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Statistical reconstruction of the GFF and KT transition

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Abstract. In this paper, we focus on the following question. Assume ϕ is a discrete Gaussian free field (GFF) on $\Lambda \subset \frac{1}{n}\mathbb{Z}^2$ and that we are given $e^{iT\phi}$, or equivalently $\phi \pmod{2\pi/T}$. Can we recover the macroscopic observables of ϕ with $o(1)$ precision? We prove that this statistical reconstruction problem undergoes the following Kosterlitz–Thouless type phase transition:

- If $T < T_{\text{rec}}^-$, one can fully recover ϕ from the knowledge of $\phi \pmod{2\pi/T}$. In this regime our proof relies on a new type of Peierls argument which we call *annealed* Peierls argument and which allows us to deal with an unknown *quenched* ground state.
- If $T > T_{\text{rec}}^+$, it is impossible to fully recover the field ϕ from the knowledge of $\phi \pmod{2\pi/T}$. To prove this result, we generalize the delocalization theorem by Fröhlich–Spencer to the case of integer-valued GFF in an inhomogeneous medium. This delocalization result is of independent interest and we give an application of our techniques to the *random-phase sine-Gordon model* in Appendix B. Also, an interesting connection with Riemann theta functions is drawn along the proof.

This statistical reconstruction problem is motivated by the two-dimensional XY and Villain models. Indeed, at low temperature T, the large scale fluctuations of these continuous spin systems are conjectured to be governed by a Gaussian free field. It is then natural to ask if one can recover the underlying macroscopic GFF from the observation of the spins of the XY or Villain model.

Another motivation for this work is that it provides us with an "integrable model" (the GFF) that undergoes a KT transition.

Keywords. Gaussian free field, KT transition, localization, phase transitions

1. Introduction

1.1. Main result

We work on the graph

$$\Lambda_n := [-1,1]^2 \cap \frac{1}{n}\mathbb{Z}^2$$

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and for functions $f, g : \Lambda_n \to \mathbb{R}$, we denote

$$\langle f, g \rangle := \sum_{x \in \Lambda_n} f(x) g(x). \tag{1.1}$$

For each $n \geq 1$, ϕ_n will denote a GFF[1] on Λ_n. Recall that for any smooth function $f : [-1, 1]^2 \to \mathbb{R}$,

$$\frac{1}{n^2} \langle \phi_n, f \rangle \to (\Phi, f) \quad \text{in law as } n \to \infty, \tag{1.2}$$

where Φ is a continuous GFF in $[-1, 1]^2$, and (Φ, f) "$:= \int \Phi(x) f(x)\, dx$". This tells us that the macroscopic observables related to ϕ_n are random variables of the form $n^{-2} \langle \phi_n, f \rangle$.

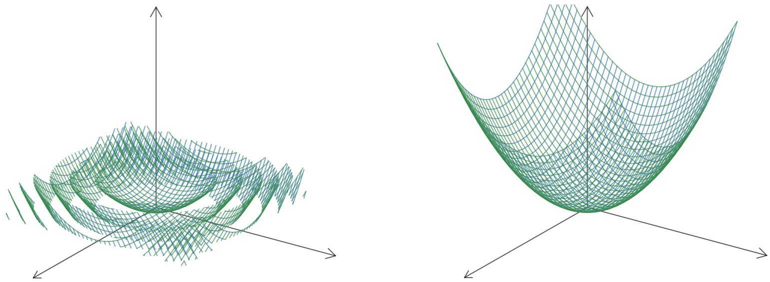

Fig. 1. If you are given the values of a function f modulo 1 (left), can you reconstruct what f is (right)? If f is smooth as in this example, surely you can. But what if f is an instance of a $2d$ Gaussian free field? Analyzing this statistical reconstruction problem is the aim of this paper.

The main focus of this paper is to understand when we can recover the full macroscopic information on ϕ_n by just knowing $\exp(iT\phi_n)$, or equivalently $\phi_n \pmod{2\pi/T}$. In Section 1.3 we will give several motivations which lead us to consider this problem. We now state our main result which shows that this statistical reconstruction problem undergoes a phase transition as T varies, which is reminiscent of the *Berezinskii–Kosterlitz–Thouless transition* (*KT transition*) (see Section 2.3).

Theorem 1.1. *Let ϕ_n be a GFF on Λ_n with Dirichlet boundary condition. Then there exist $0 < T_{\text{rec}}^- \leq T_{\text{rec}}^+ < \infty$ with the following properties:*

(a) *If $T < T_{\text{rec}}^-$, then there exists a (deterministic) reconstruction function F_T such that for any continuous function $f : [-1, 1]^2 \to \mathbb{R}$ and any $\varepsilon > 0$,*

$$\mathbb{P}[|n^{-2} \langle F_T(\exp(iT\phi_n)) - \phi_n, f \rangle| \geq \varepsilon] \to 0 \quad \textit{as } n \to \infty.$$

[1] With either free or Dirichlet boundary condition. We introduce all the relevant definitions in Section 2.

Furthermore, uniformly in n there exist constants $C, \tilde{C} > 0$ *such that for any* x, y *in* $\Lambda_n \subset [-1,1]^2$,

$$\mathbb{E}\big[\big(F_T(\exp(iT\phi_n))(x) - \phi_n(x)\big)^2\big] \le C, \tag{1.3}$$

$$\mathbb{E}\big[\big(F_T(\exp(iT\phi_n))(x) - \phi_n(x)\big)\big(F_T(\exp(iT\phi_n))(y) - \phi_n(y)\big)\big] \le e^{-\tilde{C}\|x-y\|n}. \tag{1.4}$$

(b) *If* $T > T_{\rm rec}^+$, *then for any (deterministic) function* F *and any continuous non-zero function* f *there exists* $\delta > 0$ *such that*

$$\liminf_{n\to\infty} \mathbb{P}[|n^{-2}\langle F(\exp(iT\phi_n)) - \phi_n, f\rangle| \ge \delta] > 0.$$

Also, for any $x \in (-1,1)^2$, *there exists* $c = c(T,x) > 0$ *such that for any* F,

$$\liminf_{n\to\infty} \mathbb{E}\big[\big(F(\exp(iT\phi_n))(x) - \phi_n(x)\big)^2\big] \ge c(T,x)\log n. \tag{1.5}$$

Let us remark that the recovery function F_T from the theorem refers to a non-local function of the whole field ϕ. In fact, it is completely non-local, that is, for any $x \in \Lambda_n$, $F_T(\exp(iT\phi_n))(x)$ depends on all vertices Λ_n. It is an interesting question to try to find the recovery function that minimizes this dependence.

The same result holds for a free-boundary GFF.

Theorem 1.2. *Let* ϕ_n *be a GFF on* $\Lambda_n := [-1,1]^2 \cap \frac{1}{n}\mathbb{Z}^2$ *with free boundary condition and rooted at a vertex* $x_0 \in \Lambda_n$. *Then there exist* $0 < T_{\rm rec}^- \le T_{\rm rec}^+ < \infty$ *with the following properties:*

(a) *If* $T < T_{\rm rec}^-$, *then there exists a reconstruction function* F_T *such that for any smooth function* f *with zero mean (i.e.,* $\int_{[-1,1]^2} f = 0$*) and any* $\varepsilon > 0$,

$$\mathbb{P}\big[\big|n^{-2}\big(F_T(\exp(iT\phi_n)) - \phi_n, f\big)\big| \ge \varepsilon\big] \to 0 \quad \text{as } n \to \infty.$$

(b) *If* $T > T_{\rm rec}^+$, *then for any function* F *and any smooth non-zero function* f *with zero mean there exists* $\delta > 0$ *such that*

$$\liminf_{n\to\infty} \mathbb{P}\big[\big|n^{-2}\big(F(\exp(iT\phi_n)) - \phi_n, f\big)\big| \ge \delta\big] > 0.$$

In fact, in the case of free boundary condition, one also has the equivalent statement of (1.3) and (1.5). However, there is an important difference between both boundary conditions. We do not expect the equivalent of (1.4) to be true for the free case. The main reason is that the conditional law of ϕ_n given $e^{iT\phi_n}$ may be decomposed as a convex combination of the laws of ϕ_n^k for $k \in \mathbb{Z}$, where for each of the fields ϕ_n^k one has (1.4). However, the law of ϕ_n^k is not centered (see Remark 3.8).

We now state two corollaries of the above theorems. The first one rephrases this phase transition in terms of the continuum GFF. The second one (which will give support to Conjectures 3 and 4 in Section 6) shows that one can recover macroscopic interfaces from $\phi \pmod{2\pi/T}$ when $T < T_{\rm rec}^-$.

Corollary 1.3. *Let ϕ_n be a sequence of GFFs in Λ_n such that, in probability, ϕ_n tends to a continuum GFF Φ in $[-1,1]^2$. If $T < T_{\rm rec}^-$, then the function $F_T(e^{iT\phi_n})$ converges in probability to Φ. Furthermore, if $T > T_{\rm rec}^+$ there is no deterministic function F such that $F(e^{iT\phi_n}) \to \Phi$.*

Corollary 1.4. *Let ϕ_n be a sequence of GFFs in Λ_n, and let $\eta^{(n)}$ be the Schramm–Sheffield level line*[2] *of ϕ_n. Then there exists a deterministic function L_T such that the Hausdorff distance between $L_T(\exp(iT\phi_n))$ and $\eta^{(n)}$ is $o(1)$. In particular, $L_T(\exp(iT\phi_n))$ converges in law to an* SLE_4.

Our work naturally belongs to the class of *statistical reconstruction problems* which have been the subject of an intense activity recently. For example, it shares similarities with the statistical reconstruction problems analyzed in [1, 25, 35]. In particular, in [1], the authors analyze the following problem: Imagine that each site $x \in \mathbb{Z}^d$ carries a spin or an element θ_x of a compact group $\mathfrak{G}$ (for example $\{\pm 1\}$ or $O(n)$). The question they are interested in is the following: what macroscopic information on $\{\theta_x\}_{x\in\mathbb{Z}^d}$ can be recovered from the knowledge of

$$\{\theta_i\theta_j^{-1} + \text{noise}\}_{i\sim j,\,\text{edges of }\mathbb{Z}^d}$$

where observations of neighboring spins $\theta_i\theta_j^{-1}$ are subjected to a small *noise*? Our setting is very similar in flavor, as we also have access to $\phi_n(i) - \phi_n(j)$ when $i \sim j$ except that the noise term is replaced in our case by the modulo operation, (mod $2\pi/T$). Similarly to adding a noise term, applying (mod $2\pi/T$) also reduces the information we have on $\phi_n(i) - \phi_n(j)$, and it cannot be analyzed as a convolution effect. Another difference with [1] is that our spins belong to $\mathbb{R}$ instead of a compact group $\mathfrak{G}$.

1.2. Fluctuations for integer-valued fields

Our present statistical reconstruction problem is intimately related to a generalization of the integer-valued Gaussian free field which plays a key role in the proof of the KT transition for the Villain and XY models in [21]. Let us briefly recall the classical integer-valued GFF before introducing its generalization.

For simplicity, in this subsection as well as in Sections 2.3, 4 and Appendix A, we will consider an arbitrary finite subset $\Lambda \subset \mathbb{Z}^2$, instead of the scaled box $\Lambda_n = \frac{1}{n}\mathbb{Z}^2 \cap [-1,1]^2$. This matches the setup in [21, 27].

Definition 1.5. Let $\Lambda \subset \mathbb{Z}^d$ be a finite domain.[3] An *integer-valued* GFF (IV-GFF) on Λ with *Dirichlet boundary condition*, i.e. 0 on $\partial\Lambda$, and *inverse temperature* β is a β-GFF $\{\phi(i)\}_{i\in\Lambda}$ conditioned on the singular event $\{\phi(i) \in \mathbb{Z},\ \forall i \in \Lambda\}$. Equivalently, it is the

[2] For the definition and the context used in this corollary see Section 6.2.

[3] Again, the GFF as well as the graph notations $\partial\Lambda$ etc. are defined in Section 2.

probability measure $\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda}$ on $\mathbb{Z}^\Lambda$ defined as follows:

$$\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda}(d\phi) := \frac{1}{Z} \sum_{\mathbf{m}\in\mathbb{Z}^\Lambda:\,\mathbf{m}_{|\partial\Lambda}=0} \delta_{\mathbf{m}}(d\phi)\exp\Big(-\frac{\beta}{2}\langle\nabla\phi,\nabla\phi\rangle\Big) \tag{1.6}$$

or also, to avoid any possible confusion, for any $\mathbf{m}\in\mathbb{Z}^\Lambda$ with zero boundary condition, we have $\mathbb{P}^{\mathrm{IV}}_{\beta,\Lambda}(\phi=\mathbf{m}) = \frac{1}{Z}\exp(-\frac{\beta}{2}\langle\nabla\mathbf{m},\nabla\mathbf{m}\rangle)$.

An IV-GFF with free boundary condition is defined in the same manner except that we replace $\mathbf{m}_{|\partial\Lambda}\equiv 0$ by $\mathbf{m}(x_0)=0$ for any choice of root vertex $x_0\in\Lambda$.

This integer-valued field undergoes a roughening-phase transition as T increases (i.e. as β decreases), as proved by Fröhlich–Spencer [21] (see also the very useful survey [27]). Fröhlich–Spencer proved this striking phase transition for periodic and free boundary conditions on large square boxes Λ and explained in [21, Appendix D] how to adapt their proof to the case of the Dirichlet boundary condition. Very recently, Wirth has written carefully in [45, Appendix A] the details of this extension to the Dirichlet boundary condition. We will come back to it later in Section 2.3. (See also Figure 2 for an illustration of the IV-GFF for $d=1$.)

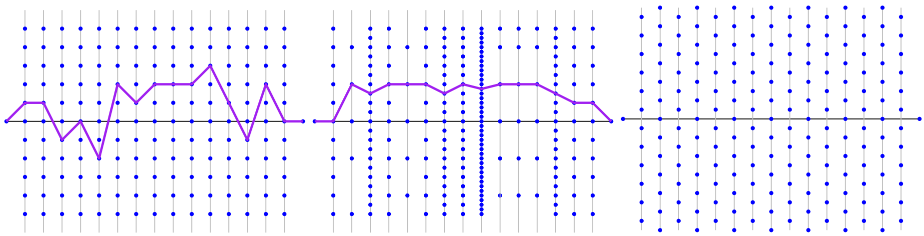

Fig. 2. Left: An instance of an IV-GFF for $d=1$ on a unit interval $\{0,\dots,n\}$ with Dirichlet boundary condition. For $d=2$, the proof of the *roughening phase transition* for the IV-GFF in [21] (and [45] for the extension to the Dirichlet boundary condition) easily generalizes to certain shifts and scaled versions on the vertical fibers $\mathbb{Z}$ as illustrated in the other pictures. Middle: Some *fibers* are $\mathbb{Z}$, some others are $2\mathbb{Z}$ and one may also add fibers with arbitrarily fine meshes $2^{-k}\mathbb{Z}$ along the interval. Right: Some fibers are $\mathbb{Z}$ for while some others are $\frac{1}{2}+\mathbb{Z}$. It is easy to check that for $d=2$, any of these can be handled with the techniques from [21].

Theorem 1.6 (Fröhlich–Spencer [21]). *There exist*[4] $0<\beta^+\le\beta^-<\infty$ *such that for any square* $\Lambda\subset\mathbb{Z}^2$*, if we consider an IV-GFF with free boundary condition rooted at* $x_0\in\Lambda$ *then we have the following dichotomy:*

- (Delocalized (rough) regime) *If* $\beta<\beta^+$*, then for any* $f:\Lambda\to\mathbb{R}$ *with* $\sum_{i\in\Lambda} f(i)=0$*,*

$$\mathbb{E}^{\mathrm{IV}}_{\beta,\Lambda}[e^{\langle\phi,f\rangle}] \ge e^{\frac{1}{4\beta}\langle f,(-\Delta)^{-1}f\rangle}.$$

[4] The choice $\beta^+\le\beta^-$ is made to highlight that these are inverse temperatures related to $T^-_{\mathrm{rec}}\le T^+_{\mathrm{rec}}$.

(*It is not hard to extract, from this Laplace transform estimate, fluctuation bounds such as* $\mathbb{E}^{\mathrm{IV}}_{\beta,\{-n,\dots,n\}^2,\,x_0=0}[\phi^2(x)] \geq \frac{c}{\beta}\log\|x\|_2$ *for any* $x \in \{-n,\dots,n\}^2$*; see for example* [27].)

- (Localized regime) *If* $\beta > \beta^-$*, then for any* $x \in \Lambda$*,*

$$\mathbb{E}^{\mathrm{IV}}_{\beta,\Lambda,x_0}[\phi^2(x)] \leq C/\beta.$$

The relationship between our statistical reconstruction problem and integer-valued fields is due to the following explicit structure of the conditional law of a GFF given its values modulo $2\pi/T$. We stick for simplicity to the case of the Dirichlet boundary condition. Let us fix $\mathbf{a} \in [0,1)^\Lambda$ satisfying $\mathbf{a}_{|\partial\Lambda} \equiv 0$. We will see in Lemma 2.8 that the conditional law of the GFF ϕ on Λ given $\phi \pmod{2\pi/T} = \frac{2\pi}{T}\mathbf{a}$ is a multiple of the following *generalized integer-valued GFF* with $\beta = \beta_T := (2\pi)^2 T^{-2}$ (see Lemma 2.8 for a precise statement).

Definition 1.7. Let $\beta > 0$ and $\mathbf{a} = \{a_i\}_{i\in\Lambda}$ with $\mathbf{a}_{|\partial\Lambda} \equiv 0$ be any collection of real numbers. We define an $\mathbf{a}$-*IV-GFF* on Λ to be a GFF $\{\phi(i)\}_{i\in\Lambda}$ (with Dirichlet boundary condition) conditioned to take its values in the shifted fibers $\{a_i + \mathbb{Z}\}_{i\in\Lambda}$ for any $i \in \Lambda$. It corresponds to the following discrete probability measure on fields:

$$\mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta,\Lambda}[d\phi] := \frac{1}{Z} \sum_{\mathbf{m}\in\mathbb{Z}^\Lambda:\,\mathbf{m}_{|\partial\Lambda}\equiv 0} \delta_{\mathbf{m}+\mathbf{a}}(d\phi)\exp\Big(-\frac{\beta}{2}\langle\nabla(\phi),\nabla(\phi)\rangle\Big).$$

Equivalently, for any $\mathbf{m} \in \mathbb{Z}^\Lambda$ with $\mathbf{m}_{|\partial\Lambda} \equiv 0$,

$$\mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta,\Lambda}[\phi = \mathbf{m}+\mathbf{a}] = \frac{1}{Z}\exp\Big(-\frac{\beta}{2}\langle\nabla(\mathbf{m}+\mathbf{a}),\nabla(\mathbf{m}+\mathbf{a})\rangle\Big). \tag{1.7}$$

Notice that if $\mathbf{a} \in \mathbb{Z}^\Lambda$, then the $\mathbf{a}$-IV-GFF is nothing but the standard IV-GFF. See Figures 2 and 3. Finally, this definition extends readily to the free boundary condition in which case $\mathbf{a} \in \mathbb{R}^\Lambda$ with $\mathbf{a}_{x_0} = 0$.

The proof of Fröhlich–Spencer [21] readily extends to some specific choices of the shift $\mathbf{a}$ which are sufficiently symmetric (i.e. any $\mathbf{a} \in \{0, 1/2\}^\Lambda$). See Figure 2 for an illustration (for $d = 1$ only) of the cases which can be analyzed using the techniques from [21] and Figure 3 for the cases which need further analysis. See also Remark 2.7. Our main result on such integer-valued fields is the following extension of the above theorem of Fröhlich and Spencer [21].

Theorem 1.8. *There exists* $\beta_c^{\mathrm{IV}} > 0$ *and a constant* $C > 0$ *such that for any square domain* $\Lambda \subset \mathbb{Z}^2$*, any* $\beta < \beta_c^{\mathrm{IV}}$*, uniformly*[5] *in* $\mathbf{a} \in \mathbb{R}^\Lambda$*, if* $\phi^{\mathbf{a}} \sim \mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta,\Lambda}$ (*with either Dirichlet or free bondary condition*) *we have:*

[5]With $\mathbf{a}_{|\partial\Lambda} \equiv 0$ for Dirichlet boundary condition and $\mathbf{a}_{x_0} = 0$ for free boundary condition.

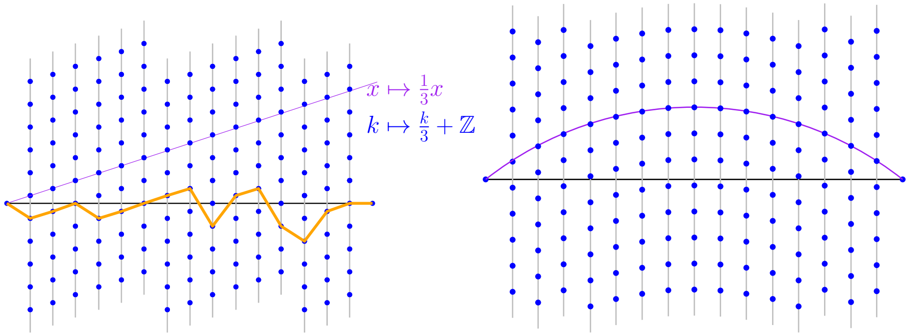

Fig. 3. As opposed to the examples in Figure 2, these choices of fibers do not satisfy the sufficient symmetries to be readily analyzed by the techniques from [21]. The simplest such example is the picture on the left where $\mathbb{Z}$ fibers are shifted by $\{0, 1/3, 2/3\}$ (here $\mathbf{a} := (k/3)_{k\in\mathbb{Z}}$). In such a case, $\sin(k\phi)$ functions appear in the Fourier transform of the periodic distribution $\sum_{i\in 1/3+\mathbb{Z}} \delta_i$ and this breaks the parity. For such linear shifts, Wirth [45] obtained some related bounds using a nice symmetrization trick (however, even in this linear case, these bounds give different control from the one we need here, see Remark 2.7). A key property used in [45] is that $x \mapsto a_x$ has to be *harmonic* in $\Lambda \setminus \partial\Lambda$. Otherwise the symmetrization technique breaks down and one cannot rely anymore on Jensen's inequality, a key step in the proof of [21]. For example the picture on the right where fibers are shifted by a quadratic curve requires an additional analysis beyond [21].

- *For any function* $f \in \mathbb{R}^\Lambda$,
$$\mathrm{Var}[\langle \phi^{\mathbf{a}}, f\rangle] \geq \frac{C}{\beta}\langle f, (-\Delta)^{-1} f\rangle,$$
where the inverse of the Laplacian is taken according to the boundary conditions
- *If* $\Lambda = \{-n, \ldots, n\}^2$ *with Dirichlet boundary condition, the variance of the field* $\phi^{\mathbf{a}}$ *at the origin satisfies*
$$\mathrm{Var}[\phi^{\mathbf{a}}(0,0)] \geq \frac{C}{\beta}\log n.$$
The analogous statement also holds for the free boundary condition.

