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Sharp quantitative stability of the planar Brunn–Minkowski inequality

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Abstract. We prove a sharp stability result for the Brunn–Minkowski inequality for $A, B \subset \mathbb{R}^2$. Assuming that the Brunn–Minkowski deficit $\delta = |A+B|^{1/2}/(|A|^{1/2}+|B|^{1/2}) - 1$ is sufficiently small in terms of $t = |A|^{1/2}/(|A|^{1/2}+|B|^{1/2})$, there exist homothetic convex sets $K_A \supset A$ and $K_B \supset B$ such that $\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \le Ct^{-1/2}\delta^{1/2}$. The key ingredient is to show for every $\varepsilon, t > 0$, if δ is sufficiently small then $|\mathrm{co}(A+B) \setminus (A+B)| \le (1+\varepsilon)(|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B|)$.

Keywords. Brunn–Minkowski, stability, planar

1. Introduction

Given measurable sets $A, B \subset \mathbb{R}^n$, the Brunn–Minkowski inequality says

$$|A+B|^{1/n} \ge |A|^{1/n} + |B|^{1/n},$$

with equality for homothetic convex sets $A = \mathrm{co}(A)$ and $B = \mathrm{co}(B)$ (less a measure 0 set). Here $A + B = \{a + b \mid a \in A,\ b \in B\}$ is the *Minkowski sum*, and $|\cdot|$ refers to the outer Lebesgue measure. Stability results for the Brunn–Minkowski inequality quantify how close A, B are to homothetic convex sets K_A, K_B in terms of

- $\delta = \delta(A, B) := \frac{|A+B|^{1/n}}{|A|^{1/n}+|B|^{1/n}} - 1$, the *Brunn–Minkowski deficit*, and
- $t = t(A, B) := \frac{|A|^{1/n}}{|A|^{1/n}+|B|^{1/n}}$, the *normalized volume ratio*.

Throughout the paper, δ and t will refer to the above quantities.

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The sharp stability question for the Brunn–Minkowski inequality, Question 1.1 below, is one of the central open problems in the study of geometric inequalities, and has been studied intensely in recent years by Barchiesi and Julin [1], Carlen and Maggi [3], Christ [4], Figalli and Jerison [5–7], Figalli, Maggi and Mooney [8], Figalli, Maggi and Pratelli [9, 10], and the present authors [12]. We provide a more detailed history of the problem in Section 1.1.

Question 1.1. For $n \geq 1$ do there exist exponents a_n, b_n such that the following is true, and if so what are the optimal exponents (prioritized in this order)? There is a constant C_n and constants $d_n(\tau) > 0$ for $\tau \in (0, \frac{1}{2}]$ such that whenever $A, B \subset \mathbb{R}^n$ are measurable sets with $t \in [\tau, 1-\tau]$ and $\delta \leq d_n(\tau)$, there exist homothetic convex sets $K_A \supset A$ and $K_B \supset B$ such that

$$\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \leq C_n \tau^{-b_n} \delta^{a_n}.$$

Prioritizing the exponents a_n, b_n in this order means that if the inequality holds for (a_n, b_n), then it also holds for (a'_n, b'_n) whenever $a_n > a'_n$ by taking $d'_n(\tau)$ sufficiently small.

For planar regions, taking $A = [0, t] \times [0, t(1+\varepsilon)]$ and $B = [0, (1-t)(1+\varepsilon)] \times [0, 1-t]$ shows that $a_2 \leq \frac{1}{2}$ and $b_2 \geq \frac{1}{2}$. Our main result, Theorem 1.2, solves the sharp stability question for planar regions $A, B \subset \mathbb{R}^2$, showing that the optimal exponents are $(a_2, b_2) = (\frac{1}{2}, \frac{1}{2})$.

Theorem 1.2. *There are computable constants $C, d(\tau) > 0$ such that if $A, B \subset \mathbb{R}^2$ are measurable sets with $t \in [\tau, 1-\tau]$ and $\delta \leq d(\tau)$, then there are homothetic convex sets $K_A \supset A$ and $K_B \supset B$ such that*

$$\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \leq C\tau^{-1/2}\delta^{1/2}.$$

Our key result in proving Theorem 1.2 is a strong generalization to arbitrary sets A, B of a conjecture [7] of Figalli and Jerison for $A = B$ that $|\mathrm{co}(A) \setminus A| = O(\delta)$ for δ sufficiently small. The original conjecture was recently proved by the present authors [12]. The generalization we now prove involves a completely different analysis to [12], and we are unaware of a similar approach used previously in the literature.

Theorem 1.3. *For all $\varepsilon, \tau > 0$ there is a computable constant $d_\tau(\varepsilon) > 0$ such that the following is true. Suppose that $A, B \subset \mathbb{R}^2$ are measurable sets with $t \in [\tau, 1-\tau]$ and $\delta \leq d_\tau(\varepsilon)$. Then*

$$|\mathrm{co}(A+B) \setminus (A+B)| \leq (1+\varepsilon)(|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B|).$$

Taking $A = B = [0,1]^2 \cup \{(0, 1+\lambda)\}$ shows that $1 + o(1)$ is optimal. By taking $\varepsilon = \tau/2$, we will deduce in Section 12 the following corollary, used to prove Theorem 1.2.

Corollary 1.4. *There is a constant C' such that*

$$\frac{|\mathrm{co}(A) \setminus A|}{|A|} + \frac{|\mathrm{co}(B) \setminus B|}{|B|} \leq C'\tau^{-1}\delta \quad \text{and} \quad \delta_{\mathrm{conv}} := \delta(\mathrm{co}(A), \mathrm{co}(B)) \leq \delta(A, B).$$

We make a note on how we apply Corollary 1.4 to deduce Theorem 1.2. We will estimate

$$\begin{aligned}
&\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \\
&\quad = \frac{|K_A \setminus \mathrm{co}(A)|}{|\mathrm{co}(A)|} \cdot \frac{|\mathrm{co}(A)|}{|A|} + \frac{|K_B \setminus \mathrm{co}(B)|}{|\mathrm{co}(B)|} \cdot \frac{|\mathrm{co}(B)|}{|B|} + \frac{|\mathrm{co}(A) \setminus A|}{|A|} + \frac{|\mathrm{co}(B) \setminus B|}{|B|} \\
&\quad \le C'' \tau^{-1/2} \delta_{\mathrm{conv}}^{1/2} + C' \tau^{-1} \delta \le C \tau^{-1/2} \delta^{1/2},
\end{aligned}$$

where the first estimate uses [10], and separately [6] to show

$$|\mathrm{co}(A)|\, |A|^{-1} \to 1 \quad \text{as } \delta \to 0.$$

In particular, the error in approximating A and B with their convex hulls is quadratically smaller than the error in approximating $\mathrm{co}(A)$ and $\mathrm{co}(B)$ with homothetic convex sets.

In order to deduce Theorem 1.2 from Theorem 1.3, even for $\tau = \frac{1}{2}$, it is insufficient to take say $1 + \varepsilon = 100$. In fact, with such a large ε the proof of Theorem 1.3 would be substantially easier. Showing the result for a suitably small ε is the primary challenge which we are able to overcome.

Example 1.5. We note that Theorem 1.3 with $\mathbb{R}^2$ replaced with $\mathbb{R}^n$ is false for any fixed $\varepsilon > 0$. To do this, we will give an example in $\mathbb{R}^3$ with equal volume sets A, B with δ arbitrarily small and with $|\mathrm{co}(A + B) \setminus (A + B)| > (1 + \varepsilon)(|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B|)$. Let T be the triangle with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(2, 0, 0)$, and let I_A, I_B be the intervals connecting $(0, 0, 0)$ to $v_A = (-\eta, 1, 0)$ and $v_B = (\eta, 1, 0)$ respectively. Let $T' = (T \setminus \{z \ge 1 - \lambda\}) \cup (1, 0, 1)$, and define

$$A = T' + I_A, \qquad B = T' + I_B.$$

Note that $\delta \to 0$ as $\lambda, \eta \to 0$. Also, $A + B = (T' + T') + (I_A + I_B)$ where $T' + T' = 2T \setminus \{z \ge 2 - \lambda\} \cup (2, 0, 2)$ and $I_A + I_B$ is a parallelogram in the xy-plane determined by the vectors v_A, v_B. Then

$$|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B| = 2\lambda^2$$

and

$$|\mathrm{co}(A + B) \setminus (A + B)| \ge |I_A + I_B| \cdot \lambda = 2\eta\lambda.$$

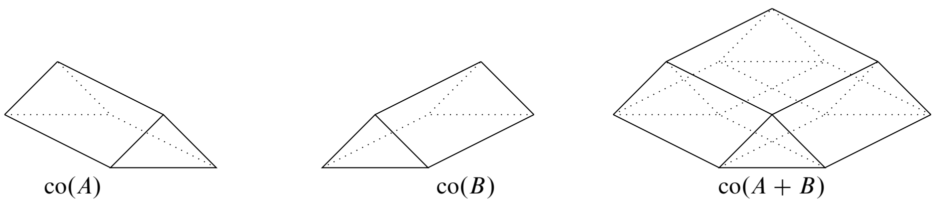

Fig. 1

Therefore, choosing $\eta > (1+\varepsilon)\lambda$, we obtain

$$|\mathrm{co}(A+B) \setminus (A+B)| = 2\eta\lambda > (1+\varepsilon)2\lambda^2 = (1+\varepsilon)(|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B|).$$

1.1. Background

In the literature, two measures for quantifying how close A, B are to homothetic convex sets have been introduced. The *Fraenkel asymmetry index* is defined to be

$$\alpha(A, B) = \inf_{x \in \mathbb{R}^n} \frac{|A \triangle (s \cdot \mathrm{co}(B) + x)|}{|A|},$$

where s satisfies $|A| = |s \cdot \mathrm{co}(B)|$. The other measure introduced by Figalli and Jerison in [6] is

$$\omega(A, B) = \min_{\substack{K_A \supset A, K_B \supset B \\ K_A, K_B \text{ homothetic convex sets}}} \max \left\{ \frac{|K_A \setminus A|}{|A|}, \frac{|K_B \setminus B|}{|B|} \right\}.$$

Providing an upper bound for ω is stronger than providing an upper bound for α as we always have $\alpha \le 2\omega$. We note that in $\mathbb{R}^2$ when A, B are both convex and δ is bounded, there is a reverse inequality (see Appendix A).

In a landmark paper, Figalli and Jerison [6, Theorem 1.3] showed the most general stability result for the Brunn–Minkowski inequality, with computable suboptimal exponents on τ and δ, and with the exponent of δ depending on τ (which we rephrase for the convenience of the reader).

Theorem 1.6 (Figalli and Jerison [6, Theorem 1.3]). *There exist computable constants $a_n(\tau), b_n$ such that the following is true. There are computable constants C_n and $d_n(\tau) > 0$ such that whenever $A, B \subset \mathbb{R}^n$ with $t \in [\tau, 1-\tau]$ and $\delta \le d_n(\tau)$, there exist homothetic convex sets $K_A \supset A$ and $K_B \supset B$ such that*

$$\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \le C_n \tau^{-b_n} \delta^{a_n(\tau)}.$$

This naturally gives rise to Question 1.1, asking for the optimal exponents of δ and τ, prioritized in this order. This question, with A, B restricted to various subclasses of geometric objects, is the subject of a large body of literature. Our main result, Theorem 1.2, proves sharp stability in the case $n = 2$ for arbitrary measurable A, B.

Prior to [6], Christ [4] had proved a non-computable non-polynomial bound involving δ and τ via a compactness argument. When A and B are convex, the optimal inequality $\alpha \le C_n \tau^{-1/2} \delta^{1/2}$ was obtained by Figalli, Maggi, and Pratelli [9,10]. When B is a ball and A is arbitrary, the optimal inequality $\alpha \le C_n \tau^{-1/2} \delta^{1/2}$ was obtained by Figalli, Maggi, and Mooney [8]. We note that this particular case is intimately connected with stability for the isoperimetric inequality. When just B is convex the (non-optimal) inequality $\alpha \le C_n \tau^{-(n+3/4)} \delta^{1/4}$ was obtained by Carlen and Maggi [3]. Finally, Barchiesi and Julin [1] showed that when just B is convex, we have the optimal inequality $\alpha \le C_n \tau^{-1/2} \delta^{1/2}$, subsuming these previous results.

Before their general result for distinct sets A, B in [6], Figalli and Jerison [5] had considered the case $A = B$ and gave a polynomial upper bound $\omega \le C_n \delta^{a_n}$. Later, in [7], they conjectured the sharp bound $\omega \le C_n \delta$ when $A = B$, and proved it in dimensions 2 and 3 using an intricate analysis which unfortunately does not extend to higher dimensions. Afterwards, Figalli and Jerison suggested a stronger conjecture that $\omega \le C_n \tau^{-1}\delta$ for A, B homothetic regions, which was proved by the present authors [12].

Finally, we note that the planar stability inequalities we consider are *not* Bonneson-style inequalities relating mixed volumes of planar convex K, L to the L-inradius and L-circumradius of K. See e.g. [2, Section 5] and separately [11] for an extensive survey of such inequalities.

1.2. Outline of paper

In Section 2, we give a reformulation of Theorem 1.3, make some simplifications and general observations, and give definitions which will be used throughout the remainder of the paper. The simplifications include assuming A, B are finite unions of polygonal regions so the vertices of $\partial \operatorname{co}(A), \partial \operatorname{co}(B)$ are contained in A, B respectively, and that they are translated in a specific way so that $\operatorname{co}(A)$ and $\operatorname{co}(B)$ contain the origin o.

In Section 3, by an averaging argument we show that $(1 - 4\tau^{-1}\sqrt{\gamma}) \operatorname{co}(A + B) \subset A + B$, where $\gamma = |\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B|$, i.e. for every $x \in \partial \operatorname{co}(A + B)$, we have $(1 - 4\tau^{-1}\sqrt{\gamma}) ox \subset A + B$.

In Section 4, we introduce a partition of $\partial \operatorname{co}(A + B)$ into *good* arcs and *bad* arcs. We think of good arcs as being the parts of the boundary of $\operatorname{co}(A + B)$ which are straight (or close to straight). We show that a very small part of the boundary $\partial \operatorname{co}(A + B)$ is covered by bad arcs.

In Section 5, we show that for x in a good arc of $\partial \operatorname{co}(A + B)$, we can in fact guarantee that $(1 - \xi\sqrt{\gamma}) ox$ lies in $A + B$ for any small ξ (provided d_τ is small). Thus $\operatorname{co}(A + B) \setminus A + B$ lies in a thickened boundary Λ of $\partial \operatorname{co}(A + B)$, which is thinner near the good arcs.

In Sections 6 and 7, we set up the following method for proving $|\operatorname{co}(A+B) \setminus (A+B)| \le (1+\varepsilon)(|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B|)$.

The edges of $\partial \operatorname{co}(A + B)$ are precisely the edges of $\partial \operatorname{co}(A)$ and $\partial \operatorname{co}(B)$ attached one after the other ordered by slope. Moreover, every edge of $\partial \operatorname{co}(A + B)$ is the Minkowski sum of an edge of $\partial \operatorname{co}(A)$ with a vertex of $\partial \operatorname{co}(B)$ or vice versa. We subdivide $\partial \operatorname{co}(A + B)$ into tiny straight arcs $\mathcal{J}$, and partition these arcs into collections $\mathcal{A}$ and $\mathcal{B}$ accordingly. We note that the arcs of $\mathcal{A}$ can be reassembled to $\partial \operatorname{co}(A)$ and the arcs of $\mathcal{B}$ can be reassembled to $\partial \operatorname{co}(B)$, in the same orders as they appear in $\partial \operatorname{co}(A + B)$.

We erect on each arc $\mathfrak{q} \in \mathcal{J}$ a parallelogram $R_\mathfrak{q}$ pointing roughly towards the origin such that these parallelograms cover the thickened boundary Λ. We ensure that we use a constant number of directions (1000 suffices) such that the $R_\mathfrak{q}$s with the same directions occur in contiguous arcs of $\partial \operatorname{co}(A + B)$. The heights of the parallelograms will be roughly on the order of $\sqrt{\gamma}$ if $\mathfrak{q}$ lies in a bad arc, and $\xi\sqrt{\gamma}$ if $\mathfrak{q}$ lies in a good arc. Each parallelogram $R_\mathfrak{q}$ with $\mathfrak{q} \in \mathcal{A}$ is the Minkowski sum of a parallelogram $R_{\mathfrak{q},A}$ erected on the corresponding segment of $\partial\,\mathrm{co}(A)$ with a vertex $p_{\mathfrak{q},B} \in \partial\,\mathrm{co}(B) \cap B$. Similarly for $\mathfrak{q} \in \mathcal{B}$.

