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Volume entropy semi-norm and systolic volume semi-norm

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Abstract. We introduce the volume entropy semi-norm and the systolic volume semi-norm in real homology and show that they satisfy functorial properties similar to the ones of the simplicial volume. Along the way, we also establish a roughly optimal upper bound on the systolic volume of the multiples of any homology class. Finally, we prove that the volume entropy semi-norm, the systolic volume semi-norm and the simplicial volume semi-norm are equivalent in every dimension.

Keywords. Volume entropy, simplicial volume, functorial geometric semi-norms, systolic volume

1. Introduction

This article deals with topology-geometry interactions and the comparison of functorial geometric semi-norms on the real homology groups of topological spaces. In his book [27, Section 5, G_+-H_+], M. Gromov pointed out directions where such geometric semi-norms might arise in connection with curvature (e.g., sectional, Ricci, scalar) properties for instance. Yet, the corresponding invariants have not been properly defined or studied as semi-norms, except for the simplicial volume which is a purely topological invariant. (General comparison results between functorial topological semi-norms in relation to the simplicial volume semi-norm have recently been established in [15, 19].) As part of this program to investigate the interactions between geometry and topology through the study of functorial geometric semi-norms, we introduce the volume entropy semi-norm and the systolic volume semi-norm in real homology and carry out a systematic study of these invariants. Both semi-norms require a substantial amount of work in order to properly define them. The volume entropy semi-norm relies on the notion of volume entropy, a geometric invariant of considerable interest closely related to the dynamics of the geodesic flow and the growth of the fundamental groups. The systolic

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volume semi-norm rests on a new asymptotically optimal estimate in systolic geometry. Both the volume entropy semi-norm and the systolic volume semi-norm share similar functorial properties with the simplicial volume semi-norm (also called the Gromov semi-norm). The equivalence of the three semi-norms in real homology is established in this article.

Let M be a connected closed m-dimensional manifold with a Riemannian metric g. Let $H \lhd \pi_1(M)$ be a normal subgroup of the fundamental group of M. The *volume entropy* (or simply *entropy*) of (M, g) relative to H, denoted by $\mathrm{ent}_H(M, g)$, is the exponential growth rate of the volume of balls in the Riemannian covering M_H corresponding to the normal subgroup $H \lhd \pi_1(M)$, that is, $\pi_1(M_H) = H$. More precisely, it is defined as

$$\mathrm{ent}_H(M, g) = \lim_{R\to\infty} \frac{1}{R} \log[\mathrm{vol}\, B_H(R)] \tag{1.1}$$

where $B_H(R)$ is a ball of radius R centered at any point in the covering M_H. The limit exists and does not depend on the center of the ball. When H is trivial, the covering M_H is the universal covering $\widetilde{M}$ of M and we simply denote its volume entropy by $\mathrm{ent}(M, g)$ without any reference to H. Note that

$$\mathrm{ent}_H(M, g) \leq \mathrm{ent}(M, g)$$

for every normal subgroup $H \lhd \pi_1(M)$. The definition extends to connected closed pseudomanifolds (see Definition 2.1), to connected finite graphs, and more generally to finite simplicial complexes with a length metric.

The importance of this notion was first noticed by Efremovich [38]. Subsequently, Shvarts [46] and Milnor [37] related the growth of the volume of balls in the universal covering $\widetilde{M}$ to the growth of the fundamental group $\pi_1(M)$ of M. Note that the volume entropy of a connected closed Riemannian manifold is positive if and only if its fundamental group has exponential growth. The connection with the dynamics of the geodesic flow was established by Dinaburg [18] and Manning [35]. More specifically, the volume entropy bounds from below the topological entropy of the geodesic flow on a connected closed Riemannian manifold and the two invariants coincide when the manifold is nonpositively curved; see [35].

The *minimal volume entropy* of a closed m-pseudomanifold M relative to a normal subgroup $H \lhd \pi_1(M)$ is defined as

$$\omega_H(M) = \inf_g \mathrm{ent}_H(M, g)\, \mathrm{vol}(M, g)^{1/m}$$

where g runs over the space of all piecewise Riemannian metrics on M. For convenience, we also introduce

$$\Omega_H(M, g) = \mathrm{ent}_H(M, g)^m\, \mathrm{vol}(M, g)$$

and

$$\Omega_H(M) = \inf_g \mathrm{ent}_H(M, g)^m\, \mathrm{vol}(M, g)$$

where g runs over the space of all piecewise Riemannian metrics on M. As previously, if H is trivial, we drop the subscript H.

As an example, the minimal volume entropy of a closed m-manifold M which carries a hyperbolic metric is attained by the hyperbolic metric and is equal to $(m-1)\operatorname{vol}(M,\mathrm{hyp})^{1/m}$; see [8, 30] for $m = 2$ and [9] for $m \geq 3$. Furthermore, the minimal volume entropy of a closed manifold which carries a negatively curved metric is positive; see [24].

For a connected closed orientable m-manifold M, the minimal volume entropy of M is a homotopy invariant (see [1]), which only depends on the image $h_*([M])$ in $H_m(\pi_1(M);\mathbb{Z})$ of the fundamental class of M by the homomorphism induced by the classifying map $h : M \to K(\pi_1(M), 1)$ of M in homology; see [13].

This homological invariance leads us to consider the volume entropy of a homology class as follows. Given a path-connected topological space X, the *volume entropy of a homology class* $\mathbf{a} \in H_m(X;\mathbb{Z})$ is defined as

$$\omega(\mathbf{a}) = \inf_{(M,f)} \omega_{\ker f_*}(M) \tag{1.2}$$

where the infimum is taken over all m-dimensional *geometric cycles* (M, f) representing $\mathbf{a}$, that is, over all maps $f : M \to X$ from an oriented connected closed m-pseudo-manifold M to X such that $f_*([M]) = \mathbf{a}$. As previously, we define

$$\Omega(\mathbf{a}) = \omega(\mathbf{a})^m.$$

For every map $f : X \to Y$ between two path-connected topological spaces and every $\mathbf{a} \in H_m(X;\mathbb{Z})$, we have

$$\Omega(f_*(\mathbf{a})) \leq \Omega(\mathbf{a}). \tag{1.3}$$

By [13, Theorem 10.2], every orientable connected closed m-manifold M with $m \geq 3$ satisfies

$$\Omega(M) = \Omega(h_*([M])) \tag{1.4}$$

where $h : M \to K(\pi_1(M), 1)$ is the classifying map of M.

The following result shows that Ω induces a pseudo-distance in homology.

Theorem 1.1. *Let X be a path-connected topological space. Then for all $\mathbf{a}, \mathbf{b}$ in $H_m(X;\mathbb{Z})$, we have*

$$\Omega(\mathbf{a} + \mathbf{b}) \leq \Omega(\mathbf{a}) + \Omega(\mathbf{b}).$$

In particular, the quantity $\Omega(\mathbf{a} - \mathbf{b})$ defines a pseudo-distance between $\mathbf{a}$ and $\mathbf{b}$ in $H_m(X;\mathbb{Z})$,

Thus, for every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$, the sequence $\Omega(k\mathbf{a})$ is subadditive. As a result, we can apply the following stabilization process and define

$$\|\mathbf{a}\|_E = \lim_{k\to\infty} \frac{\Omega(k\mathbf{a})}{k}. \tag{1.5}$$

Note that $\|\cdot\|_E$ is homogeneous, that is, $\|k\mathbf{a}\|_E = |k|\,\|\mathbf{a}\|_E$ for every $k \in \mathbb{Z}$. By homogeneity and density of $H_m(X;\mathbb{Q})$ in $H_m(X;\mathbb{R})$, this functional extends to a functional on $H_m(X;\mathbb{R})$, still denoted by $\|\cdot\|_E$.

For an orientable connected closed m-manifold M, define

$$\|M\|_E = \|[M]\|_E$$

where $[M] \in H_m(M;\mathbb{Z})$ is the fundamental class of M.

The following result, which is a direct consequence of Theorem 1.1, justifies the use of the term *volume entropy semi-norm* to designate the functional $\|\cdot\|_E$.

Corollary 1.2. *Let X be a path-connected topological space. Then the functional $\|\cdot\|_E$ is a semi-norm on $H_m(X;\mathbb{R})$.*

Functorial properties of the volume entropy semi-norm are described in Section 3, where it is shown that the volume entropy semi-norm of a closed orientable manifold depends only on the image of its fundamental class under the classifying map.

The simplicial volume is a much-studied topological invariant sharing similar properties with the volume entropy semi-norm. Let us recall its definition and its basic properties, referring to [24] for foundational constructions and results regarding this invariant. Let X be a path-connected topological space. Every real singular m-chain $c \in C_m(X;\mathbb{R})$ of X is a real linear combination of singular simplices $f_s : \Delta^m \to X$, that is,

$$c = \sum_s r_s f_s$$

where $r_s \in \mathbb{R}$. The ℓ_1-*norm* on the real chain complex is defined as

$$\|c\|_1 = \sum_s |r_s|.$$

The *simplicial volume* of a real homology class $\mathbf{a} \in H_m(X;\mathbb{R})$ is defined as

$$\|\mathbf{a}\|_\Delta = \inf_c \|c\|_1$$

where the infimum is taken over all real singular m-cycles c representing $\mathbf{a}$. The simplicial volume of an integral homology class is defined as the simplicial volume of the corresponding real homology class. It is clear that the simplicial volume $\|\cdot\|_\Delta$ is a functorial semi-norm on $H_m(X;\mathbb{R})$. This means that the real homology of a topological space with its simplicial volume semi-norm defines a functor from the category of topological spaces (whose morphisms are continuous maps) to the category of semi-normed vector spaces (whose morphisms are semi-norm-nonincreasing homomorphisms); see [10]. In other words, every continuous map between topological spaces induces a semi-norm-nonincreasing homomorphism in real homology.

As previously, for an orientable connected closed m-manifold M, we let

$$\|M\|_\Delta = \|[M]\|_\Delta$$

where $[M] \in H_m(M;\mathbb{Z})$ is the fundamental class of M. By [24, Section 3.1], the simplicial volume of M depends only on the image of its fundamental class under the classifying map.

The following inequality of M. Gromov [24, p. 37] connects the minimal volume entropy of an orientable connected closed manifold to its simplicial volume (see [41] for other topological conditions ensuring the positivity of the minimal volume entropy through a different approach). Namely, every orientable connected closed m-manifold M satisfies

$$\Omega(M) \geq c_m \|M\|_\Delta \tag{1.6}$$

for some positive constant c_m depending only on m. Extending this inequality to the semi-norm level (see Theorem 4.12), we obtain that every homology class $\mathbf{a} \in H_m(X;\mathbb{R})$ of a path-connected topological space X satisfies

$$\|\mathbf{a}\|_E \geq c_m \|\mathbf{a}\|_\Delta$$

with the same constant c_m as in (1.6).

A central question regarding the metrization of homotopy theory is to compare two given semi-norms in homology; see [27, Section 5.41]. In particular, one can ask whether a reverse inequality to (1.6) holds.

The following result affirmatively answers this question.

Theorem 1.3. *Let m be a positive integer. Then there exist two positive constants c_m and C_m such that every homology class $\mathbf{a} \in H_m(X;\mathbb{R})$ of a path-connected topological space X satisfies*

$$c_m \|\mathbf{a}\|_\Delta \leq \|\mathbf{a}\|_E \leq C_m \|\mathbf{a}\|_\Delta.$$

We immediately deduce the following corollary.

Corollary 1.4. *Let X be a path-connected topological space and $\mathbf{a} \in H_m(X;\mathbb{R})$ be a homology class. Then $\|\mathbf{a}\|_E$ vanishes if and only if $\|\mathbf{a}\|_\Delta$ vanishes.*

In particular, for every orientable connected closed manifold M, the volume entropy semi-norm $\|M\|_E$ is zero if and only if the simplicial semi-norm $\|M\|_\Delta$ is zero.

In relation with Corollary 1.4, note that we do not know whether the volume entropy of an orientable connected closed manifold with zero simplicial volume necessarily vanishes. See [6] for polyhedral counterexamples.

In this article, we also introduce the systolic volume semi-norm, whose definition rests on a new asymptotically optimal estimate in systolic geometry; see Theorem 1.5. Before stating this result, we need to introduce various notions.

Let M be a closed m-dimensional manifold or pseudomanifold with a (piecewise) Riemannian metric g. Let $f : M \to X$ be a map to a topological space X. The *systole* of M relative to f, denoted by $\mathrm{sys}_f(M,g)$, is defined as the least length of a loop γ in M whose image by f is noncontractible in X. The *systolic volume* of M relative to f is defined as

$$\sigma_f(M) = \inf_g \frac{\mathrm{vol}(M,g)}{\mathrm{sys}_f(M,g)^m} \tag{1.7}$$

where the infimum is taken over all (piecewise) Riemannian metrics g on M. When $f : M \to X$ is π_1-injective, for instance, when $f : M \to K(\pi_1(M),1)$ is the classifying map of M, we simply denote its systolic volume by $\sigma(M)$ without any reference to f. By [2, 3, 13], the systolic volume of a closed orientable manifold depends only on the image of its fundamental class under the classifying map.

As with (1.2), the *systolic volume of a homology class* $\mathbf{a} \in H_m(X;\mathbb{Z})$, where X is a path-connected topological space, is defined as

$$\sigma(\mathbf{a}) = \inf_{(M,f)} \sigma_f(M) \tag{1.8}$$

where the infimum is taken over all m-dimensional geometric cycles (M, f) representing $\mathbf{a}$.

Let us present some known estimates on the systolic volume. There exist positive constants A and B such that every closed genus g surface Σ_g satisfies

$$A \frac{g}{(\log g)^2} \leq \sigma(\Sigma_g) \leq B \frac{g}{(\log g)^2}.$$

The first inequality was established by M. Gromov [25, 26]. The second inequality was proved by P. Buser and P. Sarnak [14], who constructed hyperbolic genus g surfaces with a systole roughly equal to $\log g$.

In higher dimension, M. Gromov [25,26] related the systolic volume $\sigma(\mathbf{a})$ of a homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ to its simplicial volume $\|\mathbf{a}\|_\Delta$ through the lower bound

$$\sigma(\mathbf{a}) \geq \lambda_m \frac{\|\mathbf{a}\|_\Delta}{(\log(2 + \|\mathbf{a}\|_\Delta))^m} \tag{1.9}$$

where λ_m is a positive constant depending only on m. In particular, for every $\mathbf{a}$ in $H_m(X;\mathbb{Z})$ with nonzero simplicial volume, we have

$$\sigma(k\mathbf{a}) \geq \lambda \frac{k}{(\log k)^m} \tag{1.10}$$

where $\lambda = \lambda(\mathbf{a}) > 0$.

In a different direction, one can ask for an asymptotic upper bound on $\sigma(k\mathbf{a})$. This problem was considered in [4], where a sublinear upper bound in k was established, and in [5], where the upper bound was improved.

Using different techniques, we obtain an asymptotically optimal upper bound on $\sigma(k\mathbf{a})$. When the simplicial volume of $\mathbf{a}$ is nonzero, this upper bound shows that the lower bound (1.10) is roughly optimal in k, which positively answers a conjecture of [5].

Theorem 1.5. *Let X be a path-connected topological space. Then for every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$, there exists a constant $C = C(\mathbf{a}) > 0$ such that for every integer $k \geq 2$, we have*

$$\sigma(k\mathbf{a}) \leq C \frac{k}{(\log k)^m}.$$

This estimate allows us to define the systolic volume semi-norm in real homology of dimension $m \geq 3$ as follows. By [5, Corollary 5.3], the systolic volume induces a translation-invariant pseudo-distance ϱ on $H_m(X;\mathbb{Z})$ with $m \geq 3$, defined by $\varrho(\mathbf{a},\mathbf{b}) = \sigma(\mathbf{a}-\mathbf{b})$. Define a new translation-invariant pseudo-distance $\widehat{\varrho}$ on $H_m(X;\mathbb{Z})$ by

$$\widehat{\varrho}(\mathbf{a},\mathbf{b}) = \limsup_{k\to\infty} \frac{(\log k)^m}{k} \varrho(k\mathbf{a},k\mathbf{b}).$$

See Lemma 6.1 for further detail. Denote by

$$\widehat{\sigma}(\mathbf{a}) = \widehat{\varrho}(0,\mathbf{a})$$

the distance from the origin and apply a stabilization process to $\widehat{\sigma}$ as in (1.5). Namely, for every $\mathbf{a} \in H_m(X;\mathbb{Z})$ with $m \geq 3$, define

$$\|\mathbf{a}\|_\sigma = \lim_{k\to\infty} \frac{\widehat{\sigma}(k\mathbf{a})}{k}. \tag{1.11}$$

This functional extends to $H_m(X;\mathbb{R}) \simeq H_m(X;\mathbb{Z}) \otimes \mathbb{R}$ in a canonical way and gives rise to a semi-norm, still denoted by $\|\cdot\|_\sigma$, on $H_m(X;\mathbb{R})$, called the *systolic volume semi-norm*. Note that this definition differs from the one proposed in [27, Section 5.41].

For an orientable connected closed m-manifold M, define

$$\|M\|_\sigma = \|[M]\|_\sigma$$

where $[M] \in H_m(M;\mathbb{Z})$ is the fundamental class of M.

The systolic volume semi-norm satisfies similar functorial properties to the volume entropy semi-norm and the simplicial volume semi-norm; see Theorem 6.4.