Remark 1.9. We wish to stress that the low-temperature regime ($\beta \gg 1$) happens to be much less universal in the choice of the shift $\mathbf{a}$. Indeed, when $\Lambda = \{-n, \ldots, n\}^2$ is equipped with the Dirichlet boundary condition, and if $\phi^{\mathbf{a}} \sim \mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta,\Lambda}$, then we expect that the following different scenarios may happen (by tuning suitably $\mathbf{a}$ in each case) as $n \to \infty$ (see for example Figure 5 for scenario (1)):

(1) $\mathbb{E}[\phi^{\mathbf{a}}(0,0)] \geq 0.49n$.
(2) $\mathrm{Var}[\phi^{\mathbf{a}}(0,0)] \geq (0.49)^2 n$.
(3) $\mathrm{Var}[\phi^{\mathbf{a}}(0,0)] \leq O(1)$ and $\mathrm{Cov}[\phi^{\mathbf{a}}(x), \phi^{\mathbf{a}}(y)] \leq e^{-cn\|x-y\|_2}$.

Remark 1.10. Let us note that we do not obtain a lower bound on the Laplace transform of the integer-valued GFF, i.e., on
$$\mathbb{E}^{\mathbf{a},\mathrm{IV}}_{\beta,\Lambda}[e^{\langle \phi^{\mathbf{a}}, f\rangle}].$$

We only obtain bounds on the L^2 behavior of $\langle \phi^{\mathbf{a}}, f\rangle$. These bounds are, in fact, sufficient to detect localization vs. delocalization. This is also the case in the recent works mentioned below on localization/delocalization of integer-valued random surfaces.

As we will see in Section 4 and particularly in Appendix A, our proof of Theorem 1.8 involves an exact identity (Proposition 4.1) which is closely related to the *modular invariance* identity for Riemann theta functions (see also [2] for another use of such identities in probability). We briefly mention this connection here as it is interesting in its own right and it allows us to rephrase the Fröhlich–Spencer Theorem as well as our Theorem 1.8 easily in terms of those Riemann theta functions.

Indeed, the following function of $\mathbf{a} \in \mathbb{R}^{\Lambda\setminus\partial\Lambda}$:

$$\tilde\theta_\Lambda(\mathbf{a}) := \sum_{\mathbf{m}\in\mathbb{Z}^{\Lambda\setminus\partial\Lambda}} \exp\Big(-\frac{\beta}{2}\langle \mathbf{m}, (-\Delta)\mathbf{m}\rangle\Big)\exp(-\beta\,\mathbf{m}\cdot a)$$

can be easily written in terms of the classical Riemann theta function $\theta(\mathbf{z} \mid \Omega)$ (see (A.4)). Furthermore, one can check that for any $f : \Lambda \to \mathbb{R}$ and if $\phi^{\mathbf{a}} \sim \mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta,\Lambda}$, we have

$$\mathrm{Var}[\langle \phi^{\mathbf{a}}, f\rangle] = [\sigma\cdot\nabla_{\mathbf{a}}\,\sigma\;\cdot\nabla_{\mathbf{a}}]\log\tilde\theta_\Lambda,$$

where $\sigma := \frac{1}{\beta}(-\Delta)^{-1}f$. This expression clarifies the effect of the shift vector $\mathbf{a} \in \mathbb{R}^\Lambda$ and reveals that it plays the role of an exterior magnetic field. We may now rephrase Fröhlich–Spencer as well as our main result from this section as follows:

- (Theorem 1.6) If β is small enough, then uniformly in $\Lambda = \{-n, \dots, n\}^2$,

$$[\sigma\cdot\nabla_{\mathbf{a}}\,\sigma\;\cdot\nabla_{\mathbf{a}}]_{\mathbf{a}\equiv 0}\log\,\tilde\theta_\Lambda \geq \frac{C}{\beta}\langle f, -\Delta^{-1}f\rangle.$$

- (Theorem 1.8) If β is small enough, then uniformly in $\Lambda = \{-n, \dots, n\}^2$,

$$\inf_{\mathbf{a}\in\mathbb{R}^{\Lambda\setminus\partial\Lambda}} \sigma\cdot\nabla_{\mathbf{a}}\sigma\cdot\nabla_{\mathbf{a}}(\log\tilde\theta_\Lambda) \geq \frac{C}{\beta}\langle f, -\Delta^{-1}f\rangle.$$

Finally, let us point out that over the last few years, there have been several important works which analyzed the roughening phase transition (i.e. localization/delocalization) for other natural models of integer-valued random fields, such as the *square-ice* model and the uniformly chosen Lipschitz functions $\mathbb{Z}^2 \to \mathbb{Z}$; see in particular the recent works [13, 17, 18, 24]. These works do not rely on the Coulomb-gas techniques from [21] but rather on geometric techniques such as RSW.

1.3. Motivations behind the statistical reconstruction problem

As we will see below, one of the main reasons which lead us to consider this statistical reconstruction problem on the GFF has to do with the statistical analysis of the XY and Villain models for $d = 2$. These are celebrated models with continuous $O(2)$-symmetry. We briefly define them and refer the reader to [10, 20, 21, 27] for useful background.

Definition 1.11 (Villain and XY models). Let us fix a finite graph $\Lambda \subset \mathbb{Z}^2$ and $\beta > 0$ to be the inverse temperature. Both models are Gibbs measures on the state space $(\mathbb{S}^1)^\Lambda$. Let us parametrize this spin space via its canonical identification with $[0, 2\pi)^\Lambda$.

- *XY model* (or plane rotator model):

$$d\mathbb{P}_\beta^{\mathrm{XY}}[\{\theta_x\}_{x\in\Lambda}] \propto \prod_{i\sim j} \exp(\beta\cos(\theta_i - \theta_j)) \prod d\theta_i. \tag{1.8}$$

- *Villain model*:

$$d\mathbb{P}_\beta^{\mathrm{Villain}}[\{\theta_x\}_{x\in\Lambda}] \propto \prod_{i\sim j} \sum_{m\in\mathbb{Z}} \exp\Big(-\frac{\beta}{2}(2\pi m + \theta_i - \theta_j)^2\Big) \prod d\theta_i.$$

We may now list the main motivations which guided our work.

(1) *Extracting macroscopic random structures from XY and Villain spins.* For spin systems such as the Ising model, Potts models or percolation, which all have discrete symmetries, it is clear how to associate natural macroscopic fluctuating objects such as interfaces which may then converge to suitable SLE_κ as the mesh goes to zero. On the other hand, for spin systems with continuous symmetry such as XY or Villain models, given a realization of the Gibbs measure, say $\{\sigma_x\}_{x\in\Lambda}$ with $\sigma_x \in \mathbb{S}^1$, it is much less clear what macroscopic objects one may assign to $\{\sigma_x\}$.

One consequence of our present statistical reconstruction problem is that it gives strong evidence to the fact that it is possible to extract a macroscopic GFF ϕ_n from the observation of the spins $\{\sigma_x\}_{x\in\{-n,\dots,n\}^2}$ (up to small microscopic errors).

Indeed, at least in the case of the Villain model, it has been conjectured by Fröhlich–Spencer [22, Section 8.1] that at low temperature ($\beta \gg 1$), up to "microscopic errors", one should have

$$\{\sigma_x\}_{x\in\{-n,\dots,n\}^2} \sim \mathbb{P}_\beta^{\mathrm{Villain}} \overset{\mathrm{law}}{\approx} \left\{\exp\Big(i\frac{1}{\sqrt{\beta'}}\phi_n(x)\Big)\right\}_{x\in\{-n,\dots,n\}^2},$$

where $\beta' = \beta'(\beta)$ satisfies $|\beta' - \beta| \le e^{-C\beta}$ and where ϕ_n is a GFF on $\{-n,\dots,n\}^2$ with either free or zero boundary condition.

Once one realizes that one may extract a GFF out of the spins $\{\sigma_x\}_{x\in\Lambda_n}$, it is natural to extract level lines and flow lines from this GFF studied in [16,30,40]. Corollary 1.4 is a realization of this idea. We discuss this further in Section 6.2 where we highlight how our work led us to conjecture that when β is high enough, the natural interface for the Villain model pictured in Figure 4 should converge to an $\mathrm{SLE}(4,\rho)$ process.

(2) *A different interpretation of the KT transition.* The classical way of understanding the KT transition for spin systems such as the XY model is to notice that vortices ($\equiv$ discrete 2-forms) come into the energy balance when analyzing the Gibbs measure (1.8). The present work gives the following different interpretation of the role of the $\mathbb{S}^1$-geometry within the KT transition which does not explicitly involve vortices. When the temperature T is low, spins wiggle slowly around $\mathbb{S}^1$ and one should be able to recover a macroscopic

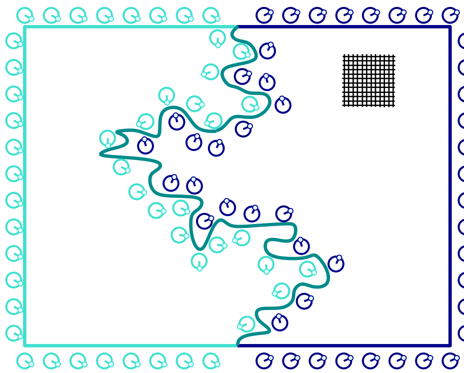

Fig. 4. Conjectures 3 and 4 in Section 6.2 predict that at low temperature, the level lines of a Villain model with $e^{i/10}$ on the right boundary and $e^{-i/10}$ on the left boundary should converge to SLE($\kappa = 4, \rho$) processes. Furthermore, the set of all interfaces should converge to the so-called ALE (see Conjecture 6). These conjectures are supported by the present statistical reconstruction problem as well as by the techniques we have used.

GFF as we have seen in item (1) above. If instead the temperature T is large, the spins start wiggling too quickly around $\mathbb{S}^1$ so that one cannot extract the whole macroscopic fluctuating Gaussian field ϕ which lives on the top of the spin field. Hence, one may interpret the KT transition in the case of the GFF as the sudden absence of a statistical estimator to recover the macroscopic field ϕ. Also, as the true XY field is not exactly given by the complex exponential of a GFF, this statistical reconstruction problem gives a new example which belongs to the KT universality class (at least conjecturally, as we have not proved anything regarding the behavior of a suitable notion of correlation length for this problem).

(3) *An integrable model for integer-valued GFF.* The main tool we use for the regime $T > T_{\text{rec}}^+$ is the proof of delocalization for the generalized IV-GFF $\phi^{\mathbf{a}} \sim \mathbb{P}_\Lambda^{\mathbf{a},\text{IV}}$ from Definition 1.7. We think of $\mathbf{a}$ as the random vector $\{a_i\}_{i\in\Lambda} \in [0,1)^\Lambda$ defined by

$$a_i := \frac{T}{2\pi}\phi_i \ (\text{mod } 1), \quad \forall i \in \Lambda,$$

where ϕ is a GFF in Λ. Hence, we may view the random measure $\mathbb{P}_\Lambda^{\mathbf{a},\text{IV}}$ as a *quenched measure* on (shifted) integer-valued fields. Interestingly, these highly non-trivial quenched measures have (by construction) a very simple *annealed measure*. Indeed, Lemma 2.8 readily implies that

$$\int \mathbb{P}_{\beta_T,\Lambda}^{\text{IV},\mathbf{a}}[d\phi]\, \mathbb{P}_T(d\mathbf{a}) = \mathbb{P}_{\beta_T}^{\text{GFF}}[d\phi], \tag{1.9}$$

where we denote by $\mathbb{P}_T(d\mathbf{a})$ the law of the above random shift $\mathbf{a}$ and $\beta_T = (2\pi)^2/T^2$. If one now assumes that some properties (such as fluctuations) are not very sensitive to $\mathbf{a}$, this identity gives a "useful laboratory" to analyze the classical integer-valued GFF (i.e. $\mathbf{a} \equiv 0$). A first illustration of this is given in Section 5 where we provide a new insight on the $\varepsilon = \varepsilon(\beta)$ correction in the bound of Fröhlich–Spencer. A second illustration is given in the item below.

(4) *Random-phase sine-Gordon model.* As pointed out to us by Tom Spencer, our work is closely related to the *random-phase sine-Gordon* model. This is a model of random interface with quenched disorder which has been studied extensively in physics and which is conjectured to exhibit a striking *super-roughening* behavior at low temperature (see for example [12, 29]). As we shall explain in Appendix B, our proof of Theorem 1.8 easily extends to the setting of the random-phase sine-Gordon model and allows us to prove $\log n$ fluctuations in the high-temperature phase of this model (see Theorem B.3).

(5) *Imaginary multiplicative chaos.* In this work we focus on lattice fields $\phi : \Lambda \to \mathbb{R}$ or $\Lambda_n \to \mathbb{R}$, but the question in the continuum is also interesting. Namely, given a Gaussian free field Φ on $[-1, 1]^2$ with zero boundary condition, can one recover Φ from $:e^{i\alpha\Phi}:$? This is the complex analog of the reconstruction procedure $:e^{\gamma\Phi}: \mapsto \Phi$ studied in [11]. We discuss this further in Section 6.1, where we show that the existence of a continuous reconstruction process in the imaginary case implies the existence of a discrete reconstruction process. However, let us highlight that even if this continuum process does exist, the discrete reconstruction process coming from it will converge much slower ($o(1)$) than the one we obtain in Theorem 1.1 using statistical mechanics (i.e. $O(1/n^2)$, see Proposition 3.9). Also, as opposed to the discrete setting where a transition arises, we do not expect in the continuum a regime where $:e^{i\alpha\Phi}:$ exists and yet the statistical reconstruction of Φ breaks down. (After the first version of this work appeared, the reconstruction problem in the continuum was solved in the beautiful recent work [5] using completely different tools.)

1.4. Idea of the proof

The first choice one needs to make in the proof is the reconstruction function F_T. We have essentially two natural choices here (see Figure 5 for an illustration of both).

(1) First, if $\mathbf{a} := \frac{T}{2\pi}(\phi \pmod{2\pi/T})$, then there is an a.s. unique *ground state*[6] for $\mathbb{P}_\beta^{\mathbf{a},\mathrm{IV}}$ which we may call

$$\hat{\mathbf{m}}(\exp(iT\phi)) = \hat{\mathbf{m}}(\exp(2i\pi\mathbf{a})) := \operatorname{argmin}_{\mathbf{m}\in\mathbb{Z}^\Lambda} \exp\left(-\frac{\beta}{2}\langle \mathbf{m}+\mathbf{a}, -\Delta(\mathbf{m}+\mathbf{a})\rangle\right).$$

It is reasonable to guess that when T is small, the field ϕ should not fluctuate much around $\hat{\mathbf{m}}(\exp(iT\phi))$.

(2) A second natural choice is to consider instead the conditional expectation of the field given $\exp(iT\phi)$.

[6] From the point of view of the statistical reconstruction, the ground state should be read as the maximum-likelihood estimator.

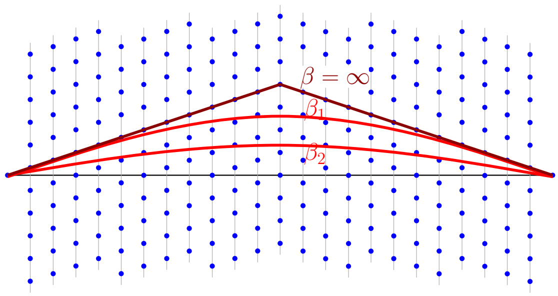

Fig. 5. Here $a_k := k/3$ on the LHS of the picture and then the slope is $-k/3$. It is easy to check that the ground state $\hat{\mathbf{m}}$ for the GFF conditioned to have its values in these shifted fibers is given by the purple curve $\beta = \infty$. Then, as β decreases, the expectation $x \mapsto \mathbb{E}_\beta^{\mathbf{a},\mathrm{IV}}[\phi^{\mathbf{a}}(x)]$ can be seen to decrease to 0.

The *quenched* groundstate $\hat{\mathbf{m}}(\exp(iT\phi))$ does not have enough symmetries to apply classical tools from Peierls theory and we are too far from the perturbative regime where *Pirogov–Sinai theory* can be used (see [20, Chapter 7]).

Therefore, for the *low-temperature regime* in the proofs of Theorems 1.1 and 1.2, we will recover the GFF given its phase via the second choice, i.e.,

$$F_T(\exp(iT\Phi))(x) := \mathbb{E}[\phi(x) \mid \exp(iT\phi)].$$

It is not easy to study this function F directly. However, for any test function f we can use Markov's inequality to see that

$$\mathbb{P}\big(|\langle \phi - F_T(\exp(iT\Phi)), f\rangle| \geq \varepsilon\big) \leq \frac{\mathbb{E}\big[\mathrm{Var}[\langle \phi, f\rangle \mid \exp(iT\phi)]\big]}{\varepsilon^2}. \tag{1.10}$$

This implies that to understand how well F approximates ϕ it is enough to bound the conditional variance of $\langle \phi, f\rangle$ given $\exp(iT\phi)$. Working with the conditional variance is much easier than to work with F directly. This is because one can study it by *coupling* two GFFs (ϕ_1, ϕ_2) such that $\exp(iT\phi_1) = \exp(iT\phi_2)$ in such a way that they are conditionally independent given $\exp(iT\phi)$ (see Definition 3.1). This is useful because

$$\mathbb{E}\big[\mathrm{Var}[\langle \phi, f\rangle \mid \exp(iT\phi)]\big] = \tfrac{1}{2}\mathbb{E}[\langle \phi_1 - \phi_2, f\rangle^2]. \tag{1.11}$$

As this function does not involve any estimate of the function F, and both ϕ_1 and ϕ_2 have the law of a GFF, we set up an appropriate *annealed Peierls argument* to show in Section 3 that (1.11) is small when T is small.

The second part of Theorems 1.1 and 1.2 also follows from similar ideas with a "statistical flavor". In fact, we are going to show that for any f there exists an $\varepsilon > 0$ such that for all n large enough,

$$\mathbb{E}\big[\mathrm{Var}[\langle \phi, f\rangle \mid \exp(iT\phi)]\big] \geq \varepsilon \mathrm{Var}[\langle \phi, f\rangle]. \tag{1.12}$$

This, together with some basic tension argument, implies that the probability that $\langle \phi_1, f \rangle$ is macroscopically different from $\langle \phi_2, f \rangle$ is uniformly positive.

To obtain (1.12), we need to modify the work of Fröhlich and Spencer [21]. In this seminal paper, the authors showed that the integer-valued GFF has variance similar to that of the GFF when the temperature is high enough. In our case, in Section 4 we will prove a result with a similar taste (Theorem 1.8) that will uniformly show that when T is high enough, for any realization of $\exp(iT\phi)$,

$$\mathrm{Var}[\langle \phi, f \rangle \mid \exp(iT\phi)] \geq \varepsilon \mathrm{Var}[\langle \phi, f \rangle]. \tag{1.13}$$

This can be read, in the context of [21], as the study of *integer-valued GFF in an inhomogeneous medium*.

2. Preliminaries

2.1. Discrete differential calculus

We start by discussing the basics of discrete differential calculus. As the only graph we work with in this paper is $\Lambda_n := [-1, 1] \cap \frac{1}{n}\mathbb{Z}^2$ with its canonical edge set, we only discuss the needed results in this framework. For simplicity we identify Λ_n with its vertex set and we denote by E_{Λ_n} its edge set. For a deeper discussion of discrete differential calculus, we refer the reader to [14].

In this section, we study two types of functions: functions on vertices, $S : \Lambda_n \to \mathbb{R}$, and functions on directed edges, $A : \vec{E} \to \mathbb{R}$. Functions on vertices can take any values, while functions on directed edges have to always satisfy

$$A(\overrightarrow{xy}) = -A(\overrightarrow{yx}). \tag{2.1}$$

Let us now present two canonical differential operators

$$\nabla S(\overrightarrow{xy}) = S(y) - S(x), \tag{2.2}$$

$$\nabla \cdot A(x) = \sum_{\overrightarrow{xy}} A(\overrightarrow{xy}). \tag{2.3}$$

Then one can write the Laplacian of S as follows:

$$\Delta S(x) = \nabla \cdot \nabla S(x) = \sum_{y \sim x} S(y) - S(x). \tag{2.4}$$

Furthermore, we say that a function $S : \Lambda_n \to \mathbb{R}$ is *harmonic* over a set $A \subseteq \Lambda_n$ if for any $x \in A$, $\Delta S(x) = 0$.

For a pair of functions $S_1, S_2 : V \to \mathbb{R}$ on vertices, or $A_1, A_2 : \vec{E} \to \mathbb{R}$ on edges, we define

$$\langle S_1, S_2 \rangle := \sum_{x \in V} S_1(x) S_2(x), \quad \langle A_1, A_2 \rangle := \frac{1}{2} \sum_{\overrightarrow{xy}} A_1(\overrightarrow{xy}) A_2(\overrightarrow{xy}).$$

Furthermore, we define

$$\langle S_1, S_2 \rangle_\nabla = \langle \nabla S_1, \nabla S_2 \rangle.$$

Let us remark that the differentials ∇ and $-\nabla\cdot$ are dual to each other, i.e.,

$$\langle \nabla S, A\rangle = \langle S, -\nabla \cdot A\rangle. \tag{2.5}$$

Thanks to this, we can easily see that $-\Delta$ is a positive definite operator.

Definition 2.1 (Inverse of the Laplacian). We fix a subset $\partial\Lambda_n \subseteq \Lambda_n$ and we call it the *boundary*. If $\partial\Lambda_n \neq \emptyset$, then for any function $S_1 : \Lambda_n \setminus \partial\Lambda_n \to \mathbb{R}$ there is a unique function $S_2 : \Lambda_n \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta S_2(x) &= S_1(x), \quad && \forall x \in V \setminus \partial\Lambda_n, \\ S_2(x) &= 0, && \forall x \in \partial\Lambda_n. \end{aligned}$$

In this case, we call $S_2 := (-\Delta^{-1})S_1$.

The inverse of the Laplacian operator can be understood thanks to the Green's function

$$G(x,\cdot) = -\Delta^{-1}(\mathbf{1}_x). \tag{2.6}$$

When necessary, we will add a superscript to make explicit the boundary conditions of G. Let us recall a classical result for the Green's function in dimension 2.

Proposition 2.2. *For the graph Λ_n and for both free and zero boundary condition and for any $x, y \in \Lambda_n$,*

$$G(x,x) = C \log(d(x,\partial\Lambda_n)) + O(1), \tag{2.7}$$

where C does not depend on any other parameter.

2.2. *The Gaussian free field*

In this subsection, we introduce the GFF and some of the properties we use throughout the paper. For a more detailed discussion of the GFF, we refer the reader to [41,42].

