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Comparison of prismatic cohomology and derived de Rham cohomology

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Abstract. We establish a comparison isomorphism between prismatic cohomology and derived de Rham cohomology respecting various structures, such as their Frobenius actions and filtrations. As an application, when X is a proper smooth formal scheme over $\mathcal{O}_K$ with K being a p-adic field, we improve Breuil–Caruso's theory on comparison between torsion crystalline cohomology and torsion étale cohomology.

Keywords. Prismatic cohomology, derived de Rham cohomology, Kisin modules, crystalline cohomology

Contents

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1. Introduction

Let k be a perfect field of characteristic $p > 0$ and K a totally ramified degree e field extension of $W(k)[1/p]$. Fix an algebraic closure $\overline{K}$ of K, denote its p-adic completion by $\mathbf{C}$, and use $\mathcal{O}_{\mathbf{C}}$ to denote its ring of integers. Let X be a smooth proper formal scheme over $\mathcal{O}_K$ with (rigid analytic) geometric generic fiber $X_{\overline{\eta}}$. Write $X_n := X \times_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Starting from [17] and [21], much effort has been put into investigating the relationship between the crystalline cohomology (and other variants) and the étale cohomology attached to X.

When $e = 1$, it was proved by Fontaine–Messing [17] and Kato [21] that if X is a proper[1] smooth scheme over $\mathcal{O}_K = W(k)$, then $\mathrm{H}^i_{\mathrm{crys}}(X_n/W_n(k))$ admits a Fontaine–Laffaille module structure when $i \leq p-1$ and the functor T_{crys} on the category of Fontaine–Laffaille modules (from Fontaine–Laffaille theory) satisfies $T_{\mathrm{crys}}(\mathrm{H}^i_{\mathrm{crys}}(X_n/W_n(k))) \simeq \mathrm{H}^i_{\mathrm{\acute{e}t}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ as G_K-modules when $i \leq p-2$.

When $e > 1$, a more complicated base ring has to be introduced. Fix a uniformizer π of K and $E = E(u) \in W(k)[u]$ the Eisenstein polynomial of π. Let S be the p-adic completion of the PD envelope of $W(k)[u]$ for the ideal (E). Note that S admits:

- a Frobenius action $\varphi : S \to S$ which extends the Frobenius φ on $W(k)$ and satisfies $\varphi(u) = u^p$;
- a filtration $\mathrm{Fil}^i S$ which is the p-complete i-th PD ideal;
- a monodromy operator $N : S \to S$ via $N(f(u)) = \frac{df}{du}(-u)$.

In [10], Breuil introduced the notion of a *Breuil module* to describe the structure of $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$, and constructed a functor $T_{\mathrm{st},\star}$ from the category of Breuil modules to the category of $\mathbb{Z}_p$-representations of G_K. Here, a Breuil module is a datum consisting of a finite S-module $\mathcal{M}$ together with a one-step filtration $\mathrm{Fil}^h \mathcal{M} \subset \mathcal{M}$, a "divided Frobenius" $\varphi_h : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$, and a monodromy operator $N : \mathcal{M} \to \mathcal{M}$ which satisfies some conditions given in Section 6.3.

Following the ideas of Breuil, Caruso proved the following.

[1] Projective in Kato's paper.

Theorem 1.1 ([13]). *Let X be a proper semistable scheme over $\mathcal{O}_K$. Then its log-crystalline cohomology $\mathrm{H}^i_{\text{log-crys}}(X_n/S_n)$ has a Breuil module structure and*

$$T_{\mathrm{st},\star}(\mathrm{H}^i_{\text{log-crys}}(X_n/S_n)) \simeq \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i) \quad \textit{as } G_K\textit{-modules}$$

for $e(i+1) < p-1$ if $n > 1$, and for $ei < p-1$ if $n = 1$.

As new cohomology theories have been introduced in [7–9], it is natural to ask whether in these new cohomology theories one can recover the aforementioned results due to Fontaine–Messing, Breuil, and Caruso, and hopefully even improve them. In this paper, we use these new cohomology theories, in particular, prismatic cohomology and derived de Rham cohomology, to study torsion crystalline cohomology, torsion étale cohomology, and their relationship. We obtain the following result:

Theorem 1.2. *Let X be a smooth proper formal scheme over $\mathcal{O}_K$ with geometric generic fiber $X_{\overline{\eta}}$, and let i be an integer satisfying $ei < p-1$. Then $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ has the structure of a Breuil module and*

$$T_{\mathrm{st},\star}(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)) \simeq \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i) \quad \textit{as } \mathbb{Z}_p[G_K]\textit{-modules.}$$

Here the additional data of the Breuil module structure is roughly given by the following:

- the filtration is given by the cohomology of the PD powers of a natural PD ideal sheaf $\mathcal{I}_{\mathrm{crys}}$ on the crystalline site $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[h]}_{\mathrm{crys}})$;
- the N is a disguise of the connection given by the crystal nature of crystalline cohomology;
- the divided Frobenius is induced by a natural map of (quasi)syntomic sheaves.

From now on, when we talk about $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$, we always implicitly think of it carrying these additional data.

Remark 1.3. (1) Let us highlight the difference between Caruso's results and our theorem above.

(a) The X in our theorem is a *smooth* proper *formal* scheme over $\mathcal{O}_K$, whereas the X in [13] is a semistable $\mathcal{O}_K$-model of a smooth proper K-variety.

(b) Our restriction on e and i is $ei < p-1$ for any n, while the restriction in [13] is $ei < p-1$ for $n = 1$ and $e(i+1) < p-1$ for $n > 1$.

(2) We actually use another functor T_S relating torsion crystalline and étale cohomology in the above theorem. But T_S and $T_{\mathrm{st},\star}$ are essentially the same; see Section 8.2.

Now let us discuss the strategy of this paper to see how prismatic cohomology and (derived) de Rham cohomology come into the picture. Let $\mathfrak{S} = W(k)[\![u]\!]$ equipped with the Frobenius morphism φ extending the (arithmetic) Frobenius φ on $W(k)$ and $\varphi(u) = u^p$. Then $(\mathfrak{S}, (E))$ is the so-called Breuil–Kisin prism. Classically, an (*étale*) *Kisin module* of height h is a finite u-torsionfree $\mathfrak{S}$-module $\mathfrak{M}$ together with a semilinear map

$\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by E^h. By definition, $\varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ admits a *Breuil–Kisin (BK) filtration* $\mathrm{Fil}^h\, \varphi^*\mathfrak{M} := (1 \otimes \varphi_{\mathfrak{M}})^{-1}(E^h\mathfrak{M})$, which plays an important technical role later. It is well-known that Kisin module theory is a powerful tool in *abstract* integral p-adic Hodge theory: the study of $\mathbb{Z}_p$-lattices in crystalline (semistable) representations and their mod p^n representations, which can been seen as the arithmetic counterpart of $\mathrm{H}^n_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}_p)$ and $\mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$. Also the relationships between Kisin modules, Galois representations and Breuil modules are known in the abstract theory. In particular, the functor $\underline{\mathcal{M}} : \mathfrak{M} \mapsto \underline{\mathcal{M}}(\mathfrak{M}) := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ sends a Kisin module $\mathfrak{M}$ of height $h \leq p-1$ to a Breuil module (without N-structures) where

$$\mathrm{Fil}^h\, \mathcal{M}(\mathfrak{M}) := \{x \in \mathcal{M}(\mathfrak{M}) \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^h\, S \otimes_{\mathfrak{S}} \mathfrak{M}\} \subset \underline{\mathcal{M}}(\mathfrak{M})$$

and $\varphi_h : \mathrm{Fil}^h\, \mathcal{M}(\mathfrak{M}) \xrightarrow{1\otimes\varphi_{\mathfrak{M}}} \mathrm{Fil}^h\, S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_h\otimes 1} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \underline{\mathcal{M}}(\mathfrak{M})$ where $\varphi_h : \mathrm{Fil}^h\, S \to S$ is defined by $\varphi_h(x) = \varphi(x)/p^h$. See Section 6.3 for more details.

It turns out that prismatic cohomology $\mathrm{H}^i_{\Delta}(X/\mathfrak{S})$ gives geometric realizations of Kisin modules, in the sense that $\mathrm{H}^i_{\Delta}(X/\mathfrak{S})$ modulo its u^∞-torsion submodule is an étale Kisin module of height i (see Sections 6.2 and 7.1 and the discussion below for more details). As suggested by the functor $\underline{\mathcal{M}}$ in the abstract theory, one naturally expects the following comparison between Breuil–Kisin prismatic cohomology and crystalline cohomology:

$$\mathrm{R}\Gamma_{\Delta}(X/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S},\varphi} S \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/S). \tag{1.4}$$

This comparison follows from [9, Theorem 5.2] and base change of prismatic cohomology. This has been pointed out to us by Koshikawa.

Inspired by the above discussion, we show the following comparison result:

Theorem 1.5 (see Theorems 3.5 and 3.11). *Let* (A, I) *be a bounded prism, and let* X *be a smooth proper* (p*-adic*) *formal scheme over* $\mathrm{Spf}(A/I)$. *Then we have a functorial isomorphism*

$$\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_{A,\varphi_A} A \otimes^{\mathbb{L}}_{A} \mathrm{dR}^{\wedge}_{(A/I)/A} \cong \mathrm{R}\Gamma(X, \mathrm{dR}^{\wedge}_{-/A}),$$

which is compatible with base change in the prism (A, I).

Here $\mathrm{dR}^{\wedge}_{-/A}$ denotes the (relative to A) p-adic derived de Rham complex introduced by Illusie [19, Chapter VIII] and studied extensively by Bhatt [3]. In fact, when A/I is p-torsionfree, this is known due to [9, Theorem 5.2]. Our proof follows closely the proof of crystalline comparison in [9].

As a consequence, the above gives several comparison results, all of which were known due to work of Bhatt, Morrow, and Scholze [7–9].

Example 1.6. By [3, Theorem 3.27], when A/I is p-torsionfree, the derived de Rham complex appearing above is given by certain crystalline cohomology. With this, we can explain what the above comparison gives in concrete situations.

(1) *BMS2/Breuil–Kisin prism:* When $(A, I) = (\mathfrak{S}, (E))$, the above comparison becomes equation (1.4), which, as mentioned above, was obtained in [9]. As a consequence, Breuil's crystalline cohomology groups $\mathrm{H}^i_{\mathrm{crys}}(X/S)$ are finitely presented S-modules; see Proposition 7.19. To the best of our knowledge, coherence of S is unknown, and we are unaware of any other means showing that these cohomology groups are finitely presented. We thank Bhatt for pointing out this application to us.

(2) *BMS1:* When $(A, I) = (A_{\mathrm{inf}}, \ker(\theta))$ is the perfect prism associated with $\mathcal{O}_{\mathbf{C}}$, then the above comparison says

$$\mathrm{R}\Gamma_{\Delta}(X/A_{\mathrm{inf}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{crys}} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(X/A_{\mathrm{inf}}).$$

Recall that [9, Theorem 17.2] states that the first base change of the left hand side gives the A_{inf}-cohomology theory constructed in [7]. Then our comparison here becomes the one established by [7, Theorem 1.8 (iii)] (see also [35]).

(3) *PD prism:* Suppose $I \subset A$ admits a PD structure γ. Then our comparison implies

$$\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_{A,\varphi_A} A \cong \mathrm{R}\Gamma_{\mathrm{crys}}(X/(A, I, \gamma)).$$

When $I = (p)$, then the above is nothing but the crystalline comparison established in [9, Theorem 1.8 (1)]. Notice here the left hand side does not depend on the choice of γ, consequently neither does the right hand side. Another class of potentially interesting PD prisms consists of $(W(S), V(1))$ for any bounded p-complete ring S.

(4) *De Rham comparison:* There is a natural map $\mathrm{gr}^0 : \mathrm{dR}^{\wedge}_{R/A} \to R^{\wedge}$ given by "quotienting out" the first Hodge filtration. Our comparison result above, after composing with this further base change, gives

$$\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_{A,\varphi_A} A \otimes^{\mathbb{L}}_{A} A/I \cong \mathrm{R}\Gamma_{\mathrm{dR}}(X/(A/I))^{\wedge};$$

here, we have used [18, Proposition 3.11] to identify the result of the right hand side under this base change. This is the de Rham comparison given by [9, Theorem 1.8 (3)].

In (1)–(3) above, the crystalline comparison [9, Theorem 5.2] also yields comparison isomorphisms. Note that there are at least two comparison isomorphisms in the above discussion, and we have just claimed that they give rise to commutative diagrams, which might worry some readers. To reassure those readers, we establish the following rigidity of p-adic derived de Rham cohomology theory.

Theorem 1.7 (see Theorem 3.14 and Remark 3.15). *Let (A, I) be a prism such that A/I is p-torsionfree. Then the functor $R \mapsto \mathrm{dR}^{\wedge}_{R/A}$ from the category of smooth (A/I)-algebras to $\mathrm{CAlg}(D(\mathrm{dR}^{\wedge}_{(A/I)/A}))$ has no automorphism. A similar statement holds for the functor $R \mapsto \mathrm{dR}^{\wedge}_{R/(A/I)}$.*

Therefore, whenever one has a diagram of functorial comparisons between various cohomology theories and p-adic derived de Rham cohomology, the diagram is always forced to be commutative. Our method of proving such rigidity is largely inspired by [6, Sections 10.3 and 10.4] and [9, Section 18]. In view of rigidity aspects of p-adic derived de Rham complexes, we would like to mention a recent result of Mondal [31]: roughly speaking, there is a *unique* deformation of de Rham cohomology from characteristic p to Artinian local rings given by crystalline cohomology (cf. [6, Theorem 10.1.2] for the case of deformation over $\mathbb{Z}_p$). Let us mention that in a recent collaboration between Mondal and the first named author [25], endomorphisms of p-adic derived de Rham cohomology are computed in various p-adic settings.

Next, we discuss compatibility of additional structures on both sides being compared in Theorem 1.5, most notably the Frobenius action and filtration. In Section 2.3 we define a natural Frobenius action on the p-adic derived de Rham complex assuming the base ring A is a p-torsionfree δ-ring. Therefore the right hand side is equipped with a Frobenius action. The left hand side admits a Frobenius action as well, by extending the Frobenius action on prismatic cohomology, as $A \to \mathrm{dR}^{\wedge}_{(A/I)/A}$ is compatible with Frobenii on them. The two Frobenii on the two sides in Theorem 1.5 agree when A is p-torsionfree; see Remark 3.6. Let us remark that these p-torsionfree conditions can most likely be relaxed, with extra work in developing the theory of "derived δ-rings". We expect the above theoretical results to hold verbatim.

