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On the Arithmetic Fundamental Lemma conjecture over a general p-adic field

Abstract. We prove the Arithmetic Fundamental Lemma conjecture over a general p-adic field with odd residue cardinality $q \geq \dim V$. Our strategy is similar to the one used by the second author in his proof of the AFL over $\mathbb{Q}_p$, but only requires the modularity of divisor generating series on the Shimura variety (as opposed to its integral model). The resulting increase in flexibility allows us to work over an arbitrary base field. To carry out our strategy, we also generalize results of Howard (2012) on CM cycle intersection and of Ehlen–Sankaran (2018) on Green function comparison from $\mathbb{Q}$ to general totally real base fields.

Keywords. Moduli space, p-divisible group, Shimura variety, intersection theory, special cycle, Green function, complex multiplication, admissible pairing

Contents

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1. Introduction

1.1. Main results

Let F/F_0 be an unramified quadratic extension of p-adic local fields with $p \neq 2$. The Arithmetic Fundamental Lemma (AFL) conjecture, first formulated by the second author in [33], is an identity relating the following two types of quantities:

(1) special values of the derivative of orbital integrals for the symmetric space

$$S_n := (\mathrm{Res}_{F/F_0} \mathrm{GL}_n)/\mathrm{GL}_{n,F_0} \simeq \{g \in \mathrm{Res}_{F/F_0} \mathrm{GL}_n \mid g\bar{g} = 1_n\},$$

for a natural action of GL_{n-1,F_0};

(2) arithmetic intersection numbers on a Rapoport–Zink formal moduli space (abbreviated as "RZ space" henceforth) of p-divisible groups with PEL type structure associated to a unitary group.

We refer to Sections 9.1–9.3 for the precise statements. The main result of this paper is a confirmation of the AFL conjecture under the assumption that the residue field of F_0 has at least n elements (see Theorem 10.1).[1] The strategy is similar to, but more refined than the one employed by the second author [35], where he shows the AFL for $F_0 = \mathbb{Q}_p$ and $p \geq n$. We describe our strategy in more detail below.

The AFL conjecture arises from the relative trace formula approach to the arithmetic Gan–Gross–Prasad conjecture (for $\mathrm{U}_n \times \mathrm{U}_{n-1}$) and we refer to [28, 35] for a more detailed introduction to its origin and its relation to the Gross–Zagier theorem [9]. In particular, the orbital integrals appear on the geometric side of Jacquet–Rallis' relative trace formula [11], while certain Rankin–Selberg L-functions appear on the spectral side. The arithmetic intersection numbers on the RZ space appear as a local component of a global intersection problem via the uniformization theorem [30]. Therefore, the AFL conjecture implies certain identities for intersection numbers on integral models of unitary Shimura varieties, which has been made precise by Rapoport, Smithling and the second author [28]. We prove their "semiglobal conjecture" in the hyperspecial case in Theorem 9.3 under the assumption of not too small residue field cardinality.

1.2. Proof of the AFL

The overall strategy is similar to [35]: We extract the local intersection number from a global one that may be accessed by using the modularity of certain generating series of cycles on unitary Shimura varieties. In order to explain these ideas and to point out the differences with [35], we now recall the arithmetic intersection pairing from [35] and then explain the variant employed here.

Let $\mathbf{F}$ be a number field and $\mathcal{X}$ a regular, flat, projective scheme over $\mathrm{Spec}\, O_{\mathbf{F}}$ with smooth generic fiber $\mathcal{X}_{\mathbf{F}}$. Denote by $\widetilde{\mathcal{Z}}_1(\mathcal{X})$ the group of 1-cycles on $\mathcal{X}$ and let $\widehat{\mathrm{Ch}}^1(\mathcal{X})$

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[1]In the more recent article [36], Z. Zhang adapts and extends the strategy of the present paper in a twofold way. First, he removes the assumption that the residue field of F_0 has at least n elements. Second, he establishes an analog of the AFL in many cases of bad reduction.

denote the arithmetic Chow group of cycles of codimension 1. The latter is isomorphic to the group $\widehat{\mathrm{Pic}}(\mathcal{X})$ of isomorphism classes of metrized line bundles. In this paper, all Chow groups have $\mathbb{Q}$-coefficients. There is an arithmetic intersection pairing (see [1, Section 2.3])

$$(\cdot,\cdot)\colon\widehat{\mathrm{Ch}}^1(\mathcal{X})\times\widetilde{\mathcal{Z}}_1(\mathcal{X})\to\mathbb{R}. \tag{1.1}$$

It is given by taking the arithmetic degree of the metrized line bundle restricted to the curve, $(\hat{\mathcal{L}},C)=\widehat{\deg}(\hat{\mathcal{L}}|_C)$. Let $\mathcal{Z}_1(\mathcal{X})$ be the quotient of $\widetilde{\mathcal{Z}}_1(\mathcal{X})$ by the subgroup generated by 1-cycles on $\mathcal{X}$ that are contained in a closed fiber and rationally trivial within that fiber. Then by [1, Proposition 2.3.1 (ii)], the above pairing factors through

$$(\cdot,\cdot)\colon\widehat{\mathrm{Ch}}^1(\mathcal{X})\times\mathcal{Z}_1(\mathcal{X})\to\mathbb{R}. \tag{1.2}$$

Our case of interest is when $\mathcal{X}/O_F$ is an integral model for a unitary Shimura variety. The Shimura datum will be defined for a hermitian F-vector space where F/F_0 is a quadratic CM extension of a totally real field that globalizes the local fields from Section 1.1. Then $\mathbf{F}$ is our notation for the reflex field.

Just as in [35], the two kinds of cycles of interest are the Kudla–Rapoport divisors (elements of $\widehat{\mathrm{Ch}}^1(\mathcal{X})$) and certain (derived) CM cycles (elements of $\mathcal{Z}_1(\mathcal{X})$). One of the key inputs in [35] was the modularity of generating series of arithmetic KR divisors as elements in $\widehat{\mathrm{Ch}}^1(\mathcal{X})$. More precisely, the modularity of the generating series of intersection numbers with a fixed CM cycle allows one to run an inductive argument. This modularity was proved by Bruinier, Howard, Kudla, Rapoport and Yang [3] when $F_0=\mathbb{Q}$ by a detailed study of the divisors of Borcherds products on the integral model in question. However, it appears that such modularity results are difficult to obtain when $F_0\neq\mathbb{Q}$ and this was why the restriction $F_0=\mathbb{Q}$ appeared in [35].

The new idea in this paper is to only use modularity of the KR generating series for the generic fiber, i.e. with coefficients in $\mathrm{Ch}^1(\mathcal{X}_{\mathbf{F}})$. This result for general F_0 is deduced by Liu [19] from its analogue for orthogonal Shimura varieties, which was proved by Yuan, S. Zhang and the second author [32]; the modularity for the generating series with coefficients in cohomology groups was shown by Kudla–Millson [15] in much greater generality and was a key ingredient in [32]. Moreover, we assume $F_0\neq\mathbb{Q}$, which implies properness for $\mathcal{X}$ and allows us to simplify the above intersection theory. For simplicity, we now additionally assume that $\mathcal{X}$ is smooth over $\mathrm{Spec}\,O_F$. Let $\mathcal{Z}_1(\mathcal{X})_{\deg=0}$ be the subgroup of $\mathcal{Z}_1(\mathcal{X})$ consisting of 1-cycles with vanishing degree on every connected component of $\mathcal{X}_{\mathbf{F}}$. Let $\widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{X})\subset\widehat{\mathrm{Ch}}^1(\mathcal{X})$ be the subgroup generated by cycles with *admissible* Green function, meaning they have harmonic curvature with respect to the naturally metrized Hodge bundle on $\mathcal{X}_{\mathbf{F}}$. Then it is easy to see (in Section 4.2) that the restriction of the pairing (4.5) to $\widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{X})\times\mathcal{Z}_1(\mathcal{X})_{\deg=0}$ factors through

$$(\cdot,\cdot)\colon\mathrm{Ch}^1(\mathcal{X}_{\mathbf{F}})\times\mathcal{Z}_1(\mathcal{X})_{\deg=0}\to\mathbb{R}, \tag{1.3}$$

where $\widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{X})\to\mathrm{Ch}^1(\mathcal{X}_{\mathbf{F}})$ is the natural map that takes the generic fiber of cycles. In effect, the pairing is computed by extending a class in $\mathrm{Ch}^1(\mathcal{X}_{\mathbf{F}})$ to one in $\widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{X})$ and then applying the pairing (4.5). Then one is in a position to apply the modularity

results from [32] for KR divisors in $\mathrm{Ch}^1(\mathcal{X}_{\mathbf{F}})$. In practice, to extract a specific local term from the global intersection number, we need to allow general level structure at a finite set of non-archimedean places S and hence we actually use an S-punctured variant of the above; see Section 4.

Just as in [35], it is also crucial to understand the archimedean local intersection, i.e. the values of Green functions at CM cycles. It turns out that the automorphic Green function constructed by Bruinier [2] is an admissible choice in the above sense. We prove that the generating series obtained by taking the difference with the Kudla Green function is modular. This was shown before by Ehlen–Sankaran [5] for $F_0 = \mathbb{Q}$. The extension to general F_0 we prove here is obtained by a rather straightforward proof that relies on the properness of $\mathcal{X}_{\mathbf{F}}$. This is carried out in Part I.

The other obstacle is that we need to modify the CM cycle to land in the smaller $\mathcal{Z}_1(\mathcal{X})_{\deg=0}$. This modification should be as simple as possible so that we keep control of the resulting modification of intersection numbers. Our strategy is to subtract a suitable linear combination of *elementary* CM-cycles. Their intersection pairings with KR divisors have been studied by various authors (e.g. by Howard [10] when $F_0 = \mathbb{Q}$ as well as earlier by Kudla–Rapoport–Yang [18]) and are essentially given by (the derivative of) Fourier coefficients of Siegel–Eisenstein series. This is carried out in Part II.

We finally mention the recent work of the first author [22] which proved the local constancy of the intersection number occurring in the AFL conjecture for all regular semi-simple elements. This allows us, both in the present paper and in [35], to spread out the AFL identity from *strongly* regular semisimple elements to all regular semisimple ones.

The strategy in this paper is quite flexible and in a future paper the authors plan to explore the possibility of a new proof of the averaged Colmez conjecture. Moreover, it seems likely that some enhancement of the arguments in Part I will also establish the modularity of generating series of arithmetic KR divisors as elements in $\widehat{\mathrm{Ch}}^1(\mathcal{X})$ under certain conditions (e.g. when $\mathcal{X}$ has good reduction everywhere).

Naturally, the present paper makes use of several results and constructions from previous works, notably [2, 28, 35]. Our policy is to recall the necessary background from these sources whenever needed to make the paper as self-contained as possible.

1.3. Notation

Notation on algebra

- For a global field $\mathbf{F}$, we denote by $\Sigma_{\mathbf{F}}$ the set of all places.

- Unless otherwise stated, F denotes a CM number field and F_0 denotes its (maximal) totally real subfield of index 2. We denote by $a \mapsto \bar{a}$ the non-trivial automorphism of F/F_0. Let $F_{0,+}$ (resp., $F_{0,\geq 0}$) be the set of totally positive (resp., semipositive) elements in F_0.

- We denote $\mathbf{H} = \mathrm{SL}_2$ and view it as an algebraic group over F_0. Denote by B the Borel subgroup of upper triangular matrices, and by N its unipotent radical. By abuse of

notation we let $\widetilde{\mathbf{H}}$ denote the metaplectic double covering of $\mathbf{H} = \mathrm{SL}_2$, even though it is not an algebraic group.

- Unless otherwise stated, we write $\mathbb{A}$, $\mathbb{A}_0$, and $\mathbb{A}_F$ for the adele rings of $\mathbb{Q}$, F_0, and F, respectively. We systematically use a subscript f for the ring of finite adeles, and a superscript p for the adeles away from the prime number p.

- For an abelian scheme A over a locally noetherian scheme S on which the prime number p is invertible, we write $\mathrm{T}_p(A)$ for the p-adic Tate module of A (regarded as a smooth $\mathbb{Z}_p$-sheaf on S) and $\mathrm{V}_p(A) := \mathrm{T}_p(A) \otimes \mathbb{Q}$ for the rational p-adic Tate module (regarded as a smooth $\mathbb{Q}_p$-sheaf on S).

- We use a superscript $\circ$ to denote the operation $- \otimes_{\mathbb{Z}} \mathbb{Q}$ on groups of homomorphisms of abelian schemes, so that for example $\mathrm{Hom}^\circ(A, A') := \mathrm{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$.

- All Chow groups and K-groups have $\mathbb{Q}$-coefficients.

- For a vector space V over a field F_0 (of characteristic not equal to 2), a quadratic form $\mathfrak{q} \colon V \to F$ has an associated symmetric bilinear pairing defined by

$$\langle x, y \rangle = \mathfrak{q}(x + y) - \mathfrak{q}(x) - \mathfrak{q}(y), \quad x, y \in V. \tag{1.4}$$

In particular,

$$\langle x, x \rangle = 2\mathfrak{q}(x). \tag{1.5}$$

For a quadratic field extension F of F_0, an F/F_0-*hermitian space* is an F_0-vector space V endowed with an F_0-linear action of F and an "F/F_0-hermitian form", i.e. a map $\langle \cdot, \cdot \rangle \colon V \times V \to F$ that is F-linear on the first factor and conjugate symmetric. Its dimension will be the dimension as an F-vector space. It induces a symmetric F_0-bilinear pairing by $(x, y) \mapsto \mathrm{tr}_{F/F_0}\langle x, y \rangle \in F_0$. In particular, the corresponding quadratic form on V is

$$\mathfrak{q}(x) = \langle x, x \rangle \in F_0. \tag{1.6}$$

We will treat V as an affine variety over F_0, and for $\xi \in F_0$ we denote by V_ξ the subscheme defined by $\mathfrak{q}(x) = \xi$.

- For an F/F_0-hermitian space V over a non-archimedean local field, and an O_F-lattice $\Lambda \subset V$ (of full rank), we denote by $\Lambda^\vee = \{x \in V \mid \langle x, \Lambda \rangle \subset O_F\}$ its dual lattice under the hermitian form.

- Let R be a commutative ring. We denote by $(\mathrm{LNSch})_{/R}$ the category of locally noetherian schemes over $\mathrm{Spec}\, R$.

- We denote by $F[t]^\circ_{\deg n}$ the set of monic polynomials α of degree n with $\alpha(0)t^n \overline{\alpha}(t^{-1}) = \alpha(t)$.

Notation on automorphic forms

- Fix the non-trivial additive character $\psi = \psi_{\mathbb{Q}} \circ \mathrm{tr}_{F_0/\mathbb{Q}} \colon F_0 \backslash \mathbb{A}_0 \to \mathbb{C}^\times$ where $\psi_{\mathbb{Q}}$ is the standard one and $\mathrm{tr}_{F_0/\mathbb{Q}} \colon F_0 \backslash \mathbb{A}_0 \to \mathbb{Q} \backslash \mathbb{A}$ is the trace map. For $\xi \in F_0$ we denote by ψ_ξ the twist $\psi_\xi(x) = \psi(\xi x)$. Let $\delta_{E/F}$ denote the different ideal for any extension E/F of number fields.

- For a smooth algebraic variety X over a local field F, we denote by $\mathcal{S}(X(F))$ the space of Schwartz functions on $X(F)$. Similarly, for a smooth algebraic variety X over a global field F, we denote by $\mathcal{S}(X(\mathbb{A}))$ the space of Schwartz functions on $X(\mathbb{A})$.

- For a quadratic space V over a number field F_0, we denote by ω the Weil representation of $\mathrm{O}(V)(\mathbb{A}_0) \times \widetilde{\mathbf{H}}(\mathbb{A}_0)$ on $\mathcal{S}(V(\mathbb{A}_0))$. If $\dim V$ is even, the representation factors through $\mathrm{O}(V)(\mathbb{A}_0) \times \mathbf{H}(\mathbb{A}_0)$. If V is an F/F_0-hermitian space, we view it as a quadratic space over F_0 and denote again by ω the Weil representation. See [35, Section 11] for an explicit description of ω.

- For $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}$, the weight k Whittaker function on $\mathrm{SL}_2(\mathbb{R})$ is defined by

$$W_\xi^{(k)}(h) = |a|^{k/2} e^{2\pi i \xi(b+ai)} \chi_k(\kappa_\theta), \tag{1.7}$$

where we write $h \in \mathrm{SL}_2(\mathbb{R})$ according to the Iwasawa decomposition

$$h = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a^{1/2} & \\ & a^{-1/2} \end{pmatrix} \kappa_\theta, \quad a \in \mathbb{R}_+, \, b \in \mathbb{R}, \tag{1.8}$$

and

$$\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2,\mathbb{R}). \tag{1.9}$$

Here the weight k character of $\mathrm{SO}(2,\mathbb{R})$, for $k \in \mathbb{Z}$, is defined by

$$\chi_k(\kappa_\theta) = e^{ik\theta}. \tag{1.10}$$

The preimage of $\mathrm{SO}(2,\mathbb{R})$ in $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is denoted by $\widetilde{\mathrm{SO}}(2,\mathbb{R})$. For $k \in \frac{1}{2}\mathbb{Z}$, we similarly define the weight k character of $\widetilde{\mathrm{SO}}(2,\mathbb{R})$ and the Whittaker function.

- $\mathcal{A}_\infty(\mathbf{H}(\mathbb{A}_0), K_\mathbf{H}, k)$ is the space of smooth functions on $\mathbf{H}(\mathbb{A}_0)$, invariant under left $\mathbf{H}(F_0)$ and right $K_\mathbf{H} \subset \mathbf{H}(\mathbb{A}_{0,f})$ translation, and of parallel weight k under the action of $\prod_{v \in \mathrm{Hom}(F_0,\mathbb{R})} \mathrm{SO}(2,\mathbb{R})$. (We do not impose any finiteness conditions on the action of the center of the universal enveloping algebra.) Similarly we define $\mathcal{A}_\infty(\widetilde{\mathbf{H}}(\mathbb{A}_0), K_{\widetilde{\mathbf{H}}}, k)$.

- For a left $N(F_0)$-invariant continuous function $\phi \colon \mathbf{H}(\mathbb{A}_0) \to \mathbb{C}$, its ξ-th Fourier coefficient for $\xi \in F_0$ is defined as the function

$$\mathbf{H}(\mathbb{A}_0) \ni h \mapsto \phi(\xi, h) := \int_{F_0 \backslash \mathbb{A}_0} \phi\left[\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} h\right] \psi_{-\xi}(b) \, db. \tag{1.11}$$

Then there is a Fourier expansion (in an absolutely convergent sum): For $h \in \mathbf{H}(\mathbb{A}_0)$,

$$\phi(h) = \sum_{\xi \in F_0} \phi(\xi, h). \tag{1.12}$$

- $\mathcal{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_\mathbf{H}, k)$ is the space of automorphic forms (with moderate growth) on $\mathbf{H}(\mathbb{A}_0)$, invariant under $K_\mathbf{H} \subset \mathbf{H}(\mathbb{A}_{0,f})$, of parallel weight k under the action of $\prod_{v \in \mathrm{Hom}(F_0,\mathbb{R})} \mathrm{SO}(2,\mathbb{R})$, and holomorphic (i.e. annihilated by the element $\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ in the complexifed Lie algebra of $\mathbf{H}(F_{0,v}) \simeq \mathrm{SL}_2(\mathbb{R})$ for every $v \in \mathrm{Hom}(F_0,\mathbb{R})$).

This is a finite-dimensional vector space over $\mathbb{C}$, and it has a $\overline{\mathbb{Q}}$-structure via the Fourier expansion. For any subfield $L \subset \mathbb{C}$ (for our applications, it suffices to assume $L \supset \overline{\mathbb{Q}}$), we define $\mathscr{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, k)_L$ as the subspace of $\mathscr{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, k)$ consisting of functions with Fourier coefficients all in L (i.e. if we write the ξ-th term in the expansion (1.12) as $\phi(\xi, h) = W_\xi^{(k)}(h_\infty)\phi_f(\xi, h_f)$, then $\phi_f(\xi, h_f) \in L$ for all $h_f \in \prod_{v \nmid \infty} \mathbf{H}(O_{F_{0,v}})$). Then for any L-vector space W, we have an L-vector space

$$\mathscr{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, k)_L \otimes_L W. \tag{1.13}$$

Part I
KR divisors generating series

2. Shimura varieties and special divisors

We work with the so-called RSZ variant of unitary Shimura varieties throughout the paper. It goes back to [17] but has been introduced systematically in [28]. Its advantage is that it admits both a PEL type moduli description and the flexible definition of cycles. This section's aim is to recall all relevant definitions and properties. In that, it overlaps considerably with [35, Sections 6 and 7]. We refer to [29] for a much more detailed survey.

2.1. Shimura varieties

Let $F/\mathbb{Q}$ be a CM field with maximal totally real subfield F_0 and non-trivial F_0-automorphism $a \mapsto \bar{a}$. Fix a CM type Φ for F, a distinguished element $\varphi_0 \in \Phi$ and a hermitian F-vector space V of dimension n such that, for $\varphi \in \Phi$,

$$\mathrm{sign}(V_\varphi) = \begin{cases} (n-1, 1) & \text{if } \varphi = \varphi_0, \\ (n, 0) & \text{if } \varphi \neq \varphi_0. \end{cases}$$

Then we consider the following algebraic groups over $\mathrm{Spec}\,\mathbb{Q}$, resp. $\mathrm{Spec}\,F_0$:

$$Z^{\mathbb{Q}} := \{z \in \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_{\mathrm{m}} \mid \mathrm{Nm}_{F/F_0}(z) \in \mathbb{G}_{\mathrm{m}}\},$$
$$G := \mathrm{U}(V), \quad \text{an algebraic group over } F_0,$$
$$G^{\mathbb{Q}} := \{g \in \mathrm{Res}_{F_0/\mathbb{Q}}\,\mathrm{GU}(V) \mid c(g) \in \mathbb{G}_{\mathrm{m}}\},$$
$$\widetilde{G} := Z^{\mathbb{Q}} \times_{\mathbb{G}_{\mathrm{m}}} G^{\mathbb{Q}} \xrightarrow{\sim} Z^{\mathbb{Q}} \times \mathrm{Res}_{F_0/\mathbb{Q}}\,G.$$

Here, $c \colon \mathrm{GU}(V) \to \mathbb{G}_{\mathrm{m},F_0}$ denotes the similitude character, while the isomorphism in the last line is given by $(z, g) \mapsto (z, z^{-1}g)$.

We now recall how to define Shimura data for the above groups. For $Z^{\mathbb{Q}}$ we take the Shimura datum defined by Φ,

$$h_{Z^{\mathbb{Q}}} \colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m}} \to Z_{\mathbb{R}}^{\mathbb{Q}} \subseteq \prod_{\varphi \in \Phi} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{\mathrm{m}}, \quad z \mapsto (z, \ldots, z). \tag{2.1}$$

Next, choose $\mathbb{C}$-bases of the V_φ, $\varphi \in \Phi$, such that the hermitian form is $\mathrm{diag}(1, \ldots, 1)$ if $\varphi \neq \varphi_0$, and $\mathrm{diag}(1, \ldots, 1, -1)$ if $\varphi = \varphi_0$. Then set

$$h: \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \, \mathbb{G}_{\mathrm{m}} \to (\mathrm{Res}_{F_0/\mathbb{Q}} \, G)_\mathbb{R} = \prod_{\varphi \in \Phi} \mathrm{U}(V_\varphi), \quad z \mapsto (h_\varphi(z))_{\varphi \in \Phi},$$

$$h_\varphi(z) = \begin{cases} \mathrm{diag}(1, \ldots, 1, \overline{z}/z), & \varphi = \varphi_0, \\ \mathrm{id}_{V_\varphi}, & \varphi \neq \varphi_0, \end{cases}$$

where the matrix notation is with respect to a fixed choice of basis. Denote by $\{h\}$ the $(\mathrm{Res}_{F_0/\mathbb{Q}} \, G)(\mathbb{R})$-conjugacy class of h. Then $(\mathrm{Res}_{F_0/\mathbb{Q}} \, G, \{h\})$ is a Shimura datum. Finally, put

$$(\widetilde{G}, \{\widetilde{h}\}) := (Z^\mathbb{Q} \times \mathrm{Res}_{F_0/\mathbb{Q}} \, G, \, h_{Z^\mathbb{Q}} \times \{h\}).$$

Let $\mathbf{F} \subseteq \mathbb{C}$ denote the reflex field for $(\widetilde{G}, \{\widetilde{h}\})$. It is the composite of the reflex fields for Φ and for the conjugacy class $\{h\}$. It always contains $\varphi_0(F)$, which allows us to view F as a subfield of $\mathbf{F}$ (see [35, (6.5)]). For open compact subgroups $K_{Z^\mathbb{Q}} \subseteq Z^\mathbb{Q}(\mathbb{A}_f)$, $K \subseteq G(\mathbb{A}_{0,f})$ and $\widetilde{K} := K_{Z^\mathbb{Q}} \times K$, we obtain Shimura varieties (over $\mathbf{F}$) denoted by

$$\mathrm{Sh}_{\widetilde{K}}(\widetilde{G}, \{\widetilde{h}\}) \simeq \mathrm{Sh}_{K_{Z^\mathbb{Q}}}(Z^\mathbb{Q}, h_{Z^\mathbb{Q}}) \times \mathrm{Sh}_K(\mathrm{Res}_{F_0/\mathbb{Q}} \, G, \{h\}). \tag{2.2}$$

Here and in the following we make the simplifying assumption that $K_{Z^\mathbb{Q}}$ and K are neat. Then $\mathrm{Sh}_{\widetilde{K}}(\widetilde{G}, \{\widetilde{h}\})$ is an actual variety instead of a stack. (This assumption will not be used in an essential way and all our results should hold in the more general setting.) From now on we write $M_{\widetilde{K}}$ or even M instead of $\mathrm{Sh}_{\widetilde{K}}(\widetilde{G}, \{\widetilde{h}\})$.

2.2. Integral models

We next recall the integral model of $M_{\widetilde{K}}$ at places where the given data are unramified. This is the special case [28, Section 5.1] and [29, Section 6.1]. Let $\Lambda \subseteq V$ be a fixed O_F-lattice and let $\mathfrak{d} \in \mathbb{Z}_{>0}$ be chosen such that:

- The $O_F[\mathfrak{d}^{-1}]$-lattice $\Lambda[\mathfrak{d}^{-1}]$ is self-dual with respect to the F-valued pairing on V.

- If $v \in \Sigma_{F_0}$ is non-archimedean with $v \nmid \mathfrak{d}$, then v has odd residue characteristic and is unramified in F.

- The level K factors as

$$K = K_\mathfrak{d} \times K^\mathfrak{d},$$

  where $K_\mathfrak{d} \subseteq G(F_{0,\mathfrak{d}})$ is any open compact subgroup and where $K^\mathfrak{d} = \mathrm{Stab}(\widehat{\Lambda}^\mathfrak{d})$ is the stabilizer of the completion $\widehat{\Lambda}$ in $G(\mathbb{A}_{0,f}^\mathfrak{d})$.

- The level $K_{Z^\mathbb{Q}}$ factors similarly as

$$K_{Z^\mathbb{Q}} = K_{Z^\mathbb{Q}, \mathfrak{d}} \times K_{Z^\mathbb{Q}}^\mathfrak{d},$$

  where $K_{Z^\mathbb{Q}, \mathfrak{d}} \subseteq Z^\mathbb{Q}(\mathbb{Q}_\mathfrak{d})$ is any open compact subgroup and where $K_{Z^\mathbb{Q}}^\mathfrak{d}$ is the maximal compact subgroup in $Z^\mathbb{Q}(\mathbb{A}_f^\mathfrak{d})$.

We next consider a moduli problem $\mathcal{M}_0$ of abelian varieties with CM by F. It is defined as the functor on $\mathrm{LNSch}_{/O_\mathbf{F}[\mathfrak{d}^{-1}]}$ that associates to S the set of isomorphism classes of $(A_0, \iota_0, \lambda_0, \overline{\eta}_0)$, where

- A_0 is an abelian scheme over S of dimension $[F_0 : \mathbb{Q}]$;
- $\iota_0 : O_F[\mathfrak{d}^{-1}] \to \mathrm{End}(A_0)[\mathfrak{d}^{-1}]$ is an $O_F[\mathfrak{d}^{-1}]$-action that satisfies the Kottwitz condition for Φ,

$$\mathrm{char}(\iota_0(a) \mid \mathrm{Lie}\, A_0) = \prod_{\varphi \in \Phi} (T - \varphi(a)) \quad \text{for all } a \in O_F[\mathfrak{d}^{-1}]; \tag{2.3}$$

- λ_0 is an away-from-$\mathfrak{d}$ principal polarization on A_0 such that $\lambda_0^{-1} \circ \iota_0(a)^\vee \circ \lambda_0 = \iota_0(\overline{a})$ for all $a \in O_F[\mathfrak{d}^{-1}]$;
- $\overline{\eta}_0$ is a $K_{Z^\mathbb{Q}, \mathfrak{d}}$-level structure. This last notion requires the choice of an auxiliary 1-dimensional hermitian $F_\mathfrak{d}$-module. It will never figure explicitly in this work, so we do not spell out the details and refer to [20, Section C.3] instead.

An isomorphism of two quadruples $(A_0, \iota_0, \lambda_0, \overline{\eta}_0)$ and $(A'_0, \iota'_0, \lambda'_0, \overline{\eta}'_0)$ is a quasi-isogeny $\varphi_0 \in \mathrm{Hom}_{O_F}(A_0, A'_0)[\mathfrak{d}^{-1}]$ such that $\varphi_0^* \lambda'_0 = \lambda_0$ and $\overline{\eta}'_0 \circ \varphi_0 = \overline{\eta}_0$. Then $\mathcal{M}_0$ is representable, finite and étale over $\mathrm{Spec}\, O_\mathbf{F}[\mathfrak{d}^{-1}]$ (see [10, Proposition 3.1.2]). Here we have used our assumption that $K_{Z^\mathbb{Q}}$ is neat; without it $\mathcal{M}_0$ would have to be defined as a stack.

Depending on F and $\mathfrak{d}$, it might happen that $\mathcal{M}_0$ is empty. This does not happen, however, if F/F_0 is ramified at some place (see [28, Remark 3.5 (ii)]) and we make that assumption throughout the paper.

Then the generic fiber M_0 of $\mathcal{M}_0$ is a disjoint union of copies of $\mathrm{Sh}_{K_{Z^\mathbb{Q}}}(Z^\mathbb{Q}, \{h_{Z^\mathbb{Q}}\})$ (see [28, Lemma 3.4] specialized to the ideal $\mathfrak{a} = O_{F_0}$). This reflects the failure of the Hasse principle for the group $Z^\mathbb{Q}$. It would be possible to work with $\mathcal{M}_0$ throughout the paper, but it is more convenient to work directly with an integral model of $\mathrm{Sh}_{K_{Z^\mathbb{Q}}}$. For this reason, we fix one of the copies $\mathrm{Sh}_{K_{Z^\mathbb{Q}}}(Z^\mathbb{Q}, \{h_{Z^\mathbb{Q}}\}) \subseteq M_0$ and *redefine* $\mathcal{M}_0$ as its closure. Note that $\mathcal{M}_0$ is still finite étale over $\mathrm{Spec}\, O_\mathbf{F}$. We also *redefine* M_0 as the generic fiber of this new $\mathcal{M}_0$.

We next define the integral model of $M = M_{\widetilde{K}}$ over $\mathrm{Spec}\, O_\mathbf{F}[\mathfrak{d}^{-1}]$.

Definition 2.1. The functor $\mathcal{M} = \mathcal{M}_{\widetilde{K}}$ associates to each scheme S in $\mathrm{LNSch}_{/O_\mathbf{F}[\mathfrak{d}^{-1}]}$ the set of isomorphism classes of $(A_0, \iota_0, \lambda_0, \overline{\eta}_0, A, \iota, \lambda, \overline{\eta})$, where

- $(A_0, \iota_0, \lambda_0, \overline{\eta}_0)$ is an object of $\mathcal{M}_0(S)$;
- A is an abelian scheme over S of dimension $n[F_0 : \mathbb{Q}]$;
- $\iota : O_F[\mathfrak{d}^{-1}] \to \mathrm{End}(A)[\mathfrak{d}^{-1}]$ is an action satisfying the Kottwitz condition of signature

$$((n-1, 1)_{\varphi_0}, (n, 0)_{\varphi \in \Phi \setminus \{\varphi_0\}})$$

on $O_F[\mathfrak{d}^{-1}]$; and
- $\lambda : A \to A^\vee$ is an away-from-$\mathfrak{d}$ principal polarization such that $\lambda^{-1} \circ \iota(a)^\vee \circ \lambda = \iota(\overline{a})$ for all $a \in O_F[\mathfrak{d}^{-1}]$;

- $\overline{\eta}$ is a $K_{\mathfrak{d}}$-orbit of isometries of hermitian modules (as smooth $F_{\mathfrak{d}} = \prod_{v|\mathfrak{d}} F_v$-sheaves on S endowed with its natural hermitian form induced by polarization)

$$\eta \colon \mathrm{V}_{\mathfrak{d}}(A_0, A) \xrightarrow{\sim} V(F_{0,\mathfrak{d}}), \tag{2.4}$$

where

$$\mathrm{V}_{\mathfrak{d}}(A_0, A) := \prod_{p|\mathfrak{d}} \mathrm{V}_p(A_0, A) \quad \text{and} \quad \mathrm{V}_p(A_0, A) = \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{Q}_p}(\mathrm{V}_p(A_0), \mathrm{V}_p(A)),$$

and

$$V(F_{0,\mathfrak{d}}) := \prod_{p|\mathfrak{d}} \mathbb{Q}_p \otimes_{\mathbb{Q}} V = \prod_{v|\mathfrak{d}} F_{0,v} \otimes_{F_0} V; \tag{2.5}$$

- finally, we impose the Eisenstein condition (see [29, Section 5.2]) for every place $v \nmid \mathfrak{d}$ of F_0; note that this condition already follows from the Kottwitz condition for v that are unramified over $\mathbb{Q}$.