Let us fix a boundary set $\partial\Lambda_n$. A *GFF with zero boundary condition on* $\partial\Lambda_n$ is a random function $\phi : V \to \mathbb{R}$ such that

$$\mathbb{P}((\phi(v) \in dx_v)_{v\in\Lambda_n}) \propto \exp\left(-\frac{\langle \phi,\phi\rangle_\nabla}{2}\right) \prod_{v\in\Lambda_n\setminus\partial\Lambda_n} dx_v \prod_{v\in\partial\Lambda_n} \delta_0(dx_v).$$

We say that ϕ is a *GFF with free boundary condition* if $\partial\Lambda_n = \{x_0\}$ for some $x_0 \in \Lambda_n$. We say that ϕ is a GFF with *zero* (or *Dirichlet*) *boundary condition* if

$$\partial\Lambda_n := \{x \in \Lambda_n : |\mathrm{Re}(x)| = 1 \text{ or } |\mathrm{Im}(x)| = 1\},$$

in other words, the points in Λ_n that are on the boundary of $[-1,1]^2$. Here, we interpret Λ_n as a subset of the complex plane $\mathbb{C}$.

An important equivalent characterization of the GFF is as the centered Gaussian process with covariance

$$\mathbb{E}[\phi(x)\phi(y)] = G(x,y),$$

where the boundary values of the Green's function are associated with the boundary values of the GFF.

A key property to understand the GFF is its Markov property.

Proposition 2.3 (Weak Markov property). *Let ϕ be a GFF in Λ_n with zero boundary condition in $\partial\Lambda_n$. Furthermore, let B be a subset of the vertices of Λ_n. Then there are independent random functions ϕ_B and ϕ^B such that $\phi = \phi_B + \phi^B$ and*

(1) *ϕ_B is harmonic in $\Lambda_n \setminus B$;*

(2) *ϕ^B is a GFF in Λ_n with zero boundary condition on $\partial\Lambda_n \cup B$.*

Let us now define a white noise on the edges of Λ_n.

Definition 2.4 (White noise). A *white noise* is a function on the directed edges of Λ_n such that $W(\vec{e})$ is a standard normal random variable independent of all other $W(\vec{e}')$ with $e \neq e'$.

The discrete gradient of the GFF has an interesting relationship to white noise. This result can be found in [4] for this setting as well as in [3] for the same decomposition in the continuous case.

Proposition 2.5. *Let ϕ be a GFF in Λ_n. Then there exists a Gaussian process $\zeta(\vec{e})$ independent of ϕ and such that*

$$W := \nabla\phi + \zeta$$

is a white noise in E. Furthermore,

$$\phi = \Delta^{-1}\nabla \cdot W.$$

2.3. *Integer-valued Gaussian free field and the KT transition*

In this section, we briefly explain how Fröhlich and Spencer proved their delocalization Theorem 1.6, because we will rely on the technology they developed (an expansion into Coulomb charges) later in Section 4. We refer the reader to the excellent review [27] from which we borrow the notations. See [27] for the relevant definitions.

For simplicity, we fix a square domain $\Lambda \subset \mathbb{Z}^2$ and we consider the case of free boundary condition rooted at some vertex $v \in \Lambda$.

The proof by Fröhlich–Spencer can essentially be decomposed into the following successive steps:

The first step is to view the singular conditioning $\{\phi_i \in 2\pi\mathbb{Z}, \forall i \in \Lambda\}$ using Fourier series[7] thanks to the identity

$$2\pi \sum_{m\in\mathbb{Z}} \delta_{2\pi m}(\phi) \equiv 1 + 2\sum_{q=1}^{\infty} \cos(q\phi).$$

[7] It is slightly more convenient to consider the GFF conditioned to live in $(2\pi\mathbb{Z})^\Lambda$ rather than $\mathbb{Z}^\Lambda$. Following [21, 27], we will stick to this convention here as well as in Section 4 and Appendix A.

To avoid dealing with infinite series, proceeding as in [27], we consider the approximate IV-GFF

$$\mathbb{P}_{\beta,\Lambda,v}[d\phi] := \frac{1}{Z_{\beta,\Lambda,v}} \prod_{i\in\Lambda} \Big(1 + 2\sum_{q=1}^{N} \cos(q\phi(i))\Big) \mathbb{P}^{\mathrm{GFF}}_{\beta,\Lambda,v}[d\phi].$$

In fact, more general measures are considered in [21, 27]: they fix a family of trigonometric polynomials $\lambda_\Lambda := (\lambda_i)_{i\in\Lambda}$ attached to each vertex $i \in \Lambda$. These trigonometric polynomials are parametrized as follows: for each $i \in \Lambda$,

$$\lambda_i(\phi) = 1 + 2\sum_{q=1}^{N} \hat\lambda_i(q)\cos(q\phi(i)).$$

Now given a family of trigonometric polynomials λ_Λ, they define

$$\mathbb{P}_{\beta,\Lambda,\lambda_\Lambda,v}[d\phi] := \frac{1}{Z_{\beta,\Lambda,,\lambda_\Lambda,v}} \prod_{i\in\Lambda} \lambda_i(\phi(i)) \mathbb{P}^{\mathrm{GFF}}_{\beta,\Lambda,v}[d\phi].$$

We mention this degree of generality to keep the same notations as in [21,27] and also so that the reader will not get confused when consulting those references. Also, this degree of generality will be useful later in Appendix B. Yet, in the present case, we will stick to the case where $\hat\lambda_i(q) = 1$ for all $i \in \Lambda$ and $1 \le q \le N$.

The second step in the proof is to fix a test function $f : \Lambda \to \mathbb{R}$ such that $\sum_{i\in\Lambda} f(i) = 0$ and to consider the Laplace transform of $\langle \phi, f\rangle$, $\mathbb{E}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\langle\phi,f\rangle}]$. As $N \to \infty$, with our choice of trigonometric polynomials,[8] this will converge to the Laplace transform $\mathbb{E}^{\mathrm{IV}}_{\beta,\Lambda}[e^{\langle\phi,f\rangle}]$.

By a simple change of variables, this Laplace transform can be rewritten as

$$\mathbb{E}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\langle\phi,f\rangle}] = \frac{1}{Z_{\beta,\Lambda,\lambda_\Lambda,v}} \exp\Big(\frac{1}{2\beta}\langle f, -\Delta^{-1} f\rangle\Big) \mathbb{E}^{\mathrm{GFF}}_{\beta,\Lambda,v}\Big[\prod_{i\in\Lambda} \lambda_i(\phi(i) + \sigma(i))\Big],$$

where the function $\sigma = \sigma_f$ will be used throughout and marked with a different color in the computations. It is defined by

$$\sigma := \frac{1}{\beta}[-\Delta]^{-1} f. \tag{2.8}$$

The main difficulty in the proof in [21] is in some sense to show that the shift σ does not have a dramatic effect compared to the exponential term $\exp(\frac{1}{2\beta}\langle f, -\Delta^{-1} f\rangle)$ so that ultimately,

$$\mathbb{E}^{\mathrm{IV}}_{\beta,\Lambda}[e^{\langle\phi,f\rangle}] \ge \exp\Big(\frac{1}{2\beta(1+\varepsilon)}\langle f, -\Delta^{-1} f\rangle\Big).$$

[8] Note that, in this section, instead of conditioning the GFF to be in $\mathbb{Z}^\Lambda$ as in Definition 1.5, we condition it to be in $(2\pi\mathbb{Z})^\Lambda$. Besides changing constants, this does not make much difference.

From such a lower bound on the Laplace transform, one can easily extract delocalization properties of the IV-GFF.

The third (and by far most difficult) *step* is to control the effect of the shift σ via a highly non-trivial expansion into Coulomb charges which enables us to rewrite the partition function as follows:

$$Z_{\beta,\Lambda,\lambda_\Lambda,v} = \sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \int \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi).$$

We refer to [21, 27] for the notations used in this expression and in particular for the concept of *charges* (i.e. $\rho : \Lambda \to \mathbb{R}$), *ensembles* (i.e. sets $\mathcal{N}$ of mutually disjoint charges ρ) etc.

One important feature of this expansion into charges is that under some (very general) assumptions on the growth of the Fourier coefficients $|\hat\lambda_i(q)|$ (see [21, (5.35)]), it can be shown that the effective activities $z(\beta,\rho,\mathcal{N})$ decay fast, namely (see [27, (1.14)]),

$$|z(\beta,\rho,\mathcal{N})| \le \exp\left(-\frac{c}{\beta}(\|\rho\|_2^2 + \log_2(\mathrm{diam}(\rho)+1))\right).$$

Hence at high temperature, the partition function corresponds to a sum of positive measures. (Also the weights $c_{\mathcal{N}}$ are positive and such that $\sum c_{\mathcal{N}} = 1$.)

Remark 2.6. In [27], the authors have introduced a slightly different definition of GFF with free boundary conditions which makes the analysis behind this decomposition into charges more pleasant (their definition handles the presence of non-neutral charges ρ very easily). One can switch to their more convenient definition in our setting since in the limit $N \to \infty$, both give the same integer-valued GFF.

This crucial third step allows us to rewrite the Laplace transform $\mathbb{E}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\langle\phi,f\rangle}]$ as follows:

$$e^{\frac{1}{2\beta}\langle f,-\Delta^{-1}f\rangle} \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \int \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle + \langle\sigma,\rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \int \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)}.$$

We now rewrite this ratio as (thus defining $Z_{\mathcal{N}}(\sigma)$ and $Z_{\mathcal{N}}(0)$)

$$\mathbb{E}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\langle\phi,f\rangle}] = e^{\frac{1}{2\beta}\langle f,-\Delta^{-1}f\rangle} \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z_{\mathcal{N}}(\sigma)}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z_{\mathcal{N}}(0)}.$$

The fourth step is an analysis, for each fixed ensemble $\mathcal{N} \in \mathcal{F}$, of the above ratio $\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)}$. Trigonometric inequalities are used here in order to obtain, for each $\mathcal{N}$,

$$\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)} \ge \exp\Big[-D_4 \sum_{\rho\in\mathcal{N}} |z(\beta,\rho,\mathcal{N})| \langle\sigma,\rho\rangle^2 \Big] \times \int \frac{e^{S(\mathcal{N},\phi)}}{Z_{\mathcal{N}}(0)} \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi),$$

where

$$S(\mathcal{N},\phi) := -\sum_{\rho\in\mathcal{N}} \frac{z(\beta,\rho,\mathcal{N})\sin(\langle\phi,\bar\rho\rangle)\sin(\langle\sigma,\rho\rangle)}{1+z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle)}. \tag{2.9}$$

Two crucial observations are made at this stage:

(1) The functional $\phi\mapsto S(\mathcal{N},\phi)$ is odd in ϕ.

(2) The measure $\prod_{\rho\in\mathcal{N}}[1+z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle)]d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)$ is invariant under $\phi\mapsto-\phi$.

Altogether this simplifies the above lower bound tremendously, since by using Jensen, one obtains readily

$$\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)} \geq \exp\Big[-D_4\sum_{\rho\in\mathcal{N}}|z(\beta,\rho,\mathcal{N})|\langle\sigma,\rho\rangle^2\Big].$$

From this lower bound together with the specific construction of the ensembles of charges $\mathcal{N}$, it is then not particularly difficult to conclude the proof with the desired lower bound

$$\mathbb{E}_{\beta,\Lambda,\lambda_\Lambda,c}[e^{\langle\phi,f\rangle}] \geq \exp\Big(\frac{1}{2\beta(1+\varepsilon)}\langle f,-\Delta^{-1}f\rangle\Big).$$

As we will see in Section 4, the effect of shifting the $\mathbb{Z}$ fibers by $\mathbf{a}\in\mathbb{R}^\Lambda$ will translate as follows:

$$\begin{aligned}\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\langle\phi,f\rangle}] &= e^{\frac{1}{2\beta}\langle f,-\Delta^{-1}f\rangle} \\ &\times \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}}\int\prod_{\rho\in\mathcal{N}}[1+z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle+\langle\sigma-\mathbf{a},\rho\rangle)]\,d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}}\int\prod_{\rho\in\mathcal{N}}[1+z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle-\langle\mathbf{a},\rho\rangle)]\,d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)}.\end{aligned}$$

The difficulty for us will be that, generically, $\mathbf{a}$ is much less regular than σ (defined in (2.8)), which thus makes the Dirichlet energy $\langle\nabla(\sigma-\mathbf{a}),\nabla(\sigma-\mathbf{a})\rangle$ typically huge. Therefore, we will not be able anymore to rely on the two symmetries above (in particular the use of Jensen is no longer possible except for very specific choices of $\mathbf{a}$; see the discussion after Definition 1.7). We will come back to this in Section 4.

Remark 2.7. The case of Dirichlet boundary condition has been outlined in Appendix D of [21] and the details of the proof appeared very recently in [45, Appendix]. The proof structure highlighted above for the free boundary condition still holds except that the decomposition into charges needs to be adapted to take into account the presence of a boundary. See [45, Appendix].

We also point out that the nice symmetrization argument used in [45] does not apply to our case (as $\mathbf{a}$ is far from being harmonic) and also because the symmetrized measure in most cases does not provide information on the fluctuations we need.

2.4. *Link with the* **a***-shifted integer-valued GFF*

In this section, we specify the link between our statistical reconstruction problem and the **a**-shifted IV-GFF introduced earlier (in Definition 1.7).

Lemma 2.8. *Let* $\Lambda \subset \mathbb{Z}^2$, $T > 0$ *and* $\mathbf{a} \in \mathbb{R}^\Lambda$ *with* $\mathbf{a}_{|\partial\Lambda} \equiv 0$*. If* ϕ *is a* 0*-boundary GFF (with inverse temperature* $\beta = 1$*) on* Λ*, then its conditional law given* $\phi \pmod{2\pi/T} = \frac{2\pi}{T}\mathbf{a}$ *is given by* $\frac{2\pi}{T}\psi$*, where* $\psi \sim \mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta_T,\Lambda}$ *and the* T*-dependent inverse temperature* β_T *is given by*

$$\beta_T := (2\pi)^2/T^2.$$

Equivalently, for any functional $F : \mathbb{R}^\Lambda \to \mathbb{R}$*,*

$$\mathbb{E}[F(\phi) \mid e^{iT\Phi} = e^{2\pi i \mathbf{a}}] = \mathbb{E}^{\mathbf{a},\mathrm{IV}}_{\beta=\beta_T,\Lambda}\left[F\left(\frac{2\pi}{T}\psi\right)\right].$$

Proof. Recall from Definition 1.7 that

$$\mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta,\Lambda}[d\phi] := \frac{1}{Z}\sum_{\mathbf{m}\in\mathbb{Z}^\Lambda:\, \mathbf{m}_{|\partial\Lambda}\equiv 0} \delta_{\mathbf{m}+\mathbf{a}}(\phi)\exp\left(-\frac{\beta}{2}\langle\nabla(\mathbf{m}+\mathbf{a}), \nabla(\mathbf{m}+\mathbf{a})\rangle\right).$$

Now, by disintegration, for any continuous[9] functional $F : \mathbb{R}^\Lambda \to \mathbb{R}$, one has

$$\begin{aligned}
&\mathbb{E}[F(\phi) \mid e^{iT\Phi} = e^{2\pi i \mathbf{a}}] \\
&\quad = \frac{\sum_{\mathbf{m}\in\mathbb{Z}^\Lambda} \exp(-\frac{1}{2}\langle \frac{2\pi}{T}(\mathbf{a}+\mathbf{m}), (-\Delta)\frac{2\pi}{T}(\mathbf{m}+\mathbf{a})\rangle) F(\frac{2\pi}{T}(m+\mathbf{a}))}{\sum_{\mathbf{m}\in\mathbb{Z}^\Lambda} \exp(-\frac{1}{2}\langle \frac{2\pi}{T}(\mathbf{a}+\mathbf{m}), (-\Delta)\frac{2\pi}{T}(\mathbf{m}+\mathbf{a})\rangle)} \\
&\quad = \frac{\sum_{\mathbf{m}\in\mathbb{Z}^\Lambda} \exp\big(-\frac{(2\pi)^2}{2T^2}\langle(\mathbf{a}+\mathbf{m}), (-\Delta)(\mathbf{m}+\mathbf{a})\rangle\big) F(\frac{2\pi}{T}(m+\mathbf{a}))}{\sum_{\mathbf{m}\in\mathbb{Z}^\Lambda} \exp\big(-\frac{(2\pi)^2}{2T^2}\langle(\mathbf{a}+\mathbf{m}), (-\Delta)(\mathbf{m}+\mathbf{a})\rangle\big)} \\
&\quad = \mathbb{E}^{\mathbf{a},\mathrm{IV}}_{\beta=\beta_T,\Lambda}\left[F\left(\frac{2\pi}{T}\phi\right)\right],
\end{aligned}$$

where we have made the slightly unusual choice $\beta_T := (2\pi)^2\, T^{-2}$ (in order to avoid dealing with $\sqrt{T}$ in most of the introduction). ■

3. Localization regime

In this section, we prove the first part of Theorems 1.1 and 1.2. That is, we show that one can recover a GFF knowing $\exp(iT\phi)$, in fact the recovery function is fairly straightforward:

$$F(\exp(iT\phi))(x) := \mathbb{E}[\phi(x) \mid \exp(iT\phi)].$$

[9] We work with continuous functionals because they characterize the law.

To show that this is the right function, we need to recall (1.10). It says that to prove the first part of Theorems 1.1 and 1.2, it is enough to show that if ϕ is a GFF in Λ_n and f a fixed smooth function in Λ_n, then

$$\mathbb{E}\big[\mathrm{Var}[\langle \phi, f\rangle \mid e^{iT\phi}]\big] = o(n^4). \tag{3.1}$$

Let us note that this approach may not look useful at first glance, as to bound this conditional variance we need to compute the conditional expectation, which is a non-trivial function of $\exp(iT\phi)$. To circumvent this issue, we write the conditional variance as follows:

$$\begin{aligned}\mathrm{Var}[\langle \phi, f\rangle \mid e^{iTh}] &= \mathbb{E}\big[(\langle \phi, f\rangle - \mathbb{E}[\langle \phi, f\rangle \mid e^{iTh}])^2 \mid e^{iT\phi}\big]\\ &= \tfrac{1}{2}\mathbb{E}[(\langle \phi_1, f\rangle - \langle \phi_2, f\rangle)^2 \mid e^{iT\phi}],\end{aligned}$$

where ϕ_1, ϕ_2 are conditionally independent given $e^{iT\phi}$. Let us be more explicit about this law.

Definition 3.1. Let ϕ be a GFF in Λ_n with any given boundary. Let (ϕ_1, ϕ_2) be a pair of GFFs in Λ_n with the same boundary condition such that a.s. $e^{iT\phi_1} = e^{iT\phi_2} = e^{iT\phi}$ and ϕ_1 is conditionally independent of ϕ_2 given $e^{iT\phi}$. In other words,

$$\mathbb{P}[(d\phi_1, d\phi_2) \mid e^{iT\phi}] \propto \prod_{i=1,2}\Big(e^{-\frac{1}{2}\langle \phi_i, \phi_i\rangle_\nabla} \prod_{x\in\Lambda_n\setminus\partial\Lambda_n}\Big(\sum_{k\in\mathbb{Z}} \delta_{2\pi k/T+\phi(x)}(d\phi_i(x))\Big)\Big).$$

To prove (3.1), we use an averaged Peierls argument.

3.1. Large gradients are costly for a GFF

The first stage to implement a Peierls argument is to show that it is costly for a GFF to have many edges with large gradients. To do this we are going to use the Markov property, i.e. Proposition 2.3. In fact, for a given deterministic set $B \subseteq \Lambda_n$, we need to understand what is the law of the norm of ϕ_B.

Lemma 3.2. *In the context of Proposition 2.3 with $B \cap \partial\Lambda_n = \emptyset$,*

(1) *the law of $\|\phi_B\|_\nabla^2$ is that of a χ^2 with $|B|$ degrees of freedom;*

(2) *the law of $\|\phi^B\|_\nabla^2$ is that of a χ^2 with $|\Lambda_n \setminus (\partial\Lambda_n \cup B)|$ degrees of freedom.*

Proof. We start by defining $\mathrm{Harm}(B)$ as the set of functions $\Lambda_n \to \mathbb{R}$ that are harmonic in $\Lambda_n \setminus (B \cup \partial\Lambda_n)$ and take value 0 on $\partial\Lambda_n$. In fact, ϕ_B is the orthogonal projection of ϕ to $\mathrm{Harm}(B)$ under the inner product $\langle \cdot, \cdot\rangle_\nabla$ (see for example [41, Section 2.6]). One can now check that the subspace $\mathrm{Harm}(B)$ has dimension $|B|$, from which (1) follows. As ϕ^B is the orthogonal projection to $\mathrm{Harm}(B)^\perp$, (2) follows by a similar reasoning, as the space of functions with zero boundary condition on $B \cup \partial\Lambda_n$ has dimension $|\Lambda_n \setminus (\partial\Lambda_n \cup B)|$. ∎

We can now use this proposition to obtain the basic input we need for a Peierls argument.

Lemma 3.3. *Let ϕ be a GFF in Λ_n with either zero or free boundary condition. Then there exist constants $\alpha, C, u_0 > 0$ independent of Λ_n such that for any finite set F of edges and all $u > u_0$,*

$$\mathbb{P}[|\phi(x) - \phi(y)| \geq u, \forall xy \in F] \leq Ce^{-\alpha u^2|F|}.$$

Proof. We use the Markov property of the GFF (Proposition 2.3) with the subset of vertices B such that $x \in B$ if there exists $xy \in F$. Let us note that $|B| \leq 2|F|$. We see that

$$\|\phi_B\|_\nabla^2 = \sum_{xy \in E} (\phi_B(y) - \phi_B(x))^2 \geq \sum_{xy \in F} (\phi(y) - \phi(x))^2 \tag{3.2}$$

has the law of a χ^2 with $|B|$ degrees of freedom. Thanks to Proposition 2.3 (1), we know that $\phi_B(y) - \phi_B(x)$ is equal to $\phi(y) - \phi(x)$. Thus, using $|B| \leq 2|F|$,

$$\begin{aligned}\mathbb{P}(|\phi(x) - \phi(y)| \geq u, \forall xy \in F) &\leq \mathbb{P}(\|\phi_B\|_\nabla^2 \geq u^2|F|)\\ &\leq \mathbb{P}(\|\phi_B\|_\nabla^2 \geq u^2|B|/2).\end{aligned}$$

We can now use Lemma 3.2 (1) to continue and see that when u is large enough,

$$\mathbb{P}(|\phi(x) - \phi(y)| \geq u, \forall xy \in F) \leq C\exp(-4\alpha u^2|B|) \leq C\exp(-\alpha u^2|F|), \tag{3.3}$$

where we have used $|F| \leq 4|B|$. ∎

3.2. *The GFFs ϕ_1 and ϕ_2 agree on a dense percolating set*

3.2.1. The 0*-boundary case.* Let ϕ be a 0-boundary GFF in Λ_n and assume we are given an instance of $e^{iT\phi}$. Let us sample two conditionally independent copies ϕ_1, ϕ_2 given $e^{iT\phi}$ as in Definition 3.1. Let us now introduce the following definition.

