{\partial \zeta_n^\sigma}(x', \zeta_n)\right| \le c \quad \text{on } (\gamma_j, \gamma_{j+1}), \tag{5.1.12}$$

or

$$\left|\frac{\partial^\sigma (f_\varkappa \circ \Psi)}{\partial \zeta_n^\sigma}(x', \zeta_n)\right| \ge c^{-1} \quad \text{on } (\gamma_j, \gamma_{j+1}). \tag{5.1.13}$$

Notice that (5.1.13) implies a constant sign of the partial derivative involved on (γ_j, γ_{j+1}).

Finally, we modify the homeomorphism Ψ with respect to the variable ζ_n by means of the smoothing homeomorphism ω with a parameter (Corollary 4.5):

$$\Phi(x', \xi_n) := \Psi\big(x', \omega(\gamma_0(x'), \dots, \gamma_s(x'); \xi_n)\big),$$

where $(x', \xi_n) \in [\delta_0, \delta_{2s}]$ and $\delta_0 \le \dots \le \delta_{2s} : D \to R$ is a sequence of continuous functions. ■

5.2. Polyhedrization of a cell by a $\mathcal{C}^p$-homeomorphism

The next lemma explains under what conditions a cell based on a simplex and bounded from below and above by $\mathcal{C}^p$-functions can be "straightened" to a polyhedron based on the same simplex by a homeomorphism of class $\mathcal{C}^p$. This together with Corollary 6.5 of the next section will be an efficient tool.

Lemma 5.2. *Let $\Delta \subset R^n$ be a simplex of dimension n, p a positive integer and let*

$$\beta_0 \le \beta_1 \le \dots \le \beta_k : \Delta \to R$$

be $\mathcal{C}^p$-functions such that for every face S of Δ and each $j \in \{0, \dots, k-1\}$ either $\beta_{j+1} - \beta_j \ne 0$ on $\mathring{S}$ or $\beta_{j+1} - \beta_j \equiv 0$ on S, and in the latter case let $\beta_{j+1} - \beta_j$ be p-flat on S. Let

$$\lambda_0 \le \lambda_1 \le \dots \le \lambda_k : \Delta \to R$$

be continuous PL*-functions such that for every face S of Δ and $j \in \{0, \dots, k\}$, $\lambda_j | S$ is affine and*

$$\beta_j \equiv \beta_{j+1} \text{ on } S \iff \lambda_j \equiv \lambda_{j+1} \text{ on } S \quad (j \in \{0, \dots, k-1\}). \tag{5.2.1}$$

Then the formula

$$\Psi(u, \zeta) = \begin{cases} \left(u, \frac{\zeta - \lambda_j(u)}{\lambda_{j+1}(u) - \lambda_j(u)}(\beta_{j+1}(u) - \beta_j(u)) + \beta_j(u)\right) & \text{if } \lambda_j(u) < \lambda_{j+1}(u), \\ (u, \beta_j(u)) & \text{if } \lambda_j(u) = \lambda_{j+1}(u), \end{cases}$$

for $(u, \zeta) \in [\lambda_j, \lambda_{j+1}]$, defines a homeomorphism of $[\lambda_0, \lambda_k]$ onto $[\beta_0, \beta_k]$ such that $\Psi(u, \lambda_j(u)) = (u, \beta_j(u))$ for $u \in \Delta$, $j \in \{0, \dots, k\}$, and for each $j \in \{0, \dots, k-1\}$, $\Psi | [\lambda_j, \lambda_{j+1}]$ is of class $\mathcal{C}^p$.

Proof. Assume that $\lambda_j < \lambda_{j+1}$ on $\mathring{\Delta}$. By a linear change of coordinates we can assume that

$$\Delta = \Big\{u \in R^n : u_\nu \geq 0 \ (\nu \in \{1, \dots, n\}),\ \sum_{\nu=1}^{n} u_\nu \leq 1\Big\}$$

and

$$\begin{aligned} S &:= \{u \in \Delta : \lambda_j(u) = \lambda_{j+1}(u)\} \\ &= \{u \in \Delta : \beta_j(u) = \beta_{j+1}(u)\} = \{u \in \Delta : u_{l+1} = \cdots = u_n = 0\}. \end{aligned}$$

Then for each $u \in \Delta$,

$$\lambda_{j+1}(u) - \lambda_j(u) = \sum_{\nu=l+1}^{n} c_\nu u_\nu, \quad \text{where } c_\nu > 0 \ (\nu \in \{l+1, \dots, n\}).$$

We want to check that

$$\frac{\partial^{|\sigma|+\rho}}{\partial u^\sigma \partial \zeta^\rho}\left[\frac{\zeta - \lambda_j(u)}{\lambda_{j+1}(u) - \lambda_j(u)}(\beta_{j+1}(u) - \beta_j(u))\right] \to 0$$

when $(\lambda_j, \lambda_{j+1}) \ni (u, \zeta) \to (u_0, \lambda_j(u_0)) \in S \times R$, $\sigma \in \mathbb{N}^n$, $\rho \in \mathbb{N}$ and $|\sigma| + \rho \leq p$.

In view of the Leibniz formula, it suffices to check that

$$(\zeta - \lambda_j(u))D^\sigma\left[\frac{1}{\lambda_{j+1} - \lambda_j}\right](u)D^\rho(\beta_{j+1} - \beta_j)(u) \to 0$$

when $\sigma, \rho \in \mathbb{N}^n$, $|\sigma| + |\rho| \leq p$ and $(u, \zeta) \to (u_0, \lambda_j(u_0))$, and

$$D^\sigma\left[\frac{1}{\lambda_{j+1} - \lambda_j}\right](u)D^\rho(\beta_{j+1} - \beta_j)(u) \to 0$$

when $\sigma, \rho \in \mathbb{N}^n$, $|\sigma| + |\rho| \leq p - 1$ and $(u, \zeta) \to (u_0, \lambda_j(u_0))$.

In the first case, by the Taylor formula,

$$\begin{aligned} &(\zeta - \lambda_j(u))D^\sigma\left[\frac{1}{\lambda_{j+1} - \lambda_j}\right](u)D^\rho(\beta_{j+1} - \beta_j)(u) \\ &\quad = (\zeta - \lambda_j(u))\frac{C}{(\lambda_{j+1}(u) - \lambda_j(u))^{|\sigma|+1}} \\ &\qquad \times \sum_{|\delta|=p-|\rho|} \frac{1}{\delta!}(u - \pi(u))^\delta D^{\rho+(0,\delta)}(\beta_{j+1} - \beta_j)(\pi(u) + \theta(u - \pi(u))), \end{aligned}$$

where $C > 0$, $\pi(u) = (u_1, \dots, u_l, 0, \dots, 0)$ and $\theta \in (0, 1)$. Consequently, with some constant $C' > 0$,

$$\begin{aligned} &\left|(\zeta - \lambda_j(u))D^\sigma\left[\frac{1}{\lambda_{j+1} - \lambda_j}\right](u)D^\rho(\beta_{j+1} - \beta_j)(u)\right| \\ &\quad \leq \frac{C'}{(\sum_{\nu=l+1}^{n} c_\nu u_\nu)^{|\sigma|}}\Big(\sum_{\nu=l+1}^{n} u_\nu\Big)^{p-|\rho|} \sup_{\substack{|\mu|=p \\ \theta\in[0,1]}} |D^\mu(\beta_{j+1} - \beta_j)(\pi(u) + \theta(u - \pi(u)))|, \end{aligned}$$

which tends to 0 as $u \to u_0$; and similarly in the second case. ■

5.3. Basic $\mathcal{C}^p$-extension lemma

The aim of this subsection is to prove Lemma 5.4 below which is a natural generalization of the extension lemma mentioned in the Introduction. To prove it we first recall the following $\mathcal{C}^1$-extension theorem (cf. [17, Proposition 2]).

Theorem 5.3 ($\mathcal{C}^1$-extension theorem)**.** *Let $f : S \to R$ be a $\mathcal{C}^1$-function defined on a cell*

$$S = \{(x', x_n) \in R^n : x' \in G,\ \varphi(x') < x_n < \psi(x')\}$$

in R^n such that G is an open subset of R^{n-1} and $\varphi < \psi : G \to R$ are of class $\mathcal{C}^1$. Assume that $\frac{\partial f}{\partial x_n}$ has a finite limit value[2] *at (almost) each point of φ (for example, when $\frac{\partial f}{\partial x_n}$ is bounded).*

Then there is a closed nowhere dense subset Z of φ such that f extends to a $\mathcal{C}^1$-function

$$f : S \cup (\varphi \setminus Z) \to R$$

where $S \cup (\varphi \setminus Z)$ is a $\mathcal{C}^1$-submanifold of R^n with boundary $\varphi \setminus Z$.

Proof. With no loss of generality we can assume that $\varphi \equiv 0$, i.e. $\varphi = G \times \{0\}$. For each $a \in G$ the set

$$\operatorname*{Lim}_{x \to (a,0)} \frac{\partial f}{\partial x_n}(x)$$

of all finite limit values of $\frac{\partial f}{\partial x_n}$ at $(a, 0)$ is a closed nonempty interval, because S satisfies the Łojasiewicz (s)-condition at points of φ. Since

$$\bigcup_{a \in G} \{a\} \times \operatorname*{Lim}_{x \to (a,0)} \frac{\partial f}{\partial x_n}(x) = \overline{\frac{\partial f}{\partial x_n}} \setminus \frac{\partial f}{\partial x_n}$$

is of dimension $n - 1$, it follows that there exists a closed nowhere dense subset E of G such that a finite limit

$$\lim_{x \to (a,0)} \frac{\partial f}{\partial x_n}(x) \quad \text{exists for each } a \in G \setminus E.$$

This implies in particular that for each $x' \in G \setminus E$ a finite limit

(5.3.1) $$g(x') := \lim_{x_n \to 0} f(x', x_n) \in R$$

exists. There exists a closed nowhere dense subset Z of G containing E such that g is of class $\mathcal{C}^1$ on $G \setminus Z$. Hence, without any loss of generality we can assume that $g \equiv 0$ and $Z = \emptyset$. Repeating the previous dimensional argument we conclude that after removing a closed nowhere dense subset from G, f extends by 0 to a continuous function on $S \cup \varphi$.

[2] An element $\alpha \in \overline{R}$ is a *limit value* of a function $g : S \to R$ at $a \in \overline{S}$ if there is an arc $\gamma : (0, 1) \to S$ such that $\lim_{t \to 0} \gamma(t) = a$ and $\lim_{t \to 0} g(\gamma(t)) = \alpha$.

Now, we will show that for any $i \in \{1, \dots, n-1\}$ the partial derivative $\partial f/\partial x_i$ extends by 0 to a continuous function defined on $S \setminus E$, where $E \subset \varphi$ and $\dim E < n-1$. With no loss of generality we assume that $i = n-1$. First we will show that

$$(5.3.2) \qquad 0 \in \operatorname*{Lim}_{x \to (a,0)} \frac{\partial f}{\partial x_{n-1}}(x) \quad \text{for all } a \in G.$$

To check this fix any $\eta > 0$ such that $B(a, \eta) := \{u \in R^{n-1} : |u - a| \le \eta\} \subset G$ and any $\varepsilon > 0$. There exists $\delta > 0$ such that $|f(x', x_n)| \le \varepsilon\eta$ when $x' \in B(a, \eta)$ and $x_n \in (0, \delta)$. By the Mean Value Theorem there exists $\theta \in (0, 1)$ such that

$$\left| \frac{\partial f}{\partial x_{n-1}}(\tilde{a}, a_{n-1} + \theta\eta, x_n) \right| = \left| \frac{f(\tilde{a}, a_{n-1} + \eta, x_n) - f(a, x_n)}{\eta} \right| \le 2\varepsilon,$$

where $a = (\tilde{a}, a_{n-1})$. This ends the proof of (5.3.2). Repeating the previous argument we conclude that

$$(5.3.3) \qquad \lim_{x \to (a,0)} \frac{\partial f}{\partial x_{n-1}}(x) = 0$$

for $a \in G \setminus Z$, where Z is a closed subset of G of dimension $< n-1$. This ends the proof of the theorem. ■

Lemma 5.4 (Basic $\mathcal{C}^p$-Extension Lemma). *Let $\Omega \subset R^k$ be an open subset, where $k \in \{0, \dots, n-1\}$, and let p be a positive integer. Let*

- *$\varphi_{k+1}, \psi_{k+1} : \Omega \to R$ be $\mathcal{C}^p$-functions such that $\varphi_{k+1} < \psi_{k+1}$;*
- *$\varphi_{k+2}, \psi_{k+2} : [\varphi_{k+1}, \psi_{k+1}) \to R$ be $\mathcal{C}^p$-functions such that*

$$\varphi_{k+2} < \psi_{k+2} \quad \text{on } (\varphi_{k+1}, \psi_{k+1}) \quad \text{and} \quad \varphi_{k+2} = \psi_{k+2} \quad \text{on } \varphi_{k+1};$$

- *$\varphi_{k+3}, \psi_{k+3} : [\varphi_{k+2}, \psi_{k+2}] \to R$ be $\mathcal{C}^p$-functions such that*

$$\varphi_{k+3} < \psi_{k+3} \quad \text{on } (\varphi_{k+2}, \psi_{k+2}) \quad \text{and} \quad \varphi_{k+3} = \psi_{k+3} \quad \text{on } \varphi_{k+2}|\varphi_{k+1};$$

...

- *$\varphi_n, \psi_n : [\varphi_{n-1}, \psi_{n-1}] \to R$ be $\mathcal{C}^p$-functions such that*

$$\varphi_n < \psi_n \quad \text{on } (\varphi_{n-1}, \psi_{n-1}) \quad \text{and} \quad \varphi_n = \psi_n \quad \text{on } \varphi_{n-1}|(\dots(\varphi_{k+2}|\varphi_{k+1})\dots).$$

Put

$$\Sigma := \big\{(x_1, \dots, x_n) \in \Omega \times R^{n-k} : \varphi_j(x_1, \dots, x_{j-1}) = x_j \ (j \in \{k+1, \dots, n\})\big\}.$$

Let $f : [\varphi_n, \psi_n] \setminus \Sigma \to R$ be a $\mathcal{C}^p$-function such that all the partial derivatives

(5.4.1) $\frac{\partial^p f}{\partial x_{k+1}^{\alpha_{k+1}} \dots \partial x_n^{\alpha_n}}$ $(|\alpha| = \alpha_{k+1} + \dots + \alpha_n = p)$ *have continuous extensions to Σ.*

Then there exists a closed subset E of Σ of dimension $< k$ such that f extends to a $\mathcal{C}^p$-function defined on $[\varphi_n, \psi_n] \setminus E$.

