J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1366

Published by EMS Press

Jian Wang

Contractible 3-manifolds and positive scalar curvature (II)

Received June 22, 2020; revised December 29, 2022

Abstract. In this article, we are interested in the question whether any complete contractible 3-manifold of positive scalar curvature is homeomorphic to $\mathbb{R}^3$. We study the fundamental group at infinity, π_1^∞, and its relationship to the existence of complete metrics of positive scalar curvature. We prove that a complete contractible 3-manifold with positive scalar curvature and trivial π_1^∞ is homeomorphic to $\mathbb{R}^3$.

Keywords. Positive scalar curvature, contractible 3-manifolds, the fundamental group at infinity

1. Introduction

This paper is the sequel of [30] and is also devoted to the study of contractible 3-manifolds which carry complete metrics of positive scalar curvature. We are mainly concerned with the following question:

Question. *Is any complete contractible* 3*-manifold of positive scalar curvature homeomorphic to* $\mathbb{R}^3$?

Gromov–Lawson [11] and Chang–Weinberger–Yu [4] independently proved that a complete contractible 3-manifold with uniformly positive scalar curvature (i.e. the scalar curvature is bounded away from zero) is homeomorphic to $\mathbb{R}^3$. The proof of Gromov and Lawson uses minimal surfaces theory, while Chang, Weinberger and Yu use K-theory.

The topological structure of contractible 3-manifolds is quite complicated. For example, Whitehead [31] and McMillan [15] showed that there are infinitely many mutually non-diffeomorphic contractible 3-manifolds, such as the Whitehead manifold.

The Geometrization Conjecture which was confirmed by Perelman [21–23] and a result of McMillan [14] tell us that a contractible 3-manifold can be written as an ascending union of handlebodies. Note that if there are infinitely many handlebodies of genus zero (i.e. 3-balls), the 3-manifold is homeomorphic to $\mathbb{R}^3$.

Jian Wang: Mathematics Department, Stony Brook University,
Stony Brook, NY 11794-3651, USA; jian.wang.4@stonybrook.edu

Mathematics Subject Classification (2020): Primary 53C20; Secondary 53C21

In [30], we consider a contractible genus 3-manifold, an ascending union of solid tori. As mentioned above, $\mathbb{R}^3$ is not genus one but genus zero, since it is an increasing union of 3-balls. In [30], it is proved that no contractible genus one 3-manifold admits a complete metric of non-negative scalar curvature.

In the present paper, we study the existence of complete metrics of positive scalar curvature and its relationship to the fundamental group at infinity.

The *fundamental group at infinity*, π_1^∞, of a path-connected space is the inverse limit of the fundamental groups of complements of compact subsets (see Definition 2.3). The triviality of the fundamental group at infinity is not equivalent to simply-connectedness at infinity. For example, the Whitehead manifold is not simply-connected at infinity but its fundamental group at infinity is trivial.

We prove the following theorem:

Theorem 1.1. *A complete contractible* 3*-manifold with positive scalar curvature and trivial* π_1^∞ *is homeomorphic to* $\mathbb{R}^3$.

However, there are uncountably many mutually non-homeomorphic contractible 3-manifolds with non-trivial π_1^∞. In Appendix C, we construct such a manifold and show that it has no complete metric of positive scalar curvature.

1.1. Handlebodies and Property (H)

Let (M, g) be a complete contractible 3-manifold of positive scalar curvature. It is an increasing union of closed handlebodies $\{N_k\}$ (see Theorem 2.7).

In the following, we suppose that M is not homeomorphic to $\mathbb{R}^3$. We may assume that none of the N_k is contained in a 3-ball (i.e. homeomorphic to a unit ball in $\mathbb{R}^3$) (see Remark 2.2). This plays a crucial role in our argument.

In the genus one case, the family $\{N_k\}$ has several good properties. For example, the maps $\pi_1(\partial N_k) \to \pi_1(\overline{M \setminus N_k})$ and $\pi_1(\partial N_k) \to \pi_1(\overline{N_k \setminus N_0})$ are both injective (see [30, Lemma 2.10]). These properties are crucial and necessary in the study of the existence of complete metrics of positive scalar curvature. Generally, the family $\{N_k\}$ may not have the above properties.

For example, the map $\pi_1(\partial N_0) \to \pi_1(\overline{M \setminus N_0})$ may not be injective. To overcome it, we use topological surgeries on N_0 and find a new handlebody to replace it. Roughly, we use the loop lemma to find an embedded disc $(D, \partial D) \subset (\overline{M \setminus N_0}, \partial N_0)$ whose boundary is a non-nullhomotopic circle in ∂N_0. The new handlebody is obtained from N_0 by attaching a closed tubular neighborhood $N_\varepsilon(D)$ of D in $\overline{M \setminus N_0}$.

We repeatedly use topological surgeries on each N_k to obtain a new family $\{R_k\}_k$ of closed handlebodies with the following properties, called *Property* (H):

(1) the map $\pi_1(\partial R_k) \to \pi_1(\overline{R_k \setminus R_0})$ is injective for $k > 0$;

(2) the map $\pi_1(\partial R_k) \to \pi_1(\overline{M \setminus R_k})$ is injective for $k \geq 0$;

(3) each R_k is homotopically trivial in R_{k+1} but not contained in a 3-ball in M;

(4) there exists an increasing sequence $\{j_k\}_k$ of integers such that $\pi_1(\partial R_k \cap \partial N_{j_k}) \to \pi_1(\partial R_k)$ is surjective.

Remark. If M is not homeomorphic to $\mathbb{R}^3$, the existence of such a family is ensured by Theorem 4.6. It is not unique. In addition, the union of such a family may not be equal to M.

For example, if $M := \bigcup_k N_k$ is a contractible genus one 3-manifold, the family $\{N_k\}$ (as above) satisfies Property (H) (see [30, Lemma 2.10]).

1.2. The vanishing property

It is classical that the geometry of minimal surfaces gives topological information on 3-manifolds. This was exhibited in Schoen–Yau's works [27, 28] as well as in Gromov–Lawson's [11].

In the genus one case, the geometry of a stable minimal surface is constrained by the geometric index (see *Property P* in [30]). In the higher genus case, the geometry of stable minimal surfaces is related to the fundamental group at infinity.

In order to clarify that relationship, let us introduce a geometric property, called the *vanishing property*. Consider a complete contractible 3-manifold (M, g) of positive scalar curvature which is not homeomorphic to $\mathbb{R}^3$. As indicated above, there is an increasing family $\{R_k\}_k$ of closed handlebodies with Property (H) (see Theorem 4.6).

A complete embedded stable minimal surface $\Sigma \subset (M, g)$ is said to satisfy the *vanishing property* with respect to the family $\{R_k\}_k$ if there is a positive integer $k(\Sigma)$ such that for $k \geq k(\Sigma)$, any circle in $\Sigma \cap \partial R_k$ is nullhomotopic in ∂R_k (see Definition 6.1).

If a complete stable minimal surface does not satisfy the vanishing property with respect to $\{R_k\}_k$, it gives a non-trivial element in the fundamental group at infinity (see Lemma 6.2). As a consequence, if π_1^∞ is trivial, each complete stable minimal surface in M has the vanishing property with respect to $\{R_k\}_k$ (see Corollary 6.3).

1.3. The idea of the proof of Theorem 1.1

Our main strategy is to argue by contradiction. Suppose that a complete contractible 3-manifold (M, g) with positive scalar curvature and trivial $\pi_1^\infty(M)$ is not homeomorphic to $\mathbb{R}^3$.

Before constructing minimal surfaces, let us introduce two notions from 3-dimensional topology. For a closed handlebody N of genus $g > 0$, a *meridian* $\gamma \subset \partial N$ of N is an embedded circle which is nullhomotopic in N but not contractible in ∂N (see Definition 3.1).

A system of meridians of N is a collection $\{\gamma^l\}_{l=1}^g$ of g distinct meridians with the property that $\partial N \setminus \coprod_{l=1}^g \gamma^l$ is homeomorphic to an open disc with some punctures. Its existence is ensured by Lemma 3.9.

Let $\{N_k\}_k$ and $\{R_k\}_k$ be as above. Since N_0 is not contained in a 3-ball (see Remark 2.2), the genus of N_k is greater than zero. The handlebody N_k has a system of meridians

$\{\gamma_k^l\}_{l=1}^{g(N_k)}$. Roughly speaking, there are $g(N_k)$ disjoint area-minimizing discs $\{\Omega_k^l\}_l$ with $\partial\Omega_k^l = \gamma_k^l$. Their existence is ensured by the works of Meeks and Yau [17, 18] when the boundary ∂N_k is mean convex, as we now explain.

We construct these discs by induction on l.

When $l = 1$, there is an embedded area-minimizing disc $\Omega_k^1 \subset N_k$ with boundary γ_k^1 (see [17, 18] or [7, Theorem 6.28]).

Suppose that there are l disjointly embedded stable minimal discs $\{\Omega_k^i\}_{i=1}^l$ with $\partial\Omega_k^i = \gamma_k^i$. Our target is to construct a stable minimal surface Ω_k^{l+1} with boundary γ_k^{l+1}.

Consider the Riemannian manifold $(T_{k,l}, g|_{T_{k,l}})$, where $T_{k,l} := N_k \setminus \coprod_{i=1}^l \Omega_k^l$. It is a handlebody of genus $g(N_k) - l$; for an example, see Figure 1.

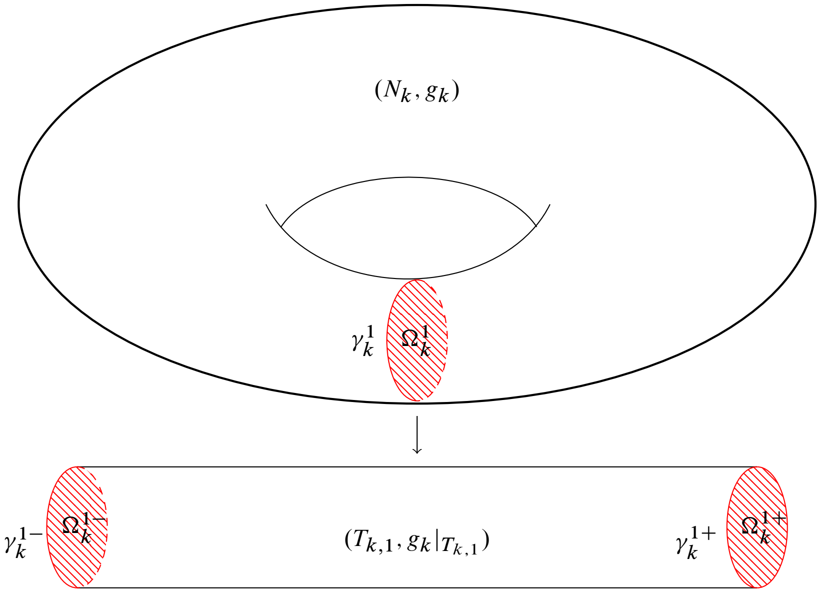

Fig. 1

The boundary of $(T_{k,l}, g|_{T_{k,l}})$ consists of $\partial N_k \setminus \coprod_{i=1}^l \gamma_k^i$ and some disjoint discs $\{\Omega_k^{i-}\}_{i=1}^l$ and $\{\Omega_k^{i+}\}_{i=1}^l$. The two discs Ω_k^{i-} and Ω_k^{i+} both come from the same minimal disc Ω_k^i. Therefore, the mean curvature of the boundary of $(T_{k,l}, g|_{T_{k,l}})$ is non-negative (see Section 5.1).

In addition, $\{\gamma_k^i\}_{i>l}$ is a system of meridians of the handlebody $(T_{k,l}, g|_{T_{k,l}})$. Then, we use the result of Meeks and Yau to find an embedded stable minimal surface $\Omega_k^{l+1} \subset T_{k,l}$ with boundary γ_k^{l+1}. The discs $\{\Omega_k^i\}_{i=1}^{l+1}$ are disjoint in N_k. This finishes the inductive construction.

If ∂N_k is not mean convex, we can deform the metric in a small neighborhood of ∂N_k so that for the new metric, it becomes mean convex. As constructed above, each Ω_k^l is stable minimal for this new metric and for the original one away from a neighborhood of ∂N_k (near N_{k-1}, for example). This is sufficient for our proof.

Define the lamination $\mathscr{L}_k := \coprod_l \Omega_k^l$ (i.e. a disjoint union of embedded surfaces). We show that each lamination $\mathscr{L}_k$ intersects the compact set R_0 (from Corollary 3.10). According to Colding–Minicozzi's theory (see [6, Appendix B]), the sequence $\{\mathscr{L}_k\}_k$ subconverges to a lamination $\mathscr{L} := \bigcup_{t\in\Lambda} L_t$ in (M,g). Note that each leaf L_t is a complete (non-compact) stable minimal surface (see Theorem 5.4).

As indicated above, since (M,g) has positive scalar curvature and $\pi_1^\infty(M)$ is trivial, each leaf L_t in $\mathscr{L}$ has the vanishing property with respect to $\{R_k\}_k$ (see Lemma 6.2 and Corollary 6.3). Furthermore, the lamination $\mathscr{L}$ also satisfies the vanishing property (see Corollary 6.5). That is,

- *there exists a positive integer k_0 such that for any $k \geq k_0$ and any $t \in \Lambda$, any circle in $L_t \cap \partial R_k$ is nullhomotopic in ∂R_k.*

The reason for this can be described as follows. We argue by contradiction. Suppose that there exists a sequence $\{k_n\}_n$ of increasing integers and a sequence $\{L_{t_n}\}$ of leaves in $\mathscr{L}$ such that $L_{t_n} \cap \partial R_{k_n}$ has at least one non-nullhomotopic circle in ∂R_{k_n} for each n.

The sequence $\{L_{t_n}\}$ smoothly subconverges to some leaf in $\mathscr{L}$. For our convenience, we may assume that $\{L_{t_n}\}$ converges to the leaf L_{t_∞}. The leaf L_{t_∞} satisfies the vanishing property (see Lemma 6.2). That is, there is a positive integer $k(L_{t_\infty})$ such that for $k \geq k(L_{t_\infty})$, any circle in $L_{t_\infty} \cap \partial R_k$ is homotopically trivial in ∂R_k.

However, since $L_{t_n} \cap \partial R_{k_n}$ has some non-nullhomotopic circle in ∂R_{k_n}, we show that for $k_n > k(L_{t_\infty})$, $L_{t_n} \cap \partial R_{k(L_{t_\infty})}$ has a meridian of $R_{k(L_{t_\infty})}$ (see Remark 4.7 and Corollary 3.8). These meridians of $R_{k(L_{t_\infty})}$ will converge to a meridian of $R_{k(L_{t_\infty})}$ which is contained in $L_{t_\infty} \cap \partial R_{k(L_{t_\infty})}$. This is in contradiction with the previous paragraph.

Let us explain how to deduce a contradiction from the vanishing property of $\mathscr{L}$.

We can show that if N_k contains R_{k_0} (for k large enough), then $\mathscr{L}_k \cap \partial R_{k_0}$ contains at least one meridian(s) of R_{k_0} (see Corollary 3.10). Since $\mathscr{L}_k$ subconverges to $\mathscr{L}$, these meridians of R_{k_0} will subconverge to a non-contractible circle in $\mathscr{L} \cap \partial R_{k_0}$. That is, some leaf L_t in $\mathscr{L}$ contains this non-nullhomotopic circle in ∂R_{k_0}. This contradicts the vanishing property of $\mathscr{L}$.

1.4. The plan of this paper

In the first part of the paper, we describe the topological properties of contractible 3-manifolds. In Section 2, we recall some notions, such as simply-connectedness at infinity, fundamental group at infinity and handlebodies. In Section 3, we introduce meridian curves and meridian discs in a handlebody. In Section 4, we introduce two types of surgeries on handlebodies. Using these surgeries, we show the existence of an increasing family of handlebodies with good properties, called Property (H).