Definition 3.4. We denote by $I := I(\phi_1, \phi_2)$ the connected component of the random set $\{x \in \Lambda_n : \phi_1(x) = \phi_2(x)\}$ connected to the boundary $\partial\Lambda_n$.

Recall that by definition, ϕ_1, ϕ_2 are GFFs with zero boundary conditions and so $\phi_1 \equiv \phi_2$ on $\partial\Lambda_n$.

Our goal in this subsection is to show via an *annealed* Peierls argument that with high probability when T is small, the random set I is *percolating* inside Λ_n. To study this, for any $x \in \Lambda_n$ *we define* $O(x)$ as the empty set if $x \in I$ and as the connected component containing x of $\Lambda \setminus I$ if $x \notin I$.

Our main observation is that having an edge connecting $O(x)$ to $\Lambda_n \setminus O(x)$ is costly in the sense that it forces either $|\nabla\phi_1(e)|$ or $|\nabla\phi_2(e)|$ to be larger than π/T. Indeed, the values of ϕ_1 and ϕ_2 are fixed modulo $2\pi T$, in other words for any $x \in \Lambda_n$ and $i \in \{1, 2\}$,

$$\phi_i(x) \in \phi(x) + \frac{2\pi}{T}\mathbb{Z}.$$

This way, if ϕ_1, ϕ_2 agree at x but disagree at $y \sim x$, this means that either $|\phi_1(x) - \phi_1(y)| > \pi/T$ or $|\phi_2(x) - \phi_x(y)| > \pi/T$. We then have the following proposition.

Proposition 3.5. *Using the definitions introduced above, for all T small enough there exist $\varpi(T) > 0$ and $C > 0$ such that*

$$\mathbb{P}(\mathrm{diam}(O(x)) \geq L) \leq C \exp(-\varpi(T)L).$$

Proof. Let us note that if $\mathrm{diam}(O(x)) \geq L$ there is a subset η of edges of length at least L such that its dual is a connected path surrounding x and for every $e \in \eta$ either $|\nabla\phi_1(e)| \geq \pi/T$ or $|\nabla\phi_2(e)| \geq \pi/T$. This implies that

$$\mathbb{P}(\mathrm{diam}(O(x)) \geq L) \leq \sum_{\substack{|\eta|\geq L \\ \eta \text{ surrounds } x}} \mathbb{P}\big(|\nabla\phi_1(e)| \geq \pi/T \text{ or } |\nabla\phi_2(e)| \geq \pi/T, \forall e \in \eta\big). \tag{3.4}$$

Let us fix η and suppose that for all $e \in \eta$, either $|\nabla\phi_1(e)| \geq \pi/T$ or $|\nabla\phi_2(e)| \geq \pi/T$. This implies that there exist $F \subseteq \eta$ and $i \in \{1,2\}$ such that for all $e \in F$ we have $|\nabla\phi_i(e)| \geq \pi/T$ and $|F| \geq \lfloor|\eta|/2\rfloor$. This implies that

$$\begin{aligned}\mathbb{P}\big(|\nabla\phi_1(e)| \geq \pi/T \text{ or } |\nabla\phi_2(e)| \geq \pi/T, \forall e \in \eta\big)& \\ &\leq 2\sum_{j=\lfloor|\eta|/2\rfloor}^{|\eta|} \sum_{\substack{F\subseteq\eta \\ |F|=j}} \mathbb{P}(|\nabla\phi(e)| \geq \pi/T, \forall e \in F) \\ &\leq 2^{|\eta|+1} \sum_{j=\lfloor|\eta|/2\rfloor}^{|\eta|} \exp(-2\tilde{\alpha} j/T^2),\end{aligned}$$

where we have used Lemma 3.3 and the fact both ϕ_1 and ϕ_2 have the law of a GFF in Λ. Additionally, $\tilde{\alpha} := \alpha\pi^2/2$. Thus, (3.4) is less than or equal to

$$\begin{aligned}C\sum_{k\geq L} \sum_{\substack{|\eta|=k \\ \eta \text{ surrounds } x}} 2^k \exp\left(-\frac{\tilde{\alpha}}{T^2}k\right) &\leq C\sum_{k\geq L} \exp(-k(\tilde{\alpha}T^{-2} - \log 2 - \log 3)) \\ &\leq \tilde{C}\exp(-L(\tilde{\alpha}T^{-2} - \log 2 - \log 3)),\end{aligned}$$

where we have used the fact that the number of η's such that $|\eta| = k$ and η surrounds x is less than $C \cdot 3^k$ and that

$$\tilde{\alpha}T^{-2} - \log 6 > 0.$$

■

3.2.2. The free-boundary case. In order to analyze the free-boundary case, we need to modify the above definitions significantly. We assume the free-boundary GFF is rooted at some vertex $x_0 \in \Lambda_n$. As in the Dirichlet case, (ϕ_1, ϕ_2) will denote two conditionally independent copies of the GFF given $e^{iT\phi}$.

The main difference with the Dirichlet case is that when T is small, it is no longer true that with high probability, ϕ_1 and ϕ_2 will agree on a large percolating set. Instead, we will find a large set, which we will call I again, together with a random integer $m_I \in \mathbb{Z}$ such that

$$\phi_1(x) = \phi_2(x) + m_I\frac{2\pi}{T}, \quad \forall i \in I.$$

Now, for each $m \in \mathbb{Z}$, let

$$\hat{I}_m := \text{largest connected component of } \left\{ x \in \Lambda_n : \phi_1(x) = \phi_2(x) + m\frac{2\pi}{T} \right\}.$$

If there are two of the same size, we choose one in a deterministic way. From these subsets $\hat{I}_m$, we define the set I and the connected components $\{O(x)\}_{x\in\Lambda_n}$ as follows:

- If there is a unique $m_0 \in \mathbb{Z}$ such that $\hat{I}_{m_0}$ has (graph) diameter larger than $n/2$, then we define
$$I := \hat{I}_{m_0},$$
and for any $x \in \Lambda_n$, we define $O(x)$ to be empty if $x \in I$ and to be the connected component of x in $\Lambda_n \setminus I$ otherwise.
- If on the other hand, one can find two integers m_1, m_2 such that both $\hat{I}_{m_1}$ and $\hat{I}_{m_2}$ have diameter greater than $n/2$, then we define
$$I := \emptyset \quad \text{and} \quad O(x) := \Lambda_n, \quad \forall x \in \Lambda_n.$$
- Furthermore, if there is no $m \in \mathbb{Z}$ such that $\hat{I}_m$ has diameter larger than $n/2$, we take $I = \emptyset$. In this case $O(x)$ is the connected component of I_{m_x} containing x where $m_x = T(\phi_1(x) - \phi_2(x))/2\pi$.

We can now state the analogue of Proposition 3.5 for the free boundary condition.

Proposition 3.6. *Let ϕ_1, ϕ_2 two free-boundary GFF such that* $\exp(iT\phi_1) = \exp(iT\phi_2)$ *and conditionally independent given* $\exp(iT\phi_1)$. *Then using the above definitions* (*for the free boundary condition*), *for all T small enough there exist $\varpi(T) > 0$ and $C > 0$ such that for all $x \in \Lambda_n$,*

$$\mathbb{P}(\operatorname{diam}(O(x)) \geq L) \leq C \exp(-\varpi(T)L). \tag{3.5}$$

Proof. The proof follows the same lines as in the Dirichlet case, as Lemma 3.3 does not care about the boundary conditions. The only difference is that we need to deal with the dichotomy entering into the definition of the set I (which does not exist for the Dirichlet case). For this, note that in order to have two sets $\hat{I}_{m_1}, \hat{I}_{m_2}$ with $m_1 \neq m_2$ and both of diameter $\geq n/2$, there must exist at least one path η in the dual graph $(\mathbb{Z}^2)^*$ which has diameter greater than $n/2$ and which satisfies the constraint that for any $e \in \eta$, either $|\nabla\phi_1(e)| \geq \pi/T$ or $|\nabla\phi_2(e)| \geq \pi/T$. By Lemma 3.3 and the same argument as in the Dirichlet case, this can only happen with probability less than $O(n)\exp(-\frac{\tilde{\alpha}}{10T^2}n)$. Note that the same argument implies that there is at most one connected component of the set $\{x \in \Lambda_n : \phi_1(x) = \phi_2(x) + m\frac{2\pi}{T}\}$ with diameter at least $n/2$.

Note that after defining I, the argument of Proposition 3.5 implies that for any point x the set $O(x)$ can only be huge with exponentially decaying probability in the diameter n. Now, note that (3.5) is obviously true as long as $L > 2n$, and thus for $L \leq 2n$ we have

$$\begin{aligned}\mathbb{P}(\operatorname{diam}(O(x)) \geq L) &\leq \mathbb{P}(\operatorname{diam}(O(x)) \geq L, I \neq \emptyset) + \mathbb{P}(I = \emptyset) \\ &\leq \frac{C}{2}e^{-\varpi(T)L} + \frac{C}{2}e^{-\frac{\varpi(T)}{2}n} \leq Ce^{-\varpi(T)L}.\end{aligned}$$

∎

3.3. The conditional variance is small for 0*-boundary GFF*

We will now prove (3.1) for a 0-boundary GFF. Let us now study the law of (ϕ_1, ϕ_2) conditionally on I and the values of ϕ_1 on I. We fix $e^{iT\phi}$, I, and the values of ϕ_1 on I and take (φ_1, φ_2) to be a possible value of (ϕ_1, ϕ_2) that satisfy the conditioning. Note that to check whether (φ_1, φ_2) is a possible realization, one just needs to check that $(\varphi_1)_{|I} = (\varphi_2)_{|I} = (\phi_1)_{|I}$, and that for any connected component O of $\Lambda_n \setminus I$, the pair (φ_1, φ_2) restricted to O locally satisfies the conditions, i.e.

$$\varphi_1(x) = \varphi_2(x) = \phi(x) \ (\mathrm{mod}\ 2\pi/T) \quad \text{for all } x \in O.$$

Furthermore, if we define $\bar{O}$ to be the graph induced by all the edges in Λ_n that have at least one vertex in O, we have

$$\mathbb{P}\big((\phi_1, \phi_2) = (\varphi_1, \varphi_2) \mid e^{iT\phi}, I, (\phi_1)_{|I}\big) \propto \prod_O e^{-\frac{1}{2}(\langle(\varphi_1)_{|\bar{O}}, (\varphi_1)_{|\bar{O}}\rangle_\nabla + \langle(\varphi_2)_{|\bar{O}}, (\varphi_2)_{|\bar{O}}\rangle_\nabla)}. \tag{3.6}$$

As a consequence of (3.6), under this conditioning if $O \neq O'$ then the law of (ϕ_1, ϕ_2) restricted to O is independent of the law of O'. Thus, $\mathbb{E}[\langle \phi_1 - \phi_2, f\rangle^2]$ is equal to

$$\begin{aligned}
&\sum_{x,y\in\Lambda_n} f(x) f(y) \mathbb{E}[(\phi_1 - \phi_2)(x)(\phi_1 - \phi_2)(y)\mathbf{1}_{O(x)=O(y)}] \\
&\leq \sum_{x,y\in\Lambda_n} |f(x)|\,|f(y)| \mathbb{E}[(\phi_1 - \phi_2)^2(x)(\phi_1 - \phi_2)^2(y)]^{1/2} \mathbb{P}(\mathbf{1}_{O(x)=O(y)})^{1/2} \\
&\leq \sum_{x,y\in\Lambda_n} |f(x)|\,|f(y)| \big(\mathbb{E}[(\phi_1 - \phi_2)^4(x)] + \mathbb{E}[(\phi_1 - \phi_2)^4(y)]\big)^{1/2} \mathbb{P}(\mathbf{1}_{O(x)=O(y)})^{1/2}.
\end{aligned} \tag{3.7}$$

We can now just bound

$$\mathbb{E}[(\phi_1 - \phi_2)^4(x)] \leq 16\mathbb{E}[\phi_1^4(x)] = 48 G_n^2(x, x).$$

Note that on the event $O(x) = O(y)$ the diameter of $O(x)$ is at least $d_{\Lambda_n}(x, y)$. Thus, there exists an absolute constant C and $\varpi(T) > 0$ such that

$$\mathbb{P}[O(x) = O(y)] \leq C \exp(-\varpi(T)\|x - y\|).$$

From the fact that $\exp(-\varpi(T)\|x - y\|)$ decreases exponentially as $\|x - y\|$ goes to infinity, we find that

$$\mathbb{E}[\langle \phi_1 - \phi_2, f\rangle^2] \leq C\|f\|_\infty^2 \sup_x G_n(x, x) n^2 \leq C\|f\|_\infty^2 n^2 \log n, \tag{3.8}$$

which proves (3.1) and gives in fact a more quantitative rate of convergence.

3.4. The conditional variance is small enough for free-boundary Gaussian free field

We will now prove (3.1) for a free-boundary GFF. The proof is very similar to that for the zero boundary condition so we are going to sketch the proof only, highlighting the differences with the Dirichlet boundary case.

Let (ϕ_1, ϕ_2) be a pair of GFFs with zero boundary condition in $\{x_0\}$ coupled as in Definition 3.1. By Proposition 3.6, there exist a (random) set I and a (random) integer m_I such that for all $y \in I$, $\Phi_1(y) = \Phi_2(y) + 2\pi m_I/T$, and furthermore for any x if we define $O(x)$ as the connected component of $\Lambda_n \setminus I$ containing x, we have the same conditional independence property of islands in this setting,

$$\mathbb{P}(\mathrm{diam}(O(x)) \geq L) \leq \exp(-\varpi(T)L). \tag{3.9}$$

Let us note that the same argument as in Section 3.3 together with the estimate of Proposition 3.7 implies that for any smooth function $f : [-1,1]^2 \to \mathbb{R}$ we have

$$\mathbb{E}[\langle \phi_1 - \phi_2 - 2\pi m_I/T, f\rangle^2] \leq C\|f\|_\infty^2 n^2 \log n.$$

Now note that for any continuous function f with $\int f = 0$, we have

$$\hat{f} := |\Lambda_n|^{-1}\langle f, 1\rangle = \|f\|_\infty o(1).$$

Thus, defining $\tilde{f}$ as $f - \hat{f}$ and noting that $\langle m_I, \tilde{f}\rangle = 0$ we have

$$\begin{aligned}\mathbb{E}[\langle \phi_1 - \phi_2, f\rangle^2] &\leq \mathbb{E}\left[\left\langle \phi_1 - \phi_2 + \frac{2\pi m_I}{T}, \tilde{f}\right\rangle^2\right] + \left(\frac{\langle f, 1\rangle}{|\Lambda_n|}\right)^2 \mathbb{E}[\langle \phi_1 - \phi_2, 1\rangle^2] \\ &\leq C\|f\|_\infty^2 n^2 \log n + C\|f\|_\infty o(1) n^4,\end{aligned} \tag{3.10}$$

which finishes the proof.

3.5. The conditional variance at a given point is bounded

In this subsection, we are going to improve the result of (3.8) for $f = \mathbf{1}_x$.

Proposition 3.7. *Let ϕ_1 and ϕ_2 be two 0-boundary (or free-boundary) GFFs coupled as in Definition 3.1. Then for all T small enough there exist $K, \tilde{K} > 0$ such that for all $n \in \mathbb{N}$ and all $x, y \in \Lambda_n$,*

$$\mathbb{E}[(\phi_1 - \phi_2)^2(x)] \leq K, \tag{3.11}$$

$$\mathbb{E}[(\phi_1 - \phi_2)(x)(\phi_1 - \phi_2)(y)\mathbf{1}_{O(x)=O(y)}] \leq K e^{-\tilde{K} d_{\Lambda_n}(x,y)}. \tag{3.12}$$

Proof. We start by proving (3.11) for ϕ a 0-boundary GFF as in Section 3.3. Let γ be a horizontal edge path connecting $\partial[-1,1]^2$ to x in Λ_n. We say that the edge e belongs to $\gamma \cap O(x)$ if $e \in \gamma$ and $e \cap O(x) \neq \emptyset$. We then have

$$(\phi_1 - \phi_2)(x) = \sum_{e \in \gamma \cap O(x)} \nabla(\phi_1 - \phi_2)(e). \tag{3.13}$$

Thus,

$$(\phi_1 - \phi_2)^2(x) \leq \sum_{e, e' \in E} \nabla(\phi_1 - \phi_2)(e)\nabla(\phi_1 - \phi_2)(e')\mathbf{1}_{e,e' \in O(x)\cap\gamma}.$$

We can now upper bound $\mathbb{E}[(\phi_1-\phi_2)^2(x)]$ by

$$\sum_{e,e'\in E}\mathbb{E}[\nabla(\phi_1-\phi_2)(e)\nabla(\phi_1-\phi_2)(e')\mathbf{1}_{e,e'\in O(x)\cap\gamma}] \\ \leq K\sup_e \mathbb{E}[(\nabla(\phi_1-\phi_2)(e))^4]^{1/2}\sum_{e,e'\in\gamma}\mathbb{P}[e,e'\in O(x)]^{1/2}.$$

We conclude (3.11) by first noting that $\mathrm{Var}(\nabla(\phi_1-\phi_2)(e))\leq 4$ thanks to Proposition 2.5, and by the fact that

$$\mathbb{P}[e,e'\in O(x)]\leq \exp\big(-\omega(T)\max\{d_{\Lambda_n}(e,x),d_{\Lambda_n}(e',x)\}\big).$$

We now prove (3.11) in the free-boundary case with 0 value at z. In this case, one needs to take an edge path γ going from x to z that only makes one turn (so that $\sum_{e,e'\in\gamma}\mathbb{P}[e,e'\in O(x)]^{1/2}$ is bounded). The same argument as before shows that

$$\mathbb{E}[(\phi_1-\phi_2)^2(x)]\leq C+2\mathbb{E}[m_I^2],$$

where $m_I:=0$ if $I=\emptyset$, while if $I\neq\emptyset$ then $m_I:=(\phi_1-\phi_2)(x')/(2\pi T)$ at a point $x'\in I$ (recall that this value is a constant in I).

To bound the variance of n, we note that we can take an edge path γ starting from z and such that it always hits I when $I\neq\emptyset$, and it only makes four turns (again so that $\sum_{e,e'\in\gamma}\mathbb{P}[e,e'\in O(z)]^{1/2}$ is bounded). By the same argument as before, one sees that

$$\mathbb{E}[m_I^2]\leq C.$$

We now prove (3.12). Note that this directly follows from

$$\mathbb{E}[(\phi_1-\phi_2)^2(y)\mathbf{1}_{O(x)=O(y)}]\leq K\exp\Big(-\frac{\varpi(T)}{2}d_{\Lambda_n}(x,y)\Big). \tag{3.14}$$

This can be done exactly as before by choosing an appropriate path γ. ■

Remark 3.8. Proposition 3.7 hides an important fact. There is a difference regarding the behavior of the (conditional) correlation function between the two different types of boundary condition we study.

To be more precise, in the case of a 0-boundary GFF, one has

$$\mathbb{E}[(\phi_1-\phi_2)(x)(\phi_1-\phi_2)(y)\mathbf{1}_{O(x)=O(y)}]=\mathbb{E}[(\phi_1-\phi_2)(x)(\phi_1-\phi_2)(y)],$$

which proves (1.4). However, in the case of free boundary conditions, one has

$$\mathbb{E}[(\phi_1-\phi_2)(x)(\phi_1-\phi_2)(y)] \\ =\mathbb{E}[(\phi_1-\phi_2)(x)(\phi_1-\phi_2)(y)\mathbf{1}_{O(x)=O(y)}]+\mathbb{E}[m_I^2\mathbf{1}_{O(x)\neq O(y)}].$$

As we do not expect that $\mathbb{E}[m_I^2]$ goes to 0 as $n\to\infty$, one can see that the (conditional) correlations do not decrease to 0 as $d_{\Lambda_n}(x, y) \to \infty$. However, it is also interesting to note that these correlations do decay exponentially to 0 if we condition not only on $e^{iT\phi}$, but also on the value of m_I. In fact, this seems to be closely related to the large-scale correlations which arise for Coulomb gases in $2d$ with free boundary conditions; see for example [19].

Note that Proposition 3.7 improves the result of (3.1) and (3.8).

Proposition 3.9. *For T small enough one has*

$$\mathbb{E}\left[\operatorname{Var}\left[\frac{1}{n^2}\langle \phi, f\rangle \;\middle|\; e^{iTh}\right]\right] \le K\frac{\|f\|_\infty^2}{n^2}. \tag{3.15}$$

Proof. The proof follows the same lines as the proof of (3.8). The main difference is that we now use (3.14). ∎

4. Delocalization regime

We start by proving the roughening transition for generalized integer-valued fields (Theorem 1.8) and then, as a corollary, extract the delocalization regime for our statistical reconstruction problem.

4.1. Proof of Theorem 1.8

In this proof, we focus on the case of the free boundary condition (as in [21, 27]); however, following [21, Appendix D] or the recent [45] (see Remark 2.7), our proof works in exactly the same way in the Dirichlet case.