The story of comparing filtrations is our main new contribution to this theory and it is quite involved. Let us rewrite the comparison:

$$\varphi^*\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_A \mathrm{dR}^{\wedge}_{(A/I)/A} \cong \mathrm{R}\Gamma(X, \mathrm{dR}^{\wedge}_{-/A}).$$

There are three natural filtrations here:

- the Nygaard filtration $\mathrm{Fil}^{\bullet}_{\mathrm{N}}(\Delta^{(1)}_{-/A})$ on $\varphi^*\mathrm{R}\Gamma_{\Delta}(X/A)$ (see [9, Section 15]);
- the I-adic filtration on A;
- the Hodge filtration $\mathrm{Fil}^{\bullet}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{-/A})$ on $\mathrm{dR}^{\wedge}_{(A/I)/A}$ and $\mathrm{R}\Gamma(X, \mathrm{dR}^{\wedge}_{-/A})$.

They are related in the following fashion.

Theorem 1.8 (see Corollary 4.23). *Let (A, I) be a prism such that A/I is p-torsionfree, and let X be a smooth proper (p-adic) formal scheme over* $\mathrm{Spf}(A/I)$.

(1) *The isomorphism in Theorem 1.5 refines to a filtered isomorphism*

$$\big(\mathrm{R}\Gamma(X, \mathrm{Fil}^{\bullet}_{\mathrm{N}}(\Delta^{(1)}_{-/A}))\big) \mathbin{\widehat{\otimes}}^{\mathbb{L}}_{(A,I^{\bullet})} (\mathcal{A}, \mathcal{I}^{[\bullet]}) \cong \mathrm{R}\Gamma(X, \mathrm{Fil}^{\bullet}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{-/A})),$$

where the left hand side denotes the p-complete derived tensor product of filtered objects over the filtered ring $(A, I^{\bullet})$ provided by the lax symmetric monoidal structure on the filtered derived infinity category.

In particular, we obtain a graded isomorphism between graded algebras

$$\mathrm{gr}^*_{\mathrm{N}}\,\mathrm{R}\Gamma(X, \Delta^{(1)}_{-/A}) \mathbin{\widehat{\otimes}}^{\mathbb{L}}_{\mathrm{Sym}^*_{A/I}(I/I^2)} \Gamma^*_{A/I}(I/I^2) \cong \mathrm{gr}^*_{\mathrm{H}}\,\mathrm{R}\Gamma(X, \mathrm{dR}^{\wedge}_{-/A}).$$

(2) *The isomorphism in Theorem 1.5 induces natural isomorphisms*

$$\mathrm{R}\Gamma(X, \Delta^{(1)}_{-/A}/\mathrm{Fil}^i_{\mathrm{N}}) \cong \mathrm{R}\Gamma(X, \mathrm{dR}^{\wedge}_{-/A}/\mathrm{Fil}^i_{\mathrm{H}}) \quad \textit{for all } i \le p.$$

Moreover, these isomorphisms are functorial in X and A.

Here $\mathcal{A}$ denotes the p-adic PD envelope of $A \twoheadrightarrow A/I$, and $\mathcal{I}^\bullet$ denotes the filtration of PD powers of the ideal $\ker(\mathcal{A} \twoheadrightarrow A/I)$. For a (somewhat) concrete description of the filtration on $\varphi^*\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_A \mathrm{dR}^{\wedge}_{(A/I)/A}$ appearing in the above theorem, we refer the readers to [18, Construction 3.9]. Note that by combining the aforementioned result of Bhatt [3, Theorem 3.27] and a classical result of Illusie [19, Corollaire VIII.2.2.8], there is a natural filtered isomorphism $(\mathrm{dR}^{\wedge}_{(A/I)/A}, \mathrm{Fil}^\bullet_{\mathrm{H}}) \cong (\mathcal{A}, \mathcal{I}^\bullet)$.

In [20, Section 2], a comparison of Nygaard and Hodge filtrations is established for a crystalline base prism $(A, I) = (W(k), (p))$, in particular his A/I is entirely p-torsion. It seems reasonable to expect the comparison of filtrations holds for general base prisms. Both Bhatt and Illusie have sketched to us an approach of resolving base prism by prisms (A, I) such that A/I is p-torsionfree, to reduce the general comparison to our theorem above. We do not pursue that direction further in this paper, as our final application only uses the comparison when the base is the Breuil–Kisin prism.

With the above general preparation, we are ready to show

$$\underline{\mathcal{M}}(\mathrm{H}^i_{\Delta}(X/\mathfrak{S})) \simeq \mathrm{H}^i_{\mathrm{crys}}(X/S)$$

(when $\mathrm{H}^i_{\Delta}(X/\mathfrak{S})$ is u-torsionfree). In order to treat p^n-torsion cohomologies in Theorem 1.2, we consider the derived mod p^n variants of the aforementioned cohomology theories. For example, we denote the p^n-torsion prismatic cohomology as $\mathrm{R}\Gamma_{\Delta}(X_n/A_n) := \mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. As pointed out by Warning 7.1, such p^n-torsion prismatic cohomology *does not only* depend on $X_n = X \times_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}$. But it is enough for our purpose to understand the p^n-torsion crystalline cohomology $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ and its relation to étale cohomology $\mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$.

Note that the cohomology groups of $\mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n)$ do fit in our setting of *generalized* Kisin module $\mathfrak{M}$ of height h (discussed in Section 6.1), i.e. a finitely generated $\mathfrak{S}$-module $\mathfrak{M}$ together with a $\varphi_{\mathfrak{S}}$-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ and an $\mathfrak{S}$-linear map $\psi : \mathfrak{M} \to \varphi^*\mathfrak{M}$ such that $\psi \circ (1 \otimes \varphi_{\mathfrak{M}}) = E^h \,\mathrm{id}_{\varphi^*\mathfrak{M}}$ and $(1 \otimes \varphi_{\mathfrak{M}}) \circ \psi = E^h \,\mathrm{id}_{\mathfrak{M}}$. The generalized Kisin module is a natural extension of the classical (étale) Kisin module discussed above allowing u-torsion. In particular, an étale Kisin module $\mathfrak{M}$ of height h is a generalized Kisin module of height h without u-torsion, where ψ is just defined by $\mathfrak{M} \simeq E^h\mathfrak{M} \overset{(1\otimes\varphi_{\mathfrak{M}})^{-1}}{\simeq} \mathrm{Fil}^h_{\mathrm{BK}}\, \varphi^*\mathfrak{M} \subset \varphi^*\mathfrak{M}$, and similarly the BK filtration can be extended to a generalized Kisin module by defining $\mathrm{Fil}^h_{\mathrm{BK}}\, \varphi^*\mathfrak{M} := \mathrm{Im}(\psi : \mathfrak{M} \to \varphi^*\mathfrak{M})$. Most importantly, $\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ is a generalized Kisin module of height i, and the BK filtration on $\varphi^*\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ exactly matches the image of the Nygaard filtration $\mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}^i_{\mathrm{N}}\, \Delta^{(1)}_n) \to \mathrm{H}^i_{\mathrm{qSyn}}(X, \Delta^{(1)}_n)$ where $\mathrm{Fil}^i_{\mathrm{N}}\, \Delta^{(1)}_n = \mathrm{Fil}^i_{\mathrm{N}}\, \Delta^{(1)}_{-/\mathfrak{S}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ and $\Delta^{(1)}_n = \Delta^{(1)}_{-/\mathfrak{S}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$; see Proposition 7.2 and Corollary 7.11. One can apply many methods in the study of étale Kisin modules to treat $\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ as well. As a consequence, we prove the following:

Theorem 1.9. *Let $A = (\mathfrak{S}, E)$ be the Breuil–Kisin prism and write $\mathfrak{M}^i_n := \mathrm{H}^i_{\Delta}(X_n/A_n)$. Let $i \le p-2$ be an integer. Then $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ has a Breuil module structure if and only if $\mathfrak{M}^j_n$ has no u-torsion for $j = i, i+1$. In that case, $\underline{\mathcal{M}}(\mathfrak{M}^i_n) \simeq \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ and $T_S(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)) \simeq \mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i)$ as G_K-modules.*

Finally, by using Caruso's Theorem 1.1 for $n = 1$, we can show that $\mathfrak{M}_n^{i+1}$ has no u-torsion if $ei < p-1$, hence deducing Theorem 1.2.

To end this introduction, let us report what we know now, a year since writing this paper, slightly beyond the case of $ei < p-1$. Recall that Breuil asked [11, Question 4.1] whether, assuming $i < p-1$, it is true that $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ always supports a Breuil module structure with associated Galois representation given by $\mathrm{H}^i_{\mathrm{et}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$. In view of our Theorem 1.9, what Breuil asked for is really some module-theoretic structure of prismatic cohomology of X: whether prismatic cohomology never has u-torsion when $i < p$. In a sequel to this paper, among other things, we study u^∞-torsion in prismatic cohomology. We obtain some results in the boundary case, that is, $ei = p-1$. Let us restrict ourselves further to the two extremes of the boundary case; our relevant findings are summarized below:

- When $i = 1$, Breuil's question amounts to vanishing of u-torsion in the first and second prismatic cohomology. The first prismatic cohomology is always u-torsionfree. The second prismatic cohomology having u-torsion is showed to be equivalent to the failure of having Albanese abelian (formal) scheme of X, by which we mean a map $X \to A$ with both central and generic fibers being the Albanese map. This was studied by Raynaud [32] and we extend some of his results using this prismatic perspective. We generalize a construction in [7, Section 2.1] to produce counterexamples to Breuil's question with $e = p-1 > 1$. In fact, the module structure of $\mathrm{H}^1_{\mathrm{crys}}(X_n/S_n)$ of this example is so pathological that it cannot possibly support a Breuil module structure. In particular, in hindsight, our Theorem 1.2 is sharp.
- When $e = 1$, this is what Fontaine–Messing [17] and Kato [21] studied. In the boundary case $i = p-1$, we are able to show that the Galois representation ρ^{p-1} attached to the Fontaine–Laffaille structure on the $(p-1)$-st crystalline cohomology is not far from the $(p-1)$-st étale cohomology of the geometric generic fiber.

The case of $ei > p-1$ remains mysterious to us. We believe the first step of investigation would be a better understanding of u^∞-torsion in prismatic cohomology, extending our results so far.

We arrange our paper as follows: after collecting rudiments on prismatic cohomology and derived de Rham cohomology in Section 2, we establish our comparison isomorphism between the two cohomologies in Section 3, together with Frobenius structures. We discuss various filtrations in Section 4 and establish a filtered comparison. We remark that the theory in Sections 2–4 accommodates quite general classes of prisms, which opens the possibilities to develop, for example, Breuil–Caruso theory for more general base rings. We hope to report the generalization in this direction in future work. Starting from Section 5, we restrict ourselves to the Breuil–Kisin prism $(\mathfrak{S}, (E))$ and focus on structures of torsion prismatic cohomology and torsion crystalline cohomology for a proper smooth formal scheme X over $\mathcal{O}_K$. In Section 5 we construct a connection ∇ on derived de Rham cohomology and hence on crystalline cohomology. Section 6 recalls classical theory of Kisin modules, Breuil modules, functors to Galois representations and the functor $\underline{\mathcal{M}}$

connecting Kisin modules and Breuil modules. Finally, Section 7 assembles all previous preparations to prove Theorem 1.9.

2. Preliminaries

From this section until Section 4, unless stated otherwise, all completions and (completed) tensor products are derived.

2.1. Transversal prisms

Lemma 2.1. *Let (A, I) be an oriented prism with $I = (d)$. The following are equivalent:*

(1) *the sequence (p, d) is Koszul regular;*

(2) *the sequence (p, d) is regular;*

(3) *the morphism $\mathbb{Z}_p[\![T]\!] \to A$ sending T to d is flat.*

Proof. (3)⇒(1): Because (3) implies that $A \otimes_{\mathbb{Z}_p[\![T]\!]} \mathbb{Z}_p[\![T]\!]/(p, T)$ is discrete.

(1)⇒(2): (1) implies that the p-torsion in A is uniquely d-divisible, and A/p has no d-torsion. On the other hand, we know that the p-torsion in A is derived d-complete, hence must vanish. Therefore (p, d) is a regular sequence.

(2)⇒(3): It suffices to show that for any prime ideal $\mathfrak{p} \subset \mathbb{Z}_p[\![T]\!]$ the derived tensor product $A \otimes_{\mathbb{Z}_p[\![T]\!]} \mathbb{Z}_p[\![T]\!]/\mathfrak{p}$ is discrete. When $\mathfrak{p}$ is the unique maximal ideal, this follows immediately from (2). So we only have to deal with height 1 primes which are always generated by a polynomial of the form

$$f = T^n + p \cdot (\text{lower order terms}),$$

and we need to show that A is f-torsionfree. Suppose $a \in A$ is an f-torsion element; modulo p we see that $\overline{a} \in A/p$ is d^n-torsion, and (2) implies that $\overline{a} = 0 \in A/p$. Therefore f-torsion in A is divisible by p. As (2) also implies that A is p-torsionfree, f-torsion in A is uniquely p-divisible. Since A is derived p-complete, we see that A must in fact be f-torsionfree. ∎

We can globalize to nonoriented prisms (A, I). The following easily follows from Lemma 2.1.

Lemma 2.2. *Let (A, I) be a prism. The following are equivalent:*

(1) *there is a (p, I)-completely faithfully flat cover by an oriented prism (A', IA'), which satisfies the equivalent conditions in Lemma 2.1;*

(2) *the ideal I is p-completely regular;*

(3) *Zariski locally, (p, I) is a regular sequence;*

(4) *the natural morphism $\mathrm{Spf}(A) \to [\mathrm{Spf}(\mathbb{Z}_p[\![T]\!])/(\mathbb{G}_{\mathrm{m}})_{\mathbb{Z}_p}]$ classified by I is flat.*

Let us explain the morphism in (4) above: Zariski locally, I is generated by a nonzerodivisor d, hence Zariski locally we get a map $\mathrm{Spf}(A) \to \mathrm{Spf}(\mathbb{Z}_p[\![T]\!])$, and on overlap these generators differ by a unit in A, hence globally we have a morphism to the quotient stack. Alternatively, we can understand this map as the composition of the universal map $\mathrm{Spf}(A) \to \Sigma$ introduced by Drinfeld [15, Section 1.2], and $\Sigma \to [\mathrm{Spf}(\mathbb{Z}_p[\![T]\!])/(\mathbb{G}_{\mathrm{m}})_{\mathbb{Z}_p}]$ induced by $W_{\mathrm{prim}} \to \mathrm{Spf}(\mathbb{Z}_p[\![T]\!])$ sending a Witt vector $(x_0, x_1, \ldots)$ to x_0.