It follows from (1.9) that every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ of a path-connected topological space X satisfies

$$\|\mathbf{a}\|_\sigma \geq \lambda_m \|\mathbf{a}\|_\Delta$$

with the same positive constant λ_m as in (1.9); see Section 7 for further detail and an alternative approach based on a comparison between the systolic volume semi-norm and the volume entropy semi-norm.

As previously, we show that the systolic volume semi-norm and the simplicial volume semi-norm are equivalent in homology.

Theorem 1.6. *Let $m \geq 3$ be an integer. Then there exist two positive constants λ_m and μ_m such that every homology class $\mathbf{a} \in H_m(X;\mathbb{R})$ of a path-connected topological space X satisfies*

$$\lambda_m \|\mathbf{a}\|_\Delta \leq \|\mathbf{a}\|_\sigma \leq \mu_m \|\mathbf{a}\|_\Delta.$$

Theorem 1.6 contrasts with the existence of a sequence of closed m-manifolds (e.g., closed hyperbolic 3-manifolds) with bounded simplicial volume and arbitrarily large systolic volume; see [40]. This illustrates the effect the double stablization process can have on the systolic volume by significantly lowering its value.

Combining the recent result [29] on the spectrum of the simplicial volume with Theorem 1.3, we immediately deduce that the volume entropy semi-norm and the systolic volume semi-norm are not bounded away from zero in dimension greater than 3. (In dimensions 2 and 3, there is a gap in the simplicial volume spectrum, and so in the volume entropy spectrum and the systolic volume spectrum by Theorem 1.3 and Theorem 1.6.) More generally, we have the following result.

Corollary 1.7. *Let $m \geq 4$ be an integer. Then the sets of all volume entropy semi-norms $\|M\|_E$ and of all systolic volume semi-norms $\|M\|_\sigma$, where M is an orientable connected closed m-manifold, are dense in* $[0, \infty)$.

In his book [27, Section 5.41], M. Gromov suggests studying some functionals of geometric nature in homology. These functionals measure the minimal volume of a singular Riemannian manifold representing a given homology class with some constraint on the metric. After a stabilization process as in (1.5) or (1.11), they should give rise to homology semi-norms. Our definitions of the volume entropy semin-norms and the systolic volume semi-norm are inspired by this general idea. However, they differ from the constructions sketched in [27, pp. 310–311], which do not consider relative volume entropy or relative systole and lead to a number of technical difficulties.

Articles about minimal volume entropy closely related to our paper include [1–3, 8, 9, 13, 24, 36, 39, 41–43]. Connections between the systolic volume and the minimal volume entropy can be found in [13, 31, 39].

This article is organized as follows. In Section 2, we establish lower and upper bounds on the minimal volume entropy of the connected sum of closed manifolds, and derive that the functional $\|\cdot\|_E$ is a semi-norm in real homology. Functorial properties of the volume entropy semi-norm are presented in Section 3. In Section 4, we show that the volume entropy semi-norm of a homology class is bounded from above and below by its simplicial volume, up to some multiplicative constants depending only on the degree of the homology class. Therefore, the volume entropy semi-norm and the simplicial volume are equivalent homology semi-norms. Our approach for the upper bound relies on a geometrization of the simplicial volume and the universal realization of homology classes established by A. Gaifullin [21, 22] regarding Steenrod's problem. More than the result about the universal realization of homology classes, we will need to retrieve combinatorial features of the construction to apply our argument leading to an upper bound on the volume entropy semi-norm of a homology class. The reverse inequality is obtained through the use of bounded cohomology by adapting M. Gromov's chain diffusion technique. In Section 5, we bound from above the systolic volume of the multiple of a given homology class. The proof relies on topological properties of the universal realizators in homology used in the previous section and on systolic estimates in geometric group theory. This optimal asymptotic estimate allows us to define the systolic volume semi-norm in Section 6. Functorial properties and comparison results for the systolic volume (semi-norm) are also presented. In Section 7, we show that the systolic volume semi-norm and the simplicial volume semi-norm are equivalent in every homology degree. In the last section, we derive the density of the volume entropy and systolic volume semi-norm spectra in dimension at least 4.

2. Entropy of connected sums

In this section, we first establish an additive formula for the functional Ω of the bouquet of simplicial complexes. We also obtain lower and upper bounds on the minimal volume entropy of the connected sum of two closed manifolds, and derive that the functional $\|\cdot\|_E$ is a semi-norm in real homology. Finally, we present a couple of applications of these estimates.

2.1. Preliminaries

Let us first recall the definition of a pseudomanifold.

Definition 2.1. A connected closed m-dimensional *pseudomanifold* is a finite simplicial complex M such that

(1) every simplex of M is a face of some m-simplex of M;

(2) every $(m-1)$-simplex of M is the face of exactly two m-simplices of M;

(3) given two m-simplices s and s' of M, there exists a finite sequence $s = s_1, s_2, \dots, s_n = s'$ of m-simplices of M such that s_i and s_{i+1} have an $(m-1)$-face in common.

The mth homology group $H_m(M;\mathbb{Z})$ of a connected closed m-dimensional pseudomanifold is either isomorphic to $\mathbb{Z}$ or trivial; see [47]. In the former case, we say that the pseudomanifold M is *orientable*.

Consider a finite simplicial complex K with a piecewise Riemannian metric g (also called polyhedral Riemannian metric). Denote by ρ the distance induced by g on K and on all the coverings of K. Let $H \lhd G$ where $G = \pi_1(K)$. The quotient group G/H acts by isometries on the H-covering K_H. Furthermore, the action of G/H on K_H is proper, discontinuous, without any fixed point. Fix $q \in K_H$. The orbit of q under the action of G/H on K_H is denoted by $q \cdot (G/H)$. Let also

$$B_H(t, q; g) = \{x \in K_H \mid \rho(q, x) \le t\}$$

be the ball of radius t centered at q in K_H.

The volume entropy of K relative to H is equal to the exponential growth rate of the number of points in the orbit of q under G/H, as stated in the following classical result; see [39, Lemma 2.3] for instance.

Proposition 2.2. *Let K be a finite simplicial complex with a piecewise Riemannian metric. Let $H \lhd G$ where $G = \pi_1(K)$. Then*

$$\mathrm{ent}_H(K, g) = \lim_{t\to\infty} \frac{1}{t} \log |B_H(t, q; g) \cap q \cdot (G/H)|. \tag{2.1}$$

2.2. Minimal volume entropy of a bouquet of simplicial complexes

Let us recall a few results established in [1]; see also [39, Lemma 3.5].

Definition 2.3. A simplicial map $f : K_1 \to K_2$ between two m-dimensional simplicial complexes is *m-monotone* if for every point x_2 in the interior of an m-simplex of K_2, the preimage $f^{-1}(x_2)$ is connected (and so is either empty or a singleton).

We will need the following comparison principle proved in [1, Section 2].

Proposition 2.4. *For $i = 1, 2$, let K_i be an m-dimensional simplicial complex and $\phi_i : \pi_1(K_i) \to G$ be an epimorphism. Suppose that there exists an m-monotone map $f : K_1 \to K_2$ such that $\phi_1 = \phi_2 \circ f_*$. Then*

$$\Omega_{H_1}(K_1) \leq \Omega_{H_2}(K_2) \quad \textit{where} \quad H_i = \ker \phi_i.$$

Actually, Proposition 2.4 is a straighforward consequence of the following result proved in [1, Section 2] and [2, Lemme 3.1], which will also be used in what follows.

Lemma 2.5. *Let $f : K_1 \to K_2$ be an m-monotone map between two m-dimensional simplicial complexes. Then for every polyhedral Riemannian metric g on K_2 and every $\varepsilon > 0$, there exists a polyhedral Riemannian metric g_ε on K_1 with*

$$\mathrm{vol}(K_1, g_\varepsilon) \leq \mathrm{vol}(K_2, g) + \varepsilon$$

such that f is nonexpanding.

Let G be a finitely presented group. For every subgroup H of G, denote by $\langle\langle H \rangle\rangle$ the normal closure of H in G.

The following result provides a formula for the minimal volume entropy of the bouquet of two simplicial complexes.

Theorem 2.6. *Let $m \geq 2$. For $i = 1, 2$, let K_i be a connected m-dimensional simplicial complex and $H_i \lhd \pi_1(K_i)$ be a normal subgroup. Then*

$$\Omega_{\langle\langle H_1 * H_2 \rangle\rangle}(K_1 \vee K_2) = \Omega_{H_1}(K_1) + \Omega_{H_2}(K_2) \tag{2.2}$$

where the basepoint of the bouquet $K_1 \vee K_2$ is a vertex.

Proof. First we prove the inequality

$$\Omega_{H_1}(K_1) + \Omega_{H_2}(K_2) \leq \Omega_{\langle\langle H_1 * H_2 \rangle\rangle}(K_1 \vee K_2). \tag{2.3}$$

Let $K = K_1 \vee K_2$. By van Kampen's theorem [28, Section 1.2], we have

$$\pi_1(K) \simeq \pi_1(K_1) * \pi_1(K_2).$$

Let g be a polyhedral Riemannian metric on K and g_i be its restriction to K_i for $i = 1, 2$. Let $\widehat{K}_i$ and $\widehat{K}$ be the normal covers corresponding to the normal subgroups $H_i \lhd \pi_1(K_i)$ and $\langle\langle H_1 * H_2 \rangle\rangle \lhd \pi_1(K)$, with the lifted metrics $\widehat{g}_i$ and $\widehat{g}$. Observe that the canonical inclusions $\widehat{K}_i \subseteq \widehat{K}$ are isometric. This implies

$$\mathrm{ent}_{H_i}(K_i, g_i) \leq \mathrm{ent}_{\langle\langle H_1 * H_2 \rangle\rangle}(K, g).$$

Thus, for every $i = 1, 2$,

$$\Omega_{H_i}(K_i) \leq \mathrm{ent}_{H_i}(K_i, g_i)^m \operatorname{vol}(K_i, g_i) \leq \mathrm{ent}_{\langle\langle H_1 * H_2\rangle\rangle}(K, g)^m \operatorname{vol}(K_i, g_i).$$

Adding the two inequalities so obtained for $i = 1, 2$, and using the relation

$$\operatorname{vol}(K, g) = \operatorname{vol}(K_1, g_1) + \operatorname{vol}(K_2, g_2),$$

we finally derive

$$\Omega_{H_1}(K_1) + \Omega_{H_2}(K_2) \leq \Omega_{\langle\langle H_1 * H_2\rangle\rangle}(K, g)$$

for every polyhedral Riemannian metric g on K. This yields inequality (2.3).

Now, let us prove the reverse inequality

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(K_1 \vee K_2) \leq \Omega_{H_1}(K_1) + \Omega_{H_2}(K_2). \tag{2.4}$$

We proceed in two steps. Without loss of generality, we can assume that the two subcomplexes K_1 and K_2 of K are glued at a common vertex. Let $p_i \in K_i$ be a vertex such that

$$K_1 \vee K_2 = K_1 \bigcup_{p_1 = p_2} K_2.$$

Define the m-dimensional simplicial complex

$$P = K_1 \bigcup_{p_1 = \{1\}} [1, 2] \bigcup_{\{2\} = p_2} K_2. \tag{2.5}$$

For the first step, let us show that

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(P) = \Omega_{\langle\langle H_1 * H_2\rangle\rangle}(K_1 \vee K_2).$$

Contracting the interval $[1, 2]$ in P to a point gives rise to an m-monotone simplicial map

$$P \to K_1 \vee K_2$$

inducing a π_1-isomorphism. By Proposition 2.4, we derive

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(P) \leq \Omega_{\langle\langle H_1 * H_2\rangle\rangle}(K_1 \vee K_2). \tag{2.6}$$

For the reverse inequality, let θ_i be a triangulation of K_i for every $i = 1, 2$. Denote by $\mathrm{St}(p_i)$ the open star of p_i for the triangulation θ_i. Let θ_i' be the triangulation of K_i which agrees with θ_i in $K_i \setminus \mathrm{St}(p_i)$ and with the semi-barycentric triangulation of $\mathrm{St}(p_i)$ in $\mathrm{St}(p_i)$ (obtained by adding a vertex at the barycenter of every simplex of $\mathrm{St}(p_i)$). The bouquet $K_1 \vee K_2$ is endowed with the triangulations given by θ_1' and θ_2'. The complex $P = K_1 \cup [1, 2] \cup K_2$ is endowed with the triangulation given by θ_1, θ_2 and the barycentric subdivision of $[1, 2]$ into $I_1 = [1, 3/2]$ and $I_2 = [3/2, 2]$.

Consider the simplicial map

$$f : K_1 \vee K_2 \to P$$

which agrees with the identity map on $K_i \setminus \mathrm{St}(p_i)$, and takes p_i to the midpoint of $[1,2]$ and all the vertices of θ_i' corresponding to the barycenters of the simplices of $\mathrm{St}(p_i)$ for the triangulation θ_i to i. By construction, the map f is m-monotone and induces a π_1-isomorphism.

The inequality obtained by applying Proposition 2.4 to f, combined with the inequality (2.6), yields the relation

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(P) = \Omega_{\langle\langle H_1 * H_2\rangle\rangle}(K_1 \vee K_2). \tag{2.7}$$

For the second step, we need to show that

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(P) \leq \Omega_{H_1}(K_1) + \Omega_{H_2}(K_2). \tag{2.8}$$

Fix $\beta_i > \Omega_{H_i}(K_i)^{1/m}$. By definition, there exists a metric h_i on K_i such that

$$\mathrm{ent}_{H_i}(K_i, h_i)^m \, \mathrm{vol}(K_i, h_i) < \beta_i^m. \tag{2.9}$$

By scale invariance, this inequality holds for every homothetic metric $\lambda_i^2 h_i$ with $\lambda_i > 0$. Choose the factors λ_1 and λ_2 so that

$$\mathrm{ent}_{H_1}(K_1, \lambda_1^2 h_1) = \mathrm{ent}_{H_2}(K_2, \lambda_2^2 h_2) \tag{2.10}$$

and

$$\mathrm{vol}(K_1, \lambda_1^2 h_1) + \mathrm{vol}(K_2, \lambda_2^2 h_2) = 1. \tag{2.11}$$

Let $\alpha = \mathrm{ent}_{H_1}(K_1, \lambda_1^2 h_1) = \mathrm{ent}_{H_2}(K_2, \lambda_2^2 h_2)$. The relations (2.9) and (2.10) combined with (2.11) show that

$$\alpha^m < \beta_1^m + \beta_2^m. \tag{2.12}$$

Consider the metric g_d on P which is defined on the three parts of P given by (2.5) as follows:

$$g_d = \begin{cases} \lambda_1^2 h_1 & \text{on } K_1, \\ 4d^2 dx^2 & \text{on } [p_1, p_2], \\ \lambda_2^2 h_2 & \text{on } K_2, \end{cases} \tag{2.13}$$

where x is the coordinate on $[p_1, p_2] = [1, 2]$ and $d > 0$ is a parameter. By construction, we have $\mathrm{length}_{g_d}([p_1, p_2]) = 2d$ and $\mathrm{vol}(P, g_d) = 1$, where the second equality comes from (2.11).

We will need the following result.

Lemma 2.7. *Let $\varepsilon > 0$. For d large enough, we have*

$$\mathrm{ent}_{\langle\langle H_1 * H_2\rangle\rangle}(P, g_d) < \alpha + \varepsilon.$$

Proof. Let $\widehat{K}_i$ and $\widehat{P}$ be the normal covers corresponding to the normal subgroups $H_i \lhd \pi_1(K_i)$ and $\langle\langle H_1 * H_2\rangle\rangle \lhd \pi_1(P)$. The cover $\widehat{P}$ of $P = K_1 \cup [p_1, p_2] \cup K_2$ can be described as follows:

(1) the cover $\widehat{P}$ decomposes into the union of the lifts of the subsets K_1 and K_2 of P, also called *leaves* of $\widehat{P}$, and the lifts of $[p_1, p_2]$;
(2) every lift of $[p_1, p_2]$ in $\widehat{P}$ is adjacent to two leaves homeomorphic to $\widehat{K}_1$ and $\widehat{K}_2$;
(3) removing a lift of $[p_1, p_2]$ from $\widehat{P}$ separates the cover into two connected components.

The group $G = \pi_1(P)/\langle\langle H_1 * H_2\rangle\rangle$ where

$$\pi_1(P) \simeq \pi_1(K_1 \vee K_2) \simeq \pi_1(K_1) * \pi_1(K_2)$$

decomposes into

$$G \simeq G_1 * G_2$$

where $G_i = \pi_1(K_i)/H_i$ (this relation is left to the reader as an exercice in group theory). With this decomposition, the action of G on $\widehat{P}$ can be described as follows. Let $F \simeq \widehat{K}_i$ be a leaf of $\widehat{P}$. The subgroup $G_i = \pi_1(K_i)/H_i$ of G acts on $F \subseteq \widehat{P}$. For every lift q_i of p_i in F, the orbit $q_i \cdot G_i$ of q_i in F is composed of all the lifts of p_i lying in F under the cover $\widehat{P} \to P$.