An isomorphism between two objects $(A_0, \iota_0, \lambda_0, \overline{\eta}_0, A, \iota, \lambda, \overline{\eta})$ and $(A_0', \iota_0', \lambda_0', \overline{\eta}_0', A', \iota', \lambda', \overline{\eta}')$ is an isomorphism $\varphi_0 \colon (A_0, \iota_0, \lambda_0, \overline{\eta}_0) \xrightarrow{\sim} (A_0', \iota_0', \lambda_0', \overline{\eta}_0')$ together with a quasi-isogeny $\varphi \in \mathrm{Hom}_{O_F}(A, A')[\mathfrak{d}^{-1}]$ such that $\varphi^*\lambda' = \lambda$ and $\overline{\eta}' \circ (\varphi_0, \varphi) = \overline{\eta}$.

Theorem 2.2 ([35, Theorem 6.2]). *The functor $\mathcal{M}_{\widetilde{K}}$ is representable. The morphism $\mathcal{M}_{\widetilde{K}} \to \mathrm{Spec}\, O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ is separated, of finite type and smooth of relative dimension $n - 1$.*

In [35], the functor $\mathcal{M}$ is defined as a stack and [35, Theorem 6.2] states that it is a Deligne–Mumford stack. The version we gave here used the assumption that $\widetilde{K}$ is neat. Moreover, the scheme $\mathcal{M}$ is projective if $F_0 \neq \mathbb{Q}$ (see [29, Theorem 4.4 (i)]), which we will assume henceforth.

By [28, Proposition 3.5], the generic fiber of $\mathcal{M}_{\widetilde{K}}$ is isomorphic to $M_{\widetilde{K}}$, the canonical model of $\mathrm{Sh}_{\widetilde{K}}(\widetilde{G}, \{\widetilde{h}\})$. (Note that the failure of the Hasse principle for $\widetilde{G}$ is already taken care of by our (re)definition of $\mathcal{M}_0$ above.)

Concerning terminology, we usually write $(A_0, A, \overline{\eta})$ for points of $\mathcal{M}$, the remaining data being implicit.

2.3. Kudla–Rapoport divisors

We next recall the definition of the Kudla–Rapoport special divisors [17]. These parametrize certain homomorphisms from CM abelian varieties occurring in $\mathcal{M}_0$ to abelian varieties parametrized by $\mathcal{M}$. They may equivalently be thought of as the loci in $\mathcal{M}$ where a CM abelian variety factor splits off in a prescribed fashion. The definition is based on the following observation. Assume that $(A_0, A, \overline{\eta}) \in \mathcal{M}(S)$ with S connected. Then one has $\mathrm{End}_F^\circ(A_0) = F$ and the space $\mathrm{Hom}_{O_F}(A_0, A)$ becomes a hermitian O_F-module with pairing

$$\langle x, y \rangle = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_F^\circ(A_0) \simeq F.$$

This space is positive definite at all archimedean places of F_0 by the positivity of the Rosati involution. Recall (see Section 1.3 for notation) that V_ξ denotes the hyperboloid

of $x \in V$ with $\langle x, x \rangle = \xi$ and that we assume that all Schwartz functions in the non-archimedean setting are $\mathbb{Q}$-valued.

Definition 2.3. Given $0 \neq \xi \in F_0$ and $\mu \in V(F_{0,\mathfrak{d}})/K_{\mathfrak{d}}$, the *KR-cycle* $\mathcal{Z}(\xi, \mu)$ is defined as follows. For $S \in \mathrm{LNSch}_{/O_{\mathbf{F}}[\mathfrak{d}^{-1}]}$, the S-points $\mathcal{Z}(\xi, \mu)(S)$ are the set of isomorphism classes of $(A_0, A, \overline{\eta}, u)$ where

- $(A_0, A, \overline{\eta}) \in \mathcal{M}(S)$;
- $u \in \mathrm{Hom}_{O_F}(A_0, A)[\mathfrak{d}^{-1}]$ with $\langle u, u \rangle = \xi$ and $\overline{\eta}(u) \in \mu$.

Two objects $(A_0, A, \overline{\eta}, u)$ and $(A_0', A', \overline{\eta}', u')$ are isomorphic if there is an isomorphism of the underlying triples, $(\varphi_0, \varphi) \colon (A_0, A, \overline{\eta}) \xrightarrow{\sim} (A_0', A', \overline{\eta}')$, such that $\varphi u \varphi_0^{-1} = u'$.

Forgetting u defines a natural morphism $i \colon \mathcal{Z}(\xi, \mu) \to \mathcal{M}$ which is finite and unramified. Moreover, $\mathcal{Z}(\xi, \mu)$ is étale locally a Cartier divisor and the morphism $\mathcal{Z}(\xi, \mu) \to \mathrm{Spec}\, O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ is flat (see [35, Proposition 7.3]). Taking its pushforward, the image of $\mathcal{Z}(\xi, \mu)$ along i defines an effective relative Cartier divisor on $\mathcal{M}$ which we still denote by $\mathcal{Z}(\xi, \mu)$. This definition extends by linearity to $\phi_{\mathfrak{d}} \in \mathcal{S}(V_{\mathfrak{d}})^{K_{\mathfrak{d}}}$,

$$\mathcal{Z}(\xi, \phi_{\mathfrak{d}}) := \sum_{\mu \in V(F_{0,\mathfrak{d}})/K_{\mathfrak{d}}} \phi_{\mathfrak{d}}(\mu) \mathcal{Z}(\xi, \mu) \in \mathcal{Z}^1(\mathcal{M}), \tag{2.6}$$

where the right hand side denotes the group of divisors with $\mathbb{Q}$-coefficients. It is non-zero only for $\xi \in F_{0,+}$ because the hermitian pairing on $\mathrm{Hom}_{O_F}(A_0, A)$ is positive definite at all archimedean places.

We finally introduce the following notation. Whenever $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))$ is a Schwartz function that factors as $\phi = \phi_{\mathfrak{d}} \otimes 1_{\widehat{\Lambda}^{\mathfrak{d}}}$, we put

$$\mathcal{Z}(\xi, \phi) := \mathcal{Z}(\xi, \phi_{\mathfrak{d}}). \tag{2.7}$$

2.4. Generating series of KR divisors

We turn to the generic fiber $M = M_{\widetilde{K}}$ again and write $Z(\xi, \mu)$, $Z(\xi, \phi_{\mathfrak{d}})$ and $Z(\xi, \phi)$ for the generic fibers of the above cycles. Since $\mathfrak{d}$ may be varied, we actually have defined a divisor $Z(\xi, \phi) \subseteq M$ for every $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$ and $\xi \in F_{0,+}$. Also put

$$Z(0, \phi) := -\phi(0)\, c_1(\boldsymbol{\omega}) \in \mathrm{Ch}^1(M_{\widetilde{K}}), \tag{2.8}$$

where $\boldsymbol{\omega}$ is the automorphic line bundle [12], and c_1 denotes the first Chern class. Note that this is just a line bundle while we have defined actual divisors when $\xi \neq 0$. Recall from [35, (11.1)] the Weil representation ω of $\mathbf{H}(\mathbb{A}_{0,f})$ on $\mathcal{S}(V(\mathbb{A}_{0,f}))^K$. We define a generating series on $\mathbf{H}(\mathbb{A}_0)$ by

$$Z(h, \phi) = Z(0, \omega(h_f)\phi) W_0^{(n)}(h_\infty) + \sum_{\xi \in F_{0,+}} Z(\xi, \omega(h_f)\phi) W_\xi^{(n)}(h_\infty), \tag{2.9}$$

where $h = (h_\infty, h_f) \in \mathbf{H}(\mathbb{A}_0)$, $h_\infty = (h_v)_{v|\infty} \in \mathbf{H}(F_{0,\mathbb{R}}) = \prod_{v|\infty} \mathrm{SL}_2(F_{0,v})$, and

$$W_\xi^{(n)}(h_\infty) = \prod_{v|\infty} W_\xi^{(n)}(h_v)$$

is the weight n Whittaker function on $\mathbf{H}(F_{0,\mathbb{R}})$ (see (1.7)). Here we recall [35, Theorem 8.1] for future reference.

Theorem 2.4. *The generating series $Z(h, \phi)$ lies in $\mathcal{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}}$ $\mathrm{Ch}^1(M_{\widetilde{K}})_{\overline{\mathbb{Q}}}$, where $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_{0,f})$ is a compact open subgroup that fixes ϕ under the Weil representation.*

We refer to (1.13) for the definition of the vector space $\mathcal{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)_{\overline{\mathbb{Q}}}$.

3. Green functions

We recall the Green function of Kudla [13], and the automorphic Green function [2]. The former is more convenient when comparing with the analytic side, while the latter is more suitable for proving (holomorphic) modularity of generating series. The difference between them was studied by Ehlen–Sankaran [5] when $F_0 = \mathbb{Q}$. Here we extend their result to a general totally real field F_0 (see Theorem 3.9). We first work in the more general setting of quadratic spaces and our notation in this section is slightly different from the rest of the paper. We then specialize the situation to the induced quadratic space structure of a hermitian space.

3.1. The set up

We recall some basic constructions from [2, Section 2]. Let V be a quadratic space over the totally real field F_0 and denote

$$\dim_{F_0} V = m + 2, \quad s_0 := \frac{\dim V - 2}{2} = \frac{m}{2}. \tag{3.1}$$

Let $G = \mathrm{SO}(V)$, a reductive group over F_0. Suppose that the signature of V is $(m, 2)$ at one distinguished archimedean place $v_0 \in \mathrm{Hom}_{\mathbb{Q}}(F_0, \mathbb{R})$, and $(m + 2, 0)$ at the remaining archimedean places. Let $\mathcal{D} = \mathcal{D}_{v_0}$ be the Grassmannian of oriented negative definite 2-dimensional subspaces of the real vector space $V_{v_0} = V \otimes_{F_0, v_0} \mathbb{R}$. Then there is an isomorphism

$$\mathcal{D} \xrightarrow{\sim} \mathrm{SO}(V_{v_0})/K_{v_0}, \tag{3.2}$$

where $K_{v_0} \simeq \mathrm{SO}(m) \times \mathrm{SO}(2)$ is the stabilizer of a fixed base point of $\mathcal{D}$. To endow $\mathcal{D}$ with a complex structure, we extend the quadratic form $\mathbb{C}$-bilinearly to $V_{v_0, \mathbb{C}} = V_{v_0} \otimes_{\mathbb{R}} \mathbb{C}$ and identify $\mathcal{D}$ with the open subdomain of a quadratic hypersurface in the projective space $\mathbb{P}^1(V_{v_0, \mathbb{C}})$ of $\mathbb{C}$-lines

$$\{\mathbb{C} \cdot z \in \mathbb{P}(V_{v_0, \mathbb{C}}) \mid \langle z, z \rangle = 0, \ \langle z, \overline{z} \rangle < 0\} \xrightarrow{\sim} \mathcal{D},$$

where the map sends z to the plane $\mathbb{R} \operatorname{Re}(z) \oplus \mathbb{R} \operatorname{Im}(z)$ with the induced orientation. The tautological line bundle on $\mathcal{D}$ has a natural hermitian metric:

$$\|z\|^2 := -\tfrac{1}{2}\langle z, \overline{z} \rangle. \tag{3.3}$$

Its first Chern form

$$\Omega = -d\,d^c \log \|z\|^2$$

is $G(\mathbb{R})$-invariant and positive. Here we recall the differential operators

$$d = \partial + \bar\partial, \quad d^c = \frac{1}{4\pi i}(\partial - \bar\partial), \quad d\,d^c = -\frac{1}{2\pi i}\partial\bar\partial.$$

The top degree wedge power Ω^m defines an invariant volume form on $\mathcal{D}$.

For a compact open subgroup $K \subset G(\mathbb{A}_{0,f})$, consider the complex analytic space

$$M_K(\mathbb{C}) = G(F_0)\backslash[\mathcal{D} \times G(\mathbb{A}_{0,f})/K].$$

We will assume that $M_K(\mathbb{C})$ is compact. When $F_0 \neq \mathbb{Q}$, this holds automatically. The tautological line bundle on $\mathcal{D}$ descends to a line bundle, called the automorphic bundle, ω_K on M_K and so does its hermitian metric (3.3), which will be called the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$ on ω_K. The first Chern form of the Petersson metric remains Ω (viewed as a $(1,1)$-form on M_K).

Let $\Delta = \Delta_{\mathcal{D}}$ be the Laplacian operator on $\mathcal{D}$ induced by the Casimir element of the Lie algebra of $SO(V_{v_0})$ and by the isomorphism (3.2). It is a constant multiple of the Laplacian operator induced by the Petersson metric. An explicit formula will be recalled below Lemma 3.1, at least for a certain class of functions.

Similar to the hermitian case in Section 2 but only over $\mathbb{C}$, there are special divisors, also denoted by $Z(\xi, \phi)$, on $M_K(\mathbb{C})$ for every $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$ and $\xi \in F_{0,+}$ [32].

3.2. Laplacian operator

We recall an explicit formula for the action of the Laplacian Δ on functions on $\mathcal{D} = SO(V_{v_0})/K_{v_0}$ that are left invariant under a subgroup H of the form H_u, the stabilizer of a non-zero vector $u \in V_{v_0}$ (not necessarily with negative norm). Define the majorant associated to the point $z \in \mathcal{D}$ (see [13, (11.14)]) as

$$R(u, z) = -\langle u_z, u_z \rangle, \tag{3.4}$$

where u_z denotes the orthogonal projection of u to z.

Lemma 3.1. *Let $\phi\colon (0, \infty) \to \mathbb{C}$ be a real analytic function and let $\widetilde{\phi}(u, z) = \phi(R(u, z))$ be the associated function on the domain $(V_{v_0} \times \mathcal{D}) \setminus \{(u, z) : u_z = 0\}$. Then*

$$\Delta\widetilde{\phi} = \widetilde{D\phi},$$

where D is the second order differential operator on $(0, \infty)$ given by (the coordinate being denoted by R)

$$D\phi = \left(4R(\langle u, u \rangle + R)\left(\frac{d}{dR}\right)^2 + 4(\langle u, u \rangle + (s_0 + 1)R)\frac{d}{dR}\right)\phi.$$

Proof. It suffices to consider the case $\langle u, u \rangle > 0$. Then the general case follows by the analyticity and the fact that the locus of (u, z) with $\langle u, u \rangle > 0$ is open (and non-empty) in each of the two connected components of the real analytic manifold $(V_{v_0} \times \mathcal{D}) \setminus \{(u, z) : u_z = 0\}$.

Now fix u such that $\alpha := \langle u, u \rangle > 0$. We introduce a normalized version of $R(u, z)$:

$$r(u, z) = \frac{R(u, z)}{\langle u, u \rangle}.$$

It satisfies the scaling invariance $r(\lambda u, z) = r(u, z)$ for $\lambda \in \mathbb{R}^\times$ and is clearly left invariant under the stabilizer H of u.

Let $z_0 \in \mathcal{D}$ be a base point such that $u \perp z_0$. Let K (resp. H) be the stabilizer of z_0 (resp. u). Let $Y_0 \in \mathfrak{g}$ (the Lie algebra of $\mathrm{SO}(V_{v_0})$) be the element Y_0 in [26, Section 1.3] (also see the proof of [4, Theorem 4.7]). We will not need its precise form, but we point out that Y_0 depends on z_0 and that the signature in *loc. cit.* is of the form $(2, m)$ rather than $(m, 2)$.

For $g = \exp(t Y_0)$, where $\exp \colon \mathfrak{g} \to \mathrm{SO}(V_{v_0})$ is the exponential map, and $z = g z_0$, one has

$$r(u, z) = \sinh^2 t$$

by [4, (4.46)]. Define

$$\Phi(g) = \phi(R(u, g z_0)), \quad g \in \mathrm{SO}(V_{v_0}) \setminus HK.$$

It is a smooth function on $\mathrm{SO}(V_{v_0}) \setminus HK$, left H-invariant and right K-invariant. By Cartan decomposition for the symmetric space $H \backslash \mathrm{SO}(V_{v_0})$, we have $\mathrm{SO}(V_{v_0}) = H \cdot \{\exp(t Y_0) : t \in \mathbb{R}\} \cdot K$. Therefore, to compute $\Delta \Phi$, it suffices to evaluate $(\Delta \Phi)(\exp(t Y_0))$. Then

$$\Phi(\exp(t Y_0)) = \phi(\alpha \sinh^2 t).$$

Denote $\varphi(t) = \phi(\alpha \sinh^2 t)$. By [26, Proposition 2.1.1] computing the radial part of the Casimir operator (we also denote it by Δ since our Laplacian is defined by the restriction of the Casimir operator), we have

$$(\Delta \Phi)(\exp(t Y_0)) = \left(\frac{d^2}{dt^2} + \mathbf{R}(t) \frac{d}{dt} \right) \varphi(t),$$

where $\mathbf{R}(t)$ denotes the function defined in *loc. cit.*,

$$\mathbf{R}(t) = m_\lambda^+ \coth t + m_\lambda^- \tanh t + 2m_{2\lambda}^+ \coth 2t + 2m_{2\lambda}^- \tanh 2t.$$

Substitute back $R = \alpha \sinh^2 t$. Then by [26, Section 2.3 and (2.5.1)] (where the substitution was $z = -\sinh^2 t$) and a simple substitution, the differential operator becomes

$$\Delta \phi(R(u, z)) = (D\phi)(R(u, z)),$$

where

$$D\phi = \left(4R(\alpha + R) \left(\frac{d}{dR} \right)^2 + 4(\alpha + (s_0 + 1)R) \frac{d}{dR} \right) \phi.$$

This completes the proof. ∎

Remark 3.2. It is natural to expect the formula to hold for all smooth ϕ by repeating the proof of [26]. In fact, the proof shows that the formula holds for smooth ϕ when we restrict the domain to a smaller subset defined by $\langle u, u \rangle > 0$.

3.3. A family of Laplacian eigenfunctions

We follow the construction of Oda–Tsuzuki and Bruinier [2, 26]. The automorphic Green function is constructed by a regularization out of the following family of functions on $\mathcal{D}$.

For $u \in V_{v_0}$, we write $u_{z^\perp} = u - u_z$ for the orthogonal projection to the (positive definite) m-dimensional space $z^\perp$. For $u \in V_{v_0}$ with $\langle u, u \rangle \neq 0$, $z \in \mathcal{D}$ and $s \in \mathbb{C}$, we define [2, Section 5, (5.4)]

$$\phi(u, z, s) = \frac{\Gamma(\frac{s}{2} + \frac{m}{4})}{\Gamma(s+1)} \left(\frac{\mathrm{q}(u)}{\mathrm{q}(u_{z^\perp})} \right)^{\frac{s}{2} + \frac{m}{4}} F\left(\frac{s}{2} + \frac{m}{4}, \frac{s}{2} - \frac{m}{4} + 1, s+1; \frac{\mathrm{q}(u)}{\mathrm{q}(u_{z^\perp})} \right). \quad (3.5)$$

Here $F(a, b, c; x)$ denotes the Gauss hypergeometric function and $\mathrm{q}(x) = \frac{1}{2}\langle x, x \rangle$. It follows from the definition that

$$\phi(u, z, s) = \phi(gu, gz, s), \quad g \in \mathrm{SO}(V_{v_0}),$$

and

$$\phi(\lambda u, z, s) = \phi(u, z, s), \quad \lambda \in \mathbb{R}^\times.$$

Moreover, it is an eigenfunction for the Laplacian [2, (5.6)][2]

$$\Delta\phi(u, z, s) = (s^2 - s_0^2)\phi(u, z, s). \quad (3.6)$$

Remark 3.3. This function is related to the secondary spherical function $\phi_s^{(2)}$ of Oda–Tsuzuki [26] as follows. Fix a base point $z_0 \in \mathcal{D}$ and let $u \in V_{v_0}$ be a non-zero vector such that $u \perp z_0$ (and hence $\langle u, u \rangle > 0$). Then, by [2, (5.5)]

$$\phi(u, gz_0, s) = \frac{-2}{\Gamma(\frac{s}{2} - \frac{s_0}{2} + 1)} \phi_s^{(2)}(g), \quad g \in \mathrm{SO}(V_{v_0}).$$

This shows that $\phi(u, z, s)$ and $\phi_s^{(2)}(g)$ determine each other. Here we recall the definition of the secondary spherical function $\phi_s^{(2)}$ in [26]. Denote by $H = H_u$ the stabilizer subgroup of $\mathrm{SO}(V_{v_0})$ of $u \in V_{v_0}$, and K the compact subgroup of $\mathrm{SO}(V_{v_0})$ fixing $z_0 \in \mathcal{D}$. Then, by [26, Proposition 2.4.2], there exists a unique family of functions $\phi_s^{(2)}$ for $\mathrm{Re}(s) > s_0$ satisfying the following properties:

(a) $\phi_s^{(2)} \colon \mathrm{SO}(V_{v_0}) \setminus HK \to \mathbb{C}$ is smooth, left H-invariant and right K-invariant;

---

[2]Note that our normalization of Δ corresponds to the Casimir element in [26], which corresponds to the $-\frac{1}{4}\Delta$ in [26] by [26, Proposition 7.2.1 (3)] with $c_{\mathrm{g}} = 2$ (see the last line in [26, proof of (7.6.2), p. 530]) and to 8 times the Δ in [2]. We also refer to the proof of [4, Theorem 4.7] for the comparison of the normalization on the Laplacian and measures.

(b) $\Delta\phi_s^{(2)} = (s^2 - s_0^2)\phi_s^{(2)}$;

(c) the function $\phi_s^{(2)}(\exp(tY_0)) - \log t$ is bounded as $t \to 0^+$;

(d) it has exponential decay near infinity: $\phi_s^{(2)}(\exp(tY_0)) = O(e^{-(\mathrm{Re}(s)+s_0)t})$ as $t \to +\infty$.

Moreover, the function $\phi_s^{(2)}$ has a meromorphic continuation to $s \in \mathbb{C}$. Here $Y_0 \in \mathfrak{g}$ is as in the proof of Lemma 3.1.

3.4. The automorphic Green function

We now recall from [2, 26] the automorphic Green function $\mathcal{G}^{\mathbf{B}}(\xi, \phi)$ for each $\xi \in F_{0,+}$, and $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$. Up to adding a constant this is the same as the Green function in [3, Section 7.2].

There is a family of smooth functions on $M_K(\mathbb{C}) \setminus \mathrm{supp}(Z(\xi, \phi))$ with a parameter $s \in \mathbb{C}$ when $\mathrm{Re}(s) > s_0$,

$$\mathcal{G}_s(\xi, \phi) = \sum \phi(g^{-1}u) \cdot (\phi(u, \cdot, s) \times 1_{gK}), \tag{3.7}$$

where the sum is over $(u, g) \in V_\xi(F_0) \times G(\mathbb{A}_{0,f})/K$. Here we recall that $\phi(u, \cdot, s)$ is defined by (3.5). The sum is absolutely convergent when $\mathrm{Re}(s) > s_0$. Up to a constant, the function $\mathcal{G}_s(\xi, \phi)$ is characterized by the differential equation [2, Theorem 5.7]

$$\Delta\mathcal{G}_s(\xi, \phi) + \frac{4s_0}{\Gamma(\frac{s}{2} - \frac{s_0}{2} + 1)}\delta_{Z(\xi,\phi)} = (s^2 - s_0^2)\mathcal{G}_s(\xi, \phi) \tag{3.8}$$

as an equality of currents of degree 0. Note that here $\delta_{Z(\xi,\phi)}$ as a current of degree 0 depends on the choice of a measure in [2, p. 201] (hence it should not be confused with the canonical current of degree $(1, 1)$, also denoted by $\delta_{Z(\xi,\phi)}$ below (3.11)).

By [2, Corollary 5.9], the above family $\mathcal{G}_s(\xi, \phi)$, viewed as distributions, admits a meromorphic continuation to $s \in \mathbb{C}$ with a simple pole at $s = s_0$ with residue equal to $A(\xi, \phi) = \frac{2}{\mathrm{vol}(M_K)} \deg Z(\xi, \phi)$ (as a locally constant function on $M_K(\mathbb{C})$, viewed as a current of degree 0, see [2, p. 201]). Then the automorphic Green function is defined as the constant term of the Laurent expansion at $s = s_0$, i.e.

$$\mathcal{G}^{\mathbf{B}}(\xi, \phi) = \lim_{s \to s_0} \left(\mathcal{G}_s(\xi, \phi) - \frac{A(\xi, \phi)}{s - s_0} \right). \tag{3.9}$$

Then by [2, Corollary 5.16],

$$dd^c \mathcal{G}^{\mathbf{B}}(\xi, \phi) + \delta_{Z(\xi,\phi)} \text{ is a harmonic smooth } (1, 1)\text{-form}. \tag{3.10}$$

It follows from (3.8) and (3.9) that we have an equality of currents

$$\Delta \mathcal{G}^{\mathbf{B}}(\xi, \phi) + 4s_0\delta_{Z(\xi,\phi)} = 2s_0 A(\xi, \phi). \tag{3.11}$$

Remark 3.4. The generating series $c_0 + \sum_{\xi \in F_{0,+}} A(\xi, \phi)q^\xi$ for a suitable constant term c_0 is essentially a Siegel–Eisenstein series $E(h_\infty, s_0, \phi)$ (evaluated at $s = s_0$) of weight $s_0 + 1 = \dim V/2$; see [14, Theorem I] (for $F_0 = \mathbb{Q}$) and [2, Section 6.2]. We will not need this fact.

3.5. Gaussian and the function Ei

Let $z_0 \in \mathcal{D}$ be a fixed base point and let $u = u_+ + u_-$ be the orthogonal decomposition with respect to $V_{v_0} = z_0^\perp \oplus z_0$. Let Φ_0 be the Gaussian function on V_{v_0},

$$\Phi_0(u) = e^{-2\pi \mathfrak{q}(u_+) + 2\pi \mathfrak{q}(u_-)}. \tag{3.12}$$

Let

$$\Phi_0^+(u) := (-4\pi \mathfrak{q}(u_+) + (\dim V/2 - 1))e^{-2\pi \mathfrak{q}(u_+) + 2\pi \mathfrak{q}(u_-)}$$
$$= (-4\pi \mathfrak{q}(u_+) + (\dim V/2 - 1))\Phi_0(u). \tag{3.13}$$

Note that the function Φ_0 (resp. Φ_0^+) has weight $\frac{\dim V}{2} - 2$ (resp. $\frac{\dim V}{2}$) under the action of $\widetilde{\mathrm{SO}}(2, \mathbb{R}) \subset \tilde{\mathbf{H}}(F_{0,v_0})$ by the Weil representation ω.

We recall the exponential integral, defined by

$$\mathrm{Ei}(-r) = -\int_r^\infty \frac{e^{-t}}{t}\, dt, \quad r > 0. \tag{3.14}$$

It has a logarithmic singularity around 0, more precisely, when $r \to 0^+$,

$$\mathrm{Ei}(-r) = \gamma + \log r + \sum_{n=1}^\infty \frac{(-r)^n}{n \cdot n!},$$

where γ is the Euler constant. We would like to compute the action of the Laplacian on the function $\mathrm{Ei}(-2\pi a R(u, z))$.

Lemma 3.5. *Let $u \in V_{v_0}$ be a non-zero vector. In terms of the Iwasawa decomposition $h = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \sqrt{a} & \\ & 1/\sqrt{a} \end{smallmatrix}\right) \in \tilde{\mathbf{H}}(F_{0,v_0})$, and $\tau = b + ai$, we have*

$$\Delta(a^{\dim V/2}\, \mathrm{Ei}(-2\pi a R(u, z))e^{2\pi i \tau \mathfrak{q}(u)}) = \omega(h, g)\Phi_0^+(u),$$

where $g \in G(F_{0,v_0}) = \mathrm{SO}(V_{v_0})$ is such that $gz_0 = z$ and Φ_0^+ is defined by (3.13).

Proof. We first compute the first and second derivatives of the function $\phi(r) := \mathrm{Ei}(-r) = -\int_1^\infty \frac{e^{-rt}}{t}dt$:
$$\phi'(r) = r^{-1}e^{-r}, \quad \phi''(r) = -r^{-2}(1 + r)e^{-r}.$$
Now we apply Lemma 3.1. To simplify notation we let $r = 2\pi a R(u, z)$:

$$\Delta \mathrm{Ei}(-2\pi a R(u, z))$$
$$= 4\left(-r(2\pi a \langle u, u \rangle + r)r^{-2}(1 + r) + (2\pi a \langle u, u \rangle + (s_0 + 1)r)r^{-1}\right)e^{-r}$$
$$= (-r + (s_0 - 2\pi a \langle u, u \rangle))e^{-r}$$
$$= (-2\pi a R(u, z) + (s_0 - 2\pi a \langle u, u \rangle))e^{-2\pi a R(u,z)}.$$

Since $\langle u, u \rangle = 2\mathfrak{q}(u) = 2\mathfrak{q}(u_z) + 2\mathfrak{q}(u_{z^\perp}) = -R(u, z) + 2\mathfrak{q}(u_{z^\perp})$, we obtain

$$\Delta \mathrm{Ei}(-2\pi a R(u, z)) = (-4\pi a \mathfrak{q}(u_{z^\perp}) + s_0)e^{-2\pi a R(u,z)}.$$

Combining (3.1), (3.13) and the formulas for the Weil representation, we complete the proof. ∎

3.6. Kudla's Green function

We now recall Kudla's Green function, defined for the orthogonal case in [13, Section 11] (see [19, Section 4B] for the unitary case).

Let $h_\infty = (h_v)_{v|\infty} \in \prod_{v|\infty} \widetilde{\mathbf{H}}(F_{0,v})$ and $h_v = \left(\begin{smallmatrix} 1 & b_v \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \sqrt{a_v} & \\ & 1/\sqrt{a_v} \end{smallmatrix}\right)\kappa_v$ in the Iwasawa decomposition. For each *non-zero* vector $u \in V_{v_0}$, denote by $\mathcal{D}_u \subseteq \mathcal{D}$ the divisor of z with $u \perp z$. Kudla [13] defined a Green function for $\mathcal{D}_u$, parametrized by h_∞ [3]

$$\mathcal{G}^{\mathbf{K}}(u, h_\infty)(z) = -\mathrm{Ei}(-2\pi a_{v_0} R(u, z)), \quad z \in \mathcal{D} \setminus \mathcal{D}_u. \tag{3.15}$$

which has logarithmic singularity along $\mathcal{D}_u$. Note that this is defined for *every* non-zero vector $u \in V_{v_0}$ (in particular, u may have null norm). If $\mathcal{D}_u$ is empty, the function is then smooth on $\mathcal{D}$. For all non-zero vectors u, we have as currents on $\mathcal{D}$, by [13, Proposition 11.1],

$$dd^c \mathcal{G}^{\mathbf{K}}(u, h_\infty) + \delta_{\mathcal{D}_u} \text{ is a smooth } (1,1)\text{-form.} \tag{3.16}$$

When $u = 0$, we set

$$\mathcal{G}^{\mathbf{K}}(0, h_\infty) = -\log|a_{v_0}|. \tag{3.17}$$

Continue to let $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$. Now we descend the Green function from $\mathcal{D}_{v_0}$ to the quotient $M_K(\mathbb{C})$: For all $\xi \in F_0$, define

$$\mathcal{G}^{\mathbf{K}}(\xi, h_\infty, \phi) = \sum \phi(g^{-1}u) \cdot (\mathcal{G}^{\mathbf{K}}(u, h_\infty) \times 1_{gK}), \tag{3.18}$$

where the sum is over $(u, g) \in V_\xi(F_0) \times G(\mathbb{A}_{0,f})/K$. This defines a Green function for the divisor $Z(\xi, \phi)$. For all $\xi \in F_0$ we have, as currents on $M_K(\mathbb{C})$,

$$dd^c \mathcal{G}^{\mathbf{K}}(\xi, h_\infty, \phi) + \delta_{Z(\xi, \phi)} \text{ is a smooth } (1,1)\text{-form,} \tag{3.19}$$

where $Z(\xi, \phi)$ is understood as zero unless $\xi \in F_{0,+}$. Define the special value at $h_\infty = 1$:

$$\mathcal{G}^{\mathbf{K}}(\xi, \phi) := \mathcal{G}^{\mathbf{K}}(\xi, 1, \phi). \tag{3.20}$$

We define the generating series of Kudla's Green functions:

$$\mathcal{G}^{\mathbf{K}}(h, \phi) = \sum_{\xi \in F_0} \mathcal{G}^{\mathbf{K}}(\xi, h_\infty, \omega(h_f)\phi) W_\xi^{(s_0+1)}(h_\infty), \quad h \in \widetilde{\mathbf{H}}(\mathbb{A}_0), \tag{3.21}$$

where the Whittaker function is defined by (1.7). Note that this generating series is understood as a formal sum over $\xi \in F_0$ and we call the terms its Fourier coefficients.