Definition 2.3. A prism (A, I) is said to be *transversal* if it satisfies the equivalent conditions in Lemma 2.2.

For the remainder of this subsection, we assume that (A, I) is a transversal prism. Denote the p-completed PD envelope of $A \twoheadrightarrow A/I$ by $\mathcal{A}$, and denote the kernel of $\mathcal{A} \twoheadrightarrow A/I$ by $\mathcal{I}$.

Example 2.4. Let us list some examples of transversal prisms.

(1) The universal oriented prism is transversal.

(2) The Breuil–Kisin prism [9, Example 1.3 (3)] is transversal. We have $A = \mathfrak{S}$ and $\mathcal{A}$ is classically denoted by S in the classical literature concerning Breuil modules.

(3) Let $\mathbf{C}$ be an algebraically closed complete non-Archimedean field extension of $\mathbb{Q}_p$. Then the perfect prism associated with $\mathcal{O}_{\mathbf{C}}$ is transversal. We have $A = A_{\mathrm{inf}}$ and $\mathcal{A} = A_{\mathrm{crys}}$.

Although $\mathcal{A}$ is usually not flat over A, it has p-completely finite Tor dimension. In the next subsection we shall see that this is a general phenomenon about the derived de Rham complex and regularity of I.

Lemma 2.5. *Let (A, I) be a transversal prism. Then $A \to \mathcal{A}$ has p-complete amplitude in $[-1, 0]$, in particular p-complete base change along $A \to \mathcal{A}$ commutes with taking totalizations in $D^{\geq 0}(A)$.*

Proof. It suffices to check the statement Zariski locally on $\mathrm{Spf}(A)$, hence we may assume the prism is oriented, say $I = (d)$. Then we may base change to A/p. So we need to check that given an $\mathbb{F}_p$-algebra R, and a nonzerodivisor $d \in R$, the divided power algebra $S = D_R(d)$ has Tor amplitude in $[-1, 0]$ over R. This follows from the fact that $d^p = 0$ in S and S is a free $R/(d^p)$-module. The commutation of tensoring and totalization now follows from [9, Lemma 4.20]. ∎

2.2. *Envelopes and derived de Rham cohomology*

Let (A, I) be a bounded prism. In this subsection we review the derived de Rham complex of simplicial A-algebras relative to A.

First we want to spell out explicitly the process of freely adjoining divided powers or delta powers of elements mentioned in [9, Sections 2.5–2.6 and 3].

Construction 2.6. (0) Recall I is locally generated by a nonzerodivisor in A. Let A_i be an affine open cover of $\mathrm{Spf}(A)$ such that $I \cdot A_i = (d_i)$ where $d_i \in I$. There is an A-algebra

$A[I \cdot x]$ obtained by glueing $A_i[x_i]$'s via $x_i = \frac{d_i}{d_j} x_j$; it has a surjection $A[I \cdot x] \twoheadrightarrow A$ obtained by glueing the maps $x_i \mapsto d_i$. Alternatively one may directly define

$$A[I \cdot x] := \bigoplus_{n \geq 0} I^n$$

with the evident surjection being the natural inclusion on each factor. It can also be seen as the ring of functions on the total space of the line bundle I^{-1} on $\mathrm{Spec}(A)$.

Similarly there is a δ-A-algebra $A\{I \cdot x\}$ obtained by glueing $A_i\{x_i\}$'s via

$$A_i\{x_i\} \otimes_{A_i} A_{ij} \xrightarrow{x_i \mapsto \frac{d_i}{d_j} x_j} A_j\{x_j\} \otimes_{A_j} A_{ij}$$

with a surjection $A\{I \cdot x\} \twoheadrightarrow A$ obtained by glueing the maps $x_i \mapsto d_i$. Alternatively one may directly define

$$A\{I \cdot x\} := \bigotimes_{m \geq 1}{}_A \Big(\bigoplus_{n \geq 0} (\delta^m(I))^n \Big)$$

with the evident surjection being the natural map on each tensor factor. This can also be seen as the ring of functions on the total space of an infinite rank vector bundle on $\mathrm{Spec}(A)$.

Note that the above construction can be generalized to the case where I is replaced by a line bundle $\mathcal{L}$ on $\mathrm{Spec}(A)$. In particular, one can make sense of $A\{I^{-1} \cdot x\}$ and $A\{\varphi(I) \cdot x\}$. We remark that there is a natural map $A\{x\} \to A\{I^{-1} \cdot y\}$ obtained by glueing the maps $x \mapsto d_i y_i$, which we shorthand as $x \mapsto y$.

(1) Let B be an A-algebra, and let $f_1, \ldots, f_r$ be a finite set of elements in B. The *simplicial B-algebra obtained by freely adjoining divided powers of f_i* is denoted by $B\langle\langle f_i \rangle\rangle$ and defined to be the derived tensor product of

$$\begin{tikzcd}
\mathbb{Z}[x_1, \ldots, x_r] \arrow[r, "x_i \mapsto f_i"] \arrow[d] & B \\
D_{\mathbb{Z}[x_1,\ldots,x_r]}(x_1, \ldots x_r) &
\end{tikzcd}$$

The *simplicial A-algebra obtained by freely adjoining divided powers of I* is denoted by $A\langle\langle I \rangle\rangle$ and defined to be the derived tensor product of

$$\begin{tikzcd}
A[I \cdot x] \arrow[r, "x_i \mapsto d_i"] \arrow[d] & A \\
D_{A[I \cdot x]}(\ker(A[I \cdot x] \twoheadrightarrow A)) &
\end{tikzcd}$$

Alternatively one may define it as the glueing of the simplicial A-algebras $A_i \otimes_{x \mapsto d_i, A[x]} D_{A[x]}(x)$.

The *simplicial B-algebra obtained by freely adjoining divided powers of I, f_i*, denoted by $B\langle\langle I, f_i\rangle\rangle$, is defined as the derived tensor product of the above two algebras over A.

(2) Let B be a δ-A-algebra, and let $f_1, \dots, f_r$ be a finite set of elements in B. We define $B\{\frac{f_i}{p}\}$ as the derived pushout of the following diagram of simplicial algebras:

$$\begin{tikzcd} A\{x_1, \dots, x_r\} \arrow[r, "x_i \mapsto f_i"] \arrow[d, "x_i \mapsto p \cdot y_i"] & B \\ A\{y_1, \dots, y_r\} & \end{tikzcd}$$

We define $A\{\varphi(I)/p\}$ as the derived pushout of

$$\begin{tikzcd} A\{I \cdot x\} \arrow[r, "\varphi"] \arrow[d, "x \mapsto p \cdot y"] & A \\ A\{I \cdot y\} & \end{tikzcd}$$

Alternatively one may define it as the glueing of the simplicial δ-A-algebras $A_i\{\frac{\varphi_A(d_i)}{p}\}$.

Analogously $B\{\frac{\varphi(I)}{p}, \frac{f_i}{p}\}$ is defined by derived tensoring the above two algebras over A.

(3) Given a sequence $(f_1, \dots, f_r)$ of elements inside a ring B, we write

$$\mathrm{dR}_B(f_1, \dots, f_r)^\wedge := \mathrm{dR}^\wedge_{\mathrm{Kos}(B; f_1, \dots, f_r)/B}$$

to denote the p-completed derived de Rham complex of $\mathrm{Kos}(B; f_1, \dots, f_r)$, viewed as a simplicial B-algebra, over B.

Similarly when B is an A-algebra, we denote

$$\mathrm{dR}_B(I)^\wedge := \mathrm{dR}^\wedge_{(B \otimes_A (A/I))/B},$$
$$\mathrm{dR}_B(I, f_i)^\wedge := \mathrm{dR}^\wedge_{(\mathrm{Kos}(B; f_1, \dots, f_r) \otimes_A (A/I))/B}.$$

Let J be an ideal in B. Then we denote

$$\mathrm{dR}_B(J)^\wedge := \mathrm{dR}^\wedge_{(B/J)/B}.$$

Here all the completions are derived p-completions.

Remark 2.7. (1) Let $B := A\{x\}^\wedge$. Note that x is (p, I)-completely regular relative to A. Using [9, Proposition 3.13], we can get a B-algebra $C := B\{\frac{x}{I}\}^\wedge$ which is locally (on $\mathrm{Spf}(A)$ as one needs to trivialize the line bundle I) given by $C = A\{y\}^\wedge$ together with the B-algebra structure $x \mapsto d \cdot y$ where d is the local generator of I. One checks immediately that, in our notation, $C \cong A\{I^{-1} \cdot y\}^\wedge$ with the B-structure given by (the (p, I)-completion of) $x \mapsto y$.

In fact, by examining the proof of [9, Proposition 3.13], one finds that in the situation described there, the algebra $B\{\frac{J}{I}\}^\wedge$ is the derived (p, I)-complete pushout of the diagram

$$\begin{tikzcd} A\{x_1, \ldots, x_r\} \arrow[r] \arrow[d, "x_i \mapsto y_i"] & B \\ A\{I^{-1} \cdot y_i\} & \end{tikzcd}$$

(2) We warn the readers that when $J = (f_1, \ldots, f_r)$ is an ideal inside B, the two simplicial B-algebras $\mathrm{dR}_B(J)^\wedge$ and $\mathrm{dR}_B(f_1, \ldots, f_r)^\wedge$ are usually different. They agree when (f_i) is a p-completely Koszul regular sequence.

Below we shall see the relation between the derived de Rham complex, divided power envelopes, and prismatic envelopes, which directly follows from [9, Section 2.5].

Lemma 2.8. (1) *Let B be an A-algebra, and let $\{f_1, \ldots, f_r\}$ be a finite set of elements of B. Then we have the following identification of derived p-complete simplicial B-algebras:*

$$\mathrm{dR}_B(f_1, \ldots, f_r)^\wedge \cong B\langle\langle f_i\rangle\rangle^\wedge.$$

Similarly we have an identification

$$\mathrm{dR}_B(I)^\wedge \cong B\langle\langle I\rangle\rangle^\wedge.$$

(2) *Let B be a δ-A-algebra, and let $\{f_1, \ldots, f_r\}$ be a finite set of elements of B. Then we have the following identification of derived p-complete simplicial B-algebras:*

$$B\langle\langle f_i\rangle\rangle^\wedge \cong B\left\{\frac{\varphi(f_i)}{p}\right\}^\wedge.$$

Similarly we have an identification

$$B\langle\langle I\rangle\rangle^\wedge \cong B\left\{\frac{\varphi(I)}{p}\right\}^\wedge.$$

Proof. By the base change property of the constructions, we reduce ourselves to the case where $B = A\{x_1, \ldots, x_r\}$ with $f_i = x_i$. Again by base change we may assume A is the initial oriented prism, in particular it is flat over $\mathbb{Z}_p$ and $I = (d)$ is generated by a nonzerodivisor. So we can focus on the case concerning a finite set of elements of B, and we may further reduce to the case where the set is a singleton.

Now the identification in (1) follows from (the limit version of) [3, Theorem 3.27] and [2, Théorème V.2.3.2]. The identification in (2) follows from [9, Lemma 2.36]. ■

We deduce a consequence concerning the Tor amplitude of $\mathrm{dR}_A(I)^\wedge$ over A, generalizing Lemma 2.5.

Lemma 2.9. *Let (A, I) be a prism. Then $A \to \mathrm{dR}_A(I)^\wedge$ has p-complete amplitude in $[-1, 0]$, in particular p-completely base changing along $A \to \mathrm{dR}_A(I)^\wedge$ commutes with taking totalizations in $D^{\geq 0}(A)$.*

Proof. We may check this statement locally on $\mathrm{Spf}(A)$, hence we may assume $I = (d)$. Next, by base change, we may assume A is the initial oriented prism, in particular we may assume it is transversal. Using Lemma 2.8 (1), we see now $\mathrm{dR}_A(I)^\wedge$ is the p-completion of the divided power envelope $D_A(I)^\wedge$. This reduces the lemma to Lemma 2.5. ∎

We also have a prototype base change formula which will be used in the next section to establish a general comparison.

Lemma 2.10. *Let (A, I) be a prism, and denote by f the composition*

$$A\{x\} \xrightarrow{\varphi_A, x \mapsto \varphi(z)} A\{z\} \to \mathrm{dR}_{A\{z\}}(I)^\wedge.$$

Then we have a base change formula

$$A\{I^{-1} \cdot x\} \widehat{\otimes}_{A\{x\}, f} \mathrm{dR}_{A\{z\}}(I)^\wedge \cong \mathrm{dR}_{A\{z\}}(I, z)^\wedge.$$

Here the completion on the left hand side is derived p-completion. As $\varphi(I) = (p)$ inside $\pi_0(\mathrm{dR}_{A\{z\}}(I)^\wedge)$, it is the same as derived (p, I)-completion when viewed as an A-complex via $\varphi_A : A \to \mathrm{dR}_A(I)^\wedge$.

Proof of Lemma 2.10. Note that by Lemma 2.8 we have identifications $\mathrm{dR}_{A\{z\}}(I)^\wedge \cong A\{z\}\{\frac{\varphi(I)}{p}\}^\wedge$ as p-complete simplicial $A\{z\}$-algebras. Similarly we can identify $\mathrm{dR}_{A\{z\}}(I, z)^\wedge$ with $A\{z\}\{\frac{\varphi(z)}{p}, \frac{\varphi(I)}{p}\}^\wedge$.

Now we look at the following diagram:

$$\begin{tikzcd}
A\{x\} \arrow[r] \arrow[d] & A\{z\} \arrow[r] \arrow[d] & A\{z\}\{\frac{\varphi(I)}{p}\} \arrow[d] \\
A\{I^{-1} \cdot x\} \arrow[r] & A\{z\}\{\frac{\varphi(z)}{\varphi(I)}\} \arrow[r] & A\{z\}\{\frac{\varphi(z)}{p}, \frac{\varphi(I)}{p}\}
\end{tikzcd}$$

The left square is a pushout diagram by definition. Hence it suffices to show that the right square, after derived p-completion, is also a pushout diagram of p-complete simplicial $A\{z\}$-algebras.

