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Simplicial generation of Chow rings of matroids

Abstract. We introduce a presentation of the Chow ring of a matroid by a new set of generators, called "simplicial generators." These generators are analogous to nef divisors on projective toric varieties, and admit a combinatorial interpretation via the theory of matroid quotients. Using this combinatorial interpretation, we (i) produce a bijection between a monomial basis of the Chow ring and a relative generalization of Schubert matroids, (ii) recover the Poincaré duality property, (iii) give a formula for the volume polynomial, which we show is log-concave in the positive orthant, and (iv) recover the validity of Hodge–Riemann relations in degree 1, which is the part of the Hodge theory of matroids that currently accounts for all combinatorial applications of the work of Adiprasito et al. (2018). Our work avoids the use of "flips," the key technical tool employed in that work.

Keywords. Matroids, tropical Hodge theory, matroid quotients, volume polynomials

1. Introduction

Chow rings of matroids were introduced in [27] as generalizations of cohomology rings of wonderful compactifications of hyperplane arrangement complements [17]:

Definition 1.0.1. Let M be a loopless matroid of rank $r = d + 1$ on a ground set E. The *Chow ring* of M is a graded ring $A^\bullet_{FY}(M) = \bigoplus_{i=0}^{d} A^i_{FY}(M)$ defined as

$$A^\bullet_{FY}(M) :=$$

$$\frac{\mathbb{R}[z_F \mid F \subseteq E \text{ a nonempty flat of } M]}{\langle z_F z_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} z_F \mid a \text{ an atom in the lattice of flats of } M\rangle}.$$

We introduce a presentation of the Chow ring by a new set of generators:

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Definition 3.2.1. Let M be a loopless matroid on E. The *simplicial presentation* $A_\triangledown^\bullet(M)$ of the Chow ring of M is the quotient of a polynomial ring $\mathbb{R}[h_F \mid F \subseteq E$ a nonempty flat of $M]$ by the kernel of the surjective map

$$\mathbb{R}[h_F \mid F \subseteq E \text{ a nonempty flat of } M] \to A_{FY}^\bullet(M) \quad \text{where} \quad h_F \mapsto -\sum_{G \supseteq F} z_G.$$

The generators h_F of $A_\triangledown^\bullet(M)$ are called the *simplicial generators* of the Chow ring of M.

The simplicial presentation allows us to

(i) interpret the simplicial generators and their monomials in $A_\triangledown^\bullet(M)$ combinatorially via the theory of matroid quotients;

(ii) recover the Poincaré duality property of $A_\triangledown^\bullet(M)$;

(iii) compute the volume polynomial of $A_\triangledown^\bullet(M)$, which we show to be Lorentzian in the sense of [10], and whose formula generalizes a main result of [50];

(iv) give a simplified proof of the Hodge–Riemann relations in degree 0 and 1 for matroids, which is the part of the Hodge theory of matroids in [1] that accounts for all currently known combinatorial applications of [1].

We achieve these by establishing combinatorial properties of the simplicial presentation that reflect the following two geometric observations:

(A) Suppose base-point-free divisors $D_1, \dots, D_s$ on a variety X generate the Chow ring $A^\bullet(X)$, and $Y \subset X$ is a subvariety. Then the pullback map $A^\bullet(X) \to A^\bullet(Y)$ can be computed via the intersections of Y with general hyperplane pullbacks of $X \to \mathbb{P}(H^0(D_i))$. If furthermore the pullback map is surjective, then these intersections generate $A^\bullet(Y)$.

(B) For base-point-free divisors $D_1, \dots, D_s$ whose sum is an ample divisor on a projective variety Y, let $\mathrm{vol}(t_1, \dots, t_s)$ be the volume polynomial $\int_Y (t_1 D_1 + \cdots + t_s D_s)^{\dim Y}$. Then $\mathrm{vol}(\underline{t})$, as a function $\mathbb{R}^s \to \mathbb{R}$, is positive and log-concave on the positive orthant $\mathbb{R}_{>0}^s$ [42, §1.6.A].

When a matroid M is realizable, its Chow ring is isomorphic to the Chow ring of a projective variety $Y_{\mathcal{R}(M)}$, which is a subvariety of a projective toric variety X_{A_n}, and the pullback $A^\bullet(X_{A_n}) \to A^\bullet(Y_{\mathcal{R}(M)})$ is surjective (see §2.3). In this case, unlike the classical presentation $A_{FY}^\bullet(M)$, the generators of our presentation $A_\triangledown^\bullet(M)$ represent *base-point-free* divisor classes on $Y_{\mathcal{R}(M)}$, obtained as pullbacks of base-point-free divisor classes on X_{A_n} that generate $A^\bullet(X_{A_n})$ (see §3.2). We demonstrate that even when M is not necessarily realizable, the generators of $A_\triangledown^\bullet(M)$ display behaviors analogous to those of base-point-free divisors on the projective varieties.

1.1. Main results

(i) *Simplicial generators and their monomials as matroid quotients.* The *principal truncation* of a matroid M by a flat F is a new matroid with bases $\{B \setminus f : B$ a basis of M and

$f \in B \cap F \neq \emptyset\}$. When M is realizable, so that $h_F \in A^1_\triangledown(M)$ represents a base-point-free divisor class on a projective variety $Y_{\mathscr{R}(M)}$, the principal truncation by F corresponds to any general hyperplane pullback of the map defined by h_F (Remarks 3.2.6 & 3.2.7). For an arbitrary matroid, the following analogue holds.

Theorem 3.2.3. *Let M be a loopless matroid. The simplicial generator $h_F \in A^\bullet_\triangledown(M)$ corresponds, via the cap product (see §2.1 for a definition), to the principal truncation of M by the flat F.*

We use this interpretation of the simplicial generators to study their monomials in $A^\bullet_\triangledown(M)$. A monomial basis for $A^\bullet_\triangledown(M)$ is obtained by carrying over a Gröbner basis computation for the classical presentation $A^\bullet_{FY}(M)$ in [27] (Proposition 3.3.2 & Corollary 3.3.3). This monomial basis, which we call the *nested basis*, now has the following combinatorial interpretation.

Definition 3.3.4. Let M' be a matroid quotient of M (that is, every flat of M' is a flat of M), denoted $f : M' \twoheadleftarrow M$. Write $n_f(S) := \mathrm{rk}_M(S) - \mathrm{rk}_{M'}(S)$ for a subset S of the ground set. An *f-cyclic flat* of f is a flat F of M' such that F is minimal (with respect to inclusion) among the flats G of M' with $n_f(G) = n_f(F)$. The matroid M' is a *relative nested quotient* of M if the f-cyclic flats form a chain.

Theorem 3.3.8. *There is a natural bijection, via the cap product, between the nested basis of $A^\bullet_\triangledown(M)$ and the set of relative nested quotients of M.*

When the matroid M is a Boolean matroid, i.e. a matroid whose ground set is a basis, the relative nested quotients of M are known as "nested matroids," studied previously in [33] in the context of Chow rings of permutohedral varieties.

(ii) *The Poincaré duality property.* We build upon the bijection in Theorem 3.3.8 to establish the following theorem, which mirrors the fact that for certain classes of varieties,[1] if $A^\bullet(X) \to A^\bullet(Y)$ is surjective for $Y \subset X$, then $A^\bullet(Y) \simeq A^\bullet(X)/\mathrm{ann}([Y])$ where $\mathrm{ann}([Y]) = \{\xi \in A^\bullet(X) \mid \xi \cdot [Y] = 0\}$.

Theorem 4.2.1. *Let M be a loopless matroid on $\{0, 1, \ldots, n\}$, and X_{A_n} the permutohedral variety of dimension n. Let Δ_M be the Bergman class of the matroid M considered as an element of $A^\bullet(X_{A_n})$. Then $A^\bullet(M) \simeq A^\bullet(X_{A_n})/\mathrm{ann}(\Delta_M)$, where $\mathrm{ann}(\Delta_M) = \{\xi \in A^\bullet(X_{A_n}) \mid \xi \cdot \Delta_M = 0\}$.*

The Poincaré duality property for $A^\bullet_\triangledown(M)$ was established in [1, Theorem 6.19] by a double induction. In our case, the Poincaré duality property is a simple algebraic consequence of Theorem 4.2.1 (Corollary 4.2.2).

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[1]These include varieties on which rational and numerical equivalence coincide, which is a feature shared by all varieties that inspire the combinatorics of this paper. See the first footnote in §2.1.

(iii) *The volume polynomial and its log-concavity.* The following formula computes the intersection numbers of the simplicial generators h_F.

Theorem 5.2.4. *For M a loopless matroid of rank $r = d + 1$, let $\int_M : A^d(M) \to \mathbb{R}$ be the degree map. For a multiset $\{F_1, \ldots, F_d\}$ of nonempty flats, we have*

$$\int_M h_{F_1} \cdots h_{F_d} = \begin{cases} 1 & \text{if } \mathrm{rk}_M\left(\bigcup_{j \in J} F_j\right) \geq |J| + 1 \text{ for every } \emptyset \subsetneq J \subseteq \{1, \ldots, d\}, \\ 0 & \text{otherwise.} \end{cases}$$

That these intersection numbers are either 1 or 0 stands in stark contrast to calculations in the classical presentation of the Chow ring of a matroid, which were carried out in [25].

The intersection numbers collect together into the "volume polynomial" of $A_\nabla^\bullet(M)$, defined in the following corollary. The corollary recovers [50, Corollary 9.4] on volumes of generalized permutohedra when M is set to be the Boolean matroid.

Corollary 5.2.5. *Let M be a loopless matroid on E of rank $d + 1$. The volume polynomial $VP_M^\nabla(\underline{t}) \in \mathbb{Q}[t_F \mid F \subseteq E \text{ a nonempty flat in } M]$ of $A_\nabla^\bullet(M)$, defined as $VP_M^\nabla(\underline{t}) := \int_M (\sum_F t_F h_F)^d$, is*

$$VP_M^\nabla(\underline{t}) = \sum_{(F_1, \ldots, F_d)} t_{F_1} \cdots t_{F_d}$$

where the sum is over ordered collections $(F_1, \ldots, F_d)$ of flats with $\mathrm{rk}_M\left(\bigcup_{j \in J} F_j\right) \geq |J| + 1$ for every $\emptyset \subsetneq J \subseteq \{1, \ldots, d\}$.

The volume polynomial VP_M^∇, as a real-valued function, is therefore positive in its positive orthant, as are volume polynomials of base-point-free divisors on projective varieties. Moreover, it displays the log-concavity behavior described in (B).

Corollary 5.3.2. *As a real-valued function, $VP_M^\nabla(\underline{t})$ is log-concave in its positive orthant.*

Log-concavity is a consequence of the following statement.

Theorem 5.3.1. *The volume polynomial $VP_M^\nabla(\underline{t})$ of a loopless matroid M is Lorentzian in the sense of [10].*

Lorentzian polynomials are multivariate polynomials characterized by two conditions: one on their supports and one on their partial derivatives (see §5.1). We show that VP_M^∇ satisfies these two conditions by using Theorem 5.2.4 to understand the support of VP_M^∇ (Proposition 5.3.3) and by using Theorem 3.2.3 to understand the partial derivatives of VP_M^∇ as volume polynomials of principal truncations of M.

(iv) *A simplified proof of the Hodge–Riemann relations in degree* 1. We use log-concavity of VP_M^∇ to give a simplified proof of the Hodge–Riemann relations in degree 1 for Chow rings of matroids:

Theorem 6.2.1. *Let M be a loopless matroid of rank $r = d + 1$ on a ground set E, and $\ell \in A^1(M)$ a combinatorially ample divisor class (see §2.2). Then the Chow ring $A^\bullet(M)$ with the degree map $\int_M$ satisfies the Kähler package in degree 0 and 1. That is, for $i \leq 1$,*

(HL$^{\leq 1}$) (hard Lefschetz in degree ≤ 1) *multiplication by ℓ*

$$L_\ell^i : A^i(M) \to A^{d-i}(M), \quad a \mapsto \ell^{d-2i}a,$$

is an isomorphism,

(HR$^{\leq 1}$) (Hodge–Riemann relations in degree ≤ 1) *the symmetric form*

$$(-1)^i Q_\ell^i : A^i(M) \times A^i(M) \to \mathbb{R}, \quad (x, y) \mapsto (-1)^i \int_M xy\ell^{d-2i},$$

is nondegenerate on $A^i(M)$ and positive-definite when restricted to the degree i primitive space $P_\ell^i := \{z \in A^i(M) \mid \ell^{d-2i+1}z = 0\}$ of ℓ.

The authors of [1] establish the Hodge–Riemann relations in all degrees by a double-inductive argument that utilizes a generalization of Bergman fans of matroids. In our case, the log-concavity of VP_M^∇ (Corollary 5.3.2) provides us a key step in the induction in degrees ≤ 1, and thus our proof of Theorem 6.2.1 is a single induction on ranks of matroids and involves only the classical Bergman fans of matroids.

The validity of the Hodge–Riemann relation in degree 1 accounts for all currently known combinatorial applications of [1], including the Heron–Rota–Welsh conjecture. Huh has posed the discovery of combinatorial applications for the Hodge–Riemann relations in higher degrees as an open problem [37].

Organization

In Section 2, we develop the necessary background on Chow rings of matroids. In Section 3, we introduce the simplicial presentation of the Chow ring, study its structural properties via principal truncations, and introduce the relative nested quotients. In Section 4, we establish the Poincaré duality property for matroids. In Section 5, we calculate intersection numbers with respect to the simplicial presentation, and prove that the volume polynomial of the Chow ring in the simplicial presentation is Lorentzian. Finally, in Section 6 we apply our results to give a simplified proof of the Hodge theory in degree 1 for matroids.

2. Preliminaries

In this section, we review relevant background material on Chow rings of matroids. Familiarity with toric and tropical geometry, from which the combinatorial notions here originate, can be helpful but is not necessary. As references we point to [29] and [15] for toric geometry, and to [31], [44, Chapter 6], and [1, §4–§5] for tropical geometry.

In §2.1, we describe Chow cohomology rings and Minkowski weights of fans, and in §2.2 we illustrate these notions in the setting of matroids. These first two subsections are purely combinatorial. In §2.3, we provide the underlying geometric picture that motivates many of the combinatorial constructions.

2.1. *Chow cohomology rings and Minkowski weights*

We give a brief account of Chow cohomology rings and Minkowski weights of smooth fans, which are combinatorial analogues of cohomology rings and homology classes of algebraic varieties.[2]

We set the following notation and definitions for rational fans over a lattice.

- Let N be a lattice of rank n, and $N^\vee$ the dual lattice. We write $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.

- For $\Sigma \subset N_\mathbb{R}$ a rational fan, let $\Sigma(k)$ be the set of k-dimensional cones of Σ.

- For a ray $\rho \in \Sigma(1)$, write $u_\rho \in N$ for the *primitive ray vector* that generates $\rho \cap N$.

- A fan Σ is *smooth* if, for all cones σ of Σ, the set of primitive ray vectors of σ can be extended to a basis of N. A smooth fan is *simplicial* in the sense that every k-dimensional cone is generated by k rays.

- A fan Σ is said to be *complete* if its support $|\Sigma|$ is equal to $N_\mathbb{R}$.

Convention. Throughout this section, we assume that $\Sigma \subset N_\mathbb{R}$ is a smooth fan of dimension d, which is not necessarily complete.

Definition 2.1.1. The *Chow cohomology ring* $A^\bullet(\Sigma)$ of Σ is a graded $\mathbb{R}$-algebra

$$A^\bullet(\Sigma) := \frac{\mathbb{R}[x_\rho \mid \rho \in \Sigma(1)]}{\langle \prod_{\rho \in S} x_\rho \mid S \subseteq \Sigma(1) \text{ do not form a cone in } \Sigma \rangle + \langle \sum_\rho m(u_\rho) x_\rho \mid m \in N^\vee \rangle}.$$

Geometrically, the ring $A^\bullet(\Sigma)$ is the Chow ring $A^\bullet(X_\Sigma)$ of the toric variety X_Σ associated to the fan Σ. See [16, §10.1] for the case where Σ is complete, and [7] or [11] for the general case. From this geometric description of $A^\bullet(\Sigma)$, or directly from the algebraic definition above, one can check that $A^\ell(\Sigma) = 0$ unless $0 \leq \ell \leq d$.

We call a linear combination $\sum_\rho c_\rho x_\rho \in \mathbb{R}[x_\rho : \rho \in \Sigma(1)]$ of the variables x_ρ a *divisor* on Σ. Divisors of special interest in algebraic geometry are ample and nef divisors.[3] They have the following combinatorial description for a complete fan Σ (i.e. a complete toric variety X_Σ).

A divisor $D = \sum_{\rho \in \Sigma(1)} c_\rho x_\rho$ on a complete fan Σ defines a piecewise-linear function $\varphi_D : N_\mathbb{R} \to \mathbb{R}$, determined by being linear on each cone of Σ with $\varphi_D(u_\rho) = c_\rho$. We

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[2]We use real coefficients for Chow cohomology rings and Minkowski weights, although Chow rings of algebraic varieties initially take integral coefficients. The algebraic varieties that motivate the constructions here—smooth complete toric varieties and wonderful compactifications—share the feature that the Chow ring, the integral cohomology ring, and the ring of algebraic cycles modulo numerical equivalence all coincide [23, Appendix C.3.4]. In particular, their Chow rings are torsion-free. In this paper, while most of our arguments work over $\mathbb{Z}$, we will always work over $\mathbb{R}$ for convenience.

[3]Base-point-free divisor classes are nef, and on toric varieties nef conversely implies base-point-free [15, Theorem 6.3.12]. In agreement with the terminology of [1], we will call certain divisors "combinatorially nef" (see §2.2) although they are furthermore base-point-free on wonderful compactifications of realizable matroids.

say that D is a *nef divisor* if φ_D is a convex function on $N_\mathbb{R}$, that is, $\varphi_D(u) + \varphi_D(u') \geq \varphi(u + u')$ for all $u, u' \in N_\mathbb{R}$. If further the inequalities $\varphi_D(u) + \varphi_D(u') \geq \varphi(u + u')$ are strict whenever u and u' are not in a common cone of Σ, we say that D is *ample*. Nef (resp. ample) divisors on Σ correspond to polytopes in $N_\mathbb{R}^\vee$ whose outer normal fans coarsen (resp. equal) Σ.

Theorem 2.1.2 ([15, Theorems 6.1.5–6.1.7]). *Let Σ be a smooth complete fan. A nef divisor $D = \sum_{\rho \in \Sigma(1)} c_\rho x_\rho$ on Σ defines a polytope $P_D \subset N_\mathbb{R}^\vee$ by*

$$P_D := \{m \in N_\mathbb{R}^\vee \mid m(u_\rho) \leq c_\rho \ \forall \rho \in \Sigma(1)\},$$

whose outer normal fan coarsens Σ. Conversely, such a polytope $P \subset N_\mathbb{R}^\vee$ defines a nef divisor

$$D_P := \sum_{\rho \in \Sigma(1)} \max\{m(u_\rho) \mid m \in P\} x_\rho.$$

A nef divisor D is ample if the outer normal fan of P_D is equal to Σ.

A divisor D defines an element $[D] \in A^1(\Sigma)$, which we call the *divisor class* (of D) on Σ. We say that a divisor class $\zeta \in A^1(\Sigma)$ is *nef* (resp. *ample*) if any choice of a divisor D representing ζ is nef (resp. ample). This is well-defined because two divisors D and D' represent the same divisor class if and only if $\varphi_D - \varphi_{D'}$ is a linear function on $N_\mathbb{R}$. In terms of polytopes, two nef divisors D and D' define the same divisor class if and only if P_D and $P_{D'}$ are parallel translates.