We would like to compute the action of the Laplacian operator on Kudla's Green functions. Recall that the theta function associated to $\Phi \in \mathcal{S}(V(\mathbb{A}_0))$ is defined as

$$\theta_\Phi(g, h) = \sum_{u \in V(F)} \omega(g, h)\Phi(u), \quad g \in G(\mathbb{A}_0), \ h \in \widetilde{\mathbf{H}}(\mathbb{A}_0).$$

---

[3] Here the constants differ slightly from those in [35, Section 12] which should be corrected as the ones here.

It is well-known that the sum is absolutely convergent and defines a smooth function on $G(\mathbb{A}_0) \times \widetilde{\mathbf{H}}(\mathbb{A}_0)$, invariant under $G(F_0) \times \mathbf{H}(F_0)$. We often omit the argument when evaluating it at $g = 1$ or $h = 1$.

Lemma 3.6. *Let $i \geq 1$. Then there exist $\phi_{i,\infty} \in \mathcal{S}(V(F_{0,\infty}))$ of parallel weight $s_0 + 1$ (with respect to the action of $\prod_{v|\infty} \mathrm{SO}(2,\mathbb{R}) \subset \prod_{v|\infty} \widetilde{\mathbf{H}}(F_v)$) such that*

$$\Delta^i \mathscr{G}^{\mathbf{K}}(h, \phi) = \begin{cases} \theta_{\phi_{i,\infty} \otimes \phi}(h) - \omega(h)(\phi_{i,\infty} \otimes \phi)(0), & i = 1, \\ \theta_{\phi_{i,\infty} \otimes \phi}(h), & i > 1. \end{cases}$$

Here and below, we will view the Fourier coefficients of $\mathscr{G}^{\mathbf{K}}(h, \phi)$ as functions on $G(\mathbb{A}_0)$ by the map $(g_\infty, g_f) \mapsto [g_{v_0} z_0, g_f] \in M_K(\mathbb{C})$ where the point z_0 is also the one used in (3.13). Moreover, the equality is understood as one between formal series, i.e. the ξ-th Fourier coefficients of the two sides are equal for every $\xi \in F_0$ and the operator Δ^i is applied on the left Fourier-coefficient-wise.

Proof. We will always take $\phi_{i,v}$ to be the standard Gaussian (3.12) for all archimedean places $v \neq v_0$. (Strictly speaking, we are defining the Gaussian on a positive definite quadratic space V_v by (3.12).) At $v = v_0$ we take

$$\phi_{i,v_0} = \Delta^{i-1} \Phi_0^+, \quad i \geq 1,$$

where Φ_0^+ is the Schwartz function defined by (3.13). Then they all have weight $s_0 + 1$.

We start with the case $i = 1$. By Lemma 3.5 and the definition (3.18), we evaluate the ξ-th term of (3.21) at $[z, g_f] \in M_K(\mathbb{C})$ where $z = g_\infty z_0 \in \mathcal{D}$ and $g_f \in G(\mathbb{A}_{0,f})$,

$$\Delta \mathscr{G}^{\mathbf{K}}(\xi, h, \phi)(z, g_f) W_\xi^{(s_0+1)}(h_\infty) = \sum_{u \in V_\xi(F_0)} \omega((g_\infty, g_f), h)(\phi_{1,\infty} \otimes \phi)(u)$$

when $\xi \neq 0$. When $\xi = 0$, we need to remove the term corresponding to the zero vector:

$$\Delta \mathscr{G}^{\mathbf{K}}(0, h, \phi)(z, g_f) W_\xi^{(s_0+1)}(h_\infty) = \sum_{u \in V_{\xi=0}(F_0), u \neq 0} \omega((g_\infty, g_f), h)(\phi_{1,\infty} \otimes \phi)(u).$$

Note that the term for $u = 0$ is independent of (g_∞, g_f),

$$\omega((g_\infty, g_f), h)(\phi_{1,\infty} \otimes \phi)(0) = \omega(h)(\phi_{1,\infty} \otimes \phi)(0).$$

This proves the case $i = 1$.

When $i > 1$, since the above term for $u = 0$ is constant in the variable (g_∞, g_f), it is annihilated by the operator Δ^{i-1}. The proof is complete. ■

3.7. Comparison of the two Green functions

We extend the theorem of Ehlen–Sankaran [5] from $\mathbb{Q}$ to any totally real field F_0. Our argument will be close to [18, Section 4.4] where the special case $\dim V = 3$ was proved.

For notational consistence, with $\mathscr{G}^{\mathbf{K}}(\xi, h_\infty, \phi)$ defined by (3.18), we define

$$\mathscr{G}^{\mathbf{B}}(\xi, h_\infty, \phi) := \mathscr{G}^{\mathbf{B}}(\xi, \phi) \tag{3.22}$$

(see (3.9)). Next we consider the difference

$$\mathscr{G}^{\mathbf{K}-\mathbf{B}}(\xi, h_\infty, \phi) := \begin{cases} \mathscr{G}^{\mathbf{K}}(\xi, h_\infty, \phi) - \mathscr{G}^{\mathbf{B}}(\xi, h_\infty, \phi), & \xi \in F_{0,+}, \\ \mathscr{G}^{\mathbf{K}}(\xi, h_\infty, \phi), & \text{otherwise,} \end{cases} \tag{3.23}$$

as a current on $M_K(\mathbb{C})$. Note that $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(\xi, h_\infty, \phi)$ is a smooth function on the complement of the support of the divisor $Z(\xi, \phi)$.

Lemma 3.7. *For every $h_\infty \in \prod_{v|\infty} \widetilde{\mathbf{H}}(F_{0,v})$, the difference $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(\xi, h_\infty, \phi)$ extends to a smooth function on $M_K(\mathbb{C})$ (and will be denoted by the same notation).*

Proof. It suffices to consider the case $\xi \in F_{0,+}$. By (3.10) and (3.19), both $\mathscr{G}^{\mathbf{B}}(\xi, \phi)$ and $\mathscr{G}^{\mathbf{K}}(\xi, h_\infty, \phi)$ define Green functions for the same divisor $Z(\xi, \phi)$. Hence the $d\,d^c$ operator sends their difference to a smooth $(1, 1)$-form (as currents). Then the elliptic regularity theorem implies that the current on the compact manifold $M_K(\mathbb{C})$ defined by $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(\xi, h_\infty, \phi)$ is represented by a smooth function. Since $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(\xi, h_\infty, \phi)$ is smooth on the complement of the support of the divisor $Z(\xi, \phi)$, it extends to a smooth function on $M_K(\mathbb{C})$. Alternatively, we may use the local expansion near the singular set $Z(\xi, \phi)$. By [2, Theorem 5.5] and the obvious analog for Kudla's Green function, the assertion follows from the fact that the difference $-\operatorname{Ei}(-2\pi a R(u, z)) - \phi(u, z, s_0)$ is smooth when z is near $\mathcal{D}_u$. This reduces to the fact that

$$(-\operatorname{Ei}(-\sinh^2 t)) - (-2\phi_2^{(s=s_0)}(\exp(t Y_0)))$$

is smooth in a neighborhood of $t = 0$. The proof is complete. ∎

Consider the space $L^2(M_K(\mathbb{C}))$ (for the fixed volume form) of L^2-functions on $M_K(\mathbb{C})$ with inner product denoted by $\langle \cdot, \cdot \rangle$. Then the subspace $\mathrm{LC}(M_K(\mathbb{C}))$ of the locally constant functions has dimension equal to the number of connected components of $M_K(\mathbb{C})$. For $f \in L^2(M_K(\mathbb{C}))$ we denote by f° the orthogonal projection to the orthogonal complement of $\mathrm{LC}(M_K(\mathbb{C}))$ in $L^2(M_K(\mathbb{C}))$. Note that $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(\xi, h_\infty, \phi)$ is smooth and hence we can define $\mathscr{G}^{\mathbf{K}-\mathbf{B},\circ}(\xi, h_\infty, \phi)$.

Similar to (3.21), we define the generating series,

$$\mathscr{G}^?(h, \phi) := \sum_{\xi \in F_0} \mathscr{G}^?(\xi, h_\infty, \omega(h_f)\phi) W_\xi^{(s_0+1)}(h_\infty), \tag{3.24}$$

where ? represents $\mathbf{B}$, $\mathbf{K}$ and $\mathbf{K} - \mathbf{B}$ respectively. We note that the definition depends on the fixed archimedean place v_0 of F_0. Similarly we set

$$\mathscr{G}^{\mathbf{K}-\mathbf{B},\circ}(h, \phi) := \sum_{\xi \in F_0} \mathscr{G}^{\mathbf{K}-\mathbf{B},\circ}(\xi, h_\infty, \omega(h_f)\phi) W_\xi^{(s_0+1)}(h_\infty). \tag{3.25}$$

Again all of them are understood as formal sums over $\xi \in F_0$.

Lemma 3.8. *Let $i \geq 1$ and let $\phi_{i,\infty} \in \mathcal{S}(V(F_{0,\infty}))$ be as in Lemma* 3.6. *Then*

$$\Delta^i \mathcal{G}^{\mathbf{K}-\mathbf{B}}(h, \phi) = \begin{cases} \theta^\circ_{\phi_{i,\infty} \otimes \phi}(h), & i = 1, \\ \theta_{\phi_{i,\infty} \otimes \phi}(h), & i > 1, \end{cases}$$

as functions on $G(\mathbb{A}_0)$. (The equality is understood as one between formal series.)

Proof. Since Δ is a self-adjoint operator, the image $\Delta \mathcal{G}^{\mathbf{K}-\mathbf{B}}(\xi, h, \phi)$ must be orthogonal to $\mathrm{LC}(M_K(\mathbb{C}))$. In particular,

$$\Delta \mathcal{G}^{\mathbf{K}-\mathbf{B}}(h, \phi) = \Delta \mathcal{G}^{\mathbf{K}-\mathbf{B},\circ}(h, \phi).$$

By (3.11) and Lemma 3.6,

$$\Delta \mathcal{G}^{\mathbf{K}-\mathbf{B}}(h, \phi) = \theta_{\phi_{1,\infty} \otimes \phi}(h)$$
$$- \omega(h)(\phi_{1,\infty} \otimes \phi)(0) - 2s_0 \sum_{\xi \in F_{0,+}} A(\xi, \omega(h_f)\phi) W^{(s_0+1)}_\xi(h_\infty).$$

Note that the subtracted terms in the RHS are all locally constant functions on $M_K(\mathbb{C})$ and hence

$$\Delta \mathcal{G}^{\mathbf{K}-\mathbf{B},\circ}(h, \phi) = \theta^\circ_{\phi_{1,\infty} \otimes \phi}(h).$$

The case of $i > 1$ now follows easily. ∎

Theorem 3.9. *Suppose that $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$ is invariant under $K_{\widetilde{\mathbf{H}}} \subset \widetilde{\mathbf{H}}(\mathbb{A}_{0,f})$ by the Weil representation. The generating series $\mathcal{G}^{\mathbf{K}-\mathbf{B},\circ}(h, \phi)$ lies in the space $\mathcal{A}_\infty(\widetilde{\mathbf{H}}(\mathbb{A}_0), K_{\widetilde{\mathbf{H}}}, \dim V/2)$, in the sense that, for every point $[z, g] \in M_K(\mathbb{C})$, the value of the generating series at $[z, g]$ is the Fourier expansion of some element in $\mathcal{A}_\infty(\widetilde{\mathbf{H}}(\mathbb{A}_0), K_{\widetilde{\mathbf{H}}}, \dim V/2)$.*

Proof. The proof is similar to that of [18, Theorem 4.4] except we need to justify the asserted properties of the function in the $\widetilde{\mathbf{H}}(\mathbb{A}_0)$-variable. Fix ϕ and denote

$$\varphi(g, h) = \mathcal{G}^{\mathbf{K}-\mathbf{B},\circ}(h, \phi)(g_\infty z_0, g_f), \quad g \in G(\mathbb{A}_0). \tag{3.26}$$

The strategy is to show that the image of $\varphi(g, h)$ under the Laplacian operator has the asserted properties and then we use the spectral theory to recover the corresponding properties of $\varphi(g, h)$.

We use the spectral theory of Δ on the compact manifold $M_K(\mathbb{C}) = \coprod_j \Gamma_j \backslash \mathcal{D}$. It suffices to consider each connected component of $M_K(\mathbb{C})$, say of the form $M = \Gamma \backslash \mathcal{D}$. Consider the space $L^2(M)$. Denote the eigenvalues of $-\Delta$ with multiplicities by

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

with the eigenfunctions

$$-\Delta \varphi_i = \lambda_i \varphi_i,$$

which are smooth and normalized by $\langle \varphi_i, \varphi_i \rangle = 1$. Here φ_0 is the constant function $\frac{1}{\mathrm{vol}(M)}$.

We *claim* that for an integer $N \geq 3(\dim_{\mathbb{R}} M/2 + 1)$, the kernel function

$$K_N(x, y) := \sum_{i>0} \frac{1}{\lambda_i^N} \varphi_i(x)\overline{\varphi}_i(y), \quad x, y \in M, \tag{3.27}$$

is a continuous function on $M \times M$. In fact, by [2, Lemma 5.11 (2)], for any integer $N_0 > \dim_{\mathbb{R}} M/2$, there exists a constant $C > 0$ such that

$$\|\varphi_i\|_{L^\infty} \leq C\lambda_i^{N_0} \quad \text{for all } i \geq 1.$$

It follows that when $N \geq 3N_0$,

$$\sum_{i>0} \frac{1}{\lambda_i^N} |\varphi_i(x)\overline{\varphi}_i(y)| \leq \sum_{i>0} \frac{1}{\lambda_i^N} \|\varphi_i\|_{L^\infty}^2 \leq C^2 \sum_{i>0} \frac{1}{\lambda_i^N} \lambda_i^{2N_0}$$

$$\leq C^2 \sum_{i>0} \frac{1}{\lambda_i^{N_0}} < \infty \quad ([2, \text{Lemma 5.11 (1)}]).$$

Therefore the sum in (3.27) converges absolutely and uniformly in $(x, y) \in M \times M$.

Let $\Phi_N = \phi_{N,\infty} \otimes \phi \in \mathcal{S}(V(\mathbb{A}_0))$ for $N \geq 1$, where $\phi_{N,\infty} \in \mathcal{S}(V(F_{0,\infty}))$ is as in Lemma 3.6. Note that $\phi_{1,\infty} = \Phi_0^+$ from (3.13). Then we have a spectral expansion

$$\theta_{\Phi_1}^\circ(g, h) = \sum_{i>0} \varphi_i(gz_0)\theta_{\Phi_1}^{\varphi_i}(h), \tag{3.28}$$

where $\theta_{\Phi_1}^{\varphi_i}$ is the "theta lifting" of φ_i, defined by

$$\theta_{\Phi_1}^{\varphi_i}(h) := \langle \theta_{\Phi_1}(\cdot, h), \varphi_i \rangle.$$

The theta function $\theta_{\Phi_j}(\cdot, \cdot)$ is smooth on $G(\mathbb{A}_0) \times \widetilde{\mathbf{H}}(\mathbb{A}_0)$, left $G(F_0) \times \mathbf{H}(F_0)$-invariant with parallel weight $\dim V/2$ under $\prod_{v|\infty} \widetilde{\mathrm{SO}}(2, \mathbb{R}) \subset \prod_{v|\infty} \widetilde{\mathbf{H}}(F_{0,v})$.

By Lemma 3.8, the function $\varphi(\cdot, h)$ is the unique solution to $\Delta\varphi = \theta_\Phi(g, h)$ and $\langle \varphi, \varphi_0 \rangle = 0$. We have

$$\varphi(g, h) = \sum_{i>0} \langle \varphi(\cdot, h), \varphi_i \rangle \varphi_i(g),$$

where by [2, Lemma 5.11 (4)] the sum is absolutely convergent, uniformly in g, by the smoothness of $\varphi(\cdot, h)$ (for every given h). The coefficients are

$$\langle \varphi(\cdot, h), \varphi_i \rangle = -\frac{1}{\lambda_i} \langle \varphi(\cdot, h), \Delta\varphi_i \rangle$$

$$= -\frac{1}{\lambda_i} \langle \Delta\varphi(\cdot, h), \varphi_i \rangle \quad \text{(self-adjointness of } \Delta)$$

$$= -\frac{1}{\lambda_i} \langle \theta_{\Phi_1}(\cdot, h), \varphi_i \rangle \quad \text{(Lemma 3.8)}$$

$$= -\frac{1}{\lambda_i} \theta_{\Phi_1}^{\varphi_i}(h).$$

It follows that

$$\varphi(g, h) = -\sum_{i>0} \frac{1}{\lambda_i} \theta_{\Phi_1}^{\varphi_i}(h) \varphi_i(g).$$

Using $\Delta^N \varphi(g, h) = \theta_{\Phi_N}(g, h)$, the same argument shows that for all $N \geq 1$,

$$\varphi(g, h) = -\sum_{i>0} \frac{1}{\lambda_i^N} \theta_{\Phi_N}^{\varphi_i}(h) \varphi_i(g).$$

When N is large enough, by the continuity of the kernel function K_N we have

$$\varphi(g, h) = \langle \Delta^N \varphi(\cdot, h), K_N(\cdot, g) \rangle.$$

It follows from this and the properties of the theta function θ_{Φ_N} stated above that the function $\varphi(g, \cdot)$ is smooth on $\widetilde{\mathbf{H}}(\mathbb{A}_0)$, and left $\mathbf{H}(F_0)$-invariant with weight $\dim V/2$. ∎

Remark 3.10. The projection of the generating series $\mathscr{G}^{\mathbf{K}}(h, \phi)$ and $\mathscr{G}^{\mathbf{B}}(h, \phi)$ in (3.24) to the finite-dimensional space $\mathrm{LC}(M_K(\mathbb{C}))$ of locally constant functions is essentially understood, by Garcia–Sankaran [7, Theorem 5.10] and Bruinier [2] respectively. We will not need these results in this paper.

Remark 3.11. In the case $F_0 = \mathbb{Q}$, the theorem of Ehlen and Sankaran shows that the function $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(h, \phi)$ has at worst "exponential growth at ∞" [5, Definition 2.8, p. 2104].

Corollary 3.12. *Let ϕ be as above. The generating function $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(h, \phi)$, when evaluated at a degree-zero (on every connected component) zero-cycle, lies in $\mathcal{A}_\infty(\widetilde{\mathbf{H}}(\mathbb{A}_0), K_{\widetilde{\mathbf{H}}}, \dim V/2)$.*

3.8. Back to hermitian spaces

We now return to the set up in Section 2. In particular, V is an F/F_0-hermitian vector space of F-dimension n. Then the two Green functions are defined similarly to the orthogonal case; see [35, Section 8.3] for details. Let ν be a place of $\mathbf{F}$ and let ν_0 be the unique place of F_0 below ν. Define $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(h, \phi)$ similarly to (3.24), as a function on $M_{\widetilde{K}} \otimes_{\mathbf{F}, \nu} \mathbb{C}$. Note that now the Weil representation of $\widetilde{\mathbf{H}}(\mathbb{A}_0)$ factors through $\mathbf{H}(\mathbb{A}_0)$.

Theorem 3.13. *Suppose that $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$ is invariant under $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_{0,f})$ by the Weil representation. Then the generating function $\mathscr{G}^{\mathbf{K}-\mathbf{B}}(h, \phi)$, when evaluated at degree-zero (on every connected component) zero-cycle, lies in $\mathcal{A}_\infty(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)$.*

Proof. This is standard by the pull-back of the assertion of Corollary 3.12 for the induced quadratic space V; cf. [19, Section 3.B]. ∎

4. Almost modularity

In this section, we describe the simple arithmetic intersection theory we use in this paper. It is modeled on [1] and takes into account the fact that our integral models are only

defined away from finitely many primes. Let S be a finite set of non-archimedean primes of a number field $\mathbf{F}$ and $\mathcal{X} \to \operatorname{Spec} O_{\mathbf{F}}[S^{-1}]$ a regular, flat and proper scheme with smooth generic fiber X.

4.1. Admissible extension

Recall that the arithmetic Chow group $\widehat{\operatorname{Ch}}^1(\mathcal{X})$ (with $\mathbb{Q}$-coefficients) is generated by arithmetic divisors, i.e. $\mathbb{Q}$-linear combinations of tuples (Z, g_Z), where Z is a divisor on $\mathcal{X}$ and $g_Z = (g_{Z,v})_{v:\mathbf{F} \to \bar{\mathbb{Q}}}$ a tuple of Green functions for the divisors $Z_v(\mathbb{C})$ on the complex manifold $X_v(\mathbb{C})$ with respect to $v:\mathbf{F} \hookrightarrow \bar{\mathbb{Q}} \subset \mathbb{C}$ (see [8, Section 3.3]). Its relations are given by the $\mathbb{Q}$-span of principal arithmetic divisors, i.e. tuples associated to rational functions $f \in \mathbf{F}(\mathcal{X})^\times$:

$$(\operatorname{div}(f), (-\log|f|_v^2)_{v\in\operatorname{Hom}_{\mathbb{Q}}(\mathbf{F},\bar{\mathbb{Q}})}). \tag{4.1}$$

If for example $\mathbf{F} = \mathbb{Q}$, then $(\mathcal{X}_p, 0) = (0, 2\log|p|)$ in $\widehat{\operatorname{Ch}}^1(\mathcal{X})$, where $\mathcal{X}_p$ is the fiber of $\mathcal{X}$ over a prime p. In case $p \in S$ and hence $\mathcal{X}_p = \emptyset$, this means $(0, 2\log|p|) = 0$ in $\widehat{\operatorname{Ch}}^1(\mathcal{X})$.

Fix a Kähler metric on each $X_v(\mathbb{C})$. It induces a Laplace operator on differential forms on $X_v(\mathbb{C})$. We call a Green function $g_{Z,v}$ *admissible* (with respect to the chosen metrics) if

$$dd^c g_{Z,v} + \delta_Z = \omega_{Z,v}, \tag{4.2}$$

where $\omega_{Z,v}$ is a *harmonic* $(1,1)$-form. An admissible Green function exists for every divisor and is unique up to adding a locally constant function, i.e. a function that is constant on each connected component of $X_v(\mathbb{C})$. We denote by $\widehat{\operatorname{Ch}}^{1,\mathrm{adm}}(\mathcal{X})$ the subgroup of $\widehat{\operatorname{Ch}}^1(\mathcal{X})$ generated by (Z, g_Z) with all $g_{Z,v}$ admissible. Note that principal arithmetic divisors are admissible since for these, $\omega_{Z,v} = 0$ in (4.2). So an arithmetic divisor defines an admissible Chow class if and only if it is itself admissible.

Consider the subgroup of $\widehat{\operatorname{Ch}}^{1,\mathrm{adm}}(\mathcal{X})$ spanned by the classes of $(0, c)$ with $c = (c_v)_v$ a tuple of locally constant functions on the $X_v(\mathbb{C})$:

$$\widehat{\operatorname{Ch}}^1_\infty(\mathcal{X}) := \operatorname{span}\{(0, c) \mid c \text{ locally constant}\} \subset \widehat{\operatorname{Ch}}^{1,\mathrm{adm}}(\mathcal{X}).$$

These are the arithmetic divisors "supported at archimedean places". Then there is a natural map

$$\frac{\widehat{\operatorname{Ch}}^{1,\mathrm{adm}}(\mathcal{X})}{\widehat{\operatorname{Ch}}^1_\infty(\mathcal{X})} \to \operatorname{Ch}^1(X), \tag{4.3}$$

sending (the equivalence class of) (Z, g_Z) to the generic fiber of Z.

Lemma 4.1. *Assume further that $\mathcal{X}$ is smooth over $\operatorname{Spec} O_{\mathbf{F}}[S^{-1}]$. Then the map* (4.3) *is an isomorphism.*

Proof. The map $\widehat{\operatorname{Ch}}^{1,\mathrm{adm}}(\mathcal{X}) \to \operatorname{Ch}^1(X)$ is surjective; its kernel is generated by all vertical classes. Those are the elements of $\widehat{\operatorname{Ch}}^1_\infty(\mathcal{X})$ and the vertical classes at the non-archimedean primes, i.e. classes of the form $(Z, 0)$ with $Z \subseteq \mathcal{X}_v$ for some non-archimedean place

$v \notin S$. For dimension reasons, any such Z is a $\mathbb{Q}$-linear combination of irreducible components of the fiber $\mathcal{X}_v$. Since $\mathcal{X}$ is assumed to be smooth, those are actually the connected components of $\mathcal{X}_v$.

The Stein factorization $\mathcal{X} \to \operatorname{Spec} R \to \operatorname{Spec} O_{\mathbf{F}}[S^{-1}]$ has the property that $R = \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ agrees with the normalization of $O_{\mathbf{F}}[S^{-1}]$ in $\mathbf{F}(\mathcal{X})$ (see [31, Tag 03H0]). The connected components of $\mathcal{X}_v$ are then in bijection with $|\operatorname{Spec} R/\mathfrak{p}_v|$. By finiteness of the class number of R, each is equivalent to an element of $\widehat{\operatorname{Ch}}^1_\infty(\mathcal{X})$, as was to be shown. ∎

4.2. S-punctured arithmetic intersection pairing

We come to the arithmetic intersection pairing (see [1, Section 2.3]). Assume in the following that $\mathcal{X}$ is smooth over $O_{\mathbf{F}}[S^{-1}]$; write

$$\mathbb{R}_S := \mathbb{R}/\operatorname{span}_{\mathbb{Q}} \{\log |q_v| \mid v \in S\}.$$

This quotient is viewed as a $\mathbb{Q}$-vector space.

Definition 4.2. Let $\widetilde{\mathcal{Z}}_1(\mathcal{X})$ be the group of 1-cycles on $\mathcal{X}$. Define $\mathcal{Z}_1(\mathcal{X})$ as the quotient of $\widetilde{\mathcal{Z}}_1(\mathcal{X})$ by the subgroup generated by 1-cycles that are rationally trivial on a closed fiber of $\mathcal{X}$. The latter means $\operatorname{div}(f) = 0$ whenever $Y \subseteq \mathcal{X}_v$ is a 2-dimensional integral closed subscheme in a special fiber and $f \in \mathbb{F}_v(Y)^\times$ a meromorphic function.

Then we have an arithmetic intersection pairing, adapting the definition in [1, Section 2.3] to our S-punctured scheme $\mathcal{X}$,

$$\widehat{\operatorname{Ch}}^1(\mathcal{X}) \times \widetilde{\mathcal{Z}}_1(\mathcal{X}) \to \mathbb{R}_S, \tag{4.4}$$

which factors through a pairing

$$(\cdot, \cdot) \colon \widehat{\operatorname{Ch}}^1(\mathcal{X}) \times \mathcal{Z}_1(\mathcal{X}) \to \mathbb{R}_S \tag{4.5}$$

by [1, Proposition 2.3.1 (ii)]. We call it the *S-punctured arithmetic intersection pairing* and next give its definition in terms of properly intersecting representatives. Assume that $Z \subseteq \mathcal{X}$ is a closed subscheme of codimension 1, viewed as a divisor, and $C \subseteq \mathcal{X}$ a closed curve such that $Z \cap C$ is artinian. Let $g_Z = (g_{Z,v})_{v:\mathbf{F}\to\overline{\mathbb{Q}}}$ be a family of Green functions for Z. Then

$$((Z, g_Z), C) = \log |\mathcal{O}_{Z\cap C}| + \frac{1}{2} \sum_{v:\mathbf{F}\to\overline{\mathbb{Q}}} g_v(C(\mathbb{C}_v)). \tag{4.6}$$

Extending this relation $\mathbb{Q}$-linearly in both arguments determines (4.5) uniquely. Let now $\mathcal{Z}_1(\mathcal{X})_{\deg=0}$ be the subgroup of $\mathcal{Z}_1(\mathcal{X})$ consisting of 1-cycles with vanishing degree on each connected component of the generic fiber of $\mathcal{X}$. Then there is an induced $\mathbb{Q}$-bilinear pairing

$$(\cdot, \cdot) \colon \frac{\widehat{\operatorname{Ch}}^1(\mathcal{X})}{\widehat{\operatorname{Ch}}^1_\infty(\mathcal{X})} \times \mathcal{Z}_1(\mathcal{X})_{\deg=0} \to \mathbb{R}_S,$$

which we may restrict to the admissible classes

$$(\cdot,\cdot)\colon \frac{\widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{X})}{\widehat{\mathrm{Ch}}^1_\infty(\mathcal{X})} \times \mathcal{Z}_1(\mathcal{X})_{\deg=0} \to \mathbb{R}_S. \tag{4.7}$$

By Lemma 4.1, we obtain a $\mathbb{Q}$-bilinear pairing

$$(\cdot,\cdot)^{\mathrm{adm}}\colon \mathrm{Ch}^1(X) \times \mathcal{Z}_1(\mathcal{X})_{\deg=0} \to \mathbb{R}_S. \tag{4.8}$$

Note again that the above are all pairings between $\mathbb{Q}$-vector spaces only.

4.3. Application: An almost modularity

We now return to the case of our Shimura variety $M = M_{\widetilde{K}}$ and the integral model $\mathcal{M} = \mathcal{M}_{\widetilde{K}}$ over Spec $O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ defined in Section 2. We endow the automorphic line bundle $\boldsymbol{\omega}$ with its Petersson metrics $\|\cdot\|_{\mathrm{Pet}}$ at the archimedean places and use the induced Kähler metric to define the notion of admissible extension in Section 4.1. Recall that we defined a divisor $Z(h,\phi) \in \mathrm{Ch}^1(M)$ for each $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$ and $h \in \mathbf{H}(\mathbb{A}_0)$ in Section 2.4. Then we can apply the pairing (4.8) to obtain an "almost modularity" result:

Theorem 4.3. *Suppose that $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$ is $K_{\mathbf{H}}$-invariant under the Weil representation. Let $z \in \mathcal{Z}_1(\mathcal{M})_{\deg=0}$. Then the generating function $\mathbf{H}(\mathbb{A}_0) \ni h \mapsto (Z(h,\phi), z)^{\mathrm{adm}}$ lies in $\mathcal{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{R}_{S,\overline{\mathbb{Q}}}$, where $\mathbb{R}_{S,\overline{\mathbb{Q}}} := \mathbb{R}_S \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$.*

Proof. This follows from Theorem 2.4. ∎

To apply this result, we still need to find a convenient lifting of $Z(\xi,\phi)$ to $\widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{M})$. We can do so whenever ϕ factors as $\phi = \phi_{\mathfrak{d}} \otimes 1_{\widehat{\Lambda}^{\mathfrak{d}}}$ as in (2.7), which we assume from now on. Then, for $\xi \in F_{0,+}$, we endow the integral KR divisor $\mathcal{Z}(\xi,\phi)$ with the automorphic Green function $\mathscr{G}^{\mathbf{B}}(\xi,\phi)$ to define an arithmetic KR divisor

$$\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi,\phi) = \big(\mathcal{Z}(\xi,\phi), (\mathscr{G}^{\mathbf{B}}_v(\xi,\phi))_v\big) \in \widehat{\mathrm{Ch}}^1(\mathcal{M}). \tag{4.9}$$

Since the automorphic Green function is admissible by (3.10), we even have $\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi,\phi) \in \widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{M})$. For $\xi = 0$, we set

$$\widehat{\mathcal{Z}}^{\mathbf{B}}(0,\phi) = -\phi(0)\, c_1(\widehat{\boldsymbol{\omega}}) \in \widehat{\mathrm{Ch}}^{1,\mathrm{adm}}(\mathcal{M}), \tag{4.10}$$

where $\widehat{\boldsymbol{\omega}} = (\boldsymbol{\omega}_{\mathcal{M}}, \|\cdot\|_{\mathrm{Pet}})$ is *any* extension of the automorphic line bundle $\boldsymbol{\omega}$ to a line bundle $\boldsymbol{\omega}_{\mathcal{M}}$ on the integral model $\mathcal{M}$, endowed with its Petersson metric.

Finally, we define a family of arithmetic divisors with Kudla's Green function: for $\xi \in F_0^\times$,

$$\widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi) := \big(\mathcal{Z}(\xi,\phi), (\mathscr{G}^{\mathbf{K}}_v(\xi, h_{v|_{F_0}}, \phi))_v\big) \in \widehat{\mathrm{Ch}}^1(\mathcal{M}), \tag{4.11}$$

and for $\xi = 0$,

$$\widehat{\mathcal{Z}}^{\mathbf{K}}(0, h_\infty, \phi) = -\phi(0) c_1(\widehat{\boldsymbol{\omega}}) + \big(0, (\mathscr{G}^{\mathbf{K}}_v(0, h_{v|_{F_0}}, \phi))_v\big), \tag{4.12}$$

where $h_\infty = (h_v)_{v|\infty} \in \mathbf{H}(F_{0,\mathbb{R}})$ (see (3.18)). The special value at $h_\infty = 1$ is denoted by

$$\widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, \phi) := \left(\mathcal{Z}(\xi, \phi), (\mathcal{G}_v^{\mathbf{K}}(\xi, \phi))_v\right) \in \widehat{\mathrm{Ch}}^1(\mathcal{M}), \quad \xi \neq 0 \tag{4.13}$$

(see (3.20)).

Part II
Intersection with CM cycles

5. CM cycles

5.1. Derived CM cycles

This section recapitulates the definition of the derived CM cycle in [35, Section 7]. Recall from Section 2 that $\mathcal{M} = \mathcal{M}_{\widetilde{K}} \to \operatorname{Spec} O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ denotes the integral model of the Shimura variety for group $\widetilde{G}$ and level $\widetilde{K}$. Let $F[t]_{\deg n}^\circ$ denote the monic polynomials α of degree n such that $\alpha(0)t^n \overline{\alpha}(t^{-1}) = \alpha(t)$. For example, the F-linear characteristic polynomial of an element of $G(F_0) = \mathrm{U}(V)(F_0)$ lies in $F[t]_{\deg n}^\circ$.