Recall from Section 2.3 and from [27, (1.13)] the following series expansion for the Laplace transform of discrete GFFs with periodic weights $\lambda_\Lambda = (\lambda_j)_{j\in\Lambda}$ (we assume the same hypothesis as in [27, Theorem 1.6]):

$$\begin{aligned}
&\mathbb{E}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\langle \phi, f\rangle}] \\
&\qquad = e^{\frac{1}{2\beta}\langle f, -\Delta^{-1} f\rangle} \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \mathbb{E}^{\mathrm{GFF}}_{\beta,\Lambda}[\prod_{\rho\in\mathcal{N}}[1 + z(\beta,\rho,\mathcal{N})\cos(\langle \phi, \bar\rho\rangle + \langle \sigma, \rho\rangle)]]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \int \prod_{\rho\in\mathcal{N}}[1 + z(\beta,\rho,\mathcal{N})\cos(\langle \phi, \bar\rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)} \\
&\qquad = e^{\frac{1}{2\beta}\langle f, -\Delta^{-1} f\rangle} \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z_{\mathcal{N}}(\sigma)}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z_{\mathcal{N}}(0)}.
\end{aligned}$$

We denote by $\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}$ or $\mu^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}$ the discrete GFF whose periodic weights are shifted by $\mathbf{a}$:

$$d\mu^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}(\phi) := \frac{1}{Z^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}} \prod_{j\in\Lambda} \lambda_j(\phi_j - a_j)\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi). \tag{4.1}$$

The shift by $\mathbf{a}$ easily translates into the following expression for the Laplace transform under $\mu^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}$:

$$\begin{aligned}
&\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\langle \phi, f\rangle}] \\
&\quad = e^{\frac{1}{2\beta}\langle f, -\Delta^{-1} f\rangle} \\
&\qquad \times \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \int \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle + \langle \sigma - \mathbf{a}, \rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \int \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)} \\
&\quad = e^{\frac{1}{2\beta}\langle f, -\Delta^{-1} f\rangle} \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z_{\mathcal{N}}(\sigma - \mathbf{a})}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z_{\mathcal{N}}(-\mathbf{a})}.
\end{aligned} \tag{4.2}$$

As the shift $\mathbf{a}$ is fixed once and for all in this proof, let us introduce the *shifted partition functions* $\{Z^{\mathbf{a}}_{\mathcal{N}}(\sigma)\}_{\mathcal{N},\sigma}$. For any $\sigma : \Lambda \to \mathbb{R}$ and any collection $\mathcal{N} \in \mathcal{F}$ of charges,

$$Z^{\mathbf{a}}_{\mathcal{N}}(\sigma) := Z_{\mathcal{N}}(\sigma - \mathbf{a}) = \int \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle + \langle \sigma - \mathbf{a}, \rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi). \tag{4.3}$$

Following the same analysis as in [27, Section 3] (or in [21, Section 5]), we obtain the following lower bound on the ratio of partition functions:

$$\begin{aligned}
\frac{Z^{\mathbf{a}}_{\mathcal{N}}(\sigma)}{Z^{\mathbf{a}}_{\mathcal{N}}(0)} \geq \exp\Big[-D_4 \sum_{\rho\in\mathcal{N}} |z(\beta,\rho,\mathcal{N})| \langle \sigma, \rho\rangle^2\Big] \\
\times \int \frac{e^{S(\mathcal{N},\mathbf{a},\phi)}}{Z^{\mathbf{a}}_{\mathcal{N}}(0)} \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi),
\end{aligned} \tag{4.4}$$

where

$$S(\mathcal{N},\mathbf{a},\phi) := -\sum_{\rho\in\mathcal{N}} \frac{z(\beta,\rho,\mathcal{N}) \sin(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle) \sin(\langle \sigma, \rho\rangle)}{1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle)}. \tag{4.5}$$

As mentioned in Section 2.3, one major observation in [21] is that $S(\mathcal{N},\phi) := S(\mathcal{N},\mathbf{a}\equiv 0,\phi) = -S(\mathcal{N},-\phi)$. Indeed, this property together with the fact that the probability measure

$$d\mathbb{P}_{\mathcal{N}}(\phi) := \frac{1}{Z_{\mathcal{N}}(0)} \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle)\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi)$$

is invariant under $\phi \mapsto -\phi$ avoids controlling terms such as $\exp(S(\mathcal{N},\phi))$ using Jensen:

$$\begin{aligned}
\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)} &\geq \exp\Big[-D_4 \sum_{\rho\in\mathcal{N}} |z(\beta,\rho,\mathcal{N})| \langle \sigma, \rho\rangle^2\Big] \times \int e^{S(\mathcal{N},\phi)} d\mathbb{P}_{\mathcal{N}}(\phi) \\
&\geq \exp\Big[-D_4 \sum_{\rho\in\mathcal{N}} |z(\beta,\rho,\mathcal{N})| \langle \sigma, \rho\rangle^2\Big] \times \exp\Big(\int S(\mathcal{N},\phi)\, d\mathbb{P}_{\mathcal{N}}(\phi)\Big) \\
&= \exp\Big[-D_4 \sum_{\rho\in\mathcal{N}} |z(\beta,\rho,\mathcal{N})| \langle \sigma, \rho\rangle^2\Big].
\end{aligned}$$

Claim 3.2 in [27] then shows that when β is sufficiently small,

$$\frac{Z_{\mathcal{N}}(\sigma)}{Z_{\mathcal{N}}(0)} \geq \exp\Big(-\frac{\varepsilon\beta}{2(1+\varepsilon)}\sum_{j\sim l}(\sigma_j-\sigma_l)^2\Big)$$
$$= \exp\Big(-\frac{\varepsilon}{2(1+\varepsilon)\beta}\langle f, -\Delta^{-1} f\rangle\Big), \tag{4.6}$$

which ended the proof in [21, 27].

In our present setting, the functional $\phi \mapsto S(\mathcal{N}, \mathbf{a}, \phi)$ introduced in (4.5) is no longer an odd functional of ϕ. Furthermore, the lower bound (4.4) suggests introducing the **a**-reweighted probability measure

$$d\mathbb{P}^{\mathbf{a}}_{\mathcal{N}}(\phi) := \frac{1}{Z^{\mathbf{a}}_{\mathcal{N}}(0)} \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle - \langle\mathbf{a},\rho\rangle)\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi),$$

which is no longer invariant under $\phi \mapsto -\phi$. This lack of symmetry does not allow us to rely on Jensen and we are left with analyzing the quantity

$$\int e^{S(\mathcal{N},\mathbf{a},\phi)}\, d\mathbb{P}^{\mathbf{a}}_{\mathcal{N}}(\phi).$$

We will not succeed in controlling the full Laplace transform but will instead extract bounds on the first and second moments from the series expansion near $\alpha \sim 0$ of the Laplace transform $\alpha \mapsto \mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\alpha\langle\phi,f\rangle}]$.

For any $\alpha \in \mathbb{R}$ (which will be colored for clarity), we have (recall (4.2), (4.4) and (4.6)) the lower bound

$$\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[e^{\alpha\langle\phi,f\rangle}] = e^{\frac{\alpha^2}{2\beta}\langle f,-\Delta^{-1}f\rangle}\frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(\alpha\sigma)}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)}$$
$$\geq e^{\frac{\alpha^2}{2\beta}\langle f,-\Delta^{-1}f\rangle}$$
$$\times \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) e^{-\frac{\varepsilon\alpha^2}{2(1+\varepsilon)\beta}\langle f,-\Delta^{-1}f\rangle} \int e^{S_\alpha(\mathcal{N},\mathbf{a},\phi)}\, d\mathbb{P}^{\mathbf{a}}_{\mathcal{N}}(\phi)}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)}, \tag{4.7}$$

where now

$$S_\alpha(\mathcal{N},\mathbf{a},\phi) = -\sum_{\rho\in\mathcal{N}} \frac{z(\beta,\rho,\mathcal{N})\sin(\langle\phi,\bar\rho\rangle - \langle\mathbf{a},\rho\rangle)\sin(\alpha\langle\sigma,\rho\rangle)}{1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle - \langle\mathbf{a},\rho\rangle)}$$
$$= -\alpha\sum_{\rho\in\mathcal{N}} \frac{z(\beta,\rho,\mathcal{N})\sin(\langle\phi,\bar\rho\rangle - \langle\mathbf{a},\rho\rangle)\langle\sigma,\rho\rangle}{1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle - \langle\mathbf{a},\rho\rangle)} + O(\alpha^3). \tag{4.8}$$

This Taylor expansion holds first because we are in the regime where β can be chosen small enough so that the denominators are uniformly $\geq 1/2$ (see [21, 27]), and second because our parameters Λ, β etc. are fixed as $\alpha \to 0$.

First order analysis. At first order in α, we find combining (4.7) and (4.8) that for any $f : \Lambda \to \mathbb{R}$ and as $\alpha \to 0$,

$$\begin{aligned}
&1 + \alpha \mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle \phi, f\rangle] + O(\alpha^2) \\
&\quad \geq (1 + O(\alpha^2)) \frac{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \int [1 + S_\alpha(\mathcal{N}, \mathbf{a}, \phi) + O(\alpha^2)]\, d\mathbb{P}^{\mathbf{a}}_{\mathcal{N}}(\phi)}{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} \\
&\quad = 1 - \alpha \frac{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}\left[\sum_{\rho \in \mathcal{N}} \frac{z(\beta,\rho,\mathcal{N}) \sin(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle) \langle \sigma, \rho\rangle}{1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle)}\right]}{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} + O(\alpha^2).
\end{aligned}$$

In particular, identifying order 1 terms (and recalling that $\sigma := \frac{1}{\beta}(-\Delta)^{-1} f$, see (2.8)), we thus have, for any $f : \Lambda \to \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle \phi, f\rangle] \\
&\quad \geq - \frac{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}\left[\sum_{\rho \in \mathcal{N}} \frac{z(\beta,\rho,\mathcal{N}) \sin(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle) \langle \frac{1}{\beta}(-\Delta)^{-1} f, \rho\rangle}{1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle)}\right]}{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)}.
\end{aligned}$$

The key observation at this stage is that for each collection $\mathcal{N}$ of charges, the functional

$$f \mapsto \hat S(\mathcal{N}, \mathbf{a}, \phi, f) := - \sum_{\rho \in \mathcal{N}} \frac{z(\beta, \rho, \mathcal{N}) \sin(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle) \langle \frac{1}{\beta}(-\Delta)^{-1} f, \rho\rangle}{1 + z(\beta, \rho, \mathcal{N}) \cos(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle)}$$

is *linear* in f. Obviously, the functional $f \mapsto \mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle \phi, f\rangle]$ is linear as well. Now by using this linearity and plugging $-f$ into the above inequality, we obtain a rather surprising exact expression for the mean value of $\langle \phi, f\rangle$ under the measure $\mu^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}$. We state this exact identity as a proposition below and we call it *modular invariance identity* for reasons which will be explained in Appendix A.

Proposition 4.1 (Modular invariance identity). *For any function f and any weights $\lambda_\Lambda = (\lambda_i)_{i\in\Lambda}$ satisfying the same hypothesis as in* [21, (5.35)] (*or equivalently in* [27, (1.9)]), *we have*

$$\begin{aligned}
&\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle \phi, f\rangle] \\
&\quad = - \frac{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}\left[\sum_{\rho \in \mathcal{N}} \frac{z(\beta,\rho,\mathcal{N}) \sin(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle) \langle \frac{1}{\beta}(-\Delta)^{-1} f, \rho\rangle}{1 + z(\beta,\rho,\mathcal{N}) \cos(\langle \phi, \bar\rho\rangle - \langle \mathbf{a}, \rho\rangle)}\right]}{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} \\
&\quad = \frac{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)\, \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}[\hat S(\mathcal{N}, \mathbf{a}, \phi, f)]}{\sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)}. \qquad (4.9)
\end{aligned}$$

Remark 4.2. This exact identity, as we shall see below, is a key step in our proof. Because it is so central and since it does not look like anything familiar, we added Appendix A to give a longer but more natural second derivation of this identity. It should not come as a surprise that our second derivation is longer, since above one relies in fact on several key parts of the proof of Fröhlich–Spencer [21]. Appendix A gives a complementary interpretation/explanation of the origin of such an identity. In particular, in Appendix A, we shall view the shift vector $\mathbf{a} = \{a_x\}_{x\in\Lambda}$ as an exterior magnetic field and we will also explain why we call this identity "modular invariance" due to a relationship to the functional equation for Riemann theta functions.

Second order analysis. The above identity for the first moment will be instrumental in bounding from below the desired second moment as we shall now see.

Again by combining (4.7) and (4.8), we find that

$$
\begin{aligned}
&1 + \alpha \mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle] + \tfrac{1}{2}\alpha^2 \mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle^2] + O(\alpha^3) \\
&\quad \geq e^{\frac{\alpha^2 \langle f, -\Delta^{-1} f\rangle}{2(1+\varepsilon)\beta}} \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}[e^{S_\alpha(\mathcal{N},\mathbf{a},\phi)}]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} \\
&\quad \geq e^{\frac{\alpha^2 \langle f, -\Delta^{-1} f\rangle}{2(1+\varepsilon)\beta}} \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}\big[1 + S_\alpha(\mathcal{N},\mathbf{a},\phi) + \tfrac{1}{2}[S_\alpha(\mathcal{N},\mathbf{a},\phi)]^2 + O(\alpha^3)\big]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} \\
&\quad \geq e^{\frac{\alpha^2 \langle f, -\Delta^{-1} f\rangle}{2(1+\varepsilon)\beta}} \\
&\qquad \times \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}\big[1 + \alpha \hat{S}(\mathcal{N},\mathbf{a},\phi,f) + \tfrac{\alpha^2}{2}[\hat{S}(\mathcal{N},\mathbf{a},\phi,f)]^2 + O(\alpha^3)\big]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} \\
&\quad = 1 + \alpha \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}[\hat{S}(\mathcal{N},\mathbf{a},\phi,f)]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} \\
&\qquad + \frac{\alpha^2}{2}\left[\frac{1}{(1+\varepsilon)\beta}\langle f, -\Delta^{-1} f\rangle + \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}\big[[\hat{S}(\mathcal{N},\mathbf{a},\phi,f)]^2\big]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)}\right] + O(\alpha^3).
\end{aligned}
$$

The first order terms are equal by Proposition 4.1 and from the second order terms, we extract the following lower bound:

$$
\begin{aligned}
&\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle^2] \\
&\qquad \geq \frac{1}{(1+\varepsilon)\beta}\langle f, -\Delta^{-1} f\rangle + \frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}\big[[\hat{S}(\mathcal{N},\mathbf{a},\phi,f)]^2\big]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)} \\
&\qquad \geq \frac{1}{(1+\varepsilon)\beta}\langle f, -\Delta^{-1} f\rangle + \left(\frac{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)\, \mathbb{E}^{\mathbf{a}}_{\mathcal{N}}[\hat{S}(\mathcal{N},\mathbf{a},\phi,f)]}{\sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)}\right)^2 \\
&\qquad = \frac{1}{(1+\varepsilon)\beta}\langle f, -\Delta^{-1} f\rangle + \mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle]^2
\end{aligned}
\tag{4.10}
$$

by first applying the Cauchy–Schwarz inequality to a suitable probability measure on the coupling $(\mathcal{N}, \phi)$, and then using Proposition 4.1 for the last equality, i.e. the modular invariance identity (4.9). This ends our proof. ∎

4.2. *Non-recovery phase* ($T > T^+_{\text{rec}}$)

In this subsection, we handle the non-recovery phases of Theorems 1.1 and 1.2.

As in Definition 3.1, let (ϕ_1, ϕ_2) be two conditionally independent instances of ϕ given $\phi \pmod{2\pi/T} = (2\pi/T)\mathbf{a}$. By Lemma 2.8, the law of (ϕ_1, ϕ_2) is given by $(2\pi/T)(\psi_1, \psi_2)$ where ψ_1, ψ_2 are independently sampled according to $\mathbb{P}^{\mathbf{a},\mathrm{IV}}_{\beta_T=(2\pi)^2/T^2}$.

Thanks to this and Theorem 1.8 with $C = 1/(1+\varepsilon)$, for any continuous function $f : [-1,1]^2 \to \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[\langle \phi_1 - \phi_2, f\rangle^2] &= \left(\frac{2\pi}{T}\right)^2 \mathbb{E}\big[\mathbb{E}^{\mathrm{IV},\mathbf{a}}_{(2\pi)^2/T^2,\Lambda}[\langle \psi_1 - \psi_2, f\rangle^2]\big] \\ &\geq \frac{2}{1+\varepsilon}\langle f, (-\Delta)^{-1} f\rangle. \end{aligned} \tag{4.11}$$

Furthermore, since both ϕ_1 and ϕ_2 are GFFs, we have

$$\mathbb{E}[\langle \phi_1 - \phi_2, f\rangle^4] \leq 12\mathbb{E}[\langle \phi_1, f\rangle^2]^2 = 12\langle f, (-\Delta)^{-1} f\rangle^2.$$

Therefore, using the Paley–Zygmund inequality we obtain

$$\mathbb{P}\big(\langle \phi_1 - \phi_2, f\rangle^2 \geq \langle f, (-\Delta)^{-1} f\rangle\big) \geq \frac{1}{24(1+\varepsilon)} > 2^{-5}.$$

Now, we use the fact that for any deterministic function F depending only on $\exp(iT\phi)$, we have $F(\exp(iT\phi_1)) = F(\exp(iT\phi_2))$. Using this we can compute

$$\begin{aligned} 2^{-5} &\leq \mathbb{P}\big(\langle \phi_1 - \phi_2, f\rangle^2 \geq \langle f, (-\Delta)^{-1} f\rangle\big) \\ &\leq \mathbb{P}\big(\langle F(\exp iT\phi_1) - \phi_i, f\rangle^2 \geq \tfrac{1}{2}\langle f, (-\Delta)^{-1} f\rangle, \text{ for some } i \in \{1,2\}\big) \\ &\leq 2\mathbb{P}\big(\langle F(\exp iT\phi_1) - \phi_1, f\rangle^2 \geq \tfrac{1}{2}\langle f, (-\Delta)^{-1} f\rangle\big). \end{aligned} \tag{4.12}$$

We conclude by noting that for any continuous non-zero function f, we have $\langle f, (-\Delta)^{-1} f\rangle \geq Cn^4$.

To finish, let us show (1.5). We start by noting that

$$\begin{aligned}\mathbb{E}[(\phi_1(x) - \phi_2(x))^2] &= \left(\frac{2\pi}{T}\right)^2 \mathbb{E}\big[\mathbb{E}^{\mathrm{IV},\mathbf{a}}_{(2\pi)^2/T^2,\Lambda}[(\psi_1(x) - \psi_2(x))^2]\big] \\ &\geq \frac{2}{1+\varepsilon} G(x,x) \geq 2c(T,x)\log n. \end{aligned}$$

Here, recall that G represents the Green's function. Note also that $c(T,x)$ depends on T as ε did depend on T.

We now see that

$$\begin{aligned} 2\mathbb{E}[(\phi_1(x) - F(\phi_1)(x))^2] &= \mathbb{E}[(\phi_1(x) - F(\phi_1)(x))^2 + (\phi_2(x) - F(\phi_1)(x))^2] \\ &\geq \mathbb{E}[(\phi_1(x) - \phi_2(x))^2] \geq 2c(T,x)\log n. \end{aligned}$$

■

We now complete this section by proving Corollary 1.3.

4.3. *Proof of Corollary* 1.3

Let us take $\phi_n \to \Phi$ in probability for the topology of the space of generalized functions. Let us now analyze the two regimes $T \ll 1$ and $T \gg 1$.

4.3.1. Small T. First, we see that $(F_T(e^{iT\phi_n}))_{n\in\mathbb{N}}$ is tight in the Sobolev space $H^{-3-\varepsilon}$. This is because $F_T(\cdot)$ is obtained via a conditional expectation and $\sup_{n\in\mathbb{N}} \mathbb{E}[\|\phi_n\|^2_{H^{-3}}] < \infty$ (see for example the proof of [7, Corollary 4.5].[10]) We now note that thanks to Theorem 1.1 (a), for any smooth function f (with zero mean if we are in the free boundary case) we have

$$\frac{1}{n^2}\langle F_T(e^{iT\phi_n}), f\rangle = \frac{1}{n^2}\big(\langle \phi_n, f\rangle + \langle F_T(e^{iT\phi_n}) - \phi_n, f\rangle\big) \to (\Phi, f).$$

From this we see that $F_T(e^{iT\phi_n})$ also converges in probability to Φ.

4.3.2. Big T. Taking any deterministic (recovery) function F, we know from Theorems 1.1 and 1.2 that for any function (with zero mean if we are in the free boundary case) $f : [0,1] \to \Lambda$,

$$\liminf_{n\to\infty} \mathbb{P}[|n^{-2}(F(\exp(iT\phi_n)) - \phi_n, f)| \geq \delta] > 0.$$

This implies that $F(\exp(iT\phi_n))$ cannot converge to Φ because $n^{-2}(\phi_n, f) \to (\Phi, f)$.

5. There is always information left

The objective of this section is to prove that for any $T > 0$, $\exp(iT\phi)$ gives non-trivial (macroscopic) information about ϕ. More precisely, in this section we quantify how much information is preserved under the operation $\phi \mapsto \phi \pmod{2\pi/T}$.

Theorem 1.8 implies that for all possible values of $\exp(iT\phi)$ and for all T large enough there exists $\varepsilon(T)$ such that

$$\mathrm{Var}[\langle \phi, f\rangle \mid \exp(iT\phi)] \geq (1-\varepsilon(T))\mathbb{E}[\langle \phi, f\rangle^2].$$

At the same time, it is clear that

$$\mathbb{E}\big[\mathrm{Var}[\langle \phi, f\rangle \mid \exp(iT\phi)]\big] \leq \mathbb{E}[\langle \phi, f\rangle^2].$$

Let us remark that it is not clear whether this $\varepsilon(T)$ is a technical constant coming from the proof or whether it is telling us something meaningful about the model. In the following proposition we show that in the average case the existence of this $\varepsilon(T)$ is not technical. In fact, in Remark 5.2 below we give its interpretation. See also Remark 5.5 for the link with the $\varepsilon = \varepsilon(T)$ correction in Fröhlich–Spencer.

Proposition 5.1. *Let $T > 0$ and ϕ be a GFF with either free or zero boundary condition in Λ_n. Then there exists $\varepsilon'(T) > 0$ such that*

$$\mathbb{E}[\mathrm{Var}\big[\langle \phi, f\rangle^2 \mid \exp(iT\phi)\big]\big] \leq (1-\varepsilon'(T))\mathbb{E}[\langle \phi, f\rangle^2]. \tag{5.1}$$

Furthermore, when $T \gg 1$, we have

$$\varepsilon'(T) \geq (1+o(1))2T^2 e^{-T^2}. \tag{5.2}$$

[10] In that case, it is proven for the Sobolev space H^{-1}, as the authors dilute the values of the discrete GFF in a given vertex, but the same argument holds for H^{-3} when this value is not diluted.

Remark 5.2. Proposition 5.1 should be interpreted in the following way:

- The field $\exp(iT\phi_n)$ gives non-trivial information on the GFF ϕ_n.

Indeed, if the statement of the proposition were not true, for any continuous function $f : [-1,1]^2 \to \mathbb{R}$ we would have

$$n^{-4}\operatorname{Var}[\langle \phi_n, f\rangle \mid \exp(iT\phi_n)] \to \iint_{[-1,1]^2\times[-1,1]^2} f(x)G(x,y)f(y)\,dx\,dy$$
$$= \lim_{n\to\infty} n^{-4}\mathbb{E}[\langle \phi_n, f\rangle],$$

where G is the continuous Green's function in $[-1,1]^2$. In the statistics world, we would say that Proposition 5.1 means that $\exp(iT\phi)$ explains at least $\varepsilon'(T)$ of the variance of ϕ.