Remark. Geometrically $[\varphi_n, \psi_n]$ can be considered as a generalized curvilinear n-dimensional angle with the k-dimensional "vertex" $\varphi_n|(\dots(\varphi_{k+2}|\varphi_{k+1})\dots)$. Notice that $[\varphi_{k+1}, \psi_{k+1})$ is assumed only left-closed, while the others are closed at both sides.

Proof of Lemma 5.4. First assume that $p = 1$. With no loss of generality we can assume that

(5.4.2) $$\varphi_{k+1} \equiv 0, \quad \varphi_{k+2}|\varphi_{k+1} \equiv 0, \; \dots, \; \varphi_n|(\dots(\varphi_{k+2}|\varphi_{k+1})\dots) \equiv 0,$$

in other words, $\Sigma = \Omega \times \{0\}^{n-k}$.

Put $y := (x_{k+1}, \dots, x_n)$. For any $a \in \Omega$ the function $f_a : [\varphi_n, \psi_n]_a \setminus \{0\} \to R$ defined by $f_a(y) := f(a, y)$ on the set $[\varphi_n, \psi_n]_a \setminus \{0\} := \{y \neq 0 : (a, y) \in [\varphi_n, \psi_n]\}$ is a $\mathcal{C}^1$-function with bounded first order partial derivatives near 0. Since $[\varphi_n, \psi_n]_a \setminus \{0\}$ is quasi-convex[3] near 0, this implies that the limit

$$g(a) := \lim_{y \to 0} f_a(y)$$

exists in R (cf. [17, Proposition 1]). Since there exists a closed subset E of Ω of dimension $< k$ such that g is of class $\mathcal{C}^1$ on $\Omega \setminus E$, we can assume with no loss of generality that g is $\mathcal{C}^1$ and then that $g \equiv 0$.

For each $a \in \Omega$ the set $\mathrm{Lim}_{x \to (a,0)} f(x)$ of all finite limit values of f at $(a, 0)$ is a closed interval containing 0, because $[\phi_n, \psi_n] \setminus \Sigma$ satisfies the Łojasiewicz (s)-condition at points of Σ. We want to show that $\mathrm{Lim}_{x \to (a,0)} f(x) = \{0\}$ for almost all $a \in \Omega$. Suppose otherwise. Then there exists a nonempty open subset G of Ω and $\varepsilon > 0$ such that $[0, \varepsilon] \subset \mathrm{Lim}_{x \to (a,0)} f(x)$ (or $[-\varepsilon, 0] \subset \mathrm{Lim}_{x \to (a,0)} f(x)$) for each $a \in G$.

Then $G \times \{0\}^{n-k} \subset \overline{f^{-1}(\varepsilon/2, \infty)}$. It follows by an analogue of the Whitney Wing Lemma (cf. [15, Section 19]) or directly by the Cell Decomposition Theorem that there exists $a \in G$ such that $\{0\}^{n-k} \subset \overline{f^{-1}(\varepsilon/2, \infty)}_a = \overline{f_a^{-1}(\varepsilon/2, \infty)}$, a contradiction.

It follows that we can assume that f extends by 0 to a continuous function defined on $[\varphi_n, \psi_n]$. Now, we will show that for any $i \in \{1, \dots, k\}$ the partial derivative $\partial f / \partial x_i$ extends by 0 to a continuous function defined on $[\varphi_n, \psi_n] \setminus E$, where $E \subset \Sigma$ and $\dim E < k$. With no loss of generality we assume that $i = k$. Suppose it is not so. Then there exists a nonempty open subset G of Ω such that

(5.4.3) $$\mathop{\mathrm{Lim}}_{x \to (a,0)} \frac{\partial f}{\partial x_k}(x) \neq \{0\} \quad \text{for all } a \in G.$$

It follows that there there exists a nonempty open subset G of Ω and $\varepsilon > 0$ such that

$$G \times \{0\}^{n-k} \subset \overline{\left(\frac{\partial f}{\partial x_k}\right)^{-1}[\varepsilon, \infty)}$$

[3] A subset A of R^m is called *quasi-convex* if there is a positive integer M such that for any two points $a_1, a_2 \in A$ there exists a (definable) continuous arc $\lambda : [0, |a_1 - a_2|] \to A$ such that $\lambda(0) = a_1$, $\lambda(|a_1 - a_2|) = a_2$ and $|\lambda'(t)| \leq M$ for any $t \in [0, |a_1 - a_2|]$ such that $\lambda'(t)$ exists. (Then λ is necessarily piecewise $\mathcal{C}^1$.)

or

$$G \times \{0\}^{n-k} \subset \overline{\left(\frac{\partial f}{\partial x_k}\right)^{-1}(-\infty, -\varepsilon]}.$$

By an analogue of the Whitney Wing Lemma or directly by the Cell Decomposition Theorem there exist a nonempty open subset G' of G and $\delta > 0$ such that $G' \times [0, \delta) \subset [\varphi_{k+1}, \psi_{k+1})$, and a continuous mapping

$$\alpha : G' \times [0, \delta) \to \left(\frac{\partial f}{\partial x_k}\right)^{-1}[\varepsilon, \infty) \tag{5.4.4}$$

such that

$$\alpha(u, x_{k+1}) = (u, x_{k+1}, \alpha_{k+2}(u, x_{k+1}), \ldots, \alpha_n(u, x_{k+1})), \tag{5.4.5}$$

where $\alpha_j(u, 0) = 0$ for each $j \in \{k+2, \ldots, n\}$ and $u \in G'$, because of (5.4.2). Since

$$\varphi_{k+2}(u, x_{k+1}) < \alpha_{k+2}(u, x_{k+1}) < \psi_{k+2}(u, x_{k+1}),$$

and

$$\begin{aligned} &\varphi_{j+1}(u, x_{k+1}, \alpha_{k+2}(u, x_{k+1}), \ldots, \alpha_j(u, x_{k+1})) < \alpha_{j+1}(u, x_{k+1}) \\ &\quad < \psi_{j+1}(u, x_{k+1}, \alpha_{k+2}(u, x_{k+1}), \ldots, \alpha_j(u, x_{k+1})) \quad \text{for } j \in \{k+2, \ldots, m\}, \end{aligned}$$

it follows that

$$\lim_{x_{k+1} \to 0} \frac{\partial \alpha_j}{\partial x_{k+1}}(u, x_{k+1}) \in R \quad \text{for all } u \in G' \text{ and } j \in \{k+2, \ldots, n\}. \tag{5.4.6}$$

By Theorem 5.3, at the expense of shrinking G' and diminishing δ, we can assume that α_j are $\mathcal{C}^1$-functions on $G' \times [0, \delta)$; in particular,

$$\lim_{x_{k+1} \to 0} \frac{\partial \alpha_j}{\partial x_k}(u, x_{k+1}) = 0 \quad \text{for all } u \in G' \text{ and } j \in \{k+2, \ldots, n\}. \tag{5.4.7}$$

It follows from (5.4.1) and (5.4.6) that for each $u \in G'$ the derivative

$$\frac{\partial (f \circ \alpha)}{\partial x_{k+1}}(u, x_{k+1})$$

is bounded when x_{k+1} is near 0. Again by Theorem 5.3, after perhaps shrinking G' and diminishing δ we can assume that $(f \circ \alpha)|G' \times [0, \delta)$ is of class $\mathcal{C}^1$; in particular,

$$\lim_{x_{k+1} \to 0} \frac{\partial (f \circ \alpha)}{\partial x_k}(u, x_{k+1}) = 0. \tag{5.4.8}$$

On the other hand,

$$\begin{aligned} \frac{\partial (f \circ \alpha)}{\partial x_k}(u, x_{k+1}) &= \frac{\partial f}{\partial x_k}(u, x_{k+1}, \alpha_{k+2}(u, x_{k+1}), \ldots, \alpha_n(u, x_{k+1})) \\ &\quad + \sum_{j=k+2}^{n} \frac{\partial f}{\partial x_j}(u, x_{k+1}, \alpha_{k+2}(u, x_{k+1}), \ldots, \alpha_n(u, x_{k+1})) \frac{\partial \alpha_j}{\partial x_k}(u, x_{k+1}), \end{aligned}$$

which, in view of (5.4.8), (5.4.1) and (5.4.7), implies that

$$\lim_{x_{k+1}\to 0} \frac{\partial f}{\partial x_k}(u, x_{k+1}, \alpha_{k+2}(u, x_{k+1}), \dots, \alpha_n(u, x_{k+1})) = 0,$$

contradicting (5.4.4). This ends the proof in the case $p = 1$.

Assume now that $p > 1$ and the lemma is true for $p - 1$. Since $[\varphi_n, \psi_n] \setminus \Sigma$ is locally quasi-convex near Σ [4] it suffices to check that all the partial derivatives

$$\frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \quad (|\beta| := \beta_1 + \dots + \beta_n \leq p) \tag{5.4.9}$$

have continuous extensions to $\Sigma \setminus E$, where E is a closed subset of Σ of dimension $< k$ (cf. [20, p. 80]). By the induction hypothesis, there exists a closed subset E of Σ of dimension $< k$ such that for each $j \in \{k + 1, \dots, n\}$ all the derivatives

$$\frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}\right), \quad \text{where } |\gamma| = p - 1,$$

have continuous extensions to $\Sigma \setminus E$. It follows from the case $p = 1$ that there exists a closed subset E' of Σ containing E of dimension $< k$ such that all the derivatives (5.4.9) have continuous extensions to $\Sigma \setminus E'$. ∎

6. Existence of strict $\mathcal{C}^p$-triangulations orthogonally flat along simplexes

We start by making the following definition.

Definition 6.1. Let Γ be an open subset of $R^k = \{(x_1, \dots, x_n) \in R^n : x_{k+1} = \dots = x_n = 0\} \subset R^n$ and let $f : D \to R^m$ be a $\mathcal{C}^p$-mapping defined on a not necessarily open but locally closed subset D of R^n such that $D \subset \overline{\operatorname{int} D}$; i.e. there exists an open neighborhood Ω of D in R^n and a $\mathcal{C}^p$-mapping $\tilde{f} : \Omega \to R^m$ such that $\tilde{f}|D = f$. Assume that $\Gamma \subset D$. We say that f *is orthogonally p-flat along Γ* if

$$\frac{\partial^{|\alpha|} f}{\partial x_{k+1}^{\alpha_{k+1}} \dots \partial x_n^{\alpha_n}}(x_1, \dots, x_k, 0, \dots, 0) = \frac{\partial^{|\alpha|} f}{\partial x_{k+1}^{\alpha_{k+1}} \dots \partial x_n^{\alpha_n}}(u, 0) = 0$$

for all $u = (x_1, \dots, x_k) \in \Gamma$ and $\alpha = (\alpha_{k+1}, \dots, \alpha_n) \in \mathbb{N}^{n-k}$ such that $1 \leq |\alpha| \leq p$. This definition generalizes in a natural way to the case when Γ is an open subset of the affine subspace $\operatorname{Aff}(\Gamma)$ generated by Γ in R^n.

Remark 6.2. If $f : D \to R^m$ is a $\mathcal{C}^p$-mapping orthogonally p-flat along $\Gamma \subset D$ and $w_1 \in \mathbb{S}^{n-1}$ is a vector orthogonal to $\operatorname{Aff}(\Gamma)$, then for each $j \in \{0, \dots, p\}$ and arbitrary $w_2, \dots, w_j \in \mathbb{S}^{n-1}$,

$$\left.\frac{\partial^j f}{\partial w_1 \dots \partial w_j}\right| \Gamma \equiv 0.$$

[4] This means that each point $u \in \Sigma$ has arbitrarily small neighborhoods U in R^n such that $U \cap [\varphi_n, \psi_n] \setminus \Sigma$ is quasi-convex.

The main theorem of the present section is the following.

Theorem 6.3. *Let $\mathcal{K}$ be any finite simplicial complex in R^n such that $|\mathcal{K}| = \overline{\operatorname{int}|\mathcal{K}|}$. Then there exists a homeomorphism $h : R^n \to R^n$ of class $\mathcal{C}^p$ such that*

(6.3.1) *$h|\mathring{\Gamma} : \mathring{\Gamma} \to \mathring{\Gamma}$ is a $\mathcal{C}^p$-diffeomorphism for each $\Gamma \in \mathcal{K}$,*

(6.3.2) *h is orthogonally p-flat along each simplex $\Gamma \in \mathcal{K}$.*

In the proof we will need the following lemma.