This construction allows us to cover the thickened boundary Λ of $\partial\,\mathrm{co}(A+B)$ with translates of small regions erected on $\partial\,\mathrm{co}(A)$ and $\partial\,\mathrm{co}(B)$ as follows:

$$\Lambda \subset \bigcup_{\mathfrak{q}\in\mathcal{A}} (R_{\mathfrak{q},A} + p_{\mathfrak{q},B}) \cup \bigcup_{\mathfrak{q}\in\mathcal{B}} (p_{\mathfrak{q},A} + R_{\mathfrak{q},B}).$$

Therefore, we can cover $\mathrm{co}(A+B) \setminus (A+B)$ as follows:

$$\mathrm{co}(A+B) \setminus (A+B) \subset \bigcup_{\mathfrak{q}\in\mathcal{A}} ((R_{\mathfrak{q},A} \setminus A) + p_{\mathfrak{q},B}) \cup \bigcup_{\mathfrak{q}\in\mathcal{B}} (p_{\mathfrak{q},A} + (R_{\mathfrak{q},B} \setminus B)).$$

If we have subsets $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$ such that $\{R_{\mathfrak{q},A}\}_{\mathfrak{q}\in\mathcal{A}'}$ are disjoint and contained in $\mathrm{co}(A)$ and analogously $\{R_{\mathfrak{q},A}\}_{\mathfrak{q}\in\mathcal{B}'}$ are disjoint and contained in $\mathrm{co}(B)$, then we obtain an inequality

$$|\mathrm{co}(A+B) \setminus (A+B)| \le |\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B| + \sum_{\mathfrak{q}\in\mathcal{A}\setminus\mathcal{A}'} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q}\in\mathcal{B}\setminus\mathcal{B}'} |R_{\mathfrak{q},B}|.$$

Hence to prove Theorem 1.3, it suffices to show that we can find such $\mathcal{A}'$ and $\mathcal{B}'$ with

$$\sum_{\mathfrak{q}\in\mathcal{A}\setminus\mathcal{A}'} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q}\in\mathcal{B}\setminus\mathcal{B}'} |R_{\mathfrak{q},B}| \le \varepsilon(|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B|).$$

In Section 8 we show that bad arcs of $\partial\,\mathrm{co}(A+B)$ are close in angular distance to the corresponding arcs in $\partial\,\mathrm{co}(A)$ and $\partial\,\mathrm{co}(B)$. This result is crucial for Sections 9 and 10 where we bound the areas of the parallelograms we have to remove to create $\mathcal{A}'$ and $\mathcal{B}'$.

In Section 9, we use Section 8 to show that parallelograms $R_{\mathfrak{q},A} \not\subset \mathrm{co}(A)$ and $R_{\mathfrak{q},B} \not\subset B$ have $\mathfrak{q}$ on a good arc. This is then used to show that the area of parallelograms not contained in $\mathrm{co}(A)$ or $\mathrm{co}(B)$ is bounded roughly by $\xi^2\gamma$.

In Section 10 we use Section 8 to show that parallelograms $R_{\mathfrak{q},A}$ and $R_{\mathfrak{r},A}$ that intersect non-trivially have at least one of $\mathfrak{q}$ and $\mathfrak{r}$ on a good arc. This allows us to remove only good parallelograms to ensure disjointness. We conclude that the area of parallelograms we need to remove is bounded by roughly $\xi\gamma$.

In Section 11 we complete the proof of Theorem 1.3 by combining our bounds to deduce the final inequality. In Section 12 we show how Theorem 1.3 implies Theorem 1.2. Finally, we add an appendix with the proof that the measures α and ω are commensurate for small δ.

2. Setup

In this section, we collect together the preliminaries we need to start proving Theorem 1.3. In Section 2.1 we introduce an equal area reformulation of Theorem 1.3. In Section 2.2 we apply a preliminary affine transformation to $\mathbb{R}^2$ and collect facts about the resulting lengths and areas. In Section 2.3 we collect the main definitions which will be used throughout the body of the paper. Finally, in Section 2.4 we collect general observations which we will use frequently throughout.

2.1. *Equal area reformulation*

We will primarily work with the equivalent equal area reformulation of Theorem 1.3 in Theorem 2.2.

Definition 2.1. For $A, B \subset \mathbb{R}^2$ measurable sets and $t \in [0,1]$, define

$$D_t = tA + (1-t)B.$$

Theorem 2.2. *For $\tau \in (0, \frac{1}{2}]$, there are constants $d_\tau = d_\tau(\varepsilon) > 0$ such that the following is true. Let $A, B \subset \mathbb{R}^2$ be measurable sets with $|A| = |B| = V$, let t be a parameter satisfying $t \in [\tau, \frac{1}{2}]$, and suppose that $|D_t| \leq (1 + d_\tau(\varepsilon))^2 V$. Then*

$$|\mathrm{co}(D_t) \setminus D_t| \leq (1+\varepsilon)(t^2 |\mathrm{co}(A) \setminus A| + (1-t)^2 |\mathrm{co}(B) \setminus B|).$$

In Theorem 2.2, t is a free parameter, which we note is the normalized volume ratio of tA and $(1-t)B$. Given the sets A, B in Theorem 1.3, A/t and $B/(1-t)$ have equal volumes, and Theorem 1.3 is equivalent to Theorem 2.2 applied with these equal volume sets.

In the equal area reformulation, we let K be the smallest convex set such that K contains a translate of A and B. We assume from now on that $A, B \subset K$. By approximation,[1] we may assume that A, B, K are unions of polygons.

2.2. *Preliminary affine transformation*

Let $T \subset K$ be the maximal area triangle, and let o be the barycenter (which we will always take to be the origin). This maximal area triangle T has the property that $T \subset K \subset -2T =: T'$, and by applying an affine transformation, we may assume that T is a unit equilateral triangle whose vertices are contained in K.

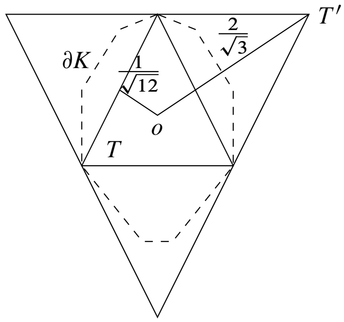

Fig. 2

[1] It is easy to show that for any fixed $d_\tau(\varepsilon)$ we must have A, B bounded. Now, approximate A, B from the inside by nested sequences of compact subsets $A_1 \subset A_2 \subset \cdots$ and $B_1 \subset B_2 \subset \cdots$. Then for each A_i, B_i approximate the pair from the outside by finite unions of polygons.

Observation 2.3. • We have $|T| = \frac{\sqrt{3}}{4}$, $|T'| = \sqrt{3}$, $|A|, |B| \in (0, \sqrt{3}]$ and $|K| \in [\frac{\sqrt{3}}{4}, \sqrt{3}]$.

• For $p \in T' \setminus T$ we have $|op| \in [\frac{1}{\sqrt{12}}, \frac{2}{\sqrt{3}}]$, and this in particular holds for $p \in \partial K$.

2.3. Definitions

We now collect the definitions we will use for the remainder of the paper.

Definition 2.4. We define

$$\gamma = t^2|\mathrm{co}(A) \setminus A| + (1-t)^2|\mathrm{co}(B) \setminus B|.$$

Definition 2.5. In a convex set C containing o, we say that a point $p \in \partial C$ is (θ, ℓ)-*bisecting* if the unique isosceles triangle $T_p(\theta, \ell)$ with angle θ at p and equal sides ℓ such that po internally bisects the corresponding angle is contained inside C.

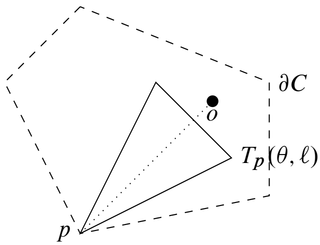

Fig. 3

Definition 2.6. Given a convex set C and a point $p \in \partial C$, we say that p is (θ, ℓ)-*good* if there are points $q, r \in C$ such that $|pq|, |pr| \geq \ell$ and $\angle qpr \geq 180° - \theta$. Any point in ∂C which is not (θ, ℓ)-good is (θ, ℓ)-*bad*.

Definition 2.7. Given a point p and a set E with $o \in \mathrm{co}(E)$, we denote by p_E the intersection of the ray op with $\partial\,\mathrm{co}(E)$.

2.4. General observations

Observation 2.8. Suppose we have subsets $R_A \subset \mathrm{co}(A)$, $R_B \subset \mathrm{co}(B)$, and $z \in \mathbb{R}^2$. Let $H = H_{-(1-t)/t,z}$ denote the negative homothety of ratio $-(1-t)/t$ through z. Then if $|R_A \cap H(R_B)| > t^{-2}\gamma$, or equivalently $|H^{-1}(R_A) \cap R_B| > (1-t)^{-2}\gamma$, then we have $z \in D_t$.[2]

[2] Note that $t^{-2}\gamma = |\mathrm{co}(A) \setminus A| + |\mathrm{co}(H(B)) \setminus H(B)|$, so there is at least one x in $R_A \cap H(R_B) \subset \mathrm{co}(A) \cap H(\mathrm{co}(B))$ which is not in $(\mathrm{co}(A) \setminus A) \cup (\mathrm{co}(H(B)) \setminus H(B))$. Thus $x \in A \cap H(B)$, and $z = tx + (1-t)H^{-1}(x) \in D_t$.

Observation 2.9. For sets A, B with common volume V, Figalli and Jerison showed (see Theorem 1.6) that for fixed τ we have $|K \setminus A|V^{-1}, |K \setminus B|V^{-1} \to 0$ as $|D_t|V^{-1} \to 1$. In particular, as $V \in (0, \sqrt{3}]$ by Observation 2.3, we have

$$|K \setminus A|, |K \setminus B|, |\mathrm{co}(A) \setminus A|, |\mathrm{co}(B) \setminus B|, \gamma \to 0 \quad \text{as } d_\tau \to 0.$$

2.5. Constants and their dependencies

Fix τ and ε. For the convenience of the reader, we describe roughly our choice of parameters throughout. First, we take $M = 1000 \in 2\mathbb{N}$ to be a universal constant and $\alpha = \frac{720°}{M} < 1°$. Next, we take ξ such that $\varepsilon \geq (\tau^2 + (1-\tau)^2)(25\tau^{-1}M\xi^2 + 16000\tau^{-1}M\xi)$. Next, we take $\theta \leq \frac{1}{2}°$ such that $\frac{1}{2}\xi^2 \sin(28°)^6/\sin(4\theta) \geq 1$, and we take ℓ such that $(\frac{1440°}{\theta} + 3)4(1 + 100t^{-1})\ell\frac{100}{99}\sqrt{12} \cdot \frac{180°}{\pi} < \frac{1}{3}\alpha$. Finally, take d_τ sufficiently small to make various statements true along the way.

3. Initial structural results

In this section, we will show three preliminary propositions which quantify how close we may assume A, B are to K, and how much of $\mathrm{co}(D_t)$ we can guarantee is covered by D_t without resorting to a finer analysis of the boundaries of the various regions.

- In Proposition 3.1 we show that for any constant $\eta \in (0, 1)$, if d_τ is sufficiently small in terms of η then
$$(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K.$$
- In Proposition 3.3 we show that if d_τ is sufficiently small, then for every $z \in \partial K$ the points z, z_A, z_B, z_{D_t} are $(59°, \frac{1}{3})$-bisecting.
- Finally, in Proposition 3.5 we show that if d_τ is sufficiently small, then
$$(1 - 4t^{-1}\sqrt{\gamma})\,\mathrm{co}(D_t) \subset D_t.$$

3.1. Showing $\mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t)$ contain a large scaled copy of K

Proposition 3.1. *For any fixed $\eta \in (0, 1)$, if d_τ is sufficiently small in terms of η, then $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$.*

To prove Proposition 3.1, we need Lemma 3.2 which guarantees that ∂K behaves well under the notion of (θ, ℓ)-bisecting from Definition 2.5.

Lemma 3.2. *Every point $p \in \partial K$ is $(60°, \frac{1}{2})$-bisecting.*

Proof. Note that the statement is trivially true if p is a vertex of ∂T (since then $T_p(60°, 1) = T \subset K$), so assume otherwise. Let x, y, z be the vertices of T and $x' = -2x$, $y' = -2y$, $z' = -2z$ the corresponding vertices of T'. Let $p = p_z$ be in the triangle xyz'. Let $p_y \in xy'z$ and $p_x \in x'yz$ be the point p_z rotated by $120°$ and $240°$ clockwise

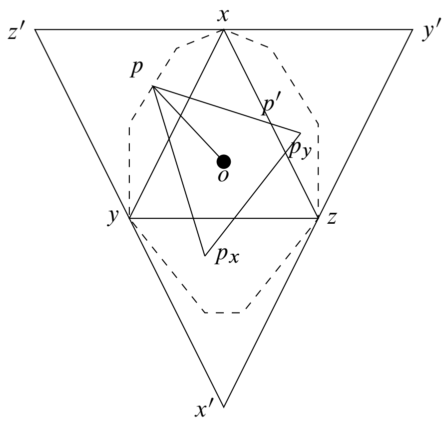

Fig. 4

around o respectively. Note that $p_x p_y p_z$ is an equilateral triangle with centre o such that $\angle o p_z p_y = 30°$. Let p' be the intersection of the segments xz and $p_z p_y$.

Note that $pp' \subset K$. We will show that $|pp'| \geq \frac{1}{2}$. Note that the points o, p, p', x are concyclic as $\angle oxp' = 30° = \angle opp'$. We have $\angle pxp' \in [60°, 120°]$, so by the law of sines, $2r = \frac{|pp'|}{\sin \angle pxp'} \leq \frac{2}{\sqrt{3}}|pp'|$, where r is the circumradius of this circle. But $2r \geq |ox| = \frac{1}{\sqrt{3}}$, so $|pp'| \geq \frac{1}{2}$. By showing a similar result for $p_z p_x$, we conclude that $T_p(60°, \frac{1}{2})$ lies in K. ■

Proof of Proposition 3.1. We prove this for $\mathrm{co}(A)$; the identical proof works for $\mathrm{co}(B)$ and then because $\mathrm{co}(D_t) = t\,\mathrm{co}(A) + (1-t)\,\mathrm{co}(B)$ we deduce the final containments. By Observation 2.9, we can take d_τ sufficiently small in terms of η so that $|K \setminus A| < \frac{\sqrt{3}}{36}\eta^2$. Let $p \in \partial K$, let $p' \in op$ be such that $|pp'| = \eta|op|$, and suppose for the sake of contradiction that $p' \notin \mathrm{co}(A)$. Then as $|op| \in [\frac{1}{\sqrt{12}}, \frac{2}{\sqrt{3}}]$ by Observation 2.3, we have $|pp'| \in [\frac{\eta}{\sqrt{12}}, \frac{2\eta}{\sqrt{3}}] = [(\frac{2}{3}\eta)h, (\frac{8}{3}\eta)h]$ where $h = \frac{\sqrt{3}}{4}$ is the height of $T_p(60°, \frac{1}{2})$. A line separating p from $\mathrm{co}(A)$ through p' cuts off from $T_p(60°, \frac{1}{2})$ an area of at least

$$\min\left(\frac{1}{2}, \left(\frac{2}{3}\eta\right)^2\right)\left|T_p(60°, \tfrac{1}{2})\right| = \frac{\sqrt{3}}{36}\eta^2$$

on the p-side, which lies in $K \setminus A$, contradicting $|K \setminus A| < \frac{\sqrt{3}}{36}\eta^2$. ■

3.2. *Showing points in ∂K, $\partial\,\mathrm{co}(A)$, $\partial\,\mathrm{co}(B)$, $\partial\,\mathrm{co}(D_t)$ are $(59°, \frac{1}{3})$-bisecting*

Proposition 3.3. *For d_τ sufficiently small, for every $z \in \partial K$ the points z, z_A, z_B, z_{D_t} are $(59°, \frac{1}{3})$-bisecting.*

Proof. By Proposition 3.1 we can take d_τ sufficiently small so that

$$(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B) \subset K$$

with $\eta = 10^{-9}$. Let C be one of $K, \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t)$. We have $T_z(60°, \frac{1}{2}) \subset K$. Let x, y denote the other two vertices of the triangle, and let $x' = (1-\eta)x$, $y' = (1-\eta)y$. Note that $x', y' \in (1-\eta)K \subset C$.

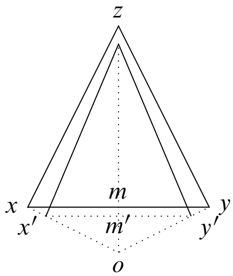

Fig. 5

Note Figure 5 is symmetric about oz. Let m be the midpoint of xy and m' be the midpoint of $x'y'$. Then $|x'm'| = \frac{1}{4}(1-\eta)$,

$$|m'z_C| \leq |mz_C| + |mm'| \leq |mz| + \eta|om| \leq \frac{\sqrt{3}}{4} + \eta\frac{2}{\sqrt{3}}$$

by Observation 2.3, and similarly $|m'z_C| \geq |mz| - |zz_C| - |m'm| \geq |mz| - \eta(|oz| + |om|) \geq \frac{\sqrt{3}}{4} - 2\eta\frac{2}{\sqrt{3}}$ (these are true even if o is inside the triangle xyz). Thus, by inspecting the right triangles $x'm'z_C$ and $y'm'z_C$, because

$$\tan(29.5°)\left(\frac{\sqrt{3}}{4} + \eta\frac{2}{\sqrt{3}}\right) < \frac{1}{4}(1-\eta) \quad \text{and} \quad \frac{1}{\cos(29.5°)}\left(\frac{\sqrt{3}}{4} - 2\eta\frac{2}{\sqrt{3}}\right) > \frac{1}{3},$$

the vertices of $T_{z_C}(59°, \frac{1}{3})$ lie in the triangle $x'y'z_C \subset C$. ∎

Corollary 3.4. *Let C be $K, \mathrm{co}(A), \mathrm{co}(B)$ or $\mathrm{co}(D_t)$. For d_τ sufficiently small, given $z \in \partial C$ and a supporting line l to C at z, we have $\angle l, zo \in (29°, 180° - 29°)$.*

3.3. Showing D_t contains a large scaled copy of $\mathrm{co}(D_t)$

Proposition 3.5. *For d_τ sufficiently small, we have*

$$(1 - 4t^{-1}\sqrt{\gamma})\,\mathrm{co}(D_t) \subset D_t.$$

In particular, if $z \in \partial\,\mathrm{co}(D_t)$ and $p \in oz$ has $|pz| \geq 5t^{-1}\sqrt{\gamma}$, then $p \in D_t$.