In the second part, we deal with minimal surfaces and related problems. In Section 5, we construct minimal laminations and consider their convergence. In Section 6, we introduce the vanishing property and study its relation to the fundamental group at infinity. This relation is clarified by Lemma 6.2.

In the third part, we give the complete proof of Theorem 1.1. In Sections 7 and 8, our proof is similar to the genus one case. In Appendix C, we construct a contractible 3-manifold with non-trivial π_1^∞. In addition, we prove that this manifold has no complete metric of positive scalar curvature.

2. Background

2.1. Simply-connectedness at infinity and π_1^∞

Definition 2.1. A topological space M is *simply-connected at infinity* if for any compact set $K \subset M$, there exists a compact set K' containing K such that the induced map $\pi_1(M \setminus K') \to \pi_1(M \setminus K)$ is trivial.

The Poincaré conjecture (see [2,3,19]) shows that any contractible 3-manifold is irreducible (i.e. any embedded 2-sphere in the 3-manifold bounds a closed 3-ball). A result of Stallings [29] tells us that the only contractible 3-manifold that is simply-connected at infinity is $\mathbb{R}^3$.

Remark 2.2. If a contractible 3-manifold M is not homeomorphic to $\mathbb{R}^3$, it is not simply-connected at infinity. That is, there is a compact set $K \subset M$ such that for any compact set $K' \subset M$ containing K, the induced map $\pi_1(M \setminus K') \to \pi_1(M \setminus K)$ is not trivial. We also know that the set K is not contained in a 3-ball in M. Indeed, if a closed 3-ball B (i.e. a closed set homeomorphic to a closed unit ball in $\mathbb{R}^3$) contains K, then van Kampen's Theorem shows that $\pi_1(M) \cong \pi_1(\overline{M \setminus B}) *_{\pi_1(\partial B)} \pi_1(B)$. In addition, $\pi_1(B)$ and $\pi_1(\partial B)$ are both trivial. Therefore, $\pi_1(\overline{M \setminus B}) \cong \pi_1(M)$ is trivial. That is, the map $\pi_1(M \setminus B) \to \pi_1(M \setminus K)$ is trivial, a contradiction.

Definition 2.3. The *fundamental group at infinity* π_1^∞ of a path-connected space is the inverse limit of the fundamental groups of complements of compact subsets.

Remark 2.4. Let M be a contractible 3-manifold. Then $\pi_1^\infty(M)$ is non-trivial if and only if there is a compact set K and a family $\{\gamma_k\}_k$ of circles in $M \setminus K$ going to infinity with the property that for each k,

- γ_k is not nullhomotopic in $M \setminus K$,
- γ_k is homotopic to γ_{k+1} in $M \setminus K$.

Such a family of circles gives a non-trivial element in $\pi_1^\infty(M)$.

A contractible n-manifold M^n with $n \geq 4$ is simply-connected at infinity if and only if $\pi_1^\infty(M^n)$ is trivial (see [4]). However, this result is not true in dimension 3.

Remark 2.5. Let M be a contractible genus one 3-manifold. Then M is not homeomorphic to $\mathbb{R}^3$. It is not simply-connected at infinity but its fundamental group at infinity is trivial. Indeed, let M be an increasing union of solid tori $\{N_k\}_{k=1}^\infty$. Lemma 2.10 in [30] shows that the induced maps $\pi_1(\partial N_k) \to \pi_1(N_k \setminus N_0)$ and $\pi_1(\partial N_k) \to \pi_1(\overline{M \setminus N_k})$ are both injective for $k > 0$, that is, the family $\{N_k\}$ is excellent (see [20, Section 2]).

Lemma 4.1 of [20, p. 33] shows that if a closed curve $\gamma \subset M \setminus N_{k+1}$ is homotopic to $\gamma' \subset N_k \setminus N_m$ in $M \setminus N_0$, then γ is contractible in $M \setminus N_0$ where $0 < m < k$.

Remark 2.4 shows that an element in $\pi_1^\infty(M)$ gives a family $\{\gamma_i\}_{i=1}^\infty$ of circles going to infinity and a compact set K. We may assume that K is equal to N_0. Each γ_i is homotopic to γ_1 in $M \setminus N_0$.

Let γ_1 be a subset of some $N_k \setminus N_0$. We choose i large enough so that γ_i is a subset of $M \setminus N_{k+1}$. We use the above statement to find that γ_i is contractible in $M \setminus N_0$. Consequently, $\pi_1^\infty(M)$ is trivial.

2.2. *Handlebodies*

Definition 2.6 ([24, p. 46]). A *closed handlebody* is any space obtained from the closed 3-ball D^3 (a 0-handle) by attaching g distinct copies of $D^2 \times [-1, 1]$ (1-handles) with the homeomorphisms identifying the $2g$ discs $D^2 \times \{\pm 1\}$ to $2g$ disjoint 2-discs on ∂D^3, all to be done in such a way that the resulting 3-manifold is orientable. The integer g is called the *genus* of the handlebody.

Let us remark that a handlebody of genus g is homeomorphic to a boundary connected sum of g solid tori. Therefore, its boundary is a compact surface of genus g (see [24, p. 46]).

From a result of McMillan [14] and the Poincaré conjecture (see [2, 3, 19]), we know the following:

Theorem 2.7 ([14, Theorem 1, p. 511]). *Any contractible* 3*-manifold can be written as an ascending union of handlebodies.*

Remark 2.8. Let M be a contractible 3-manifold. If it is not homeomorphic to $\mathbb{R}^3$, it can be written as an increasing family $\{N_k\}$ of handlebodies such that

- N_k is homotopically trivial in N_{k+1} for each k;
- none of the N_k is contained in a 3-ball (by Remark 2.2).

3. Meridians

In this section, we consider a closed handlebody N.

Definition 3.1. An embedded circle $\gamma \subset \partial N$ is called a *meridian* if γ is nullhomotopic in N, but not contractible in ∂N.

An embedded closed disc $(D, \partial D) \subset (N, \partial N)$ is called a *meridian disc* if its boundary is a meridian of N.

The disc D is a *separating meridian disc* if $N \setminus D$ is not connected. Its boundary is called a *separating meridian*.

The disc D is a *non-separating disc* if $N \setminus D$ is connected. Its boundary is called a *non-separating meridian*.

Remark. Let γ be a meridian of N. If γ is a separating meridian, it cuts ∂N into two components. The class $[\gamma]$ is equal to zero in $H_1(\partial N)$.

If γ is a non-separating meridian, then $\partial N \setminus \gamma$ is connected. The class $[\gamma]$ is a non-trivial element in $H_1(\partial N)$.

3.1. Effective meridians

Consider two closed handlebodies N' and N with $N' \subset \operatorname{Int} N$.

Definition 3.2. A meridian γ of N is called an *effective meridian relative to* N' if any meridian disc with boundary γ intersects the core of N', i.e. a deformation retraction of N' which is a 1-dimensional CW complex.

The handlebody N is called an *effective handlebody relative to* N' if any meridian of N is an effective meridian relative to N'.

Note that if N' is contained in a 3-ball $B \subset \operatorname{Int} N$, there is no effective meridian relative to N'.

In the following, we will repeatedly use the loop lemma.

Lemma 3.3 ([13, Theorem 3.1, p. 54]). *Let M be an orientable 3-manifold with boundary ∂M, not necessarily compact. If there is a map $f : (D^2, \partial D^2) \to (M, \partial M)$ such that $f|_{\partial D^2}$ is not nullhomotopic in ∂M, then there is an embedding h with the same property.*

Remark 3.4. We may assume that $h(\operatorname{Int} D^2) \subset \operatorname{Int} M$. Indeed, consider a 1-sided open neighborhood $M_\varepsilon \cong \partial M \times [0, \varepsilon)$ of ∂M in M. Shrinking the image of f into $M(\varepsilon) := M \setminus M_\varepsilon$, we find a map $f_\varepsilon : (D^2, \partial D^2) \to (M(\varepsilon), \partial M(\varepsilon))$ with the property that $f_\varepsilon(\partial D^2)$ is not nullhomotopic in $\partial M(\varepsilon)$. We use Lemma 3.3 to find an embedding h_ε with the same property. Its image stays in $(M(\varepsilon), \partial M(\varepsilon))$. Therefore, the image of h_ε is contained in $\operatorname{Int} M$.

In addition, there is an embedded circle $\gamma \subset \partial M$ which is homotopic to $h_\varepsilon(\partial D^2)$ in $\overline{M_\varepsilon}$. There is an embedded annulus $A_\varepsilon \subset \overline{M_\varepsilon}$ joining γ and $h_\varepsilon(\partial D^2)$. We find a map $h : (D^2, \partial D^2) \to (M, \partial M)$ whose image is an embedded disc (the union of A_ε and the image of h_ε). It has the same property as f and $h(\operatorname{Int} D^2) \subset \operatorname{Int} M$.

Lemma 3.5. *Let N' and N be two closed handlebodies with $N' \subset \operatorname{Int} N$. The handlebody N is an effective handlebody relative to N' if and only if the map $\pi_1(\partial N) \to \pi_1(\overline{N \setminus N'})$ is injective.*

Proof. If N is not an effective handlebody relative to N', then there is a meridian disc $(D, \partial D) \subset (N, \partial N)$ with $D \cap N' = \emptyset$. Therefore, the map $\pi_1(\partial N) \to \pi_1(\overline{N \setminus N'})$ is not injective.

Conversely, if $\pi_1(\partial N) \to \pi_1(\overline{N \setminus N'})$ is not injective, we use Lemma 3.3 to find an embedded disc $(D', \partial D') \subset (\overline{N \setminus N'}, \partial N)$ whose boundary is not contractible in ∂N. As in Remark 3.4, we may assume that $\operatorname{Int} D' \subset \operatorname{Int}(N \setminus N')$. We see that D' is a meridian disc with $D' \cap N' = \emptyset$. Therefore, N is not an effective handlebody relative to N'. This finishes the proof. ■

We now introduce some notations for circles in a disc.

Definition 3.6 (see [30, Definition 2.11]). Let $C := \{c_i\}_{i\in I}$ be a finite set of pairwise disjoint circles in the disc D^2 and $D_i \subset D^2$ the unique disc with boundary c_i. Consider $\{D_i\}_{i\in I}$ as a partially ordered set, ordered by inclusion. For each maximal element D_j in $(\{D_i\}_{i\in I}, \subset)$, its boundary c_j is called a *maximal circle* in C. For each minimal element D_j, its boundary c_j is a *minimal circle* in C.

Lemma 3.7. *Let N' and N be two closed handlebodies such that $N' \subset \operatorname{Int} N$ and $\pi_1(\partial N') \to \pi_1(\overline{N \setminus N'})$ is injective. If N is an effective handlebody relative to N', then any meridian disc $(D, \partial D) \subset (N, \partial N)$ contains a meridian of N'.*

Proof. The proof is the same as the proof of [30, Lemma 2.12]. Suppose that the closed meridian disc D intersects $\partial N'$ transversally where $\gamma := \partial D$ is a meridian of N. The intersection $D \cap \partial N'$ is a disjoint union of circles $\{c_i\}_{i\in I}$. Each c_i bounds a unique closed disc $D_i \subset \operatorname{Int} D$.

Consider the sets

$$C^{\text{non}} := \{c_i \mid c_i \text{ is not homotopically trivial in } \partial N'\},$$
$$C^{\max} := \{c_i \mid c_i \text{ is a maximal circle in } \{c_i\}_{i\in I}\}.$$

We will show that C^{non} is non-empty, and a minimal circle in C^{non} is the desired meridian.

Suppose to the contrary that C^{non} is empty. Hence, each $c_i \in C^{\max}$ is contractible in $\partial N'$ and bounds a disc $D_i' \subset \partial N'$. Consider the immersed disc

$$\hat{D} := \Big(D \setminus \bigcup_{c_i \in C^{\max}} D_i\Big) \cup \Big(\bigcup_{c_i \in C^{\max}} D_i'\Big)$$

with boundary γ. Since $\hat{D} \cap \operatorname{Int} N' = \emptyset$, we see that γ is contractible in $\overline{N \setminus N'}$.

However, Lemma 3.5 shows that the map $\pi_1(\partial N) \to \pi_1(\overline{N \setminus N'})$ is injective. That is, the circle γ is nullhomotopic in ∂N. This is in contradiction with our hypothesis that γ is non-trivial in $\pi_1(\partial N)$. We conclude that $C^{\text{non}} \neq \emptyset$.

In the following, we will prove that each minimal circle c_j in C^{non} is a required meridian. From Definition 3.1, it is sufficient to show that c_j is homotopically trivial in N'. We will construct an immersed disc $\hat{D}_j \subset N'$ with boundary c_j.

Let $C_j := \{c_i \mid c_i \subset \operatorname{Int} D_j \text{ for } i \in I\}$ and $C_j^{\max}$ be the set of maximal circles in C_j. We now have two cases: $C_j = \emptyset$ or $C_j \neq \emptyset$.

If C_j is empty, we consider the set $Z := \operatorname{Int} D_j$ and define $\hat{D}_j := \operatorname{Int} D_j$.

If C_j is not empty, then $C_j^{\max}$ is also non-empty. From the minimality of c_j in C^{non}, each $c_i \in C_j^{\max}$ is nullhomotopic in $\partial N'$ and bounds a disc $D_i'' \subset \partial N'$.

Define $Z := \operatorname{Int} D_j \setminus \bigcup_{c_i \in C_j^{\max}} D_i$ and the new disc $\hat{D}_j := Z \cup \bigcup_{c_i \in C_j^{\max}} D_i''$ with boundary c_j.

Let us explain why $\hat{D}_j$ is contained in N'. In any case, $\partial N'$ cuts N into two connected components, $N \setminus N'$ and $\operatorname{Int} N'$. The set Z is one of the components of $\operatorname{Int} D_j \setminus \partial N'$. Therefore, it must be contained in $\operatorname{Int} N'$ or $N \setminus N'$.

If Z is in $N \setminus N'$, then the disc $\hat{D}_j$ is contained in $\overline{N \setminus N'}$, so c_j is contractible in $\overline{N \setminus N'}$. However, since the induced map $\pi_1(\partial N') \to \pi_1(\overline{N \setminus N'})$ is injective, c_j is homotopically trivial in $\partial N'$, which contradicts the choice of $c_j \in C^{\text{non}}$. We conclude that Z is contained in $\operatorname{Int} N'$.

Therefore, $\hat{D}_j$ is contained in N'. That is, c_j is nullhomotopic in N'. However, it is non-trivial in $\pi_1(\partial N')$. From Definition 3.1, we conclude that $c_j \subset D$ is a meridian of N'. This finishes the proof. ∎

As a consequence, we have

Corollary 3.8. *Let N' and N be two closed handlebodies in a contractible 3-manifold M such that $N' \subset \operatorname{Int} N$ and the map $\pi_1(\partial N') \to \pi_1(\overline{M \setminus N'})$ is injective. If an embedded circle $\gamma \subset \partial N$ is not nullhomotopic in $\overline{M \setminus N'}$, then any embedded disc $D \subset M$ with boundary γ contains a meridian of N'.*

The proof is the same as that of Lemma 3.7.

3.2. Non-separating meridians

Lemma 3.9. *For a closed handlebody N of genus g, there are g disjoint non-separating meridians $\{\gamma^l\}_{l=1}^g$ such that $N \setminus \coprod_l N_{\varepsilon_l}(D_l)$ is a closed 3-ball, where D_l is a closed meridian disc with boundary γ^l and $N_{\varepsilon_l}(D_l)$ is an open neighborhood of D_l in N with small radius ε_l .*

The set $\{\gamma^l\}_{l=1}^g$ of these meridians is called a *system of meridians* of the handlebody N of genus g. It is not unique.