Denote by $\widehat{\rho}_i$ the distance on $\widehat{K}_i$ induced by $\lambda_i^2 h_i$ and denote by $\widehat{g}_d$ the metric on $\widehat{P}$ induced by g_d; see (2.13). Let $[q_1, q_2]$ be a lift of $[p_1, p_2]$ in $\widehat{P}$ and let q be the midpoint of $[q_1, q_2]$. In view of (2.1), the desired bound on $\mathrm{ent}_{\langle\langle H_1 * H_2\rangle\rangle}(P, g_d)$ will follow from a bound on $v(t; d) = |V(t; d)|$ where

$$V(t; d) = B_{\langle\langle H_1 * H_2\rangle\rangle}(t, q; g_d) \cap (q \cdot G). \tag{2.14}$$

By the normal form theorem for free product of groups (see [33]), every element $\gamma \in G \simeq G_1 * G_2$ can be uniquely written in normal form as

$$\gamma = \gamma_1 \dots \gamma_l \tag{2.15}$$

where γ_s is a nontrivial element of G_1 or G_2 for $s \in \{1, \dots, l\}$, and γ_s and γ_{s+1} do not lie in the same factor G_1 or G_2 for $s \in \{1, \dots, l-1\}$. For $s \in \{1, \dots, l\}$, denote by $i_s \in \{1, 2\}$ the index such that $\gamma_s \in G_{i_s}$. The *length* $l(\gamma)$ of γ is the number l of elements in the decomposition (2.15). It follows from the description of the cover $\widehat{P}$ (see (1)–(3)) and of the action of G_i on every leaf $F \simeq \widehat{K}_i$ of $\widehat{P}$ that every path from q to $q \cdot \gamma$, where γ is of length l, passes through the points

$$q, \dots, q_{i_s} \cdot (\gamma_{s-1} \dots \gamma_1), q_{i_s} \cdot (\gamma_s \dots \gamma_1), q_{s+1} \cdot (\gamma_{s+1} \dots \gamma_1), \dots, q \cdot \gamma$$

where s runs over $\{1, \dots, l-1\}$. Indeed, removing any of these points between q and $q \cdot \gamma$ from $\widehat{P}$ disconnects q and $q \cdot \gamma$. Since G acts by isometries on $\widehat{P}$, the $\widehat{g}_d$-distance between $q_{i_s} \cdot (\gamma_1 \dots \gamma_{s-1})$ and $q_{i_s} \cdot (\gamma_1 \dots \gamma_s)$ is equal to $\mathrm{dist}_{\widehat{g}_d}(q_{i_s}, q_{i_s} \cdot \gamma_s)$. Since the restriction of the distance $\mathrm{dist}_{\widehat{g}_d}$ to a leaf $F \simeq \widehat{K}_i$ of $\widehat{P}$ agrees with the distance $\widehat{\rho}_i$ on $\widehat{K}_i$ and since the action of G_i on F as a subgroup of G coincides with its action on $\widehat{K}_i$ under the identification $F \simeq \widehat{K}_i$, we have

$$\mathrm{dist}_{\widehat{g}_d}(q_{i_s}, q_{i_s} \cdot \gamma_s) = \widehat{\rho}_i(p_{i_s}, p_{i_s} \cdot \gamma_s).$$

Thus,

$$\begin{aligned}&\operatorname{dist}_{\hat{g}_d}(q, q\cdot\gamma)\\&\quad = d + \hat{\rho}_{i_1}(p_{i_1}, p_{i_1}\cdot\gamma_1) + 2d + \hat{\rho}_{i_2}(p_{i_2}, p_{i_2}\cdot\gamma_2) + 2d + \cdots + \hat{\rho}_{i_l}(p_{i_l}, p_{i_l}\cdot\gamma_l) + d.\end{aligned}$$

Hence,

$$\operatorname{dist}_{\hat{g}_d}(q, q\cdot\gamma) = 2dl + \sum_{s=1}^{l} \hat{\rho}_{i_s}(p_{i_s}, p_{i_s}\cdot\gamma_s). \tag{2.16}$$

To estimate the exponential growth rate of the orbit of G in $\hat{P}$, it will be useful to decompose G by the filtration induced by the length on G. Under this filtration, the group G decomposes into the disjoint union

$$G = \bigcup_{l=1}^{\infty} G^{(l)}$$

where $G^{(l)}$ is formed by the elements of G of length l. We deduce from (2.14) that

$$V(t;d) = \bigcup_{l\geq 1} V^{(l)}(t;d) \tag{2.17}$$

where

$$V^{(l)}(t;d) = B_{\langle\langle H_1 * H_2\rangle\rangle}(t, q; g_d) \cap (q\cdot G^{(l)}).$$

Since the union (2.17) is disjoint, we can write

$$v(t;d) = \sum_{l\geq 1} v^{(l)}(t;d)$$

where $v^{(l)}(t;d) = |V^{(l)}(t;d)|$.

Suppose $q\cdot\gamma \in V^{(l)}(t;d)$. Let t_s be the smallest integer greater than or equal to $\hat{\rho}_{i_s}(p_{i_s}, p_{i_s}\cdot\gamma_s)$. Then $p_{i_s}\cdot\gamma_s \in V_{i_s}(t_s)$ where

$$V_i(t) = B_{H_i}(t, p_i; \hat{\rho}_i) \cap (p_i\cdot G_i).$$

Furthermore,

$$t_s < \hat{\rho}_{i_s}(p_{i_s}, p_{i_s}\cdot\gamma_s) + 1.$$

By (2.16), this inequality leads to

$$\begin{aligned}\sum_{s=1}^{l} t_s &< \sum_{s=1}^{l} \hat{\rho}_{i_s}(p_{i_s}, p_{i_s}\cdot\gamma_s) + l\\ &< t - 2dl + l = t - (2d-1)l. \end{aligned}\tag{2.18}$$

Therefore, every element $\gamma \in G$ with $q\cdot\gamma \in V(t;d)$ decomposes into a product $\gamma = \gamma_1 \dots \gamma_l$ with $\gamma_s \in G_{i_s}$ (see (2.15)) such that $p_{i_s}\cdot\gamma_s \in V_{i_s}(t_s)$ where the integers t_s defined from γ_s satisfy (2.18). The number of elements $\gamma \in G$ of length l with $q \cdot \gamma \in V(t;d)$ and given integers t_s satisfying (2.18) is at most

$$|V_{i_1}(t_1)| \cdot |V_{i_2}(t_2)| \cdots |V_{i_l}(t_l)|.$$

By definition of α, see below (2.11), the exponential growth rate of $|V_{i_s}(t)|$ agrees with α. Thus, for every $\alpha_1 > \alpha$ (arbitrarily close to α, which will be specified afterwards), there exists $t_0 > 0$ such that $|V_{i_s}(t)| < e^{\alpha_1 t}$ for every $t > t_0$.

Let I be the subset of $L = \{1, \ldots, l\}$ given by

$$I = \{s \in L \mid t_s \leq t_0\}.$$

Let $C = \max\{V_1(t_0), V_2(t_0)\}$. For every $s \in L$, we have

$$|V_{i_s}(t_s)| \leq \begin{cases} C & \text{if } s \in I, \\ e^{\alpha_1 t_s} & \text{if } s \notin I. \end{cases}$$

These estimates yield an upper bound on the product

$$|V_{i_1}(t_1)| \cdot |V_{i_2}(t_2)| \cdots |V_{i_l}(t_l)| \leq C^{|I|} e^{\alpha_1 (\sum_{s \notin I} t_s)} \leq C^l e^{\alpha_1 (t-(2d-1)l)} \tag{2.19}$$

where the last inequality follows from $|I| \leq l$ and the bound (2.18).

Now, the number of l-uplets $\tau = (t_1, \ldots, t_l)$ with nonnegative integral coordinates satisfying (2.18) is bounded by

$$\frac{[t-(2d-1)l]^l}{l!}.$$

Combined with (2.19), this leads to

$$v^{(l)}(t;d) \leq \sum_{\tau} |V_{i_1}(t_1)| \cdot |V_{i_2}(t_2)| \cdots |V_{i_l}(t_l)| \leq C^l e^{\alpha_1 (t-(2d-1)l)} \frac{[t-(2d-1)l]^l}{l!}$$

where τ runs over all l-uplets satisfying (2.18). For $d > \frac{1}{2}$, we have $t-(2d-1)l \leq t$ and so

$$v^{(l)}(t;d) \leq C^l e^{\alpha_1 t} e^{-\alpha_1 (2d-1) l} \frac{t^l}{l!} = \frac{e^{\alpha_1 t}}{l!} \left(\frac{Cd}{e^{\alpha_1 (2d-1)}} \right)^l \left(\frac{t}{d} \right)^l.$$

For d large enough, we have

$$\frac{Cd}{e^{\alpha_1 (2d-1)}} < 1. \tag{2.20}$$

Thus,

$$v^{(l)}(t;d) \leq \frac{e^{\alpha_1 t}}{l!} \left(\frac{t}{d} \right)^l.$$

Therefore,

$$v(t;d) = \sum_{l \geq 1} v^{(l)}(t;d) \leq \sum_{l \geq 1} \frac{e^{\alpha_1 t}}{l!} \left(\frac{t}{d} \right)^l = e^{\alpha_1 t + \frac{t}{d}}.$$

Hence,

$$\lim_{t\to\infty} \frac{\log v(t;d)}{t} \leq \alpha_1 + \frac{1}{d}. \tag{2.21}$$

For $\alpha_1 < \alpha + \varepsilon$ and $d > 1/\varepsilon$ large enough so that inequality (2.20) is satisfied, we deduce from (2.21) that

$$\mathrm{ent}_{\langle\langle H_1 * H_2\rangle\rangle}(P, g_d) < \alpha + 2\varepsilon,$$

which finishes the proof of the lemma. ∎

Let us resume the proof of Theorem 2.6. Since β_i^m can be arbitrarily close to $\Omega_{H_i}(K_i)$, inequality (2.12) combined with Lemma 2.7 leads to

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(P) \leq \Omega_{H_1}(K_1) + \Omega_{H_2}(K_2).$$

Along with (2.7), this inequality yields the desired result. ∎

2.3. *Upper bound on the minimal volume entropy of connected sums*

The following result compares the minimal volume entropy of the connected sum of two (pseudo)manifolds with the minimal volume entropy of the two (pseudo)manifolds. Here, the connected sum of the two connected m-pseudomanifolds is defined in the usual way by removing an m-simplex Δ^m from each pseudomanifold and by identifying the boundary $\partial\Delta^m$ of the resulting pseudomanifolds.

Theorem 2.8. *For $i = 1, 2$, let M_i be a connected closed pseudomanifold of dimension $m \geq 3$ and $H_i \lhd \pi_1(M_i)$ be a normal subgroup. Then*

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(M_1 \sharp M_2) \leq \Omega_{H_1}(M_1) + \Omega_{H_2}(M_2). \tag{2.22}$$

Remark 2.9. In dimension 2, the result remains valid by replacing the normal subgroup $\langle\langle H_1 * H_2\rangle\rangle$ with $f^{-1}(\langle\langle H_1 * H_2\rangle\rangle)$ where $f : M_1 \sharp M_2 \to M_1 \vee M_2$ is the canonical projection.

Remark 2.10. Inequality (2.22) is the analogue for the volume entropy of a similar bound holding for the systolic volume; see [5, Proposition 3.6] and Proposition 6.6. Note however that the proof for the minimal volume entropy is more intricate.

Proof of Theorem 2.8. Consider the canonical m-monotone map

$$f : M_1 \sharp M_2 \to M_1 \vee M_2$$

obtained by collapsing the attaching sphere to a point (in order to get a simplicial map, we may have to take two barycentric subdivisions of M_1 and M_2). Since $m \geq 3$, the induced homomorphism $f_* : \pi_1(M_1 \sharp M_2) \to \pi_1(M_1 \vee M_2)$ is an isomorphism. The comparison principle (see Proposition 2.4) and Theorem 2.6 yield

$$\Omega_{\langle\langle H_1 * H_2\rangle\rangle}(M_1 \sharp M_2) \leq \Omega_{\langle\langle H_1 * H_2\rangle\rangle}(M_1 \vee M_2) = \Omega_{H_1}(M_1) + \Omega_{H_2}(M_2).$$ ∎

Corollary 2.11. *Let X be a path-connected topological space. Then for all $\mathbf{a}_1, \mathbf{a}_2$ in $H_m(X;\mathbb{Z})$, we have*

$$\Omega(\mathbf{a}_1 + \mathbf{a}_2) \leq \Omega(\mathbf{a}_1) + \Omega(\mathbf{a}_2).$$

In particular, the quantity $\Omega(\mathbf{a}_1 - \mathbf{a}_2)$ defines a pseudo-distance between $\mathbf{a}_1$ and $\mathbf{a}_2$ in $H_m(X;\mathbb{Z})$, and the functional $\|\cdot\|_E$ is a semi-norm on $H_m(X;\mathbb{R})$.

Proof. Let $\mathbf{a}_1, \mathbf{a}_2 \in H_m(X;\mathbb{R})$. Fix $\varepsilon > 0$. There exists a map $f_i : M_i \to X$ from an oriented connected closed m-pseudomanifold M_i representing $\mathbf{a}_i$ for $i = 1, 2$ such that

$$\Omega_{\ker(f_i)_*}(M_i) \leq \Omega(\mathbf{a}_i) + \varepsilon.$$

Let $M = M_1 \sharp M_2$. Consider the canonical map $f = f_1 \vee f_2 : M \to X$ obtained from f_1 and f_2 by first collapsing the attaching sphere to a point. Note that

$$\ker f_* \simeq \langle\langle \ker(f_1)_* * \ker(f_2)_* \rangle\rangle.$$

Furthermore, by Theorem 2.8, we have

$$\begin{aligned} \Omega_{\ker f_*}(M) &\leq \Omega_{\ker(f_1)_*}(M_1) + \Omega_{\ker(f_2)_*}(M_2) \\ &\leq \Omega(\mathbf{a}_1) + \Omega(\mathbf{a}_2) + 2\varepsilon. \end{aligned}$$

Hence, $\Omega(\mathbf{a}_1 + \mathbf{a}_2) \leq \Omega(\mathbf{a}_1) + \Omega(\mathbf{a}_2)$.

Replacing $\mathbf{a}_i$ with $k\mathbf{a}_i$ in the previous inequality, dividing by k and letting k go to infinity, we obtain

$$\|\mathbf{a}_1 + \mathbf{a}_2\|_E \leq \|\mathbf{a}_1\|_E + \|\mathbf{a}_2\|_E.$$

Since $\|\cdot\|_E$ is clearly homogeneous by the stabilization process (see (1.5)), the functional $\|\cdot\|_E$ is a semi-norm. ■

2.4. Lower bound on the minimal volume entropy of connected sums

In a different direction, taking the connected sum with an orientable pseudomanifold does not decrease the minimal volume entropy, as the following result shows.

Theorem 2.12. *For $i = 1, 2$, let M_i be a connected closed pseudomanifold of dimension $m \geq 3$ and $H_i \lhd \pi_1(M_i)$ be a normal subgroup. Let $H \lhd \pi_1(M_1) * \pi_1(M_2)$ be a normal subgroup such that the canonical inclusion $\pi_1(M_1) \leqslant \pi_1(M_1) * \pi_1(M_2)$ induces an inclusion*

$$\pi_1(M_1)/H_1 \leqslant (\pi_1(M_1) * \pi_1(M_2))/H. \tag{2.23}$$

Suppose M_2 is orientable. Then

$$\Omega_{H_1}(M_1) \leq \Omega_H(M_1 \sharp M_2). \tag{2.24}$$

Combining this with Theorem 2.8, we obtain

Corollary 2.13. *For $i = 1, 2$, let M_i be a connected closed pseudomanifold of dimension $m \geq 3$ and $H_i \lhd \pi_1(M_i)$ be a normal subgroup. Suppose M_2 is orientable and $\Omega_{H_2}(M_2) = 0$. Then*

$$\Omega_{\langle\langle H_1 * H_2 \rangle\rangle}(M_1 \sharp M_2) = \Omega_{H_1}(M_1).$$

In order to prove Theorem 2.12, we first establish the following result. For a CW-complex X, denote by $X(k)$ its k-skeleton.

Proposition 2.14. *Let M be an orientable connected closed pseudomanifold of dimension $m \geq 3$. Suppose that*

$$M = D^m \underset{\phi}{\cup} M(m-1) \tag{2.25}$$

is a cell decomposition with a single m-cell. Then the space

$$M \underset{M(m-2)}{\cup} \mathrm{Cone}(M(m-2))$$

obtained by gluing the cone Cone$(M(m-2))$ *over $M(m-2)$ to M along $M(m-2)$ is homotopy equivalent to a finite bouquet of spheres*

$$M \underset{M(m-2)}{\cup} \mathrm{Cone}(M(m-2)) \simeq \bigvee_s S_s^{m-1} \vee S^m.$$

Proof. We have

$$M \underset{M(m-2)}{\cup} \mathrm{Cone}(M(m-2)) \simeq M/M(m-2) \simeq \bigvee_s S_s^{m-1} \underset{\widehat{\phi}}{\cup} S^m$$

where the number of $(m-1)$-spheres S_s^{m-1} is equal to the number of $(m-1)$-cells of M and

$$\widehat{\phi} : S^{m-1} \stackrel{\phi}{\to} M(m-1) \to M(m-1)/M(m-2) \simeq \bigvee_s S_s^{m-1}$$

is the projection of the attaching map ϕ.

To derive the proposition, we need to show that ϕ is null-homotopic. Consider the triple $(M, M(m-1), M(m-2))$ and the corresponding long exact sequence with $\mathbb{Z}$-coefficients

$$\cdots \to 0 \stackrel{i_*}{\longrightarrow} H_m(M, M(m-2)) \stackrel{j_*}{\longrightarrow} H_m(M, M(m-1))$$
$$\stackrel{\partial}{\to} H_{m-1}(M(m-1), M(m-2)) \to \cdots .$$

From the long exact sequence of the pair $(M, M(m-2))$, we obtain

$$H_m(M, M(m-2)) \simeq H_m(M).$$

The orientability of M implies that

$$H_m(M, M(m-1)) \simeq H_m(M).$$

Thus, j_* is an isomorphism, which implies that $\partial = 0$.