To show Proposition 3.5, we need the following lemma.

Lemma 3.6. *For every $\eta \in (0,1)$ and d_τ sufficiently small in terms of η, we have $(1-\eta)K \subset D_t$.*

Proof. We may assume that $\eta \leq 10^{-9}$. We take d_τ sufficiently small in terms of η such that $\frac{1-\eta}{1-\eta/2}K \subset \mathrm{co}(A), \mathrm{co}(B)$ by Proposition 3.1, and $t^{-2}\gamma < \pi(\frac{1}{100}\eta)^2$ by Observation 2.9. First, we show that for every $k \in K$ we have

$$B\left((1-\eta)k, \frac{1}{100}\eta\right) \subset \mathrm{co}(A), \mathrm{co}(B).$$

We show the $\mathrm{co}(A)$ containment; the other containment's proof is identical.

Write $k = \lambda k'$ with $k' \in \partial K$ and $\lambda \in [0,1]$. Because k' is $(60°, \frac{1}{2})$-bisecting we see that

$$B\left(\left(1-\frac{\eta}{2}\right)k', \frac{\eta}{2\sqrt{12}}\sin(30°)\right) \subset T_{k'}\left(60°, \tfrac{1}{2}\right) \subset K,$$

as $|ok'| \geq \frac{1}{\sqrt{12}}$ by Observation 2.3. Thus

$$\begin{aligned} B\left((1-\eta)k', \frac{\eta}{20}\right) &\subset B\left((1-\eta)k', \frac{1-\eta}{1-\eta/2}\frac{\eta}{2\sqrt{12}}\sin(30°)\right) \\ &\subset \frac{1-\eta}{1-\eta/2}K \subset \mathrm{co}(A), \end{aligned}$$

and so $B((1-\eta)k, \frac{\lambda}{20}\eta) \subset \mathrm{co}(A)$. If $\lambda \geq \frac{1}{5}$, then $B((1-\eta)k, \frac{1}{100}\eta) \subset \mathrm{co}(A)$, as desired.

Otherwise, assume $\lambda < \frac{1}{5}$. By Observation 2.3 we have $|k'| \leq \frac{2}{\sqrt{3}}$, so it follows that $|(1-\eta)\frac{100}{99}k| + \frac{1}{99} \leq \frac{1}{\sqrt{12}}$, the distance from o to ∂T, and so $B((1-\eta)\frac{100}{99}k, \frac{1}{99}) \subset T$. Hence, $B((1-\eta)k, \frac{1}{100}) \subset \frac{99}{100}T \subset \mathrm{co}(A)$. Thus we always have $B((1-\eta)k, \frac{1}{100}\eta) \subset \mathrm{co}(A)$ as desired.

Let $k \in K$. To check that $z = (1-\eta)k = t(1-\eta)k + (1-t)(1-\eta)k \in D_t$, in the notation of Observation 2.8 we take $R_A = R_B = B((1-\eta)k, \frac{1}{100}\eta) \subset \mathrm{co}(A), \mathrm{co}(B)$. Then $|R_A \cap H_{-(1-t)/t,z}(R_B)| = |R_A| = \pi(\frac{1}{100}\eta)^2 > t^{-2}\gamma$. Hence, we conclude by Observation 2.8 that $z \in D_t$. ■

Proof of Proposition 3.5. Let $\eta = 10^{-9}$, and take d_τ sufficiently small so that Proposition 3.3 and Lemma 3.6 apply, and that $\gamma \leq \frac{t^2}{16}$ by Observation 2.9. Define $z = tx + (1-t)y \in \partial\,\mathrm{co}(D_t)$ where $x \in \partial\,\mathrm{co}(A)$ and $y \in \partial\,\mathrm{co}(B)$. We will show that $z' = (1 - 4\lambda t^{-1}\sqrt{\gamma})z$ lies in D_t for all $\lambda \in [1, \frac{t}{4\sqrt{\gamma}}]$.

By Proposition 3.3 the points x, y are $(59°, \frac{1}{3})$-bisecting. Define x', y' analogously to z', and note that $tx' + (1-t)y' = z'$ and $|xx'|, |yy'|, |zz'| \in [\frac{4}{\sqrt{12}}\lambda t^{-1}\sqrt{\gamma}, \frac{8}{\sqrt{3}}\lambda t^{-1}\sqrt{\gamma}]$, $|oz| \leq \frac{2}{\sqrt{3}}$ by Observation 2.3. Because $\frac{1}{4}|xx'|, \frac{1}{4}|yy'| \leq |zz'|$, if either $|xx'|$ or $|yy'|$ is at least $\frac{1}{100}$, then $|zz'| \geq \frac{1}{25}$, which by Lemma 3.6 implies

$$z' \in \left(1 - \frac{|zz'|}{|oz|}\right)K \subset \left(1 - \frac{\sqrt{3}}{50}\right)K \subset (1-\eta)K \subset D_t.$$

Assume now that $|xx'|, |yy'| < \frac{1}{100}$, so that the altitudes from x (resp. y) of $T_x(59°, \frac{1}{3})$ (resp. $T_y(59°, \frac{1}{3})$) exceed $2|xx'|$ (resp. $2|yy'|$). Because $\lambda \geq 1$ we have

$$|xx'|, |yy'| \geq \frac{4\sqrt{\gamma}}{\sqrt{12}}\lambda t^{-1} \geq 1.001 t^{-1}\sqrt{\frac{\gamma}{\pi}}/\sin(29.5°).$$

Together the last two sentences show that $B(x', 1.001t^{-1}\sqrt{\gamma/\pi}) \subset T_x(59°, \frac{1}{3}) \subset \mathrm{co}(A)$, and $B(y', 1.001t^{-1}\sqrt{\gamma/\pi}) \subset T_y(59°, 1/3) \subset \mathrm{co}(B)$. By applying Observation 2.8 with $R_A = B(x', 1.001t^{-1}\sqrt{\gamma/\pi})$ and $R_B = B(y', 1.001t^{-1}\sqrt{\gamma/\pi})$, we conclude that $z' \in D_t$.

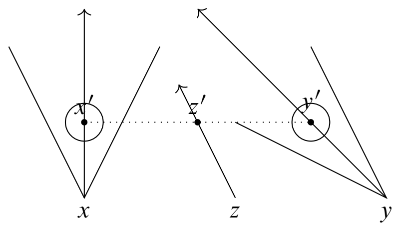

Fig. 6

Finally, $|zz'| = 4t^{-1}\sqrt{\gamma}|oz| \leq \frac{8}{\sqrt{3}}t^{-1}\sqrt{\gamma} < |pz|$, so $p \in D_t$. ■

4. Decomposing $\partial\, \mathrm{co}(D_t)$ into good arcs, and bad arcs of small total angular size

Recall that $M \in 2\mathbb{N}$ be some universal constant (1000 suffices), and set $\alpha = \frac{720°}{M} < 1°$.

Definition 4.1. For any s, we denote by $\mathcal{I}_s^{\mathrm{bad}}(\theta, \ell)$ the collection of arcs formed by the set of all points in $\partial\, \mathrm{co}(D_t)$ within Euclidean distance s of a (θ, ℓ)-bad point (which is a union of arcs). We let $\mathcal{I}_s^{\mathrm{good}}(\theta, \ell)$ denote the remaining arcs in $\partial\, \mathrm{co}(D_t)$, which we subdivide into arcs of angular length at most $\frac{1}{3}\alpha$.

Proposition 4.2. *For d_τ sufficiently small, there exists an increasing function $\ell = \ell(\theta)$ for $\theta < 180°$ such that the union of arcs $\bigcup \mathcal{I}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta, \ell)$ has total angular size at most $\frac{1}{3}\alpha$.*

Proof. Take d_τ sufficiently small so that $\frac{99}{100}K \subset \mathrm{co}(D_t)$ by Proposition 3.1.

Choose a point on $\partial\, \mathrm{co}(D_t)$, and form a polygon P inscribed in $\partial\, \mathrm{co}(D_t)$ by travelling around clockwise and picking the first vertex at distance ℓ from the previous vertex, all the way until the polygon would self-intersect, and then simply join the first and last vertex with an edge. Then all sides are of length ℓ except one side of possibly smaller size. Moreover, each vertex of the polygon is within distance ℓ of every point of the next subtended arc of $\partial\, \mathrm{co}(D_t)$.

We let $\mathcal{S}^{\mathrm{good}}$ be the collection of arcs of $\mathrm{co}(D_t)$ which arise as the arc subtended by m_2m_3, where m_1, m_2, m_3, m_4 are four consecutive vertices of the polygon P, with $|m_1m_2| = |m_2m_3| = |m_3m_4| = \ell$ and $\angle m_1m_2m_3, \angle m_2m_3m_4 \geq 180° - \theta/2$. We claim that every point $s \in \mathfrak{q} \in \mathcal{S}^{\text{good}}$ is (θ, ℓ)-good. To see this, note that the angle condition in particular implies that $\angle m_1m_2m_3, \angle m_2m_3m_4 > 90°$, so the rays m_1m_2 and m_4m_3 meet at a point r as shown in Figure 7 below.

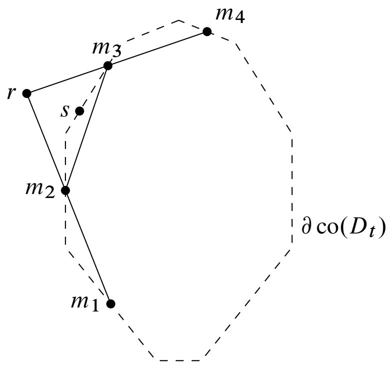

Fig. 7

We now show that m_1, m_4 realize s as a (θ, ℓ)-good point. First, note that $|m_1 s| \geq \ell = |m_1m_2|$ because $\angle m_1m_2s \geq 90°$. Similarly $|m_4s| \geq \ell = |m_3m_4|$. Finally, $\angle m_1sm_4 \geq \angle m_1rm_4 \geq 180° - \theta$, where the first inequality follows as s lies inside the triangle m_1rm_4, and the second as $\angle rm_2m_3, \angle rm_3m_2 \leq \theta/2$.

Let $\mathcal{S}^{\text{bad}}$ be the collection of remaining arcs of $\partial\,\mathrm{co}(D_t)$ subtended by sides of P which are not in $\mathcal{S}^{\text{good}}$. As the sum of the exterior angles of P is 360°, the number of interior angles which are strictly less than $180° - \theta/2$ is at most $720°/\theta$. Thus, $|\mathcal{S}^{\text{bad}}| \leq 1440°/\theta + 3$ (we add 3 for the arc subtended by the last side of the polygon and the two adjacent arcs). Note that every (θ, ℓ)-bad point is contained in an arc in $\mathcal{S}^{\text{bad}}$.

For each arc $\mathfrak{q} \in \mathcal{S}^{\text{bad}}$ let $x_{\mathfrak{q}}$ denote its clockwise starting point and $I_{\mathfrak{q}} := \partial\,\mathrm{co}(D_t) \cap B(x_{\mathfrak{q}}, (1 + 100t^{-1})\ell)$ the set of all points of $\partial\,\mathrm{co}(D_t)$ within Euclidean distance at most $(1 + 100t^{-1})\ell$ of $x_{\mathfrak{q}}$. This includes the points within Euclidean distance at most $100t^{-1}\ell$ of $\mathfrak{q}$. Let $I := \bigcup I_{\mathfrak{q}}$, so that $\bigcup \mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta, \ell) \subset I$.

Recall that $\frac{99}{100}K \subset \mathrm{co}(D_t)$, so that $\partial\,\mathrm{co}(D_t) \subset T' \setminus \frac{99}{100}T$ and thus $|ox_{\mathfrak{q}}| \geq \frac{99}{100}\frac{1}{\sqrt{12}}$ by Observation 2.3. Because $I_{\mathfrak{q}} \subset B(x_{\mathfrak{q}}, (1 + 100t^{-1})\ell)$, the angular size of $I_{\mathfrak{q}}$ is at most

$$2\sin^{-1}((1 + 100t^{-1})\ell)\frac{100}{99}\sqrt{12} \leq 4(1 + 100t^{-1})\ell\frac{100}{99}\sqrt{12} \cdot \frac{180°}{\pi}.$$

We conclude that $\bigcup \mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta, \ell) \subset I$ has angular size at most

$$\left(\frac{1440°}{\theta} + 3\right)4(1 + 100t^{-1})\ell\frac{100}{99}\sqrt{12} \cdot \frac{180°}{\pi},$$

which we can make smaller than $\frac{1}{3}\alpha$ by choosing ℓ sufficiently small. ■

Definition 4.3. We will always denote by $\ell = \ell(\theta)$ the increasing function of θ produced by the lemma above.

Observation 4.4. Every point in an arc in $\mathcal{J}_s^{\text{good}}(\theta, \ell)$ has distance at least s to all (θ, ℓ)-bad points in $\partial \operatorname{co}(D_t)$, and we have the partition (up to a finite collection of endpoints)

$$\bigsqcup \mathcal{J}_s^{\text{good}}(\theta, \ell) \sqcup \bigsqcup \mathcal{J}_s^{\text{bad}}(\theta, \ell) = \partial \operatorname{co}(D_t).$$

5. Replacing $5t^{-1}\sqrt{\gamma}$ with $\xi\sqrt{\gamma}$ on arcs in $\mathcal{J}_{2\ell}^{\text{good}}(\theta, \ell)$

This section is devoted to proving the following proposition.

Proposition 5.1. *For every $\xi \in (0, 1)$ there exists $\theta > 0$ such that for d_τ sufficiently small in terms of ξ the following is true. For every $p \in \mathfrak{q} \in \mathcal{J}_{2\ell}^{\text{good}}(\theta, \ell)$ (recalling $\ell = \ell(\theta)$) and $p' \in op$ with $|pp'| \geq \xi\sqrt{\gamma}$, we have $p' \in D_t$.*

We outline the proof of Proposition 5.1. Suppose first that p is the t-weighted average of points x_A and y_B[3] which are a distance at most ℓ apart. Then x_{D_t}, y_{D_t} are both close enough to p that by definition of $\mathcal{J}_{2\ell}^{\text{good}}(\theta, \ell)$, x_{D_t} is (θ, ℓ)-good in $\operatorname{co}(A)$ and y_{D_t} is (θ, ℓ)-good in $\operatorname{co}(B)$, which by Lemma 5.4 implies x_A, y_B are $(2\theta, \ell/2)$-good, yielding certain angular regions at x_A and y_B lying in $\operatorname{co}(A)$ and $\operatorname{co}(B)$ respectively.

If instead the distance is at least ℓ, then the triangles ox_Ay_A and oy_Bx_B serve as the large angular regions at x_A and y_B respectively.

In either case, the fact that $p \in \partial \operatorname{co}(D_t)$ implies the angular regions are in suitable directions so that Lemma 5.5 applies, showing in either case these regions are suitable for an application of Observation 2.8, and we conclude.

Lemma 5.2. *If we perturb the endpoints of a line segment of length ℓ each by an amount $r < \ell/2$, then the newly created line segment is rotated by at most $\sin^{-1}(2r/\ell)$.*

Proof. Consider two circles of radius r around the two endpoints of the segment; then the maximally rotated segment is one of the interior bitangents to these circles. ■

Lemma 5.3. *In a triangle with vertices a, b, c, suppose that $\angle acb \in (28°, 180° - 28°)$. Then the distance from c to ab is at least $\sin(14°)\min(|ac|, |bc|)$.*

Proof. Let z be the foot of the perpendicular from c to the line ab. We have either $\angle acz \leq 90° - 14°$ or $\angle bcz \leq 90° - 14°$, say the former. Then

$$|cz| = (\cos \angle azc)|ac| \geq \sin(14°)|ac|. \qquad ■$$

Lemma 5.4. *For d_τ sufficiently small in terms of θ, if x_{D_t} (resp. y_{D_t}) is (θ, ℓ)-good in $\operatorname{co}(D_t)$, then x_A is $(2\theta, \ell/2)$-good in $\operatorname{co}(A)$ (resp. y_B is $(2\theta, \ell/2)$-good in $\operatorname{co}(B)$).*

[3] Here and in the proofs of Propositions 5.1 and 8.1 we will be writing for example $x_{D_t} := (x_A)_{D_t}$ even if no point x has been defined.