Lemma 6.4. *Let*

$$\Lambda = \{(x_1, \ldots, x_k) \in R^k : \rho_i(x_1, \ldots, x_k) \geq 0 \ (i \in \{0, \ldots, k)\}$$

be a simplex of dimension k in R^k, where ρ_i are nonzero affine forms. Put

$$\sigma(u) := \frac{(\rho_0 \ldots \rho_k)(u)}{\sum_j (\rho_0 \ldots \hat{\rho}_j \ldots \rho_k)(u)} \quad \text{for } u \in \mathring{\Lambda}.$$

Then there exist constants $C_\alpha > 0$ ($\alpha \in \mathbb{N}^k$) such that

$$C_0^{-1} d(u, \partial\Lambda) \leq \sigma(u) \leq C_0 d(u, \partial\Lambda) \quad \text{for all } u \in \mathring{\Lambda}$$

and

$$|D^\alpha \sigma(u)| \leq \frac{C_\alpha}{\sigma(u)^{|\alpha|-1}} \quad \text{for all } u \in \mathring{\Lambda} \text{ and } \alpha \in \mathbb{N}^k \setminus \{0\}.$$

Proof of Lemma 6.4. Put $H_i := \rho_i^{-1}(0)$ ($i \in \{0, \ldots, k\}$). Then $d(u, \partial\Lambda) = \min_i d(u, H_i)$ and there exists $C > 0$ such that $C^{-1}\rho_i(u) \leq d(u, H_i) \leq C\rho_i(u)$ for $u \in \Lambda$. Hence

$$C^{-1} \min_i \rho_i(u) \leq d(u, \partial\Lambda) \leq C \min_i \rho_i(u).$$

For a fixed $u \in \mathring{\Lambda}$ let j be such that $\rho_j(u) = \min_i \rho_i(u)$. Then

$$\frac{1}{\rho_j(u)} \leq \frac{1}{\rho_0(u)} + \cdots + \frac{1}{\rho_k(u)} \leq \frac{k+1}{\rho_j(u)};$$

thus

$$\frac{1}{k+1} \min_i \rho_i(u) \leq \sigma(u) = \frac{1}{\frac{1}{\rho_0(u)} + \cdots + \frac{1}{\rho_k(u)}} \leq \min_i \rho_i(u); \tag{6.4.1}$$

finally,

$$\frac{1}{C(k+1)} \sigma(u) \leq d(u, \partial\Lambda) \leq C(k+1)\sigma(u).$$

There are constants a_j ($j \in \{0, \ldots, k\}$) such that

$$\begin{aligned}
\frac{\partial\sigma}{\partial x_\nu} &= \sum_i a_i \frac{(\rho_0 \ldots \hat{\rho}_i \ldots \rho_k)}{\sum_j (\rho_0 \ldots \hat{\rho}_j \ldots \rho_k)} \\
&\quad - (\rho_0 \ldots \rho_k) \Bigg(\sum_{i \neq j} a_i \frac{1}{\rho_i \rho_j} \rho_0 \ldots \rho_k \Bigg) \frac{1}{[\sum_i \rho_0 \ldots \hat{\rho}_i \ldots \rho_k]^2} \\
&= \sum_i \frac{a_i}{\rho_i} \cdot \sigma - \sum_{i \neq j} a_i \frac{1}{\rho_i \rho_j} \sigma^2.
\end{aligned}$$

By the Leibniz formula,

$$D^\alpha\left(\frac{\partial\sigma}{\partial x_\nu}\right) = \sum_i a_i \sum_{\beta\le\alpha}\binom{\alpha}{\beta} D^\beta\left(\frac{1}{\rho_i}\right) D^{\alpha-\beta}\sigma$$
$$-\sum_{i\neq j} a_i \sum_{\alpha=\beta+\gamma+\delta+\epsilon} \frac{\alpha!}{\beta!\gamma!\delta!\epsilon!} D^\beta\left(\frac{1}{\rho_i}\right) D^\gamma\left(\frac{1}{\rho_j}\right) D^\delta\sigma D^\epsilon\sigma.$$

There exist constants M_β ($\beta\in\mathbb{N}^n$) such that

$$D^\beta\left(\frac{1}{\rho_i}\right) = \frac{M_\beta}{\rho_i^{|\beta|+1}}. \tag{6.4.2}$$

By (6.4.1) and (6.4.2) and induction on the degree of the derivative,

$$\left|D^\alpha\left(\frac{\partial\sigma}{\partial x_\nu}\right)\right| \le \sum_i |a_i| \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\frac{|M_\beta|}{\rho_i^{|\beta|+1}}\,\frac{C_{\alpha-\beta}}{\sigma^{|\alpha|-|\beta|-1}}$$
$$+\sum_{i\neq j}|a_i| \sum_{\alpha=\beta+\gamma+\delta+\epsilon} \frac{\alpha!}{\beta!\gamma!\delta!\epsilon!}\,\frac{|M_\beta|}{\rho_i^{|\beta|+1}}\,\frac{|M_\gamma|}{\rho_j^{|\gamma|+1}}\,\frac{C_\delta}{\sigma^{|\delta|-1}}\,\frac{C_\epsilon}{\sigma^{|\epsilon|-1}}.$$

The lemma follows.

Proof of Theorem 6.3. Take a $\mathcal{C}^p$-function $\varphi:[0,\infty)\to[0,1]$ such that $\varphi^{(i)}(0)=0$ for each $i\in\{0,\dots,p\}$, $\varphi'(t)>0$ for $t\in(0,1)$ and $\varphi(t)=1$ for $t\in[1,\infty)$.

We will prove by induction on $k\in\{0,\dots,n-1\}$ that there exists a homeomorphism $h:R^n\to R^n$ of class $\mathcal{C}^p$ such that (6.3.1) is satisfied, while (6.3.2) is satisfied just for simplexes of dimension $\le k$.

I. Let $k=0$. Let $\{a\}\in\mathcal{K}$ and fix $r_a>0$ such that $B(a,r_a)\cap|\mathcal{K}|\subset\bigcup \mathrm{St}\{a\}$. Define

$$h_a(x) := \varphi\left(\frac{|x-a|^2}{r_a^2}\right)(x-a)+a \quad \text{for } x\in R^n.$$

Then h_a is of class $\mathcal{C}^p$ and p-flat at a. Moreover, h_a is a homeomorphism and $\mathcal{C}^p$-diffeomorphism on $R^n\setminus\{a\}$, because

$$x = a+\psi^{-1}(|h_a(x)-a|)\frac{h_a(x)-a}{|h_a(x)-a|} \quad \text{for all } x\in R^n,$$

where $\psi(t):=\varphi(t^2/r_a^2)\cdot t$ ($t\in R$) is an increasing homeomorphism of R onto R.

It is clear that $h_a(\Gamma)=\Gamma$ for each $\Gamma\in\mathcal{K}$. Now, if $a_1,\dots,a_m$ are all vertices of $\mathcal{K}$, then we put

$$h := h_{a_m}\circ\cdots\circ h_{a_1}.$$

II. Assume now that $0<k\le n-1$ and we have a $\mathcal{C}^p$-homeomorphism h satisfying (1) and (2) for all simplexes of dimension $<k$. Let $\Lambda\in\mathcal{K}$ and $\dim\Lambda=k$. With no loss of generality we can assume that $\mathring{\Lambda}$ is an open simplex in $R^k=\{(x_1,\dots,x_n): x_{k+1}=\cdots=x_n=0\}$. Put $u=(u_1,\dots,u_k)=(x_1,\dots,x_k)$ and $v=(v_1,\dots,v_{n-k})=(x_{k+1},\dots,x_n)$.

Take $\sigma : \mathring{\Lambda} \to (0,\infty)$ as in Lemma 6.4. Since $\Omega := \bigcup \mathrm{St}(\Lambda)$ is an open neighborhood of Λ in $|\mathcal{K}|$, there exists (by a Łojasiewicz inequality in PL-structure) a constant $r > 0$ such that

$$\{(u,v) \in \mathring{\Lambda} \times R^{n-k} : |v| \le r\sigma(u)\} \cap |\mathcal{K}| \subset \Omega.$$

Put $G := \{(u,v) \in \mathring{\Lambda} \times R^{n-k} : |v| < r\sigma(u)\}$. The mapping

$$g(u,v) := \begin{cases} (u, \varphi(\frac{|v|^2}{r^2\sigma^2(u)}) \cdot v) & \text{when } (u,v) \in G, \\ (u,v) & \text{when } (u,v) \in R^n \setminus G, \end{cases}$$

is a homeomorphism of R^n onto R^n such that $g|\mathring{\Gamma} : \mathring{\Gamma} \to \mathring{\Gamma}$ is a $\mathcal{C}^p$-diffeomorphism for each $\Gamma \in \mathcal{K}$. Moreover, g is of class $\mathcal{C}^p$ on $R^n \setminus \partial\Lambda$. Now define

$$H(u,v) := h(g(u,v)) \quad \text{for } (u,v) \in R^n.$$

For any $(u,v) \in G$ and $\nu \in \{1,\dots,n-k\}$,

$$\frac{\partial}{\partial v_\nu} H(u,v) = \sum_{\mu=1}^{n-k} \frac{\partial h}{\partial v_\mu}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) v_\mu \frac{2v_\nu}{r^2\sigma^2(u)} \varphi'\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big) + \frac{\partial h}{\partial v_\nu}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big)\varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big).$$

It follows by induction on $|\alpha| \in \{1,\dots,p\}$, where $\alpha = (\alpha_1,\dots,\alpha_{n-k})$, that $\frac{\partial^{|\alpha|}H}{\partial v^\alpha}$ can be expressed as a finite linear combination with real coefficients of the functions

$$\frac{\partial^{|\beta|}h}{\partial v^\beta}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) \frac{v^\gamma}{r^{2s}\sigma^{2s}(u)} \Big[\varphi^{(0)}\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)\Big]^{\nu_0} \dots \Big[\varphi^{(|\alpha|)}\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)\Big]^{\nu_{|\alpha|}},$$

where $|\beta| \in \{1,\dots,|\alpha|\}$, $|\beta| + 2s - |\gamma| = |\alpha|$, $\nu_0 + \dots + \nu_{|\alpha|} = |\beta|$ and $\nu_0 + \nu_1 + 2\nu_2 + \dots + |\alpha|\nu_{|\alpha|} \le |\alpha|$.

Hence in particular

$$\frac{\partial^{|\alpha|}H}{\partial v^\alpha}(u,v) = 0 \quad \text{when } u \in \mathring{\Lambda}, v = 0, \alpha \in \mathbb{N}^{n-k}, 1 \le |\alpha| \le p. \tag{6.3.3}$$

Now in general, if $\alpha \in \mathbb{N}^{n-k}$ and $\varkappa \in \mathbb{N}^k$ and $|\alpha| + |\varkappa| \le p$, then the derivative

$$\frac{\partial^{|\alpha|+|\varkappa|}H}{\partial v^\alpha \partial u^\varkappa}$$

is a finite linear combination with real coefficients of functions of the form

$$\frac{\partial^{|\beta|+|\lambda|}h}{\partial v^\beta \partial u^\lambda}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) \frac{v^\gamma}{\sigma^d(u)} \times \Big[\varphi^{(0)}\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)\Big]^{\nu_0} \dots \Big[\varphi^{(|\alpha|+|\varkappa|)}\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)\Big]^{\nu_{|\alpha|+|\varkappa|}} \times (D^{\varepsilon_1}\sigma(u)) \dots (D^{\varepsilon_q}\sigma(u)), \tag{6.3.4}$$

where $0 \le q \le |\alpha| + |\varkappa|$, $d \ge 0$, $|\varepsilon_1| > 0, \dots, |\varepsilon_q| > 0$, $\lambda + \varepsilon_1 + \dots + \varepsilon_q = \varkappa$, $|\beta| + d - |\gamma| = |\alpha| + q$, $\nu_0 \ge 0, \dots, \nu_{|\alpha|+|\varkappa|} \ge 0$, $d \ge |\gamma|$.

Assume now that $(u, v) \in G$ and (u, v) tends to $(u_0, 0)$ along some (definable) arc, where $u_0 \in \partial\Lambda$. Let $\Gamma_0 \in \mathcal{K}$ and $u_0 \in \mathring{\Gamma}_0$. By an orthogonal change of coordinates $u_1, \dots, u_k$ one can assume that

$$d(u, \partial\Lambda) = d(u, \Gamma) = |u_1|,$$

where $\Gamma \in \mathcal{K}$, $\dim \Gamma = k - 1$, $\Gamma \subset \{(u_1, \dots, u_k) \in R^k : u_1 = 0\}$ and $\Gamma_0 \subset \{(u_1, \dots, u_k) : u_1 = \dots = u_l = 0\}$ $(l \in \{1, \dots, k\})$.

When $\alpha \ne 0$, in a product (6.3.4) we necessarily have $\beta \ne 0$, therefore by the Taylor formula,

$$\begin{aligned}
&\left| \frac{\partial^{|\beta|+|\lambda|} h}{\partial v^\beta \partial u^\lambda}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) \right| \\
&\quad = \left| \frac{\partial^{|\beta|+|\lambda|} h}{\partial v^\beta \partial u^\lambda}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) - \frac{\partial^{|\beta|+|\lambda|} h}{\partial v^\beta \partial u^\lambda}\Big(0, u_2, \dots, u_k, 0\Big) \right| \\
&\quad = \Bigg| \sum_{\substack{\sigma+|\rho|= \\ p-|\beta|-|\lambda|}} \frac{1}{\sigma!\rho!} u_1^\sigma \Big[\varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big]^\rho \\
&\qquad\qquad \times \frac{\partial^p h}{\partial v^{\beta+\rho} \partial u^\lambda \partial u_1^\sigma}\Big(\theta u_1, u_2, \dots, u_k, \theta\varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) \Bigg|
\end{aligned}$$

for some $\theta \in (0, 1)$. Hence

$$\left| \frac{\partial^{|\beta|+|\lambda|} h}{\partial v^\beta \partial u^\lambda}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) \right| \le (\sigma(u))^{p-|\beta|-|\lambda|} \mu(u, v),$$

where $\mu(u, v) \to 0$ when $(u, v) \to (u_0, 0)$. Thus, there exists a constant $M > 0$ such that

$$\begin{aligned}
|(6.3.4)| &\le M\sigma^{p-|\beta|-|\lambda|} \mu \frac{\sigma^{|\gamma|}}{\sigma^d} \sigma^{-|\varepsilon_1|+1} \dots \sigma^{-|\varepsilon_q|+1} \\
&= M\mu\sigma^{p-|\beta|-|\lambda|+|\gamma|-d+q-|\varepsilon_1|-\dots-|\varepsilon_q|} = M\mu\sigma^{p-|\alpha|-|\varkappa|} \to 0
\end{aligned}$$

when $(u, v) \to (u_0, 0)$.