Thinking of the homology groups $H_k(M(k), M(k-1); \mathbb{Z})$ as free abelian groups with basis the k-cells e^k_α of M, the cellular boundary formula (see [28, Section 2.2]) gives

$$\partial(e^m) = \sum_s \deg(\phi_s) e^{m-1}_s$$

where $\phi_s : S^{m-1} \to M(m-1) \to S^{m-1}_s$ is the composite of the attaching map ϕ of the m-cell $e^m = D^m$ with the quotient map collapsing $M(m-1) \setminus e^{m-1}_s$ to a point. (Note that ϕ_s factorizes through $\hat{\phi}$.) Since $\partial = 0$, every map ϕ_s is contractible. Hence ϕ is null-homotopic as desired. ■

We can now prove Theorem 2.12.

Proof of Theorem 2.12. Choose a cell decomposition of M_2 with only one cell of maximal dimension m, which is coherent with the triangulation of the pseudomanifold. Denote by $\mathrm{Cone}(M_2(m-2))$ the cone over the $(m-2)$-skeleton $M_2(m-2)$ of M_2. By Proposition 2.14, the space

$$M_2 \underset{M_2(m-2)}{\cup} \mathrm{Cone}(M_2(m-2))$$

obtained by gluing the cone $\mathrm{Cone}(M_2(m-2))$ to M_2 along $M_2(m-2)$ is homotopy equivalent to a finite bouquet of spheres

$$M_2 \underset{M_2(m-2)}{\cup} \mathrm{Cone}(M_2(m-2)) \simeq \bigvee_s S^{m-1}_s \vee S^m$$

with only one m-dimensional sphere S^m. Thus, the canonical inclusion $M_1 \setminus B^m \subseteq M_1 \sharp M_2$ extends to the missing ball B^m and gives rise to an m-monotone map (see Definition 2.3)

$$f : M_1 \to M_1 \sharp M_2 \underset{M_2(m-2)}{\cup} \mathrm{Cone}(M_2(m-2)). \tag{2.26}$$

Fix $\varepsilon > 0$. Consider a metric g on $M_1 \sharp M_2$ with $\mathrm{vol}(M_1 \sharp M_2, g) = 1$ which is ε-extremal, that is,

$$\mathrm{ent}_H(M_1 \sharp M_2, g)^m \le \Omega_H(M_1 \sharp M_2) + \varepsilon. \tag{2.27}$$

Extend the metric g to a metric g' on $M_1 \sharp M_2 \underset{M_2(m-2)}{\cup} \mathrm{Cone}(M_2(m-2))$ as follows. First, observe that

$$\mathrm{Cone}(M_2(m-2)) = M_2(m-2) \times [0,1] / M_2(m-2) \times \{1\}.$$

The extension g' of g, which agrees with g on $M_1 \sharp M_2$, is defined on $\mathrm{Cone}(M_2(m-2))$ by

$$g' = \begin{cases} g_{|M_2(m-2)} + 10D\, dt^2 & \text{if } 0 \le t \le 1/2, \\ 4(1-t)^2 g_{|M_2(m-2)} + 10D\, dt^2 & \text{if } 1/2 \le t \le 1, \end{cases}$$

where $D = \mathrm{diam}(M_2(m-2), g_{|M_2(m-2)})$. Technically, the metric g' is singular, but it still induces a distance $\mathrm{dist}_{g'}$. Note also that for dimensional reasons,

$$\mathrm{vol}\Big(M_1 \sharp M_2 \underset{M_2(m-2)}{\cup} \mathrm{Cone}(M_2(m-2)), g'\Big) = \mathrm{vol}(M_1 \sharp M_2, g) = 1.$$

By construction of g', the canonical inclusion

$$i : M_1 \sharp M_2 \hookrightarrow M_1 \sharp M_2 \bigcup_{M_2(m-2)} \operatorname{Cone}(M_2(m-2)) \tag{2.28}$$

is distance-preserving. That is, for all $p_1, p_2 \in M_1 \sharp M_2$, we have

$$\operatorname{dist}_g(p_1, p_2) = \operatorname{dist}_{g'}(i(p_1), i(p_2)).$$

Since $m \geq 3$, the composite map

$$M_1 \setminus B^m \subseteq M_1 \sharp M_2 \hookrightarrow M_1 \sharp M_2 \bigcup_{M_2(m-2)} \operatorname{Cone}(M_2(m-2)) \tag{2.29}$$

induces an isomorphism between the fundamental groups. Denote by $G_1 \leqslant G = \pi_1(M_1 \sharp M_2)$ the image of $\pi_1(M_1)$ in $\pi_1(M_1 \sharp M_2)$.

Let $q_1 \in M_1 \setminus B^m \subseteq M_1 \sharp M_2$. Since $\operatorname{Cone}(M_2(m-2))$ is simply connected and the map (2.29) is distance-preserving, every loop $\gamma \subseteq M_1 \sharp M_2 \bigcup_{M_2(m-2)} \operatorname{Cone}(M_2(m-2))$ based at q_1 is homotopic to a loop $\gamma' \subseteq M_1 \sharp M_2$ based at the same point such that

$$\operatorname{length}_g(\gamma') \leq \operatorname{length}_{g'}(\gamma). \tag{2.30}$$

The group G/H acts on the cover $\widehat{M}$ of $M_1 \sharp M_2$ with fundamental group H. Similarly, the group G_1/H_1 acts on the cover $\widehat{M}'$ of $M_1 \sharp M_2 \bigcup_{M_2(m-2)} \operatorname{Cone}(M_2(m-2))$ with fundamental group H_1. Let $\widehat{g}$ and $\widehat{g}'$ be the metrics on $\widehat{M}$ and $\widehat{M}'$ induced by g and g'. Fix some lifts $q \in \widehat{M}$ and $q' \in \widehat{M}'$ of q_1. Denote by $B_H(t, q; \widehat{g})$ and $B_{H_1}(t, q'; \widehat{g}')$ the balls of $\widehat{M}$ and $\widehat{M}'$ of radius t centered at q and q'. Since $G_1/H_1 \leqslant G/H$ (see (2.23)), it follows from (2.30) that for every $t \geq 0$,

$$|B_{H_1}(t, q'; \widehat{g}') \cap q' \cdot (G_1/H_1)| \leq |B_H(t, q; \widehat{g}) \cap q \cdot (G/H)|. \tag{2.31}$$

Applying Lemma 2.5 to the m-monotone map f (see (2.26)), we derive a polyhedral Riemannian metric g_ε on M_1 such that the map f is nonexpanding and

$$\operatorname{vol}(M_1, g_\varepsilon) < \operatorname{vol}(M_1 \sharp M_2, g) + \varepsilon. \tag{2.32}$$

The group G_1/H_1 acts both on $\widehat{M}'$ and on the cover $\widehat{M}''$ of M_1 with fundamental group H_1. Let $\widehat{g}''_\varepsilon$ be the metric on $\widehat{M}''$ induced by g_ε. Fix a lift q'' of q_1 in $\widehat{M}''$. The nonexpanding map f lifts to a (G_1/H_1)-equivariant, nonexpanding map $\widehat{f} : \widehat{M}'' \to \widehat{M}'$. This implies that the ball $B_{H_1}(t, q''; \widehat{g}''_\varepsilon)$ of $\widehat{M}''$ satisfies

$$|B_{H_1}(t, q''; \widehat{g}''_\varepsilon) \cap q'' \cdot (G_1/H_1)| \leq |B_{H_1}(t, q'; \widehat{g}') \cap q' \cdot (G_1/H_1)|. \tag{2.33}$$

Combining the bounds (2.31) and (2.33), we derive the following inequalities on the exponential growth rates of the orbits of G_1/H_1 and G/H:

$$\operatorname{ent}_{H_1}(M_1, g_\varepsilon) \leq \operatorname{ent}_{H_1}\left(M_1 \sharp M_2 \bigcup_{M_2(m-2)} \operatorname{Cone}(M_2(m-2)), g'\right) \leq \operatorname{ent}_H(M_1 \sharp M_2, g).$$

Since $\mathrm{vol}(M_1 \sharp M_2, g) = 1$ and $\mathrm{vol}(M_1, g_\varepsilon) < 1 + \varepsilon$ (see (2.32)), this estimate combined with (2.27) yields the desired bound

$$\Omega_{H_1}(M_1) \leq \Omega_H(M_1 \sharp M_2). \qquad \blacksquare$$

Remark 2.15. The proof of Theorem 2.12 does not apply when M_2 is nonoriented. The conclusion is unclear in this case.

2.5. *Fundamental class of finite order*

The following result is a direct application of Theorem 2.12.

Proposition 2.16. *Let M be an oriented connected closed manifold of dimension $m \geq 3$ and $f : M \to K(\pi_1(M), 1)$ be its classifying map. Suppose $f_*([M]) \in H_m(\pi_1(M); \mathbb{Z})$ is a finite order homology class. Then*

$$\Omega(M) = 0.$$

Proof. Let $N = M \sharp \cdots \sharp M$ be the manifold obtained by taking the connected sum of k copies of M. Consider the map $F = f \vee \cdots \vee f : N \to K(\pi_1(M), 1)$ obtained by collapsing each attaching sphere to a point and by applying f to each term M in the bouquet $M \vee \cdots \vee M$. The class $F_*([N])$ is equal to $k\mathbf{a}$ where $\mathbf{a} = f_*([M])$. Suppose $k\mathbf{a} = 0$. Since N is an oriented connected closed manifold and the homomorphism $F_* : \pi_1(N) \to \pi_1(M)$ induced by F is surjective, we deduce from [13, Theorem 10.2] (see (1.4)) that

$$\Omega_{\ker F_*}(N) = 0.$$

Apply Theorem 2.12 to $M_1 = M$ and the connected sum $M_2 = M \sharp \cdots \sharp M$ of $k - 1$ copies of M by taking $H = \ker F_*$. This immediately leads to the desired result. $\blacksquare$

Remark 2.17. In order to contextualize this result, recall that under the assumption of Proposition 2.16, the volume entropy semi-norm of M (and its simplicial volume) vanishes. Proposition 2.16 asserts that the volume entropy of M also vanishes, without any stabilization process.

Example 2.18. Every manifold M with fundamental group $\mathrm{SL}(2, \mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ or $\mathrm{PSL}(2, \mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$ has zero minimal volume entropy, that is, $\Omega(M) = 0$. Indeed, the homology of an amalgamated product can be computed through a Mayer–Vietoris sequence involving the homology groups of its factors. Since the homology of every cyclic group is composed of finite groups (except in dimension zero), the same holds for the homology groups of $\mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{PSL}(2, \mathbb{Z})$; see [32, Theorem 4.1.1].

2.6. *Volume entropy semi-norm comparison*

Let us give an application of Corollary 2.13.

Theorem 2.19. *Let G be a finitely presented group and H be a finite index subgroup of G. Let $m \geq 3$. Suppose that the canonical inclusion $i : H \hookrightarrow G$ induces a monomorphism between the m-dimensional rational homology groups*

$$(i_*)_m : H_m(H;\mathbb{Q}) \to H_m(G;\mathbb{Q}).$$

Then for every homology class $\mathbf{a} \in H_m(H;\mathbb{Z})$,

$$\|\mathbf{a}\|_E = \|i_*(\mathbf{a})\|_E.$$

Proof. Still denote by $i : K(H,1) \to K(G,1)$ the classifying map induced by the canonical inclusion $i : H \hookrightarrow G$. By (1.3), for every integer $k \geq 1$, we have

$$\Omega(k i_*(\mathbf{a})) = \Omega(i_*(k\mathbf{a})) \leq \Omega(k\mathbf{a}).$$

Hence

$$\|\mathbf{a}\|_E \geq \|i_*(\mathbf{a})\|_E.$$

Thus, we only have to show the converse inequality. The idea is to start with an almost extremal geometric cycle of $K(G,1)$ representing $k i_*(\mathbf{a})$, to add handles to it to make sure it is π_1-surjective and to take a lift corresponding to the subgroup $H \leqslant G$. The resulting geometric cycle is almost extremal and represents $dk\mathbf{a}$ up to some torsion element, where $d = [G:H]$.

Denote by p the number of generators in G and by $d = [G:H]$ the index of H on G. Let

$$M_2 = (S^1 \times S^{m-1}) \sharp \cdots \sharp (S^1 \times S^{m-1})$$

be the connected sum of p copies of $S^1 \times S^{m-1}$. Let $f_2 : M_2 \to K(G,1)$ be a map inducing a surjective homomorphism between the fundamental groups. Observe that $\Omega(M_2) = 0$ and $(f_2)_*([M_2]) = 0 \in H_m(G;\mathbb{Z})$.

Fix $\varepsilon > 0$. By definition of $\|\cdot\|_E$, for every integer $k \geq 1$, there exists a map $f_1 : M_1 \to K(G,1)$ defined on an oriented connected closed m-pseudomanifold representing the class $k i_*(\mathbf{a}) \in H_m(G;\mathbb{Z})$ such that

$$\Omega_{\ker (f_1)_*}(M_1) \leq k(\|i_*(\mathbf{a})\|_E + \varepsilon).$$

Consider the connected sum $M = M_1 \sharp M_2$ and the canonical map $f = f_1 \vee f_2 : M \to K(G,1)$ obtained from f_1 and f_2 by collapsing the attaching sphere to a point. Note that

$$\ker f_* \simeq \langle\langle \ker (f_1)_* * \ker (f_2)_* \rangle\rangle.$$

Since $(f_2)_*([M_2]) = 0 \in H_m(G;\mathbb{Z})$, the map $f : M \to K(G,1)$ still represents the class $k i_*(\mathbf{a}) \in H_m(G;\mathbb{Z})$. Since $\Omega_{\ker(f_2)_*}(M_2) = 0$, we deduce from Corollary 2.13 that

$$\Omega_{\ker f_*}(M) \leq k(\|i_*(\mathbf{a})\|_E + \varepsilon). \tag{2.34}$$

Since f_2 induces a surjective homomorphism between the fundamental groups, the same holds for the map f. Let $\widehat{M}$ be the cover of M of fundamental group $f_*^{-1}(H)$.

Denote by

$$\widehat{f} : \widehat{M} \to K(H, 1)$$

the corresponding lift of f. Let $\mathbf{b} = \widehat{f}_*([\widehat{M}]) \in H_m(H; \mathbb{Z})$. Since H is of index d in G, the cover $\pi : \widehat{M} \to M$ is of degree d. Thus, $\pi_*([\widehat{M}]) = d[M]$. Still denote by $i : K(H, 1) \to K(G, 1)$ the classifying map induced by the canonical inclusion $i : H \hookrightarrow G$. It follows from the commutation relation $i \circ \widehat{f} = f \circ \pi$ that

$$i_*(\mathbf{b}) = i_*(\widehat{f}_*([\widehat{M}])) = d f_*([M]) = d k i_*(\mathbf{a}).$$

Since the canonical inclusion $i : H \hookrightarrow G$ induces a monomorphism between the m-dimensional rational homology groups, we deduce that $\mathbf{b} = dk\mathbf{a} + \mathbf{c}$ where $\mathbf{c} \in \mathrm{Tor} H_m(H; \mathbb{Z})$. Thus,

$$\|\mathbf{b}\|_E = dk \|\mathbf{a}\|_E.$$

Let g be an ε-extremal metric on M, that is,

$$\Omega_{\ker f_*}(M, g) \le \Omega_{\ker f_*}(M) + \varepsilon. \tag{2.35}$$

Denote by $\widehat{g}$ the lift of g on $\widehat{M}$. The cover of (M, g) of fundamental group $\ker f_*$ is isometric to the cover of $(\widehat{M}, \widehat{g})$ of fundamental group $\ker \widehat{f}_*$. Thus, the exponential growth rates of the volume of balls in the two coverings are equal. Since $\pi : \widehat{M} \to M$ is of degree d, we have $\mathrm{vol}(\widehat{M}) = d \, \mathrm{vol}(M)$. Therefore,

$$\Omega_{\ker \widehat{f}_*}(\widehat{M}, \widehat{g}) = d\, \Omega_{\ker f_*}(M, g).$$

Now, by construction, $(\widehat{M}, \widehat{f})$ represents $\mathbf{b}$. Hence,

$$dk \|\mathbf{a}\|_E = \|\mathbf{b}\|_E \le \Omega_{\ker \widehat{f}_*}(\widehat{M}, \widehat{g}) = d\, \Omega_{\ker f_*}(M, g).$$

This inequality combined with the bounds (2.34) and (2.35) yields

$$\|\mathbf{a}\|_E \le \|i_*(\mathbf{a})\|_E + 2\varepsilon.$$

Hence the desired inequality $\|\mathbf{a}\|_E \le \|i_*(\mathbf{a})\|_E$ by letting ε go to zero. ∎

3. Functorial properties of the volume entropy semi-norm

In this section, we present functorial properties of the volume entropy semi-norm and observe similarities to the ones satisfied by the simplicial volume.