Proof. We prove the statement for x_A; the statement for y_B is proved identically. Let $\eta = \frac{\sqrt{3}\ell}{8}\sin(\theta/2)$ (recall ℓ is defined to be a function of θ), and take d_τ sufficiently small so that $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$ by Proposition 3.1. Let w, z be the other two points in $\mathrm{co}(D_t)$ realizing x_{D_t} as (θ,ℓ)-good. Because $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(D_t) \subset K$, we have $|x_{D_t}x_A| \le \eta\frac{2}{\sqrt{3}}$. Defining $w' = (1-\eta)w \in \mathrm{co}(A)$ and $z' = (1-\eta)z \in \mathrm{co}(B)$ we have $|ww'|, |zz'| \le \eta\frac{2}{\sqrt{3}}$. Thus by Lemma 5.2, as $\sin^{-1}(\frac{4\eta}{\sqrt{3}\ell}) < \theta/2$ we have $\angle w'x_Az' \ge 180° - 2\theta$. As $|x_{D_t}x_A| + |ww'| \le \frac{4\eta}{\sqrt{3}} < \ell/2$, by the triangle inequality $|x_Aw'| \ge \ell/2$. Similarly $|x_Az'| \ge \ell/2$, so we see that w', z' realize x_A as $(2\theta, \ell/2)$-good. ■

Lemma 5.5. *Let m, n be two points and let l_m^1, l_m^2 and l_n^1, l_n^2 be pairs of rays originating at m, n, respectively and label u, v, x, y as shown in Figure 8. Assume further that $\angle unv = \angle ymu \ge 28°$. Denote $\angle num = \theta$ and $|mn| = r$. Then we have the area lower bound $|uvxy| \ge \frac{1}{2}r^2\sin(28°)^6/\sin(\theta)$.*

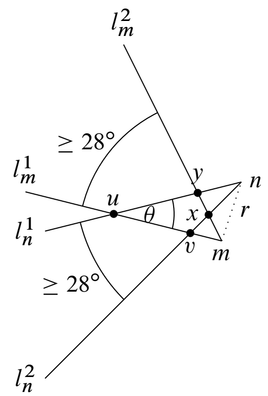

Fig. 8

Proof. First, we note that

$$|uvxy| \ge |uvy| = |umn| \cdot \frac{|uv|}{|um|} \cdot \frac{|uy|}{|un|}.$$

By the law of sines, $|um| = r\sin(\angle unm)/\sin(\theta)$ and $|un| = r\sin(\angle umn)/\sin(\theta)$. We have $\angle unm, \angle umn \ge 28°$, so as the sum of the angles of the triangle umn is $180°$, we have $\angle unm, \angle umn \in [28°, 180° - 28°]$. Therefore

$$\begin{aligned}|umn| &= \tfrac{1}{2}|um|\,|un|\sin(\theta) = \tfrac{1}{2}r^2\sin(\angle unm)\sin(\angle umn)/\sin(\theta)\\ &\ge \tfrac{1}{2}r^2\sin(28)^2/\sin(\theta).\end{aligned}$$

Next, we have

$$\begin{aligned}\frac{|uv|}{|um|} &= \frac{|unv|}{|unm|} = \frac{|nv|}{|nm|}\frac{\sin(\angle unv)}{\sin(\angle unm)} = \frac{\sin(\angle umn)\sin(\angle unv)}{\sin(\angle nvm)\sin(\angle unm)}\\ &\ge \sin(\angle umn)\sin(\angle unv) \ge \sin(28°)^2,\end{aligned}$$

and by a symmetric argument, $\frac{|uy|}{|un|} \geq \sin(28°)^2$. Multiplying the bounds, we obtain $|uvxy| \geq \frac{1}{2}r^2 \sin(28°)^6/\sin(\theta)$ as desired. ■

Proof of Proposition 5.1. We choose parameters as follows:

- $\theta \leq \frac{1}{2}^\circ$ such that $\frac{1}{2}\xi^2 \sin(28°)^6/\sin(4\theta) \geq 1$ and $\ell = \ell(\theta) \leq \frac{1}{2}$.
- Next, take $\eta = \frac{\sqrt{3}}{8}\ell \sin(\theta)$ (with this choice of η we have $(1-\eta)/\sqrt{12} \geq \ell/2$).
- Next, take γ_0 such that $5t^{-2}\sqrt{\gamma_0} \leq \frac{\ell}{20}\sin(4\theta)$.
- Finally, take $d = d_\tau$ sufficiently small so that
  - $\gamma \leq \gamma_0$ by Observation 2.9,
  - $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$ by Proposition 3.1,
  - $p' \in D_t$ if $|pp'| \geq 5t^{-1}\sqrt{\gamma_0}$ by Proposition 3.5,
  - Corollary 3.4 and Lemma 5.4 apply.

By our choice of d_τ we may assume that $|pp'| \in [\xi\sqrt{\gamma}, 5t^{-1}\sqrt{\gamma}]$. Write $p = tx_A + (1-t)y_B$ with $x_A \in \partial\,\mathrm{co}(A)$, $y_B \in \partial\,\mathrm{co}(B)$. Construct

$$A^+ = A + \overrightarrow{x_A p}, \quad B^- = B + \overrightarrow{y_B p},$$
$$o^+ = o + \overrightarrow{x_A p}, \quad o^- = o + \overrightarrow{y_B p}.$$

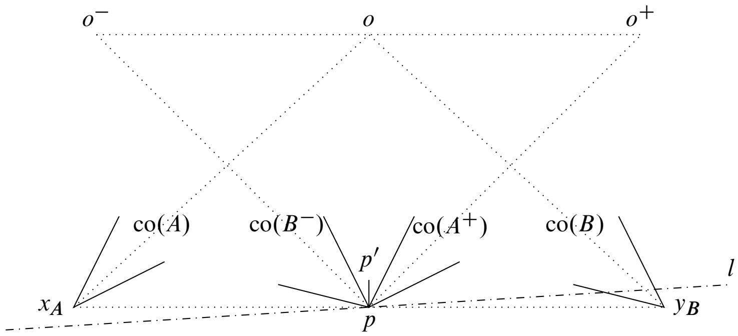

Fig. 9

Note that $o = to^+ + (1-t)o^-$ and hence p' is a point in the triangle o^+po^- such that $|pp'| \in [\xi\sqrt{\gamma}, 5t^{-1}\sqrt{\gamma}]$. It is enough to show that for any such p' we have $p' \in tA^+ + (1-t)B^-$.

Because $p \in \partial\,\mathrm{co}(D_t)$, there is a supporting line l at p to $\mathrm{co}(D_t)$, and because $\mathrm{co}(D_t)$ is the Minkowski semisum $t\,\mathrm{co}(A) + (1-t)\,\mathrm{co}(B)$, this line also leaves $\mathrm{co}(A^+), \mathrm{co}(B^-)$ on this same side as well. By Corollary 3.4 we have $\angle l, po^+, \angle l, po^- \in (29°, 180° - 29°)$.

Our goal will be to produce points $g^+ \in \mathrm{co}(A^+)$, $g^- \in \mathrm{co}(B^-)$ with $|g^+p|, |g^-p| \geq \ell/10$ as in Figure 10 where the horizontal line is l, the points appear counterclockwise in

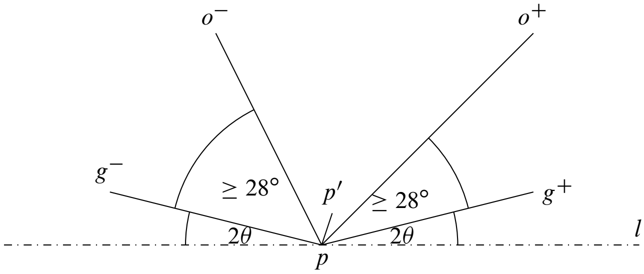

Fig. 10

the order g^+, o^+, p', o^-, g^-, and furthermore pg^+ is rotated 2θ counterclockwise from l about p, pg^- is rotated 2θ clockwise from l about p, and $\angle g^-, po^-, \angle g^+, po^+ \geq 28°$.

Claim 5.6. *If such points g^+, g^- exist then $p' \in D_t$.*

Proof. Note that $|o^+ p| = |ox_A| \geq (1-\eta)/\sqrt{12} \geq \ell/2 > \ell/10$ by Observation 2.3, and similarly $|o^- p| \geq \ell/10$. Furthermore, $|pp'| \leq 5t^{-1}\sqrt{\gamma_0} \leq \frac{\ell}{20}\sin(4\theta)$.

Let S^- denote the triangle $g^- po^-$ and S^+ denote the triangle $g^+ po^+$. Let H denote the negative homothety $H = H_{p',-(1-t)/t}$ of ratio $-(1-t)/t$ at p'. Note that the inverse homothety H^{-1} is a negative homothety with ratio $-t/(1-t)$ about p'.

First, we show that

$$|H^{-1}(S^+) \cap S^-| \geq \frac{1}{2(1-t)^2}|pp'|^2 \sin(28°)^6/\sin(4\theta).$$

This will be seen to follow from Lemma 5.5, applied with angle 4θ, $m = p$, $n = H^{-1}(p)$, $l_m^1 = pg^-$, $l_m^2 = po^-$, $l_n^1 = H^{-1}(pg^+)$ and $l_n^2 = H^{-1}(po^+)$. Let u, v, x and y be defined as in Lemma 5.5 such that $\angle num = 4\theta$.

In order to apply Lemma 5.5, we need to check that the intersection of the triangles $H^{-1}(S^+)$ and S^- contains the quadrilateral $uvxy$.

Indeed, we have

$$|un| = \sin(\angle upn)\frac{|mn|}{\sin(4\theta)} \leq \frac{\ell}{20}\cdot\frac{t}{1-t},$$

because $|mn| = \frac{1}{1-t}|pp'| \leq \frac{5}{t(1-t)}\sqrt{\gamma_0} \leq \frac{\sin(4\theta)\ell}{20}\cdot\frac{t}{1-t}$, and similarly

$$|up| \leq \frac{\ell}{20}\cdot\frac{t}{1-t}.$$

Then the triangle inequality shows that $|nv|, |py| \leq \frac{\ell}{10}\cdot\frac{t}{1-t}$ as well, and we conclude from the fact that $|H^{-1}(o^+ p)|, |H^{-1}(o^- p)|, |g^+ p|, |g^- p| \geq \frac{\ell}{10}\cdot\frac{t}{1-t}$.

Next, because $|pp'|^2 \geq \xi^2\gamma$, by our choice of θ_0 this implies that

$$|H^{-1}(S^+) \cap S^-| > (1-t)^{-2}\gamma.$$

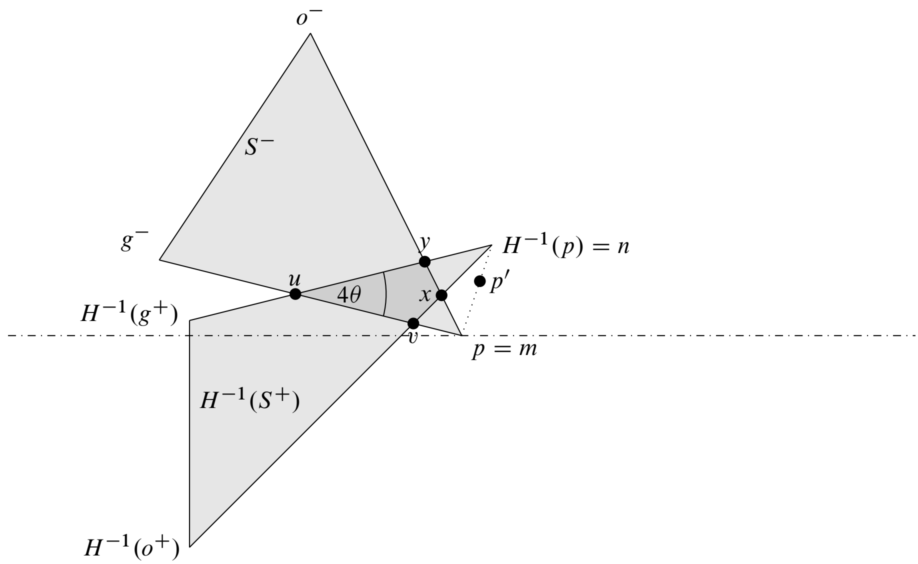

Fig. 11

Thus as

$$
\begin{aligned}
\frac{t^2}{(1-t)^2}|pg^+o^+ \setminus A^+| + |pg^-o^- \setminus B^-| &\leq \frac{t^2}{(1-t)^2}|\mathrm{co}(A^+) \setminus A^+| + |\mathrm{co}(B^-) \setminus B^-| \\
&= \frac{t^2}{(1-t)^2}|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B| \\
&= (1-t)^{-2}\gamma < |H^{-1}(S^+) \cap S^-|,
\end{aligned}
$$

a suitable modification of Observation 2.8 shows $p' \in tA^+ + (1-t)B^-$ and hence $p' \in tA + (1-t)B$. ■

Returning to the proof of the proposition, we note that exactly as at the start of the proof of Claim 5.6 we have $|po^+|, |po^-| \geq \ell/2$. We now distinguish two cases.

Case 1: $|x_A y_B| \geq \ell$. Recall the definitions of x_B and y_A from Definition 2.7. By Observation 2.3, we have $|x_A x_B|, |y_A y_B| \leq \eta\frac{2}{\sqrt{3}} \leq \ell/4$ and hence by the triangle inequality $|x_A y_A|, |x_B y_B| \geq \ell/2$.

We also have $\angle x_A y_A, x_B y_B \leq \sin^{-1}(\frac{8\eta}{\sqrt{3}\ell}) \leq \theta$ by Lemma 5.2.

Define $y_A^+ := y_A + \overrightarrow{x_A p} \in A^+, x_B^- = x_B + \overrightarrow{y_B p} \in B^-$. We have

$$|py_A^+| = |x_A y_A|, \quad |px_B^-| = |x_B y_B|,$$

and these are all $\geq \ell/2$ by the above discussion. Furthermore, $\angle y_A^+ p x_B^- = \angle x_A y_A, y_B x_B \geq \pi - \theta$, and the line l through p has $y_A^+, o^+, p', o^-, x_B^-$ on one side, appearing in this

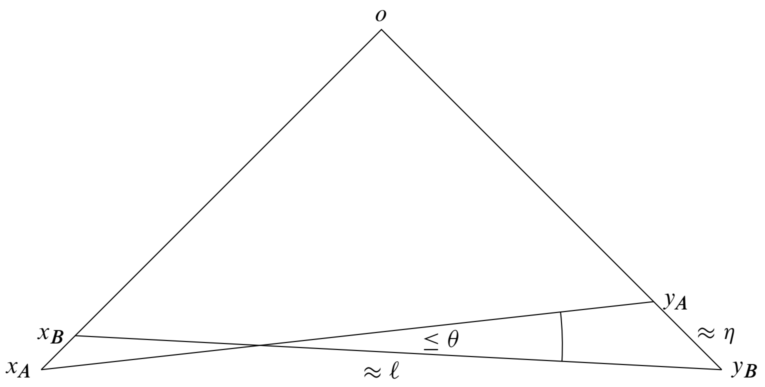

Fig. 12

order counterclockwise above l. To see this, note that as p lies on the segment $x_A y_B$, $\vec{x_A p}$ lies on the same side of the line ox_A as y_A does, so $o \notin \angle y_A^+ p o_A^+$. In particular, this implies that $\angle l, py_A^+, \angle l, px_B^- \leq \theta$.

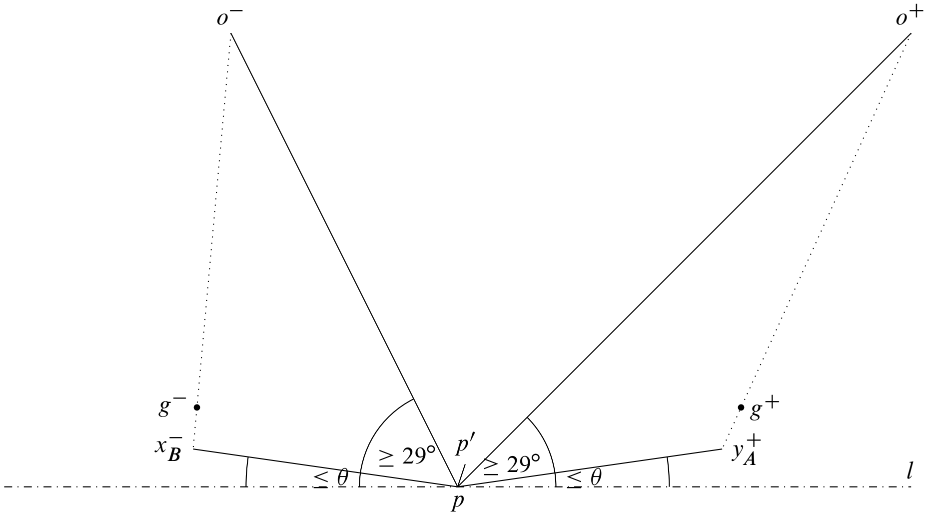

Fig. 13

Because $\angle l, po^+, \angle l, po^- \geq 29°$ and $2\theta < 29°$, we have $\angle l, py_A^+ \leq 2\theta < \angle l, po^+$ and $\angle l, px_B^- \leq 2\theta < \angle l, po^-$. These imply the existence of points

$$g^+ \in y_A^+ o^+ \subset \mathrm{co}(A^+) \quad \text{and} \quad g^- \in x_B^- o^- \subset \mathrm{co}(B^-)$$

such that $\angle l, pg^+ = \angle l, pg^- = 2\theta$. Because $\angle l, py_A^+, \angle l, px_B^- \leq \theta$ and $2\theta \leq 1°$, we have

$$\angle g^+ po^+, \angle g^- po^- \geq 29° - 2\theta \geq 28°.$$

It is clear from the construction that g^+, o^+, p', o^-, g^- also appear in this order counterclockwise above l. Finally, recall $|po^+| \geq \ell/2$, so by Lemma 5.3 as $\angle o^+ py_A^+ \in (28°, 180° - 28°)$ we have

$$|pg^+| \geq \min(|py_A^+|, |po^+|)\sin(14°) \geq \ell/10,$$

and similarly $|pg^-| \geq \ell/10$.