Remark 2.1.3. Any nef divisor class $[D] \in A^1(\Sigma)$ is *effective*, that is, it can be written as a nonnegative linear combination $D = \sum_{\rho \in \Sigma(1)} c_\rho x_\rho$ (with $c_\rho \geq 0$ for all $\rho \in \Sigma(1)$). This is an immediate consequence of Theorem 2.1.2: Given a nef divisor D, translating if necessary one can assume that the polytope P_D contains the origin in its relative interior.

With $A^\bullet(\Sigma)$ as an analogue of a cohomology ring, we describe analogues of homology groups.

Definition 2.1.4. An ℓ-dimensional *Minkowski weight* on Σ is a function $\Delta : \Sigma(\ell) \to \mathbb{R}$ such that for each $\tau \in \Sigma(\ell - 1)$, the function Δ satisfies the balancing condition

$$\sum_{\substack{\sigma \supset \tau \\ \sigma \in \Sigma(\ell)}} \Delta(\sigma) u_{\sigma \setminus \tau} \in \mathrm{span}_\mathbb{R}(\tau),$$

where $\sigma \setminus \tau$ denotes the unique ray of σ that is not in τ. The *support* of Δ, denoted $|\Delta|$, is the union of cones $\sigma \in \Sigma(\ell)$ such that $\Delta(\sigma) \neq 0$. We write $\mathrm{MW}_\ell(\Sigma)$ for the group (under addition) of ℓ-dimensional Minkowski weights on Σ.

The groups of Minkowski weights are analogues of homology groups because they are dual to the Chow cohomology ring in the following way.

Lemma 2.1.5 ([44, Theorem 6.7.5][4]). *For $0 \leq \ell \leq d$, we have an isomorphism*

$$t_\Sigma : \mathrm{MW}_\ell(\Sigma) \xrightarrow{\sim} \mathrm{Hom}(A^\ell(\Sigma), \mathbb{Z}) \quad \textit{determined by} \quad \Delta \mapsto \big((\textstyle\prod_{\rho \in \sigma(1)} x_\rho) \mapsto \Delta(\sigma)\big).$$

This isomorphism is an analogue of the Kronecker duality map in algebraic topology. We use it to define combinatorial analogues of some standard operations in algebraic topology. We define the *cap product* by

$$A^k(\Sigma) \times \mathrm{MW}_\ell(\Sigma) \to \mathrm{MW}_{\ell-k}(\Sigma), \quad (\xi, \Delta) \mapsto \xi \cap \Delta := \big(\sigma \mapsto (t_\Sigma \Delta)(\xi \cdot \textstyle\prod_{\rho \in \sigma(1)} x_\rho)\big),$$

which makes $\mathrm{MW}_\bullet(\Sigma)$ into a graded $A^\bullet(\Sigma)$-module. When Σ satisfies $\mathrm{MW}_d(\Sigma) \simeq \mathbb{R}$, the *fundamental class* Δ_Σ is defined as its generator (unique up to scaling), and the cap product with the fundamental class defines the map

$$\delta_\Sigma : A^\bullet(\Sigma) \to \mathrm{MW}_{d-\bullet}(\Sigma), \quad \xi \mapsto \xi \cap \Delta_\Sigma.$$

In particular, noting that $\mathrm{MW}_0(\Sigma) = \mathbb{R}$, the *degree map* is defined as

$$\int_\Sigma : A^d(\Sigma) \to \mathbb{R}, \quad \xi \mapsto \xi \cap \Delta_\Sigma.$$

If Σ is complete, one can check that $\mathrm{MW}_n(\Sigma) \simeq \mathbb{R}$, where the fundamental class Δ_Σ is given by $\Delta_\Sigma(\sigma) = 1$ for each cone $\sigma \in \Sigma(n)$. In this case, we have the following analogue of the Poincaré duality theorem in algebraic topology.

Theorem 2.1.6 ([31, Theorem 3.1, Proposition 4.1 (b), Theorem 4.2]). *For Σ a smooth complete fan, the cap product with the fundamental class Δ_Σ,*

$$\delta_\Sigma : A^k(\Sigma) \xrightarrow{\sim} \mathrm{MW}_{n-k}(\Sigma), \quad \xi \mapsto \xi \cap \Delta_\Sigma,$$

is an isomorphism for each $0 \leq k \leq n$. Equivalently (*by Lemma* 2.1.5), *the pairing*

$$A^k(\Sigma) \times A^{n-k}(\Sigma) \to \mathbb{R}, \quad (a, a') \mapsto \int_\Sigma aa',$$

is nondegenerate for each $0 \leq k \leq n$.

The isomorphisms in Theorem 2.1.6 make $\bigoplus_{i=0}^n \mathrm{MW}_{n-i}(\Sigma)$ into a graded ring when Σ is complete. We write $\mathrm{MW}^\bullet(\Sigma) := \mathrm{MW}_{n-\bullet}(\Sigma)$ for this graded ring. For complete fans, the resulting multiplication structure on the Minkowski weights is known as the *stable intersection*, denoted $\cap_{\mathrm{st}}$, in tropical geometry. We will only need a special case of stable intersections,[5] which we provide explicitly in the context of matroids in §3.1 (Proposition 3.1.8).

In the proof of Lemma 3.2.5, we will need the following explicit description of the map $\delta_\Sigma : A^1(\Sigma) \xrightarrow{\sim} \mathrm{MW}_{n-1}(\Sigma)$ for nef divisor classes on a complete fan Σ. It is familiar to tropical geometers as tropical hypersurfaces [44, Proposition 3.3.2 & Theorem 6.7.7].

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[4]Currently [44, Theorem 6.7.5] has a typo—it is missing $\mathrm{Hom}(\cdot, \mathbb{Z})$. The statement here was made implicitly in [30], and follows the notation of [1, Proposition 5.6].

[5]See [31], [44, §3.6], or [38] for a definition of stable intersections. It may help to note the suggestiveness of the notations here—we have $\xi \cap \Delta = \delta_\Sigma(\xi) \cap_{\mathrm{st}} \Delta$.

Proposition 2.1.7. *Let D be a nef divisor on Σ such that the corresponding polytope P_D is a lattice polytope. Then the Minkowski weight $\Delta_{P_D} := \delta_\Sigma([D]) \in \mathrm{MW}_{n-1}(\Sigma)$ given by Theorem* 2.1.6 *is defined by*

$$\Delta_{P_D}(\tau) = \begin{cases} \ell(P_D(\sigma)) & \text{if there is } \sigma \in \Sigma_{P_D}(n-1) \text{ such that } |\tau| \subseteq |\sigma|, \\ 0 & \text{otherwise,} \end{cases}$$

for each $\tau \in \Sigma(n-1)$, where $P_D(\sigma)$ is the edge of P_D corresponding to the cone $\sigma \in \Sigma_Q(n-1)$, and $\ell(P_D(\sigma))$ is its lattice length, i.e. the number of lattice points on $P_D(\sigma)$ minus 1.

We end this subsection by noting the functoriality of the constructions here. An inclusion of fans $\iota : \Sigma' \hookrightarrow \Sigma$ defines the *pullback map ι^**, which is a surjective map of graded rings

$$\iota^* : A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\Sigma'), \quad x_\rho \mapsto \begin{cases} x_\rho & \text{if } \rho \in \Sigma'(1), \\ 0 & \text{otherwise.} \end{cases}$$

Comparing the presentations of $A^\bullet(\Sigma)$ and $A^\bullet(\Sigma')$, one checks easily that this map coincides with the quotient of $A^\bullet(\Sigma)$ by the ideal $\langle x_\rho \mid \rho \in \Sigma(1) \setminus \Sigma'(1) \rangle \subset A^\bullet(\Sigma)$. Dually, a Minkowski weight Δ' on Σ' is naturally a Minkowski weight on Σ. In this case we often abuse the notation and write Δ' for both Minkowski weights.

Remark 2.1.8. Unraveling the definitions, one checks that the cap product is functorial in the following sense: The pullback map $\iota^* : A^\bullet(\Sigma) \to A^\bullet(\Sigma')$ makes $\mathrm{MW}_\bullet(\Sigma')$ into an $A^\bullet(\Sigma)$-module. Explicitly, if $\xi \in A^\bullet(\Sigma)$ and $\Delta' \in \mathrm{MW}_\ell(\Sigma')$, then $\iota^*\xi \cap \Delta' = \xi \cap \Delta'$, where Δ' on the right hand side is considered as a Minkowski weight on Σ.

2.2. Bergman classes and Chow rings of matroids

We now specialize our discussion to matroids. We begin with the braid fan, on which matroids will arise as certain Minkowski weights.

First, we fix some notations. Let $E := \{0, 1, \ldots, n\}$, and for a subset $S \subseteq E$ write $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i$, where $\mathbf{e}_0, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{Z}^E$. Let N be the lattice $N = \mathbb{Z}^E / \mathbb{Z}\mathbf{e}_E$, and write u_S for the image of $\mathbf{e}_S$ in N. The dual lattice of N is $N^\vee = (\mathbb{Z}\mathbf{e}_E)^\perp = \{(y_0, \ldots, y_n) \in \mathbb{Z}^E \mid \sum_{i=0}^n y_i = 0\}$.

The *braid fan* (of dimension n), denoted Σ_{A_n}, is the outer normal fan of the standard *permutohedron* (of dimension n), which is the polytope

$$\Pi_n := \mathrm{Conv}(w(0, 1, \ldots, n) \in \mathbb{R}^E \mid \text{all permutations } w \text{ of } E).$$

Concretely, the braid fan Σ_{A_n} is a complete fan in $N_\mathbb{R}$ whose cones are $\mathrm{Cone}(u_{S_1}, \ldots, u_{S_k}) \subset N_\mathbb{R}$, one for each chain of nonempty proper subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$. In particular, the primitive rays of Σ_{A_n} are $\{u_S \mid \emptyset \subsetneq S \subsetneq E\}$. This fan is also known as the Coxeter complex of the type A root system, hence the notation Σ_{A_n}. The associated toric variety of Σ_{A_n}, denoted X_{A_n}, is often called the *permutohedral variety* (of dimension n).

We assume familiarity with the basics of matroids, and refer to [48, 56] as general references. We fix the following notation for matroids: We write $U_{r,E}$ for the uniform matroid of rank r on E, and we set a matroid M to have

- ground set $E = \{0, 1, \ldots, n\}$,
- $\mathcal{B}(M)$ the set of bases of M,
- rk_M the rank function of M, or simply rk when the matroid in question is clear,
- $\mathcal{L}_M$ the lattice of flats of M, which we also use to denote the set of flats,
- $\mathfrak{A}(M)$ the set of atoms of $\mathcal{L}_M$, which are flats of rank 1,
- $\mathcal{S}(M)$ the set of spanning sets of M, which are subsets of E that contain a basis of M,
- $\mathrm{cl}_M(S)$ the closure of a subset $S \subseteq E$, which is the smallest flat of M containing S,
- $Q(M)$ the base polytope of M, which is the polytope $\mathrm{Conv}(\mathbf{e}_B \mid B \in \mathcal{B}(M)) \subset \mathbb{R}^E$.

Matroids define Minkowski weights on Σ_{A_n} in the following way.

Proposition 2.2.1. *Let M be a loopless matroid M of rank $r = d + 1$.*

(1) ([44, Theorem 4.4.5]) *A function $\Delta_M : \Sigma_{A_n}(d) \to \mathbb{R}$ defined by*

$$\Delta_M(\mathrm{Cone}(u_{S_1}, \ldots, u_{S_d})) = \begin{cases} 1 & \text{if } S_1, \ldots, S_d \text{ are flats of } M, \\ 0 & \text{otherwise,} \end{cases}$$

for each chain of nonempty proper subsets $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$ is a d-dimensional Minkowski weight on Σ_{A_n}.

(2) ([1, Proposition 5.2]) *Let Σ_M be the smooth fan structure on the support $|\Delta_M|$ inherited from Σ_{A_n}. That is, Σ_M is a subfan of Σ_{A_n} whose cones are $\mathrm{Cone}(u_{F_1}, \ldots, u_{F_k})$ $\subset N_{\mathbb{R}}$, one for each chain of nonempty proper flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$ of M. Then the Bergman class Δ_M is the unique d-dimensional Minkowski weight on Σ_M (up to scaling).*

Definition 2.2.2. With notations as in Proposition 2.2.1 above, we call the Minkowski weight Δ_M the *Bergman class* of M, and we call the fan Σ_M the *Bergman fan*[6] of M.

In Section 3 we will need the following description of supports of Bergman classes.

Lemma 2.2.3 ([44, Corollary 4.2.11]). *Let M be a loopless matroid, and $Q(M)$ its base polytope. The support $|\Delta_M|$ of its Bergman class is equal to the union of cones σ in the outer normal fan of $Q(M)$ satisfying the following condition: The corresponding face $Q(M)(\sigma)$ of σ is a base polytope of a loopless matroid.*

---

[6]Some define the Bergman fan of M as a coarser smooth fan structure on the support $|\Sigma_M|$. A smooth fan structure on $|\Sigma_M|$ that coarsens Σ_M corresponds to a choice of a building set on the lattice $\mathcal{L}_M$ of flats [5, 26]. Here we will always take the smooth fan structure for Σ_M as a subfan of Σ_{A_n}.

The Chow ring of a matroid is defined as the Chow cohomology ring of its Bergman fan. Explicitly, we have the following.

Definition 2.2.4. The *Chow ring* of a loopless matroid M is the graded ring

$$A^\bullet(M) := A^\bullet(\Sigma_M)$$

$$= \frac{\mathbb{R}[x_F \mid F \in \mathcal{L}_M \setminus \{\emptyset, E\}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} x_F - \sum_{G \supseteq b} x_G \mid a, b \in \mathfrak{A}(M)\rangle}.$$

We call linear combinations of the variables x_F *divisors* on M, and the elements of $A^1(M)$ *divisor classes* on M. The divisor class of $\sum_{F \supseteq a} x_F$ for any atom $a \in \mathfrak{A}(M)$ is called the *hyperplane class* of M, and is denoted $\alpha(M)$ or just α if the matroid in question is clear.

Remark 2.2.5. The ring $A^\bullet(M)$ was first studied in [27] under a slightly different presentation, which for clarity is denoted $A^\bullet_{FY}(M)$ and is given as

$$A^\bullet_{FY}(M) := \frac{\mathbb{R}[z_F \mid F \in \mathcal{L}_M \setminus \{\emptyset\}]}{\langle z_F z_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} z_F \mid a \in \mathfrak{A}(M)\rangle}.$$

That is, we have $x_F = z_F$ for every nonempty proper flat $F \in \mathcal{L}_M$, and $z_E = -\alpha$. As both presentataions $A^\bullet(M)$ and $A^\bullet_{FY}(M)$ are relevant for us, we will use the variable names x, z in a consistent manner; for example, in the summation $\sum_{F \supseteq F'} x_F$ it is implied that $F \subsetneq E$, whereas $\sum_{F \supseteq F'} z_F$ includes z_E as a summand.

Since $\mathrm{MW}(\Sigma_M) \simeq \mathbb{R}$ by Proposition 2.2.1 (2), with the Bergman class as the fundamental class, the Chow ring of a matroid M has the *degree map* $\int_M : A^d(\Sigma_M) \to \mathbb{R}$, defined by the cap product $\xi \mapsto \xi \cap \Delta_M$. Explicitly, it is determined by

$$\int_M x_{F_1} \cdots x_{F_d} = 1 \quad \text{for every maximal chain } F_1 \subsetneq \cdots \subsetneq F_d \text{ in } \mathcal{L}_M \setminus \{\emptyset, E\}.$$

Note that the braid fan Σ_{A_n} is the Bergman fan of the Boolean matroid $U_{|E|,E}$, and its fundamental class $\Delta_{\Sigma_{A_n}}$ is the Bergman class of $U_{|E|,E}$. We will thus always identify $A^\bullet(\Sigma_{A_n}) = A^\bullet(U_{|E|,E})$.

We end this subsection with a discussion of nef and ample divisors on the braid fan, and the resulting analogous notions for Bergman fans. The following characterization of nef divisors on Σ_{A_n}, which is a specialization of Theorem 2.1.2, was recognized in various works [2, 20, 47, 50]; for a modern treatment and generalization to arbitrary Coxeter root systems we point to [4].

Proposition 2.2.6. *The following are equivalent for a divisor* $D = \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S \in A^1(\Sigma_{A_n})$:

(1) *D is a nef divisor on Σ_{A_n},*

(2) *the function $c_{(\cdot)} : 2^E \to \mathbb{R}$ satisfies the* submodular property

$$c_A + c_B \geq c_{A \cup B} + c_{A \cap B} \quad \text{for all } A, B \subseteq E \text{ where } c_\emptyset = c_E = 0,$$

(3) *the normal fan of the polytope $P_D = \{m \in N_\mathbb{R}^\vee \mid m(u_S) \leq c_S \ \forall \emptyset \subsetneq S \subsetneq E\}$ coarsens Σ_{A_n},*

(4) *every edge of P_D is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in E$.*

Remark 2.2.7. Often the polytope P_D is constructed in an affine translate of $N_\mathbb{R}^\vee$ in $\mathbb{R}^E$, for which the presentation $A_{FY}^\bullet(\Sigma_{A_n})$ is useful. Given a submodular function $c_{(\cdot)} : 2^E \to \mathbb{Z}$ with $c_\emptyset = 0$ but c_E possibly nonzero, the *generalized permutohedron* associated to $c_{(\cdot)}$ is the polytope

$$P(c) := \{y \in (\mathbb{R}^E)^\vee \mid y(\mathbf{e}_E) = c_E \text{ and } y(\mathbf{e}_S) \leq c_S \ \forall \emptyset \subsetneq S \subsetneq [n]\}.$$

This polytope lives in the translate of $N_\mathbb{R}^\vee$ where the points have coordinate sum c_E. One translates $P(c)$ to $N_\mathbb{R}^\vee$ as follows. Fix an element $i \in E$. We have

$$P(c) - c_E \mathbf{e}_i = \{m \in N_\mathbb{R}^\vee \mid m(u_S) \leq c_S - c_E \alpha_S^{(i)} \ \forall S\} \subset N_\mathbb{R}^\vee,$$

where $\alpha_S^{(i)} = 1$ if $i \in S$ and 0 otherwise. Since the divisor class of $\sum_{\emptyset \subsetneq S \subsetneq E} \alpha_S^{(i)} x_S$ is the hyperplane class α in $A^1(U_{|E|,E}) = A^1(\Sigma_{A_n})$, the nef divisor class that the polytope $P(c)$ corresponds to is

$$-c_E \alpha + \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S = \sum_{\emptyset \subsetneq S \subseteq E} c_S z_S.$$

The notion of nef and ample divisors on a matroid is inherited from the braid fan. First, note that for a loopless matroid M, the inclusion of fans $\iota_M : \Sigma_M \hookrightarrow \Sigma_{A_n}$ induces the pullback map

$$\iota_M^* : A^\bullet(\Sigma_{A_n}) \to A^\bullet(M) \quad \text{defined by} \quad x_S \mapsto \begin{cases} x_S & \text{if } S \subsetneq E \text{ is a flat of } M, \\ 0 & \text{otherwise.} \end{cases}$$

When we wish to clarify whether a variable x_S is an element of $A^\bullet(M)$ or $A^\bullet(\Sigma_{A_n})$, we write

$$x_S(M) := \iota_M^* x_S,$$

in which case x_S is considered as an element of $A^\bullet(\Sigma_{A_n})$ and $x_S(M)$ of $A^\bullet(M)$.