Suppose now that $\alpha = 0$ and $\varkappa \ne 0$. Then, for each $(u, v) \in G$,

$$\frac{\partial^{|\varkappa|} H}{\partial u^\varkappa}(u, v) = \frac{\partial^{|\varkappa|} h}{\partial u^\varkappa}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) + \text{an } R\text{-linear combination of functions of the form (6.3.4), where } \beta \ne 0.$$

It follows that

$$\lim_{(u,v)\to(u_0,0)} \frac{\partial^{|\varkappa|} H}{\partial u^\varkappa}(u, v) = \lim_{(u,v)\to(u_0,0)} \frac{\partial^{|\varkappa|} h}{\partial u^\varkappa}\Big(u, \varphi\Big(\frac{|v|^2}{r^2\sigma^2(u)}\Big)v\Big) = \frac{\partial^{|\varkappa|} h}{\partial u^\varkappa}(u_0, 0).$$

We have just checked that H is of class $\mathcal{C}^p$ and is orthogonally p-flat along Γ_0, and (6.3.3) shows that it is orthogonally p-flat along Λ. We repeat the above construction for every simplex of dimension k.

Corollary 6.5. *Let $\mathcal{K}$ be a finite simplicial complex in R^n such that $|\mathcal{K}| = \overline{\operatorname{int}|\mathcal{K}|}$ and let $f : |\mathcal{K}| \to A \subset R^n$ be a homeomorphism such that for each $\Lambda \in \mathcal{K}$, $f|\Lambda$ is of class $\mathcal{C}^p$ and $f|\mathring{\Lambda} : \mathring{\Lambda} \to R^n$ is a $\mathcal{C}^p$-embedding. Let $h : R^n \to R^n$ be a homeomorphism described in Theorem 6.3. Then $(\mathcal{K}, f \circ h)$ is a strict $\mathcal{C}^p$-triangulation of A orthogonally p-flat along simplexes and such that $f(\Lambda) = (f \circ h)(\Lambda)$ for each $\Lambda \in \mathcal{K}$.*

7. Regular cells, regular θ-cells, (k, f, q)-proper regular θ-cells and convex polyhedra (k, f, q)-well situated in R^n

In this section we introduce a few auxiliary notions of technical character needed in the proof of the Main Theorem in Section 8.

Definition 7.1. A subset C of R^n is called a *regular cell in R^n* if C is a closed bounded interval $[a, b]$, where $a, b \in R$, $a < b$, when $n = 1$ and

$$C = [\alpha_{n-1}, \beta_{n-1}] := \{(x', x_n) \in R^{n-1} \times R : x' \in C',\ \alpha_{n-1}(x') \leq x_n \leq \beta_{n-1}(x')\},$$

where C' is a regular cell in R^{n-1} and $\alpha_{n-1} \leq \beta_{n-1} : C' \to R$ are continuous and $\alpha_{n-1} < \beta_{n-1}$ on $\operatorname{int} C'$ when $n > 1$.

Remark 7.2. If $C = [\alpha_{n-1}, \beta_{n-1}]$ is a regular cell in R^n, then $[\alpha_{j-1}, \beta_{j-1}] := \pi^n_j(C)$ $(j \in \{1, \dots, n\})$ is a regular cell in R^j.

Remark 7.3. If $j \in \{1, \dots, n-1\}$ and $\Psi : [\alpha_{j-1}, \beta_{j-1}] \to [\alpha_{j-1}, \beta_{j-1}]$ is a homeomorphism, then it induces, for each $\nu \in \{j+1, \dots, n\}$, a homeomorphism

$$[\alpha_{j-1}, \beta_{j-1}] \times R^{\nu-j} \ni (x_1, \dots, x_\nu) \mapsto (\Psi(x_1, \dots, x_j), x_{j+1}, \dots, x_\nu) \in [\alpha_{j-1}, \beta_{j-1}] \times R^{\nu-j},$$

which we will denote also by Ψ. It should be clear from the context what ν is. Then $\Psi^{-1}(C)$ is a regular cell such that $\pi^n_\nu(\Psi^{-1}(C)) = [\alpha_{\nu-1} \circ \Psi, \beta_{\nu-1} \circ \Psi]$ for $\nu \in \{j+1, \dots, n\}$ and $\pi^n_j(\Psi^{-1}(C)) = [\alpha_{j-1}, \beta_{j-1}]$.

Remark 7.3. If C is a regular cell in R^n, $j \in \{1, \dots, n-1\}$ and L is a regular cell in R^j such that $L \subset \pi^n_j(C)$, then $C|L := C \cap (L \times R^{n-j})$ is a regular cell in R^n contained in C.

Remark 7.4. Any convex polyhedron in R^n of dimension n is a regular cell in R^n.

Definition 7.5. Any pair $(C, \theta(C))$, where C is a regular cell in R^n and $\theta(C)$ is a closed subset of ∂C, will be called a *regular θ-cell in R^n*.

Definition 7.6. Let $(C, \theta(C))$ be a regular θ-cell in R^n, where $n \geq 2$, and let $f : B \to R^d$ be a continuous mapping defined on a subset B of R^n containing C. Let $k \in \{0, \dots, n-1\}$ and $q \in \mathbb{N}$.

We will say that $(C, \theta(C))$ is (k, f, q)*-proper* if $\dim \theta(C) \leq k$, f is of class $\mathcal{C}^q$ on $C \setminus \theta(C)$, $(\pi^n_{k+1})^{-1}(\pi^n_{k+1}(\theta(C))) \cap C = \theta(C)$ and moreover

$$\pi^n_{k+1}|\theta(C) : \theta(C) \to R^{k+1} \quad \text{is injective.}$$

Observe that if $\pi^n_j(C) = [\alpha_{j-1}, \beta_{j-1}]$ $(j \in \{1, \ldots, n\})$, then $(C, \theta(C))$ is (k, f, q)-proper if f is of class $\mathcal{C}^q$ on $C \setminus \theta(C)$, $(\pi^n_j(C), \pi^n_j(\theta(C)))$, for each $j \in \{k+1, \ldots, n-1\}$, is a regular θ-cell in R^j and

$$\alpha_j|\pi^n_j(\theta(C)) = \beta_j|\pi^n_j(\theta(C)) \quad (j \in \{k+1, \ldots, n-1\}).$$

Definition 7.7. If P is a convex polyhedron in R^n of dimension n and $f : B \to R^d$ is a continuous mapping defined in a subset $B \subset R^n$ such that $P \subset B$, we will say that P *is* (k, f, q)*-well situated in* R^n if there exists a k-dimensional face $\Sigma(P)$ of P such that $(P, \Sigma(P))$ is a regular θ-cell in R^n, which is (k, f, q)-proper.

Definition 7.8. Let $n \in \mathbb{N}, n \geq 2, k \in \{0, \ldots, n-1\}, q \in \mathbb{N}, q \geq p$.

We will say that a (k, f, q)-proper regular θ-cell $(C, \theta(C))$ is (l, m)*-prepared*, where $l \in \{k, \ldots, n-1\}$ and $m \in \{0, \ldots, p\}$, if all the partial derivatives

$$\frac{\partial^{|\varkappa|}(f|C \setminus \theta(C))}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_n^{\varkappa_n}}, \quad \text{where } \varkappa = (\varkappa_{l+1}, \ldots, \varkappa_n),$$
$$|\varkappa| := \varkappa_{l+1} + \cdots + \varkappa_n \in \{1, \ldots, p\}, \varkappa_{l+1} \in \{0, \ldots, m\},$$

extend continuously by zero to $\theta(C)$ and, for each $j \in \{l+1, \ldots, n-1\}$, the regular θ-cell $(\pi^n_j(C), \pi^n_j(\theta(C)))$ is $(k, (\alpha_j, \beta_j), q)$-proper, where $\pi^n_j(C) = [\alpha_{j-1}, \beta_{j-1}]$, and all the partial derivatives

$$\frac{\partial^{|\varkappa|}((\alpha_j, \beta_j)|\pi^n_j(C) \setminus \pi^n_j(\theta(C))}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_j^{\varkappa_j}}, \quad \text{where } \varkappa = (\varkappa_{l+1}, \ldots, \varkappa_j),$$
$$|\varkappa| := \varkappa_{l+1} + \cdots + \varkappa_j \in \{1, \ldots, p\}, \varkappa_{l+1} \in \{0, \ldots, m\},$$

extend continuously by zero to $\pi^n_j(\theta(C))$.

8. Proof of the Main Theorem

Roughly speaking, our proof will consist in $\mathcal{C}^p$-extending of the triangulating homeomorphism to faces of ever greater codimension. Codimension 1 is considered in the following proposition.

Proposition 8.1. *Assume that our Main Theorem is true in dimensions* $< n$. *Let* $\mathcal{P}$ *be a finite polyhedral complex in* R^{n-1} *and put* $D := |\mathcal{P}|$. *Let* $q_1, q \in \mathbb{Z}$ *and* $q \geq q_1 \geq p+1$. *Let* $\alpha_0 \leq \cdots \leq \alpha_r : D \to R$ *be an increasing sequence of continuous* PL*-functions such that*

$$\mathcal{K} := \left\{\overline{(\alpha_i, \alpha_{i+1})} : i \in \{0, \ldots, r-1\}\right\}$$

is a family of capsules in R^n. *Let* $\mathcal{K}_1 \subset \mathcal{K}$, $A := |\mathcal{K}|$ *and* $A_1 := |\mathcal{K}_1|$. *Let* $f = (f_1, \dots, f_d) : A_1 \to R^d$ *be a continuous mapping such that* $f|\mathring{K}$ *is of class* $\mathcal{C}^{q_1}$ *for each* $K \in \mathcal{K}_1$. *Let* $\mathcal{E}$ *be any finite family of subsets of* D.

Then there exist

(8.1.1) *a strict* $\mathcal{C}^q$*-triangulation* $(\mathcal{M}, h)$ *of* D *compatible with* $\mathcal{E}$ *such that* $|\mathcal{M}| = D$ *and* $h(\Gamma) = \Gamma$ *for every face* Γ *of each polyhedron* $P \in \mathcal{P}$,

(8.1.2) *an increasing sequence of continuous* PL*-functions*

$$\eta_0 \leq \cdots \leq \eta_k : D \to R,$$

which is a refinement of $\alpha_0, \dots, \alpha_r$, *such that the family* $\mathcal{C} := \{\overline{(\eta_j, \eta_{j+1})} : j \in \{0, \dots, k-1\}\}$ *is a family of capsules refining the family* $\mathcal{K}$,

(8.1.3) *a homeomorphism* $\Psi : [\alpha_0, \alpha_r] \to [\alpha_0, \alpha_r]$ *of the form*

$$\Psi(u, \zeta_n) = (h(u), \psi(u, \zeta_n)) \quad \textit{for } (u, \zeta_n) \in [\alpha_0, \alpha_r],$$

such that

(8.1.4) $\Psi(u, \alpha_i(u)) = (h(u), \alpha_i(h(u)))$ *for* $u \in D$ *and* $i \in \{0, \dots, r\}$;

(8.1.5) *if* $a \in C \in \mathcal{C}$, *where* $C \subset K \in \mathcal{K}_1$ *and* $f|K$ *is of class* $\mathcal{C}^{q_1}$ *in a neighborhood of* $\Psi(a)$ *in* K, *then* $\Psi|C$ *and* $f \circ \Psi|C$ *are of class* $\mathcal{C}^{q_1}$ *in a neighborhood of* a *in* C;

(8.1.6) $\Psi|\mathring{C}$ *and* $f \circ \Psi|\mathring{C}$ *are of class* $\mathcal{C}^{q_1}$ *for each* $C \in \mathcal{C}$ *such that* $C \subset K \in \mathcal{K}_1$;

(8.1.7) $\Psi|C$ *is of class* $\mathcal{C}^q$ *for each* $C \in \mathcal{C}$ *such that* $C \subset K \in \mathcal{K} \setminus \mathcal{K}_1$ *and*

$$\frac{\partial^\sigma(\psi|C)}{\partial \zeta_n^\sigma} = 0 \quad \textit{on } \partial C \textit{ for } \sigma \in \{1, \dots, p\};$$

(8.1.8) *if* $C \in \mathcal{C}$ *and* $C \subset K \in \mathcal{K}_1$, *then the derivatives*

$$\frac{\partial^\sigma(\Psi|\mathring{C})}{\partial \zeta_n^\sigma} \quad \textit{and} \quad \frac{\partial^\sigma(f \circ \Psi|\mathring{C})}{\partial \zeta_n^\sigma} \quad \textit{for } \sigma \in \{1, \dots, p\}$$

have continuous extensions by zero to the whole C;

(8.1.9) $\frac{\partial(\psi|\mathring{C})}{\partial \zeta_n} > 0$ *for each* $C \in \mathcal{C}$.

Proof. By a refinement of $\mathcal{P}$ one can assume that

(8.1.10) every function α_i is affine on each $P \in \mathcal{P}$,

(8.1.11) $\mathcal{P}$ is *compatible* with each of the sets $\{x' \in D : \alpha_i(x') = \alpha_{i+1}(x')\}$ $(i \in \{0, \dots, r-1\})$, i.e. each of these sets is a union of some $P \in \mathcal{P}$.

By Lemma 5.1, we get a sequence $\delta_0 \leq \cdots \leq \delta_k : D \to R$ of continuous functions and a homeomorphism $\Phi : [\delta_0, \delta_k] \to [\alpha_0, \alpha_r]$ with properties (5.1.1)–(5.1.6).

Now we apply the induction hypothesis. We get a strict $\mathcal{C}^q$-triangulation $(\mathcal{M}, h)$ of D such that

(8.1.12) $\mathcal{M}$ is a finite simplicial complex in R^{n-1} such that $|\mathcal{M}| = D$;

(8.1.13) $(\mathcal{M}, h)$ is compatible with each $E \in \mathcal{E}$ and with each $P \in \mathcal{P}$ (the latter follows from (8.1.14) below);

(8.1.14) $h(P) = P$ for each $P \in \mathcal{P}$; hence, each of the sets $\{x' \in D : \alpha_i(x') = \alpha_{i+1}(x')\}$ $(i \in \{0, \ldots, r-1\})$ is h-invariant (see (8.1.11));

(8.1.15) $\delta_j \circ h, \theta_j \circ h : D \to R$, where $j \in \{0, \ldots, k\}$, are of class $\mathcal{C}^q$;

(8.1.16) for all the functions $a_1, \ldots, a_{2m}$ from condition (5.1.6) the compositions $a_1 \circ h, \ldots, a_{2m} \circ h : D \to R$ are of class $\mathcal{C}^q$,

(8.1.17) $(\mathcal{M}, h)$ is compatible with each of the sets $\{x' \in D : \delta_j(x') = \delta_{j+1}(x')\}$, where $j \in \{0, \ldots, k-1\}$.