To that end, we may work Zariski locally on A, so we can assume $I = (d)$ is generated by one element. This square is the base change of the same diagram when A is the initial oriented prism, so we have reduced the task to that case. Now every ring in sight is discrete, and the p-completed square is a pushout diagram because $\varphi(d)$ and p differ by a unit inside $A\{\frac{\varphi(d)}{p}\}^\wedge \cong D_A(d)^\wedge$. ∎

In [3, Proposition 3.25], for any p-complete A-algebra B, Bhatt constructed a natural map

$$\mathscr{C}\mathrm{omp}_{B/A} : \mathrm{dR}^\wedge_{B/A} \to \mathrm{R}\Gamma_{\mathrm{crys}}(B/A).$$

Here the right hand side denotes the p-complete crystalline cohomology defined using PD thickenings of B relative to $(A, (p), \gamma)$ with $\gamma_i(p) = p^i/i!$. This natural map is functorial in $A \to B$ and agrees with Berthelot's de Rham-crystalline comparison [2, Théorème

IV.2.3.2] when it is formally smooth (viewed as a map between p-adic algebras). It is known that when both A and B are flat over $\mathbb{Z}_p$ and $A \to B$ is a p-complete local complete intersection, then the natural map above is an isomorphism [3, Theorem 3.27].

For our purpose we shall be interested in the situation where B is formally smooth over A/I, so we cannot summon the above theorem [3] to say that the natural map in this situation is an isomorphism. In fact, when $B = A/I$ the left hand side is $\mathrm{dR}_A(I)^\wedge$ and the right hand side is the classical p-adic completion of the PD envelope of A along I (compatible with the natural PD structure on $(A,(p))$), denoted as $\mathcal{A}$.[2] These two need not be the same, e.g. if $A = \mathbb{Z}_p$ and $I = (p)$, then $\mathrm{dR}_A(I)^\wedge = \mathbb{Z}_p[T^i/i!]^\wedge/(T-p)$ but $\mathcal{A} = \mathbb{Z}_p$. However. this turns out to be the only problem.

Proposition 2.11. *Let B be an A/I-algebra.*

(1) *If B is formally smooth over A/I, then we have a natural identification*

$$\mathrm{R}\Gamma_{\mathrm{crys}}(B/A) \cong \mathrm{R}\Gamma_{\mathrm{crys}}(B/\mathcal{A}),$$

where the right hand side is the usual crystalline cohomology of $\mathrm{Spf}(B)$ *over the PD base $\mathcal{A}$.*

(2) *There is a natural map*

$$\mathrm{Comp}_{B/A} : \mathrm{dR}^\wedge_{B/A} \widehat{\otimes}_{\mathrm{dR}_A(I)^\wedge} \mathcal{A} \to \mathrm{R}\Gamma_{\mathrm{crys}}(B/\mathcal{A}),$$

which is functorial in $A/I \to B$.

(3) *If B is formally smooth over A/I, then the above is an isomorphism.*

Proof. (1) is an easy consequence of the fact that B is an A/I-algebra. In fact, we only need $A/I \to B$ to be a local complete intersection. Indeed, we use the Čech–Alexander complex to compute both crystalline cohomologies, and one reduces to the following: Let P be a polynomial A-algebra with a surjection $P \twoheadrightarrow B$ of A-algebras. Then there is a naturally induced surjection $P \otimes_A \mathcal{A} \twoheadrightarrow B$ of $\mathcal{A}$-algebras, and we have an identification of PD envelopes

$$D_{(A,(p),\gamma)}(P \twoheadrightarrow B) = D_{(\mathcal{A},\mathcal{I},\gamma)}(P \otimes_A \mathcal{A} \twoheadrightarrow B).$$

(2) The functoriality of Bhatt's $\mathcal{C}\mathrm{omp}_{B/A}$ asserts that the map is compatible with the natural map $\mathrm{dR}_A(I)^\wedge \to \mathcal{A}$, hence we get our natural map $\mathrm{Comp}_{B/A}$.

(3) Choose a formal lift $\widetilde{B}$ over A (note that A is (p,I)-complete). By the functoriality of Bhatt's $\mathcal{C}\mathrm{omp}_{B/A}$, we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{dR}^\wedge_{\widetilde{B}/A} \widehat{\otimes}_A \mathcal{A} \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{crys}}(\widetilde{B}/A) \widehat{\otimes}_A \mathcal{A} \arrow[d] \\
\mathrm{dR}^\wedge_{B/A} \widehat{\otimes}_{\mathrm{dR}_A(I)^\wedge} \mathcal{A} \arrow[r] & \mathrm{R}\Gamma_{\mathrm{crys}}(B/\mathcal{A})
\end{tikzcd}$$

[2] This notation agrees with the previous subsection as we assumed (A,I) to be a transversal prism there.

The top horizontal arrow is an isomorphism by Berthelot's de Rham-crystalline comparison. The left vertical arrow is an isomorphism by the Künneth formula for the derived de Rham complex: $\mathrm{dR}^{\wedge}_{\widetilde{B}/A} \widehat{\otimes}_A \mathrm{dR}_A(I)^{\wedge} \cong \mathrm{dR}^{\wedge}_{B/A}$. The right vertical arrow is an isomorphism by the base change formula for crystalline cohomology. Therefore we conclude that the bottom horizontal arrow, which is our $\mathrm{Comp}_{B/A}$, must also be an isomorphism. ∎

The above proposition and Bhatt's results discussed before suggest that the derived de Rham complex is a substitute of crystalline cohomology. Inspired by this philosophy, we show that the derived de Rham complex only "depends on the reduction mod p of the input algebra". We need to introduce some notations first. Denote the p-adic derived de Rham complex $\mathrm{dR}^{\wedge}_{\mathbb{F}_p/\mathbb{Z}_p}$ by D. Bhatt's result implies that the natural map $\mathbb{Z}_p \to D$ admits a retraction $D \to \mathbb{Z}_p$. In Example 2.16 (1) below, one finds a detailed description of D.

Remark 2.12. In fact, one can show that D is the p-complete PD envelope of $\mathbb{Z}_p$ along the ideal (p). Moreover, under this identification one can easily see that the retraction above is unique, and is given by the fact that there is a unique PD structure on $(\mathbb{Z}_p, (p))$ (as $\mathbb{Z}_p$ has no p-torsion). Notice that when taking a PD envelope, one has to fix a PD base ring, and we always take it to be the trivial PD ring $(\mathbb{Z}_p, (0), \gamma_{\mathrm{triv}})$ when we say PD envelope without mentioning a PD base ring.

Proposition 2.13. *Let R be a ring with derived p-completion $R^{\wedge}$, and let B be a simplicial R-algebra. Then there is a natural isomorphism*

$$\mathrm{dR}^{\wedge}_{\mathrm{Kos}(B;p)/R} \widehat{\otimes}_D \mathbb{Z}_p \cong \mathrm{dR}^{\wedge}_{B/R},$$

which is functorial in $R \to B$.

Here the map $D \to \mathrm{dR}^{\wedge}_{\mathrm{Kos}(B;p)/R}$ is induced by the natural diagram

$$\begin{array}{ccc} \mathbb{F}_p & \longrightarrow & \mathrm{Kos}(B;p) = B \otimes_{\mathbb{Z}} \mathbb{F}_p \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & R \end{array}$$

Proof of Proposition 2.13. This follows from the Künneth formula for the derived de Rham complex,

$$\mathrm{dR}^{\wedge}_{\mathrm{Kos}(B;p)/R} \cong \mathrm{dR}^{\wedge}_{B/R} \widehat{\otimes}_R \mathrm{dR}_R(p)^{\wedge},$$

and the base change formula

$$\mathrm{dR}_R(p)^{\wedge} \cong D \widehat{\otimes}_{\mathbb{Z}_p} R^{\wedge}$$

as $\mathrm{Kos}(R;p) = R \otimes_{\mathbb{Z}} \mathbb{F}_p$. ∎

2.3. Frobenii

Let A be a p-torsionfree δ-ring. Using Proposition 2.13 we can define a Frobenius action on $\mathrm{dR}^{\wedge}_{B/A}$ which is functorial in (A, φ_A) and the A-algebra B.

Construction 2.14. Let A be a p-torsionfree δ-ring and B a simplicial A-algebra. Recall there is a functorial endomorphism on simplicial $\mathbb{F}_p$-algebras given by left Kan extension of the usual Frobenius on polynomial $\mathbb{F}_p$-algebras; see [29, Construction 2.2.6]. For discrete $\mathbb{F}_p$-algebras, it is just the usual Frobenius. We may view $B/p = B \otimes_A A/p$, and using the fact that φ_A on A is a lift of the Frobenius on A/p we get the following commutative diagram:

$$\begin{tikzcd} B/p \arrow[r, "\varphi_{B/p}"] & B/p \\ A \arrow[u] \arrow[r, "\varphi_A"] & A \arrow[u] \end{tikzcd}$$

It induces a Frobenius map $\widetilde{\varphi} : \mathrm{dR}^\wedge_{\mathrm{Kos}(B;p)/A} \to \mathrm{dR}^\wedge_{\mathrm{Kos}(B;p)/A}$ which is functorial in $(A \to B, \varphi_A)$.

A similar diagram for $\mathbb{Z} \to \mathbb{F}_p$ (where $A = B = \mathbb{Z}_p$) induces the identity on D, hence we have a commutative diagram

$$\begin{tikzcd} \mathrm{dR}^\wedge_{\mathrm{Kos}(B;p)/A} \arrow[rr, "\widetilde{\varphi}"] & & \mathrm{dR}^\wedge_{\mathrm{Kos}(B;p)/A} \\ & D \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Finally, we define a Frobenius map $\varphi_{B/A} : \mathrm{dR}^\wedge_{\mathrm{Kos}(B;p)/A} \widehat{\otimes}_D \mathbb{Z}_p \cong \mathrm{dR}^\wedge_{B/A} \xrightarrow{\widetilde{\varphi}\widehat{\otimes}_{\mathrm{id}_D} \mathrm{id}_{\mathbb{Z}_p}} \mathrm{dR}^\wedge_{B/A}$ which is functorial in $(A \to B, \varphi_A)$.

Remark 2.15. (1) It is conceivable that the above works for general δ-rings. In a private communication we learned from Bhatt that a δ-structure on a ring A is equivalent to specifying a commutative diagram as follows:

$$\begin{tikzcd} A/p \arrow[r, "\varphi_{A/p}"] & A/p \\ A \arrow[u] \arrow[r, "\varphi_A"] & A \arrow[u] \end{tikzcd}$$

Note that here A/p is a simplicial $\mathbb{F}_p$-algebra that has nontrivial π_1 when A is not p-torsionfree. Hence for any simplicial A-algebra B, one can also define a Frobenius on $\mathrm{dR}^\wedge_{B/A}$ as above. However, we do not work out the full story here as we do not need this great generality for our intended applications later.

(2) By letting $n \to \infty$ in [3, Proposition 3.47], one gets another construction of Frobenius on $\mathrm{dR}^\wedge_{A/\mathbb{Z}_p}$ for any $\mathbb{Z}_p$-algebra A. However, we shall see in Remark 3.15 that there is only one Frobenius that is functorial enough (in a suitable sense) on p-completed derived de Rham complexes when the base algebra is a p-torsionfree δ-algebra. In particular, our construction above agrees with Bhatt's whenever both are defined (i.e., when the base is $\mathbb{Z}_p$).

Let us work out some examples.

Example 2.16. (1) As an illustrative example, let us consider $A = \mathbb{Z}_p$ and $B = \mathbb{F}_p$. We have a derived pushout square of rings

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] & B \\ \mathbb{Z}_p[T] \arrow[u, "T \mapsto p"] \arrow[r, "T \mapsto 0"] & A \arrow[u] \end{tikzcd}$$

The bottom map is a map of δ-rings if we give $\mathbb{Z}_p[T]$ a δ-structure with $\varphi(T) = T^p$. Then we get a pushout diagram of the derived de Rham complex which says $D \cong \mathrm{dR}^{\wedge}_{\mathbb{Z}_p/\mathbb{Z}_p[T]} \widehat{\otimes}_{\mathbb{Z}_p[T]} \mathbb{Z}_p$. The latter is the same as $\mathbb{Z}_p\langle\langle T\rangle\rangle^{\wedge}/(T)$ where we have used the fact that p has divided powers in $\mathbb{Z}_p$ (hence adjoining divided powers of $T - p$ is the same as adjoining divided powers of T). It is easy to see that the Frobenius defined on $\mathrm{dR}^{\wedge}_{\mathbb{Z}_p/\mathbb{Z}_p[T]} \cong \mathbb{Z}_p\langle\langle T\rangle\rangle$ is induced by $T \mapsto T^p$ because it has to be compatible with the Frobenius on $\mathbb{Z}_p[T]$. Therefore the induced Frobenius on $\mathrm{dR}^{\wedge}_{B/A}$ is *not* the identity. This might be surprising as one would naïvely think that the Frobenius on the pair $(\mathbb{Z}_p, \mathbb{F}_p)$ is the identity, hence must induce the identity on the derived de Rham complex. However, the Frobenius on $\mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is *not* the identity (as Frobenius always kills cohomology classes in negative degrees [29, Remark 2.2.7]), and it is this Frobenius that induces a map on the derived de Rham complex. On a related note, Bhatt has pointed out to us that the identity map is also *not* a lift of the Frobenius on $D \cong \mathbb{Z}_p\langle\langle T\rangle\rangle^{\wedge}/(T)$.

(2) Let $J \subset A$ be an ideal which is Zariski locally on $\mathrm{Spec}(A)$ a colimit of ideals generated by a p-completely regular sequence. Then by Lemma 2.8 (1), we have an identification $\mathrm{dR}_A(J)^{\wedge} \cong D_A(J)^{\wedge}$. Since the Frobenius map obtained is compatible with φ_A and $D_A(J)^{\wedge}$ is p-torsionfree, we see that this pins down the Frobenius on $\mathrm{dR}_A(J)^{\wedge}$: any $\gamma_n(f)$ with $f \in J$ must be sent to $\varphi_A(f)^n/n!$. Note that f^p is divisible by p in $D_A(J)^{\wedge}$, hence $\varphi_A(f)$ is divisible by p in $D_A(J)^{\wedge}$.