The pullback map motivates the following notions regarding divisors on M. We say that a divisor (class) on M is *combinatorially nef* if it is a pullback of a nef divisor (class) on Σ_{A_n}. A *combinatorially ample* divisor (class) is defined similarly. Explicitly, a divisor $D = \sum_{F \in \mathscr{L}_M \setminus \{\emptyset, E\}} c_F \cdot x_F(M)$ is combinatorially nef if there exists a function $a_{(\cdot)} : 2^E \to \mathbb{Z}$ such that $a_\emptyset = a_E = 0$, $a_F = c_F$ for all flats $F \in \mathscr{L}_M$, and

$$a_A + a_B \geq a_{A \cup B} + a_{A \cap B} \quad \text{for all } A, B \subseteq E.$$

Say D is *ample* if the inequality is strict whenever A and B are incomparable.

Combinatorially nef (resp. ample) divisor classes on M are closed under nonnegative linear combinations, since nef (resp. ample) divisor classes on complete fans are in general. We thus let $\overline{\mathscr{K}}_M$ (resp. $\mathscr{K}_M$) be the cone in $A^1(M)$ of combinatorially nef (resp. ample) divisor classes on M, called the *combinatorially nef* (resp. *ample*) *cone* of M.

Remark 2.2.8. It follows from Remark 2.1.3 that a combinatorially nef divisor class $[D] \in A^1(M)$ is effective; that is, it can be written as $D = \sum_F c_F x_F$ where $c_F \geq 0$ for all $F \in \mathscr{L}_M \setminus \{\emptyset, E\}$.

2.3. The geometry of matroids via wonderful compactifications

We provide the underlying algebraic geometry of the combinatorial constructions in the previous two subsections.

Let M be a loopless matroid on $E = \{0, 1, \ldots, n\}$ of rank $r = d + 1$ realizable over a field $\Bbbk$, which we may assume to be algebraically closed. A *realization* $\mathscr{R}(M)$ of M consists of any of the following equivalent pieces of data:

- a list $E = \{v_0, \ldots, v_n\}$ of vectors spanning a $\Bbbk$-vector space $V \simeq \Bbbk^r$, or

- a surjection $\Bbbk^{n+1} \twoheadrightarrow V$ where $\mathbf{e}_i \mapsto v_i$, or

- an injection $\mathbb{P}V^* \hookrightarrow \mathbb{P}_{\Bbbk}^n$, dualizing the surjection $\Bbbk^{n+1} \twoheadrightarrow V$.

For a realization $\mathscr{R}(M)$ of M with $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$, the coordinate hyperplanes of $\mathbb{P}^n$ intersect with $\mathbb{P}V^*$ to give the associated *hyperplane arrangement* $\mathcal{A}_{\mathscr{R}(M)}$ on $\mathbb{P}V^*$, which is encoded by the flats of M in the following way. For each nonempty flat F of M, let L_F be a linear subspace of V^* defined by

$$L_F := \{f \in V^* \mid f(v_i) = 0 \ \forall v_i \in F\},$$

and let $\mathbb{P}L_F$ be the linear subvariety of $\mathbb{P}V^*$. The hyperplanes of $\mathcal{A}_{\mathscr{R}(M)}$ are $\{\mathbb{P}L_a\}_{a \in \mathfrak{A}(M)}$ corresponding to the atoms, and more generally, a flat F of rank c corresponds to the c-codimensional linear subvariety $\mathbb{P}L_F$.

We denote by $\mathring{Y}_{\mathscr{R}(M)}$ the hyperplane arrangement complement $\mathbb{P}V^* \setminus \bigcup \mathcal{A}_{\mathscr{R}(M)}$. It is a linear subvariety of an algebraic torus in the following way: The algebraic torus $T_N = (\Bbbk^*)^{n+1}/\Bbbk^*$ of the lattice $N = \mathbb{Z}^{n+1}/\mathbb{Z}(1, \ldots, 1)$ is the complement of the union of coordinate hyperplanes in $\mathbb{P}^n$, and hence $\mathring{Y}_{\mathscr{R}(M)}$ is the intersection of $\mathbb{P}V^*$ with T_N. The linear subvariety $\mathring{Y}_{\mathscr{R}(M)} \subset T_N$ is related to the Bergman class of M by tropicalization in the following way (see [44, §3] for tropicalizations of subvarieties of an algebraic torus).

Theorem 2.3.1. *Let $\mathscr{R}(M)$ be a realization of a loopless matroid M, and let $\mathring{Y}_{\mathscr{R}(M)} \subset T_N$ be the associated hyperplane arrangement complement. Recall that the permutohedral variety X_{A_n} is a toric variety with the torus T_N.*

(1) ([44, Theorem 4.1.11]) *The support of the tropicalization of $\mathring{Y}_{\mathscr{R}(M)}$, denoted* $\mathrm{trop}(\mathring{Y}_{\mathscr{R}(M)})$, *equals the support of the Bergman fan Δ_M.*

(2) ([44, Proposition 6.4.17 & Theorem 6.7.7]) *It follows that the closure $Y_{\mathscr{R}(M)}$ of $\mathring{Y}_{\mathscr{R}(M)}$ in the permutohedral variety X_{A_n} satisfies*

$$\delta_{\Sigma_{A_n}}([Y_{\mathscr{R}(M)}]) = \Delta_M,$$

where $\delta_{\Sigma_{A_n}}$ is the isomorphism $A^\bullet(X_{A_n}) \simeq \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ in Theorem 2.1.6.

In other words, the Bergman class Δ_M corresponds to the Chow homology class of the closure $Y_{\mathscr{R}(M)}$ of $\mathring{Y}_{\mathscr{R}(M)}$ in the permutohedral variety X_{A_n}. The variety $Y_{\mathscr{R}(M)}$ is called the *wonderful compactification* of the hyperplane arrangement complement $\mathring{Y}_{\mathscr{R}(M)}$.

Remark 2.3.2. The wonderful compactification $Y_{\mathscr{R}(M)}$ can be described in two equivalent ways [17, §3.2].

(1) The variety $Y_{\mathscr{R}(M)}$ is obtained as a series of blow-ups on $\mathbb{P}V^*$ by the following process: First blow up the points $\{\mathbb{P}L_F\}_{\mathrm{rk}(F)=\mathrm{rk}(E)-1}$, then blow up the strict transforms of the lines $\{\mathbb{P}L_F\}_{\mathrm{rk}(F)=\mathrm{rk}(E)-2}$, and continue until having blown up strict transforms of $\{\mathbb{P}L_F\}_{\mathrm{rk}(F)=1}$. We denote by $\pi_{\mathscr{R}(M)} : Y_{\mathscr{R}(M)} \to \mathbb{P}V^*$ the blow-down map.

(2) For each nonempty flat F of M, the projection away from the linear subvariety $\mathbb{P}L_F \subset \mathbb{P}V^*$ is a rational map $\mathbb{P}V^* \dashrightarrow \mathbb{P}(V^*/L_F)$. The variety $Y_{\mathscr{R}(M)}$ is then the (closure of the) graph of the rational map

$$\mathbb{P}V^* \dashrightarrow \prod_{F \in \mathscr{L}_M \setminus \{\emptyset\}} \mathbb{P}(V^*/L_F).$$

When $U_{n+1,n+1}$ is realized as the standard basis of $\Bbbk^{n+1}$, the associated wonderful compactification is the toric variety X_{A_n} of the braid fan. It is obtained from $\mathbb{P}^n$ by blowing up the coordinate points, then the (strict transforms of) coordinate lines, and so forth. Let us write $\pi_{A_n} : X_{\Sigma_{A_n}} \to \mathbb{P}^n$ for the blow-down map. Then for a realization $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$ of a loopless matroid M, Remark 2.3.2 (1) above expresses the wonderful compactification $Y_{\mathscr{R}(M)}$ as the strict transform of $\mathbb{P}V^* \subset \mathbb{P}^n$ under the sequence of blow-ups π_{A_n}. In other words, we have a diagram

$$
\begin{array}{ccc}
Y_{\mathscr{R}(M)} & \overset{\iota_M}{\hookrightarrow} & X_{\Sigma_{A_n}} \\
{\scriptstyle \pi_{\mathscr{R}(M)}}\downarrow & & \downarrow{\scriptstyle \pi_{A_n}} \\
\mathbb{P}V^* & \hookrightarrow & \mathbb{P}^n
\end{array}
$$

The boundary of $Y_{\mathscr{R}(M)} \setminus \mathring{Y}_{\mathscr{R}(M)}$ consists of the exceptional divisors $\mathcal{E}_F$ obtained by blowing up (strict transforms of) $\mathbb{P}L_F$. These divisors have simple-normal-crossings [17], and consequently the intersection theory of the boundary divisors of $Y_{\mathscr{R}(M)}$ is encoded in the matroid. More precisely, the Chow ring $A^\bullet(Y_{\mathscr{R}(M)})$ of the variety $Y_{\mathscr{R}(M)}$ is isomorphic to the Chow cohomology ring $A^\bullet(M)$ of the Bergman fan of M [27, Corollary 2].

Remark 2.3.3. We note the following geometric observations about the presentation

$$A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(M)$$
$$= \frac{\mathbb{R}[x_F \mid F \in \mathscr{L}_M \setminus \{\emptyset, E\}]}{\langle x_F x_{F'} \mid F, F' \text{ incomparable}\rangle + \langle \sum_{F \supseteq a} x_F - \sum_{G \supseteq b} x_G \mid a, b \in \mathfrak{A}(M)\rangle}.$$

(1) The variables x_F correspond to the exceptional divisors $\mathcal{E}_F$ obtained by blowing up (strict transforms of) $\mathbb{P}L_F$.

(2) The quadric relations $x_F x_{F'} = 0$ reflect the fact that two exceptional divisors from blowing up two nonintersecting linear subspaces do not intersect.

(3) The linear relations defining $A^\bullet(M)$ reflect the fact that for any atom $a \in \mathfrak{A}(M)$, we have $-z_E = \alpha(M) = \sum_{F \supseteq a} x_F = \pi^*_{\mathscr{R}(M)} h$ where $h = c_1(\mathscr{O}_{\mathbb{P} V^*}(1))$ is the hyperplane class of $\mathbb{P} V^*$.

(4) Under $A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(M)$ and $A^\bullet(X_{\Sigma_{A_n}}) \simeq A^\bullet(\Sigma_{A_n})$, the pullback map $\iota^*_M : A^\bullet(X_{\Sigma_{A_n}}) \to A^\bullet(Y_{\mathscr{R}(M)})$ along the closed embedding $\iota_M : Y_{\mathscr{R}(M)} \hookrightarrow X_{\Sigma_{A_n}}$ is the pullback map of the Chow cohomology rings of Σ_M and Σ_{A_n} induced by the inclusion of fans $\Sigma_M \hookrightarrow \Sigma_{A_n}$. In particular, the pullback map is surjective.

(5) A divisor class $D \in A^1(M)$ is combinatorially ample [nef] if and only if there exists an ample [nef] divisor class L on X_{A_n} such that $\iota^*_M L = D$. Combinatorially ample [nef] divisors are ample [nef] on the variety $Y_{\mathscr{R}(M)}$.

Remark 2.3.4 (Relation to (A)). The geometry and the combinatorics of the permutohedral variety X_{A_n} have been widely studied in various contexts including moduli spaces [6,43], convex optimization [20,47], Hopf monoids [2,18], and lattice polyhedra [50,51]. In our case, the variety X_{A_n} plays the role of the ambient variety X in (A), and the wonderful compactification $Y_{\mathscr{R}(M)}$ plays the role of the subvariety $Y \subset X$. In §3.2, we introduce base-point-free divisor classes on X_{A_n} that generate $A^\bullet(X_{A_n})$.

3. The simplicial presentation and its monomials

In this section, we introduce the paper's main object of study: a new presentation of the Chow ring of a matroid which we call the simplicial presentation $A^\bullet_\nabla(M)$ of $A^\bullet(M)$.

After a combinatorial preparation in §3.1, we introduce the simplicial presentation in §3.2 and show that multiplying by its generators corresponds to a matroid operation called principal truncations. In §3.3, we extend this correspondence to establish a combinatorial interpretation of a monomial basis of the Chow ring of a matroid.

3.1. Matroid quotients, principal truncations, and matroid intersections

We first review the relevant combinatorial notions. We point to [48, §7] and [33, §2.3] for further details.

Let M and M' be matroids on a common ground set $E = \{0, 1, \ldots, n\}$.

Definition 3.1.1. The matroid M' is a (*matroid*) *quotient* of M, written $f : M' \twoheadleftarrow M$, if every flat of M' is also a flat of M. In particular, if M and M' are loopless, then $f : M' \twoheadleftarrow M$ if and only if $\Sigma_{M'} \subseteq \Sigma_M$.

Example 3.1.2. Any matroid on the ground set E is a quotient of the Boolean matroid $U_{|E|,E}$. Any Bergman fan of a loopless matroid is a subfan of the braid fan.

Example 3.1.3 (Realizable matroid quotients). Matroid quotients model linear surjections (dually, linear injections) in the following way. Let M and M' have realizations by

$\Bbbk^E \twoheadrightarrow V$ and $\Bbbk^E \twoheadrightarrow V'$ (respectively). If there exists a commuting diagram of linear maps

$$
\begin{array}{ccc}
\Bbbk^E & \longrightarrow\!\!\!\!\!\rightarrow & V \\
\| & & \downarrow \\
\Bbbk^E & \longrightarrow\!\!\!\!\!\rightarrow & V'
\end{array}
\qquad \text{or dually} \qquad
\begin{array}{ccc}
\mathbb{P}^n_{\Bbbk} & \longleftarrow & \mathbb{P}V^* \\
\| & & \uparrow \\
\mathbb{P}^n_{\Bbbk} & \longleftarrow & \mathbb{P}V'^*
\end{array}
$$

then M' is a matroid quotient of M. Matroid quotients $M' \twoheadleftarrow M$ arising in this way are called *realizable*. We caution that a matroid quotient $M' \twoheadleftarrow M$ with both M' and M realizable over the same field need not be realizable (for an example, see [9, §1.7.5]).

For a matroid quotient $f : M' \twoheadleftarrow M$, the f-*nullity* of a subset $A \subseteq E$ is defined to be

$$
n_f(A) := \mathrm{rk}_M(A) - \mathrm{rk}_{M'}(A).
$$

We say that M' is an *elementary (matroid) quotient* of M if $n_f(E) = 1$, or equivalently $\mathrm{rk}(M') = \mathrm{rk}(M) - 1$. An elementary quotient of M corresponds to a *modular cut* $\mathcal{K}$ of M, which is a nonempty collection $\mathcal{K} \subset \mathscr{L}_M$ of flats satisfying

(1) if $F_1 \in \mathcal{K}$ and $F_1 \subset F_2$, then $F_2 \in \mathcal{K}$,

(2) if $F_1, F_2 \in \mathcal{K}$ and $\mathrm{rk}_M(F_1) + \mathrm{rk}_M(F_2) = \mathrm{rk}_M(F_1 \cup F_2) + \mathrm{rk}_M(F_1 \cap F_2)$, then $F_1 \cap F_2 \in \mathcal{K}$.

A modular cut $\mathcal{K}$ of M defines an elementary quotient $M' \twoheadleftarrow M$ by

$$
\mathscr{L}_{M'} := \{F \in \mathscr{L}_M \mid F \text{ is not covered by an element of } \mathcal{K}\} \cup \mathcal{K},
$$

where F is *covered* by G if $F \subset G$ and $\mathrm{rk}(F) = \mathrm{rk}(G) - 1$. Conversely, given an elementary quotient $f : M' \twoheadleftarrow M$, one recovers the modular cut $\mathcal{K}$ of M defining the elementary quotient by

$$
\mathcal{K} = \{F \in \mathscr{L}_{M'} \mid n_f(F) = 1\}.
$$

We write $M' \overset{\mathcal{K}}{\twoheadleftarrow} M$ to denote an elementary quotient of M given by a modular cut $\mathcal{K}$.

Example 3.1.4. Let M have a realization $\Bbbk^E \twoheadrightarrow V$. For $\mathcal{K}$ a modular cut of M, let $v_{\mathcal{K}}$ be a nonzero vector contained in $\bigcap_{F \in \mathcal{K}} \mathrm{span}_{\Bbbk}(F)$ and not contained in any span of a flat that is not in $\mathcal{K}$.[7] Dually, with the notation as in §2.3, we have a general hyperplane $H_{\mathcal{K}} = \{f \in V^* \mid f(v_{\mathcal{K}}) = 0\}$ in V^* containing $\bigcup_{F \in \mathcal{K}} L_F$. Let us consider the commuting diagram

$$
\begin{array}{ccc}
\Bbbk^E & \longrightarrow\!\!\!\!\!\rightarrow & V \\
\| & & \downarrow \\
\Bbbk^E & \longrightarrow\!\!\!\!\!\rightarrow & V/\mathrm{span}_{\Bbbk}(v_{\mathcal{K}})
\end{array}
\qquad \text{or dually} \qquad
\begin{array}{ccc}
\mathbb{P}^E_{\Bbbk} & \longleftarrow & \mathbb{P}V^* \\
\| & & \uparrow \\
\mathbb{P}^E_{\Bbbk} & \longleftarrow & \mathbb{P}H_{\mathcal{K}}
\end{array}
$$

---

[7] For such $v_{\mathcal{K}}$ to exist, the field $\Bbbk$ must be large enough, and the elementary matroid quotient defined by the modular cut $\mathcal{K}$ must be realizable.

The map $\Bbbk^E \to V/\mathrm{span}_{\Bbbk}(v_{\mathcal{K}})$ is a realization of the matroid M' of the elementary quotient $M' \overset{\mathcal{K}}{\twoheadleftarrow} M$ defined by $\mathcal{K}$. Dually, with the notation as in §2.3, the associated hyperplane arrangement $\mathcal{A}_{\mathscr{R}(M')}$ is the intersection of $\mathbb{P}H_{\mathcal{K}}$ with the coordinate hyperplanes in $\mathbb{P}^E_{\Bbbk}$. Equivalently, the hyperplane arrangement $\mathcal{A}_{\mathscr{R}(M')}$ is the intersection of $\mathbb{P}H_{\mathcal{K}}$ with the hyperplanes in the hyperplane arrangement $\mathcal{A}_{\mathscr{R}(M)}$ under the inclusion $\mathbb{P}H_{\mathcal{K}} \subset \mathbb{P}V^*$.

Of particular interest in our case is when $\mathcal{K}$ is the interval $[F, E] \subset \mathscr{L}_M$, since an interval in $\mathscr{L}_M$ is always a modular cut. We call the resulting elementary quotient, denoted $T_F(M)$, the *principal truncation* of M associated to the flat F. An explicit description of principal truncations follows.

Proposition 3.1.5 ([48, Exercise 7.2.4.]). *The principal truncation $T_F(M)$ of a matroid M associated to a flat $F \in \mathscr{L}_M$ of rank at least 1 has bases*

$$\mathcal{B}(T_F(M)) = \{B \setminus f \mid B \in \mathcal{B}(M) \text{ and } f \in B \cap F \neq \emptyset\},$$

and the flats of $T_F(M)$ partition into two sets, $\mathscr{L}_{T_F(M)} = \mathcal{K} \sqcup \mathcal{L}$, according to their f-nullities by

$$\mathcal{K} = \{G \in \mathscr{L}_{T_F(M)} \mid n_f(G) = 1\} = \{G \in \mathscr{L}_M \mid F \subseteq G\},$$
$$\mathcal{L} = \{G \in \mathscr{L}_{T_F(M)} \mid n_f(G) = 0\}$$
$$= \{G \in \mathscr{L}_M \mid G \text{ not covered by an element in } [F, E]\}.$$

Remark 3.1.6. In Example 3.1.4, if $\mathcal{K} = [F, E]$ for some flat F, then we can set $v_{\mathcal{K}} = v_F$, a general vector in $\mathrm{span}_{\Bbbk}(F)$, and dually, we can set $H_{\mathcal{K}} = H_F$, a general hyperplane in V^* containing L_F.