Assume that $(A_0, A, \overline{\eta}) \in \mathcal{M}(S)$ with S connected and that $\varphi \in \mathrm{End}_F^\circ(A)$. Pick any point $s \in S$ and a prime ℓ that is invertible in s. Then φ acts F-linearly on the ℓ-adic rational Tate module $V_\ell(A)$ and its $\mathbb{Q}_\ell \otimes_\mathbb{Q} F$-linear characteristic polynomial is known to have coefficients in F. It is moreover independent of the choice of s and ℓ and we denote it by $\mathrm{char}(\varphi)$.

Definition 5.1. Let $\alpha \in F[t]_{\deg n}^\circ$ and $\mu \in K_{\mathfrak{d}} \backslash G(F_{0,\mathfrak{d}})/K_{\mathfrak{d}}$. The *big fat CM cycle* $\mathcal{C}(\alpha, \mu) \to \mathcal{M}$ is the scheme representing the functor taking $S/O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ to the set of isomorphism classes in

$$\left\{ (A_0, A, \overline{\eta}, \varphi) \,\middle|\, \begin{array}{l} (A_0, A, \overline{\eta}) \in \mathcal{M}(S) \text{ and } \varphi \in \mathrm{End}_{O_F}(A)[\mathfrak{d}^{-1}] \text{ with} \\ \varphi^* \lambda_A = \lambda_A, \mathrm{char}(\varphi) = \alpha \text{ and } \eta \varphi \eta^{-1} \in \mu \text{ for } \eta \in \overline{\eta} \end{array} \right\}.$$

An auxiliary role is played by the Hecke correspondence Hk_μ defined by μ. It is, by definition, the scheme representing the functor taking $S/O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ to the set of isomorphism classes in

$$\left\{ (A_0, A, \overline{\eta}_A, B, \overline{\eta}_B, \varphi) \,\middle|\, \begin{array}{c} (A_0, A, \overline{\eta}_A), (A_0, B, \overline{\eta}_B) \in \mathcal{M}(S) \text{ and} \\ \varphi \in \mathrm{Hom}_{O_F}(A, B)[\mathfrak{d}^{-1}] \text{ with} \\ \varphi^* \lambda_B = \lambda_A \text{ and } \eta_B \varphi \eta_A^{-1} \in \mu \text{ for } \eta_A \in \overline{\eta}_A, \eta_B \in \overline{\eta}_B \end{array} \right\}.$$

The natural map

$$\mathrm{Hk}_\mu \to \mathcal{M} \times_{\operatorname{Spec} O_{\mathbf{F}}[\mathfrak{d}^{-1}]} \mathcal{M} \tag{5.1}$$

is then finite. It is immediate from the moduli descriptions that there is a decomposition

$$\mathrm{Hk}_\mu \times_{\mathcal{M} \times_{\operatorname{Spec} O_{\mathbf{F}}[\mathfrak{d}^{-1}]} \mathcal{M}} \Delta_{\mathcal{M}} = \coprod_{\alpha \in F[t]_{\deg n}^\circ} \mathcal{C}(\alpha, \mu). \tag{5.2}$$

As (5.1) is a finite map, only finitely many of the $\mathcal{C}(\alpha, \mu)$ are non-empty. Moreover, one sees that each map $\mathcal{C}(\alpha, \mu) \to \mathcal{M}$ is finite and unramified, which is [35, Proposition 7.9]. From $F_0 \neq \mathbb{Q}$ it follows that $\mathcal{C}(\alpha, \mu) \to \operatorname{Spec} O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ is proper.

We assume from now on that α is irreducible. Then $E := F[t]/(\alpha(t))$ is a degree n field extension of F. It comes with a distinguished root of α, namely the image of t. The Galois involution of F/F_0 extends to E by letting $t \mapsto t^{-1}$ and we denote by E_0 its fixed field. Whenever $(A_0, A, \overline{\eta}, \varphi) \in \mathcal{C}(\alpha, \mu)$ one gets a ∗-*embedding*

$$E \to \operatorname{End}^{\circ}(A), \quad t \mapsto \varphi,$$

meaning an embedding that is equivariant for the Galois involution on the left and the Rosati involution on the right. It follows from the classification of endomorphism rings of abelian varieties (see [25]) that $\mathcal{C}(\alpha, \mu) \neq \emptyset$ only for α such that E_0 is totally real and (then necessarily) E/E_0 a quadratic CM extension. We assume that this is the case and make the further assumption that α is the characteristic polynomial of some element of $G(F_0)$.

The generic fiber $C(\alpha, \mu)$ of $\mathcal{C}(\alpha, \mu)$ is then 0-dimensional as it consists of points with CM by E, but the integral structure may still be complicated, depending on the O_F-order generated by α. We set

$$^{\mathbb{L}}\mathcal{CM}(\alpha, \mu)' := [\mathcal{O}_{\mathrm{Hk}_{\mu}} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\Delta_{\mathcal{M}}}]|_{\mathcal{C}(\alpha, \mu)} \in K_0'(\mathcal{C}(\alpha, \mu)).$$

Here $K_0'(\mathcal{C}(\alpha, \mu))$ denotes the K-group of coherent sheaves with $\mathbb{Q}$-coefficients on the scheme $\mathcal{C}(\alpha, \mu)$. Concretely, $^{\mathbb{L}}\mathcal{CM}(\alpha, \mu)'$ is defined as follows. The cohomology sheaves $\mathcal{E}^i = H^i(\mathcal{O}_{\mathrm{Hk}_{\mu}} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\Delta_{\mathcal{M}}})$ have support on the intersection (5.2). Let $\mathcal{E}^i|_{\mathcal{C}(\alpha, \mu)} \subseteq \mathcal{E}^i$ denote the direct summand supported on $\mathcal{C}(\alpha, \mu)$. Then

$$^{\mathbb{L}}\mathcal{CM}(\alpha, \mu)' = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{E}^i|_{\mathcal{C}(\alpha, \mu)}],$$

where $[\mathcal{E}]$ denotes the class of $\mathcal{E}$ in $K_0'(\mathcal{C}(\alpha, \mu))$. The non-trivial statement now is that $^{\mathbb{L}}\mathcal{CM}(\alpha, \mu)'$ lies in the filtration step generated by sheaves supported on 1-dimensional subschemes, denoted as $F_1 K_0'(\mathcal{C}(\alpha, \mu))$ in [35, Appendix B]. Given a coherent sheaf $\mathcal{F}$ on $\mathcal{C}(\alpha, \mu)$ with $\leq$1-dimensional support C, one constructs a 1-cycle as follows. Let $\eta_1, \ldots, \eta_r$ be the generic points of the 1-dimensional irreducible components of C. Let C_i denote the closure of η_i with reduced scheme structure. Then take

$$\sum_{i=1}^{r} \operatorname{len}_{\mathcal{O}_{C, \eta_i}} \mathcal{F}_{\eta_i} \cdot [C_i] \in \widetilde{\mathcal{Z}}_1(\mathcal{M}).$$

Since $\mathcal{C}(\alpha, \mu)$ is itself 1-dimensional away from a finite set of primes, this descends to the K-group as a well-defined map

$$F_1 K_0'(\mathcal{C}(\alpha, \mu)) \to \mathcal{Z}_1(\mathcal{M}).$$

We define $^{\mathbb{L}}\mathcal{CM}(\alpha, \mu)$ as the image of $^{\mathbb{L}}\mathcal{CM}(\alpha, \mu)'$ in $\mathcal{Z}_1(\mathcal{M})$. This definition easily extends to a weighted version; compare [35, (7.16)].

Definition 5.2. For $\varphi \in \mathcal{S}(K_{\mathfrak{d}} \backslash G(F_{0,\mathfrak{d}})/K_{\mathfrak{d}}, \mathbb{Q})$ a Hecke function at $\mathfrak{d}$, we set

$$^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi) := \sum_{\mu \in K_{\mathfrak{d}} \backslash G(F_{0,\mathfrak{d}})/K_{\mathfrak{d}}} \varphi(\mu) \, {}^{\mathbb{L}}\mathcal{CM}(\alpha, \mu) \in \mathcal{Z}_1(\mathcal{M}).$$

5.2. Elementary CM cycles

We fix α as in the previous section, in particular we assume that E is a CM field. We write $F^1 = \{x \in \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \mid N_{F/F_0}(x) = 1\}$ and $F^{\mathbb{Q}} := Z^{\mathbb{Q}}$ in this section. We similarly define the two tori $E^1 = \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \mid N_{E/E_0}(x) = 1\}$ and $E^{\mathbb{Q}} = \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \mid N_{E/E_0}(x) \in \mathbb{G}_m\}$. Consider the product torus

$$T := F^{\mathbb{Q}} \times_{\mathbb{G}_m} E^{\mathbb{Q}} \xrightarrow{\sim} F^{\mathbb{Q}} \times E^1.$$

Given a $*$-embedding $\sigma\colon E \to \mathrm{End}(V)$, we have

$$\sigma(E^{\mathbb{Q}}) = \sigma(E) \cap G^{\mathbb{Q}}.$$

Hence any such σ induces an embedding $\sigma\colon T \to \widetilde{G}$ by taking the identity on the $F^{\mathbb{Q}}$-factors. Moreover, σ makes V into a 1-dimensional E/E_0-hermitian space by lifting the hermitian form along the trace $\mathrm{tr}_{E/F}$. It is of signature $(0, 1)$ at precisely one place $\varphi_0'\colon E_0 \to \mathbb{R}$ which has to lie above φ_0. Let Φ' denote the inverse image of Φ under the restriction map $\mathrm{Hom}(E, \mathbb{C}) \to \mathrm{Hom}(F, \mathbb{C})$, which is a CM type for E. Define a Shimura datum h_T for T where the $F^{\mathbb{Q}}$-component is (2.1) and where the E^1-component is given by

$$(h_{T,\varphi'})_{\varphi' \in \Phi'}\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to E^1_{\mathbb{R}} \xrightarrow{\sim} \prod_{\varphi' \in \Phi'} \mathbb{C}^1 \tag{5.3}$$

with

$$h_{T,\varphi'}(z) = \begin{cases} 1 & \text{if } \varphi' \neq \varphi_0', \\ \overline{z}/z & \text{if } \varphi' = \varphi_0'. \end{cases}$$

Then σ becomes a map of Shimura data, i.e. $\sigma \circ h_T \in \{\widetilde{h}\}$, and we denote by $\mathbf{E}$ the reflex field of h_T. It contains $\mathbf{F}$, as well as E via φ_0'.

Definition 5.3. Let $\sigma\colon E \to \mathrm{End}(V)$ be a $*$-embedding, $g \in G(\mathbb{A}_{0,f})$ any element, and denote by $C(\sigma, g)$ the discrete Shimura variety (over $\mathbf{E}$)

$$C(\sigma, g) := [T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\sigma^{-1}(g\widetilde{K}g^{-1})].$$

It naturally maps to the Shimura variety $\mathbf{E} \otimes_{\mathbf{F}} M_{g\widetilde{K}g^{-1}}$. Applying the Hecke isomorphism $g\widetilde{K}\colon M_{g\widetilde{K}g^{-1}} \to M_{\widetilde{K}}$ defines the so-called *elementary CM cycle*

$$C(\sigma, g) \to \mathbf{E} \otimes_{\mathbf{F}} M_{\widetilde{K}}.$$

Let $\mathcal{C}(\sigma, g) \to \mathrm{Spec}\, O_{\mathbf{E}}[\mathfrak{d}^{-1}]$ denote the integral model obtained from the normal closure of $O_{\mathbf{E}}$ in the generic fiber $C(\sigma, g)$. It defines an element of $\mathcal{Z}_1(O_{\mathbf{E}} \otimes_{O_{\mathbf{F}}} \mathcal{M})$ (see Definition 4.2).

5.3. Connected components

The group $\widetilde{G} = F^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}}(V)$ has center $Z = F^{\mathbb{Q}} \times_{\mathbb{G}_m} F^{\mathbb{Q}}$. Using the isomorphism

$$\widetilde{G} \xrightarrow{\sim} F^{\mathbb{Q}} \times G, \quad (z, g) \mapsto (z, z^{-1}g),$$

and the fact that the derived group of G is $\mathrm{SU}(V)$, one sees that Z is also the maximal abelian quotient of $\widetilde{G}$,

$$\pi \colon \widetilde{G}^{\mathrm{ab}} \xrightarrow{\sim} Z, \quad (z, g) \mapsto (z, z^{1-n} \det(g)).$$

Proposition 5.4. *Taking $\mathbb{C}$-points with respect to the inclusion $\mathbf{F} \subseteq \mathbb{C}$, the map*

$$M_{\widetilde{K}}(\mathbb{C}) = \widetilde{G}(\mathbb{Q}) \backslash X \times \widetilde{G}(\mathbb{A}_f)/\widetilde{K} \to Z(\mathbb{Q}) \backslash Z(\mathbb{A}_f)/\pi(\widetilde{K}), \quad [x, h] \mapsto \pi(h),$$

induces a bijection on connected components.

Proof. This is a direct application of [24, Theorem 5.17]. The derived group $\widetilde{G}^{\mathrm{der}}$ equals $\mathrm{SU}(V)$, which is a form of SL_n and consequently simply connected. It follows that one may apply the cited result to obtain

$$\pi_0(M_K(\mathbb{C})) \cong Z(\mathbb{Q})^{\dagger} \backslash Z(\mathbb{A}_f)/\pi(\widetilde{K}),$$

where $Z(\mathbb{Q})^{\dagger} := Z(\mathbb{Q}) \cap \mathrm{Im}(\pi|_{Z(\mathbb{R})} \colon Z(\mathbb{R}) \to Z(\mathbb{R}))$. In the case at hand, the real points $Z(\mathbb{R})$ are connected and hence the map $\pi|_{Z(\mathbb{R})}$ is surjective. It follows that $Z(\mathbb{Q})^{\dagger} = Z(\mathbb{Q})$, which concludes the proof. ∎

The degree of an elementary CM cycle $C(\sigma, g)$, viewed as a function on $\pi_0(M_{\widetilde{K}}(\mathbb{C}))$, may be understood through Proposition 5.4 as well. Namely, independently of σ, the composition $\pi \circ \sigma$ equals

$$\nu := (\mathrm{id}, N_{E/F}) \colon F^{\mathbb{Q}} \times E^1 \to F^{\mathbb{Q}} \times F^1, \tag{5.4}$$

where we have used the identification $Z \cong F^{\mathbb{Q}} \times F^1$, $(z_1, z_2) \mapsto (z_1, z_1^{-1}z_2)$, for the target. The degree of $C(\sigma, g)$ is then a scalar multiple of the characteristic function $1_{\pi(g)\mathrm{Im}(\nu(\mathbb{A}_f))\pi(\widetilde{K})}$. We now transfer this observation to big fat CM cycles.

Proposition 5.5. *The CM cycle $C(\alpha, \mu)$ is a union of elementary CM cycles in the sense that there exists a finite set $(\sigma_i, g_i)_{i \in I}$ of pairs as in Definition 5.3 such that over $\mathbb{C}$,*

$$C(\alpha, \mu) = \sum_{i \in I} C(\sigma_i, g_i). \tag{5.5}$$

Proof. The translation between moduli and group theory description of the complex points $M_{\widetilde{K}}(\mathbb{C})$ is given in [28, Theorem 3.5]: For a point $\underline{A} = (A_0, A, \overline{\eta}) \in M_{\widetilde{K}}(\mathbb{C})$, the n-dimensional F-vector space

$$W(\underline{A}) := -\mathrm{Hom}_F\big(H_1(A_0(\mathbb{C}), \mathbb{Q}), H_1(A(\mathbb{C}), \mathbb{Q})\big) \tag{5.6}$$

is isometric to V. *Transport de structure* of the complex structure on $\mathrm{Hom}_F(\mathrm{Lie}\, A_0, \mathrm{Lie}\, A)$ and the level structure $\bar{\eta}$ along *any* choice of isometry $\gamma\colon W(\underline{A}) \xrightarrow{\sim} V$ defines a point $[x, g\widetilde{K}] \in X \times \widetilde{G}(\mathbb{A}_f)/\widetilde{K}$ whose $\widetilde{G}(\mathbb{Q})$-orbit represents $\underline{A}$.

For a fixed $\underline{A}$ and a fixed isometry γ, corresponding to $[x, g\widetilde{K}]$, the F-linear, polarization-preserving quasi-endomorphisms φ of A with characteristic polynomial α are in bijection with the set

$$\{u \in G(F_0) \mid \mathrm{char}(u) = \alpha,\ uxu^{-1} = x\},$$

the bijection being $\varphi \mapsto \gamma\varphi\gamma^{-1}$. Elements $u \in G(F_0)$ with $\mathrm{char}(u) = \alpha$ in turn are in bijection with $*$-embeddings $\sigma\colon E \to \mathrm{End}(V)$ via $\sigma(t) := u$. (Here, t continues to denote the distinguished element of $E = F[t]/\alpha(t)$.) Given u, or equivalently σ, denote its centralizer in G by T_u. Then x factors over $T_{u,\mathbb{R}}$ and is, in fact, uniquely determined by the property that the composite map

$$\mathbb{C}^\times \xrightarrow{x} T_{u,\mathbb{R}} \xrightarrow{\sigma^{-1}} T_{\mathbb{R}}$$

equals (5.3). Taking the level structure into account, one obtains

$$C(\alpha, \mu) = \widetilde{G}(\mathbb{Q})\backslash\{(x, g, u)\}, \tag{5.7}$$

where $x \in X$, $g \in \widetilde{G}(\mathbb{A}_f)/\widetilde{K}$, $u \in G(F_0)$ and the $\widetilde{G}(\mathbb{Q})$-action are as follows:

(1) The characteristic polynomial of u is α. Let T_u be its centralizer; use $\sigma\colon t \mapsto u$ as above to identify it with T.

(2) The map $x\colon \mathbb{C}^\times \to \widetilde{G}_{\mathbb{R}}$ is determined by u as explained above.

(3) The level structure datum $g\widetilde{K}$ and u are compatible with μ in the sense that $u \in g\mu g^{-1}$.

(4) The action of $\widetilde{G}(\mathbb{Q})$ on such triples is componentwise. It is the usual action on x and g and the conjugation action on u.

All that is left is to rewrite (5.7) in terms of elementary CM cycles. For this we fix representatives u_i for $G(F_0)\backslash\{u \in G(F_0) \mid \mathrm{char}(u) = \alpha\}$ and decompose μ as

$$\mu = \coprod_{j \in J} \widetilde{K}m_j.$$

Let $\sigma_i\colon T \to \widetilde{G}$ be the $*$-embedding corresponding to u_i with image T_i. Let $x_i \in X$ be the point determined by σ_i. Then

$$C(\alpha, \mu) = \sum_{i,j} T_i(\mathbb{Q})\backslash\left(\sum_{g \in \widetilde{G}(\mathbb{A}_f)/\widetilde{K}} 1_{(g\widetilde{K}m_j g^{-1})}(u_i) \cdot (x_i, g, u_i)\right)$$

$$= \sum_{i,j}\ \sum_{\bar{g} \in T_i(\mathbb{A}_f)\backslash\widetilde{G}(\mathbb{A}_f)/\widetilde{K}} 1_{\bar{g}\widetilde{K}m_j\bar{g}^{-1}}(u_i) \cdot T_i(\mathbb{Q})\backslash\left(\sum_{s \in T_i(\mathbb{A}_f)/T_i(\mathbb{A}_f)\cap \bar{g}\widetilde{K}\bar{g}^{-1}} (x_i, s\bar{g}, u_i)\right).$$

Here, the intersection $T_i(\mathbb{A}_f) \cap \overline{g}\widetilde{K}\overline{g}^{-1}$ is independent of the chosen representative $\overline{g}$. The innermost sum agrees with

$$[T_i(\mathbb{Q})\backslash T_i(\mathbb{A}_f)/(T_i(\mathbb{A}_f) \cap \overline{g}\widetilde{K}\overline{g}^{-1})] \cdot \overline{g}\widetilde{K},$$

which is precisely the elementary cycle $C(\sigma_i, \overline{g})$.   ∎

Corollary 5.6. *The degree of $C(\alpha, \mu)$, as a function on the set of connected components, is constant along $\mathrm{Im}(\nu(\mathbb{A}_f))$-cosets.*

Proof. Each of the cycles $C(\sigma_i, g_i)$ in (5.5) has degree function a scalar multiple of the indicator function of a double coset $1_{\mathrm{Im}(\pi(g_i)\nu(\mathbb{A}_f))\pi(\widetilde{K})}$.   ∎

5.4. Prescribed good reduction

It is the derived cycle $^{\mathbb{L}}\mathcal{CM}(\alpha, \mu)$ that is of interest in the context of the AFL. The following result allows us to suitably modify the cycle by elementary CM cycles with maximal level at the prime one is interested in. Write $\Lambda = \Lambda^\mathfrak{d} \subseteq V$ for the $O_F[\mathfrak{d}^{-1}]$-lattice used to define the level structure away from $\mathfrak{d}$ in Section 2.2.

Proposition 5.7. *Let $p \nmid \mathfrak{d}$ be a prime. There exists a $*$-embedding $\sigma\colon E \to \mathrm{End}(V)$ such that Λ_p is $O_{E,p}$-stable. For any such σ there are pairs $\lambda_j \in \mathbb{Q}$, $g_j \in G(\mathbb{A}_{0,f}^p)$, with $j \in J$ finite, such that $C(\alpha, \mu) - \sum_{j \in J} \lambda_j \cdot C(\sigma, g_j)$ has degree zero on each connected component of $M_{\widetilde{K}}(\mathbb{C})$.*

Proof. For the first statement, it is enough to find a $*$-embedding σ such that there is *some* self-dual $O_{E,p}$-lattice in V_p. Indeed, the group $\mathrm{U}(V_p)$ acts transitively on the set of self-dual lattices in V_p. The claim then follows from the fact that $\mathrm{U}(V)$ is dense in $\mathrm{U}(V_p)$, which allows us to conjugate σ suitably.

We have assumed that α is the characteristic polynomial of an element of $G(F_0)$, so there exists some $*$-embedding σ and we fix one. Let w be a place of F_0 above p and let V_w be the corresponding factor of V_p. If w is split in F, then all places v of E_0 above w are also split (in E) and the existence of a self-dual $O_{E,w}$-lattice in V_w is automatic.

So assume that w is inert in F and consider the orthogonal decomposition

$$V_w = V_{w,\mathrm{split}} \times \prod_{v|w \text{ non-split}} V_v,$$

where the split factor is the product of all factors for places $v \mid w$ of E_0 that split in E and where the remaining factors are for the inert v. (Note that v is necessarily inert if it is non-split because w is inert in F.) We claim that a self-dual $O_{E,w}$-lattice exists if and only if all inert factors V_v are quasi-split as F_w-hermitian spaces, i.e. have discriminant in $N_{F/F_0}(F_w^\times)$. Namely in this case let $L \subseteq V_v$ be any $O_{E,v}$-lattice. Then the dual lattice $L^\vee$ is also $O_{E,v}$-stable and hence of the form $L^\vee = \pi^r L$ where $\pi \in E_v$ is a uniformizer. The index $[L : \pi^r L]$ is even since V_v is assumed to be quasi-split, but it also equals rf

where f is the inertia degree of E_v/F_w. This degree is necessarily odd, so r has to be even and $\pi^{r/2}L$ is a self-dual lattice.

Since V_w is itself quasi-split, there is an even number of "bad" non-quasi-split factors V_v. We claim that one may suitably replace those in a sense we now explain. Lifting the hermitian form along $\mathrm{tr}_{E/F}$ makes V into a 1-dimensional hermitian E-vector space. Changing its discriminant at the evenly many "bad" v just exhibited defines a new 1-dimensional E/E_0-hermitian space V'. Composing its hermitian form again with $\mathrm{tr}_{E/F}$ defines an n-dimensional hermitian F-vector space V' together with a $*$-embedding σ': $E \to \mathrm{End}(V')$.

We claim that $V \cong V'$ as hermitian F-vector spaces and that all inert factors V_v' of V' are quasi-split. Indeed, the localizations V_u and V_u' are isomorphic for all places $u \neq w$ of F_0 by construction. Since V and V' must differ at an even number of places, this already implies $V \cong V'$ over F. The claim on the inert factors of V' follows from the elementary fact that two E_v-hermitian spaces are isomorphic if and only if their restrictions to F_w along the trace of E_v/F_w are. Taking $\sigma\colon E \to \mathrm{End}(V)$ as the embedding induced from some choice of isometry $V \cong V'$ concludes the proof of the first statement in the proposition.

Now fix σ such that Λ_p is $O_{E,p}$-stable. Let

$$J := Z(\mathbb{Q})\backslash Z(\mathbb{A}_f)/\pi(\tilde{K})\mathrm{Im}(\nu(\mathbb{A}_f)).$$

For each $j \in J$ choose some element $g_j \in G(\mathbb{A}_{0,f}^p)$ lifting j. Note for the existence of g_j that $\nu|_{F\mathbb{Q}}$ is the identity and we are merely considering j up to $\mathrm{Im}(\nu(\mathbb{A}_f))$. So we only need weak approximation for F^1 (i.e. the density of $F^1(\mathbb{Q}) \subseteq F^1(\mathbb{Q}_p)$), which follows from the fact that any element of $F^1(\mathbb{Q}_p)$ is of the form $x/\sigma(x)$ with $x \in F^\times(\mathbb{Q}_p)$. Then we use the surjectivity of $\pi\colon G(\mathbb{A}_f) \to F^1(\mathbb{A}_f)$. Setting

$$\lambda_j := \frac{\deg C(\alpha,\mu)_j}{\deg C(\sigma,g_j)},$$

where $C(\alpha,\mu)_j$ is the part of the cycle contained in the connected components bundled by j, constructs the sought for (λ_j, g_j) by Corollary 5.6. ∎

6. Elementary CM cycle intersection

6.1. Main result

Throughout this section, $\sigma\colon E \to \mathrm{End}(V)$ is a $*$-embedding such that $\Lambda[\mathfrak{d}^{-1}]$ is stable under $O_E[\mathfrak{d}^{-1}]$ and $g \in G(F_{0,\mathfrak{d}})$. Let $C := C(\sigma, g)$ be the corresponding elementary CM cycle, and denote by $\mathcal{C} := \mathcal{C}(\sigma, g) \in \mathcal{Z}_1(O_{\mathbf{E}} \otimes_{O_{\mathbf{F}}} \mathcal{M})$ its integral model viewed as a cycle.

Also, $\phi = \phi_\mathfrak{d} \otimes 1_{\hat{\Lambda}^\mathfrak{d}} \in \mathcal{S}(V(\mathbb{A}_{0,f}))^K$ will always denote a Schwartz function that is the standard function away from $\mathfrak{d}$. Even though all definitions and results in the following only depend on the $\phi_\mathfrak{d}$-component, we will usually use ϕ in our notation.

The arithmetic intersection formalism from Section 4.2 provides intersection numbers

$$(\widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi),\ \mathcal{C}) \in \mathbb{R}_{\mathfrak{d}}$$

for $\xi \in F_0$ and $h_\infty \in \mathbf{H}(F_{0,\mathbb{R}})$. The result below relates these to the Fourier coefficients of a modular function. Since h_∞ is allowed to vary, it will be slightly stronger to formulate this for uniform (in h_∞) lifts to $\mathbb{R}$. To this end, denote by

$$\langle \widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi),\ \mathcal{C} \rangle \in \mathbb{R}$$

the following smooth function in h_∞. If $\xi \neq 0$, it is the intersection number in the sense of formula (4.6); Theorem 6.9 below shows that $\mathcal{Z}(\xi, \phi) \cap \mathcal{C}$ is artinian for $\xi \neq 0$, so (4.6) is applicable. For $\xi = 0$, in accordance with definition (4.12), choose any lift λ of $(-\phi(0)c_1(\widehat{\boldsymbol{\omega}}), \mathcal{C})$ to $\mathbb{R}$, then put

$$\langle \widehat{\mathcal{Z}}^{\mathbf{K}}(0, h_\infty, \phi),\ \mathcal{C} \rangle = \lambda + \big\langle \big(0, (\mathcal{G}_v^{\mathbf{K}}(0, h_{v|F_0}, \phi))_v\big),\ \mathcal{C} \big\rangle. \tag{6.1}$$

For each non-archimedean place v of F_0, we define a distinguished maximal compact open subgroup of $\mathbf{H}(F_{0,v})$ by

$$K_{\mathbf{H},v}^{\circ} := m'(c_v)^{-1}\mathbf{H}(O_{F_0,v})m'(c_v), \tag{6.2}$$

where $c_v O_{F_0,v} = \delta_{F_0/\mathbb{Q},v}$ and $m'(c_v) = \begin{pmatrix} c_v & \\ & 1 \end{pmatrix}$ (note that $m'(c_v)$ is in $\mathrm{GL}_2(F_{0,v})$ rather than $\mathrm{SL}_2(F_{0,v})$). Set $K_{\mathbf{H}}^{\mathfrak{d},\circ} = \prod_{v \nmid \mathfrak{d}} K_{\mathbf{H},v}^{\circ}$.

Theorem 6.1. *Assume that the CM type Φ is unramified at all $p \nmid \mathfrak{d}$ in the sense below. Assume further that $\phi_{\mathfrak{d}} \in \mathcal{S}(V(F_{0,\mathfrak{d}}))^{K_{\mathfrak{d}} \times K_{\mathbf{H},\mathfrak{d}}}$ is $K_{\mathbf{H},\mathfrak{d}}$-invariant under the open compact subgroup $K_{\mathbf{H},\mathfrak{d}} \subseteq \mathbf{H}(F_{0,\mathfrak{d}})$ and put*

$$K_{\mathbf{H}} = K_{\mathbf{H},\mathfrak{d}} \times K_{\mathbf{H}}^{\mathfrak{d},\circ}.$$

Then there exist coefficients $\varepsilon_\xi \in \sum_{\ell|\mathfrak{d}} \mathbb{Q} \log \ell$ when $\xi \neq 0$, resp. $\varepsilon_0 \in \mathbb{R}$, such that the generating series

$$\sum_{\xi \in F_0} [\varepsilon_\xi + \langle \widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi),\ \mathcal{C} \rangle] W_\xi^{(n)}(h_\infty) \tag{6.3}$$

is the restriction to $h_\infty \in \mathbf{H}(F_{0,\mathbb{R}})$ of some element in $\mathcal{A}_\infty(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)$.

Definition 6.2. The CM type Φ is *unramified* at a prime p if it satisfies the following condition. For any choice of isomorphism $\iota\colon \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ (equivalently, for one choice), the resulting p-adic CM type $\iota \circ \Phi$ is the inverse image under $\mathrm{Hom}(F_p, \mathbb{C}_p) \to \mathrm{Hom}(F_p^u, \mathbb{C}_p)$ of a p-adic CM type Φ^u for F_p^u. Here, F_p^u denotes the maximal subalgebra of the p-adic completion F_p that is unramified over $\mathbb{Q}_p$.

Assuming that Φ is unramified simplifies the computation of intersection numbers in that it allows one to apply the comparison isomorphism from [23]. It is also the assumption that makes the arguments of Howard [10] from the case $F_0 = \mathbb{Q}$ go through without essential changes. (Note that Φ is trivially unramified if $F_0 = \mathbb{Q}$.)

The proof of Theorem 6.1 is by computing the intersection numbers in question and comparing them with the Fourier coefficients of a known-to-be-modular analytic generating series, which is carried out in the current section and the next. The final assembly of results will happen after Proposition 7.8. The following remark provides two a priori reduction steps.

Remark 6.3. (1) An immediate relation for the intersection numbers of interest is

$$\langle \hat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi), \, \mathcal{C}(\sigma, g) \rangle_{O_{\mathbf{E}} \otimes_{O_{\mathbf{F}}} \mathcal{M}_{\widetilde{K}}} = \langle \hat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, g\phi), \, \mathcal{C}(\sigma, 1) \rangle_{O_{\mathbf{E}} \otimes_{O_{\mathbf{F}}} \mathcal{M}_{g\widetilde{K}g^{-1}}},$$

where the two intersections are taken for the Shimura variety of the indicated level (after base change to $O_{\mathbf{E}}[\mathfrak{d}^{-1}]$). In particular, we may (and will) assume that $g = 1$ from now on.

(2) Extending the base field from $\mathbf{E}$ to $\mathbf{E}'$ in the above definition multiplies the intersection numbers by $[\mathbf{E}' : \mathbf{E}]$. So we may replace $\mathbf{E}$ by a larger number field and assume in the following that all the points $C(\mathbb{C})$ are defined over $\mathbf{E}$. Moreover, we are free to compute the intersection numbers on the normalization of $\mathcal{C}$ and will take $\mathcal{C} := \coprod_{C(\mathbf{E})} \operatorname{Spec} O_{\mathbf{E}}[\mathfrak{d}^{-1}]$ in the following.