Case 2: $|x_A y_B| \leq \ell$. Then $|x_A p|, |y_B p| \leq \ell$, and we have $|x_{D_t} x_A|, |y_{D_t} y_A| \leq \frac{2}{\sqrt{3}}\eta \leq \frac{\ell}{4}$ by Observation 2.3. Thus by the triangle inequality $|x_{D_t} p|, |y_{D_t} p| \leq \frac{5}{4}\ell < 2\ell$. By definition of $\mathcal{I}_{2\ell}^{\text{good}}(\theta, \ell)$, since $p \in \mathfrak{q} \in \mathcal{I}_{2\ell}^{\text{good}}(\theta, \ell)$, the points x_{D_t}, y_{D_t} are (θ, ℓ)-good. By Lemma 5.4 we see that $x_A \in \mathrm{co}(A)$, $y_B \in \mathrm{co}(B)$ are $(2\theta, \ell/2)$-good. Therefore, there exist

$$e_1, e_2 \in \mathrm{co}(A) \quad \text{and} \quad f_1, f_2 \in \mathrm{co}(B)$$

such that

$$\angle e_1 x_A e_2, \angle f_1 y_B f_2 \geq 180° - 2\theta \quad \text{and} \quad |e_1 x_A|, |e_2 x_A|, |f_1 y_B|, |f_2 y_B| \geq \ell/2.$$

Let

$$e_1^+ = e_1 + \overrightarrow{x_A p}, \quad e_2^+ = e_2 + \overrightarrow{x_A p},$$
$$f_1^- = f_1 + \overrightarrow{y_B p}, \quad f_2^- = f_2 + \overrightarrow{y_B p},$$

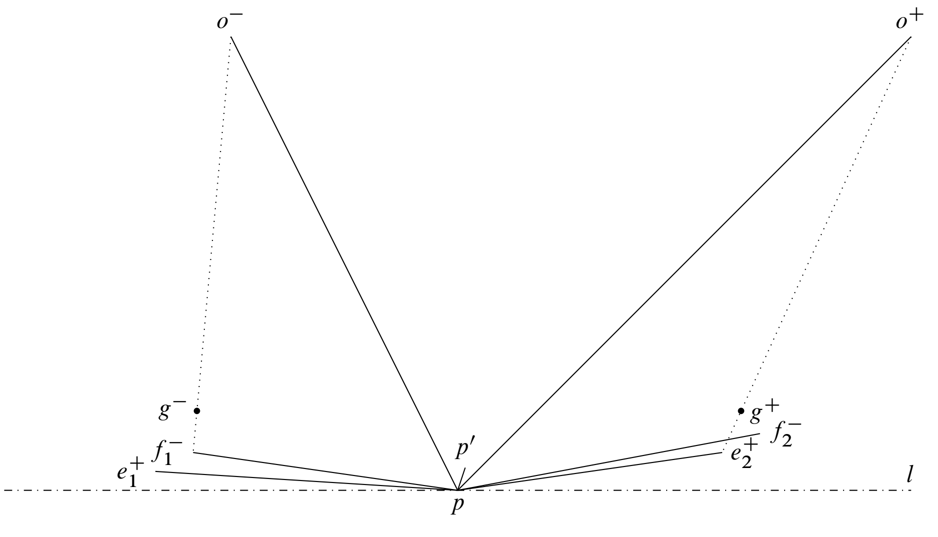

Fig. 14

such that $e_1^+, e_2^+ \in \mathrm{co}(A^+)$ and $f_1^-, f_2^- \in \mathrm{co}(B^-)$. With this notation we find that $\angle e_1^+ p e_2^+, \angle f_1^- p f_2^- \geq 180° - 2\theta$ and $|e_1^+ p|, |e_2^+ p|, |f_1^- p|, |f_2^- p| \geq \ell/2$. Recall that $\angle l, po^+, \angle l, po^- \in (29°, 180° - 29°)$.

Notice that the line l through p leaves $e_1^+, e_2^+, f_1^-, f_2^- o^+, o^-, p'$ on one side, and that up to relabelling the points, $e_2^+, o^+, p', o^-, f_1^-$ appear in this order counterclockwise above l. Note that $\angle l, e_2^+ p, \angle l, f_1^- p \leq 2\theta$. Construct points

$$g^+ \in e_2^+ o^+ \subset \mathrm{co}(A^+) \quad \text{and} \quad g^- \in f_1^- o^- \subset \mathrm{co}(B^-)$$

such that $\angle l, pg^+, \angle l, pg^- = 2\theta$ and note that $\angle g^+ po^+, \angle g^- po^- \geq 28°$ as $2\theta \leq 1°$. We can see from the construction that the points g^+, o^+, p', o^-, g^- also appear in this order counterclockwise above l. Finally, recall $|po^+| \geq \ell/2$, so by Lemma 5.3 as $\angle o^+ pe_2^+ \in (28°, 180 - 28°)$, we have

$$|pg^+| \geq \min(|pe_2^+|, |po^+|) \sin(14°) \geq \ell/10,$$

and similarly $|pg^-| \geq \ell/10$. ∎

6. Covering $\partial\, \mathrm{co}(D_t)$ with parallelograms

From now on, we let θ, ℓ depend on $\xi \in (0, 1)$ as in Proposition 5.1, and always assume that d_τ is sufficiently small so that the conclusion of Proposition 5.1 holds. We will fix ξ in terms of ε, so when we say to take d_τ sufficiently small, we implicitly will take it sufficiently small in terms of our choice of ξ.

In this section, we construct a partition $\mathcal{J}(\theta, \ell)$ of $\partial\, \mathrm{co}(D_t)$ into small straight arcs $\mathfrak{q}$, and parallelograms $R_\mathfrak{q}$ which have one side on $\mathfrak{q}$ such that

$$\mathrm{co}(D_t) \setminus D_t \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} R_\mathfrak{q}.$$

Recall that in Proposition 3.5 we showed that for d sufficiently small, D_t contains all points at radial distance $5t^{-1}\sqrt{\gamma}$ from $\partial\, \mathrm{co}(D_t)$. Furthermore, in Proposition 5.1 we improved the bound to $\xi\sqrt{\gamma}$ for points in $\partial\, \mathrm{co}(D_t)$ that belong to arcs in $\mathcal{J}_{2\ell}^{\mathrm{good}}(\theta, \ell)$.

For the remainder of the paper we will be using $\mathcal{J}_s^{\mathrm{good}}(\theta, \ell), \mathcal{J}_s^{\mathrm{bad}}(\theta, \ell)$ exclusively for $s = 2\ell, 3\ell, 100t^{-1}\ell$. Note that

$$\mathcal{J}_{2\ell}^{\mathrm{bad}}(\theta, \ell) \subset \mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta, \ell) \subset \mathcal{J}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta, \ell),$$
$$\mathcal{J}_{2\ell}^{\mathrm{good}}(\theta, \ell) \supset \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta, \ell) \supset \mathcal{J}_{100t^{-1}\ell}^{\mathrm{good}}(\theta, \ell).$$

Thus Proposition 5.1 also applies to points that belong to arcs in $\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta, \ell)$ and $\mathcal{J}_{100t^{-1}\ell}^{\mathrm{good}}(\theta, \ell)$, and Proposition 4.2 also shows that the total angular size of arcs in $\mathcal{J}_{2\ell}^{\mathrm{bad}}(\theta, \ell)$ and $\mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta, \ell)$ is at most $\frac{1}{3}\alpha$. We remark in what follows that we use

- $\mathcal{I}_{3\ell}^{\mathrm{good}}(\theta,\ell) \cup \mathcal{I}_{3\ell}^{\mathrm{bad}}(\theta,\ell)$ to determine the heights of the $R_{\mathfrak{q}}$, and
- $\mathcal{I}_{100t^{-1}\ell}^{\mathrm{good}}(\theta,\ell) \cup \mathcal{I}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta,\ell)$ to determine directions of the parallelograms $R_{\mathfrak{q}}$.

6.1. Definitions

We first refine the partitions $\mathcal{I}_s^{\mathrm{good}}(\theta,\ell) \cup \mathcal{I}_s^{\mathrm{bad}}(\theta,\ell)$ of $\partial\,\mathrm{co}(D_t)$ for $s = 2\ell, 3\ell, 100t^{-1}\ell$ into small straight segments.

Definition 6.1. Let $\mathcal{J}(\theta,\ell)$ be a partition of $\partial\,\mathrm{co}(D_t)$ formed as a common refinement to all of the sets of arcs from the partitions

$$\mathcal{I}_{2\ell}^{\mathrm{good}}(\theta,\ell) \cup \mathcal{I}_{2\ell}^{\mathrm{bad}}(\theta,\ell), \quad \mathcal{I}_{3\ell}^{\mathrm{good}}(\theta,\ell) \cup \mathcal{I}_{3\ell}^{\mathrm{bad}}(\theta,\ell), \quad \mathcal{I}_{100t^{-1}\ell}^{\mathrm{good}}(\theta,\ell) \cup \mathcal{I}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta,\ell)$$

of $\partial\,\mathrm{co}(D_t)$ into straight line segments of length at most $\xi\sqrt{\gamma}$. For $s \in \{2\ell, 3\ell, 100t^{-1}\ell\}$, define the partition $\mathcal{J}_s^{\mathrm{good}}(\theta,\ell) \cup \mathcal{J}_s^{\mathrm{bad}}(\theta,\ell)$ of $\mathcal{J}(\theta,\ell)$ by setting $\mathfrak{q} \in \mathcal{J}_s^{\mathrm{good}}(\theta,\ell)$ if and only if $\mathfrak{q} \subset \mathfrak{q}' \in \mathcal{I}_s^{\mathrm{good}}(\theta,\ell)$.

We will now in Definition 6.2 choose the vectors $v_{\mathfrak{q}}$ for $\mathfrak{q} \in \mathcal{J}(\theta,\ell)$ with direction vectors $\widehat{v}_{\mathfrak{q}}$ determined by the partition $\mathcal{I}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta,\ell) \cup \mathcal{I}_{100t^{-1}\ell}^{\mathrm{good}}(\theta,\ell)$, and with lengths determined by $\mathcal{I}_{3\ell}^{\mathrm{bad}}(\theta,\ell) \cup \mathcal{I}_{3\ell}^{\mathrm{good}}(\theta,\ell)$. Then in Definition 6.3 we form parallelograms $R_{\mathfrak{q}}$ with sides $\mathfrak{q}$ and $v_{\mathfrak{q}}$.

Definition 6.2. For an arc $\mathfrak{q} \in \mathcal{J}(\theta,\ell)$, we define a vector $v_{\mathfrak{q}}$ as follows.

- We choose the direction vector $\widehat{v}_{\mathfrak{q}}$ of $v_{\mathfrak{q}}$ as follows. Let $\mathfrak{q} \subset \mathfrak{q}' \in \mathcal{I}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta,\ell) \cup \mathcal{I}_{100t^{-1}\ell}^{\mathrm{good}}(\theta,\ell)$. If $\mathfrak{q}'$ is contained inside an angular interval $[m\alpha, (m+1)\alpha]$, we take the direction vector $\widehat{v}_{\mathfrak{q}}$ to be the inward pointing direction at angle $(m+\frac{1}{2})\alpha$. Otherwise (recalling that $\mathfrak{q}' \in \mathcal{I}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta,\ell) \cup \mathcal{I}_{100t^{-1}\ell}^{\mathrm{good}}(\theta,\ell)$ has angular length at most $\frac{1}{3}\alpha$), $\mathfrak{q}'$ overlaps a unique angle $m\alpha$, and we take $\widehat{v}_{\mathfrak{q}}$ to be the inward pointing vector at angle $m\alpha$.
- We choose the length of $v_{\mathfrak{q}}$ to be

$$\|v_{\mathfrak{q}}\| = \begin{cases} 15\sqrt{\gamma}, & \mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta,\ell), \\ 3\xi\sqrt{\gamma}, & \mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell). \end{cases}$$

For $p \in \partial\,\mathrm{co}(D_t)$, we denote $v_p = v_{\mathfrak{q}}$, where $p \in \mathfrak{q} \in \mathcal{J}(\theta,\ell)$.

Definition 6.3. For $\mathfrak{q} \in \mathcal{J}(\theta,\ell)$, let $R_{\mathfrak{q}}$ be a parallelogram with one side $\mathfrak{q}$ and one side $v_{\mathfrak{q}}$.

By construction, the directions of each of the v_p for $p \in \partial\,\mathrm{co}(D_t)$ are one of $M = 4\pi/\alpha$ directions, and the directions of the vectors are constant on arcs of $\partial\,\mathrm{co}(D_t)$ from $\mathcal{I}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta,\ell) \cup \mathcal{I}_{100t^{-1}\ell}^{\mathrm{good}}(\theta,\ell)$.

Observation 6.4. For every point $p \in \partial\,\mathrm{co}(D_t)$ we have $\angle po, v_p < \frac{1}{2}\alpha$.

6.2. *Covering $\partial\,\mathrm{co}(D_t)$ with parallelograms*

Now are able to state the main result of this section.

Proposition 6.5. *For d_τ sufficiently small, we have*

$$\mathrm{co}(D_t) \setminus D_t \subset \bigcup_{\mathfrak{q}\in\mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}.$$

We need the following observation about the unit direction vectors $\widehat{v}_{\mathfrak{q}}$ of $v_{\mathfrak{q}}$.

Lemma 6.6. *Let $p \in \partial\,\mathrm{co}(D_t)$, and $p' \in op$. Then there exists $r \in \partial\,\mathrm{co}(D_t)$, with $\widehat{v}_p = \widehat{v}_r$ and this is parallel to rp'.*

Proof. Let z be the unique point on $\partial\,\mathrm{co}(D_t)$ with zo in the direction of $\widehat{v}_p$. By Observation 6.4, the angle between $\widehat{v}_z$ and zo (which is in the direction $\widehat{v}_p$) is strictly less than $\frac{1}{2}\alpha$. As the $\widehat{v}$ angles occur in multiples of $\frac{1}{2}\alpha$, this implies $\widehat{v}_z = \widehat{v}_p$.

Let r be the unique point on $\partial\,\mathrm{co}(D_t)$ with rp' in the direction of v_p. Then r lies on the arc pz, so $\widehat{v}_p = \widehat{v}_r$ is parallel to rp'. ∎

Proof of Proposition 6.5. Assume that d_τ is sufficiently small so that we may apply Propositions 3.3 and 3.5. Given a point $p \in \partial\,\mathrm{co}(D_t)$, define the interval

$$S_p(\theta,\ell;\xi) = pp',$$

where $p' \in op$ is such that

$$|pp'| = \begin{cases} 5\sqrt{\gamma}, & p \in \mathfrak{q} \subset \mathcal{I}_{2\ell}^{\mathrm{bad}}(\theta,\ell), \\ \xi\sqrt{\gamma}, & p \in \mathfrak{q} \subset \mathcal{I}_{2\ell}^{\mathrm{good}}(\theta,\ell). \end{cases}$$

By Propositions 3.5 and 5.1 we have $(\mathrm{co}(D_t) \setminus D_t) \cap op \subset S_p(\theta,\ell,\xi)$ for all $p \in \partial\,\mathrm{co}(D_t)$. Thus denoting by

$$\Lambda(\theta,\ell;\xi) := \bigcup_{p\in\partial\,\mathrm{co}(D_t)} S_p(\theta,\ell;\xi),$$

we have

$$\mathrm{co}(D_t) \setminus D_t \subset \Lambda(\theta,\ell;\xi).$$

Fix a point $p \in \partial\,\mathrm{co}(D_t)$, and let $p' \in S_p(\theta,\ell;\xi) = op \cap \Lambda(\theta,\ell;\xi)$. It suffices to show that

$$p' \in \bigcup_{\mathfrak{q}\in\mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}.$$

Note that by Lemma 6.6 there exists a point $r' \in \partial\,\mathrm{co}(D_t)$ such that $r'p'$ is parallel to $\widehat{v}_{r'} = \widehat{v}_p$. Let r be the intersection of the line extended from the segment $\mathfrak{q}$ and the ray $p'r'$.