We end our combinatorial preparation by connecting matroid quotients to Minkowski weights on braid fans via the notion of matroid intersections. Recall that a *spanning set* of a matroid is a subset that contains a basis of the matroid.

Definition 3.1.7. The *matroid intersection*[8] of two matroids M and N on a common ground set E is a new matroid $M \wedge N$ on E whose family of spanning sets is $\mathcal{S}(M \wedge N) = \{S \cap S' \mid S \in \mathcal{S}(M), \ S' \in \mathcal{S}(N)\}$.

The matroid $M \wedge N$ is a matroid quotient of both M and N. Matroid intersection behaves well in relation to Minkowski weights in the following way. Recall that the isomorphism $A^\bullet(\Sigma_{A_n}) \simeq \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ of Theorem 2.1.6 makes $\mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ into a graded ring, with multiplication called the stable intersection $\cap_{\mathrm{st}}$. The following proposition states that stable intersections of Bergman classes are Bergman classes of matroid intersections.

---

[8]Matroid intersection, as defined here and in [12,56], is related to but different from the same terminology used in [20,41], where one considers pairwise intersections of independent sets of two matroids.

Proposition 3.1.8 ([55, Proposition 4.4], [33, Remark 2.31]). *Let M and N be two matroids on a common ground set E, and let Δ_M and Δ_N be their Bergman classes, which are Minkowski weights on Σ_{A_n}. Then*

$$\Delta_M \cap_{\mathrm{st}} \Delta_N = \begin{cases} \Delta_{M \wedge N} & \text{if } M \wedge N \text{ is loopless,} \\ 0 & \text{otherwise.} \end{cases}$$

3.2. The variables of the simplicial presentation

We now define a new presentation $A_\nabla^\bullet(M)$ of the Chow ring of a matroid M, and discuss its first properties. The key result here is that the variables of $A_\nabla^\bullet(M)$ correspond to principal truncations of M.

We prepare by noting a distinguished set of nef divisor classes on Σ_{A_n} and their polytopes, considered in [50]. For a nonempty subset S of E, let

$$\nabla_S := \mathrm{Conv}(-\mathbf{e}_i \mid i \in S) \subset \mathbb{R}^E$$

denote the negative standard simplex of S. As the edges of ∇_S are parallel translates of $\mathbf{e}_i - \mathbf{e}_j$ for $i \neq j \in S$, Proposition 2.2.6 (in the form of Remark 2.2.7) implies that ∇_S is a polytope with the corresponding nef divisor class

$$h_S := -\sum_{S \subseteq T} z_T \in A_{FY}^1(\Sigma_{A_n}).$$

These divisors were considered in [50][9] and implicitly in [33]. We now consider the presentation of $A^\bullet(M)$ given by pullbacks of these nef divisor classes corresponding to (negative) standard simplices. For M a loopless matroid on E, and $\emptyset \neq S \subseteq E$, denote $h_S(M) := \iota_M^* h_S$. If $F = \mathrm{cl}_M(S)$ is the smallest flat containing S, note that

$$h_S(M) := \iota_M^* h_S = -\sum_{S \subseteq T} z_T(M) = -\sum_{F \subseteq G \in \mathscr{L}_M} z_G(M) = \iota_M^* h_F, \qquad (1)$$

as $z_T(M) = \iota_M^* z_T = 0$ for all $T \subseteq E$ not a flat of M. By construction, the elements $h_F(M) \in A^1(M)$ are (combinatorially) nef divisor classes on M. We will simply write h_F for $h_F(M)$ when there is no confusion.

Definition 3.2.1. For M a loopless matroid on E, the *simplicial presentation* $A_\nabla^\bullet(M)$ of the Chow ring of M is the presentation of $A^\bullet(M)$ whose generators are $\{h_F\}_{F \in \mathscr{L}_M \setminus \{\emptyset\}}$ where

$$h_F := -\sum_{F \subseteq G} z_G \in A_{FY}^\bullet(M).$$

We call $\{h_F\}_{F \in \mathscr{L}_M \setminus \{\emptyset\}}$ the *simplicial generators*.

---

[9]In [50] the author uses y_S to denote the nef divisor of the standard simplex of S instead of the negative standard simplex. This difference is due to using inner normal fans instead of outer normal fans of polytopes.

The variable h here stands for "hyperplane"; for the geometric origin of the simplicial presentation see Remarks 3.2.6 and 3.2.7. The linear change of variables from $\{z_F\}_{F \in \mathscr{L}_M \setminus \{\emptyset\}}$ to $\{h_F\}_{F \in \mathscr{L}_M \setminus \{\emptyset\}}$ is evidently invertible, given by an upper triangular matrix. Explicitly, by Möbius inversion we have

$$-z_F = \sum_{F \subseteq G} \mu(F, G) h_G$$

where μ is the Möbius function on the lattice $\mathscr{L}_M$. Thus, the explicit presentation of $A_{\triangledown}^{\bullet}(M)$ is

$$A_{\triangledown}^{\bullet}(M) := \mathbb{R}[h_F \mid F \in \mathscr{L}_M \setminus \{\emptyset\}]/(I + J)$$

where

$$I = \langle h_a \mid a \in \mathfrak{A}(M) \rangle,$$
$$J = \left\langle \left(\sum_{F \subseteq G} \mu(F, G) h_G \right) \left(\sum_{F' \subseteq G'} \mu(F', G') h_{G'} \right) \;\middle|\; F, F' \text{ incomparable} \right\rangle.$$

Denote by $\mathscr{L}_M^{\geq 2}$ the set of flats of M of rank at least 2. Noting that $h_a = 0 \in A_{\triangledown}^{\bullet}(M)$ for any atom $a \in \mathfrak{A}(M)$, we define $\{h_F \mid F \in \mathscr{L}_M^{\geq 2}\}$ to be the *nontrivial simplicial generators* of the Chow ring of M. They form a basis of $A_{\triangledown}^1(M)$.

Remark 3.2.2. When the matroid M is the cyclic matroid of the complete graph K_{n-1} on $n - 1$ vertices, the Chow ring of M is the cohomology ring of the Deligne–Knudson–Mumford space $\overline{\mathcal{M}}_{0,n}$ of rational curves with n marked points [17, §4.3], [44, Theorem 6.4.12]. In this case, by using the minimal building set instead of the maximal building set, the simplicial presentation recovers the Etingof–Henriques–Kamnitzer–Rains–Singh presentation of the cohomology ring of $\overline{\mathcal{M}}_{0,n}$ [24, 54]. In this presentation, the author of [19] showed that the cohomology ring of $\overline{\mathcal{M}}_{0,n}$ is Koszul because it has a quadratic Gröbner basis. In the classical presentation, the Chow ring of any matroid with rank > 3 has no quadratic Gröbner basis.

The following theorem, which relates the variables of the simplicial presentation to principal truncations, is the key property of the simplicial presentation that we use throughout this paper. Let us first fix a notation:

Notation. For a nonempty subset $S \subseteq E$, we denote by H_S the matroid with bases

$$\mathcal{B}(H_S) := \{E \setminus i \mid i \in S\},$$

or equivalently $H_S = U_{|E \setminus S|, E \setminus S} \oplus U_{|S|-1, S}$.

Theorem 3.2.3. *Let M be a loopless matroid on E, and S a nonempty subset of E. Write F for the smallest flat of M containing S. Then $H_S \wedge M = T_F(M)$, and the nef divisor class $h_S \in A_{\triangledown}^1(\Sigma_{A_n})$ satisfies*

$$h_S \cap \Delta_M = \begin{cases} \Delta_{T_F(M)} & \text{if } \mathrm{rk}_M(S) > 1, \\ 0 & \text{otherwise.} \end{cases}$$

The theorem will follow mostly from the following lemma.

Lemma 3.2.4. *Let M and H_S be as in the statement of Theorem 3.2.3, and let F be the smallest flat of M containing S. Then*

$$H_S \wedge M = T_F(M), \tag{2}$$

and consequently

$$\Delta_{H_F} \cap_{\mathrm{st}} \Delta_M = \begin{cases} \Delta_{T_F(M)} & \textit{if } \mathrm{rk}_M(F) > 1, \\ 0 & \textit{otherwise.} \end{cases} \tag{3}$$

Proof. We first show that (3) follows from (2). A matroid is loopless if and only if the empty set $\emptyset$ is a flat. By Proposition 3.1.5, the matroid $T_F(M)$ is thus loopless if and only if $\emptyset$ is not covered by an element in the interval $[F, E]$. As claimed, this happens if and only if $\mathrm{rk}_M(F) > 1$.

We now prove (2). By definition of $H_S \wedge M$, the minimal elements in the set $\mathcal{S}(H_S \wedge M)$ of spanning sets are $B \setminus i$ where $B \in \mathcal{B}(M)$ and $i \in B \cap S \neq \emptyset$. Since minimal spanning sets are bases, we have

$$\mathcal{B}(H_S \wedge M) = \{B \setminus i \mid B \in \mathcal{B}(M), i \in B \cap S \neq \emptyset\}.$$

When $S = F$, this is the description of the bases of $T_F(M)$ in Proposition 3.1.5, so it remains to show $H_S \wedge M = H_F \wedge M$. Evidently, we have $\mathcal{B}(H_S \wedge M) \subseteq \mathcal{B}(H_F \wedge M)$ since $S \subseteq F$. For the other inclusion, suppose we have a basis $B \setminus f$ of $H_F \wedge M$ where $B \in \mathcal{B}(M)$ and $f \in B \cap F \neq \emptyset$. We show that there is an element $s \in S$ such that $B' = (B \setminus f) \cup s$ is also a basis of M, which implies that $B \setminus f = B' \setminus s$ is a basis of $H_S \wedge M$. First, because F is the closure of S, we have $\mathrm{rk}_M((B \setminus f) \cup S) = \mathrm{rk}_M((B \setminus f) \cup F)$. We also have $\mathrm{rk}_M((B \setminus f) \cup F) = \mathrm{rk}_M(B \cup F) = \mathrm{rk}_M(E)$ since B is a basis. Since $B \setminus f$ is independent in M, we thus conclude from $\mathrm{rk}_M((B \setminus f) \cup S) = \mathrm{rk}_M(E)$ that $B \setminus f$ can be extended to a basis by an element in S. That is, there exists an element $s \in S$ such that $(B \setminus f) \cup s$ is a basis of M, as desired. ∎

Proof of Theorem 3.2.3. Let $\delta_{\Sigma_{A_n}} : A^\bullet(\Sigma_{A_n}) \xrightarrow{\sim} \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ be the isomorphism map in Theorem 2.1.6. We claim that $\delta_{\Sigma_{A_n}}(h_S) = \Delta_{H_S}$, which is proved in Lemma 3.2.5 below. Our desired statement then follows immediately from Lemma 3.2.4, because $h_S \cap \Delta_M = (\delta_{\Sigma_{A_n}}(h_S)) \cap_{\mathrm{st}} \Delta_M$ by the definition of stable intersection $\cap_{\mathrm{st}}$. ∎

Lemma 3.2.5. *Let $\delta_{\Sigma_{A_n}} : A^\bullet(\Sigma_{A_n}) \xrightarrow{\sim} \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ be the isomorphism map in Theorem 2.1.6. Then*

$$\delta_{\Sigma_{A_n}}(h_S) = \Delta_{H_S}.$$

Proof. We claim that the support $|\Delta_{H_S}|$ of Δ_{H_S} is equal to the support of the $(n-1)$-skeleton of the outer normal fan of the negative standard simplex ∇_S. If this is the case, then Proposition 2.1.7 implies $\delta_{\Sigma_{A_n}}(h_S) = \Delta_{H_S}$ because all the edges of the negative standard simplex ∇_S have lattice length 1.

Now, to prove the claim, note first that the translate $\nabla_S + \mathbf{e}_E$ of ∇_S is $\mathrm{Conv}(\mathbf{e}_{E \setminus i} \mid i \in S) \subset \mathbb{R}^E$, which is equal to the base polytope $Q(H_S)$ of H_S. Since all faces of $Q(H_S)$, except for the vertices, are base polytopes of loopless matroids, by Lemma 2.2.3 the support of $|\Delta_{H_S}|$ equals the support of the $(n-1)$-dimensional skeleton of the outer normal fan of ∇_S. ∎

Theorem 3.2.3 encodes the combinatorics of the following geometric motivation for the simplicial presentation.

Remark 3.2.6. We recall the following standard fact in algebraic geometry. Let $L \subset V^*$ be an inclusion of vector spaces, so that $\mathbb{P}L$ is a linear subvariety of $\mathbb{P}V^*$. Let $X = \mathrm{Bl}_{\mathbb{P}L} \, \mathbb{P}V^*$ be the blow-up of $\mathbb{P}V^*$ along $\mathbb{P}L$, which is also the graph of the rational map $\mathbb{P}V^* \dashrightarrow \mathbb{P}(V^*/L)$, and let $\pi : X \to \mathbb{P}V^*$ be the blow-down map. Let $h = c_1(\mathcal{O}_{\mathbb{P}V^*}(1))$ denote the hyperplane class, and $\mathcal{E}$ the exceptional divisor of the blow-up. Then the map $\mathrm{Bl}_{\mathbb{P}L} \, \mathbb{P}V^* \to \mathbb{P}(V^*/L)$ corresponds to the linear series $|\mathcal{O}_X(\pi^*h - \mathcal{E})|$, so the sections of the line bundle $\mathcal{O}_X(\pi^*h - \mathcal{E})$ correspond to the hyperplanes in $\mathbb{P}V^*$ that contain $\mathbb{P}L$.

Now, suppose M has a realization $\mathscr{R}(M)$ over an algebraically closed field $\Bbbk$ as $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$, and let notations be as in §2.3. The geometry of $A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(M)$ in Remark 2.3.3 implies

$$h_F = \sum_{G \supseteq F} -z_G = -z_E - \sum_{G \supseteq F} x_G = \pi^*_{\mathscr{R}(M)} h - \sum_{G \supseteq F} \mathcal{E}_G,$$

and hence h_F represents the divisor class of the strict transform of a general hyperplane in $\mathbb{P}V^*$ containing the linear subvariety $\mathbb{P}L_F$. Thus multiplying by h_F corresponds to intersecting by a general hyperplane in $\mathbb{P}V^*$ containing $\mathbb{P}L_F$, which corresponds to the principal truncation $T_F(M)$ by Example 3.1.4 (in the form of Remark 3.1.6). More precisely, we have

$$h_F \cdot [Y_{\mathscr{R}(M)}] = [Y_{\mathscr{R}(T_F(M))}] \in A^\bullet(X_{\Sigma_{A_n}}).$$

Theorem 3.2.3 is the combinatorial mirror of this geometric observation.

Remark 3.2.7 (Relation to (A)). Suppose M has a realization $\mathscr{R}(M)$ by $\mathbb{P}V^* \hookrightarrow \mathbb{P}^n$. By the second description in Remark 2.3.2 (2), the wonderful compactification $Y_{\mathscr{R}(M)}$ is embedded in the product $\prod_{F \in \mathscr{L}_M \setminus \{\emptyset\}} \mathbb{P}(V^*/L_F)$ of projective spaces. We described h_F as a divisor class represented by the strict transform of a general hyperplane in $\mathbb{P}V^*$ containing $\mathbb{P}L_F$ in Remark 3.2.6. Alternatively, the variable h_F thus represents the base-point-free divisor obtained as the hyperplane class pullback of the map $Y_{\mathscr{R}(M)} \to \mathbb{P}(V^*/L_F)$. In other words, the divisor classes $h_S \in A^\bullet_\nabla(M)$ play the role of "base-point-free divisor classes," and we have interpreted the hyperplane class pullbacks of the maps they define as principal truncations.

Remark 3.2.8. In the classical presentation $A^\bullet(\Sigma_{A_n})$, the cap product $x_S \cap \Delta_M$ is almost never a Bergman class of a matroid—it is a Minkowski weight which may have negative weights on some cones. This reflects the geometry that the divisor x_S is effective but usually not nef.

3.3. A monomial basis of the simplicial presentation and relative nested quotients

We introduce the notion of relative nested quotients, which are relative generalizations of (loopless) Schubert matroids in matroid theory, and we show that they are in bijection with elements of a monomial basis of $A_\nabla^\bullet(M)$.

We start by producing a monomial basis of $A_{FY}^\bullet(M)$ via the Gröbner basis computation in [27]. Pick a total order $>$ on elements of $\mathscr{L}_M$ such that $F > G$ if $\mathrm{rk}_M(F) \leq \mathrm{rk}_M(G)$, and take the induced lex monomial order on $A_{FY}^\bullet(M)$. A Gröbner basis for $A_{FY}^\bullet(M)$ was given as follows.

Theorem 3.3.1 ([27, Theorem 1]). *The following form a Gröbner basis for the ideal of $A_{FY}^\bullet(M)$:*

$$
\begin{cases}
z_F z_G, & F \text{ and } G \text{ are incomparable nonempty flats}, \\
z_F (\sum_{H \geq G} z_H)^{\mathrm{rk}\,G - \mathrm{rk}\,F}, & F \subsetneq G \text{ nonempty flats}, \\
(\sum_{H \geq G} z_H)^{\mathrm{rk}\,G}, & G \text{ a nonempty flat}.
\end{cases}
$$

In [27], the authors associate a ring to an atomic lattice with a chosen "building set," and [27, Theorem 1] provides a Gröbner basis for the defining ideal of these more general rings. In our setting, the atomic lattice is the lattice of flats of a matroid, and the building set is the collection of all nonempty flats. We now note that the Gröbner basis in [27] carries over to the simplicial presentation as follows. Again, pick a total ordering $>$ of $\mathscr{L}_M$ such that if $\mathrm{rk}_M(F) \leq \mathrm{rk}_M(G)$, then $F > G$.

Proposition 3.3.2. *The following is a Gröbner basis for the defining ideal of $A_\nabla^\bullet(M)$ with respect to the lex monomial ordering induced by $>$:*

$$
\begin{cases}
(\sum_{F \subseteq G} \mu(F, G) h_G)(\sum_{F' \subseteq G'} \mu(F', G') h_{G'}), & F, F' \text{ incomparable}, \\
(\sum_{F \subseteq G} \mu(F, G) h_G) \cdot h_{F'}^{\mathrm{rk}\,F' - \mathrm{rk}\,F}, & F \subsetneq F', \\
h_F^{\mathrm{rk}\,F}, & F \in \mathscr{L}_M \setminus \{\emptyset\}.
\end{cases}
$$

Proof. Let $S_{FY} = \mathbb{R}[z_F \mid F \in \mathscr{L}_M \setminus \{\emptyset\}]$ and $S_\nabla = \mathbb{R}[h_F \mid F \in \mathscr{L}_M \setminus \{\emptyset\}]$, and define $\varphi : S_{FY} \to S_\nabla$ to be the substitution $z_F \mapsto -\sum_{F \subseteq G} \mu(F, G) h_G$.

Observe that φ is lower triangular with -1's on the diagonal when the variables z_F and h_F are written in descending order with respect to $>$. Hence, if $f \in S$ with initial monomial $z_{F_1}^{e_1} \cdots z_{F_k}^{e_k}$, then the initial monomial of $\varphi(f)$ is $h_{F_1}^{e_1} \cdots h_{F_k}^{e_k}$. The proposition now follows from the fact that the elements of the Gröbner basis above are the images under φ of the elements of the Gröbner basis given in Theorem 3.3.1. ∎

As a result, we obtain a monomial basis of $A_\nabla^\bullet(M)$.