Proof of Lemma 3.9. Pick any non-separating meridian γ^1 of N. We use Lemma 3.3 to find an embedded disc $D_1 \subset N$.

As in Remark 3.4, we may assume that $\operatorname{Int} D_1 \subset \operatorname{Int} N$. The set $N_1 := N \setminus N_\varepsilon(D_1)$ is a closed handlebody of genus $g-1$, where $N_{\varepsilon_1}(D_1)$ is the open tubular neighborhood of D_1 in N with small radius ε_1. In particular, the map $\pi_1(\partial N \cap \partial N_1) \to \pi_1(\partial N_1)$ is surjective.

Choose a non-separating meridian $\gamma^2 \subset \partial N \cap \partial N_1$ of N_1. By Lemma 3.3, there exists a meridian disc D_2 of $N_1 = N \setminus N_{\varepsilon_1}(D_1)$. The set

$$N_2 := N \setminus (N_{\varepsilon_1}(D_1) \amalg N_{\varepsilon_2}(D_2))$$

is a closed handlebody of genus $g-2$, where $N_{\varepsilon_2}(D_2)$ is an open tubular neighborhood of D_2 in N.

We repeat this process $g-2$ times and obtain g disjointly embedded discs $\{D_l\}$ such that $N \setminus \coprod_l N_{\varepsilon_l}(D_l)$ is a handlebody of genus zero (a 3-ball). The boundaries $\{\gamma^l\}_{l=1}^g$ of these discs are g distinct meridians as required. ∎

Corollary 3.10. *Let $N \subset M$, $\{\gamma^l\}$ and $\{D_l\}$ be as in Lemma 3.9, where M is a 3-manifold without boundary. If $R \subset \operatorname{Int} N$ is a closed handlebody that is not contained in a 3-ball in M, and $\pi_1(\partial R) \to \pi_1(\overline{M \setminus R})$ is injective, then $\partial R \cap \coprod_l D_l$ contains a meridian of R.*

Proof. The proof is similar to the proof of [30, Lemma 2.12]. We may assume that ∂R intersects $\coprod_l D_l$ transversally. The intersection $\partial R \cap \coprod_l D_l := \{\gamma\}_{\gamma \in C}$ has finitely many components. Set $C^{\text{non}} := \{\gamma \in C \mid \gamma$ is not homotopically trivial in $\partial R\}$.

Claim. *C^{non} is non-empty.*

Suppose that C^{non} is empty. We see that any circle in $D_l \cap \partial R$ is nullhomotopic in ∂R. As in the proof of Lemma 3.7, we get a new disc in $\overline{N \setminus R}$ with boundary γ^l. Therefore, each γ^l is nullhomotopic in $\overline{N \setminus R}$.

We use Lemma 3.3 to find a meridian disc $D_1' \subset \overline{N \setminus R}$ with boundary γ^1. As in Remark 3.4, we may assume that $\operatorname{Int} D_1' \subset \operatorname{Int} \overline{N \setminus R}$ (or $D_1' \subset N \setminus R$). Choose an open tubular neighborhood $N_{\varepsilon_1'}(D_1')$ of D_1' in $N \setminus R$ with small radius ε_1'. The set $N_1' := N \setminus N_{\varepsilon_1'}(D_1')$ is a closed handlebody of genus $g-1$ containing R.

In addition, for $l > 1$, γ^l is a non-separating meridian of N_1' but nullhomotopic in $N \setminus (N_{\varepsilon_1'}(D_1') \amalg R)$.

Repeating this process $g-1$ times, we obtain g embedded discs $\{D_l'\}_{l=1}^g$ such that

- $R \cap \coprod_l N_{\varepsilon_l'}(D_l') = \emptyset$;
- the handlebody $N \setminus \coprod_l N_{\varepsilon_l'}(D_l')$ is of genus zero (a closed 3-ball),

where $N_{\varepsilon_l'}(D_l')$ is an open tubular neighborhood of D_l' in N with small radius ε_l'.

Therefore, R is contained in the 3-ball $N \setminus \coprod_l N_{\varepsilon_l'}(D_l')$. This contradicts our hypothesis and finishes the proof of the claim.

As in the proof of Lemma 3.7, we use the injectivity assumption to show that each minimal circle in C^{non} is a required meridian. ∎

4. Effective handlebodies

4.1. Surgeries

Consider two closed handlebodies N' and N in a contractible 3-manifold M with $N' \subset \operatorname{Int} N$. We introduce two types of surgeries on handlebodies:

Type I: If there exists a meridian disc $D \subset N \setminus N'$ of N, then we consider an open tubular neighborhood $N_\varepsilon(D) \subset N \setminus N'$ of D. We then have two cases:

If D is a separating meridian disc, then $N \setminus N_\varepsilon(D)$ has two components. The closed handlebody W_1 is defined as the component containing N'.

If D is a non-separating meridian disc, then $N \setminus N_\varepsilon(D)$ is connected. The closed handlebody W_1 is defined to be $N \setminus N_\varepsilon(D)$.

Type II: Let γ be a non-contractible circle in ∂N and $D_1 \subset \overline{M \setminus N}$ an embedded disc with boundary $\gamma \subset \partial N$. We may assume that $\operatorname{Int} D_1 \subset M \setminus N$. Again we have two cases:

If γ is not homotopically trivial in N, we consider a closed tubular neighborhood $\overline{N_{\varepsilon_1}(D_1)}$ of D_1 in $\overline{M \setminus N}$. Define a new handlebody W_2 as $N \cup \overline{N_{\varepsilon_1}(D_1)}$.

If γ is nullhomotopic in N, it is a meridian of N. Consider a meridian disc D_2 of N with boundary γ and the embedded sphere $D_1 \cup_\gamma D_2$. The Poincaré conjecture (see [2, 3, 19]) implies that this sphere bounds a 3-ball B.

We can conclude that D_2 is a separating meridian disc. (Otherwise, we find a circle $\gamma_0 \subset N$ such that the intersection number of the sphere $D_1 \cup_\gamma D_2$ and γ_0 is ± 1. However, since any sphere in M contractible, the intersection number must be zero, a contradiction.) Therefore, B contains one of the components of $N \setminus D_2$ and the union $B \cup N$ is also a handlebody. The new handlebody W_2 is defined to be $B \cup N$.

Remark 4.1. For $i = 1, 2$, the genus $g(\partial W_i)$ of ∂W_i is less than $g(\partial N)$. In addition, ∂W_i is a union of $\partial W_i \cap \partial N$ and some disjoint discs. This shows that the map $\pi_1(\partial W_i \cap \partial N) \to \pi_1(\partial W_i)$ is surjective.

Lemma 4.2. *If N' is homotopically trivial in N, then N' is also homotopically trivial in W_i for each i, where W_i is obtained from the above surgeries.*

Proof. For the type II surgery, N is contained in W_2. Therefore, N' is homotopically trivial in W_2.

For the type I surgery, it is sufficient to show that any circle $c \subset N'$ bounds some (immersed) disc $\hat{D}' \subset W_1$.

The closed curve c bounds an immersed disc $D' \subset \operatorname{Int} N$. We will construct the required disc $\hat{D}' \subset W_1$ from D'.

We may assume that D' intersects $D^- \amalg D^+ := \operatorname{Int} N \cap \partial N_\varepsilon(D')$ transversally. Each component c_i of $D' \cap (D^+ \amalg D^-)$ is a circle in D' and bounds a closed subdisc $D'_i \subset D'$.

Since D^+ and D^- are two disjoint discs, each c_i is contractible in $D^+ \amalg D^-$. It also bounds a disc $D''_i \subset D^+ \amalg D^-$. Let $C^{\max}$ be the set of maximal circles of $\{c_i\}_{i \in I}$ in D'. We construct a disc

$$\hat{D}' := \Big(D' \setminus \bigcup_{c_i \in C^{\max}} D'_i\Big) \cup \bigcup_{c_i \in C^{\max}} D''_i$$

with boundary c. It lies in $\overline{N \setminus N_\varepsilon(D')}$. That is, c is contractible in W_1. Therefore, N' is homotopically trivial in W_1. ∎

4.2. Existence of effective handlebodies

In the following, M is a contractible 3-manifold.

Theorem 4.3. *Let N' and N be two closed handlebodies in M such that $N' \subset \operatorname{Int} N$ and N' is homotopically trivial in N. Then there exists a closed handlebody $R \subset M$ containing N' and such that*

(1) *the map $\pi_1(\partial R) \to \pi_1(\overline{R \setminus N'})$ is injective;*

(2) *the map $\pi_1(\partial R) \to \pi_1(\overline{M \setminus R})$ is injective;*

(3) *N' is homotopically trivial in R;*

(4) *∂R is a union of $\partial R \cap \partial N$ and some disjoint discs.*

Remark. From Lemma 3.5, R is an effective handlebody relative to N'.

Proof of Theorem 4.3. Suppose that either $i_1 : \pi_1(\partial N) \to \pi_1(\overline{N \setminus N'})$ or $i_2 : \pi_1(\partial N) \to \pi_1(\overline{M \setminus N})$ is not injective. (If both are injective, R is defined to be N.)

If i_1 is not injective, Lemma 3.3 shows that there exists a meridian disc D_1 of N with $D_1 \cap N' = \emptyset$. We do the type I surgery on N with the disc D_1 to obtain a new handlebody W.

If i_2 is not injective, we use Lemma 3.3 to find an embedded circle $\gamma \subset \partial N$ and an embedded disc $D_2 \subset \overline{M \setminus N}$ ($\operatorname{Int} D_2 \subset M \setminus N$) where $\gamma = \partial D_2$ is not nullhomotopic in ∂N. We do the type II surgery with the disc D_2 to get a new handlebody W.

In any case, we have $g(\partial W) < g(\partial N)$. The boundary ∂W is a union of $\partial W \cap \partial N$ and some disjoint discs $\{D_i'\}_i$. Therefore, $\pi_1(\partial W \cap \partial N) \to \pi_1(\partial W)$ is surjective. In addition, Lemma 4.2 implies that N' is contractible in W.

When picking a circle $\gamma' \subset \partial W$ which is not nullhomotopic in ∂W, we may assume that γ' is an embedded circle in $\partial W \cap \partial N$. Therefore, when repeating these two types of surgeries, we may assume that the new surgeries are performed away from these disjoint discs $\{D_i'\}$.

We iterate this process until we find a handlebody R satisfying (1) and (2). At each step, the genus of the handlebody obtained from the surgery is less than the original one. Therefore, this process stops in no more than $g(N)$ steps.

As above, N' is homotopically trivial in R, and ∂R is a union of $\partial R \cap \partial N$ and some disjoint discs. ∎

Remark. If N' is not contained in a 3-ball in M, then the genus of R is greater than zero.

Lemma 4.4. *Let $R \subset M$ be a closed effective handlebody relative to the closed handlebody $N' \subset \operatorname{Int} R$ and such that $\pi_1(\partial R) \to \pi_1(\overline{M \setminus R})$ is injective. If a closed handlebody N is an effective handlebody relative to $R \subset \operatorname{Int} N$, then N is an effective handlebody relative to N'.*

Proof. From Lemma 3.5, it is sufficient to show that the map $\pi_1(\partial N) \to \pi_1(\overline{N \setminus N'})$ is injective.

We use Lemma 3.5 to show that the induced map $\pi_1(\partial R) \to \pi_1(\overline{R \setminus N'})$ is injective. Since $\pi_1(\partial R) \to \pi_1(\overline{M \setminus R})$ is injective, so is $\pi_1(\partial R) \to \pi_1(\overline{N \setminus R})$.

Van Kampen's Theorem gives an isomorphism between $\pi_1(\overline{N \setminus N'})$ and

$$\pi_1(\overline{N \setminus R}) *_{\pi_1(\partial R)} \pi_1(\overline{R \setminus N'}).$$

A classical result (see [25, Theorem 11.67, p. 404]) shows that the induced map $\pi_1(\overline{N \setminus R}) \to \pi_1(\overline{N \setminus N'})$ is injective.

Lemma 3.5 shows that $\pi_1(\partial N) \to \pi_1(\overline{N \setminus R})$ is injective, hence so is the composition $\pi_1(\partial N) \to \pi_1(\overline{N \setminus R}) \to \pi_1(\overline{N \setminus N'})$. This finishes the proof. ∎

4.3. Property (H)

In the following, M is a contractible 3-manifold which is not homeomorphic to $\mathbb{R}^3$.

By Theorem 2.7, M can be written as an ascending union of handlebodies $\{N_k\}_{k=0}^{\infty}$. Each N_k is homotopically trivial in N_{k+1}. As in Section 2, we can assume that N_0 is not contained in a 3-ball in M (because M is not homeomorphic to $\mathbb{R}^3$; see Remark 2.2).

In the genus one case, the family $\{N_k\}$ has several good properties. For example, each N_k is an effective handlebody relative to N_0 and the map $\pi_1(\partial N_k) \to \pi_1(\overline{M \setminus N_k})$ is injective (see [30, Lemma 2.10]). These properties are necessary and crucial in our proof. In general, the family $\{N_k\}$ may not have these properties. To overcome this difficulty, we introduce a topological property, called Property (H).

Definition 4.5. A family $\{R_k\}_k$ of handlebodies in a contractible 3-manifold $M := \bigcup_k N_k$ is said to have *Property* (H) if

(1) the map $\pi_1(\partial R_k) \to \pi_1(\overline{R_k \setminus R_0})$ is injective for $k > 0$;
(2) the map $\pi_1(\partial R_k) \to \pi_1(\overline{M \setminus R_k})$ is injective for $k \geq 0$;
(3) each R_k is contractible in R_{k+1} but not contained in a 3-ball in M;
(4) there exists an increasing sequence $\{j_k\}_k$ of integers such that $\pi_1(\partial R_k \cap \partial N_{j_k}) \to \pi_1(\partial R_k)$ is surjective.

where $\{N_k\}$ is a fixed family of handlebodies as in Section 2.

For example, in a contractible genus one 3-manifold $M = \bigcup_k N_k$, the family $\{N_k\}$ satisfies Property (H) (see [30, Lemma 2.11]).

In the following, we will prove that if a contractible 3-manifold M is not homeomorphic to $\mathbb{R}^3$, then there is a family of handlebodies with Property (H). However, such a family is not unique.

Theorem 4.6. *If a contractible* 3*-manifold* $M := \bigcup_k N_k$ *(as above) is not homeomorphic to* $\mathbb{R}^3$*, then there is a family* $\{R_k\}_k$ *of handlebodies with Property* (H).

Remark 4.7. • The union $\bigcup_k R_k$ may not be equal to M.

- For $k > 0$, van Kampen's Theorem gives an isomorphism between $\pi_1(\overline{M \setminus R_0})$ and $\pi_1(\overline{M \setminus R_k}) *_{\pi_1(\partial R_k)} \pi_1(\overline{R_k \setminus R_0})$. By conditions (1) and (2) of Property (H), we can use [25, Theorem 11.67, p. 404] to show that the map $\pi_1(\partial R_k) \to \pi_1(\overline{M \setminus R_0})$ is injective.
- As in Theorem 4.3 (4), ∂R_k is the union of $\partial R_k \cap \partial N_{j_k}$ and disjoint discs.

Proof of Theorem 4.6. First, we construct R_0. We repeatedly apply the type II surgery to N_0, until we find a handlebody R_0 containing N_0 and such that $\pi_1(\partial R_0) \to \pi_1(\overline{M \setminus R_0})$ is injective.

From Remark 4.1, we know that, at each step, the genus of the handlebody obtained from the surgery is less than the original one. Therefore, this process stops in no more than $g(N_0)$ steps.

In addition, since N_0 is not contained in a 3-ball in M, neither is R_0.

In the following, we construct the sequence $\{R_k\}_k$ inductively.