Proof of Proposition 5.1. Let us write $F(x) := \mathbb{E}[\phi(x) \mid \exp(iT\phi_n)]$ and $\phi = \phi_n$. We are going to prove that

$$\mathbb{E}[\langle F, f\rangle^2] \geq \varepsilon \mathbb{E}[\langle \phi, f\rangle^2]. \tag{5.3}$$

This suffices because

$$\mathbb{E}\big[\operatorname{Var}[\langle \phi, f\rangle \mid \exp(iT\phi)]\big] = \mathbb{E}[\langle F-\phi, f\rangle^2]$$
$$= \mathbb{E}[\langle \phi, f\rangle^2] - \mathbb{E}[\langle F, f\rangle^2].$$

To prove (5.3), let $W = \nabla\phi + \zeta$ be as in Proposition 2.5. Then

$$\begin{aligned}\mathbb{E}[\langle F, f\rangle^2] &= \mathbb{E}\big[\mathbb{E}[\langle \phi, f\rangle \mid \exp(iT\phi)]^2\big]\\ &= \mathbb{E}\Big[\mathbb{E}\big[\mathbb{E}[\langle \phi, f\rangle \mid \exp(iT\phi), \exp(iT\zeta)]^2 \mid \exp(iTW)\big]\Big]\\ &\geq \mathbb{E}\big[\mathbb{E}[\langle \phi, f\rangle \mid \exp(iTW)]^2\big],\end{aligned}$$

where we have used Cauchy–Schwarz and the fact that $\exp(iT\zeta)$ is independent of the pair $(\langle \phi, f\rangle, \exp(iT\phi))$. Hence, it only remains to show that

$$\mathbb{E}\big[\mathbb{E}[\langle \phi, f\rangle \mid \exp(iTW)]^2\big] \geq \varepsilon \mathbb{E}[\langle \phi, f\rangle^2]. \tag{5.4}$$

Now, recall from Proposition 2.5 that $\phi = \Delta^{-1}\nabla\cdot W$ and compute

$$\begin{aligned}\mathbb{E}[\langle \phi, f\rangle \mid \exp(iTW)] &= \mathbb{E}[\langle W, -\nabla\Delta^{-1} f\rangle \mid \exp(iTW)]\\ &= -\frac{1}{2}\sum_{\vec{e}\in\vec{E}} \mathbb{E}[W(\vec{e}) \mid \exp(iTW)]\nabla\Delta^{-1} f(\vec{e})\\ &= -\frac{1}{2}\sum_{\vec{e}\in\vec{E}} \mathbb{E}[W(\vec{e}) \mid \exp(iTW(\vec{e}))]\nabla\Delta^{-1} f(\vec{e}),\end{aligned}$$

where the last line comes from the independence of the values of W on different edges, and the fact that $W(\vec{e}) = -W(\overleftarrow{e})$. The equality of this last line may seem innocent but it is the main reason why the problem simplifies when we work with the white noise.

Let us note that the random variable $\mathbb{E}[W(\vec{e}) \mid \exp(iTW(\vec{e}))]$ is centered and has the same law for all $\vec{e}$. Furthermore, it is independent for all $e \neq e'$. Let us define

$$\sigma(T) = \mathrm{Var}\big[\mathbb{E}[W(\vec{e}) \mid \exp(iTW(\vec{e}))]\big] > 0. \tag{5.5}$$

We can now compute

$$\begin{aligned}\mathbb{E}\big[\mathbb{E}[\langle \phi, f\rangle \mid \exp(iTW)]^2\big] &= 2\pi\sigma(T)\langle \nabla\Delta^{-1} f(e), \nabla\Delta^{-1} f(e)\rangle \\ &= 2\pi\sigma(T)\langle f, -\Delta^{-1} f\rangle = \sigma(T)\mathbb{E}[\langle \phi, f\rangle],\end{aligned}$$

from which we obtain (5.1).

To obtain (5.2), we remark that we set $\varepsilon'(T) = \sigma(T)$. When $T \gg 1$, one can get (5.2) by estimating (5.5) using (A.1). This is the subject of our next lemma. ■

Lemma 5.3. *As $T \to \infty$,*

$$\begin{aligned}\sigma(T) &= \mathrm{Var}\big[\mathbb{E}[W(\vec{e}) \mid \exp(iTW(\vec{e}))]\big] \\ &= 2T^2 e^{-T^2} + o(e^{-T^2}).\end{aligned} \tag{5.6}$$

Remark 5.4. Equivalently, if $Z \sim \mathcal{N}(0, \beta)$, then as $\beta \to \infty$,

$$\mathrm{Var}\big[\mathbb{E}[Z \mid Z \ (\mathrm{mod}\ 1)]\big] = 2(2\pi)^2 \beta e^{-(2\pi)^2\beta} + o(e^{-(2\pi)^2\beta}). \tag{5.7}$$

This straightforward rewriting of the lemma will happen to be useful in [23].

Proof of Lemma 5.3. Let $Z \sim \mathcal{N}(0, \beta_T^{-1})$ with $\beta_T := (2\pi)^2/T^2$ as in Lemma 2.8 so that $Z \overset{(d)}{=} \frac{T}{2\pi} W$. Then

$$\begin{aligned}\sigma(T) &= \mathrm{Var}\big[\mathbb{E}[W(\vec{e}) \mid \exp(iTW(\vec{e}))]\big] \\ &= \mathrm{Var}\big[\mathbb{E}[W(\vec{e}) \mid W(\vec{e}) \ (\mathrm{mod}\ 2\pi/T)]\big] \\ &= \beta_T \mathrm{Var}\big[\mathbb{E}[Z \mid Z \ (\mathrm{mod}\ 1)]\big] \\ &= \beta_T \mathbb{E}\big[\mathbb{E}[Z \mid Z \ (\mathrm{mod}\ 1)]^2\big].\end{aligned} \tag{5.8}$$

Notice that

$$\mathbb{E}[Z \mid Z \ (\mathrm{mod}\ 1) = a] = \frac{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta_T}{2}(n+a)^2)\cdot(n+a)}{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta_T}{2}(n+a)^2)}.$$

As $\beta_T \to 0$, it will be convenient to rely on the Jacobi identity (A.1) which plays the role of a *temperature inversion*. Below, we start by slightly rewriting this identity via a straightforward change of variable, so that it matches integer-valued fields (as opposed to fields in $2\pi\mathbb{Z}$). The following three identities are equivalent:

$$\frac{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + 2\pi a)^2)\cdot(2\pi n + 2\pi a)}{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + 2\pi a)^2)} = \frac{\frac{1}{\beta}\sum_{q\in\mathbb{Z}} e^{-\frac{q^2}{2\beta}} \sin(q\cdot 2\pi a)\cdot q}{\sum_{q\in\mathbb{Z}} e^{-\frac{q^2}{2\beta}} \cos(q\cdot 2\pi a)},$$

$$\frac{\sum_{n\in\mathbb{Z}}\exp(-\frac{(2\pi)^2\beta}{2}(n+a)^2)\cdot(n+a)}{\sum_{n\in\mathbb{Z}}\exp(-\frac{(2\pi)^2\beta}{2}(n+a)^2)} = \frac{\frac{1}{2\pi\beta}\sum_{q\in\mathbb{Z}} e^{-\frac{q^2}{2\beta}}\sin(q\cdot 2\pi a)\cdot q}{\sum_{q\in\mathbb{Z}} e^{-\frac{q^2}{2\beta}}\cos(q\cdot 2\pi a)},$$

$$\frac{\sum_{n\in\mathbb{Z}}\exp(-\frac{\beta_T}{2}(n+a)^2)\cdot(n+a)}{\sum_{n\in\mathbb{Z}}\exp(-\frac{\beta_T}{2}(n+a)^2)} = \frac{\frac{2\pi}{\beta_T}\sum_{q\in\mathbb{Z}} e^{-\frac{q^2(2\pi)^2}{2\beta_T}}\sin(q\cdot 2\pi a)\cdot q}{\sum_{q\in\mathbb{Z}} e^{-\frac{q^2(2\pi)^2}{2\beta_T}}\cos(q\cdot 2\pi a)}.$$

This rewriting of (A.1) implies the following useful expression for the conditional expectation:

$$\mathbb{E}[Z \mid Z \pmod 1 = a] = \frac{\frac{2\pi}{\beta_T}\sum_{q\in\mathbb{Z}} e^{-\frac{q^2(2\pi)^2}{2\beta_T}}\sin(q\cdot 2\pi a)\cdot q}{\sum_{q\in\mathbb{Z}} e^{-\frac{q^2(2\pi)^2}{2\beta_T}}\cos(q\cdot 2\pi a)}.$$

This readily implies

$$\begin{aligned}
\sigma(T) &= \beta_T\mathbb{E}\big[\mathbb{E}[Z \mid Z \pmod 1)]^2\big] \\
&= \beta_T\frac{(2\pi)^2}{\beta_T^2}\mathbb{E}\Big[\frac{\big(2\sin(2\pi a)e^{-\frac{(2\pi)^2}{2\beta_T}} + 4\sin(4\pi a)e^{-\frac{(2\pi)^2}{\beta_T}}\big)^2}{\big(1+2\cos(2\pi a)e^{-\frac{(2\pi)^2}{2\beta_T}}\big)^2} + o\big(e^{-\frac{(2\pi)^2}{\beta_T}}\big)\Big] \\
&= \frac{(2\pi)^2}{\beta_T}\mathbb{E}\big[4\sin^2(2\pi a)e^{-\frac{(2\pi)^2}{\beta_T}} + o\big(e^{-\frac{(2\pi)^2}{\beta_T}}\big)\big] \\
&= 2\frac{(2\pi)^2}{\beta_T} + o\big(e^{-\frac{(2\pi)^2}{\beta_T}}\big) = 2T^2e^{-T^2} + o(e^{-T^2}),
\end{aligned}$$

where we relied on the convenient abuse of notation a for the random variable $Z \pmod 1$ throughout. ■

Remark 5.5. Proposition 5.1 is one of the reasons why this model is a laboratory for IV-GFF, especially with quenched disorder. In this case, it allows us to obtain explicit lower bounds on the $\varepsilon(T)$-correction between the GFF and the integer-valued GFF with quenched disorder **a** given by a GFF (at inverse temperature β_T^{-1}) modulo 1. We will discuss such explicit bounds in more detail in [23].

6. Conjectures on T_{rec} and the interfaces of the models

The main focus of this section is to state several conjectures. However, we also prove some intermediate results which are interesting on their own and which will give support to each of these predictions. As such, this section has more mathematical content than a list of open questions.

6.1. Lower bound on the value of T_{rec}^-

The objective of this part is to justify the following conjecture:

Conjecture 1. *We have $T_{\mathrm{rec}}^- \geq 2\sqrt{\pi}$.*

We have two reasons to believe this conjecture, both related to the continuum Gaussian free field. The first reason concerns the so-called imaginary chaos and the second one is related to the flow lines of the continuum GFF.

6.1.1. Reason 1: Imaginary chaos. We will not introduce all the definitions here. We refer to [26, 28] for context and the definition. Let Φ be a 0-boundary continuum Gaussian free field in a domain $D \subseteq \mathbb{C}$ and let ν_ε^x be the uniform measure on $\partial B(x, \varepsilon)$. We normalize Φ so that if $d(x, y) \geq \varepsilon$ then

$$\mathbb{E}[(\Phi, \nu_\varepsilon^x)(\Phi, \nu_\varepsilon^y)] = G_D(x, y).$$

Note that in our normalization $G_D(x, y) \sim \frac{1}{2\pi} |\log \|x - y\||$.

Fix $\alpha \in \mathbb{R}$ and define $\mathcal{V}^\alpha$ as the imaginary chaos associated with α,

$$\mathcal{V}^\alpha = \mathcal{V}^\alpha(\Phi) := \lim_{\varepsilon \to 0} \exp\left(i\alpha\phi_\varepsilon(\cdot) + \frac{\alpha^2}{2}\mathbb{E}[\phi_\varepsilon^2(\cdot)] \right).$$

Here the limit is taken in the space of distributions, and it is only non-trivial in the case $\alpha < 2\sqrt{\pi}$. Note that our normalization is different from the one in [26, 28], in which our α corresponds to $\tilde{\alpha} = \sqrt{2}$.

We can now prove the following result.

Proposition 6.1. *Assume*

(H_1) *there exists $\hat{\alpha}$ such that for all $\alpha < \hat{\alpha}$ the GFF Φ can be measurably recovered from $\mathcal{V}^\alpha(\Phi)$, i.e., there exists a deterministic measurable function F such that a.s. $F(\mathcal{V}^\alpha) = \Phi$.*

Then $T_{\mathrm{rec}}^- \geq \hat{\alpha}$.

Remark 6.2. After the first version of this work appeared, hypothesis (H_1) was proved up to $\hat{\alpha} = 2\sqrt{\pi}$ in the recent work [5]. Let us emphasize that the imaginary case is more subtle than the same question for the real chaos analyzed in [11] as one needs to control the local fluctuations all the way to the boundary.

To prove Proposition 6.1, we need to show that the discrete imaginary chaos converges to the continuous one.

Proposition 6.3. *Let $\phi^{(n)}$ be a discrete* 0*-boundary GFF in Λ_n and let*

$$\mathcal{V}_n^\alpha(\cdot) := \exp\left(i\alpha\phi^{(n)}(\cdot) + \frac{\alpha^2}{2}\mathbb{E}[(\phi^{(n)})^2(\cdot)] \right).$$

Then for all $\alpha < 2\sqrt{\pi}$, as $n \to \infty$,

$$(\phi^{(n)}, \mathcal{V}_n^\alpha) \to (\Phi, \mathcal{V}^\alpha(\Phi)) \quad \textit{in law},$$

for the topology of generalized functions. Here Φ is a 0*-boundary GFF in* $[-1, 1]^2$.

As this section is concerned mostly with conjectures, we will only sketch the proof of this result. The main input is [26, Theorem 1.3] which states that $(\Phi, \mathcal{V}^\alpha(\Phi))$ is characterized by its moments.

Proof of Proposition 6.3. By [26, Theorem 1.3], the field $(\Phi, \mathcal{V}^\alpha(\Phi))$ is characterized by its moments, that is, by

$$\mathbb{E}\Big[\Big(\prod_i (\Phi, f_i^1)\Big)\Big(\prod_j (\mathcal{V}^\alpha, f_j^2)\Big)\Big(\prod_k \overline{(\mathcal{V}^\alpha, f_k^3)}\Big)\Big]$$
$$= \int \Big(\prod_i f_i^1(x_i)dx_i\Big)\Big(\prod_j f_j^2(y_j)dy_j\Big)\Big(\prod_k \overline{f_k^3(z_k)}dz_k\Big) C((x_i)_i, (y_j)_j, (z_k)_k), \qquad (6.1)$$

where all $f_\cdot^\ell$ are smooth functions in $[-1,1]^2$ (with zero mean if Φ is a free-boundary GFF). The function $C(\cdot,\cdot,\cdot)$ is called the correlation function of this model. By a simple (but lengthy) computation one can see that (6.1) also comes from the discrete setting

$$\mathbb{E}\Big[\Big(\prod_i n^{-2}\langle \phi_n, f_i^1\rangle\Big)\Big(\prod_j n^{-2}\langle \mathcal{V}_n^\alpha, f_j^2\rangle\Big)\Big(\prod_k \overline{n^{-2}\langle \mathcal{V}_n^\alpha, f_k^3\rangle}\Big)\Big]$$
$$\to \int \Big(\prod_i f_i^1(x_i)dx_i\Big)\Big(\prod_j f_j^2(y_j)dy_j\Big)\Big(\prod_k \overline{f_k^3(z_k)}dz_k\Big) C((x_i)_i, (y_j)_j, (z_k)_k), \qquad (6.2)$$

at least when all the functions f have disjoint supports. This can be proven by noting that C is obtained only from the Green's function and that the discrete Green's function converges to the continuum one [15, Corollary 3.11]). To finish, one needs to show that (6.2) is true for all possible fs. This can be done using the dominated convergence theorem. To see that the sum coming from the LHS of (6.2) is uniformly dominated, one applies [15, Theorem 2.5], i.e.,

$$G(x,y) = -(2\pi)^{-1}\log\Big(\frac{\|x-y\|}{n}\Big) + O(1),$$

and uses the same techniques as in [26, Section 3.2]. ∎

We can now prove Proposition 6.1.

Proof of Proposition 6.1. Let $\phi_1^{(n)}$ and $\phi_2^{(n)}$ be two 0-boundary GFFs coupled as in Definition 3.1. Thanks to Proposition 6.3, the 4-tuple

$$(\phi_1^{(n)}, \mathcal{V}_{1,n}^\alpha, \phi_2^{(n)}, \mathcal{V}_{2,n}^\alpha)$$

is tight. Let $(\Phi_1, \mathcal{V}_1^\alpha, \Phi_2, \Phi_2, \mathcal{V}_2^\alpha)$ be any accumulation point of the sequence and note that because for all $n \in \mathbb{N}$, a.s. $\mathcal{V}_{1,n}^\alpha = \mathcal{V}_{2,n}^\alpha$, we have $\mathcal{V}_1^\alpha = \mathcal{V}_2^\alpha$. This equality implies, thanks to assumption (H$_1$), that a.s. $\Phi_1 = \Phi_2$. Then, as all accumulation points are the same, we have in fact, as $n \to \infty$,

$$(\phi_1^{(n)}, \mathcal{V}_{1,n}^\alpha, \phi_2^{(n)}, \mathcal{V}_{2,n}^\alpha) \to (\Phi_1, \mathcal{V}_1^\alpha, \Phi_1, \mathcal{V}_1^\alpha) \quad \text{in distribution.}$$

Let us now take any smooth function f. For $j \in \{1,2\}$ we have

$$\sup_n \mathbb{E}\Big[\Big(\frac{1}{n^2}\langle \phi_j^{(n)}, f\rangle\Big)^4\Big] < K,$$

which implies that

$$\mathbb{E}\Big[\Big(\frac{1}{n^2}\langle \phi_1^{(n)} - \phi_2^{(n)}, f\rangle\Big)^2\Big] \to \mathbb{E}[(\Phi_1 - \Phi_1, f)^2] = 0.$$

As this implies that $\mathbb{E}[\mathrm{Var}\,\langle \phi_1^{(n)}, f\rangle \mid \exp(i\alpha\phi_1^{(n)})] = o(n^4)$, we conclude as at the beginning of Section 3. ∎

Remark 6.4. Note that even if assumption (H$_1$) is proven, this only shows that $\mathbb{E}[\mathrm{Var}\,\langle \phi_1^{(n)}, f\rangle] = o(n^4)$, which is a weaker result than $\mathbb{E}[\mathrm{Var}\,\langle \phi_1^{(n)}, f\rangle] = O(n^2)$ established in Proposition 3.9.

6.1.2. Reason 2: Flow lines. Flow lines of Gaussian free fields were introduced in [16, 40] and were studied in depth in [30–33]. Informally, a flow line can be described as the curve which is the solution of

$$\eta'(t) = e^{i(\sqrt{2\pi}\,\Phi/\chi + u)}, \quad \eta(0) = z \in \partial D,$$

where Φ is a GFF in a simply connected domain D and u is a harmonic function. For us it is important to note that the curve η should only be determined by $e^{i(\sqrt{2\pi}\,\Phi/\chi)}$. This will motivate assumption (H$_2$) below.

Flow lines can be defined using the concept of local sets [39, 44]. In other words, η is a flow line of a GFF Φ if for any stopping time τ of the natural filtration of η we have

$$\Phi = \Phi^{\eta_\tau} + h_{\eta_\tau},$$

where $\eta_\tau = \eta([0,\tau])$, Φ^{η_τ} has the law of a GFF of $D \setminus \eta_\tau$ and h_{η_τ} is a harmonic function in $D \setminus \eta_\tau$. Let us remark that in this case the function h_{η_τ} is, in fact, a measurable function of η_τ. In fact, it can be found in [30, Theorem 1.1].

A generalization of flow lines is given by the angle-varying flow lines defined in [30, Section 5.2], which can be roughly described as running a flow line with initial angle θ_1 until a stopping time[11] τ_1, and then continue with an angle θ_2 until a stopping time τ_w, and continue until finitely many iterations. These lines are called $\eta_{\theta_1\ldots\theta_\ell}^{\tau_1\ldots\tau_\ell}$ and they are a measurable function of Φ, the GFF they are coupled with (see [30, Lemma 5.6]).

In fact, [30, Proposition 5.9] shows that if $\chi \geq 1/\sqrt{2}$, there exists a countable set of angle-varying flow lines $\Big(\eta_{\theta_1^k\ldots\theta_{\ell_k}^k}^{\tau_1^k\ldots\tau_{\ell_k}^k}\Big)_{k\in\mathbb{N}}$ such that a.s.

$$\bigcup_n \eta_{\theta_1^k\ldots\theta_{\ell_k}^k}^{\tau_1^k\ldots\tau_{\ell_k}^k}$$

[11] With respect to the natural filtration of η.

is dense (because SLE_8 is a space-filling curve). Now, define $\mathbb{F}_n$ as the σ-algebra generated by $\eta_{\theta_1^{\ell n}\dots\theta_{\ell n}^n}^{\tau_1^{\ell n}\dots\tau_{\ell n}^n}$. The discussion in the paragraph before and the fact that $h_{\eta_{\theta_1^{\ell n}\dots\theta_{\ell n}^n}^{\tau_1^{\ell n}\dots\tau_{\ell n}^n}}$ is a measurable function of the set $\eta_{\theta_1^{\ell n}\dots\theta_{\ell n}^n}^{\tau_1^{\ell n}\dots\tau_{\ell n}^n}$ implies that $\mathbb{F} = \bigvee_n \mathbb{F}_n$ is equal to the σ-algebra generated by Φ (see for example [6, Lemma 2.3]). In other words, Φ is a deterministic function of $\left(\eta_{\theta_1^{\ell n}\dots\theta_{\ell n}^n}^{\tau_1^{\ell n}\dots\tau_{\ell n}^n}\right)_{n\in\mathbb{N}}$.

This allows us to show the following proposition.

Proposition 6.5. *Let ϕ_n be a 0-boundary GFF in Λ_n, and assume*

(H_2) *there exists $\hat{\chi} \geq 1/\sqrt{2}$ such that for all $\chi > \hat{\chi}$ and for any angle-varying flow line $\eta_{\theta_1\dots\theta_\ell}^{\tau_1\dots\tau_\ell}$, there exists an approximate angle-varying flow line $\eta^{(n)}$ depending on $\exp(i\sqrt{2\pi}\,\phi_n/\chi)$ such that $(\phi_n, \eta^{(n)})$ converges in law to $(\Phi, \eta_{\theta_1\dots\theta_\ell}^{\tau_1\dots\tau_\ell}(\Phi))$.*

Then $T_{\mathrm{rec}}^- \geq \sqrt{2\pi}/\chi$.

Before proving the proposition, let us recall that it is expected that the flow lines related to the discrete GFF converge to the flow lines of the continuum GFF, as this is already the case for $\chi = \infty$, the SLE_4 case [38]. If this were the case, Proposition 6.5 implies that $T_{\mathrm{rec}}^- \geq 2\sqrt{\pi}$.