Corollary 3.3.3. *For $c \in \mathbb{Z}_{\geq 0}$, a monomial $\mathbb{R}$-basis for the degree c part $A_\nabla^c(M)$ of the Chow ring $A_\nabla^\bullet(M)$ of a matroid M is*

$$
\left\{ h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \;\middle|\; \sum a_i = c, \; \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k, \; 1 \leq a_i < \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1}) \right\}.
$$

We call this basis of $A_\nabla^\bullet(M)$ the nested basis *of the Chow ring of M.*

Proof. If B is a Gröbner basis for an ideal I of a polynomial ring R over a field k, then the monomials not divisible by the initial term of any element of B are a k-basis for R/I [49, Theorem 39.6]. Applying this fact to the Gröbner basis of Proposition 3.3.2 yields the result. ∎

While the nested basis of $A_\triangledown^\bullet(M)$ looks identical to the one given for $A_{FY}^\bullet(M)$ in [27, Corollary 1], we show here that with the simplicial presentation the monomials in the basis now allow for a combinatorial interpretation as a distinguished set of matroid quotients of M.

Let $f : M' \twoheadleftarrow M$ be a matroid quotient on a ground set E.

Definition 3.3.4. An *f-cyclic flat* of f is a flat $F \in \mathscr{L}_{M'}$ such that F is minimal (with respect to inclusion) among the flats $F' \in \mathscr{L}_{M'}$ such that $n_f(F') = n_f(F)$. A matroid M' is a *relative nested quotient* of M if the f-cyclic flats of M' form a chain.

Relative nested quotients are relative generalizations of (loopless) Schubert matroids:

Example 3.3.5. If $M = U_{|E|,E}$ then any matroid M' is a quotient $f : M' \twoheadleftarrow M$. In this case, we have $n_f(A) = |A| - \mathrm{rk}_{M'}(A)$ for a subset $A \subseteq E$. So, if further a subset $B \subsetneq A$ satisfies $n_f(B) = n_f(A)$, then $|A| - |B| = \mathrm{rk}_{M'}(A) - \mathrm{rk}_{M'}(B)$, which occurs if and only if $A \setminus B$ is a set of coloops in the restriction $M'|_A$. In other words, the f-cyclic flats of M' are precisely the *cyclic flats* of M', which are flats F of M' such that $M'|_F$ has no coloops. Moreover, the relative nested quotients of $U_{|E|,E}$ are called *nested matroids*, which in the literature also go by the name of (loopless) *Schubert matroids* because they are realized by general points in the appropriate Schubert subvarieties of Grassmannians. See [33, §2.2] for more on cyclic flats and nested matroids.

The data of cyclic flats of a matroid and their ranks determine the matroid [13, Proposition 2.1]. We generalize the statement to f-cyclic flats of a matroid quotient. We first need the following fact about obtaining any matroid quotient as a sequence of elementary quotients.

Lemma 3.3.6. (1) *([36], [12, Exercise 7.20]) Any matroid quotient $f : M' \twoheadleftarrow M$ can be obtained as a sequence of elementary quotients in a canonical way called the Higgs factorization of f. The Higgs factorization of a quotient $f : M' \twoheadleftarrow M$ with $n_f(E) = c$ is a sequence of elementary quotients*

$$M' = M_0 \overset{\mathscr{K}_1}{\twoheadleftarrow} M_1 \overset{\mathscr{K}_2}{\twoheadleftarrow} \cdots \overset{\mathscr{K}_c}{\twoheadleftarrow} M_c = M$$

where the bases of M_i for $i = 1, \ldots, c$ are defined as

$$\mathscr{B}(M_i) = \{A \subseteq E \mid A \text{ spanning in } M', \text{ independent in } M, \text{ and } |A| = \mathrm{rk}(M') + i\}.$$

(2) *([39, Theorem 3.4]) The modular cuts $\mathscr{K}_i$ of the Higgs factorization are*

$$\mathscr{K}_i = \{G \in \mathscr{L}_{M_i} \mid n_f(G) \geq i\}.$$

Proposition 3.3.7. *The data of the f-cyclic flats, their f-nullities, and the matroid M determine the quotient $f : M' \lll M$. More precisely, writing $n_f(E) = c$, the data recovers the Higgs factorization $M' = M_0 \overset{\mathcal{K}_1}{\lll} M_1 \overset{\mathcal{K}_2}{\lll} \cdots \overset{\mathcal{K}_c}{\lll} M_c = M$ of f by specifying the modular cuts $\mathcal{K}_i$ to be*

$$\mathcal{K}_i = \{G \in \mathscr{L}_{M_i} \mid G \supseteq F \text{ for some } F \in \mathrm{cyc}(f) \text{ with } n_f(F) \geq i\}$$

for each $i = 1, \ldots, c$.

Proof. For each $i = 1, \ldots, c$, the modular cut $\mathcal{K}_i$ is $\{G \in \mathscr{L}_{M_i} \mid n_f(G) \geq i\}$ by Lemma 3.3.6 (2). This can equivalently be written as $\{G \in \mathscr{L}_{M_i} \mid G \supseteq F \text{ for some } F \in \mathrm{cyc}(f) \text{ with } n_f(F) \geq i\}$ by the definition of f-cyclic flats. $\blacksquare$

We now show that the nested basis of $A_\nabla^\bullet(M)$ given in Corollary 3.3.3 is in bijection with the set of relative nested quotients of M.

Theorem 3.3.8. *Let M be a loopless matroid of rank $r = d + 1$. For each $0 \leq c \leq d$, the cap product map*

$$A_\nabla^c(M) \to \mathrm{MW}_{d-c}(\Sigma_M), \quad \xi \mapsto \xi \cap \Delta_M,$$

induces a bijection between the monomial basis for $A_\nabla^c(M)$ given in Corollary 3.3.3 and the set of Bergman classes $\Delta_{M'}$ of loopless relative nested quotients $M' \lll M$ with $\mathrm{rk}(M') = \mathrm{rk}(M) - c$.

Let us first prove a lemma. For a flat F of a matroid M, and an integer $a \geq 0$, denote by $T_F{}^a(M)$ the matroid obtained from M by performing a many times the principal truncation associated to F. Such iteration of T_F is well-defined because principal truncation associated to F preserves F as a flat.

Lemma 3.3.9. *Let $f : M' \lll M$ be a matroid quotient formed by two loopless matroids, and let F be a nonempty flat of M'. Then the following hold:*

(1) *Suppose G is another flat of M' such that $G \subsetneq F$, and $a < \mathrm{rk}_{M'}(F) - \mathrm{rk}_{M'}(G)$. Then the subset G is a flat of $T_F{}^a(M')$. In particular, setting $G = \emptyset$, the matroid $T_F{}^a(M')$ is loopless if $a < \mathrm{rk}_{M'}(F)$.*

(2) *Suppose $F \subseteq S$ for every nonempty f-cyclic flat S of M'. Denote by g the matroid quotient $g : T_F(M') \lll M$. Then the set $\mathrm{cyc}(g)$ of g-cyclic flats equals $\{F\} \cup \mathrm{cyc}(f)$.*

Proof. For statement (1), we first recall the description of the flats in a principal truncation (Proposition 3.1.5): A flat G' of M' satisfying $G' \subsetneq F$ remains a flat in $T_F(M')$ if F does not cover G' in M'. Moreover, the rank of F decreases by 1 each time one performs principal truncation T_F. Hence, the condition $a < \mathrm{rk}_{M'}(F) - \mathrm{rk}_{M'}(G)$ ensures that G is not covered by F in the matroid $T_F{}^{a-1}(M')$, and so G is a flat of $T_F{}^a(M')$.

For statement (2), again by Proposition 3.1.5, the set $[F, E] = \{F' \mid F' \in \mathscr{L}_{M'} \text{ and } F' \supseteq F\}$ is exactly the set of flats of $T_F(M')$ satisfying $\mathrm{rk}_{M'}(F) - \mathrm{rk}_{T_F(M')}(F') = 1$,

and the other flats of $T_F(M')$ do not change in rank when considered as flats of M'. Thus, since F is contained in every nonempty f-cyclic flat of M', the subset F is a g-cyclic flat with $n_g(F) = n_f(F) + 1$, and if S is a nonempty f-cyclic flat with $n_f(S) = k$, then S is a g-cyclic flat with $n_g(S) = k + 1$. ∎

Proof of Theorem 3.3.8. Let $h_{F_1}^{a_1} \cdots h_{F_k}^{a_k}$ be an element of the monomial basis given in Corollary 3.3.3. By repeated application of Theorem 3.2.3 combined with Lemma 3.3.9 (1), we have

$$h_{F_1}^{a_1} \cdots h_{F_k}^{a_k} \cap \Delta_M = \Delta_{M'}$$

where M' is the loopless matroid obtained from M by a sequence of principal truncations, first by F_k repeated a_k times, then by F_{k-1} repeated a_{k-1} times, and so forth. Moreover, Lemma 3.3.9 (2) implies that $f : M' \twoheadleftarrow M$ is a matroid quotient with $\mathrm{cyc}(f) = \{\emptyset, F_1, \dots, F_k\}$ and $n_f(F_j) = \sum_{i=1}^{j} a_i$. We have thus shown that an element of the nested basis defines a loopless relative nested quotient by the cap product.

Conversely, let $f : M' \twoheadleftarrow M$ be a loopless relative nested quotient with $\mathrm{cyc}(f) = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k\}$. Define integers $a_1, \dots, a_k$ by $n_f(F_j) = \sum_{i=1}^{j} a_i$ for all $j = 1, \dots, k$. Proposition 3.3.7 implies that the f-cyclic flats and their f-nullities of a relative nested quotient $f : M' \twoheadleftarrow M$ recover the Higgs factorization

$$M' = M_0 \overset{\mathcal{K}_1}{\twoheadleftarrow} M_1 \overset{\mathcal{K}_2}{\twoheadleftarrow} \cdots \overset{\mathcal{K}_c}{\twoheadleftarrow} M_c = M$$

of f by specifying the modular cuts to be

$$\mathcal{K}_i = \{G \in \mathscr{L}_{M_i} \mid G \supseteq F \text{ for some } F \in \mathrm{cyc}(f) \text{ with } n_f(F) \geq i\}.$$

Thus, since by assumption the f-cyclic flats form a chain, the descriptions of the modular cuts $\mathcal{K}_i$ imply that the matroid M' is obtained by a sequence of principal truncations, first by F_k repeated a_k times, then by F_{k-1} repeated a_{k-1} times, and so forth, where $a_j = n_f(F_j) - n_f(F_{j-1})$ for $j > 1$, and $a_1 = n_f(F_1)$. ∎

Moreover, the bijection given in the previous theorem respects linear independence.

Proposition 3.3.10. *The elements of*

$$\{\Delta_{M'} \mid M' \text{ is a loopless relative nested quotient of } M\}$$

are linearly independent in $\mathrm{MW}_\bullet(\Sigma_{A_n})$.

Proof. The proof is a modification of the one given for nested matroids in [33, Proposition 3.2]. Fix a matroid M, and let

$$\mathscr{M} = \{\Delta_{M'} \mid M' \text{ is a rank } r \text{ loopless relative nested quotient of } M\}.$$

Write $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ for the cyclic flats of the quotient $M' \twoheadleftarrow M$, and define $\gamma : \mathscr{M} \to \mathbb{N}^r$ by

$$\gamma(\Delta_{M'})_i := \begin{cases} \mathrm{rk}_{M'}(F_i) - \mathrm{rk}_{M'}(F_{i-1}) & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

We show that there are no relations among the elements of $\mathscr{M}$ using a lexicographic induction. The following claim will be used repeatedly.

Claim. *Let* $\Delta_{M'} \in \mathscr{M}$, *let* $\mathscr{F} = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k\}$ *be the cyclic flats of* $f :$ $M' \twoheadleftarrow M$, *and let* $\mathscr{F}'$ *be a maximal chain of flats of* M' *containing* $\mathscr{F}$. *If* $\Delta_{N'} \in \mathscr{M}$ *for a quotient* $g : N' \twoheadleftarrow M$ *has the property that* $\mathscr{F}'$ *is also a maximal chain of flats of* N', *then either* $M' = N'$ *or* $\gamma(\Delta_{N'}) <_{\text{lex}} \gamma(\Delta_{M'})$.

Proof of claim. If $M' \neq N'$, then there exists a minimal $0 < j \leq k$ such that F_j is not cyclic in N'. Hence, there is a cyclic flat G of N' with $F_{j-1} \subset G \subsetneq F_j$ and $n_g(G) = n_g(F_j)$. Moreover, $G \neq F_{j-1}$ because

$$n_g(G) = n_g(F_j) = n_f(F_j) > n_f(F_{j-1}) = n_g(F_{j-1}).$$

Consequently, $\gamma(\Delta_{N'})_j < \gamma(\Delta_{M'})_j$, so $\gamma(\Delta_{N'}) <_{\text{lex}} \gamma(\Delta_{M'})$ by our choice of j. ◇

Now we prove the proposition. Suppose that $\sum_{\Delta_{M'} \in \mathscr{M}} c_{M'} \Delta_{M'} = 0$. We show that $c_{M'} = 0$ for all M'.

If $\Delta_{M'} \in \mathscr{M}$ with $\gamma(\Delta_{M'})$ lex-minimal in $\gamma(\mathscr{M})$, then the claim implies that there is a cone of Σ_{A_n} on which $\Delta_{M'}$ takes value 1 and all other elements of $\mathscr{M}$ take value 0; therefore, $c_{M'} = 0$. Otherwise, $\gamma(\Delta_{M'})$ is not lex-minimal. Suppose by induction that $c_{N'} = 0$ for all $\Delta_{N'} \in \mathscr{M}$ with $\gamma(\Delta_{N'}) <_{\text{lex}} \gamma(\Delta_{M'})$. By the claim, there is a cone of Σ_{A_n} on which $\Delta_{M'}$ takes value 1 and all elements $\Delta_{N''}$ of $\mathscr{M}$ such that $\gamma(\Delta_{N''}) \not<_{\text{lex}} \gamma(\Delta_{M'})$ take value 0; therefore, $c_{M'} = 0$, as desired. ∎

4. The Poincaré duality property

As a first application of the simplicial presentation, we establish the Poincaré duality property for Chow rings of matroids. While this was established in [1, Theorem 6.19] by a double induction, our proof is not inductive.

We review some facts about Poincaré duality algebras in §4.1, but we will only need a small portion of these facts in this section—the rest will be needed later in Section 6. We prove the Poincaré duality property of Chow rings of matroids and discuss some consequences in §4.2.

4.1. Poincaré duality algebras

We review some general algebraic notions about Poincaré duality algebras. Let $\Bbbk$ be a field.

Definition 4.1.1. A graded finite (commutative) $\Bbbk$-algebra $A^\bullet = \bigoplus_{i=0}^{d} A^i$ is a (*graded*) *Poincaré duality algebra of dimension* d if (i) $A^0 = \Bbbk$, and (ii) there exists an isomorphism $\int : A^d \xrightarrow{\sim} \Bbbk$, called the *degree map* of $A^\bullet$, such that the map

$$A^k \to \text{Hom}(A^{d-k}, \Bbbk), \quad \xi \mapsto \left(\zeta \mapsto \int \xi \cdot \zeta\right),$$

is an isomorphism for all $0 \leq k \leq d$, or equivalently the pairing

$$A^i \times A^{d-i} \to A^d \simeq \mathbb{k}, \quad (\xi, \zeta) \mapsto \int \xi \cdot \zeta,$$

is nondegenerate for all $0 \leq i \leq d$.

We write $(A^\bullet, \int)$ for a Poincaré duality algebra with a chosen degree map $\int$. In Section 6, we will often drop the degree symbol $\int$ when the context is clear. In particular, for $\zeta \in A^1$ we will often write ζ^d to mean $\int \zeta^d$. Two useful facts about Poincaré duality algebras follow. Both are straightforward to check.

Proposition 4.1.2. *Let $(A^\bullet, \int_A)$ and $(B^\bullet, \int_B)$ be Poincaré duality algebras of dimension d_A and d_B over a common field $\mathbb{k}$.*

(1) *The tensor product $(A \otimes B)^\bullet = \bigoplus_\bullet (\bigoplus_{i+j=\bullet} A^i \otimes B^j)$ is also a Poincaré duality algebra of dimension $d_A + d_B$ with degree map*

$$\int_{A \otimes B} : (A \otimes B)^{d_A + d_B} = A^{d_A} \otimes B^{d_B} \to \mathbb{k}, \quad a \otimes b \mapsto \int_A a \cdot \int_B b.$$

(2) *A surjection $A^\bullet \twoheadrightarrow B^\bullet$ of Poincaré duality algebras of the same dimension is an isomorphism.*

We will use the following construction to establish that Chow rings of matroids are Poincaré duality algebras.

Proposition 4.1.3. *If $(A^\bullet, \int)$ is a Poincaré duality algebra of dimension d, and $f \in A^\bullet$ a homogeneous element of degree k, then the $\mathbb{k}$-algebra*

$$A^\bullet / \mathrm{ann}(f), \quad \text{where} \quad \mathrm{ann}(f) = \{a \in A^\bullet \mid af = 0\},$$

is a Poincaré duality algebra of dimension $d - k$ with the induced *degree map $\int_f$ defined by $\int_f (a + \mathrm{ann}(f)) := \int af$ for $a \in A^{d-k}$.*

Proof. This is a straightforward check; see [46, Corollary I.2.3] for example. ∎

It will sometimes be convenient to identify elements of the ring $A^\bullet / \mathrm{ann}(f)$ with elements of the principal ideal $\langle f \rangle \subset A^\bullet$, with multiplication given by $af \cdot bf = (ab) \cdot f$. The construction in Proposition 4.1.3 will arise in §4.2 with f being the Bergman class of a matroid.

The rest of this subsection will not be needed until Section 6.

We describe another way the construction in Proposition 4.1.3 arises in the context of Chow cohomology rings of fans. Let Σ be a d-dimensional smooth rational fan in $N_\mathbb{R}$ for a lattice N, and let $\rho \in \Sigma(1)$ be ray. Denote by $\bar{u}$ the image of $u \in N_\mathbb{R}$ under the projection $N_\mathbb{R} \twoheadrightarrow N_\mathbb{R}/\mathrm{span}(\rho)$. The *star of Σ at ρ* is a $(d-1)$-dimensional fan in $N_\mathbb{R}/\mathrm{span}(\rho)$ defined by

$$\mathrm{star}(\rho, \Sigma) := \{\bar{\sigma} \mid \sigma \in \Sigma \text{ contains } \rho\}.$$

By definition of the Chow cohomology ring, one can check that there is a surjection $A^\bullet(\Sigma) \twoheadrightarrow A^\bullet(\operatorname{star}(\rho, \Sigma))$ determined by

$$x_{\rho'} \mapsto \begin{cases} x_{\overline{\rho'}} & \text{if } \rho' \text{ and } \rho \text{ form a cone in } \Sigma, \\ 0 & \text{otherwise,} \end{cases}$$

for each $\rho' \neq \rho$. Since

$$\langle x_{\rho'} \mid \rho' \text{ and } \rho \text{ do not form a cone in } \Sigma \rangle \subset \operatorname{ann}_{A^\bullet(\Sigma)}(x_\rho),$$

we get an induced map

$$\pi_\rho : A^\bullet(\operatorname{star}(\rho, \Sigma)) \twoheadrightarrow A^\bullet(\Sigma)/\operatorname{ann}(x_\rho).$$

In Section 6, we will use the following criterion for π_ρ to be an isomorphism.