Theorem 3.1. (1) *Let $f : X \to Y$ be a continuous map between path-connected topological spaces. Then for every $\mathbf{a} \in H_m(X; \mathbb{R})$,*

$$\|f_*(\mathbf{a})\|_E \le \|\mathbf{a}\|_E.$$

(2) *Let* $f : M \to K(\pi_1(M), 1)$ *be the classifying map of an orientable connected closed manifold* M*. Then*

$$\|f_*([M])\|_E = \|M\|_E.$$

(3) *Let* $f : M \to N$ *be a degree* d *map between oriented connected closed manifolds. Then*

$$\|M\|_E \geq |d|\, \|N\|_E.$$

(4) *Let* $f : M \to N$ *be a* d*-sheeted covering map between orientable connected closed manifolds. Then*

$$\|M\|_E = d\, \|N\|_E.$$

(5) *Let* M_1 *and* M_2 *be orientable connected closed manifolds of dimension* $m \geq 3$*. Then*

$$\|M_1 \sharp M_2\|_E \leq \|M_1\|_E + \|M_2\|_E. \tag{3.1}$$

(6) *Let* M *be an orientable connected closed* m*-manifold with a negatively curved locally symmetric metric* g_0*. Then*

$$\|M\|_E = \Omega(M, g_0).$$

In particular, if M *is a closed genus* g *surface then*

$$\|M\|_E = \pi \|M\|_\Delta = 4\pi(g-1).$$

Remark 3.2. Properties (1)–(5) are also satisfied by the simplicial volume. However, the simplicial volume is additive under connected sum in dimension at least 3; see [24]. That is, there is equality in (3.1) if one replaces the volume entropy semi-norm with the simplicial volume. This leads to the following questions. Is there equality in (3.1)? Similarly, is there equality in (2.22)? A difficulty to overcome is that, in contrast to the simplicial volume, there is no cohomological interpretation of the volume entropy semi-norm.

Remark 3.3. It follows from (3) that both the simplicial volume and the volume entropy semi-norm of an orientable connected closed manifold admitting a map to itself of degree different from 0 and ± 1 are equal to zero. In a different direction, by Theorem 4.12, neither the simplicial volume nor the volume entropy semi-norm vanishes for orientable connected closed manifolds admitting a negatively curved Riemannian metric; see [24].

Proof of Theorem 3.1. (1) Observe that if (M, φ) is a geometric cycle representing $\mathbf{a} \in H_m(X; \mathbb{R})$ then $(M, f \circ \varphi)$ is a geometric cycle representing $f_*(\mathbf{a}) \in H_m(Y; \mathbb{R})$. Moreover, $\omega_{\ker f_*}(M) \geq \omega_{\ker(\varphi \circ f)_*}(M)$ since $\ker f_* \leqslant \ker(\varphi \circ f)_*$. This immediately implies (1).

(2) For $m = 2$, the assertion is obvious since the classifying map is the identity map when $M \neq S^2$. In case $M = S^2$, all the terms of the relation vanish.

Suppose that $m \geq 3$. The inequality $\|f_*([M])\|_E \leq \|M\|_E$ follows from (1). For the reverse inequality, consider $M_k = M \sharp \cdots \sharp M$ (k copies) and the degree k map $f_k : M_k \to M$ contracting all attaching spheres to a point. By definition,

$$\Omega_{\ker(f_k)_*(M_k)} \geq \Omega(k[M]). \tag{3.2}$$

The composite map

$$F_k = f \circ f_k : M_k \to K(\pi_1(M), 1)$$

represents the class $kf_*([M]) \in H_m(\pi_1(M); \mathbb{Z})$, that is, $(F_k)_*([M_k]) = kf_*([M])$. Observe also that it is π_1-surjective. By [13, Theorem 10.2], this implies that

$$\Omega(kf_*([M])) = \Omega_{\ker (F_k)_*}(M_k). \tag{3.3}$$

Since f is π_1-injective, $\Omega_{\ker (F_k)_*}(M_k) = \Omega_{\ker (f_k)_*}(M_k)$. Combining this with (3.2) and (3.3), we derive

$$\frac{\Omega(kf_*([M]))}{k} \geq \frac{\Omega(k[M])}{k}.$$

By letting k go to infinity, we obtain $\|f_*([M])\|_E \geq \|M\|_E$, which implies (2).

(3) By definition, assertion (3) immediately follows from (1).

(4) Let (Q, ψ) be a geometric cycle representing the class $k[N] \in H_m(N; \mathbb{Z})$. By adding handles to Q and mapping them to a generating set of $\pi_1(N)$ if necessary (contracting the meridian spheres of the handles to points), we can assume that the map $\psi : Q \to N$ is π_1-surjective. By Theorem 2.8 (and Remark 2.9 when $m = 2$), adding such handles does not increase the (relative) minimal volume entropy of the geometric cycle. Now, denote by $P \to Q$ the covering of Q corresponding to the subgroup $(\psi_*)^{-1}(\mathrm{Im}\, f_*)$. Note that P is an oriented connected closed m-pseudomanifold and that the covering map $P \to Q$ is of degree d. The map $\psi : Q \to N$ lifts to a map $\varphi : P \to M$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & M \\ \downarrow & & \downarrow f \\ Q & \xrightarrow{\psi} & N \end{array}$$

commutes. Observe that the geometric cycle (P, φ) represents the class $k[M] \in H_m(M; \mathbb{Z})$.

Fix a piecewise Riemannian metric on Q and lift it to P. Since $P \to Q$ is a d-sheeted covering map and $\ker \varphi_* = \ker \psi_*$, we have $\Omega_{\ker \varphi_*}(P) \leq d\,\Omega_{\ker \psi_*}(Q)$. Thus,

$$\Omega(k[M]) \leq d\,\Omega(k[N]).$$

Dividing this inequality by k and letting k go to infinity, we obtain

$$\|M\|_E \leq d\,\|N\|_E.$$

The reverse inequality $\|M\|_E \geq d\,\|N\|_E$ follows from (3).

(5) The idea is to realize $[M_1 \sharp M_2]$ as the sum of $[M_1]$ and $[M_2]$ in a common topological space and to apply Corolllary 2.11. Let $K_i = K(\pi_1(M_i), 1)$ be a classifying space for M_i. Since $m \geq 3$, the bouquet $K = K_1 \vee K_2$ is a classifying space for $M_1 \sharp M_2$. By the Mayer–Vietoris theorem, we have

$$H_m(K; \mathbb{Z}) \simeq H_m(K_1; \mathbb{Z}) \oplus H_m(K_2; \mathbb{Z}).$$

Denote by $[M_i]_K \in H_m(K;\mathbb{Z})$ the image of the fundamental class of M_i under the homology homomorphism induced by the composite $f_i^K : M_i \to K$ of the classifying map $f_i : M_i \to K_i$ and the inclusion map $K_i \hookrightarrow K = K_1 \vee K_2$. Observe that

$$f_*([M_1 \sharp M_2]) = [M_1]_K + [M_2]_K$$

where $f : M_1 \sharp M_2 \to K$ is the classifying map of $M_1 \sharp M_2$. By (2) and the triangle inequality for the volume entropy semi-norm (see Corollary 2.11), we derive

$$\|M_1 \sharp M_2\|_E = \|f_*([M_1 \sharp M_2])\|_E \leq \|[M_1]_K\|_E + \|[M_2]_K\|_E.$$

By (1), we also have $\|[M_i]_K\|_E \leq \|[M_i]\|_E = \|M_i\|_E$. Hence the result follows.

(6) The proof proceeds from a mild improvement on the minimal volume entropy estimate for closed manifolds admitting nonzero maps onto closed negatively curved locally symmetric manifolds; see [9]. This mild improvement, leading to (3.6), was carried out in [42, Theorem 2.5] for $n \geq 3$ with the construction of a volume-nonexpanding map following [9]. Our approach is similar, except that it rests on the calibration argument of [9] (which can be applied to pseudomanifolds) and applies to both cases $n \geq 3$ and $n = 2$. We refer to [9] for the notations (accordingly renaming M to X), the definitions and further details.

Let $f : Y \to X = M$ be a map from an oriented connected closed m-pseudomanifold representing $k[X]$. The map f lifts to a map $\bar{f} : \bar{Y} \to \tilde{X}$ where $\bar{Y}$ is the covering of Y with $\pi_1(\bar{Y}) = \ker f_*$. (This is the main difference with [9, Section 8], where the map is lifted to $\tilde{Y} \to \tilde{X}$.) Given a piecewise Riemannian metric g on Y, denote by $\bar{g}$ the lifted metric on $\bar{Y}$. Fix $c > 0$. Consider the $\pi_1(Y)/\ker f_*$-equivariant map $\Psi_c : \bar{Y} \to S^\infty_+ \subset L^2(\partial\tilde{X}, d\theta)$ defined as

$$\Psi_c(y,\theta) = \left(\frac{\int_{\bar{Y}} e^{-c d_{\bar{g}}(y,z)} p_0(\bar{f}(z),\theta)\, dv_{\bar{g}}(z)}{\int_{\bar{Y}} e^{-c d_{\bar{g}}(y,z)}\, dv_{\bar{g}}(z)} \right)^{1/2}$$

where p_0 is the Poisson kernel of $(\tilde{X}, \tilde{g}_0)$; see [9]. The arguments of [9] show that the map Ψ_c is only defined when $c > \mathrm{ent}_{\ker f_*}(Y, g)$ and that

$$\sqrt{\det_{\bar{g}}(g_{\Psi_c})} \leq \frac{c^m}{(4m)^{m/2}} \tag{3.4}$$

where g_{Ψ_c} is the pull-back under Ψ_c of the Hilbert metric on $L^2(\partial\tilde{X}, d\theta)$. Loosely speaking, the equivariant map Ψ_c converges to the composite of $\bar{f}$ with the embedding $\sqrt{p_0}$ of $\tilde{X}$ into $L^2(\partial\tilde{X}, d\theta)$ given by the Poisson kernel when c goes to $\mathrm{ent}_{\ker f_*}(Y, g)$. The Poisson embedding $\sqrt{p_0}$ is an isometry up to some factor (i.e., $g_{\sqrt{p_0}} = \frac{\mathrm{ent}_{\ker f_*}(Y,g)}{4m} g_0$) admitting an equivariant calibration form in the infinite-dimensional sphere S^∞_+ as explained below.

Denote by $\pi : S^\infty_+ \to \tilde{X}$ the barycenter map, i.e., $\pi(\rho) = \mathrm{bar}(\rho^2(\theta)d\theta)$; see [9]. The following calibration result was established in [9, Proposition 5.7] for $m \geq 3$ and in [9, Theorem 6.2] for $m = 2$. Let ω_0 be the volume form on $(\tilde{X}, \tilde{g}_0)$. The $\pi_1(X)$-equivariant closed m-form $\pi^*\omega_0$ on S^∞_+ calibrates the embedding $\Phi_0 : \tilde{X} \to S^\infty_+ \subset L^2(\partial\tilde{X}, d\theta)$ defined as $\Phi_0(x) = \sqrt{p_0(x,\cdot)}$. Furthermore,

$$\operatorname{comass}(\pi^*\omega_0) = \frac{(4m)^{m/2}}{\operatorname{ent}(X, g_0)^m}. \tag{3.5}$$

Now, the map $\pi \circ \Psi_c$ is homotopic to $\pi \circ \Psi_0$, where $\Psi_0 = \Phi_0 \circ \bar{f}$, through equivariant maps from $\bar{Y}$ to $\tilde{X}$. By (3.4) and (3.5), we derive

$$|(\pi \circ \Psi_c)^*(\omega_0)| = |\Psi_c^*(\pi^*\omega_0)| \leq \left(\frac{c}{\operatorname{ent}(X, g_0)}\right)^m |\omega_{\bar{g}}|.$$

By a calibration argument, we deduce that

$$|\deg(f)| \cdot \operatorname{vol}(X, g_0) \leq \int_Y |(\pi \circ \Psi_c)^*(\omega_0)| \leq \left(\frac{c}{\operatorname{ent}(X, g_0)}\right)^m \operatorname{vol}(Y, g)$$

passing the volume form to the quotient. As c goes to $\operatorname{ent}_{\ker f_*}(Y, g)$, we obtain

$$\Omega(X, g_0) \leq \frac{\Omega_{\ker f_*}(Y, g)}{k}. \tag{3.6}$$

Taking the infimum over all piecewise Riemannian metrics g on Y and over all geometric cycles (Y, f) representing $k[X]$, we derive $\Omega(X, g_0) \leq \|X\|_E$ after letting k go to infinity.

The reverse inequality is obvious. ■

Remark 3.4. The assertion (6) can be extended to the following case. If M is an orientable connected closed $2m$-manifolds given by a compact quotient of the product of m hyperbolic planes $\mathbb{H}^2$ then

$$\|M\|_E = \Omega(M, g_0)$$

where g_0 is the unique locally symmetric metric of minimal volume entropy on M among all locally symmetric metrics of given volume. Indeed, the proof can be adapted to follow the argument of [36] based on the same calibration method as in [9]. Loosely speaking, we replace $\tilde{X}$ by the m-fold product $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$, $\partial\tilde{X}$ by the Furstenberg boundary $\partial_F(\mathbb{H}^2 \times \cdots \times \mathbb{H}^2) = \mathbb{T}^m$, $d\theta$ by the product Lebesgue probability measure on $\mathbb{T}^m$, and $p_0(x, \theta)$ by the product $p_0(x_1, \theta_1) \cdots p_0(x_m, \theta_m)$ of the Poisson kernel of $\partial\mathbb{H}^2$. The main difference is that the calibration form $\pi^*\omega_0$ of the $\pi_1(X)$-equivariant embedding Φ_0 is not constructed from the barycenter map π, but is given by a combinatorial $(2m-1)$-cocycle of $\mathbb{T}^m$; see [36] for details.

We need a couple of definitions to present the next result.

Definition 3.5. For $i = 1, 2$, let V_i be an $\mathbb{R}$-vector space endowed with a semi-norm $\|\cdot\|_i$. The tensor product $V_1 \otimes V_2$ inherits the semi-norm $\|\cdot\|_\otimes$ given by the tensor product of $\|\cdot\|_1$ and $\|\cdot\|_2$; see [44]. By definition, for every $u \in V_1 \otimes V_2$, we have

$$\|u\|_\otimes = \inf\left\{\sum_s \|x_s\|_1 \|y_s\|_2 \;\middle|\; u = \sum_s x_s \otimes y_s\right\} \tag{3.7}$$

where the infimum is taken over all representations of u by finite sums of simple tensor products.

Below, we endow the direct sum of semi-normed vector spaces with the direct sum of the semi-norms. With this convention, the graded vector space of the real homology $H_*(X;\mathbb{R})$ of a path-connected topological space X is endowed with the *graded volume entropy semi-norm* $\|\cdot\|_{*E}$ given on each homogeneous component by

$$\|\mathbf{a}\|_{mE} = \frac{1}{m^m}\|\mathbf{a}\|_E$$

for every $\mathbf{a} \in H_m(X;\mathbb{R})$.

The real homology of the direct product $X_1 \times X_2$ of path-connected topological spaces X_1 and X_2 is canonically endowed with two semi-norms. The first one is the usual volume entropy semi-norm $\|\cdot\|_{*E}$. The second one is defined via Künneth's formula

$$H_m(X_1 \times X_2;\mathbb{R}) \simeq \bigoplus_{i+j=m} H_i(X_1;\mathbb{R}) \otimes H_j(X_2;\mathbb{R})$$

as the tensor product norm (see (3.7)), of the graded volume entropy semi-norms $\|\cdot\|_{*E}^{(i)}$ on $H_*(X_i;\mathbb{R})$. It is denoted by $\|\cdot\|_E^{\otimes}$.

We can now state our next result.

Theorem 3.6. *Let X_1 and X_2 be path-connected topological spaces. Then, for every $\mathbf{a} \in H_*(X_1 \times X_2;\mathbb{R})$,*

$$\|\mathbf{a}\|_{*E} \leq \|\mathbf{a}\|_E^{\otimes}.$$

Proof. It is enough to prove the inequality for homogeneous elements. Every homology class $\mathbf{a} \in H_m(X_1 \times X_2;\mathbb{R})$ admits a representation as a sum of simple tensor products

$$\mathbf{a} = \sum_s \mathbf{x}_s \otimes \mathbf{y}_s \tag{3.8}$$

where $\mathbf{x}_s \in H_{i_s}(X_1;\mathbb{R})$ and $\mathbf{y}_s \in H_{j_s}(X_2;\mathbb{R})$ with $i_s + j_s = m$. By the triangle inequality,

$$\|\mathbf{a}\|_{*E} \leq \sum_s \|\mathbf{x}_s \otimes \mathbf{y}_s\|_{*E}. \tag{3.9}$$

By multiplying if necessary the homology class $\mathbf{a}$ by an appropriate natural number, we can suppose that all classes $\mathbf{x}_s$ and $\mathbf{y}_s$ in (3.8) are represented by closed manifolds. Proposition 2.6 of [1] implies that

$$\|\mathbf{x}_s \otimes \mathbf{y}_s\|_{*E} \leq \|\mathbf{x}_s\|_{*E}\,\|\mathbf{y}_s\|_{*E}.$$

Plugging this bound in (3.9) and minimizing over all the simple tensor product representations (3.8), we obtain the desired inequality. ∎

4. Volume entropy semi-norm and simplicial volume

In this section, we show that the volume entropy semi-norm of a homology class is bounded from above and below by its simplicial volume, up to some multiplicative constants depending only on the dimension of the homology class. Therefore, the volume entropy semi-norm and the simplicial volume are equivalent homology semi-norms.

4.1. Geometrization of the simplicial volume

Let us introduce some topological invariants.

Definition 4.1. Let K be an m-dimensional topological space supplied with a finite pseudo-triangulation (also referred to as a *pseudo-simplicial complex* or a Δ-*complex*; see [28, Section 2.1]). Loosely speaking, the space K is a finite cell complex where the closure of each cell is homeomorphic to the standard simplex of the same dimension. In comparison with usual simplicial complexes, a simplex in a pseudo-triangulation is not uniquely defined by its vertices. An m-dimensional *geometric* Δ-*cycle* is a disjoint finite union of m-dimensional Δ-complexes whose pseudo-triangulations satisfy conditions (1)–(3) of Definition 2.1.