Proof of Proposition 6.5. Let Φ be a continuous GFF with zero boundary condition. Thanks to assumption (H_2), we can define $\eta_k^{(n)}$ such that as $n \to \infty$,

$$(\phi_n, \eta_k^{(n)}) \to \left(\Phi, \eta_{\theta_1^k\dots\theta_{\ell_k}^k}^{\tau_1^k\dots\tau_{\ell_k}^k}(\Phi)\right) \tag{6.3}$$

in law. We then have

$$(\phi_n, (\eta_k^{(n)})_{k\in\mathbb{N}}) \to \left(\Phi, \left(\eta_{\theta_1^k\dots\theta_{\ell_k}^k}^{\tau_1^k\dots\tau_{\ell_k}^k}(\Phi)\right)_{k\in\mathbb{N}}\right) \tag{6.4}$$

in law for the product topology. Indeed, (6.3) implies that $(\phi_n, (\eta_k^{(n)})_{k\in\mathbb{N}})$ is tight for the product topology. We can then check, again thanks to (6.3), that any accumulation point $(\Phi, (\eta_k^{\infty})_{k\in\mathbb{N}})$ has to be such that

$$(\Phi, \eta_k^{\infty}) = \left(\Phi, \eta_{\theta_1^k\dots\theta_{\ell_k}^k}^{\tau_1^k\dots\tau_{\ell_k}^k}(\Phi)\right).$$

As a consequence,

$$(\Phi, (\eta_k^{\infty})_{k\in\mathbb{N}}) = \left(\Phi, \left(\eta_{\theta_1^k\dots\theta_{\ell_k}^k}^{\tau_1^k\dots\tau_{\ell_k}^k}(\Phi)\right)_{k\in\mathbb{N}}\right),$$

which implies (6.4).

We can now conclude in a similar way to Proposition 6.1. We take (ϕ_1, ϕ_2) coupled as in Definition 3.1 and we study the 4-tuple

$$\big(\phi_1^{(n)}, (\eta_k^n(\phi_1^{(n)}))_{k\in\mathbb{N}}, \phi_2^{(n)}, (\eta_k^n(\phi_2^{(n)})_{k\in\mathbb{N}}\big).$$

Again, this 4-tuple is tight and any accumulation point is of the form

$$\Big(\Phi_1, \Big(\eta_{\theta_1^k\ldots\theta_{\ell_k}^k}^{\tau_1^k\ldots\tau_{\ell_k}^k}(\Phi_1)\Big)_{k\in\mathbb{N}}, \Phi_2, \Big(\eta_{\theta_1^k\ldots\theta_{\ell_k}^k}^{\tau_1^k\ldots\tau_{\ell_k}^k}(\Phi_2)\Big)_{k\in\mathbb{N}}\Big).$$

Since for all $n, k \in \mathbb{N}$, we have $\eta_k^n(\phi_1^{(n)}) = \eta_k^n(\phi_2^{(n)})$, we find that at this accumulation point, a.s.,

$$\eta_{\theta_1^k\ldots\theta_{\ell_k}^k}^{\tau_1^k\ldots\tau_{\ell_k}^k}(\Phi_1) = \eta_{\theta_1^k\ldots\theta_{\ell_k}^k}^{\tau_1^k\ldots\tau_{\ell_k}^k}(\Phi_2).$$

As Φ_i is a function of this $\Big(\eta_{\theta_1^k\ldots\theta_{\ell_k}^k}^{\tau_1^k\ldots\tau_{\ell_k}^k}(\Phi_i)\Big)_{k\in\mathbb{N}}$, we see that $\Phi_1 = \Phi_2$, which implies that $(\phi_1^{(n)}, \phi_2^{(n)})$ converges in law to (Φ_1, Φ_1). By the same reasoning as at the end of the proof of Proposition 6.1 we conclude that for any continuous function f, $\mathbb{E}[\mathrm{Var}[\langle \phi^{(n)}, f\rangle \mid e^{\sqrt{2\pi}\phi^{(n)}/\chi}]] = o(n^4)$. ■

6.2. *Interfaces of* $\exp(iT\phi)$

In this section, we discuss the possible scaling limit of certain interfaces naturally appearing in $\exp(iT\phi)$ and how they may relate to the interfaces of the GFF ϕ.

6.2.1. Level lines of $\exp(iT\phi)$. In [38], the authors showed that the level line of a zero boundary GFF with a special boundary condition converges in law to an SLE_4. We believe a similar story holds for both $\exp(iT\phi)$, and more importantly for the Villain model. Let us be more explicit.

We define u_n as the bounded harmonic function in $\Lambda_n \setminus \partial\Lambda_n$ with boundary value $\lambda = \sqrt{\pi/8}$ in $\partial\Lambda_n \cap \{x : \mathrm{Re}(x) \geq 0\}$ and $-\lambda = -\sqrt{\pi/8}$ in $\partial\Lambda_n \cap \{x : \mathrm{Re}(x) < 0\}$. It is shown in [38] that if ϕ_n is a GFF in Λ_n with zero boundary condition and η the level line of $\phi + u_n$, then $\eta^{(n)}(\cdot)$ is a path in the dual of Λ_n that has the following properties (see Figure 6):

- It goes from the dual of the edge $(-i - 1/n, -i)$ to the dual of the edge $(i + 1/n, i)$.
- The primal edge associated to a dual edge in the path is such that ϕ_n is negative to its left and positive to its right.

Theorem 1.4 of [38] states that $\eta^{(n)}(\cdot)$ parametrized by capacity converges in the uniform topology to an SLE_4. This result is improved in [39] by showing that as $n \to \infty$,

$$(\phi_n, \eta^{(n)}) \to (\Phi, \eta) \quad \text{in law.}$$

Here Φ is a GFF in $[-1, 1]$ and η is the so-called level line of the continuous GFF. More precisely, η is a measurable function of Φ and the law of Φ conditioned on η is such that

$$\Phi + u_\infty = \Phi^L + \Phi^R,$$

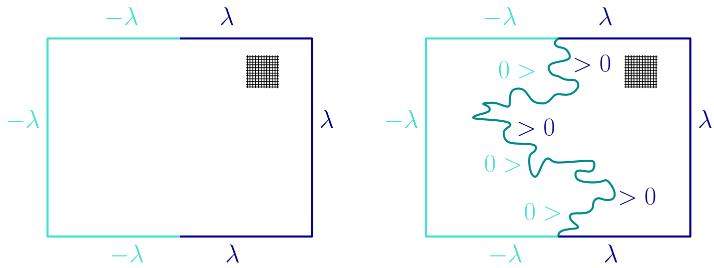

Fig. 6. Left: The boundary values of the harmonic function u_n. Right: The level line of $\phi + u_n$; note that $\phi + u_n$ takes positive values to the left and negative to the right.

where Φ^L, resp. Φ^R, is a GFF in the domain to the left, resp. right, of η with $-\lambda$, resp. λ, boundary condition (see Figure 7).

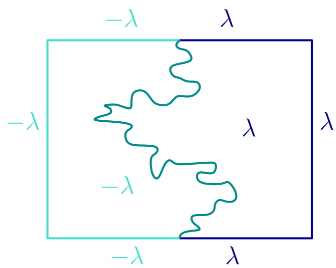

Fig. 7. The image shows how the limiting curve η separates the domain into two different domains, the left where the GFF has $-\lambda$ boundary condition and the right where its boundary condition is λ.

We now have the tools to prove Corollary 1.4.

Proof of Corollary 1.4. We assume that ϕ_n converges to a continuum GFF Φ and define $L^{(n)} = L_T^{(n)}(\exp(iT\phi_n))$ as a set parametrized by q, where

$$L^{(n)}(q) = \mathbb{E}[\eta^{(n)}(q) \mid \exp(iT\phi)].$$

Let us now prove that the set $L^{(n)}$ converges in probability to the level line η of ϕ. To do this, it is enough to show that for all q, $L^{(n)}(q)$ converges in probability to $\eta(q)$. Thanks to [38, Theorem 1.4] we know that $\eta^{(n)}(q)$ converges in law to $\eta(q)$; now it suffices to show that as $n \to \infty$,

$$\mathrm{Var}[\eta^{(n)}(q) \mid \exp(iT\phi)] \to 0 \quad \text{in probability.} \tag{6.5}$$

To do this, we use the same trick as always. Let $(\phi_1^{(n)}, \phi_2^{(n)})$ be two GFFs coupled as in Definition 3.1. We know that thanks to Theorem 1.1 (a), $(\phi_1^{(n)}, \eta_1^{(n)}, \phi_2^{(n)}, \eta_2^{(n)})$ converges in law to (Φ, η, Φ, η). Here the topology on the curves is that of the uniform distance for continuous curves. As a consequence of the convergence we find that for any $q \in Q$,

$$\mathbb{P}(\|\eta_1^{(n)}(q) - \eta_2^{(n)}(q)\| \geq \delta) \to 0 \quad \text{as } n \to \infty.$$

Since Λ_n is bounded, we conclude that

$$\mathbb{E}[\|\eta_1^{(n)}(q) - \eta_2^{(n)}(q)\|^2] \to 0 \quad \text{as } n \to \infty.$$

This concludes the proof, as it proves (6.5). ■

Corollary 1.4 gives an explicit way to recover the level line of the GFF given $e^{iT\phi_n}$. However, this recovery process does not locally depend on the field. We also believe that it is possible to recover the level line via an explicit local function of $e^{iT(\phi_n+u_n)}$, its own level line.

Now, we let T be small enough such that $T\lambda < \pi$; then the imaginary part of $\exp(iTu_n(x))$ has the same as sign as the real part of x. We also define $\eta^{(n),T}(\cdot)$, the level line of the imaginary part $\exp(iT(\phi_n + u_n))$. We conjecture the following.

Conjecture 2. *There exists a small enough T_c such that for all $T < T_c$, $\eta^{(n),T}$ converges in law to an SLE$_4$. Furthermore, $\eta^{(n)}$ and $\eta^{(n),T}$ converge to the same limit.*

Apart from Corollary 1.4, we have two other reasons to believe in this conjecture. The first one is the fact that the gradient of ϕ_n on its level line $\eta^{(n)}$ is, in mean, upper and lower bounded (see [38, Lemma 3.1]). Thus, one could expect that most edges in η have corresponding primal edges for which $Im(\exp(iT\phi_n))$ is negative on its left vertex and positive on the right one.

The second reason is that level lines do not get close to each other, nor to themselves. This can be seen in [38, Sections 3.4 and 3.5], or by understanding their scaling limit as in [43, Remark 1.5].

As we said before, we conjecture that there is a similar result for the Villain model. In fact, Fröhlich and Spencer conjectured that the Villain model at low temperature T is close to the imaginary exponential of a GFF with a slightly different temperature $T_{\text{Vil}} := T_{\text{Vil}}(T) > T$ (see [22, Section 8.1]). This allows us to interpret Conjecture 2 as follows.

Conjecture 3. *Let T be small enough and let ψ_n be a Villain model in Λ_n with temperature T and boundary values $\exp(-i\lambda\sqrt{T'_{\text{Vil}}})$ in the left side of the boundary, i.e. $\partial\Lambda_n \cap \{x : \text{Re}(x) < 0\}$, and $\exp(i\lambda\sqrt{T'_{\text{Vil}}})$ in $\partial\Lambda_n \cap \{x : \text{Re}(x) \geq 0\}$. Let $\eta^{(n)}$ be the level line of the imaginary part of ψ. Then $\eta^{(n)}$ converges in law to an SLE$_4$ (see Figure 8).*

In fact, the result should hold for more general boundary values.

Conjecture 4. *Let T be small enough and let ψ_n be a Villain model in Λ_n with temperature T and boundary values $\exp(-ia)$ in $\partial\Lambda_n \cap \{x : \text{Re}(x) < 0\}$, and $\exp(ia)$ in $\partial\Lambda_n \cap \{x : \text{Re}(x) \geq 0\}$. Let $\eta^{(n)}$ to be the level line of the imaginary part of ψ. Then for a small enough, $\eta^{(n)}$ converges in law to an SLE$_4(\rho)$ with $\rho = a/(\lambda\sqrt{t}) - 1$.*

6.2.2. Full set of interfaces. Instead of studying a single interface of a GFF, one could also study the whole set of interfaces arising from a 0-boundary GFF. These sets are called ALEs, and were introduced in [9] and further studied in [6, 8, 36].

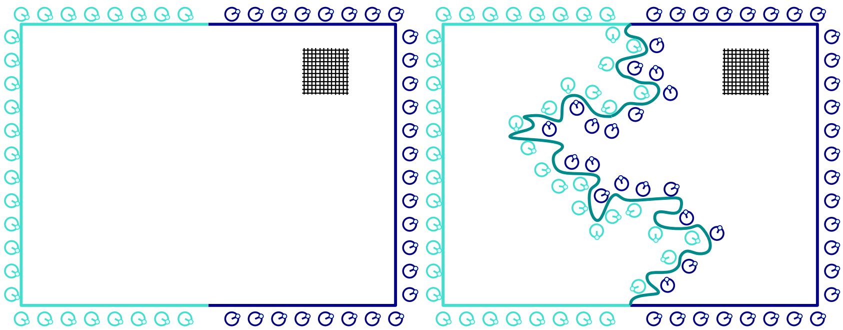

Fig. 8. Left: The boundary values of the Villain model. Right: The level line of the imaginary part of this Villain model. We believe that this line converges in law to an SLE_4 when the temperature of the system is low enough.

ALEs are characterized as the only random set $\mathbb{A}_{-\lambda,\lambda}$ such that a continuum GFF Φ can be written as

$$\Phi := \sum_{O} \Phi^{O} + \sigma^{O}\lambda, \tag{6.6}$$

where the sum is over connected components O of the complement of $\mathbb{A}_{-\lambda,\lambda}$, i.e., $[-1,1]^2 \setminus \mathbb{A}_{-\lambda,\lambda}$. Furthermore, $\sigma^O \in \{-1,1\}$ and conditionally on $\mathbb{A}_{-\lambda,\lambda}$, Φ^O is a 0-boundary GFF in O (conditionally) independent of $(\Phi_{O'})_{O'\neq O}$. The existence and uniqueness of such a set was proven in [9]. Furthermore, as shown in [8, Lemma 3.6], this set can be thought of as the union of the 0-level lines of the continuum GFF Φ.

In fact, for this discussion it is useful to define the 0-level line η of a discrete GFF going between $x \in \partial\Lambda_n$ and $y \in \partial\Lambda_n$; η is then a dual path connecting an edge containing x to an edge containing y such that for all vertices in $\Lambda_n \setminus \partial\Lambda_n$ to the left of η, one has $\phi_n(x) < 0$, and for all vertices to the right of η, $\phi_n(x) > 0$ (except maybe for points in the boundary[12]). Furthermore, let us define the discrete ALE $\mathbb{A}^n_{-\lambda,\lambda}$ as the union over all starting points and end points of the associated 0-level lines.

The 0-level line is known to converge for the Hausdorff topology by [38, Theorem 1.3], and furthermore the techniques of [39] allow us to see that it converges to the 0-level line of a continuum GFF. These techniques, together with [8, Lemma 3.6] mentioned above, allow one to show that $\mathbb{A}^n_{-\lambda,\lambda}$ converges for the Hausdorff topology to the ALE.[13]

We can now discuss similar results to those for the level lines. In particular, as before we have

[12]Note furthermore that such a line is not always unique. This should not be a problem as it is not a problem in [38].

[13]The exact argument is not written anywhere, even though this proof has been known to a small community. As the main focus of this section is not this result, but rather to shed light on this interesting direction, we will not formalize this result further here.

Proposition 6.6. *For $T < T_{\text{rec}}^-$, there exists a deterministic function $A_T(\cdot)$ such that when $\phi_n \to \Phi$, we have $A_T(e^{iT\phi_n}) \to \mathbb{A}_{-\lambda,\lambda}$.*

The problem, as before, is that we do not know whether this function A_T can be taken to be the discrete ALE associated to the imaginary part of $e^{iT\phi_n}$. This is the content of the next conjecture.

Conjecture 5. *Let T be small enough and ϕ_n a 0-boundary GFF converging to Φ. Then the ALE associated to the imaginary part of $e^{iT\phi_n}$ converges to $\mathbb{A}_{-\lambda,\lambda}$.*

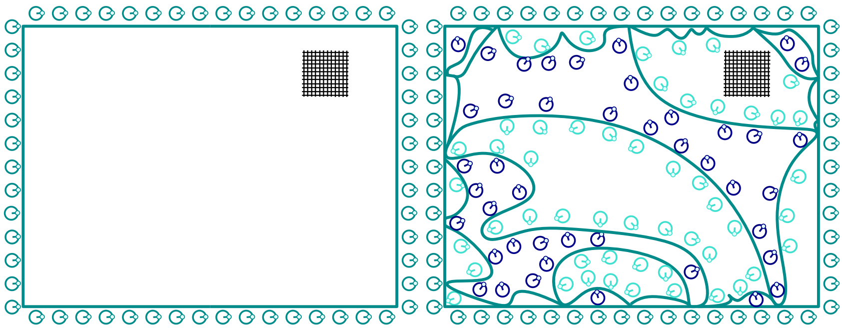

Fig. 9. Left: The boundary values of the Villain model for this case. Right: The ALE associated to the imaginary part of this Villain model. We believe that this set converges in law to the ALE $\mathbb{A}_{-\lambda,\lambda}$ when the temperature of the system is low enough. A striking consequence of this conjecture is that the law of the limiting set does not depend on the temperature, as long as the system is cold enough.

An even more daring conjecture is that the same is true for a Villain model at small enough temperature.

Conjecture 6. *Let T be small enough and let ψ_n be a Villain model in Λ_n. Then as $n \to \infty$ the discrete ALE associated to the imaginary part of ψ_n converges to $\mathbb{A}_{-\lambda,\lambda}$.*

It is interesting to note that we expect that the interfaces of the Villain model at low temperature resemble each other a lot for various β. That is, this geometry will not distinguish the temperature from which the ALE arises. However, we expect that the law inside each connected component of the complement of this ALE will look pretty different. To be more precise, we expect that the boundary conditions generated by this ALE get closer and closer to $1 = e^{i0}$ as the temperature goes to 0.

6.3. Upper bound on of T_{rec}^+

In fact, the analysis of level lines of the GFF makes us believe in the following conjecture.

Conjecture 7. *We have $T_{\text{rec}}^+ \leq 2\sqrt{2\pi}$.*

Let us note that $2\sqrt{2\pi}$ is the smallest value of T so that $\exp(iT\lambda) = \exp(-iT\lambda)$. That is, it is the value for which we could not expect to recognize the macroscopic difference between the left and the right side of the level line η introduced in Section 6.2.

The level line η is fundamental in recovering the GFF. This is shown, for example, in the construction of the free-boundary GFF given in [36].

There is another reason why we believe that one cannot recover ϕ when $T = 2\sqrt{2\pi}$. It has to do with the level set of the GFF.

Although the GFF is not a function, one can still define $\mathbb{A}_{-a,b}$. This is, informally, the (connected component connected to the boundary of the) preimage of $[-a, b]$. These sets were introduced[14] in [6, 9] and their existence is conditional on the size of the interval $[-a, b]$:

- *The set $\mathbb{A}_{-a,b}$ exists if and only if $a, b > 0$ and $a + b \geq 2\lambda$.*

The case $a + b = 2\lambda$ is special. These are the values such that $\exp(-iTa) = \exp(iTb)$. Furthermore, in [8], it is shown that these are the only values of a and b such that the following happens:

- *Fix two points $x, y \in [-1, 1]^2$ and let $O(x)$ and $O(y)$ be the connected components of $[-1, 1]^2 \setminus \mathbb{A}_{-a,b}$ containing x and y respectively. Then there is a positive probability that $O(x) \neq O(y)$ and $\partial O(x) \cap \partial O(y)$ is a continuous curve.*

This property implies that the places where the GFF takes value $-a$ and the ones where it takes value b are mesoscopically separated, i.e. they are not macroscopically far apart. As the function $x \mapsto \exp(i2\sqrt{2\pi}\, x)$ cannot distinguish between $-a$ and b, we believe it is not possible to recover $\mathbb{A}_{-a,b}$ just by knowing $\exp(i2\sqrt{2\pi}\, \phi)$. This would make it impossible to recover all the macroscopic information on the GFF.

Appendix A. Viewing the shift $\mathbf{a} = \{a_i\}_{i\in\Lambda}$ as an exterior magnetic field

The goal of this appendix is to provide a different proof of Proposition 4.1. The idea of this proof was inspired to us by an inspection of this exact identity in the simplest possible case of a Gaussian free field on a single point $\{x\}$ with Dirichlet boundary condition, namely a Gaussian $\mathcal{N}(0, 1/\beta)$. The appendix is organized as follows: first we investigate the case of one point, then we make a link with Riemann theta functions (thus explaining the name *modular invariance*) and finally we give a second proof of Proposition 4.1.

A.1. Warm up: GFF with one point and Jacobi theta function

Let us consider the GFF on a graph with two points $\{x, y\}$ with zero boundary condition at y. The partition function of the a-shifted integer-valued field (here the vector $\mathbf{a}$ is just

[14] See [37] to better understand the relationship between $\mathbb{A}_{-a,b}$ and the imaginary chaos.

one parameter which we call a) reads as follows:

$$
\begin{aligned}
Z(\beta,a) &= \int \Big(\sum_{n\in\mathbb{Z}} \delta_{2\pi n+a}(\phi)\Big) \frac{1}{\sqrt{2\pi/\beta}} e^{-\frac{\beta}{2}\phi^2}\, d\phi \\
&= \frac{1}{\sqrt{2\pi/\beta}} \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{\beta}{2}(2\pi n+a)^2\Big).
\end{aligned}
$$

In the limiting case where we plug the infinite Fourier series

$$
1 + 2\sum_{q=1}^{\infty} \cos(q(\phi-a)) \equiv 2\pi \sum_{n\in\mathbb{Z}} \delta_{2\pi n+a}(\phi)
$$

into the Fröhlich–Spencer expansion at one point, it can be checked that the identity (4.9) reads

$$
\frac{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n+a)^2)\cdot(2\pi n+a)}{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n+a)^2)} = \frac{\frac{1}{\beta}\sum_{q\in\mathbb{Z}} e^{-\frac{q^2}{2\beta}} \sin(q\cdot a)\cdot q}{\sum_{q\in\mathbb{Z}} e^{-\frac{q^2}{2\beta}} \cos(q\cdot a)}, \tag{A.1}
$$

which is correct for any $\beta > 0$ and any real $a \in (-\pi,\pi)$. (Note interestingly that it is degenerate for the LHS as $\beta \to 0$ but not for the RHS!)

One way to prove this identity is to notice its link with Jacobi's theta function. Indeed, the latter is classically defined as follows (see for example [34]):

$$
\theta(z \mid \tau) := \sum_{n\in\mathbb{Z}} \exp(i\pi n^2\tau + 2i\pi n z)
$$

for all $z \in \mathbb{C}$, $\tau \in \mathbb{H}$. Now if we plug

$$
z := i\beta a, \quad \tau := 2i\pi\beta
$$

into θ, we find

$$
\theta(z \mid \tau) = e^{\frac{\beta}{2}a^2} \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{\beta}{2}(2\pi n+a)^2\Big).
$$

Jacobi's first modular identity states that

$$
\theta\Big(\frac{z}{\tau} \,\Big|\, \frac{-1}{\tau}\Big) = \alpha\theta(z \mid \tau), \tag{A.2}
$$

where $\alpha = (-i\tau)^{1/2}\exp(\frac{\pi}{\tau} i z^2) = \sqrt{2\pi\beta}\exp(-\frac{\beta}{2}a^2)$. This identity gives

$$
\sum_{q\in\mathbb{Z}} e^{-\frac{q^2}{2\beta}} \cos(qa) = \sqrt{2\pi\beta} \sum_{n\in\mathbb{Z}} \exp\Big(-\frac{\beta}{2}(2\pi n+a)^2\Big) = \sqrt{2\pi\beta}\, Z(\beta,a),
$$

from which one can prove the identity (A.1) by taking the log-derivative in a. Note that one may also avoid using Jacobi's identity and re-prove things using a Poisson summation formula. We indicate the link here as our shift-parameter **a** which is central to our work is naturally associated to the first argument z of the theta function θ (while the second argument τ is related to the inverse temperature).