Proposition 4.1.4 ([1, Proposition 7.13]). *Suppose that the Chow cohomology ring $A^\bullet(\Sigma)$ is a Poincaré duality algebra. Then the map $\pi_\rho : A^\bullet(\operatorname{star}(\rho, \Sigma)) \twoheadrightarrow A^\bullet(\Sigma)/\operatorname{ann}(x_\rho)$ is an isomorphism if and only if $A^\bullet(\operatorname{star}(\rho, \Sigma))$ is a Poincaré duality algebra.*

Proof. The algebra $A^\bullet(\Sigma)/\operatorname{ann}(x_\rho)$ is a Poincaré duality algebra by Proposition 4.1.3. The statement thus follows from Proposition 4.1.2 (2). ∎

4.2. Poincaré duality for matroids

We show that the Chow ring $A^\bullet(M)$ of a loopless matroid M is a Poincaré duality algebra with $\int_M$ as the degree map. While this was proved in [1, Theorem 6.19], we give a noninductive proof by using the simplicial presentation.

Our main theorem of the section is the following.

Theorem 4.2.1. *Let M be a loopless matroid of rank $r = d + 1$ on a ground set $E = \{0, 1, \ldots, n\}$, and consider the Bergman class $\Delta_M \in \operatorname{MW}_d(\Sigma_{A_n})$ as an element of $A^\bullet(\Sigma_{A_n})$ via the isomorphism $A^\bullet(\Sigma_{A_n}) \simeq \operatorname{MW}_{n-\bullet}(\Sigma_{A_n})$ in Theorem 2.1.6. Then*

$$A^\bullet(M) \simeq A^\bullet(\Sigma_{A_n})/\operatorname{ann}(\Delta_M).$$

Since $A^\bullet(\Sigma_{A_n})$ is a Poincaré duality algebra (Theorem 2.1.6), Proposition 4.1.3 immediately implies the following corollary.

Corollary 4.2.2. *The Chow ring $A^\bullet(M)$ is a graded Poincaré duality algebra of dimension $\operatorname{rk}(M) - 1$ with $\int_M$ as the degree map.*

Remark 4.2.3. Because Σ_{A_n} is a smooth projective fan, there exists a purely combinatorial proof of the Poincaré duality for its Chow cohomology ring via the line shelling of the fan [28, 45]. While Bergman fans of matroids are also shellable [8], they are not complete, and the arguments of [28, 45] do not readily modify to give Poincaré duality for Chow rings of matroids.

Rephrasing Corollary 4.2.2 yields the following generalization of [33, Corollary 3.13].

Corollary 4.2.4. *For each $0 \leq c \leq d$, the cap product map*

$$A^c(M) \to \mathrm{MW}_{d-c}(\Sigma_M), \quad \xi \mapsto \xi \cap \Delta_M,$$

is an isomorphism of $\mathbb{R}$-vector spaces. Thus, the Bergman classes of relative nested quotients form a basis of $\mathrm{MW}_\bullet(\Sigma_M)$.

Proof. The first statement follows from Corollary 4.2.2 and from $\mathrm{Hom}(A^c(M), \mathbb{R}) \simeq \mathrm{MW}_{d-c}(\Sigma_M)$ (Lemma 2.1.5). Theorem 3.3.8 then implies the second statement. ∎

Remark 4.2.5 (cf. (A)). Let $Y_{\mathscr{R}(M)}$ be the wonderful compactification of a realization of M. Since the pullback map $A^\bullet(X_{\Sigma_{A_n}}) \to A^\bullet(Y_{\mathscr{R}(M)})$ along the closed embedding $Y_{\mathscr{R}(M)} \hookrightarrow X_{A_n}$ is surjective (Remark 2.3.3), we have

$$A^\bullet(M) \simeq A^\bullet(Y_{\mathscr{R}(M)}) \simeq A^\bullet(X_{\Sigma_{A_n}})/\mathrm{ann}([Y_{\mathscr{R}(M)}]).$$

Proof of Theorem 4.2.1. Recall that the isomorphism $A^\bullet(\Sigma_{A_n}) \simeq \mathrm{MW}_{n-\bullet}(\Sigma_{A_n})$ makes the set of Minkowski weights into a graded ring, denoted $\mathrm{MW}^\bullet(\Sigma_{A_n})$. Let ι_M^* be the pullback map of the inclusion $\iota_M : \Sigma_M \hookrightarrow \Sigma_{A_n}$. A formal property of cap products, given below in Lemma 4.2.6, gives us a commuting diagram of surjections

$$
\begin{array}{ccc}
A^\bullet(\Sigma_{A_n}) & \xrightarrow{\sim} & \mathrm{MW}^\bullet(\Sigma_{A_n}) \\
\iota_M^* \downarrow & & \downarrow \\
A^\bullet(M) & \longtwoheadrightarrow & \mathrm{MW}^\bullet(\Sigma_{A_n})/\mathrm{ann}(\Delta_M)
\end{array}
\qquad
\begin{array}{ccc}
\xi & \longmapsto & \xi \cap \Delta_{\Sigma_{A_n}} \\
\downarrow & & \downarrow \\
\iota_M^* \xi & \longmapsto & \xi \cap \Delta_M
\end{array}
$$

Here, we have identified the elements of $\mathrm{MW}^\bullet(\Sigma_{A_n})/\mathrm{ann}(\Delta_M)$ with the elements of the principal ideal $\langle \Delta_M \rangle \subset \mathrm{MW}^\bullet(\Sigma_{A_n})$, which is generated by Minkowski weights of the form $\xi \cap \Delta_M$ where $\xi \in A^\bullet(M)$. Proposition 3.3.10 then states that the bottom horizontal map preserves linear independence, and hence is injective as well. ∎

Lemma 4.2.6. *Let ι^* be the pullback map of an inclusion of fans $\iota : \Sigma' \hookrightarrow \Sigma$ where Σ is complete, and let $\delta_\Sigma : A^\bullet(\Sigma) \xrightarrow{\sim} \mathrm{MW}^\bullet(\Sigma)$ be the isomorphism in Theorem 2.1.6. Suppose Δ is a Minkowski weight on Σ whose support $|\Delta|$ is contained the support $|\Sigma'|$. Then we have a diagram*

$$
\begin{array}{ccc}
A^\bullet(\Sigma) & \xrightarrow{\sim} & \mathrm{MW}^\bullet(\Sigma) \\
\iota^* \downarrow & & \downarrow \\
A^\bullet(\Sigma') & \longtwoheadrightarrow & \mathrm{MW}^\bullet(\Sigma)/\mathrm{ann}(\Delta)
\end{array}
\qquad
\begin{array}{ccc}
\xi & \longmapsto & \xi \cap \Delta_\Sigma \\
\downarrow & & \downarrow \\
\iota^* \xi & \longmapsto & \xi \cap \Delta
\end{array}
$$

Proof. We need to show that the kernel $\langle x_\rho \mid \rho \in \Sigma(1) \setminus \Sigma'(1) \rangle \subset A^\bullet(\Sigma)$ of the pullback map ι^* is contained in the kernel of the map $A^\bullet(\Sigma) \xrightarrow{\sim} \mathrm{MW}^\bullet(\Sigma) \twoheadrightarrow \mathrm{MW}^\bullet(\Sigma)/\mathrm{ann}(\Delta)$. Since $|\Delta| \subseteq |\Sigma'|$, we may consider Δ as a Minkowski weight on Σ', and thus by functoriality of the cap product (Remark 2.1.8), we have $x_\rho \cap \Delta = \iota^* x_\rho \cap \Delta = 0 \cap \Delta = 0$ for $x_\rho \in A^\bullet(\Sigma)$ where $\rho \in \Sigma(1) \setminus \Sigma'(1)$. ∎

5. Log-concavity of the volume polynomial

A presentation of a graded Poincaré duality algebra $A^\bullet$ can be encoded via the Macaulay inverse system into a single polynomial VP_A, called the volume polynomial of $A^\bullet$ [22, §21.2]. In geometric contexts, the volume polynomial takes on an additional meaning:

- If $A^\bullet$ is the ring of algebraic cycles modulo numerical equivalence on a smooth projective variety, then VP_A measures degrees of ample divisors (see [21]).
- If $A^\bullet$ is the Chow cohomology ring of a complete smooth fan Σ, then VP_A measures the volumes of polytopes whose normal fans coarsen Σ (see [15, §13]).

In both geometric contexts, the volume polynomial of $A^\bullet$ is positive and log-concave on the ample cone when considered as a function $A^1 \to \mathbb{R}$.

In this section, we give a combinatorial formula for the volume polynomial VP_M^∇ of the Chow ring $A_\nabla^\bullet(M)$ of a loopless matroid M. We show that, as in the geometric cases, the volume polynomial VP_M^∇ when regarded as a function $A_\nabla^1(M) \to \mathbb{R}$ is both positive and log-concave on a subcone $\mathscr{K}_M^\nabla$ of the ample cone $\mathscr{K}_M$ generated by the nontrivial simplicial generators. While the results of [1] imply that the volume polynomial of a matroid has such properties, we give an independent proof by establishing that VP_M^∇ is a Lorentzian polynomial as defined in [10]. In §6, we build upon the results of this section to conclude that VP_M^∇ is both positive and log-concave on the entire ample cone $\mathscr{K}_M$.

5.1. Volume polynomials and Lorentzian polynomials

Here we review the notion of volume polynomials and how they generalize to Lorentzian polynomials.

One can encode a graded Poincaré duality algebra into a single polynomial called the volume polynomial as follows.

Definition 5.1.1. Let $(A^\bullet, \int)$ be a graded Poincaré duality algebra of dimension d that is generated in degree 1, with a chosen presentation $A^\bullet = \Bbbk[x_1, \ldots, x_s]/I$ and a degree map $\int : A^d \to \Bbbk$. Then its *volume polynomial* VP_A is a multivariate polynomial in $\Bbbk[t_1, \ldots, t_s]$ defined by

$$VP_A(t_1, \ldots, t_s) := \int (t_1 x_1 + \cdots + t_s x_s)^d$$

where we extend the degree map $\int$ to $A[t_1, \ldots, t_s] \to \Bbbk[t_1, \ldots, t_s]$.

If $(A^\bullet, \int)$ is a Poincaré duality algebra with a presentation $A^\bullet = \Bbbk[x_1, \ldots, x_s]/I$, then the defining ideal I can be recovered from the volume polynomial VP_A as follows [15, Lemma 13.4.7]:

$$I = \left\{ f(x_1, \ldots, x_s) \in \Bbbk[x_1, \ldots, x_s] \mid f\left(\tfrac{\partial}{\partial t_1}, \ldots, \tfrac{\partial}{\partial t_s}\right) \cdot VP_A(t_1, \ldots, t_s) = 0 \right\}.$$

In [10], the authors define Lorentzian polynomials as a generalization of volume polynomials in algebraic geometry and stable polynomials in optimization. Here we briefly summarize the relevant results.

Definition 5.1.2. A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d is *strictly Lorentzian* if its support consists of all monomials in x of degree d, all of its coefficients are positive, and any of its $(d-2)$nd order partial derivatives $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has Hessian matrix with Lorentzian signature $(+, -, -, \ldots, -)$. *Lorentzian polynomials* are polynomials that can be obtained as a limit of strictly Lorentzian polynomials.

To characterize Lorentzian polynomials, we need a combinatorial notion which generalizes the exchange axiom for matroids: a collection of points $J \subset \mathbb{Z}_{\geq 0}^n$ is *M-convex* if for any $\alpha, \beta \in J$ and $i \in [n]$ with $\alpha_i > \beta_i$ there exists $j \in [n]$ such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$. When the elements of J all have the same coordinate sum, this is equivalent to stating that the convex hull of J is a generalized permutohedron [47, Theorem 4.15].

The following characterization can be seen as a linear algebraic abstraction of the proof of Teissier–Khovanskii inequalities via the Hodge index theorem for algebraic surfaces.

Theorem 5.1.3 ([10, Theorem 5.1]). *A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d with nonnegative coefficients is Lorentzian if and only if the following two conditions are satisfied:*

(1) *The support of f is M-convex.*

(2) *The Hessian matrix of $\partial_{i_1} \cdots \partial_{i_{d-2}} f$ has at most one positive eigenvalue for any choice of $d-2$ partial derivatives.*

The Lorentzian property is preserved under nonnegative linear change of variables:

Proposition 5.1.4 ([10, Theorem 2.10]). *If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Lorentzian, then so is $f(Ax) \in \mathbb{R}[x_1, \ldots, x_m]$ for any $n \times m$ matrix A with nonnegative entries.*

Applications to log-concavity phenomena in combinatorics arise from the following properties of Lorentzian polynomials.

Theorem 5.1.5. *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial with nonnegative coefficients. The Lorentzian property of f can be characterized via log-concavity properties as follows.*

(1) ([10, Theorem 5.3]) *A homogeneous polynomial f is Lorentzian if and only if f is strongly log-concave, in the sense that if g is any partial derivative of f of any order, then either g is identically zero or $\log g$ is concave on the positive orthant $\mathbb{R}_{>0}^n$.*

(2) ([10, Example 5.2]) *If $n = 2$, so that $f = \sum_{k=0}^d a_k x_1^k x_2^{d-k}$, then f is Lorentzian if and only if $(a_0, a_1, \ldots, a_d)$ has no internal zeroes and is ultra log-concave, that is,*

$$a_{k_1} a_{k_3} \neq 0 \implies a_{k_2} \neq 0 \quad \text{for all } 0 \leq k_1 < k_2 < k_3 \leq d,$$

$$\frac{a_k^2}{\binom{d}{k}^2} \geq \frac{a_{k-1} a_{k+1}}{\binom{d}{k-1}\binom{d}{k+1}} \quad \text{for all } 0 < k < d.$$

We remark that [strictly] Lorentzian polynomials arise in classical algebraic geometry whenever one has a set of nef [ample] divisors on a smooth projective variety.

Remark 5.1.6. Let $\{D_1, \ldots, D_s\}$ be nef [ample] divisors on a smooth projective $\Bbbk$-variety X of dimension d, and $A(X)$ the Chow ring. Let $\int_X : A^d(X) \to \mathbb{R}$ be the degree map obtained as the pushforward map along the structure map $X \to \operatorname{Spec} \Bbbk$. Then

$$\operatorname{vol}_X\Big(\sum_{i=1}^s t_i D_i\Big) := \lim_{q \to \infty} \frac{\dim_\Bbbk H^0(q \sum_i t_i D_i)}{q^d/d!} = \int_X \Big(\sum_i t_i D_i\Big)^d$$

is a [strictly] Lorentzian polynomial ([42, Corollary 1.6.3 (iii)] or [10, Theorem 10.1]).

5.2. The dragon Hall–Rado formula

We prepare our formula for the volume polynomial of $A_\nabla^\bullet(M)$ by describing the combinatorial notions in [50] that we generalize to arbitrary matroids.

We first recall Hall's marriage theorem and Rado's generalization; for proofs we point to [48, §11.2]. Let $E = [n] = \{0, 1, \ldots, n\}$. A *transversal* of a collection $\{A_0, \ldots, A_m\}$ (repetitions allowed) of subsets of E is a subset $I \subseteq E$ such that there exists a bijection $\phi : \{A_0, \ldots, A_m\} \to I$ satisfying $\phi(A_i) \in A_i$ for all $0 \le i \le m$.

Hall's well-known marriage theorem [32] states that a transversal of $\{A_0, \ldots, A_n\}$ exists if and only if $|\bigcup_{j \in J} A_i| \ge |J|$ for all $J \subseteq [n]$. The following theorem of Rado gives a matroid generalization of the condition given in Hall's theorem.

Theorem 5.2.1 (Rado's theorem [52]). *Let M be a matroid on E. A family $\{A_0, \ldots, A_m\}$ of subsets of E has a transversal $I \subseteq E$ that is independent in M if and only if*

$$\operatorname{rk}_M\Big(\bigcup_{j \in J} A_j\Big) \ge |J|, \quad \forall J \subseteq [m].$$

Hall's condition can be recovered from Rado's by setting $M = U_{|E|,E}$ and $m = n$. See [48, Theorem 11.2.2] for more information and a proof of Rado's theorem. The following variant of Hall's marriage theorem was investigated by Postnikov as a combinatorial interpretation of a formula for volumes of generalized permutohedra [50, §5, §9].

Proposition 5.2.2 (Dragon marriage condition). *Let $\{A_1, \ldots, A_n\}$ be a collection of subsets of $E = \{0, 1, \ldots, n\}$. There is a transversal $I \subseteq E \setminus \{e\}$ of $\{A_1, \ldots, A_n\}$ for every $e \in E$ if and only if*

$$\Big|\bigcup_{j \in J} A_j\Big| \ge |J| + 1, \quad \forall \emptyset \subsetneq J \subseteq \{1, \ldots, n\}.$$

The dragon marriage theorem above follows easily from Hall's original marriage theorem, and conversely, one can obtain Hall's marriage theorem from the dragon marriage theorem as follows: given $A_0, \ldots, A_n \subseteq E$ as in Hall's theorem, set $E' = E \sqcup \{*\}$ and $A_i' := A_0 \sqcup \{*\}$ for each $0 \le i \le n$ and apply Postnikov's theorem to $\{A_0', \ldots, A_n'\}$.

We now consider a variant of Rado's theorem in the same spirit.

Proposition 5.2.3 (Dragon Hall–Rado condition). *Let M be a matroid on E, and let $\{A_1, \ldots, A_m\}$ be a collection of subsets of E. There is a transversal $I \subseteq E \setminus \{e\}$ of $\{A_1, \ldots, A_m\}$ for every $e \in E$ if and only if*

$$\mathrm{rk}_M \left(\bigcup_{j \in J} A_j \right) \geq |J| + 1, \quad \forall \emptyset \subsetneq J \subseteq \{1, \ldots, m\}$$

and when this condition is satisfied, we say that $\{A_1, \ldots, A_m\}$ satisfy the dragon Hall–Rado condition of M, *or* $\mathrm{DHR}(M)$ *for short.*

Proof. This follows from Theorem 5.2.1 and the observation that independent transversals $I \subseteq E \setminus \{e\}$ of $\{A_1, \ldots, A_m\}$ are the same as independent transversals of $\{A_1 \setminus \{e\}, \ldots, A_m \setminus \{e\}\}$. ∎

We can obtain Rado's theorem from the dragon Hall–Rado theorem by an argument analogous to how Hall's marriage theorem is obtained from the dragon marriage theorem. In summary, the combinatorics introduced in this subsection thus far can be schematically laid out as follows with the indicated logical implications:

$$\text{Hall's marriage theorem} \Longleftarrow \text{Rado's theorem}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\text{Dragon marriage theorem} \Longleftarrow \text{Dragon Hall–Rado theorem}$$

We are now ready to compute the intersection numbers of the variables h_F in the simplicial presentation of the Chow ring of a matroid M.

Theorem 5.2.4. *Let $\{A_1, \ldots, A_d\}$ be a collection of subsets of E, and M a loopless matroid on E of rank $d + 1$. Let $H_{A_1}, \ldots, H_{A_d}$ be matroids as defined in Proposition 3.2.4. Then*

$$M \wedge H_{A_1} \wedge \cdots \wedge H_{A_d} = U_{1,E} \iff \{A_1, \ldots, A_d\} \text{ satisfies } \mathrm{DHR}(M).$$

Thus,

$$\int_M h_{A_1}(M) \cdots h_{A_d}(M) = \begin{cases} 1 & \text{if } \{A_1, \ldots, A_d\} \text{ satisfies } \mathrm{DHR}(M), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For the first assertion, we begin by making two easy observations:

(1) $M \wedge H_S$ has a loop if and only if $\mathrm{rk}_M(S) = 1$.