6.2. The intersection

The definition of the elementary CM cycle C is in terms of Shimura data. We now explain why its points correspond to abelian varieties with $O_E[\mathfrak{d}^{-1}]$-action under the isomorphism $\operatorname{Sh}_{\widetilde{K}}(\widetilde{G}, \{\widetilde{h}\}) \cong M_{\widetilde{K}}$. To this end, we recall some details of the construction of this map that will not be used elsewhere: To a Deligne homomorphism that occurs in the conjugacy class of the Shimura datum,

$$\widetilde{h} = (h_{Z^{\mathbb{Q}}}, h) \colon \mathbb{C} \to (\mathbb{R} \otimes_{\mathbb{Q}} F) \times \operatorname{End}_{\mathbb{R} \otimes_{\mathbb{Q}} F}(\mathbb{R} \otimes_{\mathbb{Q}} V),$$

one associates the two complex tori

$$A_0' = (\mathbb{R} \otimes_{\mathbb{Z}} \Lambda_0)/\Lambda_0, \quad A' = (\mathbb{R} \otimes_{\mathbb{Z}} (\Lambda_0 \otimes_{O_F} \Lambda))/(\Lambda_0 \otimes_{O_F} \Lambda),$$

where the complex structures are given by $\widetilde{h}$. Here, Λ_0 is an invertible O_F-module together with an F-valued hermitian form $(\ ,\ )_0$ such that $\Lambda_0[\mathfrak{d}^{-1}]$ is self-dual with respect to the $\mathbb{Q}$-valued alternating pairing $\operatorname{tr}_{F/\mathbb{Q}} \zeta^{-1}(\ ,\ )_0$; the element $\zeta \in F^\times$ is an auxiliary choice of traceless element as in [28, Section 3.1]. The forms on Λ_0 and $\Lambda_0 \otimes_{O_F} \Lambda$ then define quasi-polarizations on A_0' and A' that are principal away from $\mathfrak{d}$; we will not need this and refer to [28, Section 3.4].

Given a coset $(z, g) \in \widetilde{G}(\mathbb{A}_f)/\widetilde{K}$ (see Section 2.1) as an additional datum, one then considers the isogenous quasi-polarized abelian varieties (A_0, A) with Tate modules equal to

$$z\widehat{\Lambda_0} \quad \text{and} \quad (z \otimes z^{-1}g)(\widehat{\Lambda_0} \otimes_{\widehat{O_F}} \widehat{\Lambda}).$$

Together with a suitable definition of level structure, these define the image of $[\widetilde{h}, (z, g)]$ in $M_{\widetilde{K}}(\mathbb{C}_\nu)$, where $\nu \colon E \to \mathbb{C}$ lies above φ_0. We refer to [28, Theorem 3.5] and the references there for the full construction, because we just need the following consequence. For

$(A_0, A, \overline{\eta}) \in C(\mathbf{E})$, the map $\sigma \colon E \to \mathrm{End}_F(V)$ centralizes the corresponding pair (h, g) and hence provides an action $E \to \mathrm{End}^\circ(A)$. Since $\sigma(O_E)$ is assumed to stabilize $\Lambda[\mathfrak{d}^{-1}]$, one even obtains

$$\sigma(E) \cap \mathrm{End}(A)[\mathfrak{d}^{-1}] = \sigma(O_E[\mathfrak{d}^{-1}]).$$

The universal object over $\mathcal{C}$ is obtained as the Néron model of the universal object over C and hence also has the $O_E[\mathfrak{d}^{-1}]$-action by functoriality. For $(A_0, A, \overline{\eta}) \in \mathcal{C}$, we will use the shorthand

$$L(A_0, A) := \mathrm{Hom}_{O_F}(A_0, A) \quad \text{and} \quad V(A_0, A) := \mathrm{Hom}_F^\circ(A_0, A).$$

We endow these with the E-valued hermitian form $(\,,\,)$ such that

$$\langle\,,\,\rangle = \mathrm{tr}_{E/F} \circ (\,,\,).$$

Next, we define a notion of KR divisor on $\mathcal{C}$, very similarly to [10, Definition 3.5.1].

Definition 6.4. For $\zeta \in E_0^\times$ and $\mu \subseteq V(F_{0,\mathfrak{d}})$ stable under $K_\mathfrak{d}$, let

$$\mathcal{C}(\zeta, \mu) := \left\{ (A_0, A, \overline{\eta}, x) \,\middle|\, \begin{array}{l} (A_0, A, \overline{\eta}) \in \mathcal{C} \text{ and } x \in L(A_0, A)[\mathfrak{d}^{-1}] \\ \text{with } (x, x) = \zeta \text{ and } \eta(x) \in \mu \text{ for } \eta \in \overline{\eta} \end{array} \right\}.$$

Lemma 6.5. *Each $\mathcal{C}(\zeta, \mu)$ is an artinian scheme. It is empty if ζ is not totally positive.*

Proof. The first statement is equivalent to $V(A_0, A) = 0$ for all $(A_0, A, \overline{\eta}) \in C(\mathbf{E})$. Assume for the sake of contradiction that there were $0 \neq x \colon A_0 \to A$. It would imply the existence of an O_E-linear isogeny

$$O_E \otimes_{O_F} A_0 \to A.$$

But the domain has CM type Φ', while the codomain has CM type $\{\Phi' \cup \varphi_0'\} \setminus \{\overline{\varphi_0'}\}$, so such an isogeny cannot exist.

The spaces $V(A_0, A)$ are positive definite, implying the second statement. $\blacksquare$

Since $\mathcal{C}$ is normal by definition, the $\mathcal{C}(\zeta, \mu)$ define Cartier divisors. (Using normality is actually not necessary as Proposition 6.8 below shows.) Next we define a Green function for $\mathcal{C}(\zeta, \mu)$, still assuming $\zeta \neq 0$.

Given a point $(A_0, A, \overline{\eta}) \in C(\mathbf{E}_v)$ above some place $v \colon \mathbf{E} \to \mathbb{C}$, consider the hermitian F-vector space $W(A_0, A)$ from (5.6). It comes with a hermitian E-action (because E acts on A compatibly with the polarization) and we view it as a 1-dimensional hermitian E-vector space by lifting its F-valued form along $\mathrm{tr}_{E/F}$. Let $v = v|_{E_0}$. Then $W(A_0, A)$ is the v-nearby space for V, meaning that it is negative definite at v, positive definite at all other archimedean places, and furthermore $W(A_0, A)(\mathbb{A}_{E_0,f}) \cong V(\mathbb{A}_{E_0,f})$. In fact, there is a natural equality of hermitian $\mathbb{A}_{F,f}$-modules,

$$W(A_0, A)(\mathbb{A}_{F_0,f}) = \prod_{p < \infty}' V_p(A_0, A),$$

so every choice $\eta \in \overline{\eta}$ defines an isometry $\eta \colon W(A_0, A)(\mathbb{A}_{F_0,f}) \cong V(\mathbb{A}_{F_0,f})$ of hermitian $\mathbb{A}_{F,f}$-modules. Since $(A_0, A, \overline{\eta}) \in C$, the choice can even be made E-linearly.

Definition 6.6. (1) Let $h_\infty \in \mathrm{SL}_2(E_{0,\mathbb{R}})$ be a parameter. Define the function $g(\zeta, h_\infty, \mu)$ on $C(\mathbb{C} \otimes_{\mathbb{Q}} \mathbf{E})$ as follows. Its value at a point $(A_0, A, \overline{\eta}) \in C(\mathbf{E}_v)$ is[4]

$$\sum_{x \in W(A_0, A)_\zeta} -\mathrm{Ei}(4\pi a_v v(\zeta)) \cdot 1_\mu(\eta x). \tag{6.4}$$

Here, $v = v|_{E_0}$ is the place of E_0 below v, while $\eta \in \overline{\eta}$ is any choice. The sum (6.4) is well-defined because μ is K-stable. Also, a_v is as in the Iwasawa decomposition (1.8) of the v-component of h_∞.

(2) If $\phi_{\mathfrak{d}} = \sum_i \lambda_i 1_{\mu_i} \in \mathcal{S}(V(F_{0,\mathfrak{d}}))^{K_{\mathfrak{d}}}$, set

$$\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi) := \sum_i \lambda_i \left(\mathcal{C}(\zeta, \mu_i), g(\zeta, h_\infty, \mu_i) \right).$$

Definition 6.7. The cycle for $\zeta = 0$ is again only defined as an element of $\widehat{\mathrm{Ch}}^1(\mathcal{C})$:

$$\widehat{\mathcal{C}}^{\mathbf{K}}(0, h_\infty, \phi) := -\phi(0) \left[\widehat{\omega}|_{\mathcal{C}} + \left(0, (\log |a_v|)_{v, v = v|_{E_0}} \right) \right]$$

(cf. (4.12)).

Proposition 6.8. *For every $\xi \in F_0$ and $h_\infty \in \mathrm{SL}_2(F_{0,\mathbb{R}})$, there is an identity of Arakelov divisors on $\mathcal{C}$,*

$$\widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi) \cap \mathcal{C} = \sum_{\zeta \in E_0, \, \mathrm{tr}_{E_0/F_0}(\zeta) = \xi} \widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi). \tag{6.5}$$

Proof. One is reduced to $\phi_{\mathfrak{d}} = 1_\mu$ being an indicator function. In case $\xi \neq 0$, it is immediate from the moduli description that

$$\mathcal{Z}(\xi, \mu) \cap \mathcal{C} = \coprod_{\zeta \in E_0, \, \mathrm{tr}_{E_0/F_0}(\zeta) = \xi} \mathcal{C}(\zeta, \mu),$$

proving the claim for the algebraic parts of the divisors. Still assuming $\xi \neq 0$, we turn to the archimedean components. Let $\mathcal{G}_v^{\mathbf{K}}(\xi, h_\infty, \mu)$ denote the v-component Green function of $\widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \mu)$ (see (3.18)), where $v \colon \mathbf{E} \to \mathbb{C}$. The aim is to compute $\mathcal{G}_v^{\mathbf{K}}(\xi, h_\infty, \mu)(\underline{A})$ for $\underline{A} = (A_0, A, \overline{\eta}) \in \mathcal{C}(\mathbf{E}_v)$. Set $v := v|_{E_0}$, $w := v|_{F_0}$ and recall that $W(A_0, A)$ (as E-hermitian space) is the v-nearby space for V. The value of $\mathcal{G}_v^{\mathbf{K}}$ at $\underline{A}$ is by definition

$$\sum_{u \in W(A_0, A)_\xi} -\mathrm{Ei}(4\pi a_w R(u, x)) \cdot 1_\mu(\eta u), \tag{6.6}$$

where $\eta \in \overline{\eta}$ is any choice and where x is a certain negative line in $\ell_x \subset \mathbb{R} \otimes_{F_0, w} W(A_0, A)$ that is determined by the Hodge structure on $W(A_0, A)$. (The precise definition, which we do not need, may be found in [29, Section 3].) But the Hodge structure

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[4]In this paper we use $\mathrm{Ei}(4\pi a_v v(\zeta))$ instead of $\mathrm{Ei}(2\pi a_v v(\zeta))$ because our convention differs from [33, Section 12].

on $W(A_0, A)$ is E-stable, so this line is the unique E-stable negative definite one, namely the v-eigenspace $\ell_x = \mathbb{C} \otimes_{E,v} W(A_0, A)$. Thus

$$R(u, x) = \langle \mathrm{pr}_{\ell_x} u, \mathrm{pr}_{\ell_x} u \rangle = v((u, u))$$

for every $u \in W(A_0, A)$. It follows that (6.6) may be rewritten as

$$\sum_{\zeta \in E_0,\, \mathrm{tr}_{E_0/F_0}(\zeta) = \xi} \quad \sum_{u \in W(A_0, A)_\zeta} -\mathrm{Ei}(4\pi a_v v(\zeta)) \cdot 1_\mu(\eta u)$$

as was to be shown.

The case $\xi = 0$ is handled analogously: Subtracting the term for $\zeta = 0$ from (6.5), we obtain

$$\widehat{\mathcal{Z}}^{\mathbf{K}}(0, h_\infty, \mu) \cap \mathcal{C} - \widehat{\mathcal{C}}^{\mathbf{K}}(0, h_\infty, \mu) = \left(0, \left(\sum_{(u,g)} 1_\mu(g^{-1}u) \cdot (\mathcal{G}_v^{\mathbf{K}}(u, h_\infty) \times 1_{g\widetilde{K}}) \right)_v \right)$$

$$(6.7)$$

where the v-term on the right hand side is a sum over $(u, g) \in V_{\xi=0}^{(w)}(F_0) \times \widetilde{G}(\mathbb{A}_f)/\widetilde{K}$ with $u \neq 0$ (see (3.18)). Namely, the contributions of the $(u = 0)$-summand in (3.17) and of the metrized automorphic bundle $\widehat{\omega}|_{\mathcal{C}}$ in (4.12) precisely cancel each other by Definition 6.7. The same arguments as before refine this further according to $(u, u) = \zeta \neq 0$, $\mathrm{tr}_{E_0/F_0}(\zeta) = 0$. ∎

From now on we always consider V as a 1-dimensional hermitian E-vector space. Let $\mathbb{V}$ denote the rank 1 hermitian $\mathbb{A}_E$-module which is positive definite at all archimedean places of E and which is such that $\mathbb{V}_f \cong \mathbb{A}_{E,f} \otimes_E V$. It is *incoherent* in the sense of Kudla [13]. Given $\zeta \in E_0^\times$, let $\mathrm{Diff}(\zeta, \mathbb{V})$ denote the set of places v such that $\mathbb{V}_v$ does not represent ζ. This set is always non-empty of odd cardinality since the unique adelic hermitian space that represents ζ is the adelification of $(E, \zeta N_{E/E_0})$ which is coherent. For a non-split place v of E_0, we again denote by $V^{(v)}$ the nearby hermitian E-vector space; it is the unique hermitian space that represents ζ whenever $\mathrm{Diff}(\zeta, \mathbb{V}) = \{v\}$.

We write

$$\mathbb{A}_E^1 = \{x \in \mathbb{A}_E^\times \mid N_{E/E_0}(x) = 1\},$$

and similarly for other objects where N_{E/E_0} is defined, e.g. E^1 (as earlier) or $\widehat{O}_E^1$. An important role is played by the following orbital integral. Let $\phi \in \mathcal{S}(\mathbb{V}_f)$ be as before, let $\zeta \neq 0$, assume $\mathbb{V}_{f,\zeta} \neq \emptyset$ and let $x_\zeta \in \mathbb{V}_{f,\zeta}$ be some choice. Set

$$\mathrm{Orb}(\zeta, \phi) := \int_{\mathbb{A}_{E,f}^1} \phi(t \cdot x_\zeta)\, dt, \tag{6.8}$$

where the volume is normalized by the open compact $\sigma^{-1}(K) \subseteq \mathbb{A}_{E,f}^1$. It is independent of the choice of x_ζ since the hyperboloid $\mathbb{V}_{f,\zeta}$ is a principal homogeneous space for $\mathbb{A}_{E,f}^1$. By assumption, $\phi^{\mathfrak{d}} = 1_{\widehat{\Lambda^{\mathfrak{d}}}}$ is the characteristic function for an $O_E[\mathfrak{d}^{-1}]$-lattice and $\sigma^{-1}(K^{\mathfrak{d}}) = O_E[\mathfrak{d}^{-1}]^{\widehat{}, 1}$. Hence one may factorize (6.8) as

$$\mathrm{Orb}(\zeta, \phi) = \mathrm{Orb}_{\mathfrak{d}}(\zeta, \phi_{\mathfrak{d}}) \prod_{v < \infty,\, v \nmid \mathfrak{d}} \mathrm{Orb}_v(\zeta, \phi_v), \tag{6.9}$$

where the local volumes are normalized by $\sigma^{-1}(K_{\mathfrak{d}})$ and the $O^1_{E,v}$ respectively and where the local orbital integrals are defined similarly to (6.8).

We also need the following volumes:

$$\tau(Z^{\mathbb{Q}}) := \text{vol}(Z^{\mathbb{Q}}(\mathbb{Q})\backslash Z^{\mathbb{Q}}(\mathbb{A}_f)), \quad \tau(E^1) := \text{vol}(E^1\backslash \mathbb{A}^1_{E,f}), \quad \tau(T) := \tau(Z^{\mathbb{Q}})\tau(E^1),$$
(6.10)

where the measures are normalized by $K_{Z^{\mathbb{Q}}}$ and $\sigma^{-1}(K)$, respectively. These are simply the degrees of the respective 0-dimensional Shimura varieties,

$$\tau(Z^{\mathbb{Q}}) = \deg_{\mathbf{E}}(\mathbf{E} \otimes_{\mathbf{F}} M_0) \quad \text{and} \quad \tau(T) = \deg_{\mathbf{E}} C.$$

We use the arithmetic degree notation $\widehat{\deg}\,\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi) := (\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi), \mathcal{C})$ in the following. For $\zeta \neq 0$ it is meant in the sense of (4.6), for $\zeta = 0$ it is defined by first choosing a representing metrized divisor for $\widehat{\omega}|_{\mathcal{C}}$ in Definition 6.7 and then applying (4.6). Then $\widehat{\deg}\,\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi)$ is a smooth function in h_∞ in both cases. In the case $\zeta = 0$, it is only canonical up to $\sum_{\ell|\mathfrak{d}} \mathbb{Q}\log\ell$.

Theorem 6.9. *Let $\zeta \in E_0^\times$ and $h_\infty \in \mathbf{H}(E_{0,\mathbb{R}})$.*

(1) *If $\#\text{Diff}(\zeta, \mathbb{V}) > 1$, then $\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi) = (\emptyset, 0)$.*

(2) *If $\text{Diff}(\zeta, \mathbb{V}) = \{v\}$ is a singleton with v non-archimedean and $v \nmid \mathfrak{d}$, then $\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi)$ has the form $(\mathcal{C}(\zeta, \phi), 0)$. The cycle $\mathcal{C}(\zeta, \phi)$ has support above places ν of $\mathbf{E}$ with $\nu \mid v$ only and*

$$\widehat{\deg}\,\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi) = \tau(Z^{\mathbb{Q}})[\mathbf{E} : E_0]\max\left\{0, \frac{1 + v(\delta_{E_0/F_0}\zeta)}{2}\right\}\text{Orb}(\zeta, \phi^v)\log q_v.$$
(6.11)

Here δ_{E_0/F_0} denotes the different ideal of the indicated field extension.

(3) *If $\text{Diff}(\zeta, \mathbb{V}) = \{v\}$ is a singleton with v archimedean, then $\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi)$ has the form $(\emptyset, (g_\nu)_{\nu:\mathbf{E}\to\mathbb{C}})$ where $g_\nu \neq 0$ only for $\nu \mid v$. For its degree,*

$$\widehat{\deg}\,\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi) = -\frac{\tau(Z^{\mathbb{Q}})[\mathbf{E} : E_0]}{2} \cdot \text{Ei}(4\pi a_v v(\zeta))\,\text{Orb}(\zeta, \phi).$$
(6.12)

(4) *Finally, the 0-th term is the following smooth function up to $\sum_{\ell|\mathfrak{d}} \mathbb{Q}\log\ell$:*

$$\widehat{\deg}\,\widehat{\mathcal{C}}^{\mathbf{K}}(0, h_\infty, \phi) = \widehat{\deg}\,(\widehat{\omega}|_{\mathcal{C}})\phi(0) - \frac{\tau(Z^{\mathbb{Q}})[\mathbf{E} : E_0]}{2}\tau(E^1)\phi(0)\sum_{v:E_0\to\mathbb{R}}\log|a_v|.$$
(6.13)

Only inert places may occur in case (2). So an implicit statement is that $\mathcal{C}(\zeta, \phi)$ never has support above a split place of E_0. Sections 6.3–6.5 are devoted to the proof of the theorem. Note that there is nothing to show for case (4) which follows directly from definitions. (The factor $1/2$ here is due to the normalization of the arithmetic intersection pairing in Section 4.2, resulting from the convention (4.1).)

6.3. Archimedean component

Let $\zeta \in E_0^\times$, let ν be an archimedean place of $\mathbf{E}$, and set $v := \nu|_{E_0}$. As before (see the proof of Proposition 6.8), for all $(A_0, A, \overline{\eta}) \in C(\mathbf{E}_\nu)$, $W(A_0, A) \cong V^{(v)}$. Hence $W(A_0, A)$ represents ζ if and only if $\mathrm{Diff}(\zeta, \mathbb{V}) = \{v\}$, which proves (1) of Theorem 6.9 for the archimedean component of $\widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi)$.

Now assume $\mathrm{Diff}(\zeta, \mathbb{V}) = \{v\}$. In other words, assume that $\mathbb{V}_f$ represents ζ, that $v(\zeta) < 0$ and that $v'(\zeta) > 0$ for $v' \neq v$ archimedean. Then one obtains, after a choice of a base point $c_0 \in C(\mathbf{E}_\nu)$

$$\sum_{c \in C(\mathbf{E}_\nu)} g_\nu(\zeta, h_\infty, \phi)(c) = \sum_{t \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\sigma^{-1}(\widetilde{K})} g_\nu(\zeta, h_\infty, \phi)(tc_0)$$

$$= \sum_{t \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\sigma^{-1}(\widetilde{K})} \sum_{x \in V_\zeta^{(v)}} -\mathrm{Ei}(4\pi a_v v(\zeta)) \cdot \phi(tx)$$

$$= -\tau(Z^{\mathbb{Q}}) \mathrm{Ei}(4\pi a_v v(\zeta)) \mathrm{Orb}(\zeta, \phi).$$

The last equality follows since $\mathrm{vol}(\sigma^{-1}(\widetilde{K})) = 1$ by our convention. Taking the sum over the $[\mathbf{E} : E_0]$-many different $\nu \mid v$ results in the claimed identity

$$\widehat{\deg}\, \mathcal{C}^{\mathbf{K}}(\zeta, h_\infty, \phi) = -\frac{\tau(Z^{\mathbb{Q}})[\mathbf{E} : E_0]}{2} \mathrm{Ei}(4\pi a_v v(\zeta)) \mathrm{Orb}(\zeta, \phi).$$

The factor $1/2$ here is due to the normalization of the arithmetic intersection pairing in Section 4.2, resulting from the convention (4.1). ∎

6.4. Non-archimedean components (locally)

The aim of this section is to provide some auxiliary results for p-divisible groups with CM that will be needed for statement (2) of Theorem 6.9. Our arguments are very close to those of [10], albeit disguised by the use of [23]. Let E/E_0 be an unramified quadratic extension of p-adic local fields. Let $\pi \in E_0$ be a uniformizer and let $\breve{E}$ be the completion of a maximal unramified extension of E. Denote its residue field by $\mathbb{F}$. The point of departure for our computation of deformation lengths is Gross' result on canonical liftings. Let $\mathbb{Y}$ be a formal height 1 strict O_E-module over $\mathbb{F}$ and let Y be its canonical lift to $O_{\breve{E}}$. Let also $\overline{\mathbb{Y}}$ and $\overline{Y}$ denote the same formal groups, but with the conjugate O_E-action. Write Y_k and $\overline{Y}_k$ for their reductions modulo π^{k+1}. All homomorphisms in the following are O_E-linear homomorphisms, unless indicated otherwise.

Theorem 6.10 (Gross). *Let $u \in \mathrm{Hom}(\mathbb{Y}, \overline{\mathbb{Y}})$. Then u lies in $\mathrm{Hom}(Y_k, \overline{Y}_k)$ if and only if it lies in $\pi^k \mathrm{Hom}(\mathbb{Y}, \overline{\mathbb{Y}})$.*

The underlying strict O_{E_0}-modules of the above groups are supersingular of height 2. Choosing principal polarizations of Y and $\overline{Y}$ as strict O_{E_0}-modules, compatibly with the O_E-action, one gets a hermitian form on $\mathrm{Hom}(\mathbb{Y}, \overline{\mathbb{Y}})$ by the usual rule $(x, y) := y^* \circ x$.

It is easily checked through Dieudonné theory that there is an isometry

$$\mathrm{Hom}(\mathbb{Y}, \overline{\mathbb{Y}}) \cong (O_E, \pi N_{E/E_0}).$$

Denoting by $\mathcal{Z}(u) \subseteq \mathrm{Spf}\, O_E$ the locus to which $u \colon \mathbb{Y} \to \overline{\mathbb{Y}}$ extends as a homomorphism $Y_k \to \overline{Y}_k$, Gross' result may be reformulated as

$$\mathrm{len}\, \mathcal{O}_{\mathcal{Z}(u)} = \frac{1 + v((u,u))}{2}.$$

Here v is the normalized valuation of E_0.

In order to apply this result in the context of polarized abelian varieties one needs to pass from supersingular polarized strict O_{E_0}-modules to "plain" polarized p-divisible groups with O_E-action. This has been achieved in [23] through the use of displays. Let Φ' be a p-adic CM type for E satisfying $\Phi' \sqcup \overline{\Phi'} = \mathrm{Hom}(E, \mathbb{C}_p)$ and which is the inverse image of a CM type $\Phi'^{,u}$ for the maximal subfield E^u of E unramified over $\mathbb{Q}_p$, similar to Definition 6.2. Also fix an element $\varphi'_0 \in \Phi'$ and extend it to a map $\varphi'_0 \colon \breve{E} \to \mathbb{C}_p$. Then $\varphi'_0(\breve{E})$ contains the reflex fields for Φ' and $\{\overline{\varphi'_0}\} \cup \Phi' \setminus \{\varphi'_0\}$.

By definition, a *hermitian O_E-$\mathbb{Z}_p$-module* of signature $(1,0)$ (resp. $(0,1)$) in the sense of [23] is a supersingular p-divisible group of height $[E : \mathbb{Q}_p]$ together with an O_E-action of CM type Φ' (resp. $(\Phi' \cup \{\overline{\varphi'_0}\}) \setminus \{\varphi'_0\}$) and a principal polarization as p-divisible group that is compatible with the O_E-action. By the main result [23, Theorem 3.1], the categories of such objects are equivalent to those of the polarized height 2 strict O_{E_0}-modules with O_E-action considered before. Hence Gross' result carries over in the following form.

Let X and $\overline{X}$ be hermitian O_E-$\mathbb{Z}_p$-modules of signature $(1,0)$ resp. $(0,1)$ over $\mathrm{Spf}\, O_{\breve{E}}$. (The notation is suggestive, but note that $\overline{X}$ is not the same as X with conjugate O_E-action.) Let $\mathbb{X}$ and $\overline{\mathbb{X}}$ (resp. X_k and $\overline{X}_k$) denote their reductions modulo π (resp. π^{k+1}). Endow $\mathrm{Hom}(\mathbb{X}, \overline{\mathbb{X}})$ with the usual O_E-valued hermitian form. Let $\mathcal{Z}(u) \subseteq \mathrm{Spf}\, O_{\breve{E}}$ denote the locus where the homomorphism $u \colon \mathbb{X} \to \overline{\mathbb{X}}$ lifts to a map $X_k \to \overline{X}_k$.

Corollary 6.11. *For $u \colon \mathbb{X} \to \overline{\mathbb{X}}$ a homomorphism,*

$$\mathrm{len}\, \mathcal{O}_{\mathcal{Z}(u)} = \frac{1 + v((u,u))}{2}.$$

Assume now that F/F_0 is an unramified quadratic extension with $F \subseteq E$ and $F_0 = F \cap E_0$. Assume that the CM type Φ' actually is the inverse image of a CM type Φ^u for F^u, the maximal subfield of F unramified over $\mathbb{Q}_p$. Let X_0 be a hermitian O_F-$\mathbb{Z}_p$-module over $\mathrm{Spf}\, O_{\breve{E}}$ of signature $(1,0)$ with respect to Φ, denote its reduction mod π by $\mathbb{X}_0$. Then

$$X'_0 := O_{E_0} \otimes_{O_{F_0}} X_0$$

has CM by O_E of CM type Φ', i.e is of signature $(1,0)$. Choosing a perfect symmetric O_{F_0}-linear pairing on O_{E_0} endows X'_0 with a principal polarization. By the uniqueness of hermitian O_E-$\mathbb{Z}_p$-modules of signature $(1,0)$ (see [23, Section 2]), it is isomorphic to X. With the same notational conventions as before one obtains

$$\mathrm{Hom}_{O_F}(\mathbb{X}_0, \overline{\mathbb{X}}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{X}, \overline{\mathbb{X}}), \quad u \mapsto \mathrm{id}_{O_{E_0}} \otimes u, \tag{6.14}$$

and analogous isomorphisms for the (unique) deformations of all the objects involved to $O_{\breve{E}}/\pi^{k+1}$. This allows us to also apply Gross' result to deformations of $u \in \mathrm{Hom}_{O_F}(\mathbb{X}_0, \overline{\mathbb{X}})$. Note that this is the space we care for most, i.e. it is the local analogue of $L(A_0, A)$ from Section 6.2.

The subtlety now lies in the fact that the hermitian forms in (6.14) are different. The space $\mathrm{Hom}_{O_F}(\mathbb{X}_0, \overline{\mathbb{X}})$ naturally carries the O_F-valued form $\langle x, y \rangle := y^* \circ x$. It lifts naturally to the O_E-valued form $(\, , \,)$ such that $\mathrm{tr}_{E/F}(x, y) = \langle x, y \rangle$. For later use, the following proposition also treats the case $\mathrm{Hom}_{O_F}(\mathbb{X}_0, \mathbb{X})$, to which the above formalities extend mutatis mutandis.

Proposition 6.12. *There are isomorphisms of hermitian lattices*

$$\mathrm{Hom}_{O_F}(\mathbb{X}_0, \mathbb{X}) \cong (O_E, \delta_{E_0/F_0}^{-1} N_{E/E_0}), \quad \mathrm{Hom}_{O_F}(\mathbb{X}_0, \overline{\mathbb{X}}) \cong (O_E, \pi \delta_{E_0/F_0}^{-1} N_{E/E_0}),$$

where δ_{E_0/F_0} generates the different ideal of E_0/F_0.

Proof. Denote by $M := M(\mathbb{X}_0)$ the (covariant) Dieudonné module of $\mathbb{X}_0$ and by $N := M(\mathbb{X})$ (resp. $N := M(\overline{\mathbb{X}})$) the one of $\mathbb{X}$ (resp. $\overline{\mathbb{X}}$), depending on the case one is interested in. Up to isomorphism,

$$M = O_F \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p = \bigoplus_{i : F^u \to \breve{\mathbb{Q}}_p} M_i \tag{6.15}$$

endowed with the Verschiebung

$$[V : M_{i+1} \to M_i] := \sigma^{-1} \cdot \begin{cases} p & \text{if } i \in \Phi^u, \\ 1 & \text{if } i \in \overline{\Phi^u}. \end{cases} \tag{6.16}$$

Here σ is the Frobenius of $\breve{\mathbb{Z}}_p$ and Φ^u is the CM type for F^u that induces Φ. Again up to isomorphism, there is a unique way to define a polarization on (M, V) that is compatible with the O_F-action. Namely, endow M with the $\breve{\mathbb{Z}}_p$-linear extension of a perfect, O_F-hermitian, skew-symmetric pairing

$$O_F \times O_F \to \mathbb{Z}_p. \tag{6.17}$$

These are all of the form

$$\langle m_1, m_2 \rangle := \mathrm{tr}_{F/\mathbb{Q}_p}(\delta_{F/\mathbb{Q}_p}^{-1} m_1 \overline{m_2}),$$

where $\delta_{F/\mathbb{Q}_p}$ denotes a totally imaginary generator of the different $\mathcal{D}_{F/\mathbb{Q}_p}$ and where $m \mapsto \overline{m}$ is the Galois conjugation $\mathrm{conj}_{F/F_0} \otimes \mathrm{id}_{\breve{\mathbb{Z}}_p}$.

The description of N is analogous. The underlying $\breve{\mathbb{Z}}_p$-module with O_E-action is (in both cases)

$$N = O_E \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p = \bigoplus_{j : E^u \to \breve{\mathbb{Q}}_p} N_j. \tag{6.18}$$

For $\mathbb{X}$ the Verschiebung is

$$[V : N_{j+1} \to N_j] := \sigma^{-1} \cdot \begin{cases} p & \text{if } j \in \Phi'^{,u}, \\ 1 & \text{if } j \in \overline{\Phi'^{,u}}. \end{cases} \tag{6.19}$$

In the case of $\overline{\mathbb{X}}$ it equals

$$[V : N_{j+1} \to N_j] := \sigma^{-1} \cdot \begin{cases} p & \text{if } j \in \Phi'^{,u} \setminus \{\varphi_0'\}, \\ p/\pi & \text{if } j = \varphi_0', \\ 1 & \text{if } j \in \overline{\Phi'^{,u}} \setminus \{\overline{\varphi_0'}\}, \\ \pi & \text{if } j = \overline{\varphi_0'}. \end{cases} \tag{6.20}$$

The polarization is defined as in the case of M through the choice of a totally imaginary generator $\delta_{E/\mathbb{Q}_p}$ of the different $\mathcal{D}_{E/\mathbb{Q}_p}$.

Let $1_M \in M$ and $1_N \in N$ denote the unit elements (i.e. choices of generators) in (6.15) and (6.18). Then $\mathrm{Hom}(M, N)$ identifies with $O_E \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p$ via $u \mapsto u(1_M)/1_N$. The dual map $u^\vee$ is then

$$\mathcal{D}_{E/\mathbb{Q}_p}^{-1} \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \underset{\delta_{E/\mathbb{Q}_p}}{\overset{\cong}{\cong}} O_E \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \to O_F \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \underset{\delta_{F/\mathbb{Q}_p}}{\overset{\cong}{\cong}} \mathcal{D}_{F/\mathbb{Q}}^{-1} \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p,$$

$$x \mapsto \delta_{F/\mathbb{Q}_p} \mathrm{tr}_{E/F}(\delta_{E/\mathbb{Q}_p}^{-1} \overline{x}),$$

where by $\mathrm{tr}_{E/F}$ we really mean the $\check{\mathbb{Z}}_p$-linear extension of the trace. The natural lift along $\mathrm{tr}_{E/F}$ is simply

$$x \mapsto \delta_{F/\mathbb{Q}_p} \delta_{E/\mathbb{Q}_p}^{-1} \overline{x} = \delta_{E_0/F_0}^{-1} \overline{x},$$

where we have set $\delta_{E_0/F_0}^{-1} = \delta_{F/\mathbb{Q}_p} \delta_{E/\mathbb{Q}_p}^{-1}$. This element lies in E_0 and generates the inverse different $\mathcal{D}_{E_0/F_0}^{-1}$.