The *geometric complexity* of K, denoted by $\kappa(K)$, is the number of m-simplices of K. Define the *geometric complexity* of a homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ as

$$\kappa(\mathbf{a}) = \inf_P \kappa(P)$$

where P runs over the m-dimensional geometric Δ-cycles representing $\mathbf{a}$. That is, there is a map $h : P \to X$ such that $h_*([P]) = \mathbf{a}$ where the class $[P]$ is the sum of the fundamental classes of the connected components of P with the appropriate orientations. Define also the *average geometric complexity* of $\mathbf{a}$ as

$$\kappa^\infty(\mathbf{a}) = \lim_{n\to\infty} \frac{\kappa(n\mathbf{a})}{n}. \tag{4.1}$$

Note that the function $\kappa(n\mathbf{a})$ is subadditive in n, which ensures the existence of the limit (4.1). Furthermore, $\kappa^\infty(\mathbf{a}) \leq \kappa(n\mathbf{a})/n$ for every $n \geq 1$.

We will also need the following definition extending the notion of simplicial volume to homology classes with coefficients in more general rings.

Definition 4.2. Let X be a topological space and $\mathbb{A} = \mathbb{Z}$ or $\mathbb{Q}$. For every $\mathbf{a} \in H_m(X;\mathbb{A})$, define

$$\|\mathbf{a}\|_\Delta^{\mathbb{A}} = \inf_c \|c\|_1$$

where the infimum is taken over all m-cycles $c \in C_m(X;\mathbb{A})$ with coefficients in $\mathbb{A}$ representing $\mathbf{a}$.

We present a couple of known results, including the proofs for the sake of completeness. The first result can be found in [45, Lemma 2.9]. See [27, item 5.41 (a)] for a previous statement.

Lemma 4.3. *Every homology class* $\mathbf{a} \in H_m(X;\mathbb{Z})$ *satisfies*

$$\|\mathbf{a}\|_\Delta = \|\mathbf{a}\|_\Delta^{\mathbb{Q}}.$$

Proof. Let σ (resp. σ') be a real (resp. rational) m-cycle representing the real homology class induced by $\mathbf{a}$. The difference $\sigma - \sigma'$ is the boundary of an $(m+1)$-chain

$c \in C_{m+1}(X;\mathbb{R})$, that is,

$$\sigma - \sigma' = \partial c.$$

By density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational $(m+1)$-chain c' (with the same support) such that $\|c - c'\|_1$ is arbitrarily small. Since the boundary of every $(m+1)$-simplex is formed of $m+2$ simplices of dimension m, we have

$$\|\partial z\|_1 \leq (m+2)\|z\|_1$$

for every $(m+1)$-chain $z \in C_{m+1}(X;\mathbb{R})$. Thus,

$$\|\sigma - (\sigma' + \partial c')\|_1 \leq \|\partial(c - c')\|_1 \leq (m+2)\|c - c'\|_1$$

is arbitrarily small. Therefore, the real cycle σ and the rational cycle $\sigma' + \partial c'$, which both represent the real homology class induced by $\mathbf{a}$, have arbitrarily close $\|\cdot\|_1$-semi-norms. Hence the result follows. ∎

Proposition 4.4. *Every homology class* $\mathbf{a} \in H_m(X;\mathbb{Z})$ *satisfies*

$$\|\mathbf{a}\|_\Delta^\mathbb{Z} = \kappa(\mathbf{a}), \tag{4.2}$$

$$\|\mathbf{a}\|_\Delta = \kappa^\infty(\mathbf{a}). \tag{4.3}$$

Proof. The inequality $\|\mathbf{a}\|_\Delta^\mathbb{Z} \leq \kappa(\mathbf{a})$ is obvious and the reverse inequality $\kappa(\mathbf{a}) \leq \|\mathbf{a}\|_\Delta^\mathbb{Z}$ can be found in [34, Proposition 2.1] (and also follows from [28, pp. 108–109]). This yields (4.2).

Applying the average procedure of (4.1) to the obvious inequality $\|\mathbf{a}\|_\Delta \leq \kappa(\mathbf{a})$ yields the bound $\|\mathbf{a}\|_\Delta \leq \kappa^\infty(\mathbf{a})$. For every $\varepsilon > 0$, we also have

$$\frac{\kappa(n\mathbf{a})}{n} = \frac{\|n\mathbf{a}\|_\Delta^\mathbb{Z}}{n} \leq \|\mathbf{a}\|_\Delta^\mathbb{Q} + \varepsilon$$

for some positive integer n, where the first equality follows from (4.2). Thus, $\kappa^\infty(\mathbf{a}) \leq \|\mathbf{a}\|_\Delta^\mathbb{Q} + \varepsilon$ by subadditivity of the function $\kappa(n\mathbf{a})$ with respect to n. By Lemma 4.3, this yields the bound $\kappa^\infty(\mathbf{a}) \leq \|\mathbf{a}\|_\Delta + \varepsilon$ for every $\varepsilon > 0$. This shows (4.3). ∎

4.2. Universal realization of homology classes

Let us introduce some results in geometric topology following [20–22], which rely on C. Tomei's work [49].

Definition 4.5. The *m-permutahedron* Π^m is the convex hull of the $(m+1)!$ points obtained by permutations of the coordinates of the point $(1, 2, \ldots, m+1)$ of $\mathbb{R}^{m+1}$. It is an m-dimensional simple convex polytope of $\mathbb{R}^{m+1}$ with $2^{m+1} - 2$ facets, i.e., $(m-1)$-faces, that lies in the hyperplane

$$x_1 + \cdots + x_{m+1} = \sum_{j=1}^{m+1} j = \frac{(m+1)(m+2)}{2}.$$

Here, an m-polytope is *simple* if each of its vertices is contained in exactly m facets.

From a more geometric point of view (see [23]), the m-permutahedron Π^m can be obtained by truncating the standard simplex $\Delta^m \subseteq \mathbb{R}^{m+1}$ given by

$$x_1 + \cdots + x_{m+1} = 1$$

with $x_i \geq 0$ as follows. First, truncate the vertices of Δ^m by the hyperplanes $x_i = 1 - \frac{1}{4}$. Then, truncate the edges of Δ^m by the hyperplanes

$$x_{i_1} + x_{i_2} = 1 - \left(\frac{1}{4}\right)^2.$$

At the k-th step, truncate the $(k-1)$-faces of Δ^m by the hyperplanes

$$x_{i_1} + \cdots + x_{i_k} = 1 - \left(\frac{1}{4}\right)^k.$$

The resulting polytope is combinatorially equivalent to the m-permutahedron Π^m. The faces F of Π^m correspond to the faces $\Delta = \Delta_F$ of Δ^m after truncation of which they appear.

Consider the canonical piecewise linear map $\Theta : \Pi^m \to \Delta^m$ which takes every face F of Π^m to its corresponding face Δ_F in Δ^m. More precisely, define Θ on the barycenters b_F of the faces of $\mathbb{T}^m$ by sending b_F to the barycenter of Δ_F. Then extend this map linearly to every simplex of the barycentric subdivision of $\mathbb{T}^m$; see [20]. Note that Θ is a degree 1 map which is injective in the interior of Π^m.

Definition 4.6. Consider the manifold M_0 of real symmetric tridiagonal matrices of size $m+1$ with eigenvalues $\lambda_i = i$, for $i = 1, \ldots, m+1$. Here, a matrix $A = (a_{i,j})$ is *tridiagonal* if $a_{i,j} = 0$ whenever $|i-j| > 1$. The manifold M_0 will be referred to as the *isospectral m-manifold*.

It was proved by C. Tomei [49] that the isospectral manifold M_0 is an orientable closed aspherical m-manifold. By [16, 20–22, 49], the isospectral m-manifold M_0 is tiled by 2^m copies of the m-permutahedron Π^m. More precisely, the manifold M_0 can be decomposed as

$$M_0 \simeq (\mathbb{Z}_2^m \times \Pi^m)/\sim$$

where the equivalence relation is generated by $(s, x) \sim (r_{|\omega|}s, x)$ whenever $x \in F_\omega$. Here, the elements r_i are the standard generators of $\mathbb{Z}_2^m$.

We will rely on the following universal property established by A. Gaifullin [21, 22] regarding Steenrod's problem and the realization of cycles by closed manifolds.

Theorem 4.7 ([21, 22]). *Let X be a path-connected topological space. Then for every homology class $\mathbf{a} \in H_m(X; \mathbb{Z})$, there exist a connected finite-fold covering $\widehat{M}_0 \to M_0$ of the isospectral manifold and a map $f : \widehat{M}_0 \to X$ such that*

$$f_*([\widehat{M}_0]) = q\mathbf{a}$$

for some positive integer q depending on $\mathbf{a}$.

Remark 4.8. When $m = 2$, the isospectral surface M_0 is a genus 2 surface. More precisely, the permutahedron is a hexagon, the surface M_0 is tiled with four copies of this hexagon, and these four copies surround every vertex of M_0; see [49].

4.3. Homology norm comparison: upper bound on the volume entropy semi-norm

Let us state the main theorem of this section.

Theorem 4.9. *Let m be a positive integer. Then there exists a constant $C_m > 0$ such that every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ of a path-connected topological space X satisfies*

$$\|\mathbf{a}\|_E \leq C_m \|\mathbf{a}\|_\Delta.$$

Proof. Let $\mathbf{a} \in H_m(X;\mathbb{Z})$. By Proposition 4.4 (see (4.3)), for every $\varepsilon > 0$ and every integer s large enough, there exists a map $h = h_s : P \to X$ from an m-dimensional geometric Δ-cycle $P = P_s$ such that

$$h_*([P]) = s\mathbf{a}, \tag{4.4}$$

$$s\|\mathbf{a}\|_\Delta \leq \kappa(P) \leq s(\|\mathbf{a}\|_\Delta + \varepsilon). \tag{4.5}$$

The second barycentric subdivision of P gives rise to a simplicial structure on P; see [28]. In general, the complex P is not connected. After the second barycentric subdivision, we can take the connected sum of the connected components by omitting some m-simplices and gluing together the components to obtain an orientable connected closed pseudomanifold, still denoted by P. Note that this operation does not increase the number of m-simplices. Taking a third barycentric subdivision ensures that the simplicial structure admits a regular coloring in $m + 1$ colors (that is, any two vertices connected by an edge are of distinct colors) in order to apply some constructions of [20]. Recall that the barycentric subdivision of a simplicial complex admits a regular coloring where every vertex which is the barycenter of an r-simplex of the original triangulation is of color r. The pseudomanifold P with this simplicial structure is denoted by Z. Since the barycentric subdivision of an m-simplex gives rise to $m!$ simplices of dimension m, we obtain

$$\kappa(Z) \leq (m!)^3 \kappa(P). \tag{4.6}$$

By Theorem 4.7, there exists a map $f : \widehat{M}_0 \to Z$ from a finite covering $\widehat{M}_0$ of M_0 such that

$$f_*([\widehat{M}_0]) = q[Z] \in H_m(Z;\mathbb{Z}) \tag{4.7}$$

for some positive integer q.

Consider the piecewise flat metric on M_0 where all permutahedra are isometric to the standard permutahedron Π^m with its canonical Euclidean metric. The volume of M_0 is equal to $2^m v_m$ where v_m is the Euclidean volume of Π^m.

By construction (see [20–22]), the map $f : \widehat{M}_0 \to Z$ has the following features. The map $f : \widehat{M}_0 \to Z$ takes every permutahedron Π^m of $\widehat{M}_0$ to a simplex Δ^m of Z and its restriction to Π^m agrees with the canonical piecewise linear map $\Theta : \Pi^m \to \Delta^m$ introduced in Definition 4.5. Furthermore, the number of permutahedra of the covering $\widehat{M}_0$ is equal to $q\kappa(Z)$ where q is the degree of f; see [20, end of proof of Proposition 5.3]. Therefore, the volume of $\widehat{M}_0$ satisfies

$$\mathrm{vol}(\widehat{M}_0) = q\kappa(Z)v_m \tag{4.8}$$

where v_m is the Euclidean volume of Π^m.

Consider the composite map $\varphi = h \circ f : \widehat{M}_0 \to Z \simeq P \to X$. We derive from (4.7) and (4.4) that

$$\varphi_*([\widehat{M}_0]) = qs\mathbf{a}.$$

By definition of the volume entropy semi-norm, we have

$$\|qs\mathbf{a}\|_E \leq \mathrm{ent}_\varphi(\widehat{M}_0)^m \,\mathrm{vol}(\widehat{M}_0).$$

It follows from (4.8), (4.6) and (4.5) that

$$\mathrm{vol}(\widehat{M}_0) \leq q(m!)^3 s(\|\mathbf{a}\|_\Delta + \varepsilon)v_m.$$

Since $\mathrm{ent}_\varphi(\widehat{M}_0) \leq \mathrm{ent}(M_0)$, we deduce that

$$qs\|\mathbf{a}\|_E \leq qs(m!)^3 C'_m(\|\mathbf{a}\|_\Delta + \varepsilon)$$

where $C'_m = \mathrm{ent}(M_0)^m v_m$ is a constant which only depends on m. Simplifying by qs and letting ε go to zero, we obtain

$$\|\mathbf{a}\|_E \leq C_m\|\mathbf{a}\|_\Delta$$

where $C_m = (m!)^3 C'_m$. ■

Remark 4.10. An estimate on the volume entropy $\mathrm{ent}(M_0)$ of the isospectral m-manifold provides an estimate on the constant C_m in Theorem 4.9.

Remark 4.11. A referee pointed out to us that Theorem 4.9 can be derived from [15, Proposition 7.11] combined with Theorem 3.1 (4), once the notion of the volume entropy semi-norm is well established. The current presentation allows us to make a connection with the proof of Theorem 5.1.

4.4. Homology norm comparison: lower bound on the volume entropy semi-norm

Let us show a reverse inequality to Theorem 4.9.

Theorem 4.12. *Let m be a positive integer. Then there exists a constant $c_m > 0$ such that every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ of a path-connected topological space X satisfies*

$$\|\mathbf{a}\|_E \geq c_m\|\mathbf{a}\|_\Delta.$$

In order to prove this theorem, we will need the following classical interpretation of the simplicial volume in terms of bounded cohomology; see [24] for the definitions.

Proposition 4.13. *Let X be a topological space. Then every homology class $\mathbf{a} \in H_m(X;\mathbb{R})$ satisfies*

$$\|\mathbf{a}\|_\Delta = \sup\{1/\|\alpha\|_\infty \mid \alpha \in H_b^m(X;\mathbb{R}),\ \langle \alpha, \mathbf{a}\rangle = 1\}$$

where $H_b^m(X;\mathbb{R})$ denotes the bounded cohomology of X of degree m.

The following result is a technical extension of M. Gromov's inequality (1.6).

Proposition 4.14. *Let m be a positive integer. Then there exists a constant $c_m > 0$ such that for every map $\Phi : M \to X$ from an oriented connected closed m-manifold M to a path-connected topological space X, we have*

$$\Omega_{\ker \Phi_*}(M) \geq c_m \|\Phi_*([M])\|_\Delta.$$

Proof. Fix a Riemannian metric g on M. Let $\bar{M} \to M$ be the covering of M with fundamental group $\ker \Phi_*$. The quotient group $\Gamma = \pi_1(\bar{M})/\ker \Phi_*$ acts by deck transformations on $\bar{M}$. Denote by $\mathcal{M}(\bar{M})$ the Banach space of finite (signed) measures μ on $\bar{M}$ with the norm

$$\|\mu\| = \int_{\bar{M}} |\mu|.$$

Denote also by $\mathcal{M}^+(\bar{M}) \subseteq \mathcal{M}(\bar{M})$ the cone of positive measures. Following [24, Section 2.4], a *smoothing operator* on $\bar{M}$ is a smooth Γ-equivariant map $\mathscr{S} : \bar{M} \to \mathcal{M}^+(\bar{M})$. Define

$$[\mathscr{S}] = \sup_{x \in \bar{M}} \frac{\|d_x \mathscr{S}\|}{\|\mathscr{S}(x)\|}.$$

Let $\alpha \in H_b^m(X;\mathbb{R})$ such that $\langle \alpha, \Phi_*([M])\rangle = 1$ where $\langle \cdot, \cdot \rangle$ is the bilinear pairing between cohomology and homology given by the Kronecker product. Define $\beta = \Phi^*(\alpha) \in H_b^m(M;\mathbb{R})$. Clearly, $\|\beta\|_\infty \leq \|\alpha\|_\infty$ and $\langle \beta, [M]\rangle = 1$. By [24, Proposition, p. 33], there exists a closed m-form ω on M representing the cohomology class $\beta \in H^m(M;\mathbb{R})$ such that

$$\|\omega\| \leq m! \|\beta\|_\infty [\mathscr{S}]^m \tag{4.9}$$

for every smoothing operator $\mathscr{S} : \bar{M} \to \mathcal{M}^+(\bar{M})$.

For $\lambda > \mathrm{ent}_{\ker \Phi_*}(M)$, define $\mathscr{S} = \mathscr{S}_{\lambda,R} : \bar{M} \to \mathcal{M}^+(\bar{M})$ as

$$\mathscr{S}(x) = (e^{-R d_{\bar{g}}(x,\cdot)} - e^{-\lambda R}) \mathbb{1}_{B_{\bar{g}}(x,R)}(\cdot) \mathrm{dvol}_{\bar{g}}(\cdot).$$

By [24, Section 2.5] (see also [7] for further details), there exists a positive constant A_m depending only on m such that

$$[\mathscr{S}] \leq A_m \lambda \tag{4.10}$$

for R large enough. Technically speaking, the bound is stated in [7, 24] when $\bar{M}$ is the universal covering of M, but the proof is exactly the same for intermediate coverings.