(2) ([48, Exercise 7.3.10]) For the elementary quotient $f : M \twoheadrightarrow M \wedge H_S$, by Proposition 3.1.5 we have $\{T \subseteq E \mid n_f(T) = 1\} = \{T \subseteq E \mid \mathrm{cl}_M(T) \supseteq S\}$. In particular, the rank of $T \subseteq E$ decreases by 1 in the elementary quotient if $T \supseteq S$.

For the $\Rightarrow$ direction, suppose $\mathrm{rk}_M(\bigcup_{j \in J} A_j) \leq k$ for a nonempty $J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, d\}$. Then, for $\tilde{M} := M \wedge H_{A_{j_1}} \wedge \cdots \wedge H_{A_{j_{k-1}}}$ we have $\mathrm{rk}_{\tilde{M}}(\bigcup_{j \in J} A_j) \leq k - (k-1) = 1$, so that $M \wedge \bigwedge_{j \in J} H_{A_j}$ already has a loop.

For the $\Leftarrow$ direction, we induct on d. The base case $d = 1$ is easily verified. Now, we claim that if $\{A_1, \ldots, A_d\}$ satisfies the dragon Hall–Rado condition for M, then

so does $\{A_1, \ldots, A_{d-1}\}$ for $\widetilde{M} := M \wedge H_{A_d}$. For the sake of contradiction, suppose $\{A_1, \ldots, A_{d-1}\}$ fails DHR($\widetilde{M}$), that is, without loss of generality, $\mathrm{rk}_{\widetilde{M}}(A_1 \cup \cdots \cup A_k) \leq k$ for some $1 \leq k \leq d-1$. Then we must have had $\mathrm{rk}_M(A_1 \cup \cdots \cup A_k) = k+1$ with $\mathrm{cl}_M(A_1 \cup \cdots \cup A_k) \supseteq A_d$. But then $\mathrm{rk}_M(A_1 \cup \cdots \cup A_k \cup A_d) = k+1$, violating DHR($M$).

For the second assertion, we first note that

$$\int_M h_{A_1}(M) \cdots h_{A_d}(M) = \int_{\Sigma_{A_n}} h_{A_1} \cdots h_{A_d} \cap \Delta_M.$$

Repeated application of Theorem 3.2.3 implies that $h_{A_1} \cdots h_{A_d} \cap \Delta_M$ is either 0 or the Bergman class $\Delta_{M'}$ of a loopless matroid M' of rank 1 where M' moreover equals $M \wedge H_{A_1} \wedge \cdots \wedge H_{A_d}$. Since $U_{1,E}$ is the unique loopless matroid of rank 1 on E, which defines the Bergman class $\Delta_{U_{1,E}}$ by $\Delta_{U_{1,E}}(\mathbf{0}) = 1$ where $\mathbf{0}$ is the zero-dimensional cone of Σ_{A_n}, we conclude that $\int_{\Sigma_{A_n}} h_{A_1} \cdots h_{A_d} \cap \Delta_M$ equals 1 if $\{A_1, \ldots, A_d\}$ satisfies DHR(M) and equals 0 otherwise. ∎

We obtain as an immediate corollary the promised generalization of [50, Corollary 9.4]. Recall that $\mathscr{L}_M^{\geq 2}$ denotes the flats of M of rank at least 2.

Corollary 5.2.5. *Let M be a loopless matroid on E of rank $d+1$. The volume polynomial $VP_M^{\nabla}(\underline{t}) \in \mathbb{Q}[t_F \mid F \in \mathscr{L}_M^{\geq 2}]$ of $A_{\nabla}^{\bullet}(M)$ is*

$$VP_M^{\nabla}(\underline{t}) = \sum_{(F_1, \ldots, F_d)} t_{F_1} \cdots t_{F_d}$$

where the sum is over ordered collections of nonempty flats $F_1, \ldots, F_d$ of M satisfying DHR(M). Alternatively,

$$VP_M^{\nabla}(\underline{t}) = \sum_{\{F_1^{d_1}, \ldots, F_k^{d_k}\}} \binom{d}{d_1, \ldots, d_k} t_{F_1}^{d_1} \cdots t_{F_k}^{d_k}$$

where the sum is over size d multisets $\{F_1^{d_1}, \ldots, F_k^{d_k}\}$ of flats of M satisfying DHR(M).

One recovers the following central result of [50] by setting $M = U_{|E|,E}$.

Corollary 5.2.6 ([50, Corollary 9.4]). *The volume polynomial $VP_{U_{n+1,n+1}}^{\nabla}(\underline{t})$ of $A_{\nabla}^{\bullet}(X_{A_n})$ is*

$$VP_{U_{n+1,n+1}}^{\nabla}(\underline{t}) = \sum_{(S_1, \ldots, S_n)} t_{S_1} \cdots t_{S_n}$$

where the sum is over ordered collections of nonempty subsets $S_1, \ldots, S_n$ such that $|\bigcup_{j \in J} S_j| \geq |J| + 1$ for any $\emptyset \subsetneq J \subseteq \{1, \ldots, n\}$.

The volume polynomial VP_M of the more classical presentation $A^{\bullet}(M)$ of the Chow ring of a matroid M by generators $\{x_F \mid F \in \mathscr{L}_M \setminus \{\emptyset, E\}\}$ was computed in [25].

5.3. *Volume polynomial of a matroid is Lorentzian*

Motivated by Remark 5.1.6 and the fact that the simplicial generators are combinatorially nef, we prove here that the volume polynomial VP_M^∇ of the simplicial presentation $A_\nabla^\bullet(M)$ is Lorentzian.

Theorem 5.3.1. *The volume polynomial $VP_M^\nabla \in \mathbb{R}[t_F \mid F \in \mathscr{L}_M^{\geq 2}]$ of a loopless matroid M is Lorentzian.*

As an immediate corollary, by applying Theorem 5.1.5 to Theorem 5.3.1 we obtain:

Corollary 5.3.2. *The volume polynomial VP_M^∇, as a polynomial in $\mathbb{R}[t_F \mid F \in \mathscr{L}_M^{\geq 2}]$, is strongly log-concave in the positive orthant $\mathbb{R}_{>0}^{\mathscr{L}_M^{\geq 2}}$. In other words, as a function $A^1(M) \to \mathbb{R}$, the polynomial VP_M^∇ is strongly log-concave in the interior of the cone $\mathscr{K}_M^\nabla$ generated by the nontrivial simplicial generators.*

We will show that the volume polynomial VP_M^∇ of a loopless matroid M satisfies the two conditions listed in Theorem 5.1.3. First, we see that the dragon Hall–Rado condition description of the support of VP_M^∇ implies that VP_M^∇ has M-convex support.

Proposition 5.3.3. *Let $\{F_1, \ldots, F_d\}$ and $\{G_1, \ldots, G_d\}$ be two multisets of flats of M such that both $t_{F_1} \cdots t_{F_d}$ and $t_{G_1} \cdots t_{G_d}$ are in the support of VP_M^∇. Without loss of generality, suppose G_d is a flat which appears more times in $\{G_1, \ldots, G_d\}$ than it does in $\{F_1, \ldots, F_d\}$. Then there exists another flat F_m which appears more times in $\{F_1, \ldots, F_d\}$ than it does in $\{G_1, \ldots, G_d\}$ such that $t_{F_1} \cdots t_{F_d} t_{G_d} / t_{F_m}$ is in the support of VP_M^∇.*

Proof. First note that $\{F_1, \ldots, F_d\}$ and $\{G_1, \ldots, G_d\}$ satisfy the dragon Hall–Rado condition. In this argument, we borrow standard language from (poly)matroid theory [35]. Let us call a multiset of flats $\{A_1, \ldots A_k\}$ *dependent* if $\mathrm{rk}_M(\bigcup_{j=1}^k A_j) \leq k$, and *independent* otherwise. We will also call a minimal dependent set of flats a *circuit*. We claim that the multiset of flats $\{F_1, \ldots, F_d, G_d\}$ contains a unique circuit X. The proposition will follow from this claim because we can take F_m to be any flat in X which appears more times in $\{F_1, \ldots, F_d\}$ than it does in $\{G_1, \ldots, G_d\}$, and $\{F_1, \ldots, F_d, G_d\} \setminus \{F_m\}$ will satisfy the dragon Hall–Rado condition.

To prove the claim, suppose to the contrary that $\{R_1, \ldots, R_a\}$ and $\{S_1, \ldots, S_b\}$ are two distinct circuits which are both subsets of $\{F_1, \ldots, F_d, G_d\}$. We will derive a contradiction by producing a dependent set of flats contained in $\{F_1, \ldots, F_d\}$. By assumption, $\mathrm{rk}_M(\bigcup_{j=1}^a R_j) \leq a$, and for each $1 \leq i \leq a$, we have $\mathrm{rk}_M(\bigcup_{j \neq i} R_j) \geq a$, therefore $\mathrm{rk}_M(\bigcup_{j=1}^a R_j) = a$. Similarly, $\mathrm{rk}_M(\bigcup_{j=1}^b S_j) = b$. Because $\{R_1, \ldots, R_a\}$ and $\{S_1, \ldots, S_b\}$ are not fully contained in $\{F_1, \ldots, F_d\}$, we know that $G_d \in \{R_1, \ldots, R_a\} \cap \{S_1, \ldots, S_b\}$. Let $\{R_1, \ldots, R_a\} \cap \{S_1, \ldots, S_b\} = \{T_1, \ldots, T_c\}$, where the intersection is multiset intersection, and without loss of generality assume $G_d = R_a = S_b = T_c$.

Let R and S be the joins of the elements in $\{R_1, \ldots, R_a\}$ and $\{S_1, \ldots, S_b\}$, respectively. We have $R = \bigvee_{j=1}^{a-1} R_j = \bigvee_{j=1}^a R_j$ and $S = \bigvee_{j=1}^{b-1} S_j = \bigvee_{j=1}^b S_j$, since otherwise

$\{R_1, \ldots, R_{a-1}\}$ and $\{S_1, \ldots, S_{b-1}\}$ would both be dependent, contradicting minimality. Therefore the join of the elements in the multiset $\{R_1, \ldots, R_{a-1}, S_1, \ldots, S_{b-1}\} \setminus \{T_1, \ldots, T_{c-1}\}$ is $R \vee S$. Note that the previous subtraction is multiset subtraction. Furthermore, $|\{R_1, \ldots, R_{a-1}, S_1, \ldots, S_{b-1}\} \setminus \{T_1, \ldots, T_{c-1}\}| = (a-1) + (b-1) - (c-1) = a + b - c - 1$.

By minimality of $\{R_1, \ldots, R_a\}$ and $\{S_1, \ldots, S_b\}$, we know that $\{T_1, \ldots, T_c\}$ is independent, thus $\mathrm{rk}_M(\bigcup_{j=1}^c T_j) \geq c + 1$. Submodularity then gives

$$\mathrm{rk}_M(R \cup S) \leq \mathrm{rk}_M(R) + \mathrm{rk}_M(S) - \mathrm{rk}_M(R \cap S)$$

$$\leq a + b - \mathrm{rk}_M\left(\bigcup_{j=1}^c T_j\right) \leq a + b - c - 1.$$

Therefore the multiset $\{R_1, \ldots, R_{a-1}, S_1, \ldots, S_{b-1}\} \setminus \{T_1, \ldots T_{c-1}\}$ is dependent in $\{F_1, \ldots, F_d\}$, which contradicts the fact that this set satisfied the dragon Hall–Rado condition. ∎

Remark 5.3.4. Suppose M has a realization $\mathscr{R}(M)$. By Remark 2.3.2 (2), the wonderful compactification $Y_{\mathscr{R}(M)}$ is embedded in the product $\prod_{F \in \mathscr{L}_M \setminus \{\emptyset\}} \mathbb{P}(V^*/L_F)$. Our simplicial generators are pullbacks of the hyperplane classes of the projective spaces $\mathbb{P}(V^*/L_F)$ (see Remark 3.2.7). Thus, in this case, that the support of VP_M^∇ is M-convex follows from the result of [14] that the multidegree of an irreducible variety in a product of projective spaces has M-convex support.

Proof of Theorem 5.3.1. Let M be a loopless matroid of rank $r = d + 1$. There is nothing to prove if $d = 1$, so we assume $d \geq 2$. The support of VP_M^∇ is M-convex by the previous proposition. We now show that the quadrics obtained as the $(d-2)$-th partial derivatives have the desired signature. Observe that for a flat G of rank ≥ 2, we have

$$\frac{\partial}{\partial t_G} VP_M^\nabla(t) = d \int_M h_G \cdot \left(\sum_{F \in \mathscr{L}_M^{\geq 2}} t_F h_F \right)^{d-1}$$

$$= d \int_{T_G(M)} \left(\sum_{F \in \mathscr{L}_M^{\geq 2}} t_F h_{\mathrm{cl}_{T_G(M)}(F)} \right)^{d-1}.$$

Now, suppose $\{F_1, \ldots, F_{d-2}\}$ is a multiset of size $d - 2$ consisting of flats of M with rank at least 2. If $\{F_1, \ldots, F_{d-2}\}$ does not satisfy DHR(M), then $\partial_{t_{F_1}} \cdots \partial_{t_{F_{d-2}}} VP_M^\nabla \equiv 0$, so we may assume that $\{F_1, \ldots, F_{d-2}\}$ satisfies DHR(M). One computes that

$$\partial_{t_{F_1}} \cdots \partial_{t_{F_{d-2}}} VP_M^\nabla(t) = \frac{d!}{2!} \int_{M'} \left(\sum_{F \in \mathscr{L}_M^{\geq 2}} t_F h_{\mathrm{cl}_{M'}(F)} \right)^2$$

where $M' = M \wedge H_{F_1} \wedge \cdots \wedge H_{F_{d-2}}$ is a loopless matroid of rank 3. By Proposition 5.1.4, it suffices to check that $VP_{M'}^\nabla$ is Lorentzian. For any loopless matroid M' of rank 3, the degree 1 part $A_\nabla^1(M)$ of its Chow ring has the simplicial basis $\{h_E\} \cup \{h_F \mid \mathrm{rk}_{M'}(F) = 2\}$. Noting that $\int_{M'} h_E \cdot h_E = 1$, $\int_{M'} h_E \cdot h_F = 1$, and $\int_{M'} h_F \cdot h_{F'} = 1$ if

$F \neq F'$ and 0 otherwise, the Hessian matrix of the quadratic form $VP_{M'}^{\nabla}$ is 2 times the matrix

$$
\mathscr{H} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \quad \text{which is congruent to} \quad \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}
$$

via the symmetric Gaussian elimination: Explicitly, let E_i be the elementary matrix such that multiplying E_i by a matrix A results in adding -1 times the first row of A to the i-th row of A, and let E_i^T be its transpose. Then the claimed congruence is given by $E_{|\mathscr{L}_M \setminus \{\emptyset\}|} \cdots E_3 E_2 \mathscr{H} E_2^T E_3^T \cdots E_{|\mathscr{L}_M \setminus \{\emptyset\}|}^T$. ∎

6. The Hodge theory of matroids in degrees at most 1

The *reduced characteristic polynomial* of M is defined as

$$
\bar{\chi}_M(t) := \frac{1}{t-1} \sum_{F \in \mathscr{L}_M} \mu(\emptyset, F) t^{\mathrm{rk}(M) - \mathrm{rk}(F)} = \sum_{k=0}^{d} (-1)^k \mu^k(M) t^{d-k}
$$

where $\mu(-,-)$ is the Möbius function of the lattice $\mathscr{L}_M$ and $\mu^i(M)$ is the absolute value of the i-th coefficient of $\bar{\chi}_M(t)$. The Heron–Rota–Welsh conjecture [34,53,56] stated that

$$
\mu^{k-1}(M)\mu^{k+1}(M) \leq \mu^k(M)^2 \quad \text{for } 0 < k < d.
$$

To resolve the conjecture, the authors of [1] showed that the Chow ring of a matroid satisfies properties enjoyed by the cohomology ring of a smooth projective complex variety. Namely, these are the Poincaré duality property, the hard Lefschetz property, and the Hodge–Riemann relations, which together form the "Kähler package." In particular, the Hodge–Riemann relation in degree 1 implies the Heron–Rota–Welsh conjecture [1, §9].

To prove that the Chow ring of a matroid satisfies the Hodge–Riemann relations, the authors of [1] adapt a line of argument that originally appeared in McMullen's work on simple polytopes [45]. Their method employs a double induction on the rank of the matroid and the size of an order filter on the matroid's lattice of flats: the outer induction on rank shows that the Hodge–Riemann relations hold for all ample classes if they hold for a single ample class, and the inner induction on the size of an order filter is then used to construct an ample class for which the Hodge–Riemann relations can be verified.

In this section, we independently establish the Hodge–Riemann relations in degree 1 using a similar argument. As we have established in the previous section that the volume polynomial VP_M^{∇} of a matroid M is strongly log-concave in the subcone $\mathscr{K}_M^{\nabla}$ of the ample cone $\mathscr{K}_M$, we are able to avoid working with generalized Bergman fans associated

to order filters and the flipping operation which interpolates between them. Thus our proof involves only classical Bergman fans associated to matroids and takes the form of a single induction on rank alone.

6.1. The Kähler package in degree 1 and log-concavity

We begin by discussing here the statements of the Kähler package, and how in degree 1 they relate to log-concavity. We then provide some generalities on the inductive paradigm for proving Kähler package for Chow cohomology rings of fans, similar to one in [1] adapted from the earlier work [45].

Definition 6.1.1. Let $(A^\bullet, \int)$ be a Poincaré duality $\Bbbk$-algebra of dimension d with degree map $\int$. For $\ell \in A^1$ and $0 \le i \le \lfloor d/2 \rfloor$, we define L_ℓ^k to be the Lefschetz operator

$$L_\ell^i : A^i \to A^{d-i}, \quad a \mapsto \ell^{d-2i} a,$$

and define Q_ℓ^i to be the Hodge–Riemann symmetric bilinear form

$$Q_\ell^i : A^i \times A^i \to \Bbbk, \quad (x, y) \mapsto \int xy\ell^{d-2i}.$$

We define the set of *degree i primitive classes of ℓ* to be $P_\ell^i := \{x \in A^k \mid x\ell^{d-2i+1} = 0\}$.

Definition 6.1.2. Let $(A^\bullet, \int)$ be a Poincaré duality $\mathbb{R}$-algebra of dimension d, and let $\ell \in A^1$. For $0 \le i \le \lfloor d/2 \rfloor$, we say that $(A^\bullet, \int)$ satisfies

- HL_ℓ^i if L_ℓ^i induces an isomorphism between A^i and A^{d-i},
- HR_ℓ^i if the symmetric form $(-1)^i Q_\ell^i$ is positive-definite when restricted to P_ℓ^i.

Moreover, for $\mathcal{K}$ a convex cone in A^1, we say that $(A^\bullet, \int, \mathcal{K})$ satisfies the *hard Lefschetz property* ($\mathrm{HL}_\mathcal{K}^i$), resp. the *Hodge–Riemann relation* ($\mathrm{HR}_\mathcal{K}^i$), in degree i if $A^\bullet$ satisfies HL_ℓ^i, resp. HR_ℓ^i, for all $\ell \in \mathcal{K}$.

The Poincaré duality property (PD) of $(A^\bullet, \int)$ implies that the form Q_ℓ^i is nondegenerate if and only if HL_ℓ^i holds. The properties (PD), (HL), and (HR) together are called the *Kähler package* for a graded ring $A^\bullet$. We will write $\mathrm{HL}^{\le i}$ and $\mathrm{HR}^{\le i}$ to mean hard Lefschetz property and Hodge–Riemann in degrees at most i, respectively. The relation between log-concavity and the Kähler package in degree ≤ 1 was realized in various contexts; for a survey we point to [37]. We will only need the following, adapted from [10, Proposition 5.6]. It also appeared in [3, §2.3], and is a consequence of the Cauchy interlacing theorem.