When $k = 1$, we pick a handlebody N_{j_1} containing R_0 such that R_0 is homotopically trivial in N_{j_1}. Its existence is ensured by the following fact:

Because R_0 is compact, there is some handlebody N_{j_1-1} containing R_0. Since N_{j_1-1} is homotopically trivial in N_{j_1}, R_0 is contained in N_{j_1} and contractible in N_{j_1}.

By Theorem 4.3, there exists a handlebody R_1 containing R_0 such that

- $\pi_1(\partial R_1) \to \pi_1(\overline{R_1 \setminus R_0})$ is injective;
- $\pi_1(\partial R_1) \to \pi_1(\overline{M \setminus R_1})$ is injective;
- R_0 is contractible in R_1;
- ∂R_1 is a union of $\partial R_1 \cap \partial N_{j_1}$ and some disjoint closed discs. Therefore, the map $\pi_1(\partial R_1 \cap \partial N_{j_1}) \to \pi_1(\partial R_1)$ is surjective.

In particular, since R_0 is not contained in a 3-ball in M, neither is R_1.

Suppose that there exists a handlebody R_{k-1} and a positive integer j_{k-1} satisfying (1)–(4) of Property (H).

Just as in the case of N_{j_1}, there exists a handlebody N_{j_k} containing R_{k-1} such that R_{k-1} is homotopically trivial in N_{j_k}. We use Theorem 4.3 to find an effective handlebody R_k relative to R_{k-1} satisfying (2)–(4).

Since the map $\pi_1(\partial R_{k-1}) \to \pi_1(\overline{R_{k-1} \setminus R_0})$ is injective, R_{k-1} is an effective handlebody relative to R_0 (by Lemma 3.5). Lemma 4.4 shows that R_k is an effective handlebody relative to R_0. We apply Lemma 3.5 again and conclude that R_k also satisfies (1). This finishes the proof. ∎

5. Minimal surfaces and laminations

In Sections 5 and 6, we will talk about minimal surfaces in contractible 3-manifolds. In the following two sections, we make the following assumptions:

- (M, g) is a complete 3-manifold which is not homeomorphic to $\mathbb{R}^3$;
- M is an increasing union of handlebodies $\{N_k\}_{k=0}^{\infty}$.
- N_k is homotopically trivial in N_{k+1} for each k and N_0 is not contained in a 3-ball in M (see Remark 2.2).

In addition, for each k, the genus of N_k is greater than zero. (If not, there is some handlebody N_k of genus zero, so a 3-ball, contrary to assumption.)

5.1. *Minimal laminations*

From Lemma 3.9, each N_k has a system of meridians $\{\gamma_k^l\}_{l=1}^{g(N_k)}$, where $g(N_k)$ is the genus of N_k. Our target is to construct a lamination $\mathscr{L}_k := \bigcup_l \Omega_k^l \subset N_k$ (i.e. a disjoint union of embedded surfaces) with $\partial \Omega_k^l = \gamma_k^l$ and "good" properties.

As in [30], we use a result of Meeks and Yau (see [7, Theorem 6.26, p. 244]) to construct them. However, it requires a geometric condition that the boundary of N_k is mean convex. Therefore we construct a new metric g_k over N_k such that

- $g_k|_{N_{k-1}} = g|_{N_{k-1}}$;
- ∂N_k is mean convex for g_k.

As in [30, Section 5.1], the metric g_k is constructed as follows:

- *Let $h(t)$ be a positive smooth function on $\mathbb{R}$ such that $h(t) = 1$, for any $t \in \mathbb{R} \setminus [-\varepsilon, \varepsilon]$. Consider the function $f(x) := h(d(x, \partial N_k))$ and the metric $g_k := f^2 g|_{N_k}$. For the metric g_k, the mean curvature $\hat{H}(x)$ of ∂N_k is*

$$\hat{H}(x) = h^{-1}(0)(H(x) + 2h'(0)h^{-1}(0)).$$

Choosing ε small enough and h with $h(0) = 2$ and $h'(0) > 2\max_{x\in\partial N_k} |H(x)| + 2$, one gets the required metric g_k.

Let us describe the inductive construction of $\{\Omega_k^i\}_{i=1}^{g(N_k)}$.

When $l = 1$, there is an embedded area-minimizing disc $\Omega_k^1 \subset N_k$ with boundary γ_k^1 for the metric g_k (see [7, Theorem 6.28]).

Suppose that there are l disjointly embedded stable minimal discs $\{\Omega_k^i\}_{i=1}^l$ with $\partial\Omega_k^i = \gamma_k^i$.

Consider the Riemannian manifold $(T_{k,l}, g_k|_{H_{k,l}})$, where $T_{k,l} := N_k \setminus \coprod_{i=1}^l \Omega_k^i$. It is a handlebody of genus $g(N_k) - l$ (for an example, see Figure 1).

The boundary of $(T_{k,l}, g_k|_{T_{k,l}})$ consists of two different parts. One is $\partial N_k \setminus \coprod_{i=1}^l \gamma_l^i$. The mean curvature is non-negative on this part. The other consists of $2l$ disjoint discs $\{\Omega_k^{i-}\}_{i=1}^l$ and $\{\Omega_k^{i+}\}_{i=1}^l$. The two discs Ω_k^{i-} and Ω_k^{i+} are two sides of the same minimal disc Ω_k^i. The mean curvature vanishes on these discs.

Therefore, the mean curvature of the boundary of $(T_{k,l}, g_k|_{H_{k,l}})$ is non-negative. In addition, $\{\gamma_k^i\}_{i>l}$ is a system of meridians of the handlebody $(T_{k,l}, g_k|_{T_{k,l}})$.

Then, we use the result of Meeks and Yau (see [7, Theorem 6.28]) to find an embedded stable minimal surface Ω_k^{l+1} in the closure of $(T_{k,l}, g_k|_{T_{k,l}})$ with boundary γ_k^{l+1}. The disc Ω_k^{l+1} intersects the boundary of $(T_{k,l}, g_k|_{T_{k,l}})$ transversally. Hence, $\operatorname{Int}\Omega_k^{l+1}$ is contained in $\operatorname{Int} T_{k,l}$. That is, $\{\Omega_k^i\}_{i=1}^{l+1}$ are disjoint stable minimal surfaces for g_k.

This finishes the inductive construction.

To sum up, there exist $g(N_k)$ disjointly embedded meridian discs $\{\Omega_k^l\}_{l=1}^{g_k}$. Define the lamination $\mathscr{L}_k$ by $\coprod_l \Omega_k^l$. It is a stable minimal lamination for the new metric g_k and for the original one away from ∂N_k (near N_{k-1}, for example).

The set $\mathscr{L}_k \cap N_{k-1}$ is a stable minimal lamination in (M, g). Each leaf has its boundary contained in ∂N_{k-1}.

We know that each lamination $\mathscr{L}_k$ intersects N_0. Indeed, if $\mathscr{L}_k \cap N_0 = \emptyset$, we choose a tubular neighborhood $N(\mathscr{L}_k)$ in N_k with small radius so that $N(\mathscr{L}_k) \cap N_0$ is also empty. That is, N_0 lies in the handlebody $N_k \setminus N(\mathscr{L}_k)$ of genus zero (i.e. a 3-ball), contrary to assumption.

5.2. Limits of laminations

First, we recall a classical convergence theorem for minimal surfaces.

Definition 5.1. In a complete Riemannian 3-manifold (M, g), a sequence $\{\Sigma_n\}$ of immersed minimal surfaces *converges smoothly with finite multiplicity* (at most m) to an immersed minimal surface Σ if for each point p of Σ, there is a disc neighborhood D in Σ of p, an integer m and a neighborhood U of D in M (consisting of geodesics of M orthogonal to D and centered at such points of D) such that for n large enough, each Σ_n intersects U in at most m connected components. Each component is a graph over D in the geodesic coordinates. Moreover, each component converges to D in the $C^{2,\alpha}$-topology as $n \to \infty$.

Note that if each Σ_n is embedded, the surface Σ is also embedded. The *multiplicity* at p is equal to the number of connected components of $\Sigma_n \cap U$ for n large enough. It remains constant on each component of Σ.

Remark 5.2. Let $\{\Sigma_n\}_n$ be a family of properly embedded minimal surfaces converging to a minimal surface Σ with finite multiplicity. Fix a compact simply-connected subset $D \subset \Sigma$. Let U be the tubular neighborhood of D in M with radius ε and $\pi : U \to D$ be the projection from U onto D. It follows that $\pi|_{\Sigma_n \cap U} : \Sigma_n \cap U \to D$ is an m-sheeted covering map for ε small enough and n large enough, where m is the multiplicity.

Therefore, the restriction of π to each component of $\Sigma_n \cap U$ is also a covering map. Since D is simply-connected, the restriction is bijective. Therefore, each component of $\Sigma_n \cap U$ is a normal graph over D.

Theorem 5.3 (see [1, Compactness Theorem, p. 96] and [16, Theorem 4.37, p. 49]). *Let $\{\Sigma_k\}_{k\in\mathbb{N}}$ be a family of properly embedded minimal surfaces in a 3-manifold M^3 such that each Σ_k intersects a given compact set K_0 and for any compact set K in M, there are three constants $C_1 = C_1(K) > 0$, $C_2 = C_2(K) > 0$ and $j_0 = j_0(K) \in \mathbb{N}$ such that for each $k \geq j_0$,*

(1) *$|A_{\Sigma_k}|^2 \leq C_1$ on $K \cap \Sigma_k$, where $|A_{\Sigma_k}|^2$ is the square length of the second fundamental form of Σ_k;*

(2) $\mathrm{Area}(\Sigma_k \cap K) \leq C_2$.

Then, after passing to a subsequence, Σ_k converges to a properly embedded minimal surface with finite multiplicity in the C^∞-topology.

Note that the limit surface may be disconnected.

Let us consider the sequence $\{\mathscr{L}_k\}$ and its limit. However, this sequence may not satisfy condition (2) in Theorem 5.3 (see [30, Section 5.2]).

Therefore, $\{\mathscr{L}_k\}$ may not subconverge with finite multiplicity. To overcome this, we consider the convergence to a lamination. Colding–Minicozzi's theory [6] shows that the sequence $\{\mathscr{L}_k\}$ subconverges. More precisely, from [6, Proposition B.1, p. 610], this sequence subconverges to a minimal lamination $\mathscr{L}$ in (M, g). Furthermore, we have

Theorem 5.4 ([30, Theorems 5.8 and 5.9, p. 18]). *If (M, g) has positive scalar curvature, then each leaf in $\mathscr{L}$ is a complete (non-compact) stable minimal surface.*

5.3. Properness of the limit surfaces

To sum up, there is a family $\{\mathscr{L}_k\}_k$ of laminations subconverging to a lamination $\mathscr{L}$. Each leaf in $\mathscr{L}$ is a complete (non-compact) embedded stable minimal surface in (M, g) (see Theorem 5.4).

The remaining question is whether each leaf is properly embedded. The following theorem gives an answer.

Theorem 5.5 ([30, Theorem 5.10, p. 18]). *Let (M, g) be a complete oriented 3-manifold with positive scalar curvature $\kappa(x)$. Assume that Σ is a complete non-compact stable minimal surface in M. Then*

$$\int_\Sigma \kappa(x)\, dv \leq 2\pi,$$

where dv is the volume form of the induced metric ds^2 over Σ. Moreover, if Σ is an embedded surface, then Σ is proper.

We will prove this in Appendix B.

6. The vanishing property

Let (M, g) be a complete contractible Riemannian 3-manifold of positive scalar curvature and $\Sigma \subset (M, g)$ a complete (non-compact) embedded stable minimal surface. From [28, Theorem 2, p. 211] and Theorem 5.5, the surface Σ is a properly embedded plane (i.e. diffeomorphic to $\mathbb{R}^2$).

In the genus one case, the geometry of such a stable minimal surface is constrained by Property P (see [30, Theorem 4.2]). In general, its geometry is related to the fundamental group at infinity.

Let (M, g) and $\{N_k\}$ be as in Section 5. Theorem 4.6 gives an increasing family $\{R_k\}_k$ of closed handlebodies with Property (H).

Definition 6.1. A complete embedded stable minimal surface $\Sigma \subset (M, g)$ is said to satisfy the *vanishing property* with respect to $\{R_k\}_k$ if there exists a positive integer $k(\Sigma)$ such that for any $k \geq k(\Sigma)$, any circle in $\Sigma \cap \partial R_k$ is contractible in ∂R_k.

Let $\mathscr{L} \subset (M, g)$ be a stable minimal lamination where each leaf is a complete (non-compact) stable minimal surface. Then $\mathscr{L}$ is said to have the *vanishing property* with respect to $\{R_k\}_k$ if there is a positive integer k_0 such that for any $k \geq k_0$ and each leaf L_t in $\mathscr{L}$, any circle in $L_t \cap \partial R_k$ is contractible in ∂R_k.

We will prove in Corollary 6.3 and Theorem 6.4 that if $\pi_1^\infty(M)$ is trivial, then any stable minimal lamination has the vanishing property with respect to $\{R_k\}_k$.

Lemma 6.2. *Let (M, g) be a complete contractible Riemannian 3-manifold with positive scalar curvature $\kappa(x) > 0$ and $\{R_k\}_k$ a family of handlebodies with Property* (H). *If there is a complete embedded stable minimal surface Σ which does not satisfy the vanishing property with respect to $\{R_k\}_k$, then $\pi_1^\infty(M)$ is non-trivial.*

Roughly, there is a sequence of non-trivial circles in Σ going to infinity. This sequence gives a non-trivial element in $\pi_1^\infty(M)$.

Proof. Since Σ does not satisfy the vanishing property with respect to $\{R_k\}$, there is an increasing sequence $\{k_n\}_n$ of integers such that for each k_n, there is a circle $\gamma_n \subset \Sigma \cap \partial R_{k_n}$ which is not nullhomotopic in ∂R_{k_n}. By [28, Theorem 2, p. 211], Σ is diffeomorphic to $\mathbb{R}^2$. Each γ_n bounds a unique closed disc $D_n \subset \Sigma$.

Remark. The circle γ_n may not be a meridian of R_{k_n}. Property (H) implies that the map $\pi_1(\partial R_{k_n}) \to \pi_1(\overline{M \setminus R_{k_n}})$ is injective (see Definition 4.5). Corollary 3.8 implies that D_n contains at least one meridian of R_{k_n}.

Without loss of generality, we may assume that γ_n is a meridian of R_{k_n} and Int D_n contains no meridian of R_{k_n}. (If not, we can replace γ_n by the meridian in Int D_n).

Since $\{\gamma_n\}_n$ is a collection of disjointly embedded circles in Σ, for each n and n' one of the following holds:

$$(1)\ D_n \subset D_{n'}; \quad (2)\ D_{n'} \subset D_n; \quad (3)\ D_n \cap D_{n'} = \emptyset.$$

Based on our assumption, we know that

$(*)$ for any $n' > n$, $D_n \subset D_{n'}$ or $D_n \cap D_{n'} = \emptyset$.

Indeed, if not, $D_{n'}$ is a subset of D_n. We use the argument in the above remark to find a meridian curve in $D_{n'} \cap \partial R_{k_n} \subset \mathrm{Int}\, D_n$, in contradiction with the above assumption.

We will first show that there is an increasing subsequence of $\{D_n\}$ and use the subsequence to find a non-trivial element in $\pi_1^\infty(M)$.

Step 1: The existence of an ascending subsequence of $\{D_n\}$. Suppose that there is no ascending subsequence of $\{D_n\}$. Consider the partially ordered set $(\{D_n\}_n, \subset)$. Let $C_{\min}$ be the set of minimal elements in $(\{D_n\}_n, \subset)$. The discs in $C_{\min}$ are disjoint in Σ.

If $C_{\min}$ is finite, set $n_0 := \max\{n \mid D_n \in C_{\min}\}$. From $(*)$ above, $\{D_n\}_{n>n_0}$ is an increasing subsequence, which contradicts our hypothesis. Thus, $C_{\min}$ is infinite, so there is a subsequence $\{D_{n_s}\}_s$ of disjointly embedded discs.