By passing to the barycentric subdivision we can have in addition

(8.1.18) for each $j \in \{0, \ldots, k-1\}$ and each simplex $\Delta \in \mathcal{M}$, if $\delta_j \circ h \not\equiv \delta_{j+1} \circ h$ on Δ, then $\delta_j(h(w)) < \delta_{j+1}(h(w))$ for some vertex w of Δ,

and by Corollary 6.5,

(8.1.19) $(\mathcal{M}, h)$ is a strict $\mathcal{C}^q$-triangulation orthogonally q-flat along simplexes.

Define a homeomorphism

$$\Phi^* : [\delta_0 \circ h, \delta_k \circ h] \to [\alpha_0, \alpha_r]$$

by

(8.1.20) $$\Phi^*(u, \xi_n) := (h(u), \varphi(h(u), \xi_n)) = (h(u), \varphi^*(u, \xi_n)).$$

Then

(8.1.21) the sequence $\theta_j \circ h$ $(j \in \{0, \ldots, k\})$ is a refinement of $\alpha_0 \circ h, \ldots, \alpha_r \circ h$;

(8.1.22) $\mathcal{L}^* := \{\overline{(\delta_j \circ h, \delta_{j+1} \circ h)} : j \in \{0, \ldots, k-1\}\}$ is a family of capsules in R^n such that $\{\Phi^*(L^*) : L^* \in \mathcal{L}^*\} = \{\Phi(L) : L \in \mathcal{L}\}$ is a refinement of $\mathcal{K}$.

Put $\mathcal{L}_1^* := \{L^* \in \mathcal{L}^* : \Phi^*(L^*) \subset K \text{ for some } K \in \mathcal{K}_1\}$. Then

(8.1.23) for any $L^* \in \mathcal{L}_1^*$, $\Phi^*|\mathring{L}^*$ and $f \circ \Phi^*|\mathring{L}^*$ are of class $\mathcal{C}^{q_1}$ (by (5.1.6) and (8.1.16)), $\frac{\partial \varphi^*}{\partial \xi_n} > 0$ on $\mathring{L}^*$ and all the derivatives $\frac{\partial^\sigma(\Phi^*|\mathring{L}^*)}{\partial \xi_n^\sigma}$, $\frac{\partial^\sigma(f \circ \Phi^*|\mathring{L}^*)}{\partial \xi_n^\sigma}$, where $\sigma \in \{1, \ldots, p\}$, have continuous extensions by zero to L^*;

(8.1.24) for any $L^* \in \mathcal{L}^* \setminus \mathcal{L}_1^*$, $\Phi^*|L^*$ is of class $\mathcal{C}^q$, by (5.1.6) and (8.1.16), $\frac{\partial \varphi^*}{\partial \xi_n} > 0$ on $\mathring{L}^*$ and the derivatives $\frac{\partial^\sigma(\Phi^*|L^*)}{\partial \xi_n^\sigma}$ for $\sigma \in \{1, \ldots, p\}$ are equal to zero on ∂L^*;

(8.1.25) if $L^* \in \mathcal{L}_1^*$, $b \in \partial L^*$ and $\Phi^*(L^*) \subset K \in \mathcal{K}_1$ and $f|K$ is of class $\mathcal{C}^{q_1}$ in a neighborhood of $\Phi^*(b)$ in K, then $\Phi^*|L^*$ and $f \circ \Phi^*|L^*$ are of class $\mathcal{C}^{q_1}$ in a neighborhood of b in L^*.

Now we want to replace the $\mathcal{C}^q$-functions $\delta_j \circ h$ by continuous PL-functions defined on D by using Lemma 5.2. Therefore we want to find continuous PL-functions, affine when restricted to any simplex $S \in \mathcal{M}$,

$$\eta_0 \leq \cdots \leq \eta_k : D \to R,$$

such that for each $j \in \{0, \dots, k-1\}$,

$$\{u \in D : (\delta_j \circ h)(u) = (\delta_{j+1} \circ h)(u)\} = \{u \in D : \eta_j(u) = \eta_{j+1}(u)\}. \tag{8.1.26}$$

For any continuous function $\beta : D \to R$ define a continuous PL-function $\beta^\sharp : D \to R$ by

$$\beta^\sharp(\lambda_0 v_0 + \dots + \lambda_s v_s) := \lambda_0 \beta(v_0) + \dots + \lambda_s \beta(v_s),$$

where $(v_0, \dots, v_s) \in \mathcal{M}$ is a simplex with vertices $v_0, \dots, v_s$ and where $\lambda_0, \dots, \lambda_s \geq 0$ and $\lambda_0 + \dots + \lambda_s = 1$.

In view of (8.1.17) and (8.1.18),

$$\begin{aligned} \delta_j \circ h(u) < \delta_{j+1} \circ h(u) &\iff \theta_j \circ h(u) < \theta_{j+1} \circ h(u) \\ &\iff (\theta_j \circ h)^\sharp(u) < (\theta_{j+1} \circ h)^\sharp(u), \end{aligned} \tag{8.1.27}$$

for any $u \in D$ and $j \in \{0, \dots, k-1\}$.

By (8.1.27), $(\theta_j \circ h)^\sharp$ are continuous PL-functions, affine on simplexes and satisfying (8.1.26). However, they might not be a refinement of $\alpha_0, \dots, \alpha_r$, so some improvement is necessary.

By (8.1.21), $(\theta_j \circ h)^\sharp$ $(j \in \{0, \dots, k\})$ are a refinement of $(\alpha_i \circ h)^\sharp$ $(i \in \{0, \dots, r\})$. By (8.1.14) and (8.1.27), for each $i \in \{0, \dots, r-1\}$,

$$\begin{aligned} \{u \in D : (\alpha_i \circ h)^\sharp(u) = (\alpha_{i+1} \circ h)^\sharp(u)\} &= \{u \in D : (\alpha_i \circ h)(u) = (\alpha_{i+1} \circ h)(u)\} \\ &= \{u \in D : \alpha_i(u) = \alpha_{i+1}(u)\}. \end{aligned}$$

This shows that we can define homeomorphisms

$$\begin{aligned} &H_i : [(\alpha_i \circ h)^\sharp, (\alpha_{i+1} \circ h)^\sharp] \to [\alpha_i, \alpha_{i+1}], \\ &H_i\big(u, \tau((\alpha_{i+1} \circ h)^\sharp(u) - (\alpha_i \circ h)^\sharp(u)) + (\alpha_i \circ h)^\sharp(u)\big) \\ &\qquad\qquad = (u, \tau(\alpha_{i+1}(u) - \alpha_i(u)) + \alpha_i(u)), \end{aligned}$$

where $\tau \in [0, 1]$, $i \in \{0, \dots, r-1\}$. Gluing them together gives a homeomorphism

$$H := \bigcup_{i=0}^{r-1} H_i : [(\alpha_0 \circ h)^\sharp, (\alpha_r \circ h)^\sharp] \to [\alpha_0, \alpha_r]$$

strictly increasing with respect to the last variable. Finally, we put

$$\eta_j := (H((\theta_j \circ h)^\sharp))^\sharp \quad (j \in \{0, \dots, k\});$$

these functions refine $\alpha_0, \dots, \alpha_r$, according to (8.1.10). ∎

Corollary 8.2. *In addition to the properties stated in Proposition 8.1, the homeomorphism Ψ can be found such that $\Psi(\Gamma) = \Gamma$ for any face Γ of any $K \in \mathcal{K}$.*

Proof. Indeed, it is enough to assume that the polyhedral complex $\mathcal{P}$ is such that for each $P \in \mathcal{P}$ each of the functions α_i is affine on P. ∎

The next proposition contains the inductive step towards preparation of partial derivatives of the triangulating homeomorphism in order to use the Basic $\mathcal{C}^p$-Extension Lemma 5.4.

Proposition 8.3. *Assume that the Main Theorem is true in dimensions $< n$, where $n \geq 2$. Let $k \in \{0, \dots, n-2\}$, $l \in \{k, \dots, n-2\}$, $m \in \{0, \dots, p-1\}$, $q \in \mathbb{N}$ and $q \geq p - m + 1$. Let*

$$\sigma_0 \leq \dots \leq \sigma_r : D \to R \quad (r \geq 1)$$

be a sequence of continuous PL*-functions defined on a convex polyhedron D in R^l of dimension l such that*

$$\mathcal{K} := \left\{\overline{(\sigma_j, \sigma_{j+1})} : j \in \{0, \dots, r-1\}\right\}$$

is a family of convex PL*-capsules in R^{l+1}. Let $\mathcal{K}_1 \subset \mathcal{K}$ be a subfamily of capsules such that for each $K \in \mathcal{K}_1$ there is a regular θ-cell $(C(K), \theta(C(K)))$ in R^n such that $\pi^n_{l+1}(C(K)) = K$ and $(C(K), \theta(C(K)))$ is (k, f, q)-proper and (l, m)-prepared.*

Then there exist

(8.3.1) *a sequence of continuous* PL*-functions*

$$\eta_0 \leq \dots \leq \eta_s : D \to R \quad (s \geq r)$$

refining $\sigma_0 \leq \dots \leq \sigma_r$ such that $\mathcal{L} := \{\overline{(\eta_j, \eta_{j+1})} : j \in \{0, \dots, s-1\}\}$ is a family of convex PL*-capsules refining $\mathcal{K}$, and*

(8.3.2) *a homeomorphism $\Psi : [\sigma_0, \sigma_r] \to [\sigma_0, \sigma_r]$ with $\Psi(\Gamma) = \Gamma$ for any face Γ of any $K \in \mathcal{K}$,*

such that

(8.3.3) *$\Psi|L$ is of class $\mathcal{C}^q$ for any $L \in \mathcal{L}$ such that $L \subset K \in \mathcal{K} \setminus \mathcal{K}_1$,*

(8.3.4) *$(\Psi^{-1}(C(K))|L, \Psi^{-1}(\theta(C(K)))|L)$, where $L \in \mathcal{L}$, $L \subset K \in \mathcal{K}_1$, is a family of regular θ-cells which are $(k, (f \circ \Psi, \Psi), q - p + m)$-proper and $(l, m+1)$-prepared.*

Proof. In this proof we focus on the case where $\mathcal{K} = \mathcal{K}_1 = \{K\}$ is just one PL-capsule. In the general case the argument given below should be applied to every PL-capsule $K \in \mathcal{K}_1$ separately, while Proposition 8.1 applied later in the proof will take care of the capsules $K \in \mathcal{K} \setminus \mathcal{K}_1$. Put $C := C(K)$. Clearly, $\pi^n_{l+1}(\theta(C)) \subset \partial K$.

For any $\varkappa = (\varkappa_{l+2}, \dots, \varkappa_n)$ such that $|\varkappa| = \varkappa_{l+2} + \dots + \varkappa_n \in \{1, \dots, p - m - 1\}$ and any $j \in \{l+1, \dots, n-1\}$, the following functions defined on $K \setminus \pi^n_{l+1}(\theta(C))$ are all continuous:

(8.3.5)

$$(x', x_{l+1}) \mapsto \sup\left\{\left|\frac{\partial^{m+1+|\varkappa|} f}{\partial x_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \dots \partial x_n^{\varkappa_n}}(x', x_{l+1}, \dots, x_n)\right| : (x', \dots, x_n) \in C\right\},$$

(8.3.6)
$$(x', x_{l+1}) \mapsto \sup\left\{\left|\frac{\partial^{m+1+|\varkappa|}\alpha_j}{\partial x_{l+1}^{m+1}\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_j^{\varkappa_j}}(x', x_{l+1}, \dots, x_j)\right| : (x', \dots, x_j) \in \pi_j^n(C)\right\},$$

(8.3.7)
$$(x', x_{l+1}) \mapsto \sup\left\{\left|\frac{\partial^{m+1+|\varkappa|}\beta_j}{\partial x_{l+1}^{m+1}\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_j^{\varkappa_j}}(x', x_{l+1}, \dots, x_j)\right| : (x', \dots, x_j) \in \pi_j^n(C)\right\}.$$

By Proposition 2.5, Remark 2.6 and Lemma 2.7, there exists a refinement $\mathcal{K}'$ of $\mathcal{K}$ into convex PL-capsules such that on each $K' \in \mathcal{K}'$ each of the functions (8.3.5)–(8.3.7) is either bounded by 2 from above (**case (I)**) or bounded by 1 from below (**case (II)**). Without any loss of generality we will assume that $\mathcal{K}' = \mathcal{K} = \{K\}$; hence, for each of the functions (8.3.5)–(8.3.7) we have either case (I) or case (II) on K.

Observe that in case (II) for (8.3.5) we can have detectors on $K \setminus \pi_{l+1}^n(\theta(C))$ (cf. Proposition 3.3) which are $\mathcal{C}^q$-mappings, because α_j, β_j $(j \in \{l+1, \dots, n-1\})$ are of class $\mathcal{C}^q$ on $\pi_j^n(C) \setminus \pi_j^n(\theta(C))$. Hence, there are a finite number of $\mathcal{C}^q$-mappings $\{\omega_\mu\}_\mu = \{(\omega_{\mu,l+2}, \dots, \omega_{\mu,n})\}_\mu$ such that

$$K \setminus \pi_{l+1}^n(\theta(C)) \ni (x', x_{l+1}) \mapsto (x', x_{l+1}, \omega_\mu(x', x_{l+1})) \in C$$

and

$$\left|\frac{\partial^{m+1+|\varkappa|} f}{\partial x_{l+1}^{m+1}\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_n^{\varkappa_n}}(x', x_{l+1}, x_{l+2}, \dots, x_n)\right| \leq 2\max_\mu \left|\frac{\partial^{m+1+|\varkappa|} f}{\partial x_{l+1}^{m+1}\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_n^{\varkappa_n}}(x', x_{l+1}, \omega_\mu(x', x_{l+1}))\right| \tag{8.3.8}$$

for any $(x', x_{l+1}, x_{l+2}, \dots, x_n) \in C \setminus \theta(C)$.