(3) Let A be p-complete, and let $B = A\langle X^{1/p^\infty}\rangle$. Since $A \to B$ is relatively perfect modulo p, there is a unique lift of the Frobenius φ_B on B covering the Frobenius on A and it is given by $\varphi_B(X^i) = X^{i \cdot p}$. By [18, Proposition 3.4 (1)], we see that the natural map to the 0-th graded piece of the Hodge filtration induces an isomorphism $\mathrm{dR}^{\wedge}_{B/A} \cong B$. Applying the functoriality of Construction 2.14 to the map of triples $(A \to B, \varphi_A) \to (B \to B, \varphi_B)$, we see that the Frobenius on $\mathrm{dR}^{\wedge}_{B/A} \cong B$ must be φ_B.

When the map $A \to B$ is a surjection with good regularity properties, we have seen in Lemma 2.8 that one can express $\mathrm{dR}^{\wedge}_{B/A}$ in terms of prismatic envelopes. Since prismatic envelopes are δ-rings, they possess a Frobenius map by design. We can use this to give an alternative construction of the Frobenius for derived de Rham cohomology of certain regular A-algebras relative to A. To that end, we need to first establish a sheaf property for derived de Rham cohomology.

Proposition 2.17. *Let S be an R-algebra. Assume:*

- *the cotangent complex $\mathbb{L}_{S/R} \in D(S)$ has p-complete Tor amplitude in $[-1, 0]$;*
- *the Frobenius twist of S/p (relative to R/p) is in $D^{\geq -m}(\mathbb{F}_p)$.*

Consider the category $\mathcal{C}$ consisting of triangles $R \to P \to S$ with P being an ind-polynomial R-algebra, equipped with a nondiscrete topology. Let $\mathrm{dR}^{\wedge}_{S/\_}$ be the sheaf that associates any triangle $R \to P \to S$ with $\mathrm{dR}^{\wedge}_{S/P}$. Then:

(1) *For any $R \to P \to S$, $\mathrm{dR}^{\wedge}_{S/P}$ is in $D^{\geq -m}(R)$.*

(2) *The natural map $\mathrm{dR}^{\wedge}_{S/R} \to \lim_{\mathcal{C}} \mathrm{dR}^{\wedge}_{S/P}$ is an isomorphism.*

(3) *For any $R \to P \to S$ with $P \twoheadrightarrow S$ surjective, the natural map $\mathrm{dR}^{\wedge}_{S/R} \to \lim_{\Delta} \mathrm{dR}^{\wedge}_{S/P_\bullet}$ is an isomorphism. Here*
$$P_n := P^{\otimes_R (n+1)} \quad \text{for any } [n] \in \Delta,$$
with induced maps $P_n \twoheadrightarrow S$.

Proof. We shall prove this by reduction modulo p. Hence we may assume R and S are simplicial $\mathbb{F}_p$-algebras.

For (1) we use the conjugate filtrations on the derived de Rham complex. Since $\mathbb{L}_{S/R}$ has Tor amplitude in $[-1, 0]$, so is $\mathbb{L}_{S^{(1,P)}/P}$ where $S^{(1,P)}$ is the Frobenius twist of S (relative to P). The above estimate shows that the graded pieces of the conjugate filtration has Tor amplitude at least 0 over $S^{(1,P)}$. Since $S^{(1)}$ is assumed to be in $D^{\geq -m}(\mathbb{F}_p)$ and the relative Frobenius for P is flat, we see that all the graded pieces of the conjugate filtration live in $D^{\geq -m}(R)$.

Note that $P \twoheadrightarrow S$ is surjective if and only if $R \to P \to S$ is weakly final in $\mathcal{C}$. Since these $\mathrm{dR}^{\wedge}_{S/P}$ are cohomologically uniformly bounded below, [5, Lecture V, Lemma 4.3] (see also [34, Tag 07JM]) reduces (2) to (3).

Lastly, to show (3) we appeal to the conjugate filtration again. Since the graded pieces of the conjugate filtration are cohomologically uniformly bounded below by our proof of (1) above, it suffices to show $\mathbb{L}_{S^{(1)}/R} \to \lim_{\Delta} \mathbb{L}_{S^{(1,\bullet)}/P_\bullet}$ is an isomorphism, where $S^{(1,n)}$ is the Frobenius twist of S (relative to P_n). This follows easily from the fact that $\lim_{\Delta} \mathbb{L}_{P_\bullet/R} \cong 0$. ∎

The above proposition gives us a way to describe the Frobenius action of the p-completed derived de Rham complex in more cases than those listed in Example 2.16.

Proposition 2.18. *Let A be a p-torsionfree p-complete δ-algebra, and let $I \subset A$ be an ideal which is Zariski locally on* Spec(A) *generated by a p-completely regular element. Let B be a p-completely smooth A/I-algebra.*

(1) *For any (p, I)-completely ind-polynomial A-algebra P with a surjection $P \twoheadrightarrow B$, the kernel J is Zariski locally on* Spf(P) *a colimit of ideals generated by a p-completely regular sequence.*

(2) *For any $A \to P \to B$ as in* (1), *$\mathrm{dR}^{\wedge}_{B/P}$ is an ordinary algebra.*

(3) *For any (p, I)-completely free δ-A-algebra F with a surjection $F \twoheadrightarrow B$, there is a unique δ-algebra structure on $\mathrm{dR}^{\wedge}_{B/F}$ compatible with that on F. With this δ-structure, we have an identification*
$$\mathrm{dR}^{\wedge}_{B/F} \cong F\left\{\frac{\varphi_F(J)}{p}\right\}^{\wedge}.$$

(4) *Consider the category* $\mathcal{C}$ *of all triples* $A \to F \twoheadrightarrow B$ *as in* (3)*. Then*

$$\mathrm{dR}^{\wedge}_{B/A} \cong \lim_{\mathcal{C}} \mathrm{dR}^{\wedge}_{B/F}\,.$$

In fact, it suffices to take the limit over the Čech nerve of one such $F \twoheadrightarrow B$*. Together with* (3) *we get a natural Frobenius action on* $\mathrm{dR}^{\wedge}_{B/A}$*.*

(5) *The Frobenius on* $\mathrm{dR}^{\wedge}_{B/A}$ *obtained in* (4) *agrees with the one in Construction* 2.14*.*

The notation $F\{\frac{\varphi_F(J)}{p}\}^\wedge$ is defined analogously to [9, Corollary 3.14]. Using the fact that J is Zariski locally given by an ind-p-completely regular ideal, we may define $F\{\frac{\varphi_F(J)}{p}\}^\wedge$ as the glueing of the colimit of $F\{\frac{\varphi_F(f_i)}{p}\}^\wedge$, where (f_i) is the ind-regular sequence generating J on a Zariski open set.

Proof of Proposition 2.18*.* (1) follows easily from the fact that B is formally smooth over A/I and I is Zariski locally generated by a p-completely regular element.

(2) follows from the argument of Proposition 2.17 (1). Indeed, we set $R = A$ and $S = B$. The Frobenius twist of B/p is smooth over $A/(\varphi_A(I), p) = A/(I^p, p)$, and the latter is an ordinary algebra. Hence in our situation, we have $m = 0$ in the assumptions of Proposition 2.17. This shows that $\mathrm{dR}^{\wedge}_{B/P}$ is in $D^{\geq 0}$. Using the conjugate filtration again, it is easy to see that the p-completed derived de Rham complex of any surjection must be in $D^{\leq 0}$. Hence our $\mathrm{dR}^{\wedge}_{B/P}$ must in fact be an ordinary algebra.

(3) essentially follows from (1) and Lemma 2.8. Indeed, by description of J, we see that

$$\mathrm{dR}^{\wedge}_{B/F} \cong D_F(J)^\wedge.$$

Since J is Zariski locally an ind-p-completely regular ideal, we see that $D_F(J)^\wedge$ is p-torsionfree, hence having a δ-structure is equivalent to having a lift of Frobenius. The argument in Example 2.16 (2) tells us that there is at most one Frobenius structure on it compatible with that on F. Lastly, Lemma 2.8 shows that we can put a δ-structure on it by identifying

$$\mathrm{dR}^{\wedge}_{B/F} \cong D_F(J)^\wedge \cong F\left\{\frac{\varphi_F(J)}{p}\right\}.$$

(4) follows from Proposition 2.17 (2, 3).

As for (5), it suffices to notice that for any of these $A \to F \twoheadrightarrow B$ the two Frobenii defined on $\mathrm{dR}^{\wedge}_{B/F}$ agree and they are both functorial in $A \to F \twoheadrightarrow B$. ■

The following is similar to Proposition 2.17, and will be used in the next section.

Proposition 2.19. *Let* (A, I) *be a bounded prism. Let* R *be a formally smooth* A/I*-algebra. Let* $\mathcal{C}$ *be the category of all triples* $A \to P \twoheadrightarrow R$ *where* P *is a* p*-completed polynomial algebra over* A*. Associated with such a triple is the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & F \arrow[d] \\ A/I \arrow[r] & R \arrow[r] & S \end{tikzcd}$$

where F is the p-completed free δ-A-algebra associated with P, and S is the p-completed tensor product $R \mathbin{\hat{\otimes}_P} F$.

(1) *Choose an object $A \to P \twoheadrightarrow R$, and consider the n-th self-fiber product $A \to P^n := P^{\hat{\otimes}_A n} \twoheadrightarrow R$ for any positive integer n. Then the associated p-completed free δ-A-algebra is $F^n := F^{\hat{\otimes}_A n}$, and we have*

$$R \mathbin{\hat{\otimes}_{P^n}} F^n \cong S^{\hat{\otimes}_R n},$$

which we shall denote by S^n below.

(2) *Choose an object $A \to P \twoheadrightarrow R$. Then the natural map*

$$\mathrm{dR}^{\wedge}_{R/A} \to \lim_{[n]\in\Delta} \mathrm{dR}^{\wedge}_{S^n/F^n}$$

is an isomorphism.

(3) *The natural map*

$$\mathrm{dR}^{\wedge}_{R/A} \to \lim_{\mathcal{C}} \mathrm{dR}^{\wedge}_{S/F}$$

is an isomorphism.

Notice that we do not need to assume that A is p-torsionfree here.

Proof. For (1), if P is p-completely adjoining a set T of variables, then F is p-completely adjoining the set $\coprod_{\mathbb{N}} T$ of variables, where t in the i-th component represents $\delta^i(x_t)$. The statement on the fiber product and the associated F^n is clear. As for the statement about S^n, just notice that we have the pushout diagrams

$$\begin{tikzcd}
P^n \arrow[r] \arrow[d] & R^n := R^{\hat{\otimes}_A n} \arrow[r] \arrow[d] & R \arrow[d] \\
F^n \arrow[r] & S^{\hat{\otimes}_A n} \arrow[r] & S^n := S^{\hat{\otimes}_R n}
\end{tikzcd}$$

To prove (2), we may reduce modulo p. Note that $A \to F$ and $R \to S$ are p-completely faithfully flat. In a similar manner to the proof of Proposition 2.17 (3), using the conjugate filtration, plus the distinguished triangle of the cotangent complex, and fpqc descent of the cotangent complex (see [8, Theorem 3.1]), one can show that this natural map is an isomorphism.

(3) follows from (2) in the same way as Proposition 2.17 (2) follows from Proposition 2.17 (3). ∎

Remark 2.20. Similar to Proposition 2.18, assume A is p-torsionfree. Then the $\mathrm{dR}^{\wedge}_{S/F}$ appearing above are discrete rings, and we can equip them with a natural δ-structure. By the same proof of Proposition 2.18 the induced Frobenius on $\mathrm{dR}^{\wedge}_{R/A}$ agrees with the one provided by Construction 2.14.

We shall see in Remark 3.15 (1) that if (A, I) is a transversal prism, then there is only one Frobenius in a strong sense. So all these different constructions must give rise to the same map.

2.4. Naïve comparison

Consider the composition $f : A \xrightarrow{\varphi_A} A \to \mathcal{A}$; it induces a morphism of prisms which we still denote by $f : (A, I) \to (\mathcal{A}, (p))$. Let $\mathcal{X}$ be a p-completely smooth affine formal scheme over $\mathrm{Spf}(A/I)$. Now by the base change formula of prismatic cohomology [9, Theorem 1.8 (5)], we have

$$\mathrm{R\Gamma}_{\Delta}(\mathcal{X}/A) \mathbin{\widehat{\otimes}}_{A,f} \mathcal{A} \cong \mathrm{R\Gamma}_{\Delta}(\mathcal{Y}/\mathcal{A}),$$

where $\mathcal{Y} = \mathcal{X} \times_{\mathrm{Spf}(A/I),f} \mathrm{Spec}(\mathcal{A}/p)$.

Then the crystalline comparison of prismatic cohomology [9, Theorem 1.8 (1)] gives

$$\begin{aligned}\varphi_{\mathcal{A}}^*(\mathrm{R\Gamma}_{\Delta}(\mathcal{X}/A) \mathbin{\widehat{\otimes}}_{A,f} \mathcal{A}) &\cong \varphi_{\mathcal{A}}^*(\mathrm{R\Gamma}_{\Delta}(\mathcal{Y}/\mathcal{A})) \cong \mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{Y}/\mathcal{A}) \\ &\cong \varphi_{\mathcal{A}}^*(\mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{X}/\mathcal{A})). \end{aligned} \tag{$\boxdot$}$$

Here the last isomorphism comes from the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads] \arrow[d, "\varphi_{\mathcal{A}}"'] & A/(I,p) \arrow[d, "f"] \\ \mathcal{A} \arrow[r, two heads] & \mathcal{A}/p \end{tikzcd}$$

In the following, we aim at getting a Frobenius descent of the isomorphism obtained in ($\boxdot$); see Remark 3.8.

3. Comparing prismatic and derived de Rham cohomology

Let (A, I) be a bounded prism. Let X be a p-adic formal scheme which is formally smooth over $\mathrm{Spf}(A/I)$. In this section we shall establish a functorial comparison between the prismatic cohomology $\mathrm{R\Gamma}_{\Delta}(X/A)$ and the derived de Rham cohomology $\mathrm{dR}^{\wedge}_{X/A}$.

3.1. The comparison

First we need to comment on an error in the construction of Čech–Alexander complex in [9, Construction 4.16]. We learned this subtlety from Bhatt who was informed by Koshikawa. The issue is as follows, with notation as in *loc. cit.*: Suppose $D \to D/ID \leftarrow R$ is an object in $(R/A)_{\Delta}$. Then one needs to exhibit a morphism $(B\{\frac{J}{I}\}^{\wedge} \to D)$ in $(R/A)_{\Delta}$. The argument was along the following lines: by the universal property it suffices to exhibit a map $B \to D$ sending J into ID, which amounts to filling in the dotted arrow (of δ-rings)

$$\begin{tikzcd} R \arrow[r] & D/ID \\ B \arrow[u] \arrow[r, dashed] & D \arrow[u] \end{tikzcd}$$

that makes the diagram commutative. At first sight this seems easy, as B is a free δ-ring in a set of variables: we just lift images of those variables under $B \to R \to D/ID$ to D to get a map of δ-rings. But there is no way a general lift will make the above diagram commutative for the δ's of those variables.