From Remark 4.7, the map $\pi_1(\partial R_{k_{n_s}}) \to \pi_1(M \setminus R_0)$ is injective. Then the disc D_{n_s} intersects R_0. Choose $x_s \in R_0 \cap D_{n_s}$ and $r_0 = \frac{1}{2}\min\{i_0, r\}$, where $r := d^M(\partial R_0, \partial R_1)$ and $i_0 := \inf_{x \in R_1} \mathrm{Inj}_M(x)$. Hence, the geodesic ball $B(x_s, r_0) \subset M$ lies in R_1.

We apply [17, Lemma 1, p. 445] to the minimal surface $D_{n_s} \cap R_1$ in $(R_1, \partial R_1)$ and obtain

$$\mathrm{Area}(D_{n_s} \cap B(x_s, r_0)) \geq C_1(K, i_0, r_0),$$

where C_1 is a constant independent of n_s and K is the bound of the sectional curvature on R_1. By Theorem 5.5 this leads to a contradiction:

$$\begin{aligned}2\pi &\geq \int_\Sigma \kappa\, dv \geq \int_{R_1\cap\Sigma} \kappa\, dv \geq \sum_s \int_{D_{n_s}\cap B(x_s,r_0)} \kappa\, dv\\ &\geq \sum_s C\ \mathrm{Area}(D_{n_s}\cap B(x_s,r_0)) \geq \sum_s CC_1 = \infty,\end{aligned}$$

where $C := \inf_{x\in R_1} \kappa(x) > 0$.

Thus there is an ascending subsequence of $\{D_n\}_n$. From now on, we abuse the notation and write $\{D_n\}$ for an ascending subsequence.

Step 2: $\pi_1^\infty(M)$ is non-trivial.

Claim. *There is an integer n_0 such that for $n \geq n_0$, $(D_n \setminus D_{n-1}) \cap R_0$ is empty.*

Towards a contradiction, suppose that there exists an increasing sequence $\{n_l\}$ of integers such that $D_{n_l} \setminus D_{n_{l-1}}$ intersects R_0.

Choose $x_l \in (D_{n_l} \setminus D_{n_{l-1}}) \cap R_0$. Hence, the geodesic ball $B(x_l, r_0) \subset M$ is contained in R_1, where r_0 is as above. We again apply [17, Lemma 1, p. 445] to the minimal surface $(D_{n_l} \setminus D_{n_{l-1}}) \cap R_1$ in $(R_1, \partial R_1)$ to get

$$\mathrm{Area}((D_{n_l} \setminus D_{n_l-1}) \cap B(x_l, r_0)) \geq C_1(K, i_0, r_0).$$

From Theorem 5.5, one gets a contradiction:

$$\begin{aligned}2\pi &\geq \int_\Sigma \kappa\, dv \geq \int_{R_1\cap\Sigma} \kappa\, dv \geq \sum_l \int_{(D_{n_l}\setminus D_{n_{l-1}})\cap B(x_l,r_0)} \kappa\, dv\\ &\geq \sum_l C\ \mathrm{Area}((D_{n_l} \setminus D_{n_{l-1}}) \cap B(x_l, r_0)) \geq C\sum_l C_1 = \infty.\end{aligned}$$

This completes the proof of the claim.

Therefore, for $n > n_0$, γ_n is homotopic to γ_{n_0} in $M \setminus R_0$ and is not nullhomotopic in $M \setminus R_0$.

Because $\bigcup_k R_k$ may not be equal to M, the sequence $\{\gamma_n\}_{n>n_0}$ of circles may not go to infinity. In order to overcome it, we replace it by a new family $\{\gamma'_n\}_{n>n_0}$ of circles going to infinity.

The map $\pi_1(\partial R_{k_n} \cap \partial N_{j_{k_n}}) \to \pi_1(\partial R_{k_n})$ is surjective (see Theorem 4.6 and Definition 4.5). Hence, we can find a circle $\gamma'_n \subset \partial N_{j_{k_n}} \cap \partial R_{k_n}$ which is homotopic to γ_n in ∂R_{k_n}. The sequence $\{\gamma'_n\}_{n\geq n_0}$ goes to infinity.

The sequence $\{\gamma'_n\}$ also has the property that for $n > n_0$,

- γ'_n is homotopic to γ'_{n+1} in $M \setminus R_0$;
- γ'_n is not nullhomotopic in $M \setminus R_0$.

From Remark 2.4, $\pi_1^\infty(M)$ is not trivial. ∎

As a corollary, we have

Corollary 6.3. *Let (M, g) be a complete Riemannian 3-manifold of positive scalar curvature and $\{R_k\}_k$ a family of handlebodies with Property* (H). *If $\pi_1^\infty(M)$ is trivial, then any complete stable minimal surface in (M, g) has the vanishing property with respect to $\{R_k\}_k$.*

Theorem 6.4. *Let (M, g) be a complete Riemannian 3-manifold of positive scalar curvature and $\{R_k\}_k$ a family of handlebodies with Property* (H). *If each leaf in a lamination $\mathscr{L}$ is a complete (non-compact) stable minimal surface satisfying the vanishing property with respect to $\{R_k\}_k$, then $\mathscr{L}$ also has the vanishing property with respect to $\{R_k\}_k$.*

Proof. Suppose for contradiction that there exists a sequence $\{L_{t_n}\}$ of leaves in $\mathscr{L}$ and an increasing sequence $\{k_n\}_n$ of integers such that for each n some circle $\gamma_n \subset L_{t_n} \cap \partial R_{k_n}$ is not homotopically trivial in ∂R_{k_n}.

The leaf L_{t_n} is a complete (non-compact) stable minimal surface. From [28, Theorem 2, p. 211], it is diffeomorphic to $\mathbb{R}^2$. The circle γ_n bounds a unique closed disc $D_n \subset L_{t_n}$. Since γ_n is not nullhomotopic in $\overline{M \setminus R_0}$ (see Remark 4.7), the disc D_n intersects R_0.

Step 1: The sequence $\{L_{t_n}\}_n$ subconverges smoothly with finite multiplicity. Since each L_{t_n} is a stable minimal surface, we use [26, Theorem 3, p. 122] to show that, for a fixed compact set $K \subset M$, there exists a constant $C_1 = C_1(K, M, g)$ such that

$$|A_{L_{t_n}}|^2 \leq C_1 \quad \text{on } K \cap L_{t_n},$$

where $|A_{L_{t_n}}|^2$ is the squared norm of the second fundamental form of L_{t_n}.

From Theorem 5.5, $\int_{L_{t_n}} \kappa \, dv \leq 2\pi$, hence

$$\operatorname{Area}(K \cap L_{t_n}) \leq 2\pi \Big(\inf_{x \in K} \kappa(x)\Big)^{-1}.$$

From Theorem 5.3, $\{L_{t_n}\}_n$ smoothly subconverges to a sublamination $\mathscr{L}'$ of $\mathscr{L}$ with finite multiplicity. In addition, $\mathscr{L}'$ is also properly embedded (see Theorem 5.3).

The lamination $\mathscr{L}'$ may have infinitely many components. Let $\mathscr{L}'' \subset \mathscr{L}'$ be a set of leaves intersecting R_0. Since $\mathscr{L}'$ is properly embedded, $\mathscr{L}''$ has finitely many leaves.

Since each leaf L_t in $\mathscr{L}'$ is homeomorphic to $\mathbb{R}^2$, an embedded circle $\gamma \subset \partial R_k \cap L_t$ bounds a unique disc $D \subset L_t$ for $k > 0$.

If L_t is in $\mathscr{L}' \setminus \mathscr{L}''$, the intersection $D \cap R_0$ is empty, so γ is homotopically trivial in $\overline{M \setminus R_0}$. Since the map $\pi_1(\partial R_k) \to \pi_1(\overline{M \setminus R_0})$ is injective for $k > 0$ (see Remark 4.7), γ is nullhomotopic in ∂R_k.

Therefore, for any $k > 0$ and any leaf $L_t \in \mathscr{L}' \setminus \mathscr{L}''$, any circle in $L_t \cap \partial R_k$ is homotopically trivial in ∂R_k.

Step 2: The vanishing property gives a contradiction. From now on, we abuse the notation and write $\{L_{t_n}\}$ for a convergent sequence. In addition, we assume the lamination $\mathscr{L}''$ equals $\coprod_{s=1}^m L_{t_s}$ ($\mathscr{L}''$ has finitely many leaves).

The vanishing property gives an integer $k(L_{t_s})$ for L_{t_s}. For $k \geq \sum_{s=1}^m k(L_{t_s})$, any circle in $\mathscr{L}'' \cap \partial R_k$ is contractible in ∂R_k. From the above fact, for $k > 0$, any closed curve in $(\mathscr{L}' \setminus \mathscr{L}'') \cap \partial R_k$ is also homotopically trivial in ∂R_k.

Therefore, for any $k \geq \sum_{s=1}^m k(L_{t_s})$, any circle in $\mathscr{L}' \cap \partial R_k$ is homotopically trivial in ∂R_k.

We fix the integer $k \geq \sum_{s=1}^m k(L_{t_s})$ and have the following:

Claim. *For n large enough, any circle in $L_{t_n} \cap \partial R_k$ is homotopically trivial in ∂R_k.*

We may assume that $\mathscr{L}'$ intersects ∂R_k transversally. Since $\mathscr{L}'$ is properly embedded, $\mathscr{L}' \cap \partial R_k$ has finitely many components. Each component of $\mathscr{L}' \cap \partial R_k$ is an embedded circle. From the above fact, it is homotopically trivial in ∂R_k. That is,

$$\pi_1(\mathscr{L}' \cap \partial R_k) \to \pi_1(\partial R_k) \text{ is the trivial map.}$$

Choose an open tubular neighborhood U of $\mathscr{L}' \cap \partial R_k$ in ∂R_k. It is homotopy equivalent to $\mathscr{L}' \cap \partial R_k$ in ∂R_k. Thus, the map $\pi_1(U) \to \pi_1(\partial R_k)$ is also trivial.

Since $\{L_{t_n}\}$ converges to $\mathscr{L}'$, $L_{t_n} \cap \partial R_k$ is contained in U for n large enough. Hence, $\pi_1(L_{t_n} \cap \partial R_k) \to \pi_1(\partial N_k)$ is trivial, so any circle in $L_{t_n} \cap \partial R_k$ is homotopically trivial in ∂R_k. The claim follows.

The boundary $\gamma_n \subset \partial R_{k_n}$ of D_n is non-contractible in ∂R_{k_n}. It is also non-contractible in $\overline{M \setminus R_0}$ (see Remark 4.7). If $k_n > k$, we use Corollary 3.8 to find a meridian $\gamma' \subset L_{t_n} \cap \partial R_k$ of R_k. This is in contradiction with the above claim. ∎

As a consequence, we have

Corollary 6.5. *Let (M, g) be a complete contractible Riemannian manifold of positive scalar curvature and $\{R_k\}_k$ a family of handlebodies with Property* (H). *If $\pi_1^\infty(M)$ is trivial, then any complete stable minimal lamination in (M, g) has the vanishing property with respect to $\{R_k\}_k$.*

7. Proof of main theorems

In Sections 7 and 8, we will complete the proof of Theorem 1.1. We make the following assumptions:

- (M, g) is a complete contractible 3-manifold with trivial $\pi_1^\infty(M)$ and with positive scalar curvature.
- M is an increasing union of closed handlebodies $\{N_k\}_k$.
- M is not homeomorphic to $\mathbb{R}^3$.
- As in Remark 2.2, we may assume that each N_k is homotopically trivial in N_{k+1} and none of the N_k is contained in a 3-ball (see Remark 2.2). In addition, the genus of N_k is greater than zero for $k > 0$.

- Each N_k has a system of meridians $\{\gamma_k^l\}_{l=1}^{g(N_k)}$. There is a lamination $\mathscr{L}_k := \coprod_l \Omega_k^l \subset N_k$, where each leaf Ω_k^l is a disc with boundary $\gamma_k^l \subset \partial N_k$.
- As in Section 5, $\mathscr{L}_k$ subconverges to a lamination $\mathscr{L} := \bigcup_{t\in\Lambda} L_t$, where each leaf L_t is a complete (non-compact) stable minimal surface.

Towards a contradiction, we suppose that (M, g) has positive scalar curvature. Each leaf in $\mathscr{L}$ is a properly embedded plane (see Theorem 5.5 and a result of Schoen and Yau [28]).

We now study the lamination $\mathscr{L}$ and its relationship to the vanishing property.

We now prove that there is an ascending family $\{R_k\}_k$ of handlebodies satisfying Property (H) and such that

(a) the lamination $\mathscr{L}$ has the vanishing property with respect to $\{R_k\}_k$;

(b) for each k and any N_j containing R_k, the intersection $\mathscr{L}_j \cap \partial R_k$ contains at least one meridian of R_k.

Indeed, since M is not homeomorphic to $\mathbb{R}^3$, we use Theorem 4.6 to find $\{R_k\}_k$. Since $\pi_1^\infty(M)$ is trivial, Corollary 6.5 shows that $\mathscr{L}$ has the vanishing property with respect to this family.

None of the R_k is contained in a 3-ball (see Definition 4.5). Together with Property (H), we use Corollary 3.10 to show that if N_j contains R_k, then $\mathscr{L}_j \cap \partial R_k$ contains at least one meridian of R_k.

Remark 7.1. In Section 8, our proof requires that ∂R_k intersects some leaf L_t transversally. In order to overcome it, we will deform the handlebody R_k in a small tubular neighborhood of ∂R_k so that the boundary of the new handlebody intersects L_t transversally.

This new handlebody also satisfies (a) and (b). Indeed, for any handlebody R'_k obtained by deforming R_k, the maps $\pi_1(\partial R'_k) \to \pi_1(\overline{R'_k \setminus R_0})$ and $\pi_1(\partial R'_k) \to \pi_1(\overline{M \setminus R'_k})$ are both injective. The proofs of items (a) and (b) just depend on that injectivity. Hence, items (a) and (b) are also true for R'_k.

We use item (a) to show that there is an integer $k_0 > 0$ such that for any $k \geq k_0$ and any leaf L_t in $\mathscr{L}$, any circle in $L_t \cap \partial R_k$ is nullhomotopic in ∂R_k.

This fact implies a covering lemma that we will prove in Section 8.

Lemma 7.2. *For any $k \geq k_0$, $\mathscr{L} \cap \partial R_k(\varepsilon)$ is contained in a disjoint union of finitely many closed discs in $\partial R_k(\varepsilon)$, where $R_k(\varepsilon) := R_k \setminus N_\varepsilon(\partial R_k)$, and $N_\varepsilon(\partial R_k)$ is some tubular neighborhood of ∂R_k in R_k.*

We now use the covering lemma to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that a complete contractible 3-manifold (M, g), with positive scalar curvature and trivial π_1^∞ is not homeomorphic to $\mathbb{R}^3$. As above, there is an ascending family $\{R_k\}_k$ of handlebodies with Property (H) such that

(a) the lamination $\mathscr{L}$ has the vanishing property with respect to $\{R_k\}_k$;

(b) for each k and any N_j containing $R_k(\varepsilon)$, the intersection $\mathscr{L}_j \cap \partial R_k(\varepsilon)$ contains at least one meridian of $R_k(\varepsilon)$.

The vanishing property implies Lemma 7.2 (we will show this in Section 8). That is, $\mathscr{L} \cap \partial R_k(\varepsilon)$ is in the union of the disjoint closed discs $\{D_i\}_{i=1}^s$ for $k \geq k_0$.

Choose an open neighborhood U of the closed set $\mathscr{L} \cap R_{k+1}$ such that $U \cap \partial R_k(\varepsilon)$ is contained in a disjoint union $\coprod_{i=1}^s D_i'$, where D_i' is an open tubular neighborhood of D_i in $\partial R_k(\varepsilon)$. Each D_i' is an open disc in $\partial R_k(\varepsilon)$.