Note that $\angle rpp' \in (29°, 180° - 29°)$ by Corollary 3.4, and $\angle pp'r < \frac{1}{2}\alpha$ by Observation 6.4, so $\angle prp' \in (29° - \frac{1}{2}\alpha, 180° - 29°)$. Thus by the law of sines,

$$|r'p'| \le |rp'| = \frac{\sin(\angle rpp')}{\sin(\angle prp')}|pp'| \le 3|pp'|.$$

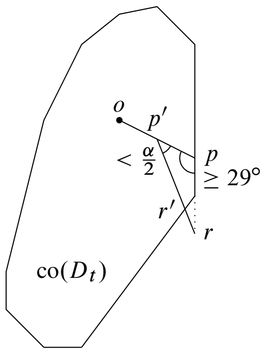

Fig. 15

If $\mathfrak{q} \in \mathcal{J}_{2\ell}^{\mathrm{good}}(\theta, \ell)$, then $|pp'| \leq \xi\sqrt{\gamma}$, so $|r'p'| \leq 3\xi\sqrt{\gamma}$, and letting $r' \in \mathfrak{r} \in \mathcal{J}(\theta, \ell)$ we have $p' \in R_{\mathfrak{r}} \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}$.

Alternatively if $\mathfrak{q} \in \mathcal{J}_{2\ell}^{\mathrm{bad}}(\theta, \ell)$ then $|pp'| \leq 5\sqrt{\gamma}$. Note that $|pr'| \leq |pp'| + |rp'| \leq 4|pp'| \leq \ell$, so r' is in an arc $\mathfrak{r} \in \mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta, \ell)$. Hence, $|r'p'| \leq 15\sqrt{\gamma}$, implying $p' \in R_{\mathfrak{r}} \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}$. ■

7. Preimages of the $R_{\mathfrak{q}}$ associated to A and B

By Proposition 6.5, for d_τ sufficiently small we have

$$\mathrm{co}(D_t) \setminus D_t \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} (R_{\mathfrak{q}} \setminus D_t).$$

The boundary of $\mathrm{co}(D_t)$ is composed of translates of edges from $\partial\,\mathrm{co}(A)$ scaled by a factor of t and of edges from $\partial\,\mathrm{co}(B)$ scaled by a factor of $1 - t$. If an edge of $\mathrm{co}(A)$ is parallel to an edge of $\mathrm{co}(B)$ then there is an ambiguity in how we do this; we fix one such decomposition from now on.

Definition 7.1. Let $\mathcal{J}(\theta, \ell) = \mathcal{A} \sqcup \mathcal{B}$ be the partition defined as follows. For every arc $\mathfrak{q} \in \mathcal{J}(\theta, \ell)$ (which is straight by construction), we let $\mathfrak{q} \in \mathcal{A}$ if $\mathfrak{q}$ is on a translated t-scaled edge from $\partial\,\mathrm{co}(A)$, and we let $\mathfrak{q} \in \mathcal{B}$ if $\mathfrak{q}$ is on a translated $(1-t)$-scaled edge from $\partial\,\mathrm{co}(B)$.

Definition 7.2. For $\mathfrak{q} \in \mathcal{A}$, let $p_{\mathfrak{q},B} \in \partial\,\mathrm{co}(B)$ and $R_{\mathfrak{q},A} \subset \mathbb{R}^2$ be the parallelogram with edge $\mathfrak{q}_A \subset \partial\,\mathrm{co}(A)$ such that

$$R_{\mathfrak{q}} = tR_{\mathfrak{q},A} + (1-t)p_{\mathfrak{q},B}.$$

Similarly, for $\mathfrak{q} \in \mathcal{B}$, let $p_{\mathfrak{q},A} \in \partial\,\mathrm{co}(A)$ and $R_{\mathfrak{q},B} \subset \mathbb{R}^2$ be the parallelogram with edge $\mathfrak{q}_B \subset \partial\,\mathrm{co}(B)$ such that

$$R_{\mathfrak{q}} = tp_{\mathfrak{q},A} + (1-t)R_{\mathfrak{q},B}.$$

Remark 7.3. The parallelogram $R_{\mathfrak{q},A}$ (resp. $R_{\mathfrak{q},B}$) may not be entirely contained inside $\mathrm{co}(A)$ (resp. $\mathrm{co}(B)$), and the various $R_{\mathfrak{q},A}$ with $\mathfrak{q} \in \mathcal{A}$ (respectively $R_{\mathfrak{q},B}$ with $\mathfrak{q} \in \mathcal{B}$) may not be disjoint.

Proposition 7.4. *For d_τ sufficiently small, we have*

$$|\mathrm{co}(D_t) \setminus D_t| \leq t^2 \sum_{\mathfrak{q}\in\mathcal{A}} |R_{\mathfrak{q},A} \setminus A| + (1-t)^2 \sum_{\mathfrak{q}\in\mathcal{B}} |R_{\mathfrak{q},B} \setminus B|.$$

Proof. Assume d_τ is sufficiently small that the conclusion of Proposition 6.5 holds. Then

$$\mathrm{co}(D_t) \setminus D_t \subset \bigcup_{\mathfrak{q}\in\mathcal{J}(\theta,\ell)} (R_{\mathfrak{q}} \setminus D_t).$$

The result then follows from the fact that

- if $\mathfrak{q} \in \mathcal{A}$ then $|R_{\mathfrak{q}} \setminus D_t| \leq |R_{\mathfrak{q}} \setminus (tA + (1-t)p_{\mathfrak{q},B})| = t^2 |R_{\mathfrak{q},A} \setminus A|$,
- if $\mathfrak{q} \in \mathcal{B}$ then $|R_{\mathfrak{q}} \setminus D_t| \leq |R_{\mathfrak{q}} \setminus (tp_{\mathfrak{q},A} + (1-t)B)| = (1-t)^2 |R_{\mathfrak{q},B} \setminus B|$. ∎

From Proposition 7.4, we see that if the preimages in A, B of these regions were disjoint and contained in $\mathrm{co}(A)$ and $\mathrm{co}(B)$, then we would immediately obtain $|\mathrm{co}(D_t) \setminus D_t| \leq t^2 |\mathrm{co}(A) \setminus A| + (1-t)^2 |\mathrm{co}(B) \setminus B|$.

Our goal will be to remove certain $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ to ensure that all the remaining parallelograms are disjoint and are entirely contained in $\mathrm{co}(A)$ and $\mathrm{co}(B)$, so that the total area of the $R_{\mathfrak{q},A}$ with $\mathfrak{q} \in \mathcal{A}$ that were removed is at most $\varepsilon|\mathrm{co}(A) \setminus A|$, and the total area of the $R_{\mathfrak{q},B}$ with $\mathfrak{q} \in \mathcal{B}$ that were removed is at most $\varepsilon|\mathrm{co}(B) \setminus B|$. This will imply Theorem 2.2.

8. Far away weighted averages in $\partial\,\mathrm{co}(D_t)$ lie in $\mathcal{J}^{\mathrm{good}}_{3\ell}(\theta, \ell)$

We now show that points on the $\partial\,\mathrm{co}(D_t)$ which are the t-weighted averages of points from $\partial\,\mathrm{co}(A)$, $\partial\,\mathrm{co}(B)$ that are at distance at least $20t^{-1}\ell$ lie in arcs from $\mathcal{J}^{\mathrm{good}}_{3\ell}(\theta, \ell)$.

The main application will be to show that for parallelograms $R_{\mathfrak{q}}$ with $\mathfrak{q} \in \mathcal{J}^{\mathrm{bad}}_{3\ell}(\theta, \ell)$, we know that the point and parallelogram or parallelogram and point in $\mathrm{co}(A)$ and $\mathrm{co}(B)$ whose t-weighted average gives $R_{\mathfrak{q}}$ are close to each other.

Proposition 8.1. *For d_τ sufficiently small, if $p \in \partial\,\mathrm{co}(D_t)$ with $p = tx_A + (1-t)y_B$, where $x_A \in \partial\,\mathrm{co}(A)$, $y_B \in \partial\,\mathrm{co}(B)$ and $|x_A y_B| \geq 20t^{-1}\ell$, then $p \in \mathfrak{q} \in \mathcal{J}^{\mathrm{good}}_{3\ell}(\theta, \ell)$.*

Proof. Let $\eta = \min(10\sqrt{3}\sin(\frac{\theta}{4}), \frac{\sqrt{3}}{2}\ell)$. Assume d_τ is sufficiently small so that the conclusion of Corollary 3.4 holds, and $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$ by Proposition 3.1. We will first show that x_{D_t} and y_{D_t} realize p as a $(\frac{1}{2}\theta, 19\ell)$-good point.

For the angle, note that by Observation 2.3 we have

$$\angle x_{D_t} p x_A \leq \sin^{-1}\left(\frac{|x_A x_{D_t}|}{|x_A p|}\right) \leq \sin^{-1}\left(\frac{\eta |o x_A|}{20t^{-1}\ell}\right) \leq \sin^{-1}\left(\frac{\eta}{10\sqrt{3}\,t^{-1}\ell}\right) \leq \frac{\theta}{4}$$

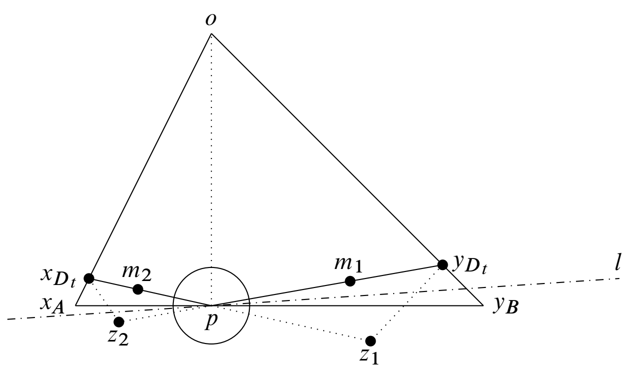

Fig. 16

and similarly $\angle y_{D_t} p y_B \leq \sin^{-1}\left(\frac{|y_B y_{D_t}|}{|y_B p|}\right) \leq \frac{\theta}{4}$. For the lengths, notice that $|x_{D_t} x_A| \leq \eta |o x_A| \leq \frac{\sqrt{3}}{2}\ell |o x_A| \leq \ell$ and similarly $|y_{D_t} y_A| \leq \ell$, so by the triangle inequality we have

$$
\begin{aligned}
|p x_{D_t}| &\geq |p x_A| - |x_{D_t} x_A| = (1-t)|x_A y_B| - |x_{D_t} x_A| \geq 20\ell - \ell = 19\ell, \\
|p y_{D_t}| &\geq |p y_B| - |y_{D_t} y_B| = t|x_A y_B| - |y_{D_t} y_A| \geq 20\ell - \ell = 19\ell.
\end{aligned}
$$

Now, we show that $p \in \mathfrak{q} \in \mathcal{I}_{3\ell}^{\text{good}}(\theta, \ell)$ by showing that if $p' \in \partial\, \mathrm{co}(D_t)$ and $|pp'| \leq 3\ell$, then p' is (θ, ℓ)-good. Denote by l the supporting line to $\mathrm{co}(D_t)$ at p, and note by Corollary 3.4 that $\angle l, op \in (29°, 180° - 29°)$. The line l intersects either the interior of the angle $\angle x_{D_t} p x_A$ or $\angle y_{D_t} p y_B$, so since we have already shown that $\angle x_{D_t} p x_A, \angle y_{D_t} p y_B \leq \theta/4$, we find that $x_A y_B$ makes an angle of at most $\theta/4$ with l. In particular, $\angle o p x_A, \angle o p y_B \in (29° - \theta/4, 180° - 29° + \theta/4) \subset (28°, 180° - 28°)$. Thus we may apply Lemma 5.3 to the triangles $x_A p o$ and $y_B p o$ to conclude that the distance from p to the lines $o x_A$ and $o y_B$ is at least

$$
\sin(14°) \min(|p x_A|, |po|, |p y_B|) \geq \sin(14°) 20\ell > 3\ell.
$$

Because $o x_{D_t} p y_{D_t} \subset \mathrm{co}(D_t)$, we conclude that p' lies outside of the angle $x_{D_t} p y_{D_t}$ (and because $p' \in \mathrm{co}(D_t)$, it lies on the same side of l as x_{D_t}, y_{D_t}).

Let z_1 be in the ray $x_{D_t} p$ extended past p such that $|z_1 p| = |z_1 y_{D_t}|$. Note that as $p z_1 y_{D_t}$ is isosceles, $\angle p z_1 y_{D_t} \geq \pi - \theta$, and note that $\angle y_{D_t} p z_1 \leq \theta/2$. Analogously let z_2 be the point at $p y_{D_t}$ which has $|z_2 x_{D_t}| = |z_2 p|$, so that $\angle p z_2 x_{D_t} \geq \pi - \theta$ and $\angle x_{D_t} p z_2 \leq \theta/2$. Finally, let m_1 be the midpoint of $p y_{D_t}$, and let m_2 be the midpoint of $p x_{D_t}$, so that $\angle p m_1 z_1 = \angle p m_2 z_2 = 90°$.

We claim that $p' \in p m_1 z_1 \cup p m_2 z_2$. First, note that by the above, p' lies in either the angular region $\angle m_1 p z_1$ or $\angle m_2 p z_2$. Thus as $p m_1 z_1, p m_2 z_2$ are right triangles, it suffices to note that $|p m_1|, |p m_2| \geq \frac{19}{2}\ell > 3\ell$. Therefore, $p' \in p m_1 z_1 \cup p m_2 z_2 \subset p y_{D_t} z_1 \cup p x_{D_t} z_2$. Hence, $\angle y_{D_t} p' x_{D_t} \geq \pi - \theta$ and p' is (θ, ℓ)-good since $|p' x_{D_t}|, |p' y_{D_t}| \geq 19\ell - 3\ell > \ell$ by the triangle inequality. ∎

9. Bound on parallelograms jutting out of co(A), co(B)

We will now show that the $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ which are not entirely contained in $\mathrm{co}(A)$ and $\mathrm{co}(B)$ have negligible total area.

Proposition 9.1. *For d_τ sufficiently small, we have*

$$\sum_{\mathfrak{q}\in\mathcal{A},\, R_{\mathfrak{q},A}\not\subset \mathrm{co}(A)} \left|R_{\mathfrak{q},A}\right| \le 25t^{-1}M\xi^2\gamma, \qquad \sum_{\mathfrak{q}\in\mathcal{B},\, R_{\mathfrak{q},B}\not\subset \mathrm{co}(B)} \left|R_{\mathfrak{q},B}\right| \le 25t^{-1}M\xi^2\gamma.$$

To prove this proposition, we first use Proposition 8.1 to show that for such parallelograms we have $\mathfrak{q} \in \mathcal{I}_{3\ell}^{\mathrm{good}}(\theta,\ell)$.

Lemma 9.2. *For d_τ sufficiently small, if either $\mathfrak{q}\in\mathcal{A}$ and $R_{\mathfrak{q},A}\not\subset \mathrm{co}(A)$, or $\mathfrak{q}\in\mathcal{B}$ and $R_{\mathfrak{q},B}\not\subset\mathrm{co}(B)$, then $\mathfrak{q}\in\mathcal{I}_{3\ell}^{\mathrm{good}}(\theta,\ell)$.*

Proof. The cases $\mathfrak{q}\in\mathcal{A}$ and $\mathfrak{q}\in\mathcal{B}$ are proved identically, so we will now suppose that $\mathfrak{q}\in\mathcal{A}$. Assume d_τ is sufficiently small so that Propositions 3.3 and 8.1 are true. Recall that we defined $p_{\mathfrak{q},B}\in\partial\,\mathrm{co}(B)$ and $\mathfrak{q}_A\subset\mathrm{co}(A)$ such that $\mathfrak{q}=t\mathfrak{q}_A+(1-t)p_{\mathfrak{q},B}$.

We first show that there exists a point $p_A\in\mathfrak{q}_A$ such that $\angle p_A o, v_{\mathfrak{q}}\ge 29°$. Indeed, by Proposition 3.3 we know that every point in $x\in\mathfrak{q}_A$ is $(59°,\frac{1}{3})$-bisecting in $\mathrm{co}(A)$. For $x\in\mathfrak{q}_A$, let $x'=x+t^{-1}v_{\mathfrak{q}}$, which lies on the opposite side of $\partial R_{\mathfrak{q},A}$ to x. Note that $|xx'|\le\frac{1}{10}$, so if $\angle ox, v_q\le 29°$, then $xx'\subset T_x(58°,\frac{1}{3})$. Hence, as $R_{\mathfrak{q},A}=\bigcup_{x\in\mathfrak{q}_A}xx'\not\subset$ $\mathrm{co}(A)$ but $\bigcup_{x\in\mathfrak{q}_A}T_x(58°,\frac{1}{3})\subset\mathrm{co}(A)$, we find a point $p_A\in\mathfrak{q}_A$ with $\angle p_A o, v_{\mathfrak{q}}\ge 29°$.