Below we describe a fix that we learned from Bhatt. Recall that the forgetful functor from δ-A-algebras to A-algebras admits a left adjoint [9, Remark 2.7]. One checks the following easily:

- Given a derived (p, I)-completed polynomial A-algebra P which is freely generated by a set of variables, apply this left adjoint to get a derived (p, I)-completed free δ-A-algebra F generated by the same set of variables.
- This left adjoint commutes with completed tensor product.

In particular, the natural map $P \to F$ is (p, I)-completely ind-smooth.

Construction 3.1 (Čech–Alexander complex for prismatic cohomology). Let R be a p-completely smooth A/I-algebra. Let P be a derived (p, I)-completed polynomial A-algebra along with a surjection $P \twoheadrightarrow R$, and let J be the kernel. Associated with the triple $A \to P \twoheadrightarrow R$ is a δ-A-algebra $F\{\frac{JF}{I}\}^\wedge$, obtained by applying [9, Corollary 3.14]. We make three claims about this construction.

Claim 3.2. (1) *The δ-A-algebra $F\{\frac{JF}{I}\}^\wedge$ is naturally an object in $(R/A)_{\Delta}$;*

(2) *as such, it is weakly initial in $(R/A)_{\Delta}$;*

(3) *if there is a set of triples $A \to P_i \twoheadrightarrow R$, then the coproduct of the associated $F_i\{\frac{J_i F_i}{I}\}^\wedge$ in $(R/A)_{\Delta}$ is given by the δ-A-algebra associated with the triple $A \to \widehat{\bigotimes}_A P_i \twoheadrightarrow R$ where the second map is given by the completed tensor product of those $P_i \twoheadrightarrow R$ maps.*

Let us postpone the verification of these claims and continue with the construction. At this point we may simply follow the rest of [9, Construction 4.16]. Form the derived (p, I)-completed Čech nerve $P^\bullet$ of $A \to P$, and let $J^\bullet \subset P^\bullet$ be the kernel of the augmentation map $P^\bullet \to P \to R$. By the first claim above, we get a cosimplicial object $(F^\bullet\{\frac{J^\bullet F^\bullet}{I}\}^\wedge)$ in $(R/A)_{\Delta}$. The third claim above shows that this is the Čech nerve of $F\{\frac{JF}{I}\}^\wedge$ in $(R/A)_{\Delta}$, and according to the second claim the object $F\{\frac{JF}{I}\}^\wedge$ covers the final object of the topos $\mathrm{Shv}((R/A)_{\Delta})$. Therefore $\Delta_{R/A}$ is computed by $F^\bullet\{\frac{J^\bullet F^\bullet}{I}\}^\wedge$.

This construction commutes with base change of the prism (A, I). When (A, I) is fixed, this construction can be carried out in a way which is strictly functorial in R, by setting P to be the completed polynomial A-algebra generated by the underlying set of R.

Proof of Claim 3.2. (1) Form the pushout diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ P & \longrightarrow & F \end{array}$$

Denote $F\{\frac{JF}{I}\}^\wedge$ by C^0; by its defining property there is a natural map $S \cong F/JF \to C^0/IC^0$. Hence C^0 gives rise to a diagram $(C^0 \to C^0/IC^0 \leftarrow S \leftarrow R)$ which is an object in $(R/A)_{\mathbb{\Delta}}$.

(2, 3) These follow from chasing through universal properties. Let $(D \to D/ID \leftarrow R)$ be an object in $(R/A)_{\mathbb{\Delta}}$. We have the following chain of equivalences:

$$\begin{aligned} &F\left\{\frac{JF}{I}\right\}^\wedge \to D \text{ in } (R/A)_{\mathbb{\Delta}} \\ &\qquad \iff \text{a map } F \to D \text{ of } \delta\text{-}A\text{-algebras such that } JF \text{ is mapped into } ID \\ &\qquad \iff \text{a map } P \to D \text{ of } A\text{-algebras such that } J \text{ is mapped into } ID. \end{aligned}$$

It is easy to see that the last statement is equivalent to filling in the dotted arrow in the diagram

$$\begin{tikzcd} R \arrow[r] & D/ID \\ P \arrow[u] \arrow[r, dashed] & D \arrow[u] \end{tikzcd}$$

of A-algebras, making the diagram commutative. Note that there is no requirement from δ-ring considerations here. Now one checks claims (2) and (3) easily. ■

With the above preparatory discussion, we are ready to compare prismatic cohomology and derived de Rham cohomology. The key computation we need is the following.

Lemma 3.3 (Comparing prismatic and PD envelopes for regular sequences). *Let B be a (p, I)-completely flat δ-A-algebra, and let $f_1, \dots, f_r \in B$ be a (p, I)-completely regular sequence. Write $J = (I, f_1, \dots, f_r) \subset B$. Then we have a natural identification of p-completely flat $\mathrm{dR}_A(I)^\wedge$-algebras:*

$$B\left\{\frac{J}{I}\right\}^\wedge \widehat{\otimes}_{B,\varphi_B} B \,\widehat{\otimes}_A\, \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}_B(J)^\wedge.$$

Here the $B\{\frac{J}{I}\}^\wedge$ is as in [9, Proposition 3.13], which is (p, I)-completely flat over A. Let us clarify the various completions involved in the left hand side. First we take the derived (p, I)-complete tensor product, and then we take the derived $(p, \varphi(I))$-complete tensor product, which is the same as the derived p-complete tensor product since $\varphi(I) = (p)$ in $\pi_0(\mathrm{dR}_A(I)^\wedge)$.

Proof of Lemma 3.3. Recall that in the proof of [9, Proposition 3.13], as also explained in Remark 2.7, $B\{\frac{J}{I}\}^\wedge$ is constructed as the p-complete pushout of the diagram

$$\begin{tikzcd} A\{x_1, \dots, x_r\} \arrow[r, "x_i \mapsto f_i"] \arrow[d] & B \\ A\{x_1, \dots, x_r\}\{\frac{x_i}{I}\} & \end{tikzcd}$$

The left hand side in this lemma is therefore given by pushing out the above diagram further along

$$f_B : B \xrightarrow{\varphi_B} B \to B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge.$$

The composition

$$A\{x_1, \dots, x_r\} \xrightarrow{x_i \mapsto f_i} B \xrightarrow{f_B} B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge$$

can now be factored as

$$A\{x_1, \dots, x_r\} \xrightarrow{\varphi_A, x_i \mapsto \varphi(z_i)} A\{z_1, \dots, z_r\} \to \mathrm{dR}_{A\{z_1,\dots,z_r\}}(I)^\wedge \to B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge,$$

where the last map sends z_i to $f_i \otimes 1$. Hence the left hand side becomes the p-complete outer pushout of the following diagram with solid arrows:

$$\begin{tikzcd}
A\{x_1, \dots, x_r\} \arrow[r] \arrow[d] & \mathrm{dR}_{A\{z_1,\dots,z_r\}}(I)^\wedge \arrow[r] \arrow[d, dotted] & B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \arrow[d, dotted] \\
A\{x_1, \dots, x_r\}\{\frac{x_i}{I}\} \arrow[r, dotted] & \mathrm{dR}_{A\{z_1,\dots,z_r\}}(I, z_1, \dots, z_r)^\wedge \arrow[r, dotted] & \mathrm{dR}_{B \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge}(f_i \otimes 1)^\wedge \cong \mathrm{dR}_B(J)^\wedge
\end{tikzcd}$$

Using (the multi-variable version of) Lemma 2.10 we see that the left square above is a p-complete pushout. The base change property of the derived de Rham complex now shows that the right square is also a p-complete pushout. Here the isomorphism of the right bottom corner follows from the fact that $(I, f_1, \dots, f_r)$ is a Koszul regular sequence in B. ∎

Just like [9, Proposition 3.13] implies [9, Corollary 3.14], our Lemma 3.3 gives the following.

Lemma 3.4. *Let R be a p-completely smooth A/I-algebra. Let P be a p-completed polynomial algebra over A, and let $P \twoheadrightarrow R$ be a surjection of A-algebras with kernel J. Consider the diagram*

$$\begin{tikzcd}
A/I \arrow[r] & R \arrow[r] & S \\
A \arrow[r] \arrow[u] & P \arrow[r] \arrow[u] & F \arrow[u]
\end{tikzcd}$$

where F is the p-completed free δ-A-algebra associated with P, and S is the p-completed tensor product $R \widehat{\otimes}_P F$. Then we have a natural identification of p-completely flat $\mathrm{dR}_A(I)^\wedge$-algebras:

$$F\left\{\frac{J \cdot F}{I}\right\} \widehat{\otimes}_{F,\varphi_F} F \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}^\wedge_{S/F}.$$

Proof. Zariski locally on $\mathrm{Spf}(P)$ and $\mathrm{Spf}(F)$, the kernel J and $J \cdot F$ are colimits of the form considered in Lemma 3.3. Also note that $F/J \cdot F \cong S$, so by definition we have $\mathrm{dR}_F(J \cdot F)^\wedge \cong \mathrm{dR}^\wedge_{S/F}$.

Since formation of the p-complete derived de Rham complex commutes with taking p-complete colimit (of the algebra over A) and descends from p-completely flat covers, we may glue the local isomorphisms obtained in Lemma 3.3 and take a colimit to get our identification here. ∎

Using this comparison of the prismatic envelope and the derived de Rham complex, we get a comparison between prismatic and derived de Rham cohomology as follows.

Theorem 3.5. *Let (A, I) be a bounded prism. For any p-completely smooth A/I-algebra R, there is a natural isomorphism*

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A,\varphi_A} A \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}^\wedge_{R/A} \quad \text{in } \mathrm{CAlg}(A),$$

which is functorial in $A/I \to R$ and satisfies base change in (A, I).

Let us emphasize again that when (A, I) is transversal, this follows from [9, Theorem 5.2].

Proof of Theorem 3.5. Let us first construct the desired natural morphism

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A,\varphi_A} A \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \to \mathrm{dR}^\wedge_{R/A}.$$

Given any triple $A \to P \twoheadrightarrow R$ as in the setting of Lemma 3.4, we have a natural morphism

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A,\varphi_A} A \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \to F\left\{\frac{J \cdot F}{I}\right\}^\wedge \widehat{\otimes}_{F,\varphi_F} F \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}^\wedge_{S/F},$$

which is functorial in $A \to P \twoheadrightarrow R$. By Proposition 2.19 (3), the limit of the right hand side over all triples $A \to P \twoheadrightarrow R$ is just $\mathrm{dR}^\wedge_{R/A}$, hence we get the desired natural morphism. It is functorial in $A/I \to R$ and satisfies base change in (A, I).

Now we need to show the natural arrow constructed above is a natural isomorphism. Let us make more reductions. It suffices to check this is an isomorphism after a faithfully flat cover, and since both sides commute with base change in A, we may Zariski localize on A, hence we may first reduce to the case where A is oriented, i.e., $I = (d)$. Observe that both sides are the left Kan extensions of their restrictions to the category of polynomial A-algebras, so it suffices to show that the above arrow is a natural isomorphism for algebras of the form $R = A/I[X_1, \ldots, X_n]^\wedge$, which is the base change of p-complete polynomial algebras over the universal oriented prism. Hence we can reduce further to the case that A is the universal oriented prism. In particular, we may assume that (A, I) is transversal and that φ_A is flat.

Lastly, we shall prove the statement under the assumption that (A, I) is transversal and that φ_A is flat. Choose a (p, I)-completely polynomial A algebra P with a surjection of A-algebras $P \twoheadrightarrow R$, and form the cosimplicial object $(F^\bullet\{\frac{J^\bullet F^\bullet}{I}\}^\wedge)$ in $(R/A)_{\Delta}$ computing $\Delta_{R/A}$ as in Construction 3.1. Notice that we have an identification of cosimplicial (p, I)-complete algebras $A \xrightarrow{\simeq} F^\bullet$.

Since we have reduced ourselves to the case where (A, I) is transversal and that φ_A is flat, using Lemma 2.5, the natural morphism considered above gives rise to the following identification:

$$\begin{aligned} &\mathrm{R}\Gamma_{\Delta}(R/A)\,\widehat{\otimes}_{A,\varphi_A} A\,\widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \lim_{\Delta}\Bigg(\bigg(F^\bullet\Big\{\frac{F^\bullet\cdot J^\bullet}{I}\Big\}^\wedge\bigg)\,\widehat{\otimes}_{A,\varphi_A} A\,\widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge\Bigg) \\ &\cong \lim_{\Delta}\Bigg(\bigg(F^\bullet\Big\{\frac{F^\bullet\cdot J^\bullet}{I}\Big\}^\wedge\bigg)\,\widehat{\otimes}_{F^\bullet,\varphi_{F^\bullet}} F^\bullet\,\widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge\Bigg) \cong \lim_{[n]\in\Delta} \mathrm{dR}^\wedge_{S\widehat{\otimes}_{R^n}/F^n} \cong \mathrm{dR}^\wedge_{R/A} \end{aligned}$$

(⚁)

as desired. Let us comment on the identifications above. Here we have used the cosimplicial replacement $(A,\varphi_A)\xrightarrow{\simeq}(F^\bullet,\varphi_{F^\bullet})$ in the second identification. The second-to-last identification is provided by Lemma 3.4, and the last identification is because of Proposition 2.19. ∎

Remark 3.6. In this paper we have only defined the Frobenius action on $\mathrm{dR}^\wedge_{-/A}$ under the assumption that A is a p-torsionfree δ-ring. Now suppose (A,I) is a p-torsionfree prism; by Remark 2.20, we see that the chain of identifications in (⚁) is compatible with the Frobenius. Consequently, the identification in Theorem 3.5 is compatible with the Frobenius in a functorial manner.

We expect however that one can remove the p-torsionfree condition with additional work, developing the framework of "derived δ-rings". However, since the primary interest of this paper is in the case of p-torsionfree prisms, we do not pursue that level of generality here.

Below we deduce two consequences from Theorem 3.5.