In order to complete the proof, we need to identify the subspace $\mathrm{Hom}((M, V), (N, V)) \subseteq O_E \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p$. In case $N = M(\mathbb{X})$ this is $O_E \otimes_{\mathbb{Z}_p} 1$. An O_E-module generator is given by the identity and the above shows

$$\mathrm{Hom}_{O_F}(\mathbb{X}_0, \mathbb{X}) \cong (O_E, \delta_{E_0/F_0}^{-1} N_{E/E_0}).$$

To understand the case $N = M(\overline{\mathbb{X}})$ we identify $\{j : E^u \to \mathbb{C}_p\}$ with $\mathbb{Z}/2f\mathbb{Z}$, where $2f = [E^u : \mathbb{Q}_p]$, so that $\sigma \circ j = j + 1$ and $\varphi_0'|_{E^u} = 0$. Then $\mathrm{Hom}((M, V), (N, V))$ is generated by $x = (x_j)_{j \in \mathbb{Z}/2f} \in O_E \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p$ with

$$x_j = \begin{cases} \pi & \text{if } j \in \{0, \ldots, f-1\}, \\ 1 & \text{if } j \in \{f, \ldots, 2f-1\}. \end{cases}$$

It follows that

$$\mathrm{Hom}_{O_F}(\mathbb{X}_0, \overline{\mathbb{X}}) \cong (O_E, \pi \delta_{E_0/F_0}^{-1} N_{E/E_0})$$

as claimed. ∎

Combining the proposition with the comparison isomorphism (6.14) and Gross' formula one obtains

Corollary 6.13. *For* $u\colon \mathbb{X}_0 \to \overline{\mathbb{X}}$ *a homomorphism,*

$$\operatorname{len} \mathcal{O}_{\mathcal{Z}(u)} = \frac{1 + v(\delta_{E_0/F_0}(u,u))}{2}.$$

6.5. Non-archimedean components (globally)

We return to the global setting. Let $\zeta \in E_0^{\times}$, let ν be a non-archimedean place of $\mathbf{E}$ not dividing $\mathfrak{d}$ and set $v := \nu|_{E_0}$. Also let $\mathbb{F}$ denote an algebraic closure of the residue field $\mathbb{F}_\nu$.

Proposition 6.14. *For* $(A_0, A, \overline{\eta}) \in \mathcal{C}(\mathbb{F})$ *the following hold:*

(1) *If v is split in E, then $V(A_0, A) = 0$.*

(2) *If v is inert in E, then $V(A_0, A) \cong V^{(v)}$.*

Proof. Recall that $\mathbf{E}$ is by definition a subfield of $\mathbb{C}$. Choose an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ which induces the prime ν. This allows us to view Φ and Φ' as p-adic CM types. Let w be an arbitrary prime of E over p and let w_0, u and u_0 denote its restrictions to E_0, F and F_0, respectively. The isotypic factors $A_0[u^\infty]$ and $A[w^\infty]$ have to be isoclinic p-divisible groups since they have CM by a field. Their slopes are necessarily

$$\lambda_u := \frac{\#\,\Phi \cap \operatorname{Hom}(F_u, \overline{\mathbb{Q}}_p)}{\#\,\operatorname{Hom}(F_u, \overline{\mathbb{Q}}_p)}, \quad \text{resp.} \quad \lambda_w := \frac{\#\,(\{\overline{\varphi_0'}\} \cup \Phi' \setminus \{\varphi_0'\}) \cap \operatorname{Hom}(E_w, \overline{\mathbb{Q}}_p)}{\#\,\operatorname{Hom}(E_w, \overline{\mathbb{Q}}_p)}.$$
$$(6.21)$$

In general one has

$$\frac{\#\,\Phi \cap \operatorname{Hom}(F_u, \overline{\mathbb{Q}}_p)}{\#\,\operatorname{Hom}(F_u, \overline{\mathbb{Q}}_p)} = \frac{\#\,\Phi' \cap \operatorname{Hom}(E_w, \overline{\mathbb{Q}}_p)}{\#\,\operatorname{Hom}(E_w, \overline{\mathbb{Q}}_p)}$$

since Φ' is the inverse image of Φ under the restriction map $\operatorname{Hom}(E, \mathbb{C}) \to \operatorname{Hom}(F, \mathbb{C})$. One deduces the following properties:

- If w/w_0 is inert, then u/u_0 is also inert and $\lambda_w = \lambda_u = 1/2$.
- If w/w_0 is split and $w_0 \neq v$, then $\lambda_w = \lambda_u$ and $\lambda_w + \lambda_{\overline{w}} = 1$. Indeed, the addition of φ_0' and removal of $\overline{\varphi_0'}$ in (6.21) does not come in since φ_0' lies above v.
- If w/w_0 is split and $w_0 = v$, then

$$\#\,(\{\overline{\varphi_0'}\} \cup \Phi' \setminus \{\varphi_0'\}) \cap \operatorname{Hom}(E_w, \overline{\mathbb{Q}}_p) = \#\,\Phi' \cap \operatorname{Hom}(E_w, \overline{\mathbb{Q}}_p) \pm 1.$$

In particular, $\lambda_w \neq \lambda_u$ and one obtains statement (1) of the proposition.

Now assume that v is inert. The first aim is to show $V(A_0, A) \neq 0$. The above computation of slopes shows that $A_0[v^\infty]$ and $A[v^\infty]$ are supersingular. In other words, they are (isomorphic to) the O_F-$\mathbb{Z}_p$-module $\mathbb{X}_0$ and O_E-$\mathbb{Z}_p$-module $\overline{\mathbb{X}}$ from Proposition 6.12, respectively. It follows that there is an O_E-linear isogeny $O_E \otimes_{O_F} A_0[v^\infty] \to A[v^\infty]$. It extends to an isogeny $O_E \otimes_{O_F} A_0[p^\infty] \to A[p^\infty]$ by the above computations of slopes.

Set $A'_0 := O_E \otimes_{O_F} A_0$, which is an abelian variety over $\mathbb{F}$ with CM by O_E of CM type Φ'. The argument just given shows that $A'_0[p^\infty]$ and $A[p^\infty]$ are O_E-linearly isogenous. This does not yet imply that A'_0 and A are isogenous, but it shows at least that A'_0 is isogenous to some A' with CM by O_E of CM type Φ. Then A' together with its CM deforms to characteristic 0. But over $\mathbb{C}$, any two O_E-CM abelian varieties with the same CM type are isogenous. This finally proves that A'_0 and A are isogenous and hence $V(A_0, A) \neq 0$.

For dimension reasons one obtains

$$\mathbb{Q}_\ell \otimes_\mathbb{Q} V(A_0, A) \cong \begin{cases} \operatorname{Hom}_{F_\ell}(\mathrm{V}_\ell(A_0), \mathrm{V}_\ell(A)), & \ell \neq p, \\ \operatorname{Hom}^\circ_{F_p}(A_0[p^\infty], A[p^\infty]), & \ell = p. \end{cases}$$

In particular, $V(A_0, A)$ has to be of dimension 1 over E. It is positive definite by the positivity of the Rosati involution. Its localizations at $\ell \neq p$ are isomorphic to V_ℓ (viewed as 1-dimensional E-vector space) by the existence of a level structure.

This leaves us to check $V_{w_0}^{(v)} \cong V(A_0, A)_{w_0}$ for w_0 above p. If w_0 is inert in E with $u_0 := w_0|_{F_0}$, the statement $\operatorname{Hom}(A_0[u_0^\infty], A[w_0^\infty]) = V_{w_0}^{(v)}$ is precisely Proposition 6.12. (Both cases of the proposition occur, depending on whether $v = w_0$ or not.) For w_0 split in E, there is nothing to show since there is only a single isomorphism class of 1-dimensional hermitian spaces. ∎

Proof of Theorem 6.9 (1, 2). Part (1) is implied by the previous proposition: $\mathbb{F} \otimes_{O_E[\mathfrak{d}^{-1}]} \mathcal{C}(\zeta, \phi) \neq \emptyset$ only if ζ is represented by $V^{(v)}$, which can only happen if v is inert and $\operatorname{Diff}(\zeta, \mathbb{V}) = \{v\}$.

Assume that this is the case, and let $c = (A_0, A, \bar{\eta}, u) \in \mathcal{C}(\zeta, \phi)(\mathbb{F})$ be any point. The completed local ring of $\mathcal{C}(\zeta, \phi)$ in c is $O_{\breve{E}_v}$, which is finite free of degree equal to the ramification index $e(v/v)$ over $O_{\breve{E}_v}$. Applying the Serre–Tate Theorem and Corollary 6.13, one obtains

$$\operatorname{len} \mathcal{O}_{\mathcal{C}(\zeta, \phi), c} = e(v/v) \cdot \frac{1 + v(\delta_{E_0/F_0}\zeta)}{2}.$$

The length is in particular independent of c and all that is left to do is to count the points of $\mathcal{C}(\zeta, \phi)(\mathbb{F})$.

For c as above, the level structure yields a coset of identifications

$$\sigma^{-1}(K^p)\eta^p : \mathbb{A}_{E, f}^p \otimes_E V(A_0, A) \cong \mathbb{V}_f^p.$$

By Proposition 6.12, it can naturally be extended to an identification

$$\sigma^{-1}(K^v)\eta^v : \mathbb{A}_{E, f}^v \otimes_E V(A_0, A) \cong \mathbb{V}_f^v$$

by looking at p-divisible groups. By "natural" we mean that $x \in V(A_0, A)$ lies in $\operatorname{Hom}(A_0, A)[\mathfrak{d}^{-1}]$ if and only if $\eta^{v, \mathfrak{b}}(x) \in \widehat{\Lambda}^{v, \mathfrak{b}}$ and if it is a homomorphism at v. The latter is equivalent to $v(\delta_{E_0/F_0}\zeta) > 0$ by Proposition 6.12. It follows that for a quasi-homomorphism $x \in V(A_0, A)_\zeta$, the length of $\mathcal{C}(\zeta, \phi)$ at the point $(A_0, A, \bar{\eta}, x)$ is

$$e(v/v)\phi^v(\eta^v(x)) \max \left\{ 0, \frac{1 + v(\delta_{E_0/F_0}\zeta)}{2} \right\}.$$

The set of such x forms a simply transitive E^1-orbit, while $\mathcal{C}(\zeta, \phi)(\mathbb{F})$ is a simply transitive $T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\sigma^{-1}(\tilde{K})$-orbit. So

$$
\widehat{\deg}_v \, \mathcal{C}(\zeta, \phi) = \tau(Z^{\mathbb{Q}}) e(v/v) \, \max\left\{0, \frac{1 + v(\delta_{E_0/F_0}\zeta)}{2}\right\} \cdot \log q_v
$$

$$
\times \sum_{t \in E^1 \backslash \mathbb{A}_f^1/\sigma^{-1}(K)} \sum_{x \in V(A_0, A)_\zeta} \phi^v(tx)
$$

$$
= \tau(Z^{\mathbb{Q}}) e(v/v) \, \max\left\{0, \frac{1 + v(\delta_{E_0/F_0}\zeta)}{2}\right\} \cdot \mathrm{Orb}(\zeta, \phi^v) \cdot \log q_v. \quad (6.22)
$$

Summing over all $v \mid v$ one obtains the claimed expression

$$
\widehat{\deg}_v \mathcal{C}(\zeta, \phi) = \tau(Z^{\mathbb{Q}})[\mathbf{E} : E_0] \cdot \max\left\{0, \frac{1 + v(\delta_{E_0/F_0}\zeta)}{2}\right\} \cdot \mathrm{Orb}(\zeta, \phi^v) \log q_v. \qquad \blacksquare
$$

7. Comparison with analytic generating series

7.1. The analytic generating series

Consider the split E_0-quadratic space $V' = E_0 \times E_0$ with its quadratic form

$$
u' = (u_1, u_2) \mapsto \mathfrak{q}(u') = u_1 u_2.
$$

The special orthogonal group $\mathrm{SO}(V', \mathfrak{q})$ can be identified with the E_0-group $G' := \mathrm{GL}_{1, E_0}$, via the action on V' by $g \cdot (u_1, u_2) = (g^{-1}u_1, gu_2)$.

Denote by $\chi \colon \mathbb{A}_{E_0}^\times/E_0^\times \to \{\pm 1\}$ the quadratic character associated to E/E_0 by class field theory. For $\phi' \in \mathcal{S}(V'(\mathbb{A}_{E_0}))$ we recall from [35, Section 12] that one defines a regularized integral

$$
J(\phi', s) = \int_{[G']} \left(\sum_{u' \in V'(E_0)} \phi'(g^{-1} \cdot u') \right) |g|^s \chi(g) \, dg. \qquad (7.1)
$$

It decomposes as a sum

$$
J(\phi', s) = \mathrm{Orb}(0_\pm, \phi', s) + \sum_{\zeta = \mathfrak{q}(u') \in E_0^\times} \mathrm{Orb}(\zeta, \phi', s), \qquad (7.2)
$$

where for $\zeta = \mathfrak{q}(u'_\zeta) \neq 0$,

$$
\mathrm{Orb}(\zeta, \phi', s) := \int_{G'(\mathbb{A}_{E_0})} \phi'(g^{-1} \cdot u'_\zeta) |g|^s \chi(g) \, dg. \qquad (7.3)
$$

(Note that this notation slightly differs from [35, Section 12], where it was denoted by $\mathrm{Orb}(u'_\zeta, \phi', s)$.) We refer to *loc. cit.* for the definition of nilpotent orbital integrals $\mathrm{Orb}(0_\pm, \phi', s) := \mathrm{Orb}(0_+, \phi', s) + \mathrm{Orb}(0_-, \phi', s)$.

Define the analytic generating function

$$J(h, \phi', s) := J(\omega(h)\phi', s), \quad h \in \mathbf{H}(\mathbb{A}_{E_0}).$$

We recall from [35, Theorem 12.9] for future reference.

Theorem 7.1. *The function* $\mathbf{H}(\mathbb{A}_{E_0}) \times \mathbb{C} \ni (h, s) \mapsto J(h, \phi', s)$ *is smooth, left* $\mathbf{H}(E_0)$-*invariant, and entire in* $s \in \mathbb{C}$.

Remark 7.2. The integral (7.1) can be viewed as the theta lifting for the pair

$$(SO(V', \mathfrak{q}), SL_2),$$

from the automorphic representation $\chi| \cdot |^s$ of $SO(V') \simeq GL_1$ to SL_2 (see [35, Remark 12.10]). Therefore, $J(h, \phi', s)$ should be a degenerate Eisenstein series for the induced representation $\mathrm{Ind}_{B(\mathbb{A}_{E_0})}^{\mathbf{H}(\mathbb{A}_{E_0})}(\chi| \cdot |^s)$ (B the Borel subgroup of upper triangular matrices). However, it may not be associated to a standard section in our application below.

7.2. Local results

Let E_0 be a local field and E/E_0 a (possibly split) quadratic extension. Denote by $\chi: E_0^\times \to \{\pm 1\}$ the corresponding character from local class field theory. For $\zeta \in E_0^\times$, we define the normalized local orbital integral with a transfer factor,

$$\mathrm{Orb}(\zeta, \phi', s) = \chi(u_1)|u_1|^{-s} \int_{E_0^\times} \phi'(u_1 t^{-1}, t u_2)|t|^s \chi(t) \, d^\times t,$$

where $(u_1, u_2) \in E_0 \times E_0$ is any choice such that $\zeta = u_1 u_2$. Set

$$\mathrm{Orb}(\zeta, \phi') = \mathrm{Orb}(\zeta, \phi', 0),$$

$$\partial \mathrm{Orb}(\zeta, \phi') = \frac{d}{ds}\bigg|_{s=0} \mathrm{Orb}(\zeta, \phi', s).$$

For $h \in \mathbf{H}(E_0)$, we set

$$\mathrm{Orb}(\zeta, h, \phi', s) := \mathrm{Orb}(\zeta, \omega(h)\phi', s)$$

and similarly for the other orbital integrals.

Now let E_0 be a local p-adic field and q the cardinality of its residue field. Let E/E_0 be an unramified quadratic field extension. Let v be the normalized valuation on E_0 and $| \cdot | = q^{v(\cdot)}$ the normalized absolute value. For each $c \in E_0^\times$, consider the characteristic function

$$\phi'_c = 1_{O_{E_0} \times c \cdot O_{E_0}} \in \mathcal{S}(V'(E_0)). \tag{7.4}$$

Normalize the Haar measure on $E_0^\times$ so that $\mathrm{vol}(O_{E_0}^\times) = 1$.

Proposition 7.3. *We have*

$$\mathrm{Orb}(\zeta, \phi_c') = \begin{cases} 1 & \text{if } v(c^{-1}\zeta) \ge 0 \text{ and } \equiv 0 \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

When $v(\zeta) - v(c)$ is odd,

$$\partial\mathrm{Orb}(\zeta, \phi_c') = -\max\left\{0, \frac{1 + v(c^{-1}\zeta)}{2}\right\} \log q. \tag{7.5}$$

Proof. This follows by a straightforward computation. ∎

We recall from [33, Section 12] the analogous result for the archimedean extension $E\mathbb{C}$ of $E_0 = \mathbb{R}$, the standard character $\psi_0(b) := \exp(2\pi i b)$. Now we have $V' = \mathbb{R} \times \mathbb{R}$ and let

$$\phi'(x, y) = 2^{-1}(x + y)e^{-\pi(x^2 + y^2)} \in \mathcal{S}(\mathbb{R} \times \mathbb{R}). \tag{7.6}$$

Write $h \in \mathrm{SL}_2(\mathbb{R})$ according to the Iwasawa decomposition (see (1.9))

$$h = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a^{1/2} & \\ & a^{-1/2} \end{pmatrix} \kappa_\theta, \quad a \in \mathbb{R}_+, b \in \mathbb{R}.$$

Normalize the Haar measure on $E_0^\times = \mathbb{R}^\times$ to be $dt/|t|$ where dt is the Lebesgue measure on $\mathbb{R}$. We recall the following result from [33, Lemma 12.5].[5]

Proposition 7.4. *Let $\zeta \in \mathbb{R}^\times$. Then*

$$\mathrm{Orb}(\zeta, h, \phi') = \chi_1(\kappa_\theta) \begin{cases} a^{1/2} e^{2\pi i \zeta(b + ia)}, & \zeta > 0, \\ 0, & \zeta < 0, \end{cases}$$

and when $\zeta < 0$,

$$\partial\mathrm{Orb}(\zeta, h, \phi') = \tfrac{1}{2}\chi_1(\kappa_\theta) a^{1/2} e^{2\pi i \zeta(b + ia)} \, \mathrm{Ei}(-4\pi a|\zeta|). \tag{7.7}$$

Here the character χ_1 is defined by (1.10).

7.3. Comparison

From now on, F_0, F, E_0 and E again denote the global fields as in previous sections. Given an element $c \in (\mathbb{A}_{E_0,f}^{\flat})^\times$, denote by $\mathbb{V}_c$ the rank 1 adelic hermitian space

$$\mathbb{V}_c := V_{\flat} \times (\mathbb{A}_{E,f}^{\flat}, cx\overline{y}) \times \mathbb{V}_\infty,$$

where $\mathbb{V}_\infty$ is positive definite at all archimedean places, and $V_{\flat}$ is the base change to $F_{\flat}$ from the F/F_0-hermitian space V fixed in Section 2.2, which is now viewed as an E/E_0-

---

[5]Here the Gaussian functions differ slightly from those in [35, Section 12]; the latter should be corrected accordingly.

hermitian space (see the paragraph after the proof of Proposition 6.8). Define a Schwarz function $\phi_c \in \mathcal{S}(\mathbb{V}_c)$ as

$$\phi_c := \phi_{\mathfrak{d}} \otimes 1^{\mathfrak{d}}_{\hat{\mathcal{O}}_E} \otimes \phi_\infty$$

where $\phi_{\mathfrak{d}}$ is from the definition (2.6) of the KR-generating series and where ϕ_∞ is the Gaussian

$$\phi_\infty(x) = e^{-2\pi \operatorname{tr}_{E_0/\mathbb{Q}} \langle x,x \rangle}.$$

For $v \mid \infty$, we normalize the local Haar measure on E_v^1 by $\operatorname{vol}(E_v^1) = 1$.

Choose $\phi' = \prod_v \phi'_v$ such that for every v, ϕ'_v is a transfer of ϕ_v (and the zero function on the other isomorphism class of hermitian space), i.e. they satisfy

$$\operatorname{Orb}(\zeta, \phi'_v) = \begin{cases} \operatorname{Orb}(\zeta, \phi_v) & \text{if } \mathbb{V}_v \text{ represents } \zeta, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.5. (1) *Let $v \nmid \mathfrak{d}$ and $v < \infty$. Then*

$$\operatorname{Orb}(\zeta, \phi_v) = \begin{cases} 1 & \text{if } v(\delta_{E_0/F_0}\zeta) \geq 0 \text{ and } \equiv 0 \bmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{7.8}$$

(2) *Let $v \mid \infty$. Then, for $h = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} a^{1/2} & \\ & a^{-1/2} \end{pmatrix} \kappa_\theta \in \operatorname{SL}_2(\mathbb{R})$ (see (1.9)), we have*

$$\operatorname{Orb}(\zeta, h_v, \phi_v) = \chi_1(\kappa_\theta)\begin{cases} a^{1/2}e^{2\pi i\zeta(b+ia)} & \text{if } \zeta \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.9}$$

Proof. This follows by a straightforward computation. ■

By (7.8) and Proposition 7.3 we can and do choose ϕ'_v as ϕ'_{c_v} defined by (7.4) when v is unramified (including the split case which is easy to verify); by (7.9) and Proposition 7.4 we can and do choose ϕ'_v as (7.6) when $v \mid \infty$.

Let δ_{E_0/F_0}, $\delta_{E_0/\mathbb{Q}}$ and $\delta_{F_0/\mathbb{Q}}$ denote the different ideals of the indicated field extensions; we will also view them as elements in the respective rings of ideles. We now apply the previous considerations in the case of the incoherent adelic hermitian space $\mathbb{V}$ from Theorem 6.9. It is of the form $\mathbb{V}_c$ for $c \in \mathbb{A}_{E_0,f}^{\mathfrak{d},\times}$ satisfying

$$c\hat{\mathcal{O}}_{E_0}^{\mathfrak{d}} = \delta_{E_0/F_0}^{-1}\hat{\mathcal{O}}_{E_0}^{\mathfrak{d}}.$$

The Schwartz function ϕ considered in the same theorem agrees with ϕ_c for a suitable choice of isomorphism $\mathbb{V} \cong \mathbb{V}_c$.

Set

$$\partial J(h, \phi') := \frac{d}{ds}\bigg|_{s=0} J(h, \phi', s), \quad h \in \mathbf{H}(\mathbb{A}_{E_0}). \tag{7.10}$$

Write

$$\partial J(h, \phi') = \sum_{\zeta \in E_0} \partial J(\zeta, h, \phi')$$

for its Fourier expansion (cf. (1.12)).

Theorem 7.6. (1) *For all $0 \neq \zeta$ with $\mathrm{Diff}(\zeta, \mathbb{V}) = \{v\}$ a singleton and $v \nmid \mathfrak{d}$, we have equalities in $\mathbb{R}$:*

$$2\tau(Z^{\mathbb{Q}})[\mathbf{E} : E] \cdot \partial J(\zeta, h_\infty, \phi') = -\widehat{\deg} \, \widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi) W_\zeta^{(1)}(h_\infty). \tag{7.11}$$

(2) *If instead $0 \neq \zeta$ with $\mathrm{Diff}(\zeta, \mathbb{V}) = \{v\}$ and $v \mid \mathfrak{d}$, then $\partial J(\zeta, h_\infty, \phi') = d_\zeta \, W_\zeta^{(1)}(h_\infty)$ for a constant d_ζ such that*

$$2\tau(Z^{\mathbb{Q}})[\mathbf{E} : E] \cdot d_\zeta \in \mathbb{Q} \log q_v \subseteq \sum_{\ell \mid \mathfrak{d}} \mathbb{Q} \log \ell.$$

(3) *When $\zeta = 0$, there is a constant $d_0^{\mathrm{corr}} \in \mathbb{R}$ such that*

$$2\tau(Z^{\mathbb{Q}})[\mathbf{E} : E] \cdot \partial J(0, h_\infty, \phi') = [d_0^{\mathrm{corr}} - \widehat{\deg} \, \widehat{\mathcal{C}}^{\mathbf{K}}(0, h_\infty, \phi)] W_0^{(1)}(h_\infty).$$

Proof. By the choice of ϕ' and ϕ, we have $\partial J(\zeta, h_\infty, \phi') \neq 0$ only for $\zeta = 0$ or ζ with $\mathrm{Diff}(\zeta, \mathbb{V}) = \{v\}$ a singleton.

We first consider the case $\mathrm{Diff}(\mathbb{V}, \zeta) = \{v\}$ with $v \nmid \mathfrak{d}$ non-archimedean. Then v is necessarily inert in E and $v(\delta_{E_0/F_0}\zeta)$ is odd. If $v(\delta_{E_0/F_0}\zeta) \leq 0$, then both sides in statement (1) vanish by Theorem 6.9 and Proposition 7.3. In case $v(\delta_{E_0/F_0}\zeta) \geq 1$ we would like to apply the identity (7.5) to $c = \delta_{E_0/F_0}^{-1}$ and to obtain

$$\begin{aligned}
\partial J(\zeta, h_\infty, \phi') &= \partial \mathrm{Orb}(\zeta, \phi_v') \, \mathrm{Orb}(\zeta, h_\infty, \phi'^v) \\
&= -\max\left\{0, \frac{1 + v(\delta_{E_0/F_0}\zeta)}{2}\right\} \mathrm{Orb}(\zeta, \phi'^v) \log q_v.
\end{aligned}$$

Since ζ is totally positive, by (7.9) the archimedean components give us

$$\prod_{v \mid \infty} \mathrm{Orb}(\zeta, h_v, \phi_v') = W_\zeta^{(1)}(h_\infty). \tag{7.12}$$

Taking into account (6.11), statement (1) (for non-archimedean v) follows.

We now assume v is archimedean. Then ζ is negative exactly at v. By (7.7), we have

$$\begin{aligned}
\partial J(\zeta, h_\infty, \phi') &= \partial \mathrm{Orb}(\zeta, h_v, \phi_v') \, \mathrm{Orb}(\zeta, h_\infty^v, \phi'^v) \\
&= \tfrac{1}{2} a_v^{1/2} \, e^{2\pi i v(\zeta)(b_v + i a_v)} \, \mathrm{Ei}(4\pi a_v v(\zeta)) \, \mathrm{Orb}(\zeta, h_\infty^v, \phi'^v).
\end{aligned}$$

Now statement (1) (for archimedean v) follows from Theorem 6.9 (3) and (7.9).

Coming to part (2), we now assume $\mathrm{Diff}(\mathbb{V}, \zeta) = \{v\}$ with $v \mid \mathfrak{d}$. Then we still have

$$\partial J(\zeta, h_\infty, \phi') = \partial \mathrm{Orb}(\zeta, \phi_v') \, \mathrm{Orb}(\zeta, h_\infty, \phi'^v).$$

By (7.12), we may set $d_\zeta = \partial \mathrm{Orb}(\zeta, \phi_v') \, \mathrm{Orb}(\zeta, \phi'^{v,\infty})$. Clearly, $\mathrm{Orb}(\zeta, \phi_v', s)$ is a polynomial in q_v^s and q_v^{-s} with $\mathbb{Q}$-coefficients. Hence

$$\partial \mathrm{Orb}(\zeta, \phi_v') \in \mathbb{Q} \log q_v.$$

Similarly we have $\mathrm{Orb}(\zeta, \phi'^{v,\infty}) \in \mathbb{Q}$ and hence the term d_ζ lies in $\mathbb{Q} \log q_v$.

We come to claim (3) on the 0-th Fourier coefficient. This essentially follows from the last part of the proof of [35, Proposition 14.5]. More precisely, by *loc. cit.* we have

$$\partial J(0, h_\infty, \phi') = \Big(\mathrm{Orb}(0_+, \phi') \sum_{v: E_0 \to \mathbb{R}} \log |a_v| + c\Big) W_0^{(1)}(h_\infty),$$

for some constant $c \in \mathbb{R}$, and

$$\mathrm{Orb}(0_+, \phi') = -\mathrm{Orb}(0_-, \phi') = \tfrac{1}{2}\tau(E^1)\phi(0).$$

(This last identity was a result of Jacquet.) We obtain

$$\partial J(0, h_\infty, \phi') = \Big(\tfrac{1}{2}\tau(E^1) \cdot \phi(0) \sum_{v: E_0 \to \mathbb{R}} \log |a_v| + c\Big) W_0^{(1)}(h_\infty).$$

By Theorem 6.9 (4), statement (3) is proven. ∎

Lemma 7.7. *Let $v \nmid \mathfrak{d}$ be a non-archimedean place of F_0 (in particular, v is unramified in F/F_0).*

(1) *Let $\phi'_v = \prod_{w|v, w \in \Sigma_{E_0}} \phi'_w$ be as above. Then ϕ'_v is $K^\circ_{\mathbf{H}, v}$-invariant under the Weil representation.*

(2) *Let $\phi_v = \prod_{w|v, w \in \Sigma_{E_0}} \phi_w$ be as above. Then ϕ_v is $K^\circ_{\mathbf{H}, v}$-invariant under the Weil representation.*

Here $K^\circ_{\mathbf{H}, v}$ is defined by (6.2), and the group $\mathbf{H}(F_{0,v})$ is viewed as a natural subgroup of $\mathbf{H}(E_0 \otimes_{F_0} F_{0,v})$.

Proof. We only prove (1) since the proof applies verbatim to (2). Note that the group $K^\circ_{\mathbf{H}, v}$ is generated by the matrices

$$m(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad n(b) := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad n^-(b') := \begin{pmatrix} 1 & \\ b' & 1 \end{pmatrix},$$

where $a \in O^\times_{F_0, v}$, $b \in \delta^{-1}_{F_0/\mathbb{Q}, v}$, $b' \in \delta_{F_0/\mathbb{Q}, v}$. The invariance under such $m(a)$ is clear. Note that our function $\otimes_{w|v} \phi'_{w, c_w}$ defined by (7.4) for a generator $c_w \in E^\times_{0,w}$ of the fractional ideal $\delta^{-1}_{E_0/F_0, w}$ is then invariant under $n(b)$ when $\psi_v(\mathrm{tr}_{E_0/F_0}(b\,(c_w)_{w|v})) = 1$. This equality is equivalent to $\psi_{\mathbb{Q}_p}(\mathrm{tr}_{E_0/\mathbb{Q}}(b\,(c_w)_{w|v})) = 1$, which holds if $b\delta^{-1}_{E_0/F_0, w} \subseteq \delta^{-1}_{E_0/\mathbb{Q}, w}$ for every $w \mid v$, i.e. if $b \in \delta^{-1}_{F_0/\mathbb{Q}, v}$. The invariance for such $n^-(b')$ is proved similarly by considering the Fourier transform $\widehat{\phi}'_{w, c_w}$, which is $1_{c_w^{-1} O_{E_0, w} \times O_{E_0, w}}$ up to a constant multiple. ∎

Proposition 7.8. *Let $\phi' = \phi'_\mathfrak{d} \otimes \phi'^\mathfrak{d}$ be as above and suppose that $\phi'_\mathfrak{d}$ is $K_{\mathbf{H}, \mathfrak{d}}$-invariant under the Weil representation. Then the function*

$$\mathbf{H}(\mathbb{A}_0) \ni h \mapsto \partial J(h, \phi')$$

lies in $\mathcal{A}_\infty(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)$ with $K_{\mathbf{H}} = K_{\mathbf{H}, \mathfrak{d}} \times K^{\mathfrak{d}, \circ}_{\mathbf{H}}$.

Proof. By Theorem 7.1, ∂J is smooth. The weight n condition follows from Proposition 7.4. The $K_{\mathbf{H}}$-invariance follows from Lemma 7.7. ∎

Proof of Theorem 6.1. Given $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))$ as in Theorem 6.1, choose a matching function $\phi' \in \mathcal{S}(V'(\mathbb{A}_{E_0}))$ as in Theorem 7.6. Define

$$f(h) := -2\tau(Z^{\mathbb{Q}})[\mathbf{E}:E] \cdot \partial J(h, \phi'), \quad h \in \mathbf{H}(\mathbb{A}_0).$$

Then $f \in \mathcal{A}_{\infty}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)$ by Proposition 7.8 and this is also the space that appears in Theorem 6.1. Restricting to $\mathbf{H}(F_{0,\infty})$, Theorem 7.6 provides the Fourier expansion

$$f(h_\infty) = \sum_{\xi \in F_0} \Big(\varepsilon_\xi + \sum_{\mathrm{tr}_{E_0/F_0}(\zeta)=\xi} \widehat{\deg}\, \widehat{\mathcal{C}}^{\mathbf{K}}(\zeta, h_\infty, \phi) \Big) \cdot W_\xi^{(n)}(h_\infty)$$

for correcting scalars $\varepsilon_\xi \in \sum_{\ell \mid \mathfrak{d}} \mathbb{Q} \log \ell$ when $\xi \neq 0$, resp. $\varepsilon_0 \in \mathbb{R}$. The inner sum equals

$$\langle \widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi),\ \mathcal{C} \rangle$$

by Proposition 6.8, and the proof of Theorem 6.1 is complete. ∎

Part III
Application to the AFL

8. Orbital integrals

In this paper, we mainly consider the semi-Lie algebra version of the AFL and hence require the analytic notions from [35] only in this setting. We refer to [35, Section 2] and [34] for the group setting and its relation to the definitions here.