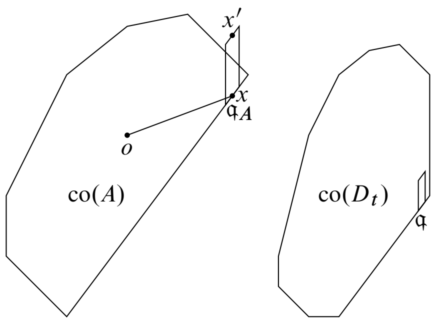

Fig. 17

Let $z=tp_A+(1-t)p_{\mathfrak{q},B}\in\mathfrak{q}$. By Observation 6.4, $\angle zo, v_{\mathfrak{q}}\le\frac{1}{2}\alpha$. Hence $\angle p_A oz\ge 29°-\frac{1}{2}\alpha\ge 28°$, so $|p_A z|\ge\sin(28°)|oz|>\frac{1}{100}$, so as z lies on the segment $p_A p_{\mathfrak{q},B}$, we have $|p_A p_{\mathfrak{q},B}|>\frac{1}{100}$. Note that by definition of $\ell=\ell(\theta)$ in Definition 4.3, we have $20t^{-1}\ell\le\frac{1}{100}$. Therefore, by Proposition 8.1 applied with $x_A=p_A$ and $y_B=p_{\mathfrak{q},B}$, we have $z\in\mathfrak{q}\in\mathcal{I}_{3\ell}^{\mathrm{good}}(\theta,\ell)$. ∎

We now know that parallelograms $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ which escape $\mathrm{co}(A)$ and $\mathrm{co}(B)$ have small height, since they are supported on arcs from $\mathcal{I}_{3\ell}^{\mathrm{good}}(\theta,\ell)$. By showing that such arcs with a constant direction v_p have small total length, we will obtain Proposition 9.1 (recalling M is the number of distinct v_p).

Proof of Proposition 9.1. The proof below works for the $\text{co}(B)$ inequality verbatim, so we focus on proving the $\text{co}(A)$ inequality. Take d_τ sufficiently small so that the conclusion of Proposition 3.3 holds, and so that $t^{-1}3\xi\sqrt{\gamma} \leq \frac{1}{4}\sin(1^\circ)$ by Observation 2.9.

By Lemma 9.2, all $\mathfrak{q} \in \mathcal{A}$ with $R_{\mathfrak{q},A} \not\subset \text{co}(A)$ are in $\mathcal{J}_{3\ell}^{\text{good}}(\theta, \ell)$. Fix one of the $\leq M$ vectors v with $|v| = 3\xi\sqrt{\gamma}$. It suffices to show

$$\sum_{\mathfrak{q}\in\mathcal{A},\, v_\mathfrak{q}=v,\, R_{\mathfrak{q},A}\not\subset\text{co}(A)} |R_{\mathfrak{q},A}| \leq 25t^{-1}\xi^2\gamma.$$

Recall that by construction v was chosen not parallel to any edge of $\text{co}(A)$. Let l, l' be the two lines in the direction v which are tangent to $\text{co}(A)$, and let y and y' be the points of contact with $\text{co}(A)$. Note that every line in the direction v between y and y' intersects each of the arcs $\partial\,\text{co}(A) \setminus \{y, y'\}$ exactly once. As $\text{co}(A)$ is convex, the cross-sectional slices in the v-direction satisfy unimodality. Hence there are exactly two pairs (x_1, x_2) and (x_1', x_2') of points in the two different arcs of $\partial\,\text{co}(A) \setminus \{y, y'\}$ such that we have the equality of vectors $x_1x_2 = x_1'x_2' = t^{-1}v$ – we let (x_1, x_2) be the pair closer to y.

We will show that the lengths of the two minor arcs in $\text{co}(A)$ between x_1x_2 and between $x_1'x_2'$ are both of length at most $24t^{-1}\sqrt{\gamma}$. We show this for x_1x_2 as the other case will be identical.

Note that $T_y(56^\circ, \frac{1}{4}) \subset T_y(59^\circ, \frac{1}{3}) \subset \text{co}(A)$. Let $z \in oy$ be such that $|yz| = t^{-1}3\xi\sqrt{\gamma} \leq \frac{1}{4}\sin(1^\circ)$ and denote by z_1, z_2 the intersections of the extensions of the arms of $T_y(56^\circ, \frac{1}{4})$ with the line through z with direction vector v. We will show that the line x_1x_2 is closer to y than the line z_1z_2 by showing that $|z_1z_2| \geq |x_1x_2|$ and applying unimodality.

Note that $\angle z_1yz = 28^\circ$ and $\angle z_1zy \in (29^\circ, 180^\circ - 29^\circ)$. Hence $\angle yz_1z \in (1^\circ, 180^\circ - 57^\circ)$ so $\sin\angle yz_1z \geq \sin(1^\circ)$. Thus by the law of sines,

$$|yz_1| = \frac{\sin\angle z_1zy}{\sin\angle yz_1z}|yz| \leq \frac{|yz|}{\sin 1^\circ} \leq \frac{1}{4}.$$

Hence $z_1 \in T_y(56^\circ, \frac{1}{4})$ and by a similar argument we obtain $z_2 \in T_y(56^\circ, \frac{1}{4})$.

Now,

$$|z_1z_2| \geq |z_1z| = \frac{\sin 28^\circ}{\sin\angle yz_1z}|yz| \geq \sin(28^\circ)|yz| = t^{-1}3\xi\sqrt{\gamma} = |x_1x_2|.$$

Thus by unimodality, the line x_1x_2 is closer than the line z_1z_2 to y, so denoting $x = oy \cap x_1x_2$ we see that x lies in the segment yz. Hence

$$|yx| \leq |yz| = t^{-1}3\xi\sqrt{\gamma}.$$

Note that there are up to two arcs $\mathfrak{q}_A$ which contain one of the points x_1, x_1', and as each arc in $\mathcal{J}(\theta, \ell)$ has length at most $\xi\sqrt{\gamma}$ by construction, the total length of these arcs is at most $2t^{-1}\xi\sqrt{\gamma}$.

If $v_\mathfrak{q} = v$ and $R_{\mathfrak{q},A} \not\subset \text{co}(A)$, then $\mathfrak{q}_A$ is contained in the arc of $\partial\,\text{co}(A) \setminus \{y, y'\}$ containing x_1, x_1', and $\mathfrak{q}_A$ intersects either the minor arc subtended by x_1y or the one

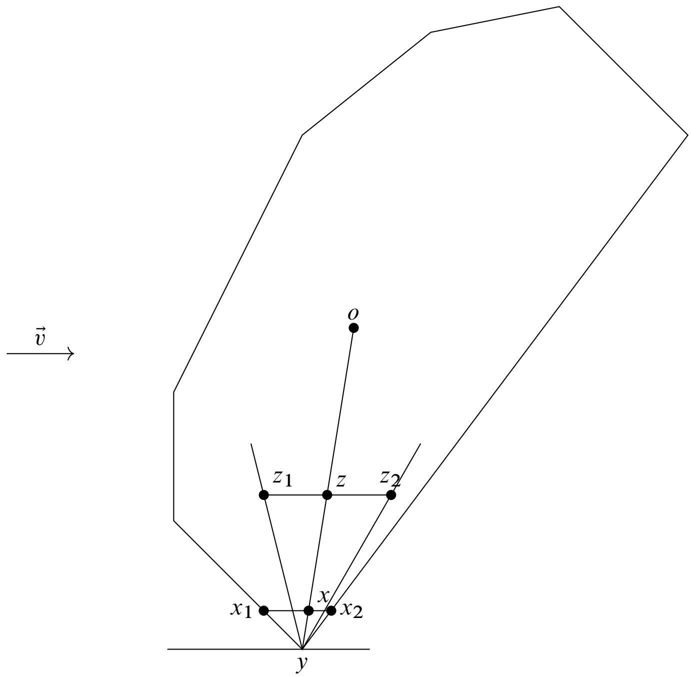

Fig. 18

subtended by $x_1'y'$. Indeed, let $\tilde{l}$ be the supporting line of $\mathfrak{q}$. Then for any point $p \in \mathfrak{q}$, by Proposition 3.3 we have $\angle po, \tilde{l} \in (29°, 180° - 29°)$, and by Observation 6.4, $\angle po, v_{\mathfrak{q}} \leq \alpha/2$. Hence $v_{\mathfrak{q}}$ lies on the same side of $\tilde{l}$ as $\mathrm{co}(D_t)$. Therefore $v_{\mathfrak{q}}$ lies on the same side of the supporting line $\tilde{l}_A$ to $\mathfrak{q}_A$ as $\mathrm{co}(A)$, so $\mathfrak{q}_A$ lies in the arc of $\mathrm{co}(A) \setminus \{y, y'\}$ that contains x_1, x_1'. Now, if $\mathfrak{q}_A$ does not intersect the minor arc $x_1 y$ or $x_1' y'$, then by unimodality, the v cross-sectional lengths of $\mathrm{co}(A)$ on the arc $\mathfrak{q}_A$ exceed $3\xi t^{-1}\sqrt{\gamma} = \|t^{-1}v\|$, which implies $R_{\mathfrak{q}_A}$ is contained inside $\mathrm{co}(A)$.

Hence, the total width (measured in the direction $v^{\perp}$) of such parallelograms $R_{\mathfrak{q},A}$ in direction v which are not contained in $\mathrm{co}(A)$ is at most $2 \cdot t^{-1}3\xi\sqrt{\gamma} + 2t^{-1}\xi\sqrt{\gamma} = 8t^{-1}\xi\sqrt{\gamma}$.

Because all of the arcs $\mathfrak{q}$ we are considering lie in $\mathcal{I}_{3\ell}^{\mathrm{good}}(\theta, \ell)$, the total area of such parallelograms is then at most

$$(8t^{-1}\xi\sqrt{\gamma})(3\xi\sqrt{\gamma}) = 24t^{-1}\xi^2\gamma. \qquad \blacksquare$$

10. Bounding overlapping parallelograms

We will now show that the $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ which we remove to guarantee non-overlapping have negligible area.

Proposition 10.1. *For d_τ sufficiently small, if $\mathfrak{q}, \mathfrak{q}' \in \mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta, \ell) \cap \mathcal{A}$, then we have $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| = 0$, and if $\mathfrak{q}, \mathfrak{q}' \in \mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta, \ell) \cap \mathcal{B}$, then $|R_{\mathfrak{q},B} \cap R_{\mathfrak{q}',B}| = 0$.*

Because of Proposition 10.1, it will suffice to bound overlaps between parallelograms supported on arcs in $\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta, \ell)$ with all other parallelograms.

Proposition 10.2. *For d_τ sufficiently small, we have*

$$\sum_{\mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell) \cap \mathcal{A} \text{ and } \exists \mathfrak{q}' \in \mathcal{A} \setminus \{\mathfrak{q}\} \text{ with } |R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| > 0} |R_{\mathfrak{q},A}| \le 16000 t^{-1} M \xi \gamma$$

and similarly with B and $\mathcal{B}$.

Proof of Proposition 10.1. The proof we give works verbatim for B and $\mathcal{B}$, so we focus on A and $\mathcal{A}$. We take d_τ sufficiently small such that the implications in Proposition 8.1 hold, and such that $\sqrt{\gamma} \le \ell$ by Observation 2.9. Because $\mathfrak{q}, \mathfrak{q}' \in \mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta, \ell)$, we have $\|v_{\mathfrak{q}}\| = \|v_{\mathfrak{q}'}\| = 15\sqrt{\gamma}$. Consider the arcs $\mathfrak{r}, \mathfrak{r}' \in \mathcal{J}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta, \ell)$ such that $\mathfrak{q} \subset \mathfrak{r}$ and $\mathfrak{q}' \subset \mathfrak{r}'$. If $\mathfrak{r} = \mathfrak{r}'$ then $v_{\mathfrak{q}} = v_{\mathfrak{q}'}$ so $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| = 0$.

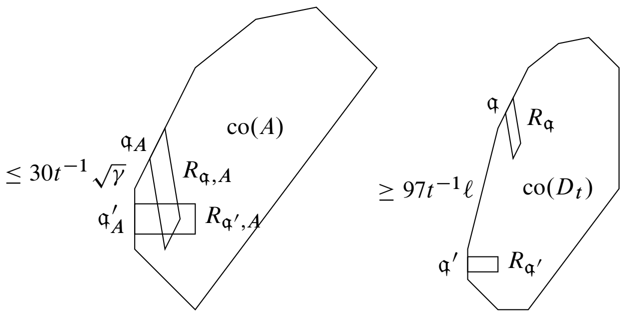

Fig. 19

Assume now that $\mathfrak{r} \ne \mathfrak{r}'$. In this case, the distance between $\mathfrak{q}$ and $\mathfrak{q}'$ is at least $97t^{-1}\ell$. Indeed, otherwise there exist points $p \in \mathfrak{q}$ and $p' \in \mathfrak{q}'$ such that $|pp'| \le 97t^{-1}\ell$. Let x be a (θ, ℓ)-bad point such that $|xp| \le 3\ell$. Then $B(x, 100t^{-1}\ell)$ contains p, and by the triangle inequality it also contains p'. This implies p, p' are contained in the same arc of $\mathcal{J}_{100t^{-1}\ell}^{\mathrm{bad}}(\theta, \ell)$, so $\mathfrak{r} = \mathfrak{r}'$, a contradiction.

Assume for the sake of contradiction that $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| > 0$. Then there exists a point $z \in R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}$. Since z is within distance $t^{-1}\|v_{\mathfrak{q}}\| = 15t^{-1}\sqrt{\gamma}$ of $\mathfrak{q}_A$ and within distance $t^{-1}\|v_{\mathfrak{q}'}\| = 15t^{-1}\sqrt{\gamma}$ of $\mathfrak{q}'_A$, we see by the triangle inequality that the distance between $\mathfrak{q}_A$ and $\mathfrak{q}'_A$ is at most $30t^{-1}\sqrt{\gamma} \le 30t^{-1}\ell$.

By the above, either there exist $p \in \mathfrak{q}$ and $z_A \in \mathfrak{q}_A$ such that $|pz_A| \ge 33t^{-1}\ell$, or there exist $p' \in \mathfrak{q}'$ and $z'_A \in \mathfrak{q}'_A$ such that $|p'z'_A| \ge 33t^{-1}\ell$. Suppose without loss of generality the first case holds. Then $p = tx_A + (1-t)y_B$ for some $x_A \in \mathfrak{q}$ and $y_B = p_{\mathfrak{q},B}$, and $|x_A z_A| \le \xi t^{-1}\sqrt{\gamma}$ since this is an upper bound for the length of $\mathfrak{q}_A$. Therefore,

$$|x_A y_B| \ge |x_A p| \ge |pz| - |x_A z| \ge 20t^{-1}\ell,$$

so by Proposition 8.1, $p \in \mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta, \ell)$, a contradiction. ∎

Proof of Proposition 10.2. The proof we give works verbatim for B and $\mathcal{B}$, so we focus on A and $\mathcal{A}$. Assume d_τ is sufficiently small so that the conclusion of Corollary 3.4 is true, and such that $\frac{99}{100}K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$ by Proposition 3.1. Fix one of the M directions v. Consider all arcs $\mathfrak{q} \in \mathcal{J}(\theta, \ell) \cap \mathcal{A}$ with direction vector $\widehat{v}_{\mathfrak{q}} = v$. Let $\mathfrak{r}_A$ be the union of the corresponding arcs $\mathfrak{q}_A$. Note that $\mathfrak{r}_A$ forms a connected arc of $\partial\,\mathrm{co}(A)$. Let x and x' be the endpoints of this arc.

For any point $z \in \mathfrak{r}_A$, we claim that $|xz| \le \frac{9}{\sin(14°)}\,\mathrm{dist}(z, ox)$. Indeed, by Lemma 5.3, since $|xz| \le 9|oz|$ (this follows as the diameter of $\mathrm{co}(A) \subset T'$ is at most $2/\sqrt{3}$ by Observation 2.3, and $|oz| \ge \frac{99}{100}\frac{1}{\sqrt{12}}$) it suffices to show that $\angle ozx \in (28°, 180° - 28°)$. By Corollary 3.4, we know that the supporting lines l_x, l_z to $\mathrm{co}(A)$ at x, z make an angle of at most $180° - 29°$ with ox, oz respectively. Therefore, we have $\angle ozx, oxz \le 180° - 29°$. By Observation 6.4, ox, oz each make an angle of at most $\frac{1}{2}\alpha$ with v. Therefore, $\angle xoz \le \alpha$. Because the sum of the angles in xoz is $180°$, this implies that $\angle ozx \in (29° - \alpha, 180° - 29°) \subset (28°, 180° - 28°)$.

Every y outside of $\mathfrak{r}_A$ is either on the opposite side of ox or on the opposite side of oy to $\mathfrak{r}_A$. This implies that $\min(zx, zx') \le \frac{9}{\sin(14°)}|yz|$ as y lies either on the other side of ox or on the other side of ox' to z.

We claim that if $R_{\mathfrak{q},A}$ with $\mathfrak{q}_A \subset \mathfrak{r}_A$ intersects some $R_{\mathfrak{q}',A}$ in positive area, then $\mathfrak{q}_A, \mathfrak{q}'_A \subset (B(x, 1200t^{-1}\sqrt{\gamma}) \cup B(x', 1200t^{-1}\sqrt{\gamma}))$. Indeed, first note that if $\mathfrak{q}'_A \subset \mathfrak{r}_A$, then $\widehat{v}_{\mathfrak{q}} = \widehat{v}_{\mathfrak{q}'}$, forbidding a positive area intersection. Hence $\mathfrak{q}_A$ lies outside of $\mathfrak{r}_A$. Note that if $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| > 0$, then the distance between $\mathfrak{q}_A$ and $\mathfrak{q}'_A$ is at most $30t^{-1}\sqrt{\gamma}$ by the triangle inequality (as the heights of these parallelograms are each at most $15t^{-1}\sqrt{\gamma}$). From this, we conclude that

$$\min(\mathrm{dist}(\mathfrak{q}_A, x), \mathrm{dist}(\mathfrak{q}_A, x')) \le \frac{9}{\sin(14°)}30t^{-1}\sqrt{\gamma} \le 1199t^{-1}\sqrt{\gamma}.$$

Because

$$|\mathfrak{q}_A| \le \xi t^{-1}\sqrt{\gamma} \le t^{-1}\sqrt{\gamma},$$

the conclusion follows.