Similarly, if for (8.3.6) (respectively, (8.3.7)) we have case (II), then

$$\left|\frac{\partial^{m+1+|\varkappa|}\alpha_j}{\partial x_{l+1}^{m+1}\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_j^{\varkappa_j}}(x', x_{l+1}, x_{l+2}, \dots, x_j)\right| \leq 2\max_\mu \left|\frac{\partial^{m+1+|\varkappa|}\alpha_j}{\partial x_{l+1}^{m+1}\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_j^{\varkappa_j}}(x', x_{l+1}, \omega_\mu(x', x_{l+1}))\right| \tag{8.3.9}$$

(respectively, with β_j in place of α_j) for any $(x', x_{l+1}, \dots, x_j) \in \pi_j^n(C) \setminus \pi_j^n(\theta(C))$.

Notice that since

$$\alpha_j(x', x_{l+1}, \omega_{\mu,l+2}(x', x_{l+1}), \dots, \omega_{\mu j}(x', x_{l+1})) \leq \omega_{\mu,j+1}(x', x_{l+1}) \leq \beta_j(x', x_{l+1}, \omega_{\mu,l+2}(x', x_{l+1}), \dots, \omega_{\mu j}(x', x_{l+1})),$$

and $\alpha_j = \beta_j$ on $\pi_j^n(\theta(C))$, all ω_μ have continuous extensions to $\pi_{l+1}^n(\theta(C))$.

The next step is to apply Proposition 8.1 to all the following mappings of class $\mathcal{C}^{q-p+m+1}$ on $K \setminus \pi^n_{l+1}(\theta(C))$ and extending continuously by zero to $\pi^n_{l+1}(\theta(C))$:

$$(x', x_{l+1}) \mapsto \frac{\partial^{|\varkappa|} f}{\partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}}(x', x_{l+1}, \omega_\mu(x', x_{l+1})) \in R^d,$$

$$(x', x_{l+1}) \mapsto \frac{\partial^{|\varkappa|} \alpha_j}{\partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_j^{\varkappa_j}}(x', x_{l+1}, \omega_\mu(x', x_{l+1})) \in R,$$

$$(x', x_{l+1}) \mapsto \frac{\partial^{|\varkappa|} \beta_j}{\partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_j^{\varkappa_j}}(x', x_{l+1}, \omega_\mu(x', x_{l+1})) \in R$$

and all $(x', x_{l+1}) \mapsto \omega_{\mu j}(x', x_{l+1}) \in R$. Hence, denoting $K = [\sigma_0, \sigma_1]$, where $\sigma_0 \le \sigma_1 : D \to R$, there exists a strict $\mathcal{C}^q$-triangulation $(\mathcal{M}, h)$ of D such that $|\mathcal{M}| = D$, there exists a refinement

$$\sigma_0 = \eta_0 \le \eta_1 \le \cdots \le \eta_s = \sigma_1 : D \to R \quad \text{of } \sigma_0 \le \sigma_1$$

such that $\mathcal{L} := \{\overline{(\eta_\nu, \eta_{\nu+1})} : \nu \in \{0, \ldots, s-1\}\}$ is a family of convex PL-capsules refining K, and there exists a homeomorphism $\Psi : K \to K$ of the form $\Psi(u, \xi_{l+1}) = (h(u), \psi(u, \xi_{l+1}))$ such that $\Psi(\Gamma) = \Gamma$ for any face Γ of K, Ψ is of class $\mathcal{C}^{q-p+m+1}$ on $L \setminus L \cap \Psi^{-1}(\pi^n_{l+1}(\theta(C)))$ for each $L \in \mathcal{L}$, and all the mappings

$$\frac{\partial^\rho(\Psi|\mathring{L})}{\partial \xi_{l+1}^\rho}, \quad \frac{\partial^\rho(\omega_\nu \circ \Psi|\mathring{L})}{\partial \xi_{l+1}^\rho} \quad (\rho \in \{1, \ldots, q-p+m\}),$$

$$\mathring{L} \ni (u, \xi_{l+1}) \mapsto \frac{\partial^{m+1}}{\partial \xi_{l+1}^{m+1}}\left[\frac{\partial^{|\varkappa|} f}{\partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}}(h(u), \psi(u, \xi_{l+1}), \omega_\nu(h(u), \psi(u, \xi_{l+1})))\right]$$

extend continuously by zero to $\partial L \supset L \cap \Psi^{-1}(\theta(C)) = \Psi^{-1}(\theta(C))|L$.

Hence, if we have case (II) for (8.3.5), then in view of our assumption of (l, m)-preparation, for any $(u, \xi_{l+1}) \in \mathring{L}$,

$$\begin{aligned}
&\frac{\partial^{m+1}}{\partial \xi_{l+1}^{m+1}}\left[\frac{\partial^{|\varkappa|} f}{\partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}}(h(u), \psi(u, \xi_{l+1}), \omega_\nu(h(u), \psi(u, \xi_{l+1})))\right] \\
&\quad = \frac{\partial^{m+1+|\varkappa|} f}{\partial x_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}}(h(u), \psi(u, \xi_{l+1}), \omega_\nu(h(u), \psi(u, \xi_{l+1})))\left[\frac{\partial \psi}{\partial \xi_{l+1}}\right]^{m+1} \\
&\qquad + \text{a function extending continuously by zero to } \partial L.
\end{aligned}$$

Consequently,

$$\frac{\partial^{m+1+|\varkappa|} f}{\partial x_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}}(\Psi(u, \xi_{l+1}), \omega_\nu(\Psi(u, \xi_{l+1})))\left[\frac{\partial \psi}{\partial \xi_{l+1}}(u, \xi_{l+1})\right]^{m+1}$$

extends continuously by zero to ∂L, and by (8.3.8),

$$\frac{\partial^{m+1+|\varkappa|} f}{\partial x_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}}(\Psi(u, \xi_{l+1}), x_{l+2}, \ldots, x_n)\left[\frac{\partial \psi}{\partial \xi_{l+1}}(u, \xi_{l+1})\right]^{m+1}$$

for $(u, \xi_{l+1}, x_{l+2}, \ldots, x_n) \in \Psi^{-1}(C)|L \setminus \Psi^{-1}(\theta(C))|L$ extends continuously by zero to $\Psi^{-1}(\theta(C))|L$.

On the other hand, by (l, m)-preparation,

$$\begin{aligned}(8.3.10)\qquad & \frac{\partial^{m+1+|\varkappa|}}{\partial \xi_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}} [f(\Psi(u, \xi_{l+1}), x_{l+2}, \ldots, x_n)] \\ &= \frac{\partial^{m+1+|\varkappa|} f}{\partial x_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}} (\Psi(u, \xi_{l+1}), x_{l+2}, \ldots, x_n) \left[\frac{\partial \psi}{\partial \xi_{l+1}} (u, \xi_{l+1}) \right]^{m+1} \\ &\quad + \text{a function extending continuously by zero to } \Psi^{-1}(\theta(C))|L.\end{aligned}$$

It follows that

$$\frac{\partial^{m+1+|\varkappa|}(f \circ \Psi)}{\partial \xi_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}} (u, \xi_{l+1}, x_{l+2}, \ldots, x_n)$$

extends continuously by zero to $\Psi^{-1}(\theta(C))|L$.

If we have case (I) for (8.3.5), there is no need to employ detectors. Again by (l, m)-preparation we can write (8.3.10), from which we conclude that

$$\frac{\partial^{m+1+|\varkappa|}(f \circ \Psi)}{\partial \xi_{l+1}^{m+1} \partial x_{l+2}^{\varkappa_{l+2}} \ldots \partial x_n^{\varkappa_n}} (u, \xi_{l+1}, x_{l+2}, \ldots, x_n)$$

extends continuously by zero to $\Psi^{-1}(\theta(C))|L$, because of the factor

$$\left[\frac{\partial \psi}{\partial \xi_{l+1}} (u, \xi_{l+1}) \right]^{m+1}.$$

A similar argument concerns the functions α_j and β_j. The proof of Proposition 8.3 is now complete. ∎

In the next proposition we describe the procedure of passing from l to $l - 1$.

Proposition 8.4. *Assume that $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0, \ldots, n-1\}$, $l \in \{k+1, \ldots, n\}$, $q \in \mathbb{N}$ and $q \geq p$. Let $\sigma_0 \leq \cdots \leq \sigma_r : D \to R$ $(r \geq 1)$ be a sequence of continuous* PL*-functions defined on a convex polyhedron D in R^l of dimension l such that*

$$\mathcal{K} := \left\{\overline{(\sigma_i, \sigma_{i+1})} : i \in \{0, \ldots, r-1\}\right\}$$

is a family of convex PL*-capsules in R^{l+1}. Let $\mathcal{K}_1 \subset \mathcal{K}$ be a subfamily of capsules such that for each $K \in \mathcal{K}_1$ there is a regular θ-cell $(C(K), \theta(C(K)))$ in R^n such that $\pi^n_{l+1}(C(K)) = K$ and $(C(K), \theta(C(K)))$ is (k, f, q)-proper and (l, p)-prepared.*

Then, after an arbitrarily small linear change of coordinates in R^l, there exists a sequence $\delta_0 \leq \cdots \leq \delta_s : D' \to R$ $(s \geq 1)$ of continuous PL*-functions defined on a convex polyhedron D' in R^{l-1} such that*

$$\mathcal{K}' = \left\{\overline{(\delta_j, \delta_{j+1})} : j \in \{0, \ldots, s-1\}\right\}$$

is a family of convex PL*-capsules in R^l such that*

(8.4.1) $\bigcup \mathcal{K}' = D'$;

(8.4.2) *for each* $K' \in \mathcal{K}'$ *and* $i \in \{0, \dots, r\}$, σ_i *is affine on* K';

(8.4.3) $\mathcal{N} := \{\overline{(\sigma_i|K', \sigma_{i+1}|K')} : K' \in \mathcal{K}', \sigma_i < \sigma_{i+1}$ *on* $\mathring{K}'\}$ *is a family of regular cells in* R^l, *refining* $\mathcal{K}$ *and such that for each* $K \in \mathcal{K}_1$ *and* $N \in \mathcal{N}$, *if* $N \subset K$, *then* $(C(K)|N, \theta(C(K))|N)$ *is* (k, f, q)*-proper and* $(l-1, 0)$*-prepared.*

Proof. Let $\mathcal{T}$ be a simplicial (or any polyhedral) complex in R^l such that $|\mathcal{T}| = D$ and each σ_i is affine on each $\Delta \in \mathcal{T}$. Then, for each $i \in \{0, \dots, r-1\}$ and $\Delta \in \mathcal{T}$, either $\sigma_i < \sigma_{i+1}$ on $\mathring{\Delta}$ or $\sigma_i \equiv \sigma_{i+1}$ on Δ. Then all $\Delta \in \mathcal{T}$ will become convex PL-capsules in R^l, after perhaps a small linear change of coordinates in R^l. Now we use Remark 2.3. ■

As a result of iterative application of Proposition 8.3 interlaced with Proposition 8.4 we get the required $\mathcal{C}^p$-extension, but the final homeomorphism is not defined on a polyhedron of a simplicial complex, but on some regular cell in R^n, and now the task is to build a strict $\mathcal{C}^p$-triangulation of this regular cell through a simplicial complex refining the initial polyhedral complex and in such a way that the triangulating homeomorphism composed with the previous one transforms any face of any initial polyhedron onto itself. Roughly speaking, this is done by the induction hypothesis that our theorems are true for $n-1$ and precisely described in the proof of Proposition 8.5 below. Recall that for any convex polyhedron P in R^n and $k \in \{0, \dots, n\}$, $P^{(k)}$ stands for the k-dimensional skeleton of P.

Proposition 8.5. *Assume that the Main Theorem is true in dimensions* $< n$, *where* $n \geq 2$. *Fix integers* $k \in \{0, \dots, n-1\}$, $q \geq (n-1-k)\binom{p}{2} + p + 1$ *and* $\tilde{q} \geq q$. *Let* $\mathcal{P}$ *be a polyhedral complex in* R^n *such that* $D_n := |\mathcal{P}|$ *is a convex polyhedron of dimension* n *and let* $\mathcal{P}_1 \subset \mathcal{P}$ *be such that* $|\mathcal{P}_1|$ *is of constant dimension* n. *Assume that* $f : |\mathcal{P}_1| \to R^d$ *is a continuous mapping such that each* $P \in \mathcal{P}_1$ *is* (k, f, q)*-well situated in* R^n.

Then there exists a $\mathcal{C}^p$*-triangulation* $(\mathcal{T}, h)$ *of* D_n *such that* $\mathcal{T}$ *is a refinement of* $\mathcal{P}$; *for each* $\Delta \in \mathcal{T}$ *of dimension* n, *if* $\Delta \subset P \in \mathcal{P}_1$, *then* $(f \circ h, h)|\Delta \setminus \Delta^{(k-1)}$ *is of class* $\mathcal{C}^p$; *if* $\Delta \subset P \in \mathcal{P} \setminus \mathcal{P}_1$, *then* $h|\Delta$ *is of class* $\mathcal{C}^{\tilde{q}}$; *and finally* $h(\Gamma) = \Gamma$ *for any face* Γ *of any* $P \in \mathcal{P}$.