The argument we have just outlined bares some resemblance to the fact that, in an Ising model with an exterior magnetic field h, one can compute the average magnetization as a derivative with respect to h of the free energy $\log Z$. In our context, we have used the fact that

$$\frac{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + a)^2) \cdot (2\pi n + a)}{\sum_{n\in\mathbb{Z}} \exp(-\frac{\beta}{2}(2\pi n + a)^2)} = -\frac{1}{\beta}\partial_a \log(Z(\beta, a)).$$

This suggests that our key identity (4.9) for a general domain $\Lambda \subset \mathbb{Z}^2$ should be reminiscent of recovering the average magnetic field of the Ising model from the derivative in h of its free energy $\log Z$. We implement this idea in the rest of the appendix by viewing the vector shift $\mathbf{a} = \{a_i\}_{i\in\Lambda}$ acting as an external magnetic field. We prove Proposition 4.1 along these lines in two steps:

Section A.2: First, as in the case of one point, we work in the limiting case of infinite Fourier series at each vertex $x \in \Lambda$. This makes the analogy with the Ising model clearer and makes a connection with the modular invariance of certain Riemann theta functions. From the intuition gathered here, we notice that the key identity (4.9) is an appropriate log-derivative with respect to **a**, namely $-\langle \sigma, \nabla_{\mathbf{a}} \log Z\rangle = -\langle \frac{1}{\beta}(-\Delta)^{-1} f, \nabla_{\mathbf{a}} \log Z\rangle$.

Section A.3: In the second part, we work in the finite cut-off case. Here it is not so clear how to recognize the integral against $\langle f, \phi\rangle$ on the RHS of the identity (4.9). The reason is that expansion into charges from [21] (and particularly the effect of the complex translation under spin waves) somehow obfuscates the readability of $\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle \phi, f\rangle]$. To end the proof, we first get around the blurring effect caused by the expansion into charges from [21] (using the matching of partition functions before and after expansions into charges) and then connect with an actual average of $\langle \phi, f\rangle$ by Gaussian integration by parts.

A.2. *Riemann theta function and* **a***-shifted integer-valued GFF*

In this section, we implicitly rely on expansions into infinitely many charge configurations in [21,27] by attaching to each vertex $i \in \Lambda$ the infinite trigonometric series

$$\lambda_i(\phi_i) = 1 + 2\sum_{k=1}^{\infty} \cos(k(\phi_i - a_i)).$$

We will not properly justify here that these series are well defined as our goal is to justify properly in the next section the key identity (4.9) which holds in the finite cut-off case

$$\lambda_i(\phi_i) = 1 + 2\sum_{k=1}^{N} \cos(k(\phi_i - a_i)),$$

with N large.

Let us introduce the following two partition functions in the general case of $\Lambda \subset \mathbb{Z}^2$ with, say, Dirichlet boundary condition:

$$Z(\beta, \mathbf{a}) := \frac{1}{\sqrt{(2\pi\beta^{-1})^{|\Lambda|}\det(-\Delta)^{-1}}} \sum_{m\in\mathbb{Z}^\Lambda} \exp\Big(-\frac{\beta}{2}\sum_{i\sim j}\big(2\pi(m_i - m_j) + a_i - a_j\big)^2\Big)$$

$$\begin{aligned}\tilde{Z}(\beta, \mathbf{a}) &= \sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0) \\ &= \sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} \int \prod_{\rho\in\mathcal{N}} [1 + z(\beta,\rho,\mathcal{N})\cos(\langle\phi,\bar\rho\rangle - \langle\mathbf{a},\rho\rangle)]\, d\mu^{\mathrm{GFF}}_{\beta,\Lambda,v}(\phi).\end{aligned}$$

(As hinted above, $\mathcal{F}$ must be an infinite set of charge configurations here.)

The expansion from Fröhlich–Spencer (in this limiting case) reads as follows: for all $\beta < \beta_0$ and $\mathbf{a} \in \mathbb{R}^\Lambda$,

$$Z(\beta, \mathbf{a}) = \tilde{Z}(\beta, \mathbf{a}). \tag{A.3}$$

Inspired by the analogy with Ising, we now compute, for any $g : \Lambda \to \mathbb{R}$,

$$\begin{aligned}\langle g, \nabla_{\mathbf{a}} \log Z(\beta, \mathbf{a})\rangle &= \sum_{i\in\Lambda} g_i \partial_{a_i} \log Z(\beta, \mathbf{a}) \\ &= -\beta \frac{\sum_{i\in\Lambda} g_i \sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2} \sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)}{\sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2}} \\ &= \beta \frac{\sum_{i\in\Lambda}\sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2} [\Delta(2\pi m + \mathbf{a})]_i g_i}{\sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2}} \\ &= \beta \frac{\sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2} \langle g, \Delta(2\pi m + \mathbf{a})\rangle}{\sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2}}.\end{aligned}$$

Choose (as in [21,27])

$$g = \sigma := \frac{1}{\beta} - \Delta^{-1} f.$$

This gives

$$\begin{aligned}\langle \sigma, \nabla_{\mathbf{a}} \log Z(\beta, \mathbf{a})\rangle &= -\frac{\sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2} \langle f, 2\pi m + \mathbf{a}\rangle}{\sum_{m\in\mathbb{Z}^\Lambda} e^{-\frac{\beta}{2}\sum_{i\sim j}(2\pi(m_i-m_j)+a_i-a_j)^2}} \\ &= -\mathbb{E}^{\mathrm{IV}}_{\beta,\Lambda,\mathbf{a}}[\langle\phi, f\rangle].\end{aligned}$$

Now, from (A.3), we know that for any function $g : \Lambda \to \mathbb{R}$,

$$\langle g, \nabla_{\mathbf{a}} \log Z(\beta, \mathbf{a})\rangle = \langle g, \nabla_{\mathbf{a}} \log \tilde{Z}(\beta, \mathbf{a})\rangle.$$

This implies, with $g = \sigma$, the following formula for $\mathbb{E}^{\mathrm{IV}}_{\beta,\Lambda,\mathbf{a}}[\langle\phi, f\rangle]$:

$$\mathbb{E}^{\mathrm{IV}}_{\beta,\Lambda,\mathbf{a}}[\langle\phi, f\rangle] = -\langle\sigma, \nabla_{\mathbf{a}} \log \tilde{Z}(\beta, \mathbf{a})\rangle.$$

Let us then compute $\nabla_{\mathbf{a}}$ and check that it gives the desired identity:

$$
\begin{aligned}
&\langle \sigma, \nabla_{\mathbf{a}} \log \tilde{Z}(\beta, \mathbf{a}) \rangle \\
&\quad = \frac{1}{\tilde{Z}(\beta, \mathbf{a})} \sum_i \sigma_i \sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} 1_{i \in \rho = \rho_i \in \mathcal{N}} \int z(\beta, \rho, \mathcal{N}) [- \sin(\langle \phi, \bar{\rho} \rangle - \langle \mathbf{a}, \rho \rangle) - \rho_i] \\
&\quad \times \prod_{\rho \in \mathcal{N} \setminus \{\rho_i\}} [1 + z(\beta, \rho, \mathcal{N}) \cos(\langle \phi, \bar{\rho} \rangle - \langle \mathbf{a}, \rho \rangle)] \, d\mu^{\mathrm{GFF}}_{\beta, \Lambda, v}(\phi) \\
&\quad = \frac{1}{\tilde{Z}(\beta, \mathbf{a})} \sum_{\mathcal{N} \in \mathcal{F}} c_{\mathcal{N}} \int \left(\sum_{\rho \in \mathcal{N}} \frac{z(\beta, \rho, \mathcal{N}) \sin(\langle \phi, \bar{\rho} \rangle - \langle \mathbf{a}, \rho \rangle) \langle \sigma, \rho \rangle}{1 + z(\beta, \rho, \mathcal{N}) \cos(\langle \phi, \bar{\rho} \rangle - \langle \mathbf{a}, \rho \rangle)} \right) \\
&\quad \times \prod_{\rho \in \mathcal{N}} [1 + z(\beta, \rho, \mathcal{N}) \cos(\langle \phi, \bar{\rho} \rangle - \langle \mathbf{a}, \rho \rangle)] \, d\mu^{\mathrm{GFF}}_{\beta, \Lambda, v}(\phi)
\end{aligned}
$$

and we thus recover the RHS of (4.9) in the limiting case of infinite trigonometric polynomials at each site.

Let us briefly highlight now the link with Riemann theta functions which we believe illustrates what is beneath the identity (4.9). It is not hard to rewrite the partition function

$$
Z(\beta, \mathbf{a}) = \frac{1}{\sqrt{(2\pi\beta^{-1})^{|\Lambda|} \det(-\Delta)^{-1}}} \sum_{m \in \mathbb{Z}^\Lambda} \exp\left(-\frac{\beta}{2} \sum_{i \sim j} (2\pi(m_i - m_j) + a_i - a_j)^2 \right)
$$

as a theta function in several variables (i.e., a Riemann theta function). The latter generalized theta functions may be defined as follows (see for example [34]): for any $g \geq 1$, $\mathbf{z} = (z_1, \ldots, z_g) \in \mathbb{C}^g$ and a symmetric $g \times g$ complex matrix Ω whose imaginary part is positive definite, set

$$
\theta(\mathbf{z} \mid \Omega) := \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp(\pi i \, \mathbf{m}^T \Omega \mathbf{m} + 2 i \pi \, \mathbf{m} \cdot \mathbf{z}). \tag{A.4}
$$

The Riemann theta functions therefore match exactly our model when

$$
\mathbf{z} := i\beta(-\Delta)\mathbf{a}, \quad \Omega := 2i\pi\beta(-\Delta).
$$

We claim that the identity (4.9) is reminiscent of the suitable log-derivative (i.e. taking $F(\mathbf{a}) \mapsto -\langle \sigma, \nabla_{\mathbf{a}} F(\mathbf{a}) \rangle$) of the modular invariance identity for Riemann theta functions (see for [34, Example 5.1]) which states that

$$
\theta(\Omega^{-1}\mathbf{z} \mid -\Omega^{-1}) = \sqrt{\det(-i\Omega)} \exp(i\pi \mathbf{z}^T \Omega^{-1} \mathbf{z}) \theta(\mathbf{z} \mid \Omega). \tag{A.5}
$$

A.3. Blurring effect of the decomposition into charges

In this subsection, we work with finite cut-off Fourier series (and therefore do not need to worry about convergence of series) and we end our alternative proof of Proposition 4.1.

By the same computation as the one outlined above for the infinite trigonometric series, the RHS in (4.9) is given by

$$
-\langle \sigma, \nabla_{\mathbf{a}} \log \tilde{Z}_N(\beta, \mathbf{a}) \rangle
$$

where $\tilde{Z}_N(\beta, a)$ denotes the rewriting by Fröhlich–Spencer of the finite cut-off partition function (i.e. $\tilde{Z}_N(\beta, \mathbf{a}) = \sum_{\mathcal{N}\in\mathcal{F}} c_{\mathcal{N}} Z^{\mathbf{a}}_{\mathcal{N}}(0)$). The key point here is that it is not clear how to recognize the expectation $\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle]$ from the above log-derivative. To make that identification easier, one should rely instead on an easier expression of $\tilde{Z}_N(\beta, \mathbf{a})$. Indeed, we will instead work with the initial expression of the partition function before subtle expansions into charges are made. Namely, we consider

$$\hat{Z}_N(\beta, \mathbf{a}) := \int \prod_{x\in\Lambda} \Big(1 + 2\sum_{k=1}^N \cos(k(\phi_x - a_x))\Big)\, d\mathbb{P}^{\mathrm{GFF}}_{\beta,\Lambda}$$

and we then compute

$$\begin{aligned}
&-\langle \sigma, \nabla_{\mathbf{a}} \log \tilde{Z}(\beta, \mathbf{a})\rangle = -\langle \sigma, \nabla_{\mathbf{a}} \log \hat{Z}_N(\beta, \mathbf{a})\rangle \\
&= -\sum_{i\in\Lambda} \sigma_i \frac{\mathbb{E}^{\mathrm{GFF}}_{\beta}[2(\sum_{k=1}^N k\sin(k(\phi_i - a_i)))\prod_{x\in\Lambda\setminus i}(1 + 2\sum_{k=1}^N \cos(k(\phi_x - a_x)))]}{\hat{Z}_N(\beta, \mathbf{a})}.
\end{aligned}$$

We now wish to compare this with an expression for $\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle]$:

$$\begin{aligned}
\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle] &= \mathbb{E}^{\mathrm{GFF}}_{\beta,\Lambda}\Big[\prod_{x\in\Lambda}\Big(1 + 2\sum_{k=1}^N \cos(k(\phi_x - a_x))\Big)\langle\phi, f\rangle\Big] \\
&= \sum_{i\in\Lambda} f_i \mathbb{E}^{\mathrm{GFF}}_{\beta,\Lambda}\Big[\prod_{x\in\Lambda}\Big(1 + 2\sum_{k=1}^N \cos(k(\phi_x - a_x))\Big)\phi_i\Big] \\
&= \sum_{i\in\Lambda} f_i \sum_{j\in\Lambda} \langle \phi_i \phi_j\rangle^{\mathrm{GFF}}_{\beta} \mathbb{E}^{\mathrm{GFF}}_{\beta,\Lambda}\Big[\partial_j \prod_{x\in\Lambda}\Big(1 + 2\sum_{k=1}^N \cos(k(\phi_x - a_x))\Big)\Big],
\end{aligned}$$

by Gaussian integration by parts. Continuing, this gives us

$$\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle] = -\sum_{i\in\Lambda} f_i \frac{1}{\beta}(-\Delta)^{-1}(i,j)\Psi(j),$$

where

$$\Psi(j) := 2\mathbb{E}^{\mathrm{GFF}}_{\beta,\Lambda}\Big[\sum_{k=1}^N \sin(k(\phi_j - a_j))k \prod_{x\in\Lambda\setminus j}\Big(1 + 2\sum_{k=1}^N \cos(k(\phi_x - a_x))\Big)\Big].$$

Hence

$$\begin{aligned}
\mathbb{E}^{\mathbf{a}}_{\beta,\Lambda,\lambda_\Lambda,v}[\langle\phi, f\rangle] &= -\sum_{i\in\Lambda} f_i \frac{1}{\beta}[-\Delta^{-1}]\Psi(i) = -\frac{1}{\beta}\langle f, (-\Delta^{-1})\Psi\rangle = -\Big\langle \frac{1}{\beta}(-\Delta^{-1})f, \Psi\Big\rangle \\
&= -\sum_{i\in\Lambda} \sigma_i \mathbb{E}^{\mathrm{GFF}}_{\beta,\Lambda}\Big[2\sum_{k=1}^N \sin(k(\phi_i - a_i))k \prod_{x\in\Lambda\setminus i}\Big(1 + 2\sum_{k=1}^N \cos(k(\phi_x - a_x))\Big)\Big],
\end{aligned}$$

which ends our proof because we have obtained, as desired, the same expression as for $-\langle\sigma, \nabla_{\mathbf{a}} \log \hat{Z}_N(\beta, \mathbf{a})\rangle$. ∎

Appendix B. Link with the random-phase sine-Gordon model

As pointed out to us by Tom Spencer, our work turns out to be closely related to the *random-phase sine-Gordon* model which has been studied extensively in the physics literature [12, 29] and which we now introduce.

Definition B.1. Let $\Lambda \subset \mathbb{Z}^2$ be a finite domain. Let $z \in [0,\infty]$ (this is called the *activity*) and $\mathbf{a} = \{a_i\}_{i\in\Lambda}$ be a *quenched disorder* on the vertices given by i.i.d. random variables a_i uniform in $[0, 2\pi)$.

We equip the domain Λ with either Dirichlet or free boundary conditions. The *random-phase sine-Gordon model* is the following quenched disorder probability measure on fields $\{\phi_i\}_{i\in\Lambda}$:

$$\mathbb{P}^{\mathbf{a}\text{-SG}}_{\beta,z,\Lambda}[d\phi] := \frac{1}{Z^{\mathbf{a}\text{-SG}}_{\beta,z,\Lambda}} \exp\Big(z\sum_{i\in\Lambda}\cos(\phi_i - a_i)\Big)\mathbb{P}^{\mathrm{GFF}}_{\beta,\Lambda}[d\phi].$$

Remark B.2. (1) Note that if we let $z \to \infty$, the measure $\mathbb{P}^{\mathbf{a}\text{-SG}}_{\beta,\Lambda}$ converges to the $\mathbf{a}$-shifted IV-GFF on Λ (with $\mathbf{a} \in [0, 2\pi)^\Lambda$).

(2) If $z \to \infty$ and if the disorder $\mathbf{a}$, instead of being uniform in $[0, 2\pi)^\Lambda$, is sampled as follows:

$$\mathbf{a} := \varphi \ (\mathrm{mod}\ 2\pi) \quad \text{with } \varphi \sim \mathbb{P}^{\mathrm{GFF}}_{\beta},$$

then the annealed law $\int \mathbb{P}(d\mathbf{a})\mathbb{P}^{\mathbf{a}\text{-SG}}_{\beta,\infty,\Lambda}[d\phi]$ is very simple and is given by $\mathbb{P}^{\mathrm{GFF}}_{\beta}[d\phi]$.

When the disorder $\mathbf{a}$ is uniform, it turns out that the annealed law is very different from the law of a GFF. Indeed, in a series of works including [12, 29], the following *roughening/super-roughening* phase transition has been predicted:

- If the temperature is high enough, then on large domains $\Lambda_n := \{-n,\dots,n\}^2$, it is predicted that the random phase sine-Gordon model will fluctuate as a GFF, namely for any fixed $z \in [0,\infty]$ and β small enough,

$$\mathbb{E}_{\mathbf{a}}\big[\mathbb{E}^{\mathbf{a}-\mathrm{SG}}_{\beta,z,\Lambda_n}[\phi^2(0)]\big] \asymp_{n\to\infty} \log n.$$

- On the other hand, if the temperature is low enough, fluctuations are predicted to be larger! The following super-roughening behavior is predicted (see [12, 29]): for any fixed positive activity $z \in (0,\infty]$ (note that here $z > 0$ is required) and β high enough,

$$\mathbb{E}_{\mathbf{a}}\big[\mathbb{E}^{\mathbf{a}-\mathrm{SG}}_{\beta,z,\Lambda_n}[\phi^2(0)]\big] \asymp_{n\to\infty} (\log n)^2.$$

Our present work does not allow us to investigate the more surprising low temperature phase with expected $(\log n)^2$ variances (see for example Remark 1.9). Yet, it enables us to prove rigorously that the fluctuations for the *random-phase sine-Gordon model* in the high temperature regime are at least as large as for a GFF. (Note that with the quenched disorder $\mathbf{a}$, one cannot rely on classical correlation inequalities such as Ginibre.) Namely, a very mild generalization of the proof of Theorem 1.8 yields the following result (which also clarifies the link between high enough temperature and the choice of activity z).

For simplicity, we state our result for zero boundary conditions around Λ_n (but the analogous statement also holds for free boundary conditions by considering $\phi(a) - \phi(b)$ for two distant points a, b in the bulk).

Theorem B.3. *There exists β_0 such for all $\beta < \beta_0$ and all $z \in [0, \infty]$, uniformly in the disorder $\mathbf{a} \in [0, 2\pi)^{\Lambda_n}$, we have*

$$\mathrm{Var}^{\mathbf{a}\text{-SG}}_{\beta,z,\Lambda_n}[\phi(0)] \geq \Omega(1) \log n.$$

This implies in particular the following lower bound for the fluctuations of random phase sine-Gordon (i.e. with a quenched disorder $\mathbf{a} \sim$ i.i.d.) when $\beta < \beta_0$:

$$\mathbb{E}_{\mathbf{a}}\big[\mathbb{E}^{\mathbf{a}\text{-SG}}_{\beta,z,\Lambda_n}[\phi^2(0)]\big] \geq \Omega(1) \log n.$$

Sketch of proof. Since Theorem 1.8 is stated uniformly in the disorder $\mathbf{a}$, it implies the limiting case $z = \infty$. It remains to notice that the proof also handles the case of finite activities $z \in [0, \infty)$ using the following minor modifications. Indeed, recall that we wrote the proof of Theorem 1.8 for general trigonometric polynomials

$$\lambda_i(\phi) = 1 + 2 \sum_{q=1}^{N} \hat{\lambda}_i(q) \cos(q\phi(i)),$$

where the set of weights $\lambda_\Lambda = (\lambda_i)_{i \in \Lambda}$ is assumed to satisfy the same hypothesis as in [21, (5.35)] (or equivalently [27, (1.9)]).

In our present setting, at any site $i \in \Lambda$, we need to work with the following periodic function:

$$\phi_i \mapsto e^{z\cos(\phi_i - a_i)} = \sum_{q \in \mathbb{Z}} \alpha(q) \cos(q(\phi_i - a_i))$$

with

$$\alpha(q) = \frac{1}{2\pi} \int_0^{2\pi} e^{-iq\theta} e^{z\cos\theta}\, d\theta.$$

It is sufficient to notice that $0 < \alpha(0) \leq e^z$ and $|\alpha(q)| \leq \alpha(0)$ for each $q \in \mathbb{Z}$. Indeed, we may rewrite our periodic function as follows:

$$e^{z\cos(\phi_i - a_i)} = \alpha(0)\Big(1 + 2\sum_{q \geq 1} \hat{\lambda}(q) \cos(q(\phi_i - a_i))\Big),$$

where $|\hat{\lambda}(q)| \leq 1$. Since the conditions on $\hat{\lambda}(q)$ in [21, 27] are conditions on the growth of these coefficients, it is immediate to see that this "sine-Gordon" trigonometric polynomial satisfies the conditions required to run the same proof as for Theorem 1.8. To fully match the setup in that proof, one can absorb the multiplicative constant $\alpha(q)$ in the trigonometric polynomial into the partition function without any impact on the fluctuations. Also, we wrote the proof as in [27], with a cut-off N on large frequencies (i.e. looking at $1 + 2\sum_{q=1}^{N} \hat{\lambda}(q)$). The same limiting argument $N \to \infty$ as in [27] applies here. The rest of the proof (in particular the analysis of the first and second moments in Section 4.1) is identical in this setting. This ends this extension of Theorem 1.8 to the case of the random-phase sine-Gordon model. ∎

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