Integrating ω on M using the relation $\langle \omega, [M]\rangle = 1$ and the combination of (4.9) with the bounds $\|\beta\|_\infty \leq \|\alpha\|_\infty$ and (4.10), we obtain

$$1 \leq m!(A_m)^m \|\alpha\|_\infty \mathrm{ent}_{\ker \Phi_*}(M)^m \,\mathrm{vol}(M).$$

Hence,

$$c_m \|\Phi_*([M])\|_\Delta \leq \Omega_{\ker \Phi_*}(M)$$

by Proposition 4.13, with $c_m = (m!(A_m)^m)^{-1}$ where A_m is the multiplicative constant in (4.10). ∎

We can now proceed to the proof of Theorem 4.12.

Proof of Theorem 4.12. For every $\varepsilon > 0$, there exists a positive integer k such that

$$\Omega(k\mathbf{a}) \leq k(\|\mathbf{a}\|_E + \varepsilon).$$

Thus, there exists a map $\varphi : P \to X$ defined on an oriented connected closed m-pseudo-manifold P such that $\varphi_*([P]) = k\mathbf{a}$ and

$$\Omega_{\ker \varphi_*}(P) \leq \Omega(k\mathbf{a}) + \varepsilon \leq k\|\mathbf{a}\|_E + (k+1)\varepsilon. \tag{4.11}$$

By Thom's theorem [48], there exists a map $f : M \to P$ defined on an oriented connected closed m-manifold M such that

$$f_*([M]) = d[P] \in H_m(P; \mathbb{Z})$$

for some suitable nonzero integer d. Extend $f : M \to P$ by handle attachments to a π_1-surjective map $f' : M' \to P$ where

$$M' = M \sharp \Big(\sharp_{i=1}^{l} S^1 \times S^{m-1} \Big).$$

Clearly, $f'_*([M']) = f_*([M]) = d[P]$. By [1, Proposition 2.2], we have

$$\Omega_{\ker \Phi_*}(M') \leq d\,\Omega_{\ker \varphi_*}(P) \tag{4.12}$$

where $\Phi : M' \to X$ is the composite map $\Phi = \varphi \circ f'$.

Now observe that $\Phi_*([M']) = dk\mathbf{a}$. By Proposition 4.14, we derive

$$c_m dk \|\mathbf{a}\|_\Delta \leq \Omega_{\ker \Phi_*}(M'). \tag{4.13}$$

Combining (4.11)–(4.13), dividing by dk and letting ε go to zero, we obtain

$$c_m \|\mathbf{a}\|_\Delta \leq \|\mathbf{a}\|_E$$

as desired. ∎

Remark 4.15. By a density argument, it follows from Theorems 4.9 and 4.12 that the volume entropy semi-norm and the simplicial volume semi-norm are equivalent in real homology, and not only in integral homology. A natural question would be to determine whether the two semi-norms are proportional in every degree, though we do not have any strong evidence for whether they are or not.

5. Systolic volume of a multiple homology class

The following asymptotically optimal upper bound on the systolic volume of the multiples of a given homology class positively answers a conjecture of [5], where a sublinear upper bound was established.

Theorem 5.1. *Let m be a positive integer. For every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ of a path-connected topological space X, there exists a constant $C = C(\mathbf{a}) > 0$ such that for every $k \geq 2$, we have*

$$\sigma(k\mathbf{a}) \leq C \frac{k}{(\log k)^m}.$$

The proof of Theorem 5.1 rests on some systolic estimates in geometric group theory based on the following notion.

Definition 5.2. Let G be a finitely generated group and S be a finite generating set of G. Denote by d_S the word distance induced by S. For every finite index subgroup $\Gamma \leqslant G$, define

$$\mathrm{sys}(\Gamma, d_S) = \inf_{\gamma \in \Gamma \setminus \{e\}} d_S(e, \gamma).$$

The systolic growth of finitely generated linear groups has been described by K. Bou-Rabee and Y. Cornulier; see [11]. Originally stated in terms of residual girth rather than systolic growth, their result can be written as follows.

Theorem 5.3. *Let G be a finitely generated linear group over a field and let S be a finite symmetric generating set of G. Then there exist a constant $C_0 > 0$ and an infinite sequence of subgroups $\Gamma_k \leqslant G$ of finite index k such that*

$$\mathrm{sys}(\Gamma_k, d_S) \geq C_0 \log k.$$

Remark 5.4. A similar estimate has been previously stated without proof by M. Gromov for finitely generated subgroups G of $\mathrm{SL}_d(\mathbb{Z})$ under the extra assumption that no unipotent element lies in G; see [26, Elementary Lemma, p. 334].

We need to review some features of the isospectral m-manifold M_0 introduced in Definition 4.6.

It was proved by C. Tomei [49] that M_0 is an orientable closed aspherical m-manifold. By M. Davis [16], its fundamental group $G = \pi_1(M_0)$ is isomorphic to a torsion-free subgroup of finite index of the Coxeter group

$$\begin{aligned} W = \langle s_1, \dots, s_m, r_1, \dots, r_m \mid s_i^2 = r_i^2 = 1,\ s_i s_j = s_j s_i \text{ for } |i-j| > 1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, r_i r_j = r_j r_i,\ s_i r_j = r_j s_i \text{ for } i \neq j \rangle. \end{aligned}$$

Recall that J. Tits showed that every Coxeter group admits a faithful linear representation in a finite-dimensional vector space; see [12, Chapter V, Section 4, Corollary 2]. Thus, the group G is linear. This is an important feature in view of Theorem 5.3.

We can now proceed to the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $G = \pi_1(M_0)$. Fix a finite symmetric generating set S of G once and for all. The metric on M_0 induced by the Hilbert–Schmidt metric (also called the Frobenius metric) on the space $\mathcal{M}_{m+1}(\mathbb{R})$ of square matrices of size $m+1$ lifts to a metric d_0 on the universal covering $\widetilde{M}_0$ of M_0. (Here, the choice of the metric does not matter. We simply fix one once and for all.) Since M_0 is compact, its universal covering $\widetilde{M}_0$ is quasi-isometric to (G, d_S); see [17, Section IV.B, Theorem 23]. More precisely, there exist some constants $A_0 > 1$ and $B_0 > 0$ such that for every $\gamma \in G$ and every $x \in \widetilde{M}_0$, we have

$$A_0^{-1} d_S(e, \gamma) - B_0 \leq d_0(x, \gamma \cdot x) \leq A_0 d_S(e, \gamma) + B_0. \tag{5.1}$$

Note that A_0 and B_0 only depend on m.

By Theorem 4.7, there exist a map $f : \widehat{M}_0 \to X$ from a finite covering $\widehat{M}_0$ of M_0 and a positive integer q such that

$$f_*([\widehat{M}_0]) = q[\mathbf{a}]. \tag{5.2}$$

Let $\Gamma \leqslant \widehat{G} := \pi_1(\widehat{M}_0)$ be a finite index subgroup of $\widehat{G}$. Denote by $f_\Gamma : \widetilde{M}_0/\Gamma \to X$ the lift of $f : \widehat{M}_0 \to X$ under the canonical projection $\pi_\Gamma : \widetilde{M}_0/\Gamma \to \widehat{M}_0$. By the first inequality of (5.1), we have

$$A_0^{-1} \operatorname{sys}(\Gamma, d_S) - B_0 \leq \operatorname{sys}(\widetilde{M}_0/\Gamma) \leq \operatorname{sys}_{f_\Gamma}(\widetilde{M}_0/\Gamma). \tag{5.3}$$

Now, apply Theorem 5.3 about the systolic growth of linear groups to the finitely generated linear group $\widehat{G}$. Thus, there exists a sequence of subgroups $\Gamma_k \leqslant \widehat{G}$ of finite index $[\widehat{G} : \Gamma_k] = k \geq 2$ such that

$$\operatorname{sys}(\Gamma_k, d_S) \geq C_0 \log k \tag{5.4}$$

for some $C_0 > 0$ which does not depend on k.

Let $\widehat{M}_k = \widetilde{M}_0/\Gamma_k$. We deduce from (5.3) that

$$\operatorname{sys}_{f_k}(\widehat{M}_k) \geq A_0^{-1} C_0 \log k - B_0 \geq D_0 \log k \tag{5.5}$$

where $f_k : \widehat{M}_k \to X$ is the lift of $f : \widehat{M}_0 \to X$ and $D_0 > 0$ does not depend on k. Since Γ_k is of index k in $\widehat{G}$ and the map $f : \widehat{M}_0 \to X$ represents $q\mathbf{a}$, we deduce that

$$(f_k)_*([\widehat{M}_k]) = kq\mathbf{a}.$$

Since $\operatorname{vol}(\widehat{M}_k) = k \operatorname{vol}(\widehat{M}_0)$, this yields the inequalities

$$\sigma(kq\mathbf{a}) \leq \sigma_{f_k}(\widehat{M}_k) \leq C' \frac{k}{(\log k)^m}$$

for every $k \geq 2$, where $C' = C'(\mathbf{a})$ does not depend on k. Since σ is subadditive (see [5, Proposition 3.6]), we derive

$$\sigma(k\mathbf{a}) \leq \sigma\left(\left\lceil \frac{k}{q} \right\rceil q\mathbf{a}\right) \leq C \frac{k}{(\log k)^m}$$

for every $k \geq 2$, where $C = C(\mathbf{a})$ does not depend on k. ∎

6. Systolic volume semi-norm and functorial properties

In this section, we define the systolic volume semi-norm and present its functorial properties along with some comparison results.

6.1. Systolic volume semi-norm

Theorem 5.1 allows us to define the systolic volume semi-norm in real homology of dimension $m \geq 3$. This definition is based on the following observation, whose proof is left to the reader.

Lemma 6.1. *Let $\mathfrak{M}$ be a $\mathbb{Z}$-module endowed with a translation-invariant pseudo-distance ϱ. Given a function $h : \mathbb{N} \to \mathbb{R}_+$ with $\lim_{k\to\infty} h(k) = \infty$, suppose that for every $\mathbf{a} \in \mathfrak{M}$, there is a positive constant $C = C(\mathbf{a})$ such that $\varrho(0, k\mathbf{a}) \leq C\, h(k)$ for every $k \in \mathbb{N}$. Then*

$$\widehat{\varrho}(\mathbf{a}, \mathbf{b}) = \limsup_{k\to\infty} \frac{\varrho(k\mathbf{a}, k\mathbf{b})}{h(k)} \tag{6.1}$$

defines a translation-invariant pseudo-distance on $\mathfrak{M}$.

Let X be a path-connected topological space. Apply Lemma 6.1 with $h(k) = \frac{k}{(\log k)^m}$ (see Theorem 5.1) to the translation-invariant pseudo-distance ϱ defined on $H_m(X;\mathbb{Z})$ with $m \geq 3$ by $\varrho(\mathbf{a}, \mathbf{b}) = \sigma(\mathbf{a} - \mathbf{b})$; see [5, Corollary 5.3]. This yields a new translation-invariant pseudo-distance $\widehat{\varrho}$ on $H_m(X;\mathbb{Z})$. Define the *systolic volume semi-norm* of $\mathbf{a} \in H_m(X;\mathbb{Z})$ as

$$\|\mathbf{a}\|_\sigma = \lim_{k\to\infty} \frac{\widehat{\sigma}(k\mathbf{a})}{k} \tag{6.2}$$

where $\widehat{\sigma}(\mathbf{a}) = \widehat{\varrho}(0, \mathbf{a})$.

Remark 6.2. The behavior of $\sigma(k\mathbf{a})$ as a function of k can be quite irregular; see [5, Section 5.4]. This suggests it may not be possible to replace the lim sup in (6.1) by the usual limit for $\varrho(\mathbf{a}, \mathbf{b}) = \sigma(\mathbf{a} - \mathbf{b})$. It is also unclear whether, though unlikely that the second stablization process in (6.2) can be omitted in the definition of the systolic volume semi-norm.

The following lemma is useful to establish upper bounds on the systolic volume semi-norm.

Lemma 6.3. *Let $\mathbf{a}, \mathbf{b} \in H_m(X;\mathbb{Z})$ with $m \geq 3$, where X is a path-connected topological space.*

(1) *Suppose that there exists $\lambda \geq 0$ such that*

$$\sigma(k\mathbf{a}) \leq \lambda \sigma(k\mathbf{b})$$

for every integer $k \geq 1$. Then $\|\mathbf{a}\|_\sigma \leq \lambda \|\mathbf{b}\|_\sigma$.

(2) *Suppose that there exists* $\sigma_0 \geq 0$ *such that*

$$\sigma(k\mathbf{a}) \leq \frac{k}{(\log k)^m}\sigma_0$$

for every integer $k \geq 2$. *Then* $\|\mathbf{a}\|_\sigma \leq \sigma_0$.

Proof. (1) By assumption, replacing k by kp, we have $\sigma(kp\mathbf{a}) \leq \lambda\sigma(kp\mathbf{b})$. Dividing this inequality by $\frac{k}{(\log k)^m}$ and letting k go to infinity, we obtain $\widehat{\sigma}(p\mathbf{a}) \leq \lambda\widehat{\sigma}(p\mathbf{b})$. Dividing this inequality by p and letting p go to infinity, we derive the desired bound $\|\mathbf{a}\|_\sigma \leq \lambda\|\mathbf{b}\|_\sigma$.

(2) By assumption, replacing k by kp, we have

$$\sigma(kp\mathbf{a}) \leq \frac{kp}{(\log k + \log p)^m}\sigma_0.$$

Dividing this inequality by $\frac{k}{(\log k)^m}$ and letting k go to infinity, we derive $\widehat{\sigma}(p\mathbf{a}) \leq p\sigma_0$. Dividing this inequality by p and letting p go to infinity yields $\|\mathbf{a}\|_\sigma \leq \sigma_0$ as desired. ■

6.2. Functorial properties of the systolic volume semi-norm

As in Section 3, we establish functorial properties of the systolic volume semi-norm similar to the ones satisfied by the simplicial volume.

Theorem 6.4. *Let* $m \geq 3$ *be an integer.*

(1) *Let* $f : X \to Y$ *be a continuous map between path-connected topological spaces. Then for every* $\mathbf{a} \in H_m(X;\mathbb{R})$,

$$\|f_*(\mathbf{a})\|_\sigma \leq \|\mathbf{a}\|_\sigma.$$

(2) *Let* $f : M \to K(\pi_1(M), 1)$ *be the classifying map of an orientable connected closed manifold* M. *Then*

$$\|f_*([M])\|_\sigma = \|M\|_\sigma.$$

(3) *Let* $f : M \to N$ *be a degree* d *map between oriented connected closed manifolds. Then*

$$\|M\|_\sigma \geq |d|\|N\|_\sigma.$$

(4) *Let* $f : M \to N$ *be a* d*-sheeted covering map between orientable connected closed manifolds. Then*

$$\|M\|_\sigma = d\|N\|_\sigma.$$

(5) *Let* M_1 *and* M_2 *be orientable connected closed manifolds of dimension* $m \geq 3$. *Then*

$$\|M_1 \sharp M_2\|_\sigma \leq \|M_1\|_\sigma + \|M_2\|_\sigma. \tag{6.3}$$

Remark 6.5. As in Remark 3.2, we can ask whether equality holds in (6.3). Here is a reasoning suggesting this question might be rather subtle. By [4, Theorem A], for every essential m-manifold M_1 with $m \geq 4$, there exists an essential m-manifold M_2 such that

$$\sigma(M_1 \sharp M_2) < \sigma(M_1) + \sigma(M_2).$$

Now, it is unclear whether this strict inequality subsists or not under the double stabilization process in the definition of the systolic volume semi-norm. More generally, the inequalities (3.1), (6.3) and the one for the simplicial volume reflect convexity properties of the semi-norms. Even if the three semi-norms are equivalent (as we will see), we have no strong evidence that the volume entropy semi-norm and the systolic volume semi-norm satisfy the same additivity property as the simplicial volume.

Proof of Theorem 6.4. We argue as in the proof of Theorem 3.1, pointing out only the differences.

(1) Observe that if (M, φ) is a geometric cycle representing $\mathbf{a} \in H_m(X; \mathbb{R})$ then $(M, f \circ \varphi)$ is a geometric cycle representing $f_*(\mathbf{a}) \in H_m(Y; \mathbb{R})$. Since every $(f \circ \varphi)$-noncontractible loop of M is f-noncontractible, we get $\text{sys}_f(M) \leq \text{sys}_{f\circ\varphi}(M)$. Hence, $\sigma_f(M) \geq \sigma_{f\circ\varphi}(M)$. This implies that $\sigma(f_*(\mathbf{a})) \leq \sigma(\mathbf{a})$. Replacing $\mathbf{a}$ by $k\mathbf{a}$ and applying Lemma 6.3 (1), we obtain the desired inequality $\|f_*(\mathbf{a})\|_\sigma \leq \|\mathbf{a}\|_\sigma$.