Since $\mathscr{L}_k$ subconverges to $\mathscr{L}$, there exists an integer j, large enough, satisfying

$$(1)\ \mathscr{L}_j \cap R_{k+1} \subset U; \quad (2)\ R_k(\varepsilon) \text{ is contained in } N_j.$$

Therefore, $\mathscr{L}_j \cap \partial R_k(\varepsilon)$ is contained in $U \cap \partial R_k(\varepsilon) \subset \coprod_i D_i'$. Then the induced map $\pi_1(\mathscr{L}_j \cap \partial R_k(\varepsilon)) \to \pi_1(\coprod_i D_i') \to \pi_1(\partial R_k(\varepsilon))$ is trivial. We conclude that any circle in $\mathscr{L}_j \cap \partial R_k(\varepsilon)$ is homotopically trivial in $\partial R_k(\varepsilon)$.

However, from (b), there exists a meridian $\gamma \subset \mathscr{L}_j \cap \partial R_k(\varepsilon)$ of $R_k(\varepsilon)$. This contradicts the previous paragraph and finishes the proof of Theorem 1.1. ∎

8. The proof of Lemma 7.2

This section is the same as Section 7 of [30]. In order to prove Lemma 7.2, we introduce a set S and prove its finiteness, which will imply Lemma 7.2. We begin with two topological lemmas.

Lemma 8.1. *Let $(\Omega, \partial\Omega) \subset (N, \partial N)$ be a 2-sided embedded disc with some closed subdiscs removed, where N is a closed handlebody of genus $g > 0$. Assume that each circle γ_i is contractible in ∂N, where $\partial\Omega = \coprod_i \gamma_i$. Then $N \setminus \Omega$ has two connected components. Moreover, there is a unique component B such that the induced map $\pi_1(B) \to \pi_1(N)$ is trivial.*

We will show the lemma in Appendix A.

Lemma 8.2. *Let $(\Omega_1, \partial\Omega_1)$ and $(\Omega_2, \partial\Omega_2)$ be two disjoint surfaces as in Lemma 8.1. Assume that for each $t = 1, 2$, $N \setminus \Omega_t$ has a unique component B_t such that the map $\pi_1(B_t) \to \pi_1(N)$ is trivial. Then one of the following holds:*

$$(1)\ B_1 \cap B_2 = \emptyset; \quad (2)\ B_1 \subset B_2; \quad (3)\ B_2 \subset B_1.$$

The proof is the same as the proof of [30, Lemma 7.2].

8.1. Definition of the set S

Let (M, g), $\{N_k\}_k$ and $\mathscr{L} := \coprod_{t\in\Lambda} L_t$ be as at the beginning of Section 7. As in Section 7, there is a family $\{R_k\}_k$ of handlebodies such that

- *there is a positive integer k_0 such that for each $k \geq k_0$ and each $t \in \Lambda$, each circle in $L_t \cap \partial R_k$ is homotopically trivial in ∂R_k.*

In the following, we will work on the open handlebody $\operatorname{Int} R_k$ and construct the set S, for a fixed integer $k \geq k_0$.

8.1.1. Set-up for defining S. Let $\{\Sigma_i^t\}_{i\in I_t}$ be the set of components of $L_t \cap \operatorname{Int} R_k$ for each $t \in \Lambda$. (It may be empty.) We will show that for each component Σ_i^t, $R_k \setminus \overline{\Sigma_i^t}$ has a unique component B_i^t such that $\pi_1(B_i^t) \to \pi_1(R_k)$ is trivial.

If L_t intersects ∂R_k transversally, the boundary $\partial\Sigma_i^t \subset L_t \cap \partial R_k$ is the union of some disjointly embedded circles. From the vanishing property, any circle in the boundary $\partial\Sigma_i^t \subset L_t \cap \partial R_k$ is homotopically trivial in ∂R_k.

In addition, since L_t is homeomorphic to $\mathbb{R}^2$, Σ_i^t is homeomorphic to an open disc with finitely many punctures. By Lemma 8.1, $R_k \setminus \overline{\Sigma_i^t}$ has a unique component B_i^t such that $\pi_1(B_i^t) \to \pi_1(R_k)$ is trivial.

In general, L_t may not intersect ∂R_k transversally. To overcome it, we will deform the surface ∂R_k. More precisely, for the leaf L_t, there is a new handlebody $\tilde{R}_k[\varepsilon_t]$ containing R_k such that L_t intersects $\partial\tilde{R}[\varepsilon_t]$ transversally, where $\tilde{R}_k[\varepsilon_t]$ is a closed tubular neighborhood of R_k in M.

We consider the component $\tilde{\Sigma}_i^t$ of $L_t \cap \operatorname{Int} \tilde{R}_k[\varepsilon_t]$ containing Σ_i^t. As above, $\tilde{R}_k[\varepsilon_t] \setminus \overline{\tilde{\Sigma}_i^t}$ has a unique component $\tilde{B}_i^t$ such that the map $\pi_1(\tilde{B}_i^t) \to \pi_1(\tilde{R}_k[\varepsilon_t])$ is trivial.

Choose the component B_i^t of $\tilde{B}_i^t \cap R_k$ whose boundary contains Σ_i^t. It is a component of $R_k \setminus \overline{\Sigma_i^t}$. In addition, the map $\pi_1(B_i^t) \to \pi_1(\tilde{B}_i^t) \to \pi_1(\tilde{R}_k[\varepsilon_t])$ is trivial. Since R_k and $\tilde{R}_k[\varepsilon_t]$ are homotopy equivalent, the map $\pi_1(B_i^t) \to \pi_1(R_k)$ is also trivial. This finishes the construction of B_i^t.

8.1.2. The properties of S. From Lemma 8.2, for any B_i^t and $B_{i'}^{t'}$, one of the following holds:

$$(1)\ B_i^t \cap B_{i'}^{t'} = \emptyset; \quad (2)\ B_i^t \subset B_{i'}^{t'}; \quad (3)\ B_{i'}^{t'} \subset B_i^t,$$

where $t, t' \in \Lambda$, $i \in I_t$ and $i' \in I_{t'}$. Then $(\{B_i^t\}_{t\in\Lambda, i\in I_t}, \subset)$ is a partially ordered set.

We now consider the set $\{B_j\}_{j\in J}$ of maximal elements. However, this set may be infinite.

Definition 8.3. Set

$$S := \{B_j \mid B_j \cap R_k(\varepsilon/2) \neq \emptyset \text{ for any } j \in J\},$$

where $R_k(\varepsilon/2)$ is $R_k \setminus N_{\varepsilon/2}(\partial R_k)$ and $N_{\varepsilon/2}(\partial R_k)$ is a 2-sided tubular neighborhood of ∂R_k with radius $\varepsilon/2$.

Proposition 8.4. *Let Σ_i^t be a component of $L_t \cap \operatorname{Int} R_k$ and let B_i^t be as above. If B_i^t is an element in S, then $\Sigma_i^t \cap R_k(\varepsilon/2)$ is non-empty.*

The proof is the same as that of [30, Proposition 6.5].

Proposition 8.5. *$R_k(\varepsilon)\cap\mathscr{L}\subset\bigcup_{B_j\in S}\overline{B}_j\cap R_k(\varepsilon)$. Moreover, $\mathscr{L}\cap\partial R_k(\varepsilon)\subset\bigcup_{B_j\in S}\overline{B}_j\cap\partial R_k(\varepsilon)$.*

The proof is the same as that of [30, Proposition 6.6].

8.2. The finiteness of S

The set $\partial B_j\cap\text{Int}\,R_k$ equals some $\Sigma_i^t\subset L_t$ for $t\in\Lambda$. Set $S_t:=\{B_j\in S\mid\partial B_j\cap\text{Int}\,R_k\subset L_t\}$. Then $S=\coprod_{t\in\Lambda}S_t$. Note that each $B_j\in S_t$ is a B_i^t for some $i\in I_t$.

In this subsection, we first show that each S_t is finite. Then, we argue that $\{S_t\}_{t\in\Lambda}$ contains at most finitely many non-empty sets. These imply the finiteness of S.

Lemma 8.6. *Each S_t is finite.*

Proof. Suppose that S_t is infinite for some t.

For each $B_j\in S_t$, there exists an $i\in I_t$ such that B_j is equal to B_i^t, where B_i^t is a component of $R_k\setminus\overline{\Sigma_i^t}$ and Σ_i^t is a component of $L_t\cap\text{Int}\,R_k$. By Proposition 8.4, $\Sigma_i^t\cap R_k(\varepsilon/2)$ is non-empty.

Choose $x_j\in\Sigma_i^t\cap R_k(\varepsilon/2)$ and $r_0=\frac{1}{2}\min\{\varepsilon/2,i_0\}$, where $i_0:=\inf_{x\in R_k}\text{Inj}_M(x)$. Then the geodesic ball $B(x_j,r_0)$ in M is contained in R_k.

We apply [17, Lemma 1, p. 445] to the minimal surface $(\Sigma_i^t,\partial\Sigma_i^t)\subset(R_k,\partial R_k)$ to find that

$$\text{Area}(\Sigma_i^t\cap B(x_j,r_0))\geq C(r_0,i_0,K)$$

where $K=\sup_{x\in R_k}|K_M|$. By Theorem 5.5, this leads to a contradiction:

$$\begin{aligned}2\pi\geq\int_{L_t}\kappa(x)\,dv&\geq\sum_{B_j\in S_t}\int_{\Sigma_i^t}\kappa(x)\,dv\geq\sum_{B_j\in S_t}\int_{\Sigma_i^t\cap B(x_j,r_0)}\kappa(x)\,dv\\&\geq\inf_{x\in R_k}\kappa(x)\cdot\sum_{B_j\in S_t}\text{Area}(\Sigma_i^t\cap B(x_j,r_0))\\&\geq C\cdot\inf_{x\in R_k}\kappa(x)\cdot|S_t|=\infty.\end{aligned}$$

This finishes the proof. ∎

Lemma 8.7. *$\{S_t\}_{t\in\Lambda}$ contains at most finitely many non-empty sets.*

Proof. Suppose that there exists a sequence $\{S_{t_n}\}_{n\in\mathbb{N}}$ of non-empty sets. For any $B_{j_{t_n}}\in S_{t_n}$, there is some $i_n\in I_{t_n}$ such that $B_{j_{t_n}}$ equals $B_{i_n}^{t_n}$ where $B_{i_n}^{t_n}$ is a component of $R_k\setminus\overline{\Sigma_{i_n}^{t_n}}$ and $\Sigma_{i_n}^{t_n}$ is component of $L_{t_n}\cap\text{Int}\,R_k$. Note that $\pi_1(B_{i_n}^{t_n})\to\pi_1(R_k)$ is trivial.

By Proposition 8.4, $\Sigma_{i_n}^{t_n}\cap R_k(\varepsilon/2)$ is not empty. Pick a point p_{t_n} in $\Sigma_{i_n}^{t_n}\cap R_k(\varepsilon/2)$.

Step 1: $\{L_{t_n}\}$ subconverges to a lamination $\mathscr{L}'\subset\mathscr{L}$ with finite multiplicity. Since L_{t_n} is a stable minimal surface, by [26, Theorem 3, p. 122], for any compact set $K\subset M$, there is a constant $C_1 := C_1(K, M, g)$ such that

$$|A_{L_{t_n}}|^2 \leq C_1 \quad \text{on } K \cap L_{t_n}.$$

From Theorem 5.5, $\int_{L_{t_n}} \kappa(x)\, dv \leq 2\pi$. Hence,

$$\mathrm{Area}(K \cap L_{t_n}) \leq 2\pi \Big(\inf_{x\in K} \kappa(x)\Big)^{-1}.$$

We use Theorem 5.3 to find a subsequence of $\{L_{t_n}\}$ subconverging to a properly embedded lamination $\mathscr{L}'$ with finite multiplicity. Since $\mathscr{L}$ is a closed set in M, $\mathscr{L}' \subset \mathscr{L}$ is a sublamination.

From now on, we abuse notation and write $\{L_{t_n}\}$ and $\{p_{t_n}\}$ for the convergent subsequence.

Step 2: $\{\Sigma_{i_n}^{t_n}\}$ converges with multiplicity 1. Let L_{t_∞} be the unique component of $\mathscr{L}'$ passing through p_∞, where $p_\infty = \lim_{n\to\infty} p_{t_n}$. The limit of $\{\Sigma_{i_n}^{t_n}\}$ is the component Σ_∞ of $L_{t_\infty} \cap R_k$ passing through p_∞, where $\Sigma_{i_n}^{t_n}$ is the unique component of $R_k \cap L_{t_n}$ passing through p_{t_n}.

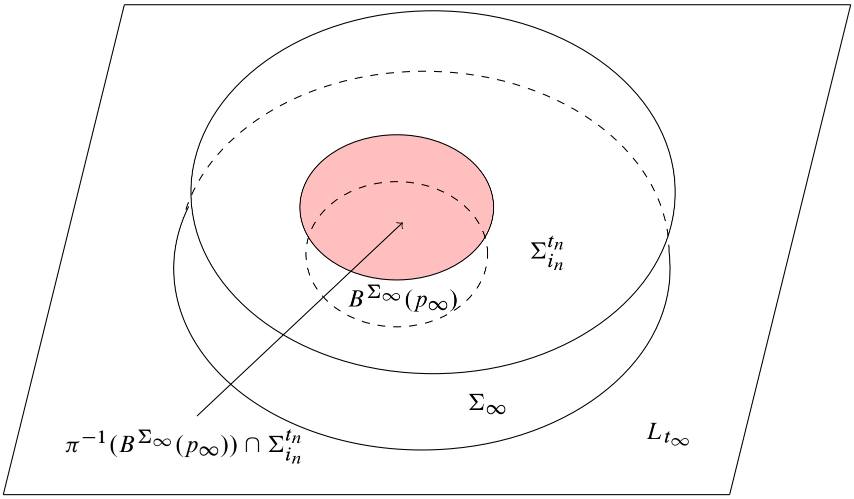

Fig. 2

Let $D \subset L_{t_\infty}$ be a simply-connected subset satisfying $\Sigma_\infty \subset D$. (Its existence is ensured by the fact that L_∞ is homeomorphic to $\mathbb{R}^2$.) Since $\{L_{t_n}\}$ smoothly converges to L_{t_∞} with finite multiplicity, there exists $\varepsilon_1 > 0$ and an integer n_0 such that

$$\Sigma_{i_n}^{t_n} \subset D(\varepsilon_1) \quad \text{for } n > n_0,$$

where $D(\varepsilon_1)$ is the tubular neighborhood of D with radius ε_1 in M (see Definition 5.1).

Let $\pi : D(\varepsilon_1) \to D$ be the projection. From Remark 5.2, we know that for n large enough, the restriction of π to each component of $L_{t_n} \cap D(\varepsilon_1)$ is injective.

Hence, $\pi|_{\Sigma_{i_n}^{t_n}} : \Sigma_{i_n}^{t_n} \to D$ is injective. That is, $\Sigma_{i_n}^{t_n}$ is a normal graph over a subset of D. Therefore, $\{\Sigma_{i_n}^{t_n}\}$ converges to Σ_∞ with multiplicity 1 (see Definition 5.1). That is, there is a geodesic disc $B^{\Sigma_\infty}(p_\infty) \subset \Sigma_\infty$ centered at p_∞ with small radius such that

(∗∗) the set $\pi^{-1}(B^{\Sigma_\infty}(p_\infty)) \cap \Sigma_{i_n}^{t_n}$ is connected and a normal graph over $B^{\Sigma_\infty}(p_\infty)$, for large n.