Corollary 3.7. *Let (A,I) be a bounded prism. For any p-completely smooth A/I-algebra R, there is a natural isomorphism*

$$\mathrm{R}\Gamma_{\Delta}(R/A)\,\widehat{\otimes}_{A,\varphi_A} A\,\widehat{\otimes}_A \mathcal{A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/\mathcal{A}) \quad \text{in } \mathrm{CAlg}(A),$$

which is functorial in $A/I\to R$ and satisfies base change in (A,I).

Proof. This follows from Theorem 3.5: simply base change along the morphism $\mathrm{dR}_A(I)^\wedge\to\mathcal{A}$, and by Proposition 2.11 we have

$$\mathrm{dR}^\wedge_{R/A}\,\widehat{\otimes}_{\mathrm{dR}_A(I)^\wedge}\mathcal{A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/\mathcal{A}).$$ ∎

Remark 3.8. By diagram chasing, one verifies that the diagram of isomorphisms

$$\begin{tikzcd} \varphi^*_{\mathcal{A}}(\mathrm{R}\Gamma_{\Delta}(R/A)\,\widehat{\otimes}_{A,\varphi_A} A\,\widehat{\otimes}_A \mathcal{A}) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \varphi^*_{\mathcal{A}}(\mathrm{R}\Gamma_{\mathrm{crys}}(R/A)) \arrow[d, "\epsilon"] \\ \varphi^*_{\mathcal{A}}(\mathrm{R}\Gamma_{\Delta}((R\,\widehat{\otimes}_{A,\varphi_A}\mathcal{A})/\mathcal{A})) \arrow[r, "\gamma"] & \mathrm{R}\Gamma_{\mathrm{crys}}((R/p\otimes_{A/(p,I),\varphi_A} A/(p,I))/\mathcal{A}) \end{tikzcd}$$

is commutative, since all comparisons here are expressed in terms of various explicit envelopes. Here the arrows are as follows:

(1) α is the Frobenius pullback of the arrow in Corollary 3.7;

(2) β is the base change of prisms $\varphi_A : (A, I) \to (\mathcal{A}, p)$;

(3) γ is the crystalline comparison for crystalline prisms [9, Theorem 5.2];

(4) ϵ is the base change of crystalline cohomology.

A bounded prism (A, I) is called a *PD prism* if there is a PD structure γ on I, compatible with the canonical one on (p).

Corollary 3.9. *Let (A, I, γ) be a bounded PD prism. Then for any p-completely smooth A/I-algebra R, there is a natural isomorphism*

$$\mathrm{R}\Gamma_{\Delta}(R/A) \mathbin{\widehat{\otimes}}_{A,\varphi_A} A \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/(A, I, \gamma)) \quad \text{in } \mathrm{CAlg}(A),$$

which is functorial in $A/I \to R$ and satisfies base change in (A, I).

Here $\mathrm{R}\Gamma_{\mathrm{crys}}(-/(A, I, \gamma))$ denotes the crystalline cohomology with respect to the p-adic PD base (A, I, γ).

Proof. The additional PD structure gives us a section $\mathcal{A} \to A$, which makes the composition

$$A \to \mathrm{dR}_A(I)^\wedge \to \mathcal{A} \to A$$

the identity. Take the functorial isomorphism in Corollary 3.7, and base change further along $\mathcal{A} \to A$ to get

$$\mathrm{R}\Gamma_{\Delta}(R/A) \mathbin{\widehat{\otimes}}_{A,\varphi_A} A \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/\mathcal{A}) \mathbin{\widehat{\otimes}}_{\mathcal{A}} A;$$

the latter is naturally isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(R/(A, I, \gamma))$ due to base change in crystalline cohomology. ■

Remark 3.10. (1) Any derived p-complete δ-ring A with bounded p-torsion together with the ideal (p) is a PD prism. In this situation, our Corollary 3.9 is simply the crystalline comparison in [9, Theorem 1.8 (1)].

(2) The left hand side of this comparison does not depend on the PD structure γ on I, whereas the right hand side *a priori* does. Therefore this comparison tells us that the right hand side does not depend on the PD structure γ either.

We can "globalize" these comparisons to general quasicompact quasiseparated smooth formal schemes over $\mathrm{Spf}(A/I)$.

Theorem 3.11. *Let (A, I) be a bounded prism. Let $X \to \mathrm{Spf}(A/I)$ be a quasicompact quasiseparated smooth morphism of formal schemes. Then we have natural isomorphisms in $\mathrm{CAlg}(A)$:*

$$\mathrm{R}\Gamma_{\Delta}(X/A) \mathbin{\widehat{\otimes}}_{A,\varphi_A} A \mathbin{\widehat{\otimes}}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{R}\Gamma(X, \mathrm{dR}^\wedge_{-/A}),$$
$$\mathrm{R}\Gamma_{\Delta}(X/A) \mathbin{\widehat{\otimes}}_{A,\varphi_A} A \mathbin{\widehat{\otimes}}_A \mathcal{A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(X/\mathcal{A}).$$

If (A, I, γ) is a PD prism, then we have a natural isomorphism in $\mathrm{CAlg}(A)$*:*

$$\mathrm{R}\Gamma_{\mathbb{\Delta}}(X/A) \mathbin{\widehat{\otimes}}_{A,\varphi_A} A \cong \mathrm{R}\Gamma_{\mathrm{crys}}(X/(A, I, \gamma)).$$

All the isomorphisms above satisfy base change in (A, I)*. Moreover, if X is also proper over* $\mathrm{Spf}(A/I)$*, then all the completed tensor products above may be replaced by tensor products.*

Proof. Since X is assumed to be quasicompact and quasiseparated, these cohomologies are computed as finite limits of the corresponding cohomologies of affine opens of X. Because completed tensor product commutes with finite limit, the comparisons here follow from Theorem 3.5, Corollary 3.7, and Corollary 3.9.

To justify the replacement of completed tensor products with tensor products, just note that $\mathrm{R}\Gamma_{\mathbb{\Delta}}(X/A)$ is a perfect complex of A-modules for smooth proper $X \to \mathrm{Spf}(A/I)$; see the last sentence of [9, Theorem 1.8]. ∎

3.2. Functorial endomorphisms of the derived de Rham complex

Throughout this subsection, we assume that (A, I) is a transversal prism, in particular $\mathrm{dR}_A(I)^\wedge \cong \mathcal{A}$ and $\mathrm{dR}^\wedge_{R/A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(R/\mathcal{A})$ where R is any p-adic formally smooth A/I-algebra (see Proposition 2.11).

In this subsection, we aim at understanding all functorial endomorphisms of the derived de Rham complex functor, under this transversality assumption.

In particular, we shall see that the functorial isomorphism

$$\mathrm{R}\Gamma_{\mathbb{\Delta}}(R/A) \mathbin{\widehat{\otimes}}_{A,\varphi_A} A \mathbin{\widehat{\otimes}}_A \mathrm{dR}_A(I)^\wedge \to \mathrm{dR}^\wedge_{R/A}$$

appearing in Theorem 3.5 is unique if we assume that (A, I) is a transversal prism. In order to show this, we need to first extend the natural isomorphism to a larger class of A/I-algebras.

Construction 3.12 (cf. [9, Construction 7.6] and [8, Example 5.12]). Fix a bounded prism (A, I), and consider the functor $R \mapsto \mathrm{dR}^\wedge_{R/A}$ on p-completely smooth A/I-algebras R valued in the category of commutative algebras in the ∞-category of p-complete objects in $\mathcal{D}(A)$. Left Kan extend it to all derived p-complete simplicial A/I-algebras, which results in nothing other than the p-adic derived de Rham complex relative to A, still denoted by $\mathrm{dR}^\wedge_{R/A}$. Let us record some properties of this construction:

(1) Since R is an A/I-algebra, $\mathrm{dR}^\wedge_{R/A}$ is naturally a $\mathrm{dR}^\wedge_{(A/I)/A}$-algebra. Hence we may actually view the functor as taking values in the category $\mathrm{CAlg}(\mathrm{dR}_A(I)^\wedge)$.

(2) The formation of $\mathrm{dR}^\wedge_{R/A}$ commutes with base change in A.

(3) Below we shall see that, following the reasoning of [8, Theorem 3.1 and Example 5.12], the assignment $R \mapsto \mathrm{dR}^\wedge_{R/A}$ defines a sheaf on the relative quasisyntomic site $\mathrm{qSyn}_{A/I}$.

(4) By left Kan extending the natural isomorphism obtained in Theorem 3.5, we get an isomorphism of sheaves

$$\Delta^{(1)}_{R/A} \widehat{\otimes}_A \mathrm{dR}_A(I)^\wedge \cong \mathrm{dR}^\wedge_{R/A},$$

which is compatible with base change in A. Here $\Delta^{(1)}_{R/A} := \Delta_{R/A} \widehat{\otimes}_{A,\varphi_A} A$ is the Frobenius pullback of the derived prismatic cohomology.

(5) Moreover, if we assume that (A, I) is a transversal prism, then for any R which is large quasisyntomic over A/I, the value $\mathrm{dR}^\wedge_{R/A}$ is p-completely flat over A and lives in cohomological degree 0.

Let us justify claim (3) above.

Proposition 3.13. *The assignment* $R \mapsto \mathrm{dR}^\wedge_{R/A}$ *defines a sheaf on the relative quasisyntomic site* $\mathrm{qSyn}_{A/I}$.

Proof. Let $R \to S$ be a quasisyntomic cover of objects in $\mathrm{qSyn}_{A/I}$, with Čech nerve $S^\bullet$. Our task is to show $\mathrm{dR}^\wedge_{R/A} = \lim_{\Delta^{\mathrm{op}}} \mathrm{dR}^\wedge_{S^\bullet/A}$. Since both sides are p-complete, we may check this after taking the derived complex modulo p. Below we shall always use $-/p$ to denote "derived modulo p".

Now we closely follow the argument in [8, Example 5.12], correcting a typo thereof. First, there is a functorial exhaustive increasing $\mathbb{N}$-index filtration, i.e., the conjugate filtration, on $\mathrm{dR}_{R/A}/p \cong \mathrm{dR}_{(R/p)/(A/p)}$ with graded pieces given by $(\bigwedge^i_{(R/p)^{(1)}} \mathbb{L}_{(R/p)^{(1)}/(A/p)})[-i]$ (and similarly for $\mathrm{dR}^\wedge_{(S^\bullet/p)/(A/p)}$). Here $(-/p)^{(1)}$ denotes the base change along the Frobenius on A/p, and the *loc. cit.* has a typo of not adding this Frobenius twist. For a discussion of p-complete derived de Rham complex and conjugate filtration in the realm of animated rings, we refer the readers to [23, pp. 33–35].

Let us look at the following diagram (with its $S^\bullet$ analogs in mind):

$$\begin{tikzcd}
R \arrow[r] & R/p \arrow[r, "\varphi_{A/p}"] & (R/p)^{(1)} \\
A/I \arrow[r] \arrow[u] & (A/I)/p \arrow[r, "\varphi_{A/p}"] \arrow[u] & ((A/I)/p)^{(1)} \cong (A/I^p)/p \arrow[u] \\
A \arrow[r] \arrow[u] & A/p \arrow[r, "\varphi_{A/p}"] \arrow[u] & A/p \arrow[u]
\end{tikzcd}$$

Note that every square above is Cartesian. The base change property of the cotangent complex implies that the graded pieces $(\bigwedge^i_{(R/p)^{(1)}} \mathbb{L}_{(R/p)^{(1)}/(A/p)})[-i]$ (and their $S^\bullet$ analogs) can be identified with either

(1) $\bigwedge^i_R \mathbb{L}_{R/A}[-i] \otimes_R \varphi_{A/p,*}(R/p)^{(1)}$, or

(2) $\bigwedge^i_R \mathbb{L}_{R/A} \otimes_A \varphi_{A/p,*}(A/p)$,

where the A-module (resp. R-module) structure on $\varphi_{A/p,*}(A/p)$ (resp. $\varphi_{A/p,*}(R/p)^{(1)}$) is given by the top and bottom row of the above diagram.

The identification (1) above implies that all these graded pieces live in $D^{\geq -1}$. Indeed, $\bigwedge^i_R \mathbb{L}_{R/A}[-i]$ as an R-module has Tor-amplitude in $[0, i]$, and $(R/p)^{(1)}$ is flat over $((A/I)/p)^{(1)} \cong (A/I^p)/p$ which lives in $[-1, 0]$, so $(R/p)^{(1)}$ lives in $D^{\geq -1}$. Similar statements for $S^\bullet$ hold as well. Hence it remains to check that these graded pieces satisfy the descent property; here we are using the reasoning of [8, last sentence of Example 5.12]. Now using the identification (2) above, we are reduced to flat descent for "tensored" wedge powers of the cotangent complex; see [25, Proposition 3.2] (which is itself a generalization of [8, Theorem 3.1]). ∎

Recall that an A/I-algebra is called *large quasisyntomic over A/I* (see [9, Definition 15.1]) if

- $A/I \to R$ is quasisyntomic;
- there is a surjection $A/I\langle X_j^{1/p^\infty} \mid j \in J\rangle \twoheadrightarrow R$ where J is a set.

The following is inspired by [6, Sections 10.3 and 10.4], and our proof is a modification of the reasoning there.

Theorem 3.14. *Let (A, I) be a transversal prism, and assume that* $\mathrm{Spf}(A/I)$ *is connected.*

(1) *The mapping space*

$$\mathrm{End}_{\mathrm{Shv}(\mathrm{qSyn}_{A/I}, \mathrm{CAlg}(\mathcal{A}))}(\mathrm{dR}^\wedge_{-/A}, \mathrm{dR}^\wedge_{-/A})$$

has contractible components given by a submonoid in $\mathbb{N}$. In particular, the automorphism space has only one contractible component given by the identity.

(2) *The automorphism space*

$$\mathrm{Aut}_{\mathrm{Shv}(\mathrm{qSyn}_{A/I}, \mathrm{CAlg}(A/I))}(\mathrm{dR}^\wedge_{-/(A/I)}, \mathrm{dR}^\wedge_{-/(A/I)})$$

has only one contractible component given by the identity.

Since $\mathcal{A}/p \to A/(I, p)$ is a locally nilpotent thickening, we see that $\mathrm{Spf}(\mathcal{A})$ is also connected. In particular, the only idempotents in $\mathcal{A}$ are 0 and 1. It is easy to see that the statements concerning automorphism spaces for these functors hold true without the connectedness assumption, as on each connected component the automorphism must be the identity.