Proposition 6.1.3. *Let $A^\bullet$ be a Poincaré duality $\mathbb{R}$-algebra of dimension d with degree map $\int$, and $\mathcal{K}$ a convex cone in A^1. Suppose $(A^\bullet, \int, \mathcal{K})$ satisfy $\mathrm{HL}_\mathcal{K}^0$ and $\mathrm{HR}_\mathcal{K}^0$. Then the following are equivalent:*

(1) *The volume function* $\mathrm{vol} : A^1 \to \mathbb{R}, \ell \mapsto \int \ell^d$, *is log-concave on $\mathcal{K}$.*

(2) *For any $\ell \in \mathcal{K}$, the symmetric form Q_ℓ^1 has exactly one positive eigenvalue.*

In particular, if the volume polynomial VP_A of $A^\bullet = \mathbb{R}[x_1, \ldots, x_s]/I$ is Lorentzian, then $(A^\bullet, \int, \mathcal{K})$ satisfies $\mathrm{HR}^{\leq 1}_{\mathcal{K}}$ where $\mathcal{K}$ is the interior of $\mathrm{Cone}(x_1, \ldots, x_s)$, provided that $A^\bullet$ satisfies $\mathrm{HL}^{\leq 1}_{\mathcal{K}}$.

We now turn to an inductive paradigm for establishing (HL) and (HR). We assume all Poincaré duality algebras to be over $\mathbb{R}$. We begin by noting an easy linear algebraic observation also made in [1, Proposition 7.16].

Proposition 6.1.4. *Let $(A^\bullet, \int, \mathcal{K})$ be a Poincaré duality algebra which satisfies $\mathrm{HL}^i_{\mathcal{K}}$ for $\mathcal{K}$ in a convex cone in A^1. Suppose that $(A^\bullet, \int)$ satisfies HR^i_ℓ for some $\ell \in \mathcal{K}$. Then $A^\bullet$ satisfies $\mathrm{HR}^i_{\mathcal{K}}$.*

Proof. Let $\ell' \in \mathcal{K}$, and let $l(t) = t\ell + (1-t)\ell'$ for $t \in [0, 1]$ be a line segment connecting ℓ and ℓ'. By convexity of $\mathcal{K}$, we know that every point on l is in $\mathcal{K}$. If the signature of the bilinear pairing $Q^i_{l(t)}$ changes along $l(t)$ starting at ℓ, then it must degenerate at some point $l(t_0)$ for $t_0 \in [0, 1]$, but this violates $\mathrm{HL}^i_{\mathcal{K}}$. ∎

We now note how properties (HL) and (HR) behave under tensor products and quotients by annihilators of an element. While these are adapted from [1, §7] where they are phrased in terms of Chow cohomology rings of fans, since we restrict ourselves Kähler package up to degree 1, we can provide here easier and more direct proofs for general Poincaré duality algebras.

Proposition 6.1.5. *Let $(A^\bullet, \int_A)$ and $(B^\bullet, \int_B)$ be two Poincaré duality algebras of dimension $d_A \geq 1$ and $d_B \geq 1$. Suppose that $A^\bullet$ and $B^\bullet$ satisfy $\mathrm{HR}^{\leq 1}_{\ell_A}$ and $\mathrm{HR}^{\leq 1}_{\ell_B}$, respectively, then $((A \otimes B)^\bullet, \int_{A \otimes B})$ satisfies $\mathrm{HR}^{\leq 1}_{\ell_A \otimes 1 + 1 \otimes \ell_B}$.*

Before giving the proof, we remark that if $d_A = 0$ then $(A \otimes B)^\bullet \simeq B^\bullet$ (and likewise if $d_B = 0$) so that the statement in the proposition is trivially satisfied after suitable modifications. If $d_A = 1$ (or $d_B = 1$), so that $\mathrm{HR}^1_{\ell_A}$ (or $\mathrm{HR}^1_{\ell_B}$) is vacuous, our proof below implies that $(A^\bullet \otimes B^\bullet)$ satisfies $\mathrm{HR}^1_{\ell_A \otimes 1 + 1 \otimes \ell_B}$ if $\mathrm{HR}^0_{\ell_A}$ and $\mathrm{HR}^0_{\ell_B}$ are satisfied.

Proof of Proposition 6.1.5. Set $\ell := \ell_A \otimes 1 + 1 \otimes \ell_B$. First, note that HR^0_ℓ follows easily from the description of the Poincaré duality algebra $(A \otimes B)^\bullet$ in Proposition 4.1.2 (1). Now, we verify that $(A \otimes B)^\bullet$ satisfies HR^1_ℓ. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be orthonormal bases for $P^1_{\ell_A}$ and $P^1_{\ell_B}$, respectively. Then

$$A^1 \cong \bigoplus_{i=1}^{m} \langle v_i \rangle \oplus \langle \ell_A \rangle \quad \text{and} \quad B^1 \cong \bigoplus_{i=1}^{n} \langle w_i \rangle \oplus \langle \ell_B \rangle.$$

Noting that $(A \otimes B)^\bullet$ is a Poincaré duality algebra of dimension $d = d_A + d_B$, we expand

$$\ell^{d-2} = ((\ell_A \otimes 1) + (1 \otimes \ell_B))^{d-2} = \sum_{i=0}^{d-2} \binom{d-2}{i} (\ell_A^i \otimes \ell_B^{d-i-2}).$$

The symmetric matrix for Q_ℓ^1 with respect to the above basis is given by

$$Q_\ell^1(a,b) = \begin{cases} -\binom{d-2}{d_A}, & a = b = v_i \otimes 1, \\ -\binom{d-2}{d_B}, & a = b = 1 \otimes w_j, \\ \lambda\binom{d-2}{d_A-2}, & a = b = \ell_A \otimes 1, \\ \lambda\binom{d-2}{d_A}, & a = b = 1 \otimes \ell_B, \\ \lambda\binom{d-2}{d_A-1} & a = \ell_A \otimes 1 \text{ and } b = 1 \otimes \ell_B, \\ \lambda\binom{d-2}{d_B-1}, & a = 1 \otimes \ell_B \text{ and } b = \ell_A \otimes 1, \\ 0, & a = v_i \otimes 1 \text{ and } b = 1 \otimes w_j \text{ or } 1 \otimes \ell_B, \\ 0, & a = v_i \otimes 1 \text{ or } \ell_A \otimes 1 \text{ and } b = 1 \otimes w_j, \end{cases}$$

where $\lambda := (\int_A \ell_A^{d_A})(\int_B \ell_B^{d_B})$.

So the matrix $Q_\ell^1(a, b)$ is a block matrix comprised of three blocks. By $\mathrm{HR}_{\ell_A}^1$ and $\mathrm{HR}_{\ell_B}^1$, the first two blocks are negative identity matrices induced by $\{v_i \otimes 1\} \times \{v_i \otimes 1\}$ and $\{1 \otimes w_j\} \times \{1 \otimes w_j\}$. The third and only nontrivial block is induced by $\{\ell_A \otimes 1, 1 \otimes \ell_B\} \times \{\ell_A \otimes 1, 1 \otimes \ell_B\}$, which gives the 2×2 matrix

$$M = \lambda \begin{bmatrix} \binom{d-2}{d_A-2} & \binom{d-2}{d_A-1} \\ \binom{d-2}{d_A-1} & \binom{d-2}{d_A} \end{bmatrix}.$$

One computes that $\det(M) < 0$, and hence M has signature $(+, -)$. We conclude that $Q_\ell^1(a, b)$ is nondegenerate and has exactly one positive eigenvalue, completing the proof. $\blacksquare$

Proposition 6.1.6. *Let $(A^\bullet = \mathbb{R}[x_1, \ldots, x_s]/I, \int)$ be a Poincaré duality algebra of dimension d, and let $\ell \in A^1$ be an effective divisor—that is, a nonnegative linear combination of $\{x_1, \ldots, x_s\}$. Denote by ℓ_k the image of ℓ in $A^\bullet/\mathrm{ann}(x_k)$. For $0 \leq i \leq \lfloor (d-1)/2 \rfloor$, if $(A^\bullet/\mathrm{ann}(x_k), \int_{x_k})$ satisfies $\mathrm{HR}_{\ell_k}^i$ for every $k = 1, \ldots, s$, then $(A^\bullet, \int)$ satisfies HL_ℓ^i.*

Proof. Let $\ell = \sum_{k=1}^s c_k x_k$ with $c_k \in \mathbb{R}_{\geq 0}$, and suppose $\ell^{d-2i} f = 0$ for some $f \in A^i$. We will show that $f = 0$ necessarily. Let f_i be the image of f in $A^\bullet/\mathrm{ann}(x_i)$. As $0 = \ell^{d-2i} f$, we have $0 = \ell_k^{d-2i} f_k$, and since $A^\bullet/\mathrm{ann}(x_k)$ is a Poincaré duality algebra of dimension $d - 1$, we conclude that f_k belongs to the primitive space $P_{\ell_k}^i$. Now, we note that for all k,

$$0 = \int \ell^{d-2i} f^2 = \int \left(\sum_k c_k x_k\right) \ell^{d-2i-1} f^2 = \sum_k \left(\int_{x_k} c_k \ell_k^{d-2i-1} f_k^2\right)$$

and $\int_{x_k} c_k \ell_k^{d-2i-1} f_k^2 \leq 0$ by $\mathrm{HR}_{\ell_k}^i$. Moreover, as $Q_{\ell_k}^1$ is negative-definite on $P_{\ell_k}^i$, we conclude each f_k is 0, that is, $x_k f = 0$ for all $k = 1, \ldots, s$. Since $\{x_1, \ldots, x_s\}$ generate $A^\bullet$, the Poincaré duality property of $A^\bullet$ implies that if $f \neq 0$ then there exists a polynomial $g(x)$ of degree $d - i$ such that $\int g(x) f \neq 0$, and hence we conclude that $f = 0$. $\blacksquare$

6.2. *Kähler package in degree at most* 1 *for matroids*

We now specialize our discussion to the setting of matroids, and establish Kähler package in degree at most 1 for Chow rings of matroids. As a consequence, we recover the proof of the Heron–Rota–Welsh conjecture as in [1].

Theorem 6.2.1. *The Chow ring of a matroid* $(A^\bullet(M), \int_M, \mathscr{K}_M)$ *satisfies* $\mathrm{HL}^{\leq 1}_{\mathscr{K}_M}$ *and* $\mathrm{HR}^{\leq 1}_{\mathscr{K}_M}$.

We will prove the theorem by induction on the rank of the matroid. The key combinatorial observation that allows one to reduce the rank is the following, adapted from [1, Proposition 3.5]. It underlies the well-known Hopf algebraic structure for the lattice of flats of a matroid; see [2, 40] for a detailed discussion of such structures.

Lemma 6.2.2. *Let M be a loopless matroid, and F a nonempty proper flat of M. Let ρ_F be the ray corresponding to F in the Bergman fan Σ_M of M. Then*

(1) $\mathrm{star}(\Sigma_M, \rho_F) \simeq \Sigma_{M|F} \times \Sigma_{M/F}$, *and consequently*

(2) *we have an isomorphism of Poincaré duality algebras*

$$A^\bullet(M)/\mathrm{ann}(x_F) \simeq (A(M|F) \otimes A(M/F))^\bullet$$

such that if $\ell \in \mathscr{K}_M$ then its image in $A^\bullet(M)/\mathrm{ann}(x_F)$ is in $(\mathscr{K}_{M|F} \otimes 1) \oplus (1 \otimes \mathscr{K}_{M/F})$.

Proof. A face of Σ_M is in $\mathrm{star}(\Sigma_M, F)$ if and only if it corresponds to a flag of flats which contains F. Any such flag naturally factors as the concatenation of two flags, one with maximal element strictly contained in F, and the other with minimal element F. This geometrically corresponds to the factorization of fans in (1).

For (2), first note that $M|F$ and M/F are loopless since F is a flat. Then, combine Propositions 4.1.2 and 4.1.4 with the easily verifiable fact that $A^\bullet(\Sigma \times \Sigma') \simeq (A(\Sigma) \otimes A(\Sigma'))^\bullet$ for rational fans Σ and Σ'. This proves the isomorphism $A^\bullet(M)/\mathrm{ann}(x_F) \simeq (A(M|F) \otimes A(M/F))^\bullet$. Lastly, the statement about $\ell \in \mathscr{K}_M$ follows from the fact that submodular functions on lattices remain submodular under restriction to intervals in the lattice. ∎

The remaining key part of the induction in the proof of Theorem 6.2.1, in light of Proposition 6.1.4, is to establish HR^1_ℓ for some divisor $\ell \in \mathscr{K}_M$. In [1] the authors employ the method of order filters and flips for this purpose; in our case, the Lorentzian property of the volume polynomial provides the desired key step.

Lemma 6.2.3. *Let M be a loopless matroid of rank $r = d + 1 \geq 2$, and recall that $\mathscr{K}_M^\nabla$ is the interior of the cone generated by the nontrivial simplicial generators of $A^\bullet_\nabla(M)$. (It is a subcone of $\mathscr{K}_M$.) For any $\ell \in \mathscr{K}_M^\nabla$, we have $\int_M \ell^d > 0$. If further $r = d + 1 \geq 3$, then the form Q^1_ℓ has exactly one positive eigenvalue. In particular, $\mathrm{HR}^1_{\mathscr{K}_M}$ is satisfied for M with rank 3.*

Proof. The statement $\int_M \ell^d > 0$ follows from our dragon Hall–Rado formula (Corollary 5.2.5). When the rank of M is at least 3, that the form Q_ℓ^1 has exactly one positive eigenvalue follows from combining Theorem 5.3.1 and Proposition 6.1.3. Lastly, when rank of M equals 3, the bilinear form Q_ℓ^1 is the Poincaré pairing $A^1(M) \times A^{2-1}(M) \to A^2(M)$, which is nondegenerate and independent of $\ell \in \mathscr{K}_M^\nabla$. Thus, we deduce $\mathrm{HR}_{\ell'}^1$ for any element $\ell' \in \mathscr{K}_M$ since $Q_{\ell'}^1 = Q_\ell^1$. ∎

Proof of Theorem 6.2.1. We proceed by induction on the rank of the matroid M. The base case consists of rank ≤ 2 matroids for $\mathrm{HL}_{\mathscr{K}_M}^0$ & $\mathrm{HR}_{\mathscr{K}_M}^0$ and rank ≤ 3 for $\mathrm{HL}_{\mathscr{K}_M}^1$ & $\mathrm{HR}_{\mathscr{K}_M}^1$. For these cases, the stated properties are either vacuous or easily verified with the use of Lemma 6.2.3. Let now M be a loopless matroid of rank $r = d + 1$ on a ground set E.

Observe that $\mathrm{HL}_{\mathscr{K}_M}^0$ and $\mathrm{HR}_{\mathscr{K}_M}^0$ both hold if and only if $\int_M \ell^d > 0$ for all $\ell \in \mathscr{K}_M$. Moreover, given HL_ℓ^1, the property HR_ℓ^1 holds if and only if Q_ℓ^1 has exactly one positive eigenvalue. Combined with Lemma 6.2.3 and Proposition 6.1.4, these facts imply that proving $\mathrm{HL}_{\mathscr{K}_M}^{\leq 1}$ is sufficient to establish $\mathrm{HR}_{\mathscr{K}_M}^{\leq 1}$. By Remark 2.2.8, any element $\ell \in \mathscr{K}_M$ can be written as a nonnegative linear combination of $\{x_F \mid F \in \mathscr{L}_M \setminus \{\emptyset, E\}\}$; therefore, by Proposition 6.1.6, to establish $\mathrm{HL}_{\mathscr{K}_M}^{\leq 1}$, it suffices in turn to prove $\mathrm{HR}^{\leq 1}$ for $A^\bullet(M)/\mathrm{ann}(x_F)$ for every nonempty proper flat F. Finally, $A^\bullet(M)/\mathrm{ann}(x_F) \simeq (A(M|F) \otimes A(M/F))^\bullet$ by Lemma 6.2.2 (2), so by the induction hypothesis and Proposition 6.1.5, the proof is complete. ∎

We conclude by recounting the argument in [1, §9] that the Hodge–Riemann relations in degree 1 imply the Heron–Rota–Welsh conjecture.

Lemma 6.2.4 ([1, Lemma 9.6]). *Let $\ell_1, \ell_2 \in A^1(M)$. If ℓ_2 is nef, then*

$$\left(\int_M \ell_1^2 \ell_2^{d-2}\right)\left(\int_M \ell_2^2 \ell_2^{d-2}\right) \leq \left(\int_M \ell_1 \ell_2 \ell_2^{d-2}\right)^2.$$

Proof. Suppose first that ℓ_2 is ample. By Theorem 6.2.1, $A^\bullet(M)$ satisfies $\mathrm{HL}_{\ell_2}^{\leq 1}$, so we obtain a decomposition $A^1(M) \cong \langle \ell_2 \rangle \oplus P_{\ell_2}^1$ that is orthogonal with respect to the Hodge–Riemann form $Q_{\ell_2}^1$. By $\mathrm{HR}_{\ell_2}^{\leq 1}$, $Q_{\ell_2}^1$ is positive-definite on $P_{\ell_2}^1$ and negative-definite on $\langle \ell_2 \rangle$; therefore, the restriction of $Q_{\ell_2}^1$ to the subspace $\langle \ell_1, \ell_2 \rangle \subset A^1(M)$ is neither positive-definite nor negative-definite, so

$$\left(\int_M \ell_1^2 \ell_2^{d-2}\right)\left(\int_M \ell_2^2 \ell_2^{d-2}\right) < \left(\int_M \ell_1 \ell_2 \ell_2^{d-2}\right)^2.$$

If ℓ_2 is merely nef rather than ample, then for any ample element ℓ, the class $\ell_2(t) := \ell_2 + t\ell$ is ample for all $t > 0$. An ample ℓ exists by Lemma 6.2.3. Taking the limit as $t \to 0$ in the inequality

$$\left(\int_M \ell_1^2 \ell_2(t)^{d-2}\right)\left(\int_M \ell_2(t)^2 \ell_2(t)^{d-2}\right) < \left(\int_M \ell_1 \ell_2(t) \ell_2(t)^{d-2}\right)^2$$

yields the desired inequality. ∎

Corollary 6.2.5. *For each* $0 < k < d$,

$$\mu^{k-1}(M)\mu^{k+1}(M) \leq \mu^k(M)^2.$$

Proof. This proof is reproduced from [1, Proposition 9.8]. We proceed by induction on $\mathrm{rk}(M)$. When $k < d - 1$, the induction hypothesis applied to the truncation $T_E(M)$ implies the inequality because the absolute values of the lower coefficients of $\overline{\chi}_{T_E(M)}$ are the same as those of $\overline{\chi}_M$. Now, consider $k = d - 1$. For any $i \in E$, denote $\alpha := \sum_{i \in F} x_F \in A^1(M)$ and $\beta := \sum_{i \notin F} x_F$. Both α and β are independent of the choice of i and are nef. Proposition 9.5 of [1] states that $\mu^k(M) = \int_M \alpha^{d-k}\beta^k$; therefore, the desired inequality is

$$\left(\int_M \alpha^2 \beta^{d-2}\right)\left(\int_M \beta^2 \beta^{d-2}\right) \leq \left(\int_M \alpha\beta\beta^{d-2}\right)^2.$$

Since α and β are nef, this inequality holds by Lemma 6.2.4. ∎

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