Step 3: Getting a contradiction. There exists a neighborhood U of p_∞ and a coordinate map Φ such that each component of $\Phi(\mathscr{L} \cap U)$ is $\mathbb{R}^2 \times \{x\} \cap \Phi(U)$ for some $x \in \mathbb{R}$ (see the definition of the lamination in [6, Appendix B, 609–612]). Choose the disc $B^{\Sigma_\infty}(p_\infty)$ and ε_1 small enough such that $\pi^{-1}(B^{\Sigma_\infty}(p_\infty)) \subset U$. We may assume that $U = \pi^{-1}(B^{\Sigma_\infty}(p_\infty))$.

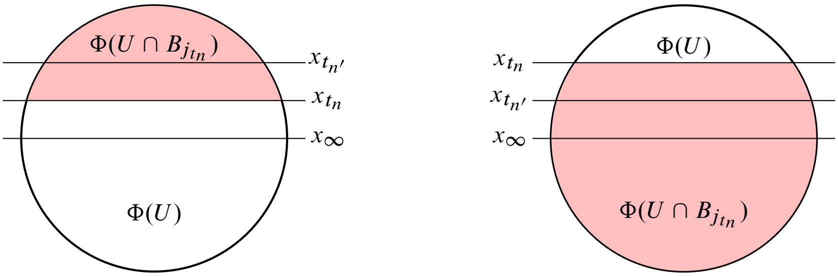

Fig. 3

From (∗∗), $\Sigma_{i_n}^{t_n} \cap U \subset L_{t_n}$ is connected and a graph over $B^{\Sigma_\infty}(p_\infty)$, for n large enough. Since $\partial B_{j_{t_n}} \cap U \subset L_{t_n}$ equals $\Sigma_{i_n}^{t_n} \cap U$, it is also connected. Therefore $\Phi(\partial B_{j_{t_n}} \cap U)$ equals $\mathbb{R}^2 \times \{x_{t_n}\} \cap \Phi(U)$ for some $x_{t_n} \in \mathbb{R}$. In addition, $\Phi(\Sigma_\infty \cap U)$ equals $\mathbb{R}^2 \times \{x_\infty\} \cap \Phi(U)$ for some $x_\infty \in \mathbb{R}$. Since $\lim_{n\to\infty} p_{t_n} = p_\infty$, we have $\lim_{n\to\infty} x_{t_n} = x_\infty$.

The set $U \setminus \partial B_{j_{t_n}}$ has two components. Therefore, $\Phi(B_{j_{t_n}} \cap U)$ is either $\Phi(U) \cap \{x \mid x_3 > x_{t_n}\}$ or $\Phi(U) \cap \{x \mid x_3 < x_{t_n}\}$. For n large enough, there exists some $n' \neq n$ such that $\mathbb{R}^2 \times \{x_{t_{n'}}\} \cap \Phi(U) \subset \Phi(B_{j_{t_n}} \cap U)$. This implies that $B_{j_{t_n}} \cap B_{j_{t_{n'}}}$ is non-empty.

Since S consists of maximal elements in $(\{B_i^t\}, \subset)$, the set $B_{j_{t_n}} \cap B_{j_{t_{n'}}}$ must be empty, which leads to a contradiction. This finishes the proof. ∎

8.3. *The finiteness of S implies Lemma 7.2*

We will now explain how to deduce Lemma 7.2 from the finiteness of S.

Proof of Lemma 7.2. Since S is finite, we may assume that ∂B_j intersects $\partial R_k(\varepsilon)$ transversally for each $B_j \in S$. Note that each B_j is equal to some B_i^t and $\partial B_j \cap \partial R_k(\varepsilon)$

equals $\Sigma_i^t \cap \partial R_k(\varepsilon)$. Since each Σ_i^t is properly embedded, $\{c_i\}_{i\in I} := \bigcup_{B_j\in S} \partial B_j \cap \partial R_k(\varepsilon)$ has finitely many components. Each component is an embedded circle.

The vanishing property of $\mathscr{L}$ and Remark 7.1 show that each c_i is contractible in $\partial R_k(\varepsilon)$ and bounds a unique closed disc $D_i \subset \partial R_k(\varepsilon)$ (since $k \geq k_0$). The set $(D_i, \subset)$ is partially ordered. Let $\{D_{i'}\}_{i'\in I'}$ be the set of maximal elements. The set I' is finite .

Since the boundary of $\bar{B}_j \cap \partial R_k(\varepsilon)$ is a subset of $\partial B_j \cap \partial R_k(\varepsilon) \subset \coprod_{i\in I} c_i$, it is contained in $\coprod_{i'\in I'} D_{i'}$ for each $B_j \in S$.

Next we show that for any $B_j \in S$, $\bar{B}_j \cap \partial R_k(\varepsilon)$ is contained in $\coprod_{i'\in I'} D_{i'}$.

If not, $\partial R_k(\varepsilon) \setminus \coprod_{i'\in I'} D_{i'}$ is contained in $\bar{B}_j \cap \partial R_k(\varepsilon)$ for some $B_j \in S$. This implies that the composition $\pi_1(\partial R_k(\varepsilon) \setminus \coprod_{i'\in I'} D_{i'}) \to \pi_1(\bar{B}_j) \to \pi_1(R_k)$ is not the zero map, which contradicts the fact that the induced map $\pi_1(\bar{B}_j) \to \pi_1(R_k)$ is trivial. We conclude that for each $B_j \in S$, $\bar{B}_j \cap \partial R_k(\varepsilon)$ is contained in $\coprod_{i'\in I'} D_{i'}$.

Therefore, $\bigcup_{B_j\in S} \bar{B}_j \cap \partial R_k(\varepsilon)$ is contained in $\coprod_{i'\in I'} D_{i'}$. From Proposition 8.5, $\mathscr{L} \cap \partial R_k(\varepsilon)$ is contained in a disjoint union of finitely many discs $\{D_{i'}\}_{i'\in I'}$. This completes the proof. ∎

Appendix A

Lemma 8.1. *Let $(\Omega, \partial\Omega) \subset (N, \partial N)$ be a 2-sided embedded disc with some closed subdiscs removed, where N is a closed handlebody of genus $g > 0$. Assume that each circle γ_i is contractible in ∂N, where $\partial\Omega = \coprod_i \gamma_i$. Then $N \setminus \Omega$ has two connected components. Moreover, there is a unique component B such that the induced map $\pi_1(B) \to \pi_1(N)$ is trivial.*

Proof. As in [30, proof of Lemma 7.1], we find that Ω cuts N into two components, B_1 and B_2.

Note that each embedded circle γ_i is contractible in ∂N and bounds a unique closed disc $D_i \subset \partial N$. Consider the surface $\hat{\Omega} := \Omega \cup \bigcup_i D_i$. It is an immersed 2-sphere in N, so that the map $\pi_1(\Omega) \to \pi_1(\hat{\Omega})$ is trivial. Therefore, $\pi_1(\Omega) \to \pi_1(N)$ is trivial.

In the following, we show the existence of B.

Consider the set $\{D_i\}$ to be partially ordered by inclusion. Then $\bigcup_i D_i$ is equal to a disjoint union of maximal elements in $(\{D_i\}, \subset)$. The set $\partial N \setminus \bigcup_i D_i$ is a compact surface with some punctures.

Therefore, the induced map $\pi_1(\partial N \setminus \bigcup_i D_i) \to \pi_1(\partial N)$ is surjective. In addition, the induced map $\pi_1(\partial N) \to \pi_1(N)$ is also surjective. We can conclude that the composition $\pi_1(\partial N \setminus \bigcup_i D_i) \to \pi_1(N)$ of these two maps is also surjective.

The set $\partial N \setminus \bigcup_i D_i$ is contained in one of the two components, B_1 or B_2, of $N \setminus \Omega$. Without loss of generality, we may assume that B_1 contains $\partial N \setminus \bigcup_i D_i$. From the above, the induced map $\pi_1(B_1) \to \pi_1(N)$ is surjective.

Let G_i be the image of $\pi_1(B_i) \to \pi_1(N)$. Van Kampen's Theorem gives an isomorphism between $\pi_1(N)$ and $\pi_1(B_1) *_{\pi_1(\Omega)} \pi_1(B_2)$. Since the image of $\pi_1(\Omega) \to \pi_1(N)$ is trivial, $\pi_1(N)$ is isomorphic to $G_1 * G_2$. Grushko's Theorem [12] shows that

$\operatorname{rank}(G_1) + \operatorname{rank}(G_2) = \operatorname{rank}(\pi_1(N))$. (The rank of a group is the smallest cardinality of a generating set for the group.) From the last paragraph, the image, G_1, of the map $\pi_1(B_1) \to \pi_1(N)$ is isomorphic to $\pi_1(N)$. That is, $\operatorname{rank}(G_1) = \operatorname{rank}(\pi_1(N))$. Therefore, $\operatorname{rank}(G_2)$ is zero, so G_2 is a trivial group. We find that $B := B_2$ is as required.

The uniqueness is proved as in the genus one case (see [30, proof of Lemma 7.1]). ∎

Appendix B

Theorem 5.5. *Let (M, g) be a complete oriented 3-manifold with positive scalar curvature $\kappa(x)$. Assume that Σ is a complete (non-compact) stable minimal surface in M. Then*

$$\int_\Sigma \kappa(x)\, dv \le 2\pi,$$

where dv is the volume form of the induced metric ds^2 over Σ. Moreover, if Σ is an embedded surface, then Σ is proper.

Proof. By [28, Theorem 2, p. 211], Σ is conformally diffeomorphic to $\mathbb{R}^2$.

Consider the Jacobi operator $L := \Delta_\Sigma - K_\Sigma + (\kappa(x) + \frac{1}{2}|A|^2)$, where K_Σ is the Gaussian curvature of the metric ds^2 and Δ_Σ is the Laplace–Beltrami operator of (Σ, ds^2). From [9, Theorem 1, p. 201], there exists a positive function u on Σ satisfying $L(u) = 0$, since Σ is a stable minimal surface.

Consider the metric $d\tilde{s}^2 := u^2 ds^2$. Let $\tilde{K}_\Sigma$ be its sectional curvature and $d\tilde{v}$ its volume form. We know that

$$\tilde{K}_\Sigma = u^{-2}(K_\Sigma - \Delta_\Sigma \log u) \quad \text{and} \quad d\tilde{v} = u^2 dv.$$

By Fischer-Colbrie's work [8, Theorem 1, p. 126], $(\Sigma, d\tilde{s}^2)$ is a complete surface with non-negative sectional curvature $\tilde{K}_\Sigma \ge 0$. By the Cohn-Vossen inequality [5],

$$\int_\Sigma \tilde{K}_\Sigma\, d\tilde{v} \le 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

Since $L(u) = 0$, one has $\int_{B^\Sigma(0,R)} L(u)u^{-1} dv = 0$, where $B^\Sigma(0, R)$ is the geodesic ball in (Σ, ds^2) centered at $0 \in \Sigma$ with radius R. We deduce that

$$\begin{aligned}
\int_{B^\Sigma(0,R)} \big(\kappa(x) + \tfrac{1}{2}|A|^2\big)\, dv &= \int_{B^\Sigma(0,R)} (K_\Sigma - u^{-1}\Delta_\Sigma u)\, dv \\
&= \int_{B^\Sigma(0,R)} \big(K_\Sigma - (\Delta_\Sigma \log u + u^{-2}|\nabla u|)\big)\, dv \\
&\le \int_{B^\Sigma(0,R)} u^{-2}(K_\Sigma - \Delta_\Sigma \log u)u^2\, dv \\
&= \int_{B^\Sigma(0,R))} \tilde{K}_\Sigma\, d\tilde{v} \le \int_\Sigma \tilde{K}_\Sigma\, d\tilde{v}.
\end{aligned}$$

We know that $\chi(\Sigma) = 1$, since Σ is diffeomorphic to $\mathbb{R}^2$. Combining the two inequalities above and letting $R \to \infty$, we obtain

$$\int_\Sigma \left(\kappa(x) + \tfrac{1}{2}|A|^2\right) dv \le 2\pi.$$

Suppose that Σ is not proper. There is an accumulation point p of Σ such that the set $B(p, r/2) \cap \Sigma$ is a non-compact closed set in Σ. Hence, it is unbounded in (Σ, ds^2). Thus, there is a sequence $\{p_k\}$ of points in $\Sigma \cap B(p, r/2)$ going to infinity in (Σ, ds^2).

Therefore, we may assume that the geodesic discs $\{B^\Sigma(p_k, r/2)\}_k$ in Σ are disjoint.

Set $r_0 := \frac{1}{2}\min\{r, i_0\}$ and $K := \sup_{x \in B(p,r)} |K_M(x)|$ where $i_0 := \inf_{x \in B(p,r)} \mathrm{Inj}_M(x)$ and K_M is the sectional curvature. The geodesic disc $B^\Sigma(p_k, r_0/2)$ is in $B(p, r)$.

Applying [10, Appendix, Theorem 3, p. 139] to the geodesic disc $B^\Sigma(p_k, r_0/2) \subset B(p, r)$, we have

$$\mathrm{Area}(B^\Sigma(p_k, r_0/2)) \ge C(i_0, r_0, K).$$

This leads to a contradiction:

$$\begin{aligned} 2\pi &\ge \int_\Sigma \kappa(x)\, dv \ge \int_{B(p,r)\cap\Sigma} \kappa(x)\, dv \ge \sum_k \int_{B^\Sigma(p_k, r_0/2)} \kappa(x) \\ &\ge \inf_{x \in B(p,r)} \kappa(x) \cdot \sum_k \mathrm{Area}(B^\Sigma(p_k, r_0/2)) \\ &\ge \inf_{x \in B(p,r)} \kappa(x) \cdot \sum_k C = \infty. \end{aligned}$$

■

Appendix C: Example

There are infinitely many contractible 3-manifolds with non-trivial fundamental group at infinity. In this Appendix, we construct such a 3-manifold M and analyse its topology. We will prove that this 3-manifold has no complete metric of positive scalar curvature.

C.1. The construction of M

Before constructing the 3-manifold, let us introduce a definition. A handlebody $N \subset \mathbb{S}^3$ of genus g is said to be *unknotted* in $\mathbb{S}^3$ if its complement in $\mathbb{S}^3$ is a handlebody of genus g.

Choose an unknotted handlebody $W_0 \subset \mathbb{S}^3$ of genus 2. Take a second handlebody $W_1 \subset \mathrm{Int}\, W_0$ of genus 2 which is a tubular neighborhood of the curve in Figure 4. Then, embed another handlebody W_2 of genus 2 inside W_1 in the same way as W_1 lies in W_0, and so on infinitely many times. Thus, we obtain a decreasing family $\{W_k\}$ of handlebodies of genus 2.

The manifold M is defined as $M := \mathbb{S}^3 \setminus \bigcap_{k=0}^{\infty} W_k$. It is an open manifold.

We see that each W_k is unknotted in $\mathbb{S}^3$, so the complement N_k of W_k in $\mathbb{S}^3$ is a handlebody of genus 2. Therefore, M can be written as an increasing union of handle-

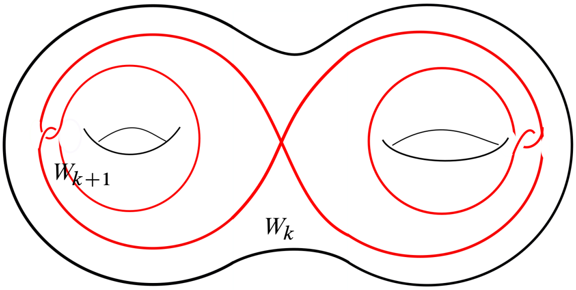

Fig. 4

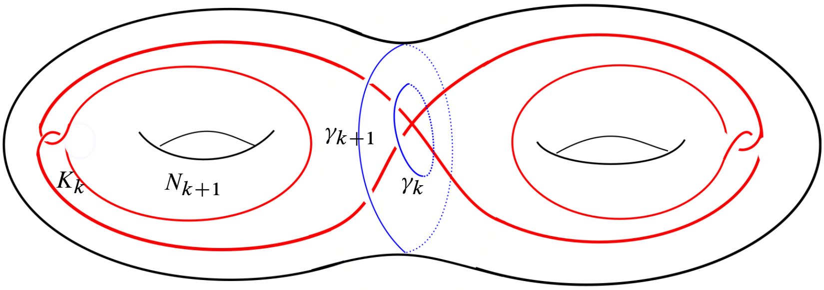

Fig. 5

bodies $\{N_k\}_k$ of genus 2. Furthermore, each N_k lies in N_{k+1} as in Figure 5. (The set K_k is the core of N_k.)