8.1. Orbit matching

Consider the symmetric space

$$S_n := \{ \gamma \in \mathrm{Res}_{F/F_0} \mathrm{GL}_n \mid \gamma\overline{\gamma} = 1_n \}, \tag{8.1}$$

and the F_0-vector space

$$V_n' = F_0^n \times (F_0^n)^*, \tag{8.2}$$

where $(F_0^n)^* = \mathrm{Hom}_{F_0}(F_0^n, F_0)$ denotes the F_0-linear dual space. For convenience we will identify F_0^n (resp. $(F_0^n)^*$) with the space of column vectors (resp. row vectors). With the tautological pairing, we will view V_n' as an $(F_0 \times F_0)/F_0$-hermitian space. Let

$$G' := \mathrm{GL}_{n,F_0},$$

and consider the (diagonal) action of G' on the product $S_n \times V_n'$,

$$h \cdot (\gamma, (u_1, u_2)) = (h^{-1}\gamma h, (h^{-1}u_1, u_2 h)).$$

Next, whenever V is an F/F_0-hermitian space of dimension n, we let $U(V)$ act diagonally on the product $U(V) \times V$. (Here, we view V as a $2n$-dimensional vector space over F_0.)

There is a natural bijection of orbit spaces of *regular semisimple* elements (see [35, Section 2.2])

$$\coprod_V [(U(V) \times V)(F_0)]_{rs} \xrightarrow{\sim} [(S_n \times V_n')(F_0)]_{rs}, \tag{8.3}$$

where the disjoint union runs over the set of isometry classes of F/F_0-hermitian spaces V of dimension n. Here the left (resp. right) hand side denotes the orbits under the action of the group $U(V)(F_0)$ (resp. $GL_n(F_0)$). The bijection is called the *matching relation* between regular semisimple orbits and denoted by $(g, u) \leftrightarrow (\gamma, u')$.

For $\alpha \in F[t]_{\deg n}^\circ$, we denote by $S_n(\alpha)$ the subscheme of S_n consisting of elements with characteristic polynomial α. For $\xi \in F_0$, we let $V_{n,\xi}'$ denote the subscheme of V_n' defined by $u_2 u_1 = \xi$. We denote by $[(S_n(\alpha) \times V_n')(F_0)]$ and $[(S_n(\alpha) \times V_{n,\xi}')(F_0)]$ the sets of $GL_n(F_0)$-orbits. Similar notation applies to unitary groups.

8.2. Orbital integral matching: Smooth transfer

We recall the definition of orbital integrals from [35, Section 11]; see also [27, Section 2.2]. Now let F/F_0 be a quadratic extension of local fields of characteristic zero (the split $F = F_0 \times F_0$ is similar and simpler). Let

$$\eta = \eta_{F/F_0} \colon F_0^\times \to \{\pm 1\}$$

be the quadratic character associated to F/F_0 by local class field theory. Here and thereafter we will also denote by η the character $\eta \circ \det$ of $GL_n(F_0)$.

For $(\gamma, u') \in (S_n \times V_n')(F_0)_{rs}$, $\Phi' \in S((S_n \times V_n')(F_0))$, and $s \in \mathbb{C}$, we define the orbital integral

$$\mathrm{Orb}((\gamma, u'), \Phi', s) := \int_{GL_n(F_0)} \Phi'(h \cdot (\gamma, u'))|\det h|^s \eta(h) \, dh, \tag{8.4}$$

with special values

$$\mathrm{Orb}((\gamma, u'), \Phi') := \omega(\gamma, u') \, \mathrm{Orb}((\gamma, u'), \Phi', 0), \tag{8.5}$$

$$\partial \mathrm{Orb}((\gamma, u'), \Phi') := \omega(\gamma, u') \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}((\gamma, u'), \Phi', s). \tag{8.6}$$

Here, $\omega(\gamma, u') \in \{\pm 1\}$ is the so-called transfer factor from [35, (2.13)]. We emphasize that, unlike in [34], the transfer factor is already included in the special values here.

On the unitary side, for $(g, u) \in (U(V) \times V)(F_0)_{rs}$ and $\Phi \in S((U(V) \times V)(F_0))$, we define

$$\mathrm{Orb}((g, u), \Phi) := \int_{U(V)(F_0)} \Phi(h \cdot (g, u)) \, dh. \tag{8.7}$$

For simplicity let us now assume that F is non-archimedean. Then there are exactly two isometry classes of F/F_0-hermitian spaces with dimension n, denoted by V_0 and V_1.

When F/F_0 is unramified, we will assume that V_0 has a self-dual lattice. Then the orbit bijection (8.3) specializes to

$$[(\mathrm{U}(V_0) \times V_0)(F_0)]_{\mathrm{rs}} \sqcup [(\mathrm{U}(V_1) \times V_1)(F_0)]_{\mathrm{rs}} \xrightarrow{\sim} [(S_n \times V'_n)(F_0)]_{\mathrm{rs}}.$$

Definition 8.1. A function $\Phi' \in \mathcal{S}((S_n \times V_n)(F_0))$ and a pair of functions $(\Phi_0, \Phi_1) \in \mathcal{S}((\mathrm{U}(V_0) \times V_0)(F_0)) \times \mathcal{S}((\mathrm{U}(V_1) \times V_1)(F_0))$ are (smooth) *transfers* of each other if for each $i \in \{0, 1\}$ and each $(g, u) \in (\mathrm{U}(V_i) \times V_i)(F_0)_{\mathrm{rs}}$,

$$\mathrm{Orb}((g, u), \Phi_i) = \mathrm{Orb}((\gamma, u'), \Phi') \tag{8.8}$$

whenever $(\gamma, u') \in (S_n \times V'_n)(F_0)_{\mathrm{rs}}$ matches (g, u).

Definition 8.2. For a fixed $V \in \{V_0, V_1\}$ and a regular semisimple $\alpha \in F[t]^{\circ}_{\deg n}$, we say that Φ' *partially* (relative to α) *transfers* to $\Phi \in \mathcal{S}(\mathrm{U}(V) \times V)(F_0)$ if we only require the equality (8.8) in Definition 8.1 to hold for matching orbits $(\gamma, u') \in (S_n(\alpha) \times V'_n)(F_0)_{\mathrm{rs}}$ and $(g, u) \in (\mathrm{U}(V)(\alpha) \times V)(F_0)_{\mathrm{rs}}$; and $\mathrm{Orb}((\gamma, u'), \Phi') = 0$ for any other $(\gamma, u') \in (S_n(\alpha) \times V'_n)(F_0)_{\mathrm{rs}}$.

Smooth transfers exist for p-adic fields and commute with the Weil representation (see [35, Appendix A]). For archimedean local fields, a specific partial transfer of the Gaussian test functions has been constructed in [35, Section 12].

8.3. Modular analytic generating functions

Now we return to the global situation. Our goal in this subsection is to define an analytic generating series (of orbital integrals) that will match the generating series defined later by intersection numbers. Our proof of the AFL conjecture will be based on a comparison of the two generating series.

Recall from Section 5.1 that we have fixed an irreducible $\alpha \in F[t]^{\circ}_{\deg n}$ such that the field $E = F[t]/(\alpha)$ is a CM extension of a totally real subfield E_0. Then $S_n(\alpha)(F_0)$ consists of exactly one $G'(F_0)$-orbit and we fix a representative $\gamma \in S_n(\alpha)(F_0)$.

Via the action of E_0 induced by γ, the vector space V'_n carries the structure of a rank 1 free $(E_0 \times E_0)/E_0$-hermitian module. Let

$$\mathfrak{q}' \colon V'_n \to E_0 \tag{8.9}$$

be the associated quadratic form over E_0.

Now, for every $v \mid \infty$, we fix the archimedean $\Phi'_v \in \mathcal{S}((S_n \times V')(F_{0,v}))$ to be the (partial) Gaussian test function constructed in [35, (12.10), (12.11)] relative to γ. For $h \in \mathbf{H}(\mathbb{A}_0)$, $\Phi' = \otimes_v \Phi'_v \in \mathcal{S}((S_n \times V')(\mathbb{A}_0))$ with Φ'_v the fixed Gaussian test function for $v \mid \infty$, and $s \in \mathbb{C}$, one defines a regularized integral in *loc. cit.*,

$$J(h, \Phi', s) = \int_{[G']} \Bigg(\sum_{(\gamma, u') \in (S_n(\alpha) \times V')(F_0)} r(g)\omega(h)\Phi'(\gamma, u') \Bigg) |\det(g)|^s_{F_0} \eta(g)\, dg. \tag{8.10}$$

Here the Weil representation ω of $\mathbf{H}(\mathbb{A}_0)$ is through the factor V' with its natural quadratic form. By [35, Theorem 12.14], the function $\mathbf{H}(\mathbb{A}_0) \times \mathbb{C} \ni (h, s) \mapsto J(h, \Phi', s)$ is smooth in (h, s), entire in $s \in \mathbb{C}$, and left invariant under $\mathbf{H}(F_0)$. There is an expansion according to the orbits

$$
J(h, \Phi', s) = \sum_{\substack{(\gamma, u') \in [(S_n(\alpha) \times V')(F_0)] \\ u' \neq 0}} \mathrm{Orb}((\gamma, u'), \omega(h)\Phi', s)
$$

$$
= \mathrm{Orb}((\gamma, 0_{\pm}), \omega(h)\Phi', s) + \sum_{(\gamma, u') \in [(S_n(\alpha) \times V')(F_0)]_{\mathrm{rs}}} \mathrm{Orb}((\gamma, u'), \omega(h)\Phi', s). \tag{8.11}
$$

Here we refer to [35, Section 12.6] for the regularization defining $\mathrm{Orb}((\gamma, 0_{\pm}), \omega(h)\Phi', s)$, and the orbit of $(\gamma, 0)$ has no contribution essentially because the character η is non-trivial when restricted to its stabilizer (see the proof of [35, Theorem 12.9]). For regular semisimple (γ, u') and decomposable $\Phi' = \otimes_v \Phi'_v \in \mathcal{S}((S_n \times V')(\mathbb{A}_0))$, we have

$$
\mathrm{Orb}((\gamma, u'), \Phi', s) := \prod_{v \in \Sigma_{F_0}} \mathrm{Orb}((\gamma, u'), \Phi'_v, s), \tag{8.12}
$$

where the local orbital integrals are defined by (8.4). Then, for $\xi \in F_0^{\times}$, the ξ-th Fourier coefficient of $J(\cdot, \Phi', s)$ (see (1.11)) is equal to

$$
\sum_{(\gamma, u') \in [(S_n(\alpha) \times V'_{\xi})(F_0)]} \mathrm{Orb}((\gamma, u'), \omega(h)\Phi', s). \tag{8.13}
$$

8.4. The decomposition of the special value at $s = 0$

We set the special value at $s = 0$,

$$
J(h, \Phi') := J(h, \Phi', 0).
$$
$$
\mathrm{Orb}((\gamma, u'), \omega(h)\Phi') := \mathrm{Orb}((\gamma, u'), \omega(h)\Phi', 0).
$$

Then the decomposition (8.11) specializes to

$$
J(h, \Phi') = \sum_{\substack{(\gamma, u') \in [(S_n(\alpha) \times V')(F_0)] \\ u' \neq 0}} \mathrm{Orb}((\gamma, u'), \omega(h)\Phi'). \tag{8.14}
$$

We set the special value of the first derivative at $s = 0$,

$$
\partial J(h, \Phi') := \frac{d}{ds}\Big|_{s=0} J(h, \Phi', s),
$$
$$
\partial \mathrm{Orb}((\gamma, u'), \Phi'_v) := \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}((\gamma, u'), \Phi'_v, s). \tag{8.15}
$$

Lemma 8.3. *Suppose that Φ' is $K_{\mathbf{H}}$-invariant under the Weil representation. Then we have $\partial J(-, \Phi') \in \mathcal{A}_{\infty}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)$.*

Proof. By [35, Theorem 12.14], the function $\mathbf{H}(\mathbb{A}_0) \times \mathbb{C} \ni (h, s) \mapsto J(h, \Phi', s)$ is smooth. It follows that $\partial J(-, \Phi')$ is smooth. ∎

Now we introduce

$$\partial J_v(h, \Phi') := \partial J_v(\omega(h)\Phi'), \quad \text{where}$$

$$\partial J_v(\Phi') := \sum_{\substack{(\gamma, u') \in [(S_n(\alpha) \times V')(F_0)] \\ u' \neq 0}} \partial \mathrm{Orb}((\gamma, u'), \Phi'_v) \cdot \mathrm{Orb}((\gamma, u'), \Phi'^v). \tag{8.16}$$

Then by Leibniz's rule, we obtain a decomposition

$$\partial J(h, \Phi') = \partial \mathrm{Orb}(0_\pm, \omega(h)\Phi') + \sum_v \partial J_v(h, \Phi'), \tag{8.17}$$

where the nilpotent term $\partial \mathrm{Orb}(0_\pm, \omega(h)\Phi')$ is defined in [35, Section 12.7] and we do not need its precise form in this paper.

8.5. *Preparation for the comparison*

Let V be the n-dimensional F/F_0-hermitian space we used to define the Shimura variety $\mathrm{Sh}_{\widetilde{K}}(\widetilde{G}, \{\widetilde{h}\})$ in Section 2.1.

Let $\Phi = \otimes_{v < \infty} \Phi_v \in \mathcal{S}((\mathrm{U}(V) \times V)(\mathbb{A}_{0,f}))$ be a pure tensor. Let $\Phi' = \otimes_v \Phi'_v \in \mathcal{S}((S_n \times V'_n)(\mathbb{A}_0))$ be a pure tensor such that

- for every $v \mid \infty$, Φ'_v is the fixed partial Gaussian test function;
- for every non-archimedean v, Φ'_v partially (relative to α) transfers to Φ_v (Definition 8.2).

If v is a place of F_0 that splits in F (necessarily non-archimedean), then $\partial J_v(h, \Phi') = 0$. If v is non-split (including the archimedean places), let $V^{(v)}$ be the v-nearby hermitian space of V. We recall that $V^{(v)} \simeq V$ if v is the unique archimedean place φ_0 where V has signature $(n-1, 1)$ (see Section 2.1), and otherwise $V^{(v)}$ is characterized by the following conditions:

- for all w other than v and φ_0, we have $V_w^{(v)} \simeq V_w$;
- $V_{\varphi_0}^{(v)}$ is positive definite;
- $V_v^{(v)}$ has signature $(n-1, 1)$ if v is archimedean.

Then the subsum of regular semisimple terms in $\partial J_v(h, \Phi')$ (see (8.16)) is a sum over orbits $(\gamma, u') \in [(S_n(\alpha) \times V')(F_0)]_{\mathrm{rs}}$ matching orbits in $[(\mathrm{U}(V^{(v)}) \times V^{(v)})(F_0)]_{\mathrm{rs}}$. Moreover, there is a Fourier expansion (see (1.11))

$$\partial J_v(h, \Phi') = \sum_{\xi \in F_0} \partial J_v(\xi, h, \Phi'), \tag{8.18}$$

where

$$\partial J_v(\xi, h, \Phi') = \sum_{\substack{(\gamma, u') \in [(S_n(\alpha) \times V'_\xi)(F_0)] \\ u' \neq 0}} \partial \mathrm{Orb}((\gamma, u'), \omega(h_v)\Phi'_v) \cdot \mathrm{Orb}((\gamma, u'), \omega(h^v)\Phi'^v). \tag{8.19}$$

Then we set the value at $h = 1$ normalized by the Whittaker function at the archimedean place:

$$\partial J_v(\xi, \Phi') := W_\xi^{(n)}(1)^{-1} \, \partial J_v(\xi, 1, \Phi'), \tag{8.20}$$

where $W_\xi^{(n)}(h_\infty) := \prod_{v|\infty} W_{v,\xi}^{(n)}(h_v)$ and each local factor $W_{v,\xi}^{(n)}$ is defined by (1.7).

Now let $v \nmid \mathfrak{d}$ be a non-archimedean non-split place. By (8.18) and (8.19), and the fact that Φ'_∞ is a (partial) Gaussian test function, we have $\partial J_v(\xi, h, \Phi') = 0$ unless $\xi \geq 0$ and

$$\partial J_v(h, \Phi') = \sum_{\xi \in F_0, \xi \geq 0} \partial J_v(\xi, \omega(h_f)\Phi') W_\xi^{(n)}(h_\infty). \tag{8.21}$$

Moreover, (8.20) becomes

$$\partial J_v(\xi, \Phi') = \sum_{\substack{(\gamma, u') \in [(S_n(\alpha) \times V'_\xi)(F_0)] \\ u' \neq 0}} \partial \mathrm{Orb}((\gamma, u'), \Phi'_v) \cdot \mathrm{Orb}((\gamma, u'), \Phi'^{v,\infty}). \tag{8.22}$$

Finally, we introduce

$$\partial J(\xi, \Phi') := \sum_v \partial J_v(\xi, \Phi'), \tag{8.23}$$

where the local terms vanish for all but finitely many places v of F_0 (for a fixed pair (ξ, Φ')).

9. Intersection numbers and AFL

In this section we recall the relevant RZ spaces, the local analog of CM cycles and local KR divisors, and the statement of the AFL conjecture. Then we globalize the intersection problem to the Shimura varieties we introduced in Part I. We also state an application to the arithmetic intersection conjecture of Rapoport, Smithling and the second author [28], assuming the AFL conjecture that will be proved in the next section.

9.1. Unitary RZ spaces

Let F/F_0 be an unramified quadratic extension of p-adic local fields with p odd and let $n \geq 1$. In this section, we recall the definition of the Rapoport–Zink formal moduli scheme $\mathcal{N}_n = \mathcal{N}_{n,F/F_0}$ associated to the unitary group for the quasi-split n-dimensional hermitian F-vector space; see [27, Section 4] and [35, Section 3].

Let $\breve{F}$ denote the completion of a maximal unramified extension of F. For an Spf $O_{\breve{F}}$-scheme S (i.e. an $O_{\breve{F}}$-scheme on which p is locally nilpotent), we consider triples (X, ι, λ), where

- X is a p-divisible group of absolute height $2nd$ and dimension n over S, where $d := [F_0 : \mathbb{Q}_p]$,
- ι is an action of O_F such that the induced action of O_{F_0} on Lie X is via the structure morphism $O_{F_0} \to \mathcal{O}_S$,
- λ is a principal (O_{F_0}-relative) polarization.

Hence $(X, \iota|_{O_{F_0}})$ is a strict O_{F_0}-module of relative height $2n$ and dimension n. We require that the Rosati involution Ros_λ induces on O_F the non-trivial Galois automorphism in $\mathrm{Gal}(F/F_0)$, denoted by $O_F \ni a \mapsto \bar{a}$, and that the *Kottwitz condition* of signature $(n-1, 1)$ is satisfied, i.e.

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}\, X; T) = (T-a)^{n-1}(T-\bar{a}) \in \mathcal{O}_S[T] \quad \text{for all } a \in O_F. \tag{9.1}$$

An isomorphism $(X, \iota, \lambda) \xrightarrow{\sim} (X', \iota', \lambda')$ between two such triples is an O_F-linear isomorphism $\varphi \colon X \xrightarrow{\sim} X'$ such that $\varphi^*(\lambda') = \lambda$.

Over the residue field $\mathbb{F}$ of $O_{\breve{F}}$, there is a triple $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ such that $\mathbb{X}_n$ is supersingular, unique up to O_F-linear quasi-isogeny compatible with the polarization. We fix such a triple and call it the *framing object*. Then $\mathcal{N}_n$ by definition represents the functor over $\mathrm{Spf}\, O_{\breve{F}}$ that associates to each S the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ over S, where the last entry is an O_F-linear quasi-isogeny of height zero defined over the special fiber $\bar{S} := S \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathrm{Spec}\, \mathbb{F}$,

$$\rho \colon X \times_S \bar{S} \to \mathbb{X}_n \times_{\mathrm{Spec}\, \mathbb{F}} \bar{S},$$

such that $\rho^*((\lambda_{\mathbb{X}_n})_{\bar{S}}) = \lambda_{\bar{S}}$. The map ρ is called the *framing*. The formal scheme $\mathcal{N}_n$ is formally locally of finite type and formally smooth over $\mathrm{Spf}\, O_{\breve{F}}$ of relative dimension $n-1$.

The group of quasi-automorphisms of the framing object is

$$\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n}) = \{g \in \mathrm{End}_F^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}) \mid g^\vee \circ \lambda_{\mathbb{X}_n} \circ g = \lambda_{\mathbb{X}_n}\}.$$

The condition $g^\vee \circ \lambda_{\mathbb{X}_n} \circ g = \lambda_{\mathbb{X}_n}$ may also be formulated as $gg^* = \mathrm{id}$, where $g \mapsto g^* = \mathrm{Ros}_{\lambda_{\mathbb{X}_n}}(g)$ denotes the Rosati involution. Then $\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ acts on $\mathcal{N}_n$ by changing the framing:

$$g \cdot (X, \iota, \lambda, \rho) = (X, \iota, \lambda, g \circ \rho).$$

Another description is as follows. Taking $n = 1$, there is a unique triple $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ over $\mathbb{F}$ with signature $(1, 0)$. Set

$$\mathbb{V}_n := \mathrm{Hom}_{O_F}^\circ(\mathbb{E}, \mathbb{X}_n), \tag{9.2}$$

which is an n-dimensional hermitian F-vector space with respect to the hermitian form

$$\langle x, y \rangle = \lambda_{\mathbb{E}}^{-1} \circ y^\vee \circ \lambda_{\mathbb{X}_n} \circ x \in \mathrm{End}_F^\circ(\mathbb{E}) \simeq F.$$

It is the unique (up to isomorphism) n-dimensional hermitian space that does *not* contain a self-dual O_F-lattice. Then there is a natural isomorphism

$$\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n}) \cong U(\mathbb{V}_n)(F_0), \tag{9.3}$$

where $\mathrm{Aut}^\circ(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ acts by composition on $\mathbb{V}_n$.

9.2. *Local intersection numbers*

Now we introduce the local intersection numbers $\mathrm{Int}(g, u)$. In [16], Kudla and Rapoport have defined for every non-zero $u \in \mathbb{V}_n$ a certain divisor $\mathcal{Z}(u)$ on $\mathcal{N}_n$, the so-called (local) *KR divisor*. For its definition, note that $\mathcal{N}_1 \cong \mathrm{Spf}\, O_{\breve{F}}$, so $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ deforms to a unique triple $(\mathcal{E}, \iota_{\mathcal{E}}, \lambda_{\mathcal{E}})$ over $O_{\breve{F}}$, called its *canonical lift*. (This is the universal object over $\mathcal{N}_1$ with Galois conjugated O_F-action.) Then $\mathcal{Z}(u)$ is defined as the locus where the quasi-homomorphism $u \colon \mathbb{E} \to \mathbb{X}_n$ lifts to a homomorphism from $\mathcal{E}$ to the universal object over $\mathcal{N}_n$. By [16, Proposition 3.5], $\mathcal{Z}(u)$ is a relative divisor (or empty). It is the local analog of the global divisor considered in Section 2.3. It follows from the definition that if $g \in \mathrm{U}(\mathbb{V}_n)(F_0)$, then

$$g\mathcal{Z}(u) = \mathcal{Z}(gu). \tag{9.4}$$

For simplicity we will write $\mathcal{N}_n \times \mathcal{N}_n$ for the fiber product $\mathcal{N}_n \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_n$. For $g \in \mathrm{U}(\mathbb{V}_n)(F_0)$, let $\Gamma_g \subseteq \mathcal{N}_n \times \mathcal{N}_n$ be the graph of the automorphism of $\mathcal{N}_n$ induced by g. The fixed point locus of g is defined as the intersection

$$\mathcal{N}_n^g := \Gamma_g \cap \Delta_{\mathcal{N}_n}, \tag{9.5}$$

viewed as a closed formal subscheme of $\mathcal{N}_n$. We also form the *derived fixed point locus*, denoted by $\mathbb{L}\mathcal{N}_n^g$, i.e. the derived tensor product

$$\mathbb{L}\mathcal{N}_n^g := \Gamma_g \overset{\mathbb{L}}{\cap} \Delta_{\mathcal{N}_n} := \mathcal{O}_{\Gamma_g} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{N}_n \times \mathcal{N}_n}} \mathcal{O}_{\Delta_{\mathcal{N}_n}} \tag{9.6}$$

viewed as an element in $K_0^{\mathcal{N}_n^g}(\mathcal{N}_n)$ as defined in [35, Appendix B]. The fixed point locus and its derived version are the local analogs of the global CM cycle and its derived version in Section 5.1.

For a pair $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F_0)_{\mathrm{rs}}$, we now set

$$\mathrm{Int}(g, u) := \langle \mathcal{Z}(u), \mathbb{L}\mathcal{N}_n^g \rangle_{\mathcal{N}_n} := \chi(\mathcal{N}_n, \mathcal{O}_{\mathcal{Z}(u)} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{N}_n}} \mathbb{L}\mathcal{N}_n^g). \tag{9.7}$$

Here, for a finite complex $\mathcal{F}$ of coherent $\mathcal{O}_{\mathcal{N}_n}$-modules, we define its Euler–Poincaré characteristic as

$$\chi(\mathcal{N}_n, \mathcal{F}) = \sum_{i,j} (-1)^{i+j} \mathrm{len}_{O_{\breve{F}}} H^j(\mathcal{N}_n, H_i(\mathcal{F}))$$

if the lengths are all finite.

When (g, u) is regular semisimple, the intersection $\mathcal{Z}(u) \cap \mathcal{N}_n^g$ is a proper *scheme* over $\mathrm{Spf}\, O_{\breve{F}}$ and hence the right hand side of (9.7) is finite. The number $\mathrm{Int}(g, u)$ depends only on the $\mathrm{U}(\mathbb{V}_n)(F_0)$-orbit of (g, u). There is an equivalent definition that does not involve the derived fixed point locus $\mathbb{L}\mathcal{N}_n^g$ (see [35, Remark 3.1]):

$$\mathrm{Int}(g, u) = \chi(\mathcal{N}_n \times \mathcal{N}_n, \mathcal{O}_{\Gamma_g} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{N}_n \times \mathcal{N}_n}} \mathcal{O}_{\Delta_{\mathcal{Z}(u)}}). \tag{9.8}$$

9.3. The AFL conjecture

The Arithmetic Fundamental Lemma conjecture states that $\mathrm{Int}(g, u)$ equals a derivative of the orbital defined in (8.4).

Conjecture 9.1 (AFL, semi-Lie algebra version). *Suppose that* $(\gamma, u') \in (S_n \times V_n')(F_0)_{\mathrm{rs}}$ *matches the element* $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F_0)_{\mathrm{rs}}$. *Then*

$$\partial \mathrm{Orb}((\gamma, u'), 1_{(S_n \times V_n')(O_{F_0})}) = - \mathrm{Int}(g, u) \cdot \log q.$$

The case of a fixed u with $(u, u) \in O_{F_0}^\times$ has been studied first historically (cf. [33]), and is known as the *group version* of the AFL; see also [35, Conjecture 3.2]. In that case there is a natural isomorphism

$$\mathcal{Z}(u) \simeq \mathcal{N}_{n-1}.$$

By [35, Proposition 4.12], the intersection number (9.8) is equal to the one considered in [33],

$$\chi(\mathcal{N}_{n-1} \times \mathcal{N}_n, \mathcal{O}_{(1,g) \cdot \Delta_{\mathcal{N}_{n-1}}} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{N}_{n-1} \times \mathcal{N}_n}} \mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}}),$$

the orbital integral reduces to the one in *loc. cit.* as well, and hence the group version for $\mathbb{V}_{n+1}$ is equivalent to Conjecture 9.1, at least if $q_{F_0} \geq n + 1$.

9.4. Global intersection numbers

We now recall the relation of local and global intersection numbers which relies on the non-archimedean uniformization for $\mathcal{M}$. We resume the notation from Section 2. Let $\nu \nmid \mathfrak{d}$ be a non-archimedean place of $\mathbf{F}$ and set $v = \nu|_{F_0}$. Assume that v is inert in F. There is a uniformization along the basic locus (see [28, proof of Theorem 8.15])

$$\widehat{\mathcal{M}_{O_{\breve{\mathbf{F}}_\nu}}} = \widetilde{G}^{(v)}(\mathbb{Q}) \backslash [\mathcal{N}' \times \widetilde{G}(\mathbb{A}_f^p)/K_{\widetilde{G}}^p]. \tag{9.9}$$

Here the left hand side is the formal completion of the base change $\mathcal{M} \otimes \mathcal{O}_{\breve{\mathbf{F}}_\nu}$ along the basic locus of the geometric special fiber $\mathcal{M} \otimes \overline{\mathbb{F}}_\nu$. The basic locus here is the closed subspace consisting of $(A_0, A, \overline{\eta})$ where $A[v^\infty]$ is a supersingular p-divisible group. On the right hand side, $\widetilde{G}^{(v)}$ is the analog of $\widetilde{G}$, but for the nearby hermitian space $V^{(v)}$. The formal scheme $\mathcal{N}'$ is an RZ space of PEL type in [30] whose precise definition we omit. We now make the additional assumption that the CM type Φ is unramified at p (see Definition 6.2). Then the comparison isomorphism [23, Theorem 3.1] applies to $\mathcal{N}'$ and (9.9) may be rewritten as

$$\widehat{\mathcal{M}_{O_{\breve{\mathbf{F}}_\nu}}} = \widetilde{G}^{(v)}(\mathbb{Q}) \backslash [\mathcal{N}_{n, F_v/F_{0,v}} \times \widetilde{G}(\mathbb{A}_f^v)/\widetilde{K}^v]. \tag{9.10}$$

Here $\mathcal{N}_{n, F_v/F_{0,v}}$ is the RZ space from Section 9.1 for the quadratic extension $F_v/F_{0,v}$ and, by abuse of notation,

$$\widetilde{G}(\mathbb{A}_f^v)/\widetilde{K}^v = \widetilde{G}(\mathbb{A}_f^p)/\widetilde{K}^p \times (Z^{\mathbb{Q}}(\mathbb{Q}_p)/K_{Z^{\mathbb{Q}}, p}) \times \prod_{w|p, w \neq v} G(F_{0,w})/K_w,$$

where the product is over places w of F_0 above p. (In [28] it is assumed that v is unramified over $\mathbb{Q}$ which explains why the assumption on the CM type Φ does not figure explicitly.)

Now we come to the global intersection numbers. Let $\alpha \in F[t]^\circ_{\deg n}$ be a fixed irreducible polynomial, $\varphi = \varphi_{\mathfrak{d}} \otimes 1_{K^{\mathfrak{d}}} \in \mathcal{S}(K \backslash G(\mathbb{A}_{0,f})/K)$ and $\phi = \phi_{\mathfrak{d}} \otimes 1_{\widehat{\Lambda}^{\mathfrak{d}}} \in \mathcal{S}(V(\mathbb{A}_{0,f}))$. Set $\Phi := \varphi \otimes \phi$. For each $\xi \in F_0$, equation (4.5) defines an intersection number of $\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi, \phi)$ and $^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)$ in $\mathbb{R}_{\mathfrak{d}}$. (Recall that $^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)$ is defined in Definition 5.2.) However, due to the passage to $\widehat{\mathrm{Ch}}^1$ in (4.5), the information of the individual place-by-place contributions is lost. We assume $\xi \neq 0$ from now on. Then we can work with the representing divisor $\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi, \phi)$ directly to define a more specific intersection number

$$\mathrm{Int}(\xi, \Phi) := \frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{F} : F]} \big(\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi, \phi), \, ^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi) \big) \in \mathbb{R} \qquad (9.11)$$

that lifts the $\mathbb{R}_{\mathfrak{d}}$-valued intersection number from (4.5). It is defined as a sum of an archimedean and a non-archimedean part. The *archimedean contribution* is defined as usual by evaluating the Green function of $\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi, \phi)$ on $^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)_{\mathbf{F}}$. This is well-defined since the relations in the definition of $\mathcal{Z}_1(\mathcal{M})$ are supported in special fibers only (see Definition 4.2), so that the horizontal part of $^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)$ is unambiguous. The *non-archimedean contribution* is the arithmetic Euler–Poincaré characteristic

$$\chi \big(\mathcal{M}, \mathcal{O}_{\mathcal{Z}(\xi, \phi)} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{M}}} {}^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi) \big). \qquad (9.12)$$

The cycles do not intersect in the generic fiber by [35, Theorem 9.2 (i)], so (9.12) is a finite number. With this definition of $\mathrm{Int}(\xi, \Phi)$, we even obtain a well-defined place-by-place decomposition

$$\mathrm{Int}(\xi, \Phi) = \sum_{v \in \Sigma_{F_0}, \, v \nmid \mathfrak{d}} \mathrm{Int}_v(\xi, \Phi).$$

Note that the local factor vanishes for all but finitely many v (for a fixed pair (ξ, Φ)). The v-term here is the sum of the contributions of all $\nu \in \Sigma_{\mathbf{F}}$ lying above v. Part (b) of the following theorem relies on the non-archimedean uniformization. Also see Theorem 6.9 (1, 2) for the analogous result for elementary CM cycles.