Proof. After an arbitrarily small linear change of coordinates, all $P \in \mathcal{P}$ become PL-capsules, so by Remark 2.6 and Lemma 2.7, there exists a sequence of continuous PL-functions

$$\sigma_{n-1,0} \leq \cdots \leq \sigma_{n-1,r_{n-1}} : D_{n-1} \to R \quad (r_{n-1} \geq 1),$$

where $D_{n-1} = \pi^n_{n-1}(D_n)$, such that

$$\mathcal{K}_n := \left\{\overline{(\sigma_{n-1,j}, \sigma_{n-1,j+1})} : j \in \{0, \dots, r_{n-1}-1\}\right\}$$

is a family of convex capsules refining $\mathcal{P}$. By Proposition 8.1, there exists a sequence of continuous PL-functions

$$\eta_{n-1,0} \leq \cdots \leq \eta_{n-1,s_{n-1}} : D_{n-1} \to R \quad (s_{n-1} \geq 1)$$

and a homeomorphism $\Psi_n : D_n \to D_n$ such that

$$\mathcal{L}_n := \{\overline{(\eta_{n-1,\nu}, \eta_{n-1,\nu+1})} : \nu \in \{0, \dots, s_{n-1} - 1\}\}$$

is a family of capsules refining $\mathcal{K}_n$, $\Psi_n(\Lambda) = \Lambda$ for any face Λ of any $K_n \in \mathcal{K}_n$ (hence also $\Psi_n(\Lambda) = \Lambda$ for any face Λ of any $P \in \mathcal{P}$), $\Psi_n|L_n$ is of class $\mathcal{C}^{\tilde{q}}$ for any $L_n \in \mathcal{L}_n$ such that $L_n \subset P \in \mathcal{P} \setminus \mathcal{P}_1$, while for each $L_n \in \mathcal{L}_n$ such that $L_n \subset P \in \mathcal{P}_1$, $(L_n, \Sigma(P) \cap L_n)$ is a regular $(k, (f \circ \Psi_n, \Psi_n), q)$-proper θ-cell in R^n and

$$\frac{\partial^\rho (\Psi_n|\mathring{L})}{\partial x_n^\rho}, \quad \frac{\partial^\rho (f \circ \Psi_n|\mathring{L})}{\partial x_n^\rho}, \quad \text{for } \rho \in \{1, \dots, q-1\},$$

extend continuously by zero to $\partial L_n \supset \Sigma(P) \cap L_n$.

After a small linear change of coordinates in R^{n-1}, we can find a sequence of continuous PL-functions

$$\sigma_{n-2,0} \le \dots \le \sigma_{n-2,r_{n-2}} : D_{n-2} \to R \quad (r_{n-2} \ge 1),$$

where $D_{n-2} = \pi^n_{n-2}(D_n)$, such that

$$\mathcal{K}_{n-1} := \{\overline{(\sigma_{n-2,j}, \sigma_{n-2,j+1})} : j \in \{0, \dots, r_{n-2} - 1\}\}$$

is a family of convex PL-capsules in R^{n-1} such that every $\eta_{n-1,\nu}$ is affine on each $K_{n-1} \in \mathcal{K}_{n-1}$.

Notice that $(L_n|K_{n-1}, \theta(L_n)|K_{n-1})$, where $\theta(L_n) := \Sigma(P) \cap L_n$, $L_n \subset P \in \mathcal{P}_1$, is a $(k, (f \circ \Psi_n, \Psi_n), q)$-proper θ-cell, $(n-1, p)$-prepared.

Now, using Proposition 8.3 interlaced with Proposition 8.4, we continue by descending induction defining sequences $\mathcal{L}_{n-1}, \mathcal{K}_{n-2}, \mathcal{K}_{n-3}, \mathcal{L}_{n-3}, \dots, \mathcal{K}_{k+1}, \mathcal{L}_{k+1}$ of families of convex PL-capsules and homeomorphisms

$$\Psi_{n-1} : D_{n-1} \to D_{n-1}, \dots, \Psi_{k+1} : D_{k+1} \to D_{k+1} \quad (\text{where } D_i := \pi^n_i(D_n))$$

such that

$$\mathcal{K}_i := \{\overline{(\sigma_{i-1,j}, \sigma_{i-1,j+1})} : j \in \{0, \dots, r_{i-1} - 1\}\},$$
$$\mathcal{L}_i := \{\overline{(\eta_{i-1,\nu}, \eta_{i-1,\nu+1})} : \nu \in \{0, \dots, s_{i-1} - 1\}\},$$

$\mathcal{L}_i$ refines $\mathcal{K}_i$, $|\mathcal{K}_i| = D_i$, every $\eta_{i-1,\nu}$ is affine on each $K_{i-1} \in \mathcal{K}_{i-1}$, and $\Psi_i(\Gamma) = \Gamma$ for any face Γ of any $K_i \in \mathcal{K}_i$. Moreover,

$$\Big(\Psi_i^{-1}\big(\dots(\Psi_{n-1}^{-1}(L_n)|L_{n-1})\dots\big)|L_i, \Psi_i^{-1}\big(\dots(\Psi_{n-1}^{-1}(\theta(L_n))|L_{n-1})\dots\big)|L_i\Big)$$

is a regular θ-cell, $\big(k, (f \circ \Psi_n \circ \Psi_{n-1} \circ \dots \circ \Psi_i, \Psi_n \circ \Psi_{n-1} \circ \dots \circ \Psi_i), q - (n-i)\binom{p}{2}\big)$-proper and $(i-1, p)$-prepared, whenever

$$L_i \subset \pi^{i+1}_i(L_{i+1}), L_{i+1} \subset \pi^{i+2}_{i+1}(L_{i+2}), \dots, L_{n-1} \subset \pi^n_{n-1}(L_n), L_n \subset P \in \mathcal{P}_1,$$

while if

$$L_i \subset \pi_i^{i+1}(L_{i+1}), L_{i+1} \subset \pi_{i+1}^{i+2}(L_{i+2}), \ldots, L_{n-1} \subset \pi_{n-1}^n(L_n), L_n \subset P \in \mathcal{P} \setminus \mathcal{P}_1,$$

then $\Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_i$ is of class $\mathcal{C}^{\tilde{q}}$ on

$$\Psi_i^{-1}\big(\ldots(\Psi_{n-1}^{-1}(L_n)|L_{n-1})\ldots\big)|L_i.$$

(In fact, each Ψ_i $(i \in \{n-1, \ldots, k+1\})$ is a composition of appropriate p homeomorphisms.)

It will also be convenient to take a family $\mathcal{K}_k$ of convex PL-capsules in R^k partitioning $D_k = \pi_k^n(D_n)$ in such a way that every $\eta_{k\nu}$ is affine on each $K_k \in \mathcal{K}_k$ and then for any $K_k \in \mathcal{K}_k$ we have either $\eta_{k\nu} < \eta_{k,\nu+1}$ on $\mathring{K}_k$ or $\eta_{k\nu} \equiv \eta_{k,\nu+1}$ on K_k.

Now we apply the Basic $\mathcal{C}^p$-Extension Lemma 5.4. It follows that the mapping

$$(f \circ \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_{k+1}, \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_{k+1}),$$

when restricted to

$$\Psi_{k+1}^{-1}\big(\ldots(\Psi_{n-1}^{-1}(L_n)|L_{n-1})\ldots\big)|L_{k+1},$$

is of class $\mathcal{C}^p$ on

$$\Psi_{k+1}^{-1}\big(\ldots(\Psi_{n-1}^{-1}(\theta(L_n))|L_{n-1})\ldots\big)|L_{k+1},$$

off an exceptional closed subset of the latter of dimension $< k$, which can be identified with a closed subset of ∂L_{k+1}. Hence, by our induction assumption, we can find a strict $\mathcal{C}^{\tilde{q}}$-triangulation $(\mathcal{T}_k, h_k)$ of D_k, compatible with (the projections π_k^{k+1} to D_k of) all these exceptional subsets. Additionally, we assume that $\mathcal{T}_k$ is a refinement of $\mathcal{K}_k$ and h_k is orthogonally $\tilde{q}$-flat along simplexes (see Section 6), and $h_k(\Lambda) = \Lambda$ for any face Λ of any $K_k \in \mathcal{K}_k$.

It follows that if $T_k \in \mathcal{T}_k$, $\dim T_k = k$ and

$$h_k(T_k) \subset \pi_k^{k+1}\Big(\Psi_{k+1}^{-1}\big(\ldots(\Psi_{n-1}^{-1}(\theta(L_n))|L_{n-1})\ldots\big)|L_{k+1}\Big)$$

then

$$(f \circ \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_{k+1} \circ h_k, \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_{k+1} \circ h_k),$$

when restricted to

$$h_k^{-1}\Big(\Psi_{k+1}^{-1}\big(\ldots(\Psi_{n-1}^{-1}(L_n)|L_{n-1})\ldots\big)|L_{k+1}\Big)|\mathring{T}_k,$$

is of class $\mathcal{C}^p$ on

$$h_k^{-1}\Big(\Psi_{k+1}^{-1}\big(\ldots(\Psi_{n-1}^{-1}(\theta(L_n))|L_{n-1})\ldots\big)|L_{k+1}\Big)|\mathring{T}_k.$$

Now we want to extend the triangulation $(\mathcal{T}_k, h_k)$ to a $\mathcal{C}^{\tilde{q}}$-triangulation of the domain of the homeomorphism $\Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_{k+1} \circ h_k$ which is a regular cell in R^n. To this end, we put

$$\tilde{\Psi}_{i+1} := \Psi_{i+1} \circ \cdots \circ \Psi_{k+1} \circ h_k \quad \text{for } i \in \{k, \ldots, n-1\}.$$

Then $\tilde{\Psi}_{i+1} : \tilde{D}_{i+1} \to D_{i+1}$, where

$$\tilde{D}_{i+1} := \{(u, \xi_{k+1}, \ldots, \xi_{i+1}) \in D_k \times R^{i+1-k} : (\tilde{\Psi}_i(u, \xi_{k+1}, \ldots, \xi_i), \xi_{i+1}) \in D_{i+1}\}$$

and $\tilde{\Psi}_k := h_k$.

For each $i \in \{k, \ldots, n-1\}$, we will define a strict $\mathcal{C}^{\tilde{q}}$-triangulation $(\mathcal{T}_{i+1}, h_{i+1})$, where $h_{i+1} : D_{i+1} \to \tilde{D}_{i+1}$, such that $\mathcal{T}_{i+1}$ is a refinement of $\mathcal{L}_{i+1}$, $(\tilde{\Psi}_{i+1} \circ h_{i+1})(\Gamma) = \Gamma$ for any face Γ of any $K_{i+1} \in \mathcal{K}_{i+1}$, and moreover $\tilde{\Psi}_{i+1} \circ h_{i+1}$ is of class $\mathcal{C}^{\tilde{q}}$ when $i \in \{k, \ldots, n-2\}$.

For $i = k$, we first put

(8.5.1)
$$h_{k+1}\big(u, \tau\eta_{k\nu}(u) + (1-\tau)\eta_{k,\nu+1}(u)\big) := \big(u, \tau\eta_{k\nu}(h_k(u)) + (1-\tau)\eta_{k,\nu+1}(h_k(u))\big),$$

where $\nu \in \{0, \ldots, s_k - 1\}$, $u \in D_k$ and $\tau \in [0, 1]$. Observe that the definition is correct, because if $u \in D_k$ and $\eta_{k\nu}(u) = \eta_{k,\nu+1}(u)$, then there exists a face Γ of some $K_k \in \mathcal{K}_k$ such that $u \in \mathring{\Gamma}$. Since $\eta_{k\nu} \leq \eta_{k,\nu+1}$ and $\eta_{k\nu}, \eta_{k,\nu+1}$ are affine on Γ, it follows that $\eta_{k\nu} \equiv \eta_{k,\nu+1}$ on Γ; so $\eta_{k\nu}(h_k(u)) = \eta_{k,\nu+1}(h_k(u))$ since $h_k(\Gamma) = \Gamma$.

Now observe that for any face Λ of any polyhedron $[\eta_{k\nu}, \eta_{k,\nu+1}]|K_k$, we have $(h_k \circ h_{k+1})(\Lambda) = \Lambda$. Indeed, then $\Gamma := \pi_k^{k+1}(\Lambda)$ is a face of K_k, $\Lambda = [\eta_{k\nu}, \eta_{k,\nu+1}]|\Gamma$ and $h_k(\Gamma) = \Gamma$; hence, $(h_k \circ h_{k+1})(\Lambda) = \Lambda$ in view of (8.5.1). It follows that for any face Λ of any K_{k+1}, $(\tilde{\Psi}_{k+1} \circ h_{k+1})(\Lambda) = \Psi_{k+1}(h_k \circ h_{k+1}(\Lambda)) = \Psi_{k+1}(\Lambda) = \Lambda$, because K_{k+1} is a union of some polyhedra $[\eta_{k\nu}, \eta_{k,\nu+1}]|K_k$.

In order to turn h_{k+1} into a $\mathcal{C}^{\tilde{q}}$-triangulation of $\tilde{D}_{k+1}$, we take a simplicial complex $\mathcal{T}_{k+1}$ refining all polyhedra $[\eta_{k\nu}, \eta_{k,\nu+1}]|T_k$, where $T_k \in \mathcal{T}_k$. It is clear that this triangulation is compatible with all $L_{k+1} \in \mathcal{L}_{k+1}$ and is of class $\mathcal{C}^{\tilde{q}}$ on simplexes. To make it strict $\mathcal{C}^{\tilde{q}}$ and orthogonally $\tilde{q}$-flat along simplexes we apply Corollary 6.5.

Notice that if $T_{k+1} \in \mathcal{T}_{k+1}$ is a simplex of dimension $k+1$ and $(\tilde{\Psi}_k \circ h_{k+1})(T_{k+1}) \subset L_{k+1} \in \mathcal{L}_{k+1}$ and

$$L_{k+1} \subset \pi_{k+1}^{k+2}(L_{k+2}), \ \ldots, \ L_{n-1} \subset \pi_{n-1}^{n}(L_n), \quad L_n \subset P \in \mathcal{P}_1,$$

then

$$(f \circ \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \tilde{\Psi}_{k+1} \circ h_{k+1}, \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \tilde{\Psi}_{k+1} \circ h_{k+1}),$$

when restricted to

$$h_{k+1}^{-1}\Big(\tilde{\Psi}_{k+1}^{-1}\big(\ldots(\Psi_{n-1}^{-1}(\theta(L_n))|L_{n-1})\ldots\big)\Big)|T_{k+1},$$

is of class $\mathcal{C}^p$ except possibly on the faces of T_{k+1} of dimension $< k$.