(2) We consider the degree k map $f_k : M_k \to M$ defined on the connected sum $M_k = M \sharp \cdots \sharp M$ of k copies of M. By definition, $\sigma_{f_k}(M_k) \geq \sigma(k[M])$. Consider also the composite map $F_k = f \circ f_k : M_k \to K(\pi_1(M), 1)$, where $(F_k)_*([M_k]) = kf_*([M])$. Note that F_k is π_1-surjective. By [13, Theorem 10.2], this implies that $\sigma_{F_k}(M_k) = \sigma(kf_*([M]))$. Since f is π_1-injective, $\text{sys}_{F_k}(M_k) = \text{sys}_{f_k}(M_k)$. Hence, $\sigma_{F_k}(M_k) = \sigma_{f_k}(M_k)$. Combining the previous estimates, we obtain

$$\sigma(kf_*([M])) \geq \sigma(k[M]).$$

By Lemma 6.3 (1), we derive $\|f_*([M])\|_\sigma \geq \|M\|_\sigma$. Since the reverse inequality follows from (1), this implies (2).

(3) By definition, assertion (3) immediately follows from (1).

(4) Construct a geometric cycle (Q, ψ) representing $k[N] \in H_m(N; \mathbb{Z})$ with $\psi : Q \to N$ π_1-surjective, whose (relative) systolic volume $\sigma_\psi(Q)$ is arbitrarily close to $\sigma(k[N])$. Consider the geometric cycle (P, φ) representing $k[M] \in H_m(M; \mathbb{Z})$ where P is the cover of Q corresponding to the subgroup $(\psi_*)^{-1}(\text{Im} f_*)$ and $\varphi : P \to M$ is the lift of $\psi : Q \to N$. Since $P \to Q$ is a d-sheeted covering map and $\ker \varphi_* = \ker \psi_*$, we have $\sigma_\varphi(P) \leq d\sigma_\psi(Q)$. Thus,

$$\sigma(k[M]) \leq d\sigma(k[N]).$$

By Lemma 6.3 (1), we derive the inequality $\|M\|_\sigma \leq d\|N\|_\sigma$. The reverse inequality follows from (3).

(5) The proof is similar to the proof of point (5) in Theorem 3.1. Simply replace the volume entropy semi-norm $\|\cdot\|_E$ by the systolic volume semi-norm $\|\cdot\|_\sigma$. ■

6.3. *Systolic volume comparison*

We present analogues of comparison results obtained for the minimal volume entropy (semi-norm) in Section 2 to the systolic volume (semi-norm) case.

It is convenient to introduce the following definitions. Let M be a connected closed m-manifold with a Riemannian metric g. Let $H \lhd \pi_1(M)$ be a normal subgroup. Define the *systole* of M *relative to* H, denoted by $\mathrm{sys}_H(M, g)$, as the length of the shortest loop of M whose homotopy class does not lie in H. As in (1.7), define the *systolic volume* of M *relative to* H as

$$\sigma_H(M) = \inf_g \frac{\mathrm{vol}(M, g)}{\mathrm{sys}_H(M, g)^m}$$

where the infimum is taken over all (piecewise) Riemannian metrics g on M. This is a slight modification of the definition (1.7). For $f : M \to X$ and $H = \ker f_*$, the invariants σ_f and σ_H coincide. Note that the definition of σ_H extends to finite simplicial complexes.

Though [5, Proposition 3.6] is stated for the absolute systolic volume, its short proof based on the comparison principle [2, Proposition 3.2] can easily be adapted to cover the relative case. Thus, we obtain

Proposition 6.6. *For $i = 1, 2$, let M_i be a connected pseudomanifold of dimension $m \geq 3$ and let $M_i \lhd \pi_1(M_i)$ be a normal subgroup. Then*

$$\sigma_{\langle\langle H_1 * H_2 \rangle\rangle}(M_1 \sharp M_2) \leq \sigma_{H_1}(M_1) + \sigma_{H_2}(M_2).$$

Theorem 2.12 holds true if one replaces the relative minimal entropy to the power m, namely Ω_H, with the relative systolic volume σ_H. (A previous version valid for "admissible" pseudomanifolds can be found in [5, Corollary 3.5].) More precisely, we have

Theorem 6.7. *For $i = 1, 2$, let M_i be a connected closed pseudomanifold of dimension $m \geq 3$ and $H_i \lhd \pi_1(M_i)$ be a normal subgroup. Let $H \lhd \pi_1(M_1) * \pi_1(M_2)$ be a normal subgroup such that the canonical inclusion $\pi_1(M_1) \leqslant \pi_1(M_1) * \pi_1(M_2)$ induces an inclusion*

$$\pi_1(M_1)/H_1 \leqslant (\pi_1(M_1) * \pi_1(M_2))/H.$$

Suppose M_2 is orientable. Then

$$\sigma_{H_1}(M_1) \leq \sigma_H(M_1 \sharp M_2).$$

Proof. The proof is similar to the one of Theorem 2.12, except that the bound on the volume entropy $\mathrm{ent}_{H_1}(M_1, g_\varepsilon)$ at the end of the proof of Theorem 2.12 should be replaced with

$$\begin{aligned} \mathrm{sys}_{H_1}(M_1, g_\varepsilon) &\geq \mathrm{sys}_{H_1}\Big(M_1 \sharp M_2 \bigcup_{M_2(m-2)} \mathrm{Cone}(M_2(m-2)), g'\Big) \\ &\geq \mathrm{sys}_H(M_1 \sharp M_2, g). \end{aligned}$$ ■

As previously, combining Theorem 6.7 and Proposition 6.6, we obtain the following analogue of Corollary 2.13 for the (relative) systolic volume. In the case of "admissible" pseudomanifolds, it follows from [5, Corollary 3.5 and Proposition 3.6].

Corollary 6.8. *For $i = 1, 2$, let M_i be a connected closed pseudomanifold of dimension $m \geq 3$ and let $H_i \lhd \pi_1(M_i)$ be a normal subgroup. Suppose M_2 is orientable and $\sigma_{H_2}(M_2) = 0$. Then*

$$\sigma_{\langle\langle H_1 * H_2 \rangle\rangle}(M_1 \sharp M_2) = \sigma_{H_1}(M_1).$$

The analogue of Theorem 2.19 for the systolic volume semi-norm holds true. Note however that its proof differs from the one of Theorem 2.19. This is due to the fact that the systolic volume and the minimal volume entropy do not have the same behavior under finite coverings.

Theorem 6.9. *Let G be a finitely presented group and H be a finite index subgroup of G. Let $m \geq 3$. Suppose that the canonical inclusion $i : H \hookrightarrow G$ induces a monomorphism between the m-dimensional rational homology groups*

$$(i_*)_m : H_m(H; \mathbb{Q}) \to H_m(G; \mathbb{Q}).$$

Then for every homology class $\mathbf{a} \in H_m(H; \mathbb{Z})$,

$$\|\mathbf{a}\|_\sigma = \|i_*(\mathbf{a})\|_\sigma.$$

Proof. Let $\mathbf{a} \in H_m(H; \mathbb{Z})$. By Thom's theorem [48], there exists an integer $q \geq 1$ such that $q i_*(\mathbf{a}) \in H_m(G; \mathbb{Z})$ can be represented by a geometric cycle $f : M \to K(G, 1)$ where M is a closed m-manifold. By adding handles if necessary, we can further assume that $f_* : \pi_1(M) \to G$ is surjective. Arguing as in the proof of Theorem 2.19, we construct a commutative diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\hat{f}} & K(H,1) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & K(G,1) \end{array}$$

where the vertical maps are d-sheeted coverings, such that

$$f_*([M]) = q i_*(\mathbf{a}) \quad \text{and} \quad \hat{f}_*([\widehat{M}]) = dq\mathbf{a} + \mathbf{c}$$

where $\mathbf{c} \in \operatorname{Tor} H_m(H; \mathbb{Z})$.

Denote by t the order of the torsion class $\mathbf{c}$. For every integer $k \geq 1$, consider the connected sum M_k of kt copies of M and the map $f_k : M_k \to K(G, 1)$ obtained by collapsing the attaching spheres of the connected sum M_k to a point and by applying $f : M \to K(G, 1)$ to each term M of the bouquet so obtained. By construction, the map $f_k : M_k \to K(G, 1)$ is π_1-surjective with

$$(f_k)_*([M_k]) = kt f_*([M]) = ktq i_*(\mathbf{a}).$$

Since M_k is a closed manifold and $f_k : M_k \to K(G, 1)$ is π_1-surjective, we infer from [2, 3] or [13, Theorem 10.2] that

$$\sigma_{f_k}(M_k) = \sigma(ktq i_*(\mathbf{a})).$$

The lift $\widehat{f_k} : \widehat{M}_k \to K(H,1)$ of $f_k : M_k \to K(G,1)$ to the corresponding covers $\widehat{M}_k$ of M_k represents

$$(\widehat{f_k})_*([\widehat{M}_k]) = kt\,\widehat{f}([\widehat{M}]) = kt(dq\mathbf{a} + \mathbf{c}) = ktdq\mathbf{a}.$$

Since the vertical map $\widehat{M}_k \to M_k$ is a d-sheeted covering, we have $\sigma_{\widehat{f_k}}(\widehat{M}_k) \leq d\sigma_{f_k}(M_f)$. Hence,

$$\sigma(ktdq\mathbf{a}) \leq \sigma_{\widehat{f_k}}(\widehat{M}_k) \leq d\sigma_{f_k}(M_k) = d\sigma(ktqi_*(\mathbf{a}))$$

for every $k \geq 1$. By Lemma 6.3 (1) and the homogeneity of the systolic volume semi-norm, we deduce that $\|\mathbf{a}\|_\sigma \leq \|i_*(\mathbf{a})\|_\sigma$.

The reverse inequality $\|i_*(\mathbf{a})\|_\sigma \leq \|\mathbf{a}\|_\sigma$ follows from Theorem 6.4 (1). ■

7. Systolic volume semi-norm and simplicial volume semi-norm

In this section, we show that the systolic volume semi-norm and the simplicial volume semi-norm are equivalent in real homology.

Let us show that the systolic volume semi-norm bounds from above the simplicial volume semi-norm and/or the volume entropy semi-norm (up to a multiplicative constant). Recall that every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ with $m \geq 3$, where X is a path-connected topological space, satisfies

$$\sigma(\mathbf{a}) \geq \lambda_m \frac{\|\mathbf{a}\|_\Delta}{(\log(2 + \|\mathbf{a}\|_\Delta))^m}$$

where λ_m is a positive constant depending only on m; see (1.9). Therefore,

$$\frac{(\log k)^m}{k}\sigma(k\mathbf{a}) \geq \lambda_m \|\mathbf{a}\|_\Delta \left(\frac{\log k}{\log(2 + k\|\mathbf{a}\|_\Delta)}\right)^m.$$

Letting k go to infinity, we obtain $\widehat{\sigma}(\mathbf{a}) \geq \lambda_m \|\mathbf{a}\|_\Delta$ (arguing separately for $\|\mathbf{a}\|_\Delta$ being zero and nonzero). Thus, $\|\mathbf{a}\|_\sigma \geq \lambda_m \|\mathbf{a}\|_\Delta$.

Alternatively, every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ satisfies

$$\sigma(\mathbf{a}) \geq \lambda'_m \frac{\Omega(\mathbf{a})}{(\log(2 + \Omega(\mathbf{a})))^m}$$

for some constant $\lambda'_m > 0$ depending only on m; see [13, 39]. Therefore,

$$\frac{(\log k)^m}{k}\sigma(k\mathbf{a}) \geq \lambda'_m \frac{\Omega(k\mathbf{a})}{k} \left(\frac{\log k}{\log(2 + k\Omega(\mathbf{a}))}\right)^m.$$

Again, letting k go to infinity, we obtain $\widehat{\sigma}(\mathbf{a}) \geq \lambda'_m \|\mathbf{a}\|_E$ (arguing separately for $\Omega(\mathbf{a})$ zero and nonzero). Thus, $\|\mathbf{a}\|_\sigma \geq \lambda'_m \|\mathbf{a}\|_E$.

We establish a reverse inequality in the following theorem.

Theorem 7.1. *Let $m \geq 3$ be an integer. Then there exists a constant $\mu_m > 0$ such that every homology class $\mathbf{a} \in H_m(X;\mathbb{Z})$ of a path-connected topological space X satisfies*

$$\|\mathbf{a}\|_\sigma \leq \mu_m \|\mathbf{a}\|_\Delta.$$

Proof. We argue as in the proof of Theorems 4.9 and 5.1. Let $\mathbf{a} \in H_m(X;\mathbb{Z})$. For every $\varepsilon > 0$ and every integer s large enough, we consider the map $h : Z \to X$ from a closed connected simplicial m-pseudomanifold Z to X such that

$$h_*([Z]) = s\mathbf{a}, \quad \kappa(Z) \leq s(m!)^3(\|\mathbf{a}\|_\Delta + \varepsilon)$$

obtained at the beginning of the proof of Theorem 4.9. By Theorem 4.7, there exists a map $f : \widehat{M}_0 \to Z$ from a finite covering $\widehat{M}_0$ of M_0 such that

$$f_*([\widehat{M}_0]) = q[Z] \in H_m(Z;\mathbb{Z})$$

for some positive integer q. Applying Theorem 5.3 as in (5.4) yields a sequence of subgroups Γ_k in the finitely generated linear group $\widehat{G} := \pi_1(\widehat{M}_0)$ with finite index $[\widehat{G} : \Gamma_k] = k \geq 2$ such that

$$\mathrm{sys}(\Gamma_k, d_S) \geq C_0 \log k$$

for some positive constant C_0 (which does not depend on k), by applying Theorem 5.3 as in (5.4). Denote by $f_k : \widehat{M}_k \to Z$ the lift of $f : \widehat{M}_0 \to Z$ to the cover $\widehat{M}_k = \widetilde{M}_0/\Gamma_k$ corresponding to the subgroup $\Gamma_k \leqslant \widehat{G}$. Since Γ_k is of index k in $\widehat{G}$, we derive

$$(f_k)_*([\widehat{M}_k]) = kq[Z].$$

Define $\varphi_k = h \circ f_k : \widehat{M}_k \to Z \to X$. Observe that

$$(\varphi_k)_*([\widehat{M}_k]) = kqs\mathbf{a}. \tag{7.1}$$

As in (5.5), we have

$$\mathrm{sys}_{\varphi_k}(\widehat{M}_k) \geq \mathrm{sys}_{f_k}(\widehat{M}_k) \geq D_0 \log k \tag{7.2}$$

where D_0 is a positive constant (which does not depend on k or $\mathbf{a}$). As in (4.8), we also have

$$\mathrm{vol}(\widehat{M}_k) = k\,\mathrm{vol}(\widehat{M}_0) = kq\kappa(Z)v_m \leq kqs(m!)^3(\|\mathbf{a}\|_\Delta + \varepsilon)v_m. \tag{7.3}$$

It follows from (7.1)–(7.3) that

$$\sigma(qks\mathbf{a}) \leq \sigma_{\varphi_k}(\widehat{M}_k) \leq \frac{kqs(m!)^3}{D_0^m(\log k)^m}(\|\mathbf{a}\|_\Delta + \varepsilon)v_m.$$

By Lemma 6.3 (2), this implies

$$qs\|\mathbf{a}\|_\sigma \leq qs\frac{(m!)^3}{D_0^m}(\|\mathbf{a}\|_\Delta + \varepsilon)v_m.$$

Hence, $\|\mathbf{a}\|_\sigma \leq \mu_n \|\mathbf{a}\|_\Delta$ where $\mu_n = \frac{(m!)^3}{D_0^m}v_m$. ■

Remark 7.2. By a density argument, the systolic volume semi-norm and the simplicial volume semi-norm are equivalent in real homology, and not only in integral homology.

8. Density of the volume entropy and systolic volume semi-norms of manifolds

The following density result can be deduced from the equivalence of the semi-norms given by Theorems 1.3 and 1.6 together with the recent work [29].

Corollary 8.1. *Let $m \geq 4$ be an integer. Then the sets of all volume entropy semi-norms $\|M\|_E$ and of all systolic volume semi-norms $\|M\|_\sigma$, where M is an orientable connected closed m-manifold, are dense in $[0, \infty)$.*

Proof. Denote by $\|\cdot\|_*$ the volume entropy semi-norm $\|\cdot\|_E$ or the systolic volume semi-norm $\|\cdot\|_\sigma$. Fix $\varepsilon > 0$. By [29, Theorem A], there exists an orientable connected closed m-manifold M with $0 < \|M\|_\Delta < \min\{\varepsilon/C_m, \varepsilon/\mu_m\}$ where C_m and μ_m are the positive constants in Theorems 1.3 and 1.6. Define

$$M_k = M \sharp \cdots \sharp M \quad (k \text{ copies}).$$

By additivity of the simplicial volume under connected sums in dimension at least 3 (see [24]), we have $\|M_k\|_\Delta = k\|M\|_\Delta$. It follows from the equivalence of the semi-norms (see Theorems 1.3 and 1.6) that the sequence $\|M_k\|_*$ starts from the interval $(0, \varepsilon)$, with $0 < \|M\|_* < \varepsilon$, and goes to infinity. Now, the map $M_{k+1} \to M_k$ collapsing one copy of M to a point is of degree 1. By the functorial properties of the volume entropy semi-norm, namely (3) and (5) of Theorems 3.1 and 6.4, we derive

$$\|M_k\|_* \leq \|M_{k+1}\|_* \leq \|M_k\|_* + \|M\|_*.$$

Thus, the sequence $\|M_k\|_*$ is nondecreasing and increases by at most $\|M\|_* < \varepsilon$ at each step. We deduce from the properties of the sequence $\|M_k\|_*$ that every subinterval of $[0, \infty)$ of length ε contains at least one term $\|M_k\|_*$, proving the desired density result. ∎

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