Theorem 9.2 ([35, Theorem 9.4]). *Let v be a non-archimedean place of $\mathbf{F}$ and $v = \nu|_{F_0}$.*

(a) *If v is split in F and $\xi \neq 0$, then $(\mathcal{C}(\alpha, \varphi) \cap \mathcal{Z}(\xi, \phi))_v = \emptyset$. In particular $\mathrm{Int}_v(\xi, \phi)$ $= 0$.*

(b) *Assume v is inert in F and $\xi \neq 0$. Then*

$$\mathrm{Int}_v(\xi, \Phi) = 2 \log q_v \sum_{(g,u) \in [(G^{(v)}(\alpha) \times V^{(v)}_\xi)(F_0)]} \mathrm{Int}(g, u) \cdot \mathrm{Orb}((g, u), \Phi^v). \quad (9.13)$$

Here $G^{(v)} = \mathrm{U}(V^{(v)})$ and $G^{(v)}(\alpha)$ is the subvariety of elements with characteristic polynomial α. Moreover, $\mathrm{Int}(g, u)$ is the local intersection number from (9.7).

There is an analogous result for archimedean ν which is proved via complex uniformization (cf. Theorem 6.9 (3)). Since this is identically the same as [35, Corollary 10.3], we will not repeat it here.

9.5. *Application to diagonal intersection*

In [28], Rapoport, Smithling and the second author study an intersection problem of arithmetic diagonal cycles that is motivated by the Arithmetic Gan–Gross–Prasad Conjecture (see [6,33]). They formulate a conjectural formula for the local contributions to the resulting intersection numbers (see [28, Conjecture 8.13 (i)]). These local terms decompose just as $\mathrm{Int}_v(\xi, \Phi)$ does in (9.13) above. In the hyperspecial case of [28], this reduces their conjecture to the AFL. Anticipating our proof of the AFL for $q_v \geq n$ (in Theorem 10.1 below), we formulate here the implied intersection identity in the diagonal setting.

Let $u \in V$ be a non-isotropic vector with orthogonal complement W. Define the following groups (over $\mathrm{Spec}\,\mathbb{Q}$ and $\mathrm{Spec}\,F_0$, respectively):

$$
\begin{aligned}
H &:= U(W), \quad \text{an algebraic group over } F_0, \\
H^{\mathbb{Q}} &:= \{g \in \mathrm{Res}_{F_0/\mathbb{Q}}\,\mathrm{GU}(W) \mid c(g) \in \mathbb{G}_{\mathrm{m}}\}, \quad c \text{ the similitude factor,} \\
\widetilde{H} &:= Z^{\mathbb{Q}} \times_{\mathbb{G}_{\mathrm{m}}} H^{\mathbb{Q}} \cong Z^{\mathbb{Q}} \times \mathrm{Res}_{F_0/\mathbb{Q}} H, \\
\widetilde{HG} &:= Z^{\mathbb{Q}} \times_{\mathbb{G}_{\mathrm{m}}} H^{\mathbb{Q}} \times_{\mathbb{G}_{\mathrm{m}}} G^{\mathbb{Q}} \cong Z^{\mathbb{Q}} \times \mathrm{Res}_{F_0/\mathbb{Q}} H \times \mathrm{Res}_{F_0/\mathbb{Q}} G.
\end{aligned}
$$

Let $\{h_H\}$ and $\{h_{\widetilde{H}}\}$ be the Shimura data for H and $\widetilde{H}$ defined in Section 2.1 with W instead of V. The product $\{h_{Z^{\mathbb{Q}}} \times h_H \times h_G\}$ defines a Shimura datum for $\widetilde{HG}$. The reflex field for $\widetilde{H}$ and $\widetilde{HG}$ is again $\mathbf{F}$. Let $K_H \subseteq H(\mathbb{A}_{0,f})$ be a level subgroup and let $K_{\widetilde{H}} := K_{Z^{\mathbb{Q}}} \times K_H$ as well as $K_{\widetilde{HG}} := K_{Z^{\mathbb{Q}}} \times K_H \times K_G$. Then, for the respective Shimura varieties over $\mathrm{Spec}\,\mathbf{F}$,

$$
\mathrm{Sh}_{K_{\widetilde{HG}}}(\widetilde{HG}, \{h_{\widetilde{HG}}\}) = \mathrm{Sh}_{K_{\widetilde{H}}}(\widetilde{H}, \{h_{\widetilde{H}}\}) \times_{\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, h_{Z^{\mathbb{Q}}})} \mathrm{Sh}_{K_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\}).
$$

Here we wrote $K_{\widetilde{G}} := \widetilde{K}$ and $\{h_{\widetilde{G}}\} := \{\widetilde{h}\}$ to make the notation more coherent. There is an embedding $H \hookrightarrow G$ that identifies H with the stabilizer of u in G. Recall that $K_G^{\mathfrak{d}} = \mathrm{Stab}(\widehat{\Lambda}^{\mathfrak{d}})$ is the stabilizer of a self-dual $O_F[\mathfrak{d}^{-1}]$-lattice in V. We now impose

- $u \in \widehat{\Lambda}^{\mathfrak{d}}$ with $(u, u) \in O_F[\mathfrak{d}^{-1}]^{\times}$;

- $K_H = H \cap K_G$.

In particular, the data of W and K_H satisfy the good reduction assumptions away from $\mathfrak{d}$ in Section 2.2 and we obtain a smooth integral model $\mathcal{M}_{\widetilde{H}}$ over $O_{\mathbf{F}}[\mathfrak{d}^{-1}]$ for the Shimura variety of $\widetilde{H}$. It parametrizes tuples $(A_0, \iota_0, \lambda_0, \overline{\eta}_0, A, \iota, \lambda, \overline{\eta})$ in the sense of Definition 2.1, but for W instead of V and $n-1$ instead of n. The closed immersion of Shimura varieties for $\widetilde{H}$ and $\widetilde{G}$ extends to a closed immersion of integral models with moduli description

$$
\mathcal{M}_{\widetilde{H}} \to \mathcal{M}_{\widetilde{G}}, \quad (A_0, A, \overline{\eta}) \mapsto (A_0, A_0 \times A, (\mathrm{id}_{A_0} \mapsto u) \times \overline{\eta}). \tag{9.14}
$$

The product

$$\mathcal{M}_{\widetilde{HG}} := \mathcal{M}_{\widetilde{H}} \times_{\mathcal{M}_0} \mathcal{M}_{\widetilde{G}}$$

is a smooth integral model for $\mathrm{Sh}_{K_{\widetilde{HG}}}(\widetilde{HG}, \{h_{\widetilde{HG}}\})$. The definition of the Hecke correspondence (5.1) can be made for $\mathcal{M}_{\widetilde{H}}$ in exactly the same way; it moreover extends by linearity from the case of a coset μ to the Hecke algebra at $\mathfrak{d}$. Taking the product, we obtain for every function of the form

$$\varphi_\mathfrak{d} = 1_{K_{Z^\mathbb{Q},\mathfrak{d}}} \otimes \varphi_{H,\mathfrak{d}} \otimes \varphi_{G,\mathfrak{d}} \in \mathcal{S}(K_{\widetilde{HG},\mathfrak{d}} \backslash \widetilde{HG}(\mathbb{Q}_\mathfrak{d})/K_{\widetilde{HG},\mathfrak{d}})$$

a Hecke operator

$$R(\varphi_\mathfrak{d}): \mathrm{Hk}_{\varphi_\mathfrak{d}} \to \mathcal{M}_{\widetilde{HG}} \times_{\mathcal{M}_0} \mathcal{M}_{\widetilde{HG}}.$$

(Only the component $\varphi_{H,\mathfrak{d}} \otimes \varphi_{G,\mathfrak{d}}$ is used to define the correspondence; the trivial $1_{K_{Z^\mathbb{Q}}}$ acts trivially.) When we write $R(\varphi)$ with $\varphi \in \mathcal{S}(K_{\widetilde{HG}} \backslash \widetilde{HG}(\mathbb{A}_f)/K_{\widetilde{HG}})$ in the following, it is understood that $\varphi = \varphi_\mathfrak{d} \otimes \varphi^\mathfrak{d}$ with $\varphi^\mathfrak{d} = 1_{K_{\widetilde{HG}}^\mathfrak{d}}$ standard and $R(\varphi) = R(\varphi_\mathfrak{d})$. This completes the setting for the definition of the semiglobal intersection number from [28, (8.18)] in the hyperspecial case. Let $v \mid \mathfrak{d}$ be a place of F_0 that is inert in F. We define, for $\nu \in \Sigma_\mathbf{F}$ with $\nu \mid v$,

$$\mathrm{Int}_\nu^\natural(\varphi) := \langle R(\varphi)^* \mathcal{M}_{\widetilde{H}}, \mathcal{M}_{\widetilde{H}} \rangle_{\mathcal{M}_{\widetilde{HG}}, \nu},$$

$$\mathrm{Int}_v(\varphi) := \frac{1}{\tau(Z^\mathbb{Q})[\mathbf{F} : F]} \sum_{\nu \mid v} \mathrm{Int}_\nu^\natural(\varphi). \tag{9.15}$$

Here, the map $\mathcal{M}_{\widetilde{H}} \to \mathcal{M}_{\widetilde{HG}}$ is the graph of (9.14). The following is [28, Conjecture 8.13] in the present notation for v. We refer to [28, Section 7] for the notions of transfer and of Gaussian test function in the present setting.

Theorem 9.3. *Assume that $q_v \geq n$ and that the CM type Φ is unramified at p as in Definition 6.2. Let $\varphi = \varphi_\mathfrak{d} \otimes \varphi^\mathfrak{d}$ be as above with $\varphi^\mathfrak{d}$ standard. Assume that $\varphi_\mathfrak{d}$ is completely decomposed, $\varphi_\mathfrak{d} = \otimes_{w \mid \mathfrak{d}} \varphi_w$, and let $\varphi' = \otimes_w \varphi'_w \in \mathcal{S}(G'(\mathbb{A}_{F_0}))$ be a Gaussian test function such that $\otimes_{w < \infty} \varphi'_w$ is a smooth transfer of φ. Assume that for some prime ℓ prime to v and some place λ above ℓ, both φ_λ and φ'_λ have regular support. Then*

$$\mathrm{Int}_v(\varphi) = -\partial J_v(\varphi').$$

Here $G' = \mathrm{Res}_{F/F_0}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n)$ and ∂J_v is the v-th component of the global distribution ∂J on $G'(\mathbb{A}_{F_0})$ defined in [28, Section 7.2]. (Note that this distribution ∂J_v is related to but not the same as the ∂J_v-distribution defined in Sections 8.3 and 8.4.)

Proof of Theorem 9.3. The statement was already shown for $n \leq 3$ and v unramified over p in [28, Theorem 8.15]. As long as the CM type Φ is unramified, the proof applies for arbitrary n and reduces the statement to the AFL identity. This we prove in the next section whenever $q_v \geq n$. ∎

10. Proof of the AFL

This last section is devoted to the proof of the AFL of Section 9.3, at least for not too small residue cardinality:

Theorem 10.1. *Conjecture* 9.1 *holds under the assumption that the residue cardinality of F_0 is $\geq n$.*

10.1. Globalization

Starting from the data of the AFL identity in question (an unramified quadratic extension of p-adic local fields, the integer n and the regular semisimple pair (g, u)), we choose a quadratic extension F/F_0 of number fields with F CM and F_0 totally real such that

- F_0 has a place v above p which is inert in F and such that $F_v/F_{0,v}$ is the extension of local fields one we would like to prove an AFL identity for;
- all places $w \neq v$ of F_0 above p are split in F.

Next, choose an n-dimensional F/F_0-hermitian vector space V such that

- the signature of V is $(n-1, 1)$ at a unique place $\varphi_0 \colon F_0 \to \mathbb{R}$ and $(n, 0)$ otherwise;
- the localization V_p contains a self-dual $O_{F,p}$-lattice Λ_p.

Let $V^{(v)}$ be the v-nearby positive definite hermitian space of V and choose an isomorphism $V_v^{(v)} \cong \mathbb{V}_n$ to view $(g, u) \in (G^{(v)} \times V^{(v)})(F_{0,v})$. Here we recall that $\mathbb{V}_n$ is defined by (9.2). Note that such an isomorphism is only unique up to $G^{(v)}(F_{0,v})$, so it is really the orbit $G^{(v)}(F_{0,v}) \cdot (g, u)$ that is canonical. The main result of [22] is that the intersection number $\mathrm{Int}(g, u)$ is locally constant (for the p-adic topology) in (g, u). On the analytic side, it is a priori clear that $\partial \mathrm{Orb}(\gamma, u')$ is locally constant in (γ, u'). Moreover, the matching relation preserves p-adic closeness since it is defined through the invariants of (g, u) and (γ, u') (see Section 8.2). It follows that we may replace (g, u) by a suitable p-adically close enough *global* pair with the following additional properties:

- The characteristic polynomial α of g (now an element of $F[t]_{\deg n}^\circ$) is irreducible over F.
- The self-pairing $\xi := \langle u, u \rangle$ (now an element of F_0) is non-zero.
- For all places $w \neq v$ of F_0 above p, the "standard" orbital integral for the orbit of (g, u) does not vanish,

$$\mathrm{Orb}((g, u), 1_{\mathrm{Stab}(\Lambda_w)} \otimes 1_{\Lambda_w}) \neq 0, \tag{10.1}$$

where $\Lambda_w \subseteq \Lambda_p$ is the factor corresponding to w.

Next, we choose a CM type Φ for F which we assume to be unramified at p in the sense of Definition 6.2. For example, one may first choose an unramified p-adic CM type $\Phi \subseteq \mathrm{Hom}(F, \mathbb{C}_p)$ and then transport it via any choice of isomorphism $\mathbb{C} \cong \mathbb{C}_p$. Enlarging $\mathfrak{d}$ while keeping $p \nmid \mathfrak{d}$, we assume that Φ is unramified at all $\ell \nmid \mathfrak{d}$. The choice of Φ also defines an extension of φ_0 to F. Finally, we complete the data so far to a Shimura datum as in Section 2, subject only to the condition that $p \nmid \mathfrak{d}$.

Proposition 10.2 ([35, proof of Lemma 13.7 and Proposition 13.8]). *Assume that for all non-archimedean places w of F_0 with $w \nmid v\mathfrak{d}$,*

$$\mathrm{Orb}((g, u), 1_{K_w} \otimes 1_{\Lambda_w}) \neq 0. \tag{10.2}$$

Fix a compact subset Ω_v of $(G^{(v)} \times V^{(v)})(F_{0,v})$ containing (g, u). Then, after possibly shrinking $K_{\mathfrak{d}}$, there exist choices for $\phi^v = \phi_{\mathfrak{d}} \otimes 1_{\widehat{\Lambda}^{v\mathfrak{d}}} \in \mathcal{S}(V(\mathbb{A}_{0,f}^v))$ and $\varphi^v = \varphi_{\mathfrak{d}} \otimes 1_{K^{v\mathfrak{d}}} \in \mathcal{S}(K^v \backslash G(\mathbb{A}_{0,f}^v)/K^v)$ such that the following are equivalent for $(\widetilde{g}, \widetilde{u}) \in (G^{(v)}(\alpha) \times V_\xi^{(v)})(F_0)$:

(1) *The $G^{(v)}(F_{0,v})$-orbit of $(\widetilde{g}, \widetilde{u})$ in $(G^{(v)} \times V^{(v)})(F_{0,v})$ intersects non-trivially with Ω_v, and the away-from-v orbital integral $\mathrm{Orb}((\widetilde{g}, \widetilde{u}), \Phi^v)$ is non-vanishing with $\Phi^v = \phi^v \otimes \varphi^v$.*

(2) *There is an equality of orbits $G^{(v)}(F_0) \cdot (\widetilde{g}, \widetilde{u}) = G^{(v)}(F_0) \cdot (g, u)$.*

Enlarging $\mathfrak{d}$, we may assume that condition (10.2) is satisfied. Since we already imposed (10.1), we may do so while maintaining $p \nmid \mathfrak{d}$. For example, $\mathfrak{d}$ now also contains all places $w \neq v$ where α or ξ are not integral. We may find and fix a compact Ω_v as in the proposition such that if an orbit in $(G^{(v)}(\alpha) \times V_\xi^{(v)})(F_0)$ contributes non-trivially to (9.13), or matches an orbit in $(S_n(\alpha) \times V_\xi')(F_0)$ that contributes non-trivially to (8.22), then it intersects Ω_v. (Loosely speaking, existence of Ω_v follows from the fact that the local terms $\mathrm{Int}(g, u)$ resp. $\partial \mathrm{Orb}((\gamma, u'), \Phi_v')$ in the two formulas can only be non-zero when (g, u) resp. (γ, u') satisfy an integrality condition. We refer to [35, Section 13.4 and Corollary 14.8] for the precise argument.) Let $K_{\mathfrak{d}}$ and $\Phi_{\mathfrak{d}}$ be as in the proposition. Let $\Phi' \in \mathcal{S}((S_n \times V')(\mathbb{A}_0))$ be a Gaussian transfer of Φ as in Section 8.5. It follows that, for $(\gamma, u') \in (S_n(\alpha) \times V_\xi')(F_0)$ such that locally at v the orbit of (γ, u') matches one in Ω_v, the away-from-v orbital integral $\mathrm{Orb}((\gamma, u'), \Phi'^{,v})$ is non-vanishing if and only if $\mathrm{GL}_n(F_0) \cdot (\gamma, u')$ is the matching orbit of (g, u).

Corollary 10.3. *The AFL identity for (g, u) and (γ, u') holds if and only if*

$$2\partial J_v(\xi, \Phi') = -\mathrm{Int}_v(\xi, \Phi). \tag{10.3}$$

Proof. This follows by comparing the two local-global decomposition formulas (8.22) and (9.13), each of them containing (at most) one non-zero term. ∎

This idea may be developed further. Namely, for w non-archimedean,

$$2\partial J_w(\xi, \Phi'), \mathrm{Int}_w(\xi, \Phi) \in \mathbb{Q} \log q_w,$$

so both terms in Corollary 10.3 lie in $\mathbb{Q} \log p$. Note that $\mathrm{Int}_w(\xi, \phi) = 0$ for all $w \neq v$ above p since these were assumed to be split. Moreover, there is a known identity of archimedean contributions: by [35, Lemma 14.4], for an archimedean place w,

$$2\partial J_w(\xi, \Phi') = -\mathrm{Int}_w^{\mathbf{K}}(\xi, \Phi).$$

It follows that

$$2\partial J_w(\xi, \Phi') + \mathrm{Int}_w^{\mathbf{K}-\mathbf{B}}(\xi, \Phi) = -\mathrm{Int}_w(\xi, \Phi), \quad w \mid \infty.$$

Using the linear independence (over $\mathbb{Q}$) of the log ℓ for ℓ prime, we find that (10.3), and hence the AFL identity in question, follow from the identity

$$2\partial J(\xi, \Phi') + \mathrm{Int}^{\mathbf{K}-\mathbf{B}}(\xi, \Phi) = -\mathrm{Int}(\xi, \Phi) \quad \text{in } \mathbb{R}_{\mathfrak{d}}, \tag{10.4}$$

where $\partial J(\xi, \Phi')$ is defined in (8.23).

10.2. Modular forms

The way to proceed in [35] was to use the fact that all three terms in (10.4) are known to be the Fourier coefficients of modular forms if $F_0 = \mathbb{Q}$. For the ∂J-term this was worked out in [35, Section 9], for the $\mathrm{Int}^{\mathbf{K}-\mathbf{B}}$-term this is due to Ehlen–Sankaran [5], for the Int-term it is the main result of [3]. Only the statement for ∂J is currently known to carry over without change. For the other two terms we will instead apply the almost modularity results from Section 4. This, however, requires modifying $^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)$, since Theorems 3.9 and 4.3 only apply to pairings with 1-cycles of degree 0.

By Proposition 5.7, there exist a family of $*$-embeddings $\sigma_i \colon E \to \mathrm{End}(V)$ with $\sigma_i(O_E)$ stabilizing Λ_p, Hecke operators $g_i \in G(F_{0,\mathfrak{d}})$ and scalars $\lambda_i \in \mathbb{Q}$ such that

$$C(\alpha, \varphi)^\circ := C(\alpha, \varphi) - \sum_i \lambda_i C(\sigma_i, g_i)$$

has degree 0 on every connected component of M. Let $\mathbf{E}$ be an extension of $\mathbf{F}$ over which the $C(\sigma_i, g_i)$ are defined and set

$$\mathcal{C} := \sum_i \lambda_i \mathcal{C}(\sigma_i, g_i),$$

where the integral models $\mathcal{C}(\sigma_i, g_i)$ of the $C(\sigma_i, g_i)$ are the normal ones (see the paragraph after Definition 5.3). Define the modified CM cycle

$$^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)^\circ = {}^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi) - \mathcal{C} \in \mathcal{Z}_1(O_{\mathbf{E}} \otimes_{O_{\mathbf{F}}} \mathcal{M}).$$

At this point, we again enlarge $\mathfrak{d}$ (but still keep K unchanged and maintain the condition $p \nmid \mathfrak{d}$) if necessary so that we can assume that $\Lambda[\mathfrak{d}^{-1}]$ is $\sigma_i(O_E)$-stable for all i.

Apply the pairing (4.8) and define

$$\mathrm{Int}(h, \Phi)^\circ := \frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]} \big(Z(h, \phi), {}^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)^\circ \big)^{\mathrm{adm}}. \tag{10.5}$$

For the rest of the section we let $K_{\mathbf{H}, \mathfrak{d}}$ be a compact open subgroup of $\mathbf{H}(F_{0,\mathfrak{d}})$ that fixes $\phi'_{\mathfrak{d}}, \phi_{\mathfrak{d}}$ and $\Phi'_{\mathfrak{d}}$ under the Weil representation. Recall $K_{\mathbf{H}}^{\mathfrak{d},\circ} = \prod_{v \nmid \mathfrak{d}} K_{\mathbf{H}, v}^\circ$ from (6.2). Then Lemma 7.7 (2) shows that ϕ_v is invariant under $K_{\mathbf{H}, v}^\circ$ for every $v \nmid \mathfrak{d}$.

Corollary 10.4 (to Theorem 4.3). *The function* $\mathrm{Int}(-, \Phi)^\circ$ *lies in* $\mathcal{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)_{\overline{\mathbb{Q}}}$
$\otimes_{\overline{\mathbb{Q}}} \mathbb{R}_{\mathfrak{d},\overline{\mathbb{Q}}}$ *with* $K_{\mathbf{H}} = K_{\mathbf{H},\mathfrak{d}} \times K_{\mathbf{H}}^{\mathfrak{d},\circ}$ *and* $\mathbb{R}_{\mathfrak{d},\overline{\mathbb{Q}}} := \mathbb{R}_{S,\overline{\mathbb{Q}}}$ *for* S *the set of places* $v \mid \mathfrak{d}$.

We can spell out the Fourier coefficients $\mathrm{Int}(\xi, h_\infty, \Phi)^\circ$ using the arithmetic KR divisors $\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi, \phi)$ (see (4.9)) on the integral model $\mathcal{M}$, and using the arithmetic intersection pairing (4.8)

$$\mathrm{Int}(\xi, \Phi)^\circ := \frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]} \big(\widehat{\mathcal{Z}}^{\mathbf{B}}(\xi, \phi), {}^{\mathbb{L}}\mathcal{CM}(\alpha, \varphi)^\circ\big) \tag{10.6}$$

for $\xi \in F_{0,+}$.

Now we define the corresponding automorphic form on the analytic side. Let

$$\partial J(h, \phi') = \frac{1}{[\mathbf{E} : F]} \sum_i \lambda_i \partial J(h, \phi'_i)$$

be the linear combination of (the restriction to $\mathbf{H}(\mathbb{A}_0)$ of) ∂J from (7.10). In particular, by Theorem 7.6 (and Proposition 6.8), the ξ-th Fourier coefficient with $\xi \neq 0$ is given by

$$2\partial J(\xi, h_\infty, \phi') = -\frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]} \big(\widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, h_\infty, \phi), \mathcal{C}\big) W_\xi^{(n)}(h_\infty) \tag{10.7}$$

up to $\sum_{\ell \mid \mathfrak{d}} \mathbb{Q} \log \ell$. Define the modified analytic generating function:

$$\partial J(h, \Phi')^\circ = \partial J(h, \Phi') - \partial J(h, \phi'), \quad h \in \mathbf{H}(\mathbb{A}_0). \tag{10.8}$$

Then we consider

$$\partial J_{\mathrm{hol}}(h) := 2\partial J(h, \Phi')^\circ + \mathrm{Int}^{\mathbf{K}-\mathbf{B}}(h, \Phi)^\circ, \tag{10.9}$$

where the last summand denotes the evaluation of the difference of the two Green functions at the modified CM cycle (see (3.24)),

$$\mathrm{Int}^{\mathbf{K}-\mathbf{B}}(h, \Phi)^\circ := \frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]} \sum_{v \in \Sigma_{\mathbf{E}},\, v \mid \infty} \mathcal{G}_v^{\mathbf{K}-\mathbf{B}}(h, \phi)(C(\alpha, \varphi)^\circ).$$

Proposition 10.5. *The function* $\partial J_{\mathrm{hol}}(h)$ *lies in* $\mathcal{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{R}_{\mathfrak{d},\overline{\mathbb{Q}}}$ *with* $K_{\mathbf{H}} = K_{\mathbf{H},\mathfrak{d}} \times K_{\mathbf{H}}^{\mathfrak{d},\circ}$.

Proof. First we note that each summand in (10.9) belongs to $\mathcal{A}_\infty(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)$. For $\partial J(h, \phi')$ the statement is Proposition 7.8. For $\partial J(h, \Phi')$ this is Lemma 8.3, together with the fact that the same proof of Lemma 7.7 shows that Φ'_v is invariant under $K_{\mathbf{H},v}^\circ$ for every $v \nmid \mathfrak{d}$. For $\mathrm{Int}^{\mathbf{K}-\mathbf{B}}(h, \Phi)$ it is a consequence of Theorem 3.13 and the fact that Φ_v is invariant under $K_{\mathbf{H},v}^\circ$ for every $v \nmid \mathfrak{d}$.

To complete the proof we need to show that the Fourier expansion of $\partial J_{\mathrm{hol}}$ takes the form

$$\partial J_{\mathrm{hol}}(h) = \sum_{\xi \in F_0,\, \xi \geq 0} A_\xi(h_f) W_\xi^{(n)}(h_\infty), \quad A_\xi \in \mathbb{R}_{\mathfrak{d}}, \tag{10.10}$$

where $A_\xi(h_f) = 0$ unless ξ lies in a fractional ideal of F_0 (depending on $h_f \in \mathbf{H}(\mathbb{A}_{0,f})$). Similar to the proof of [35, Proposition 14.5], using the fact that the Weil representation commutes with smooth transfer, it suffices to consider the case $h_f = 1$.

The proof of [35, Proposition 14.5] applies for a general totally real field F_0 and shows that the sum

$$2\partial J(h_\infty, \Phi') + \frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]} \sum_{\nu \in \Sigma_{\mathbf{E}}, \, \nu | \infty} \mathscr{G}_\nu^{\mathbf{K}-\mathbf{B}}(h_\infty, \phi)(C(\alpha, \varphi))$$

has the desired form of (10.10). It remains to show that

$$2\partial J(h_\infty, \phi') + \frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]} \sum_{\nu \in \Sigma_{\mathbf{E}}, \, \nu | \infty} \mathscr{G}_\nu^{\mathbf{K}-\mathbf{B}}(h_\infty, \phi)(C)$$

takes the form of (10.10). By definition, the automorphic Green functions $\mathscr{G}_\nu^{\mathbf{B}}(\xi, h_\infty, \phi)$ are actually independent of h_∞ (see (3.22)), and hence $\mathscr{G}_\nu^{\mathbf{B}}(h_\infty, \phi)$ takes the form of (10.10). So by (3.23) and (3.24), it remains to show that

$$2\partial J(h_\infty, \phi') + \frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]} \sum_{\nu \in \Sigma_{\mathbf{E}}, \, \nu | \infty} \mathscr{G}_\nu^{\mathbf{K}}(h_\infty, \phi)(C)$$

takes the form of (10.10). But here both terms arise by diagonal restriction from generating functions over E_0 to which we may apply Theorem 7.6 (together with Proposition 6.8). The proof is now complete. ∎

10.3. Comparison

For $\xi \in F_{0,+}$, recall that our aim is to show the identity

$$2\partial J(\xi, \Phi') + \mathrm{Int}^{\mathbf{K}-\mathbf{B}}(\xi, \Phi) = -\mathrm{Int}(\xi, \Phi) \quad \text{in } \mathbb{R}_{\mathfrak{d}}$$

(see (10.4)). Subtracting the known identity (10.7) evaluated at $h_\infty = 1$,

$$2\partial J(\xi, \phi') = -\frac{1}{\tau(Z^{\mathbb{Q}})[\mathbf{E} : F]}(\widehat{\mathcal{Z}}^{\mathbf{K}}(\xi, \phi), \mathcal{C}) \quad \text{in } \mathbb{R}_{\mathfrak{d}} \tag{10.11}$$

(see (3.20) and (4.13)), we see that the AFL identity in question would follow from

$$\partial J_{\mathrm{hol}}(\xi) = -\mathrm{Int}(\xi, \Phi)^\circ,$$

where the left hand side is the ξ-th Fourier coefficient of (10.9). Slightly enlarging $\mathfrak{d}$, we now assume it is divided by all places with residue cardinality $< n$.

Theorem 10.6. *There is an equality in $\mathscr{A}_{\mathrm{hol}}(\mathbf{H}(\mathbb{A}_0), K_{\mathbf{H}}, n)_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{R}_{\mathfrak{d}, \overline{\mathbb{Q}}}$:*

$$\partial J_{\mathrm{hol}}(h) = -\mathrm{Int}(h, \Phi)^\circ. \tag{10.12}$$

In particular, the AFL holds whenever $q_\nu \geq n$.

Proof. Let B be the product of all primes $\ell \nmid \mathfrak{d}$ where $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} O_F[t]/(\alpha(t))$ is not the maximal order in the étale $\mathbb{Q}_\ell$-algebra $E_\ell = \mathbb{Q}_\ell \otimes_{\mathbb{Q}} E$. (Note that the case of interest is typically $p \mid B$.) By [35, Lemma 13.6] applied to $\partial J_{\mathrm{hol}}(h) + \mathrm{Int}(h, \Phi)^\circ$ and B, it suffices to show the equality of Fourier cofficients (see (1.11) for notation)

$$\partial J_{\mathrm{hol}}(\xi, h_\infty) = -\mathrm{Int}(\xi, h_\infty, \Phi)^\circ$$

for all $\xi \in F_0^\times$ such that $w(\xi) = 0$ for all $w \mid B$ (here we write $w(\xi)$ for the valuation of ξ associated to the place w). Since both sides are holomorphic of the same weight (see Proposition 10.5 and Corollary 10.4), this is equivalent to

$$\partial J_{\mathrm{hol}}(\xi) = -\mathrm{Int}(\xi, \Phi)^\circ$$

for all $\xi \in F_0^\times$ such that $w(\xi) = 0$ for all $w \mid B$.

By an inductive argument, we assume from now on that the AFL Conjecture holds in all cases of residue cardinality (of F_0) $\geq n$ and dimension (of the hermitian space V) smaller than n. It is now sufficient to verify the following claim.

Claim. *For ξ with $w(\xi) = 0$ for all $w \mid B$, we have $\partial J_{\mathrm{hol}}(\xi) = -\mathrm{Int}(\xi, \Phi)^\circ$ in $\mathbb{R}_{\mathfrak{d}}$.*

The CM identity (10.11) may be added again, which leaves us with

$$2\partial J(\xi, \Phi') + \mathrm{Int}^{\mathbf{K}-\mathbf{B}}(\xi, \Phi) \overset{?}{=} \mathrm{Int}(\xi, \Phi),$$

which may be rewritten as

$$2\partial J(\xi, \Phi') \overset{?}{=} -\mathrm{Int}^{\mathbf{K}}(\xi, \Phi).$$

From here on, the argument is again as in [35, Theorem 14.6]. Write

$$2\partial J(\xi, \Phi') + \mathrm{Int}^{\mathbf{K}}(\xi, \Phi) = \sum_{w \nmid \mathfrak{d},\, w \nmid \infty} \left(2\partial J_w(\xi, \Phi') + \mathrm{Int}_w(\xi, \Phi) \right)$$
$$+ \sum_{w \nmid \mathfrak{d},\, w \mid \infty} \left(2\partial J_w(\xi, \Phi') + \mathrm{Int}_w^{\mathbf{K}}(\xi, \Phi) \right).$$

The archimedean terms are known to vanish by [35, Lemma 14.4]. The non-archimedean terms may be rewritten using the local-global decompositions (8.22) and (9.13):

$$2\partial J_w(\xi, \Phi') + \mathrm{Int}_w(\xi, \Phi)$$
$$= \sum_{\substack{(g,u) \in [(G^{(w)}(\alpha) \times V_\xi^{(w)})(F_0)] \\ (\gamma, u') \leftrightarrow (g, u)}} 2\left(\partial \mathrm{Orb}((\gamma, u'), \Phi_w') + \mathrm{Int}_w(g, u) \right) \mathrm{Orb}((g, u), \Phi^w) \log q_w.$$

The assumption on ξ and B implies that at least one of the following two cases holds:

(1) If $w \nmid \xi$, then (u, u) is a w-adic unit. The AFL identity $\partial \mathrm{Orb}((\gamma, u'), \Phi_w') = -\mathrm{Int}_w(g, u)$ is known to reduce to a similar AFL identity in dimension $n-1$, which holds by our induction hypothesis. The reduction statement is [35, Proposition 4.12].

(2) If $w \mid \xi$, then $w \nmid B$ so $O_F[t]/(\alpha(t))$ is a maximal order at w. In this case the above AFL identity is also known [35, Proposition 3.10].

This finishes the proof of the Claim, of the theorem and of the AFL for $q_v \geq n$. ∎

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