We continue by induction. Suppose we have already defined a strict $\mathcal{C}^{\tilde{q}}$-triangulation $(\mathcal{T}_{i+1}, h_{i+1})$ of $\tilde{D}_{i+1}$, where $\mathcal{T}_{i+1}$ is a refinement of all $[\eta_{i\nu}, \eta_{i,\nu+1}]|T_i$, with $T_i \in \mathcal{T}_i$, such that h_{i+1} is orthogonally $\tilde{q}$-flat along simplexes, $\tilde{\Psi}_i \circ h_{i+1}$ is compatible with all $L_{i+1} \in \mathcal{L}_{i+1}$, $\tilde{\Psi}_{i+1} \circ h_{i+1}$ is of class $\mathcal{C}^{\tilde{q}}$ if $i \in \{k, \ldots, n-2\}$, $(\tilde{\Psi}_{i+1} \circ H_{i+1})(\Gamma) = \Gamma$ for any face Γ of any $K_{i+1} \in \mathcal{K}_{i+1}$, and finally

$$(f \circ \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \tilde{\Psi}_{i+1} \circ h_{i+1}, \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \tilde{\Psi}_{i+1} \circ h_{i+1}),$$

when restricted to

$$h_{i+1}^{-1}\Big(\tilde{\Psi}_{i+1}^{-1}\big(\dots(\Psi_{n-1}^{-1}(\theta(L_n))|L_{n-1})\dots\big)\Big)|T_{i+1},$$

is of class $\mathcal{C}^p$ except possibly on the faces of T_{i+1} of dimension $< k$ when

$(\tilde{\Psi}_i \circ h_{i+1})(T_{i+1}) \subset L_{i+1}, L_{i+1} \subset \pi_{i+1}^{i+1}(L_{i+2}), \dots, L_{n-1} \subset \pi_{n-1}^{n}(L_n), L_n \subset P \in \mathcal{P}_1.$

Now we define $h_{i+2} : D_{i+2} \to \tilde{D}_{i+2}$ first by the formula

$$\begin{aligned} &h_{i+2}(w, \tau\eta_{i+1,\nu}(w) + (1-\tau)\eta_{i+1,\nu+1}(w)) \\ &\quad := \big(h_{i+1}(w), \tau\eta_{i+1\nu}(\tilde{\Psi}_{i+1}(h_{i+1}(w))) + (1-\tau)\eta_{i+1,\nu+1}(\tilde{\Psi}_{i+1}(h_{i+1}(w)))\big), \end{aligned}$$

where $w \in D_{i+1}$, $\nu \in \{0, \dots, s_{i+1}-1\}$ and $\tau \in [0,1]$. We check easily, by the same argument as for h_{k+1}, that h_{i+2}, after appropriate modifications analogous to those for h_{k+1}, satisfies all the required conditions.

Consequently, we obtain a strict $\mathcal{C}^{\tilde{q}}$-triangulation $(\mathcal{T}_n, h_n)$ with $h_n : D_n \to \tilde{D}_n$ such that $\tilde{\Psi}_{n-1} \circ h_n$ is of class $\mathcal{C}^{\tilde{q}}$, $\mathcal{T}_n$ is a refinement of $\mathcal{L}_n$ (so of $\mathcal{K}_n$ as well), $\tilde{\Psi}_{n-1} \circ h_n$ is of class $\mathcal{C}^{\tilde{q}}$, $\tilde{\Psi}_{n-1} \circ h_n$ is compatible with $\mathcal{L}_n$, $(\tilde{\Psi}_n \circ h_n)(\Gamma) = \Gamma$ for any face Γ of any face $K_n \in \mathcal{K}_n$, and

$$(f \circ (\tilde{\Psi}_n \circ h_n), \tilde{\Psi}_n \circ h_n)$$

is of class $\mathcal{C}^p$ when restricted to any $T_n \in \mathcal{T}_n$, except on the faces of T_n of dimension $< k$, assuming that $(\tilde{\Psi}_{n-1} \circ h_n)(T_n) \subset P \in \mathcal{P}_1$.

Finally, observe that if $(\tilde{\Psi}_{n-1} \circ h_n)(T_n) \subset P \in \mathcal{P} \setminus \mathcal{P}_1$, then $\tilde{\Psi}_n \circ h_n|T_n = \Psi_n \circ (\tilde{\Psi}_{n-1} \circ h_n)|T_n$ is of class $\mathcal{C}^{\tilde{q}}$. Now the proof of Proposition 8.5 is complete. ■

Corollary 8.6. *Assume that the Main Theorem is true in dimensions $< n$, where $n \geq 2$. Let $k \in \{0, \dots, n-1\}$ and let $q \geq (n-1-k)\binom{p}{2} + p + 1$ be an integer. Let $\mathcal{P}$ be a polyhedral complex in R^n such that $|\mathcal{P}|$ is a convex polyhedron of dimension n. Let $f : |\mathcal{P}| \to R^d$ be a continuous mapping such that for each $P \in \mathcal{P}$ the restriction $f|P \setminus P^{(k)}$ is of class $\mathcal{C}^q$.*

Then there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $|\mathcal{P}|$ such that $\mathcal{T}$ is a refinement of $\mathcal{P}$, $h(\Gamma) = \Gamma$ for any face Γ of any $P \in \mathcal{P}$, and for each simplex $\Delta \in \mathcal{T}$ the restrictions $h|\Delta \setminus \Delta^{(k-1)}$, $f \circ h|\Delta \setminus \Delta^{(k-1)}$ are of class $\mathcal{C}^p$.

Proof. By barycentric subdivision we reduce to the situation where for each $P \in \mathcal{P}$ of dimension n, there exists a face $\Sigma(P)$ of P of dimension k such that $f|P \setminus \Sigma(P)$ is of class $\mathcal{C}^q$. There are a finite number of orthogonal bases $\mathbf{v}_1, \dots, \mathbf{v}_s$ in R^n such that each $P \in \mathcal{P}$ is (k, f, q)-well situated in R^n with respect to some basis $\mathbf{v}_i$ ($i \in \{1, \dots, s\}$). Thus, we can represent (the set of all polyhedra of dimension n belonging to) $\mathcal{P}$ as a pairwise disjoint union

$$\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_s,$$

where each $P \in \mathcal{P}_i$ is (k, f, q)-well situated in R^n with respect to $\mathbf{v}_i$.

By Proposition 8.5, there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}_1, h_1)$ of $|\mathcal{P}|$ such that $\mathcal{T}_1$ is a refinement of $\mathcal{P}$, for each $\Delta_1 \in \mathcal{T}_1$ of dimension n, if $\Delta_1 \subset P \in \mathcal{P}_1$, then the restrictions

$$h_1|\Delta_1 \setminus \Delta_1^{(k-1)}, \quad f \circ h_1|\Delta_1 \setminus \Delta_1^{(k-1)}$$

are of class $\mathcal{C}^p$, if $\Delta_1 \subset P \in \mathcal{P} \setminus \mathcal{P}_1$, then the restriction $h_1|\Delta_1$ is of class C^q, and $h_1(\Gamma) = \Gamma$ for any face Γ of any $P \in \mathcal{P}$.

Put

$$\mathcal{T}_{1i} := \{\Delta_1 \in \mathcal{T}_1 : \dim \Delta_1 = n,\ \Delta_1 \subset P \in \mathcal{P}_i\} \quad (i \in \{1, \dots, s\}).$$

Observe that if $\Delta_1 \in \mathcal{T}_{1i}$, where $i \geq 2$ and $\Delta_1 \subset P \in \mathcal{P}_i$, then picking $\Sigma(\Delta_1)$ to be a k-dimensional face of Δ_1 which contains $\Delta_1 \cap \Sigma(P)$, we see that

$$h_1|\Delta_1 \setminus \Sigma(\Delta_1), \quad f \circ h_1|\Delta_1 \setminus \Sigma(\Delta_1)$$

are of class $\mathcal{C}^q$; hence, Δ_1 is $(k, (h_1, f \circ h_1), q)$-well situated in R^n with respect to the basis $\mathbf{v}_i$.

By Proposition 8.5, there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}_2, h_2)$ of $|\mathcal{P}|$ such that $\mathcal{T}_2$ is a refinement of $\mathcal{T}_1$, for each $\Delta_2 \in \mathcal{T}_2$ of dimension n, if $\Delta_2 \subset \Delta_1 \in \mathcal{T}_{12}$, then

$$h_1 \circ h_2|\Delta_2 \setminus \Delta_2^{(k-1)}, \quad f \circ h_1 \circ h_2|\Delta_2 \setminus \Delta_2^{(k-1)} \tag{6.8.1}$$

are of class $\mathcal{C}^p$, while if $\Delta_2 \subset \Delta_1 \in \mathcal{T}_1 \setminus \mathcal{T}_{12}$ the restriction $h_2|\Delta_2$ is of class $\mathcal{C}^q$, and $h_2(\Gamma_1) = \Gamma_1$ for any face Γ_1 of any $\Delta_1 \in \mathcal{T}_1$. Clearly, the mappings (6.8.1) are of class $\mathcal{C}^p$ when $\Delta_2 \subset \Delta_1 \in \mathcal{T}_{11}$ as well.

Put

$$\mathcal{T}_{2i} := \{\Delta_2 \in \mathcal{T}_2 : \dim \Delta_2 = n,\ \Delta_2 \subset \Delta_1 \in \mathcal{T}_{1i}\} \quad (i \in \{1, \dots, s\}).$$

Observe that if $\Delta_2 \in \mathcal{T}_{2i}$, where $i \geq 3$, then Δ_2 is $(k, (f \circ h_1 \circ h_2, h_1 \circ h_2), q)$-well situated in R^n with respect to the basis $\mathbf{v}_i$.

It is clear how to continue the above process, which at the final s-th step gives the required triangulation $(\mathcal{T}, h) = (\mathcal{T}_s, h_1 \circ \cdots \circ h_s)$. ■

Corollary 8.7. *Let p be a positive integer and let $q_1, \dots, q_n$ be integers such that*

$$q_1 \geq (n-1)\binom{p}{2} + p + 1, \quad q_2 \geq (n-2)\binom{q_1}{2} + q_1 + 1, \ \dots,$$
$$q_n \geq 0\binom{q_{n-1}}{2} + q_{n-1} + 1 = q_{n-1} + 1.$$

Let $\mathcal{P}$ be a polyhedral complex in R^n such that $|\mathcal{P}|$ is a convex polyhedron of dimension n. Let $f : |\mathcal{P}| \to R^d$ be a continuous mapping such that for each $P \in \mathcal{P}$ the restriction $f|P \setminus P^{(n-1)}$ is of class $\mathcal{C}^{q_n}$.

Then there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ such that $\mathcal{T}$ is a refinement of $\mathcal{P}$, $h(\Gamma) = \Gamma$ for any face Γ of any $P \in \mathcal{P}$, and $f \circ h$ is of class $\mathcal{C}^p$.

Proof. By Corollary 8.6 applied n times we obtain a $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $|\mathcal{P}|$ such that $\mathcal{T}$ is a refinement of $\mathcal{P}$, $h(\Gamma) = \Gamma$ for any face Γ of any $P \in \mathcal{P}$, and for each simplex $\Delta \in \mathcal{T}$ of dimension n the restrictions $h|\Delta$ and $f \circ h|\Delta$ are of class $\mathcal{C}^p$. We now improve h using Corollary 6.5. ∎

Corollary 8.7 ends the proof of the Main Theorem as well as of the Strict $\mathcal{C}^p$-Refinement Theorem, since it is classical that there exists a $\mathcal{C}^{q_n}$-triangulation $(\mathcal{T}, h)$ of $A = \mathrm{dom}(f)$ such that $f \circ h|\mathring{\Delta}$ is of class $\mathcal{C}^{q_n}$ for each simplex $\Delta \in \mathcal{T}$.

9. An application to approximation theory

Fernando and Ghiloni [7] proved the following approximation theorem.

Theorem 9.1 ([7, Corollary 1.5])**.** *Let A be a definable, closed, bounded subset of R^n and let $\mathcal{T}$ be a finite simplicial complex in R^m. Let $f : A \to |\mathcal{T}|$ be a definable continuous mapping.*

Then for any positive integer p and any $\varepsilon \in R$ such that $\varepsilon > 0$ there exists a $\mathcal{C}^p$-mapping $g : A \to |\mathcal{T}|$ such that

$$|f(x) - g(x)| \le \varepsilon \quad \textit{for all } x \in A,$$

where $|(y_1, \dots, y_m)| := (\sum_{i=1}^m y_i^2)^{1/2}$.

In fact, [7] contains the proof of Theorem 9.1 only in the semialgebraic case and $R = \mathbb{R}$ (the field of real numbers), but it is easy to check that the same proof, with obvious modifications, holds true in our general context.

The existence of strict $\mathcal{C}^p$-triangulations allows us to improve the last theorem.

Theorem 9.2. *Let A and B be any definable, closed bounded subsets of R^n and of R^m, respectively. Let $f : A \to B$ be a definable continuous mapping.*

Then for any positive integer p and any $\varepsilon \in R$ such that $\varepsilon > 0$ there exists a $\mathcal{C}^p$-mapping $g : A \to B$ such that

$$|f(x) - g(x)| \le \varepsilon \quad \textit{for all } x \in A.$$

Proof. Let $(\mathcal{T}, h)$ be a strict $\mathcal{C}^p$-triangulation of B; hence $h : |\mathcal{T}| \to B$ is a homeomorphism of class $\mathcal{C}^p$. Since h is uniformly continuous, there exists $\delta > 0$ such that for each pair $u, w \in |\mathcal{T}|$, if $|u - w| \le \delta$, then $|h(u) - h(w)| \le \varepsilon$. By Theorem 9.1 there exists a $\mathcal{C}^p$-mapping $g : A \to |\mathcal{T}|$ such that

$$|h^{-1} \circ f(x) - g(x)| \le \delta \quad \text{for all } x \in A.$$

Hence,

$$|f(x) - h \circ g(x)| \le \varepsilon \quad \text{for all } x \in A,$$

and $h \circ g : A \to B$ is of class $\mathcal{C}^p$ as a composition of two $\mathcal{C}^p$-mappings.

Acknowledgments. The author thanks the anonymous referee for valuable comments which greatly improved the previous version of the article. He is also grateful to the participants of the Singularity Seminar at the Jagiellonian University and especially to Wojciech Kucharz, Krzysztof Nowak and Artur Piękosz, who commented on the first version. Thanks are also due to Jerzy Trzeciak for his help in making final adjustments.

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