Each N_k is homotopically trivial in N_{k+1}. We can conclude that M is a contractible 3-manifold.

C.2. The topological property of M

In this part, we first show that the fundamental group at infinity of M is non-trivial. As a consequence, M is not homeomorphic to $\mathbb{R}^3$. In the manifold M, there is a properly embedded plane. This plane cuts M into two Whitehead manifolds.

First, we see from Figure 4 that W_k is an effective handlebody relative to W_{k+1} for each k. From Lemma 3.5, the map $\pi_1(\partial W_k) \to \pi_1(\overline{W_k \setminus W_{k+1}})$ is injective. In addition, the set $\overline{W_k \setminus W_{k+1}}$ is equal to $\overline{N_{k+1} \setminus N_k}$. Therefore, we conclude that for each k, the map $\pi_1(\partial N_k) \to \pi_1(\overline{N_{k+1} \setminus N_k})$ is injective.

Second, from Figure 5, we see that each N_{k+1} is an effective handlebody relative to N_k. By Lemma 3.5, the map $\pi_1(\partial N_{k+1}) \to \pi_1(\overline{N_{k+1} \setminus N_k})$ is injective.

As in the genus 1 case, for each k, the maps $\pi_1(\partial N_k) \to \pi_1(\overline{M \setminus N_k})$ and $\pi_1(\partial N_k) \to \pi_1(\overline{N_k \setminus N_0})$ are both injective. That is, the family $\{N_k\}$ has Property (H).

Pick the separating meridian $\gamma_k \subset \partial N_k$ as in Figure 5 . From Figure 5, for each k, γ_k is homotopic to γ_{k+1} in $\overline{N_{k+1} \setminus N_k}$. Since $\{N_k\}$ satisfies Property (H), the map $\pi_1(\partial N_k) \to \pi_1(\overline{M \setminus N_0})$ is injective (see Remark 4.7). That is, for $k > 0$, γ_k is non-contractible in $M \setminus N_0$.

From Remark 2.4, the sequence $\{\gamma_k\}$ gives a non-trivial element in $\pi_1^\infty(M)$. Since $\pi_1^\infty(M)$ is non-trivial, M is not simply-connected at infinity. In particular, M is not homeomorphic to $\mathbb{R}^3$.

Next, we construct the properly embedded plane in M from the sequence $\{\gamma_k\}_k$.

Choose an embedded annulus $A_k \subset \overline{N_{k+1} \setminus N_k}$ with boundary $\gamma_k \amalg \gamma_{k+1}$. Let $D_0 \subset N_0$ be a meridian disc with boundary γ_0. We define the plane P as

$$P := \bigcup_{k\geq 0} A_k \cup D_0.$$

The plane P cuts M into two contractible 3-manifolds M' and M''. In addition, the intersection $P \cap N_k$ is a separating meridian disc of N_k with boundary γ_k.

From the sequence $\{N_k\}$, we obtain two increasing families, $\{N_k'\}$ and $\{N_k''\}$, of solid tori in M satisfying

- $M' = \bigcup_k N_k'$ and $M'' = \bigcup_k N_k''$;
- $N_k \setminus (N_k' \amalg N_k'')$ is a tubular neighborhood of the meridian disc $P \cap N_k$.

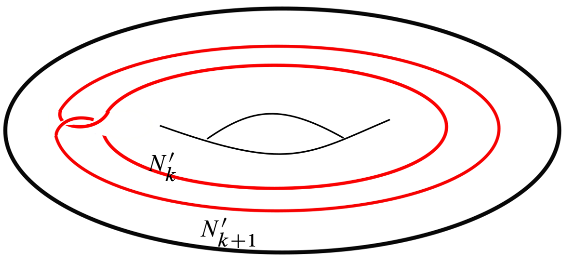

Fig. 6

Furthermore, each N_k' is embedded into N_{k+1}' as in Figure 6. We see that M' is homeomorphic to the Whitehead manifold. Similarly, the contractible 3-manifold M'' is also homeomorphic to the Whitehead manifold. Therefore, P cuts M into two Whitehead manifolds.

C.3. Non-existence of PSC metrics

In this subsection, we show that the manifold M has no complete metric of positive scalar curvature.

Suppose for contradiction that M has a complete metric of positive scalar curvature. As in Section 5.1, there is a family $\{\mathscr{L}_k\}_k$ of laminations subconverging to a stable minimal lamination $\mathscr{L} := \bigcup_{t\in\Lambda} L_t$.

Since $\pi_1^\infty(M)$ is non-trivial, some leaf in $\mathscr{L}$ may not satisfy the vanishing property for $\{N_k\}_k$. To overcome it, we attempt to find a new family of handlebodies with Property (H).

We know that $M' = \bigcup_k N'_k$ is homeomorphic to the Whitehead manifold. The geometric index $I(N'_k, N'_{k+1})$ is equal to 2 (see [30, Section 2]). From [30, Lemma 2.10], the maps $\pi_1(\partial N'_k) \to \pi_1(\overline{M \setminus N'_0})$ and $\pi_1(\partial N'_k) \to \pi_1(\overline{N'_k \setminus N'_0})$ are both injective. Therefore, the family $\{N'_k\}$ satisfies Property (H).

In addition, each leaf L_t in $\mathscr{L}$ satisfies Property P (see [30, Definition 3.3]). That is, for any circle $\gamma \subset L_t \cap \partial N'_k$, one of the following holds:

- γ is homotopically trivial in $\partial N'_k$;
- for $l \leq k$, $D \cap \operatorname{Int} N'_l$ has at least $I(N'_l, N'_k)$ components intersecting N_0,

where $D \subset L_t$ is the unique disc with boundary γ and the geometric index $I(N'_l, N'_k)$ is equal to 2^{k-l}.

In the following, we consider the geometry of the leaves intersecting M'. We have the following claim.

Claim. *$\mathscr{L}$ satisfies the vanishing property with respect to $\{N'_k\}_k$.*

The proof of this claim is the same as that of [30, Lemma 6.1]. Towards a contradiction, suppose that there exists an increasing sequence $\{k_n\}_n$ of integers such that

- *for each k_n, there exists a minimal surface L_{t_n} in $\{L_t\}_{t\in\Lambda}$ and an embedded curve $c_{k_n} \subset L_{t_n} \cap \partial N'_{k_n}$ which is not contractible in $\partial N'_{k_n}$.*

Since $\lim_{n\to\infty} k_n = \infty$, we have $\lim_{n\to\infty} I(N'_1, N'_{k_n}) = \infty$.

Since (M, g) has positive scalar curvature, L_{t_n} is homeomorphic to $\mathbb{R}^2$. Then there exists a unique disc $D_n \subset L_{t_n}$ with boundary c_{k_n}. From the above property, we see that $D_n \cap N'_1$ has at least $I(N'_1, N'_{k_n})$ components intersecting N'_0, denoted by $\{\Sigma_j\}_{j=1}^m$.

Define $r := d^M(\partial N'_0, \partial N'_1)$, $C := \inf_{x\in N'_1} \kappa(x)$, $K := \sup_{x\in N'_1} |K_M|$ and $i_0 := \inf_{x\in N'_1} \operatorname{Inj}_M(x)$, where K_M is the sectional curvature of (M, g) and $\operatorname{Inj}_M(x)$ is the injectivity radius of (M, g) at x.

Choose $r_0 = \frac{1}{2}\min\{i_0, r\}$ and $x_j \in \Sigma_j \cap N'_0$. Then $B(x_j, r_0)$ is in N'_1. We apply [17, Lemma 1, p. 445] to the minimal surface $(\Sigma_j, \partial\Sigma_j) \subset (N'_1, \partial N'_1)$. Hence,

$$\operatorname{Area}(\Sigma_j \cap B(x_j, r_0)) \geq C_1(K, i_0, r_0).$$

From Theorem 5.5, we have

$$\begin{aligned} 2\pi &\geq \int_{L_{t_n}} \kappa(x)\, dv \geq \sum_{j=1}^m \int_{\Sigma_j} \kappa(x)\, dv \geq \sum_{j=1}^m \int_{\Sigma_j \cap B(x_j, r_0)} \kappa(x)\, dv \\ &\geq \sum_{j=1}^m C \operatorname{Area}(\Sigma_j \cap B(x_j, r_0)) \geq CC_1 m \geq CC_1 I(N'_1, N'_{k_n}). \end{aligned}$$

This contradicts the fact that $\lim_{n\to\infty} I(N'_1, N'_{k_n}) = \infty$ and completes the proof of the claim.

In addition, since none of the N'_k is contained in a 3-ball, we use Corollary 3.10 to find that if N_j contains N'_k, then $\mathscr{L}_j \cap \partial N'_k$ contains at least one meridian of R_k.

To sum up, the family $\{N'_k\}_k$ satisfies (a) and (b) of Section 7. That is,

(a) $\mathscr{L}$ satisfies the vanishing property for $\{N'_k\}_k$;

(b) if N_j contains N'_k, then $\mathscr{L}_j \cap \partial N'_k$ contains at least one meridian of N'_k.

The remaining proof is the same as the proof of Theorem 1.1 in Sections 7 and 8.

Acknowledgments. I would like to express my deep gratitude to my supervisor Gérard Besson for suggesting this problem, also for his endless support and constant encouragement. I also thank Harold Rosenberg for many enlightening discussions about minimal surfaces theory. I am thankful to Laurent Mazet for his interest and for many valuable conversations. I am also very grateful to the referees for many helpful comments that improved the exposition of this paper.

Funding. This research is supported by ERC Advanced Grant 320939, GETOM and partially by the Special Priority Program SPP 2026 "Geometry at infinity".

References

[1] Anderson, M. T.: Curvature estimates for minimal surfaces in 3-manifolds. Ann. Sci. École Norm. Sup. (4) **18**, 89–105 (1985) Zbl 0578.49027 MR 803196

[2] Bessières, L., Besson, G., Maillot, S., Boileau, M., Porti, J.: Geometrisation of 3-manifolds. EMS Tracts in Mathematics 13, European Mathematical Society, Zürich (2010) Zbl 1244.57003 MR 2683385

[3] Cao, H.-D., Zhu, X.-P.: A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton–Perelman theory of the Ricci flow. Asian J. Math. **10**, 165–492 (2006) Zbl 1200.53057 MR 2233789

[4] Chang, S., Weinberger, S., Yu, G.: Taming 3-manifolds using scalar curvature. Geom. Dedicata **148**, 3–14 (2010) Zbl 1201.53047 MR 2721617

[5] Cohn-Vossen, S.: Kürzeste Wege und Totalkrümmung auf Flächen. Compos. Math. **2**, 69–133 (1935) Zbl 0011.22501 MR 1556908

[6] Colding, T. H., Minicozzi, W. P., II: The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected. Ann. of Math. (2) **160**, 573–615 (2004) Zbl 1076.53069 MR 2123933

[7] Colding, T. H., Minicozzi, W. P., II: A course in minimal surfaces. Graduate Studies in Mathematics 121, American Mathematical Society, Providence, RI (2011) Zbl 1242.53007 MR 2780140

[8] Fischer-Colbrie, D.: On complete minimal surfaces with finite Morse index in three-manifolds. Invent. Math. **82**, 121–132 (1985) Zbl 0573.53038 MR 808112

[9] Fischer-Colbrie, D., Schoen, R.: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. Comm. Pure Appl. Math. **33**, 199–211 (1980) Zbl 0439.53060 MR 562550

[10] Frensel, K. R.: Stable complete surfaces with constant mean curvature. Bol. Soc. Brasil. Mat. (N.S.) **27**, 129–144 (1996) Zbl 0890.53054 MR 1418929

[11] Gromov, M., Lawson, H. B., Jr.: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. Inst. Hautes Études Sci. Publ. Math. **58**, 83–196 (1983) Zbl 0538.53047 MR 720933

[12] Grushko, I. A.: On the bases of a free product of groups. Mat. Sb. **8**, 169–182 (1940) (in Russian) JFM 66.0066.02

[13] Hatcher, A.: Notes on basic 3-manifold topology (2000)

[14] McMillan, D. R., Jr.: Cartesian products of contractible open manifolds. Bull. Amer. Math. Soc. **67**, 510–514 (1961) Zbl 0116.40802 MR 131280

[15] McMillan, D. R., Jr.: Some contractible open 3-manifolds. Trans. Amer. Math. Soc. **102**, 373–382 (1962) Zbl 0111.18701 MR 137105

[16] Meeks, W. H., III, Ros, A., Rosenberg, H.: The global theory of minimal surfaces in flat spaces. Lecture Notes in Mathematics 1775, Springer, Berlin; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence (2002) Zbl 0983.00044 MR 1901611

[17] Meeks, W. H., III, Yau, S. T.: Topology of three-dimensional manifolds and the embedding problems in minimal surface theory. Ann. of Math. (2) **112**, 441–484 (1980) Zbl 0458.57007 MR 595203

[18] Meeks, W. W., III, Yau, S. T.: The existence of embedded minimal surfaces and the problem of uniqueness. Math. Z. **179**, 151–168 (1982) Zbl 0479.49026 MR 645492

[19] Morgan, J., Tian, G.: Ricci flow and the Poincaré conjecture. Clay Mathematics Monographs 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA (2007) Zbl 1179.57045 MR 2334563

[20] Myers, R.: Contractible open 3-manifolds which non-trivially cover only non-compact 3-manifolds. Topology **38**, 85–94 (1999) Zbl 0926.57017 MR 1644087

[21] Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159 (2002) Zbl 1130.53001

[22] Perelman, G.: Ricci flow with surgery on three-manifolds. arXiv:math/0303109 (2002) Zbl 1130.53002

[23] Perelman, G.: Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arXiv:math/0307245 (2003) Zbl 1130.53003

[24] Rolfsen, D.: Knots and links. Mathematics Lecture Series 7, Publish or Perish, Berkeley, CA (1976) Zbl 0339.55004 MR 0515288

[25] Rotman, J. J.: An introduction to the theory of groups. 4th ed., Graduate Texts in Mathematics 148, Springer, New York (1995) Zbl 0944.00001 MR 1307623

[26] Schoen, R.: Estimates for stable minimal surfaces in three-dimensional manifolds. In: Seminar on minimal submanifolds, Annals of Mathematics Studies 103, Princeton University Press, Princeton, NJ, 111–126 (1983) Zbl 0532.53042 MR 795231

[27] Schoen, R., Yau, S. T.: On the structure of manifolds with positive scalar curvature. Manuscripta Math. **28**, 159–183 (1979) Zbl 0423.53032 MR 535700

[28] Schoen, R., Yau, S. T.: Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature. In: Seminar on Differential Geometry, Annals of Mathematics Studies 102, Princeton University Press, Princeton, NJ, 209–228 (1982) Zbl 0481.53036 MR 645740

[29] Stallings, J.: Group theory and three-dimensional manifolds. Yale Mathematical Monographs 4, Yale University Press, New Haven, CN (1971) Zbl 0241.57001 MR 0415622

[30] Wang, J.: Contractible 3-manifolds and positive scalar curvature (I). J. Differential Geom. (to appear); arXiv:1901.04605

[31] Whitehead, J.: A certain open manifold whose group is unity. Quart. J. Math. **6**, 268–279 (1935) JFM 61.0607.01