The length of $\partial\,\mathrm{co}(A) \cap (B(x, 1200t^{-1}\sqrt{\gamma}) \cup B(x', 1200t^{-1}\sqrt{\gamma}))$ is at most $4800\pi t^{-1}\sqrt{\gamma}$, the sum of the perimeters of the two balls. Hence for each direction v we have

$$\sum_{\substack{\mathfrak{q} \in \mathcal{J}^{\mathrm{good}}_{3\ell}(\theta,\ell) \cap \mathcal{A},\, \widehat{v}_{\mathfrak{q}} = v, \\ \exists \mathfrak{q}' \in \mathcal{A} \setminus \{\mathfrak{q}\} \text{ with } |R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| > 0}} |R_{\mathfrak{q},A}| \le 4800\pi t^{-1}\sqrt{\gamma} \cdot \xi\sqrt{\gamma} = 16000t^{-1}\xi\gamma. \qquad \blacksquare$$

11. Proofs of Theorems 1.3 and 2.2

With all the machinery in place, we are now ready to tackle Theorem 2.2. We note that Theorems 1.3 and 2.2 are formally equivalent by replacing A with $\frac{1}{t}A$ and B with $\frac{1}{1-t}B$.

Proof of Theorem 2.2. Fix $\varepsilon > 0$ and choose ξ such that

$$\varepsilon \geq (t^2 + (1-t)^2)(25t^{-1}M\xi^2 + 16000t^{-1}M\xi).$$

Choose θ depending on ξ given by Proposition 5.1. Choose ℓ depending on θ given by Proposition 4.2. Recall that M, α are universal constants chosen above. Finally, take d_τ sufficiently small so that the conclusions of Propositions 4.2, 7.4, 9.1, 10.1 and 10.2 hold. Recall by Proposition 7.4 that

$$|\mathrm{co}(D_t) \setminus D_t| \leq t^2 \sum_{\mathfrak{q} \in \mathcal{A}} |R_{\mathfrak{q},A} \setminus A| + (1-t)^2 \sum_{\mathfrak{q} \in \mathcal{B}} |R_{\mathfrak{q},B} \setminus B|.$$

We split the first summand on the right into three parts: one for those $\mathfrak{q}$ such that $R_{\mathfrak{q},A} \not\subset \mathrm{co}(A)$ (collect them in a set X_A), one for those $\mathfrak{q} \in \mathcal{I}_{3\ell}^{\mathrm{good}}(\theta, \ell)$ such that $R_{\mathfrak{q},A}$ intersects non-trivially $R_{\mathfrak{q},A}$ for some $\mathfrak{q}' \neq \mathfrak{q}$ (collect them in a set Y_A), and all the other $\mathfrak{q}$ (collect them in a set Z_A). Note that the $R_{\mathfrak{q},A}$ in the last sum are disjoint by Proposition 10.1 and contained in $\mathrm{co}(A)$, so $\sum_{\mathfrak{q} \in Z_A} |R_{\mathfrak{q},A} \setminus A| \leq |\mathrm{co}(A) \setminus A|$. Combining Propositions 9.1 and 10.2 we find

$$\begin{aligned}\sum_{\mathfrak{q} \in \mathcal{A}} |R_{\mathfrak{q},A} \setminus A| &\leq \sum_{\mathfrak{q} \in X_A} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q} \in Y_A} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q} \in Z_A} |R_{\mathfrak{q},A} \setminus A| \\ &\leq 25t^{-1}M\xi^2\gamma + 16000t^{-1}M\xi\gamma + |\mathrm{co}(A) \setminus A|.\end{aligned}$$

We similarly obtain

$$\sum_{\mathfrak{q} \in \mathcal{B}} |R_{\mathfrak{q},B} \setminus B| \leq 25t^{-1}M\xi^2\gamma + 16000t^{-1}M\xi\gamma + |\mathrm{co}(B) \setminus B|.$$

Hence, (recalling $\gamma = t^2|\mathrm{co}(A) \setminus A| + (1-t)^2|\mathrm{co}(B) \setminus B|$), we have

$$\begin{aligned}|\mathrm{co}(D_t) \setminus D_t| &\leq (t^2 + (1-t)^2)(25t^{-1}M\xi^2 + 16000t^{-1}M\xi)\gamma \\ &\quad + t^2|\mathrm{co}(A) \setminus A| + (1-t)^2|\mathrm{co}(B) \setminus B| \\ &\leq (1+\varepsilon)\big(t^2|\mathrm{co}(A) \setminus A| + (1-t)^2|\mathrm{co}(B) \setminus B|\big).\end{aligned}$$ ■

12. Proof that Theorem 1.3 implies Theorem 1.2

Finally, what remains is to deduce Theorem 1.2. Note that we now return to A and B with unequal areas. Along the way, we will show Corollary 1.4.

Proof that Theorem 1.3 *implies Theorem* 1.2. By [9, 10] and Appendix A, there is a constant $\widetilde{C}$ such that

$$\frac{|K_A \setminus \mathrm{co}(A)|}{|\mathrm{co}(A)|} + \frac{|K_B \setminus \mathrm{co}(B)|}{|\mathrm{co}(B)|} \leq \widetilde{C}\tau_{\mathrm{conv}}^{-1/2}\sqrt{\delta_{\mathrm{conv}}},$$

where $\delta_{\text{conv}} = \frac{|\text{co}(A+B)|^{1/2}}{|\text{co}(A)|^{1/2}+|\text{co}(B)|^{1/2}} - 1$, and

$$t_{\text{conv}} = \frac{|\text{co}(A)|^{1/2}}{|\text{co}(A)|^{1/2} + |\text{co}(B)|^{1/2}} \in [\tau_{\text{conv}}, 1 - \tau_{\text{conv}}].$$

Also, by Theorem 1.6 by taking d_τ sufficiently small, we may assume that $|\text{co}(A)|/|A|$, $|\text{co}(B)|/|B|$, and $|\text{co}(A + B)|/|A + B|$ are as close to 1 as we like, so in particular we may assume that $\tau_{\text{conv}}^{-1} \le 2\tau^{-1}$. Thus it suffices to prove that

$$\delta_{\text{conv}} \le \delta \quad \text{and} \quad \frac{|\text{co}(A) \setminus A|}{|\text{co}(A)|} + \frac{|\text{co}(B) \setminus B|}{|\text{co}(B)|} \le 5\tau^{-1}\delta. \tag{$*$}$$

We have

$$\begin{aligned}
\delta - \delta_{\text{conv}} &\ge \frac{|A|^{1/2} + |B|^{1/2}}{|\text{co}(A)|^{1/2} + |\text{co}(B)|^{1/2}}\delta - \delta_{\text{conv}} \\
&= C\big(|\text{co}(A)|^{1/2} - |A|^{1/2} + |\text{co}(B)|^{1/2} - |B|^{1/2} - (|\text{co}(A+B)|^{1/2} - |A+B|^{1/2})\big) \\
&= C\left(\frac{|\text{co}(A)\setminus A|}{|\text{co}(A)|^{1/2}+|A|^{1/2}} + \frac{|\text{co}(B)\setminus B|}{|\text{co}(B)|^{1/2}+|B|^{1/2}} - \frac{|\text{co}(A+B)\setminus(A+B)|}{|\text{co}(A+B)|^{1/2}+|A+B|^{1/2}}\right) \\
&\ge C\left(\frac{|\text{co}(A)\setminus A|}{|\text{co}(A)|^{1/2}+|A|^{1/2}} + \frac{|\text{co}(B)\setminus B|}{|\text{co}(B)|^{1/2}+|B|^{1/2}} - \frac{(1+\varepsilon)(|\text{co}(A)\setminus A|+|\text{co}(B)\setminus B|)}{|\text{co}(A+B)|^{1/2}+|A+B|^{1/2}}\right)
\end{aligned}$$

with $C = \frac{1}{|\text{co}(A)|^{1/2}+|\text{co}(B)|^{1/2}}$. Suppose $t \le 1/2$ and take $\varepsilon = \tau/2$. We can write the last line as $m_A \frac{|\text{co}(A)\setminus A|}{|\text{co}(A)|} + m_B \frac{|\text{co}(B)\setminus B|}{|\text{co}(B)|}$ with

$$\begin{aligned}
m_A &= tC_A\left(\frac{1}{\frac{|\text{co}(A)|^{1/2}}{|A|^{1/2}} + 1} - \frac{1}{\frac{|\text{co}(A+B)|^{1/2}}{|A+B|^{1/2}} + 1} \cdot \frac{(1+\varepsilon)t}{(1+\delta)}\right) \\
&\ge tC_A\left(\frac{1}{\frac{|\text{co}(A)|^{1/2}}{|A|^{1/2}} + 1} - \frac{1}{\frac{|\text{co}(A+B)|^{1/2}}{|A+B|^{1/2}} + 1} \cdot \frac{3}{4}\right)
\end{aligned}$$

with $C_A = \frac{|\text{co}(A)|}{|A|} \cdot \frac{|A|^{1/2}+|B|^{1/2}}{|\text{co}(A)|^{1/2}+|\text{co}(B)|^{1/2}}$, and

$$\begin{aligned}
m_B &= (1-t)C_B\left(\frac{1}{\frac{|\text{co}(B)|^{1/2}}{|B|^{1/2}} + 1} - \frac{1}{\frac{|\text{co}(A+B)|^{1/2}}{|A+B|^{1/2}} + 1} \cdot \frac{(1+\varepsilon)(1-t)}{(1+\delta)}\right) \\
&\ge (1-t)C_B\left(\frac{1}{\frac{|\text{co}(B)|^{1/2}}{|B|^{1/2}} + 1} - \frac{1}{\frac{|\text{co}(A+B)|^{1/2}}{|A+B|^{1/2}} + 1} \cdot \left(1 - \frac{\tau}{2}\right)\right).
\end{aligned}$$

with $C_B = \frac{|\text{co}(B)|}{|B|} \cdot \frac{|A|^{1/2}+|B|^{1/2}}{|\text{co}(A)|^{1/2}+|\text{co}(B)|^{1/2}}$. Both of these are at least $\frac{1}{5}\tau$ assuming d_τ is sufficiently small. Thus we get

$$\delta - \delta_{\text{conv}} \ge \frac{1}{5}\tau\left(\frac{|\text{co}(A) \setminus A|}{|\text{co}(A)|} + \frac{|\text{co}(B) \setminus B|}{|\text{co}(B)|}\right)$$

which shows ($*$). ∎

Appendix A. Equivalence of the measures ω and α

In this appendix, we show that in two dimensions the measures ω and α are commensurate for convex sets when d_τ is sufficiently small. Recall from the introduction that we always have $\alpha \leq 2\omega$.

Proposition A.1. *For all $\tau \in (0, \frac{1}{2}]$, there exists a $d_\tau > 0$ such that the following holds. If $E, F \subset \mathbb{R}^2$ are convex with $t(E, F) \in [\tau, 1-\tau]$ and $\delta(E, F) \leq d_\tau$, then*

$$\omega(E, F) \leq 21\alpha(E, F).$$

Proof. Let d_τ be sufficiently small so that by [9], $\alpha(E, F) \leq \frac{1}{10}$. We never use any other property of $\delta(E, F)$ or $t(E, F)$. The quantitites ω, α are invariant under affine transformations of E and F separately, so by applying these transforms we can take E, F to have equal volumes, translated so that $\alpha(E, F) = |E \triangle F|/|E|$. After a further affine transformation, we may assume that the maximal triangle $T \subset E \cap F$ is a unit equilateral triangle. Note that because T is maximal, we have $T \subset E \cap F \subset -2T$. Take $K = \mathrm{co}(E \cup F)$. Note that $|E \triangle F| \leq \frac{1}{18}|E \cap F| \leq \frac{1}{18}|-2T| \leq \frac{1}{2}$.

First, we claim that $E, F \subset 10C$. Indeed, if any point $x \in E$ lies in $\partial 10T$ then $|E \triangle F| \geq |\mathrm{co}(x \cup T) \setminus (-2T)| \geq 1$, a contradiction.

To show $\omega(E, F) \leq 11\alpha(E, F)$, it suffices to prove

$$|(K \setminus (A \cup B))| \leq 10|A \triangle B|.$$

Indeed, if this is true, then

$$\begin{aligned}|E| \cdot \omega(E, F) &\leq |K \setminus E| + |K \setminus F| = 2|K \setminus (E \cup F)| + |E \triangle F| \\ &\leq 21|E \triangle F| = |E| \cdot 21\alpha(E, F).\end{aligned}$$

We consider the triangle opq with p, q consecutive vertices of K. These triangles partition the area of K, so it suffices to show for each such triangle that

$$|(K \setminus (E \cup F)) \cap opq| \leq 10|(E \triangle F) \cap opq|.$$

To see this, we note that if $p, q \in E$ or $p, q \in F$ then the left-hand side is zero and the inequality holds. Suppose now that $p \in E$ and $q \in F$ (the other case is identical). Then there must be a point $i \in \partial\,\mathrm{co}(A) \cap \partial\,\mathrm{co}(F)$ which lies in the triangle opq. Let q' be the intersection of the ray pi with the segment oq, and let p' be the intersection of the ray qi with op. Because $o, p \in E$ we also have $p' \in E$, and similarly $q' \in F$. We note that $E, F \subset 10C$ implies $|op'| \geq \frac{1}{10}|oq|$ and $|oq'| \geq \frac{1}{10}|oq|$.

If any point x in the strict interior $(qiq')^\circ$ lies in E, then i lies in the strict interior of $xpo \subset E$, contradicting that i lies on ∂E. Also, $qiq' \subset oqi \subset F$. Thus $(qiq')^\circ \subset E \triangle F$. Similarly $(pip')^\circ \subset E \triangle F$. Finally, we note that $(K \setminus (E \cup F)) \cap opq \subset piq$, so it suffices to show that

$$|piq| \leq 10(|pip'| + |qiq'|).$$

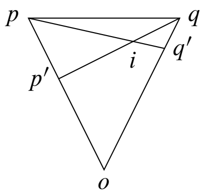

Fig. 20

To show this, suppose without loss of generality that $|oiq| \le |oip|$. Then $\frac{|piq|}{|oiq|} = \frac{|pip'|}{|oip'|}$, so

$$|piq| = |pip'|\frac{|oiq|}{|oip'|} \le |pip'|\frac{|oip|}{|oip'|} = |pip'|\frac{|op|}{|op'|} \le 10|pip'|. \qquad \blacksquare$$

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References

[1] Barchiesi, M., Julin, V.: Robustness of the Gaussian concentration inequality and the Brunn–Minkowski inequality. Calc. Var. Partial Differential Equations **56**, art. 80, 12 pp. (2017) Zbl 1378.60042 MR 3646982

[2] Böröczky, K. J., Lutwak, E., Yang, D., Zhang, G.: The log-Brunn–Minkowski inequality. Adv. Math. **231**, 1974–1997 (2012) Zbl 1258.52005 MR 2964630

[3] Carlen, E., Maggi, F.: Stability for the Brunn–Minkowski and Riesz rearrangement inequalities, with applications to Gaussian concentration and finite range non-local isoperimetry. Canad. J. Math. **69**, 1036–1063 (2017) Zbl 1379.26021 MR 3693147

[4] Christ, M.: Near equality in the Brunn–Minkowski inequality. arXiv:1207.5062 (2012)

[5] Figalli, A., Jerison, D.: Quantitative stability for sumsets in $\mathbb{R}^n$. J. Eur. Math. Soc. **17**, 1079–1106 (2015) Zbl 1325.49052 MR 3346689

[6] Figalli, A., Jerison, D.: Quantitative stability for the Brunn–Minkowski inequality. Adv. Math. **314**, 1–47 (2017) Zbl 1380.52010 MR 3658711

[7] Figalli, A., Jerison, D.: A sharp Freiman type estimate for semisums in two and three dimensional Euclidean spaces. Ann. Sci. École Norm. Sup. (4) **54**, 235–257 (2021) Zbl 1482.11139 MR 4245865

[8] Figalli, A., Maggi, F., Mooney, C.: The sharp quantitative Euclidean concentration inequality. Cambridge J. Math. **6**, 59–87 (2018) Zbl 1385.39005 MR 3786098

[9] Figalli, A., Maggi, F., Pratelli, A.: A refined Brunn–Minkowski inequality for convex sets. Ann. Inst. H. Poincaré C Anal. Non Linéaire **26**, 2511–2519 (2009) Zbl 1192.52015 MR 2569906

[10] Figalli, A., Maggi, F., Pratelli, A.: A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math. **182**, 167–211 (2010) Zbl 1196.49033 MR 2672283

[11] Osserman, R.: Bonnesen-style isoperimetric inequalities. Amer. Math. Monthly **86**, 1–29 (1979) MR 519520

[12] van Hintum, P., Spink, H., Tiba, M.: Sharp stability of Brunn–Minkowski for homothetic regions. J. Eur. Math. Soc. **24**, 4207–4223 (2022) Zbl 1501.52007 MR 4493623