Proof of Theorem 3.14. The assertion that all components are contractible follows from the fact that on the basis of large quasisyntomic A/I-algebras, the sheaves $\mathrm{dR}^\wedge_{-/A}$ and $\mathrm{dR}^\wedge_{-/(A/I)}$ are discrete.

All we need to check is that there are not many functorial endomorphisms (resp. automorphisms) for these two sheaves. Since (2) has the same proof as (1), let us only present the proof of (1) here. To simplify notation, let us denote the set of functorial endomorphisms by $\mathrm{End}(\mathrm{dR}^\wedge_{-/A})$. By restriction, any functorial endomorphism induces a functorial endomorphism of the functor restricted to the subcategory of A/I-algebras of the form $A/I\langle X_h^{1/p^\infty} \mid h \in H\rangle$ for some set H. We denote the latter monoidal space by $\mathrm{End}(\mathrm{dR}^\wedge_{-/A}|_{\mathrm{perf}})$; all of its components are also contractible by the same reasoning. By definition there is a natural map

$$\mathrm{res} : \mathrm{End}(\mathrm{dR}^\wedge_{-/A}) \to \mathrm{End}(\mathrm{dR}^\wedge_{-/A}|_{\mathrm{perf}})$$

of monoids.

Now we make the following three claims:

- the natural map res is injective;
- the monoid $\mathrm{End}(\mathrm{dR}^\wedge_{-/A}|_{\mathrm{perf}})$ is a submonoid of $\mathbb{Z}$;
- the image of res is contained in $\mathbb{N}$.

To show that res is injective, we need to show that any functorial endomorphism of $\mathrm{dR}^\wedge_{-/A}$ is determined by its restriction to the algebras of the form $A/I\langle X_h^{1/p^\infty} \mid h \in H\rangle$ for some set H. To see this, notice that $\mathrm{qSyn}_{A/I}$ has a basis given by large quasisyntomic A/I-algebras. Any large quasisyntomic A/I-algebra S, by definition, admits a surjection from an algebra of the form $A/I\langle X_l^{1/p^\infty} \mid l \in L\rangle$ for some set L. By choosing a set $\{f_j \mid j \in J\}$ of generators of the kernel, we may form a surjection (cf. [9, proof of Proposition 7.10])

$$S' := A/I\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J\rangle/(Y_j - f_j \mid j \in J)^\wedge \twoheadrightarrow S : Y_j^m \mapsto 0.$$

This induces a surjection $\mathbb{L}_{S'/A}[-1] \twoheadrightarrow \mathbb{L}_{S/A}[-1]$ of shifted cotangent complexes, therefore it induces a surjection $\mathrm{dR}^\wedge_{S'/A} \twoheadrightarrow \mathrm{dR}^\wedge_{S/A}$ of p-adic derived de Rham complexes. For any such S', we have

$$\mathrm{dR}^\wedge_{S'/A} \cong D_{\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} | l\in L,\, j\in J\rangle}(Y_j - f_j \mid j \in J)^\wedge,$$

i.e., p-completely adjoining divided powers of $Y_j - f_j$ for all $j \in J$ to $\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L, j \in J\rangle$. Since $\mathcal{A}$ is p-torsionfree, any endomorphism of $\mathrm{dR}^\wedge_{S'/A}$ is determined by its restriction to $\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L,\ j \in J\rangle$. Lastly, we know that applying $\mathrm{dR}^\wedge_{-/A}$ to the map

$$A/I\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L,\ j \in J\rangle \to S'$$

exactly induces the natural map

$$\mathcal{A}\langle X_l^{1/p^\infty}, Y_j^{1/p^\infty} \mid l \in L,\ j \in J\rangle \to \mathrm{dR}_{S'/A}.$$

Therefore we know that any functorial endomorphism of $\mathrm{dR}^\wedge_{-/A}$ must be determined by its restriction to algebras of the form $A/I\langle X_h^{1/p^\infty} \mid h \in H\rangle$.

Next, let us show that $\mathrm{End}(\mathrm{dR}^\wedge_{-/A}|_{\mathrm{perf}})$ is a submonoid of integers. Consider a functorial endomorphism f. It is determined by its restriction to the one-variable "perfect" A/I-algebra $R = A/I\langle X^{1/p^\infty}\rangle$. We know $\mathrm{dR}^\wedge_{R/A} \cong \mathcal{A}\langle X^{1/p^\infty}\rangle$. Suppose $f(x) = \sum_{i\in\mathbb{N}[1/p]} a_i X^i \in \mathcal{A}\langle X^{1/p^\infty}\rangle$. Consider the map $R \to S := A/I\langle Y^{1/p^\infty}, Z^{1/p^\infty}\rangle$ sending X^i to $Y^i Z^i$. This map induces the corresponding map $\mathcal{A}\langle X^{1/p^\infty}\rangle \to$

$\mathcal{A}\langle Y^{1/p^\infty}, Z^{1/p^\infty}\rangle$ which also sends X^i to $Y^i Z^i$. Now the functoriality of the endomorphism tells us that $f(YZ) = f(Y) \cdot f(Z)$. We immediately get $a_i^2 = a_i$ and $a_i \cdot a_j = 0$ for any pair of distinct indices $i, j \in \mathbb{N}[1/p]$. By the connectedness of $\mathrm{Spf}(A/I)$, there is at most one index $i \in \mathbb{N}[1/p]$ with nonzero $a_i = 1$. To see there is at least one nonzero a_i, we use the map $R \to A/I$ given by $X^i \mapsto 1$ for all $i \in \mathbb{N}[1/p]$.

We want to show that the $i \in \mathbb{N}[1/p]$ obtained in the previous paragraph, defining the functorial endomorphism f, must in fact lie in $p^{\mathbb{Z}}$. Assume $i = \ell/p^N$ where ℓ is an integer coprime to p. Now we consider the map $R \to S$ given by $X \mapsto \lim_n (Y^{1/p^n} + Z^{1/p^n})^{p^n}$. It induces a map of $\mathrm{dR}^\wedge_{-/A}$ with the image of X given by the same formula. Functoriality of f implies that

$$\left(\lim_n (Y^{1/p^n} + Z^{1/p^n})^{p^{n-N}}\right)^\ell = \lim_n (Y^{\ell/p^{n-N}} + Z^{\ell/p^{n-N}})^{p^n}.$$

Reduction modulo p yields

$$(Y^{1/p^N} + Z^{1/p^N})^\ell = Y^{\ell/p^N} + Z^{\ell/p^N} \in \mathbb{F}_p[Y^{1/p^\infty}, Z^{1/p^\infty}],$$

forcing $\ell = 1$. Therefore we see that $\mathrm{End}(\mathrm{dR}^\wedge_{-/A}|_{\mathrm{perf}}) \subset p^{\mathbb{Z}}$, i.e., it is a submonoid in $\mathbb{Z}$.

Finally, let us prove the image of res lands in $p^{\mathbb{N}}$. We want to rule out negative powers of p. To that end consider $R \to R/(X)$, which induces the map of the p-adic derived de Rham complex:

$$\widetilde{R} := \mathcal{A}\langle X^{1/p^\infty}\rangle \to \widetilde{S} := D_{\mathcal{A}\langle X^{1/p^\infty}\rangle}(X)^\wedge.$$

Here the latter denotes the p-complete PD envelope of the former along the ideal X, and this is the natural map. Take a positive integer j; we need to argue that $X \mapsto X^{1/p^j}$ on $\widetilde{R}$ does not extend to an endomorphism of $\widetilde{S}$. Suppose otherwise; then the extended endomorphism of $\widetilde{S}$ must send X^p to $X^{p^{1-j}}$, but X^p is divisible by p in $\widetilde{S}$ whereas $X^{p^{1-j}}$ is not (here we use the fact that $j > 0$), hence we get a contradiction.

The only invertible element in the additive monoid $\mathbb{N}$ is 0, corresponding to $X \mapsto X^{(p^0)} = X$, hence the only functorial automorphism of $\mathrm{dR}^\wedge_{-/A}$ is the identity. ■

Remark 3.15. Let (A, I) be a transversal prism.

(1) By the same argument, there are not many functorial homomorphisms from $\varphi_A^* \mathrm{dR}^\wedge_{-/A}$ to $\mathrm{dR}^\wedge_{-/A}$. Similarly, these are determined by restriction to $R = A/I\langle X^{1/p^\infty}\rangle$. If we require that the restriction sends X to X^p, then there is a unique one given by the Frobenius constructed in Section 2.3. Therefore in a strong sense, there is a unique Frobenius.

(2) Due to the previous remark, we see that the comparison in Theorem 3.5 must be compatible with the Frobenius.

(3) It is unclear which positive integer i, corresponding to $X \mapsto X^{p^i}$, can occur as a functorial endomorphism. If $A/(p, I)$ has transcendental (relative to $\mathbb{F}_p$) elements, then none of these can occur. This can be seen by considering the map $R \to R/(X - a)$ for some lift a of the transcendental element $\overline{a} \in A/(p, I)$.

Consequently, we get the following uniqueness of the functorial comparison established in Theorem 3.5; the readers should compare it with [9, Section 18].

Corollary 3.16. *Fix a transversal prism (A, I). There is a unique natural isomorphism of p-complete commutative algebra objects in $\mathcal{D}(\mathcal{A})$:*

$$\mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A,\varphi_A} A \widehat{\otimes}_A \mathcal{A} \to \mathrm{R}\Gamma_{\mathrm{crys}}(R/A),$$

which is functorial in the p-completely smooth A/I-algebra R.

Proof. The existence part is given by Theorem 3.5; we need to show uniqueness. Suppose there are two such functorial isomorphisms. Composing one with the inverse of the other, we get a natural automorphism of the functor $\mathrm{dR}_{-/A} \cong \mathrm{R}\Gamma_{\mathrm{crys}}(-/A)$ on smooth A/I-algebras. By left Kan extension, this will induce a natural automorphism of the functor $\mathrm{dR}_{-/A}$ on quasisyntomic A/I-algebras. By Theorem 3.14, this automorphism must be the identity. ■

Corollary 3.17. *Let* **C** *be an algebraically closed complete non-Archimedean field extension of $\mathbb{Q}_p$, and let (A, I) be the associated perfect prism (denoted by $(A_{\mathrm{inf}}, \ker(\theta))$ in the literature). Then the comparison in Theorem 3.5 is compatible with the crystalline comparison over $\mathcal{A} = A_{\mathrm{crys}}$ of the $A\Omega$-theory obtained in* [7]*. Concretely, the following diagram of isomorphisms is commutative:*

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\Delta}(R/A) \widehat{\otimes}_{A,\varphi_A} A \widehat{\otimes}_A \mathcal{A} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{crys}}(R/A) \\ \downarrow & & \downarrow \\ A\Omega(R) \widehat{\otimes}_A \mathcal{A} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{crys}}((R/p)/A) \end{array}$$

where the left vertical arrow is given by [9, Theorem 17.2] *and the bottom horizontal arrow is given by* [7, Theorem 12.1] *or* [35].

Proof. This follows from the uniqueness statement in Corollary 3.16. ■

Both sides of the isomorphism obtained in Theorem 3.5 after completely tensoring $\mathcal{A}/\mathcal{I} \cong A/I$ over $\mathcal{A}$, are naturally isomorphic to $\mathrm{dR}^{\wedge}_{R/(A/I)}$. For the left hand side this follows from the de Rham comparison of (the Frobenius pullback of) the prismatic cohomology:

$$\Delta^{(1)}_{R/A} \widehat{\otimes}_A \mathcal{A} \widehat{\otimes}_{\mathcal{A}} A/I \cong \Delta^{(1)}_{R/A} \widehat{\otimes}_A A/I \cong \mathrm{dR}^{\wedge}_{R/(A/I)},$$

where the last equality follows from [9, Theorem 6.4 or Corollary 15.4]. For the right hand side this is just the base change of the derived de Rham complex (or base change of crystalline cohomology and the comparison of de Rham and crystalline cohomology for smooth morphisms). We observe that a similar argument forces these natural isomorphisms to be compatible with each other.

Corollary 3.18. *Let (A, I) be a transversal prism. The following triangle of natural isomorphisms is commutative:*

$$\begin{tikzcd} & \mathrm{dR}^{\wedge}_{R/(A/I)} & \\ {\mathbb{\Delta}^{(1)}_{R/A} \widehat{\otimes}_A \mathcal{A} \widehat{\otimes}_{\mathcal{A}} A/I} \arrow[ur, "\cong"] \arrow[rr, "\cong"] & & {\mathrm{dR}^{\wedge}_{R/A} \widehat{\otimes}_{\mathcal{A}} A/I} \arrow[ul, "\cong"'] \end{tikzcd}$$

Proof. Observe that all three natural isomorphisms are functorial in R, hence going around the circle produces a functorial automorphism of $\mathrm{dR}^{\wedge}_{R/(A/I)}$.

Now we argue as in the proof of Corollary 3.16: by Theorem 3.14 this functorial automorphism must be the identity, so the above diagram commutes functorially. ■

4. Filtrations

Throughout this section, we assume that (A, I) is a transversal prism and let $(\mathcal{A}, \mathcal{I})$ be the p-adic PD envelope of A along I. By Theorem 3.5, for any p-completely smooth A/I-algebra R we have a functorial isomorphism

$$\varphi^*(\mathrm{R}\Gamma_{\mathbb{\Delta}}(R/A)) \widehat{\otimes}_A \mathcal{A} \cong \mathrm{dR}^{\wedge}_{R/A}\,.$$

All objects involved here have interesting filtrations: the Nygaard filtration on $\varphi^*(\mathrm{R}\Gamma_{\mathbb{\Delta}}(R/A))$, the I-adic filtration on A, the PD ideal filtration $\mathcal{I}^{[\bullet]}$ on $\mathcal{A}$, and the Hodge filtration on $\mathrm{dR}^{\wedge}_{R/A}$. In this section, we discuss how these filtrations are related.

Unless otherwise specified, R will denote a general A/I-algebra, and S a large quasisyntomic A/I-algebra (see the discussion right after Construction 3.12).

Let us briefly recall how these filtrations are defined and their properties.

4.1. Hodge filtration on $\mathrm{dR}^{\wedge}_{R/A}$

Recall that $\mathrm{R}\Gamma_{\mathrm{crys}}(R/A)$ is the cohomology of the structure sheaf $\mathcal{O}_{\mathrm{crys}}$ on the (absolute) crystalline site $(R/A)_{\mathrm{crys}}$. The crystalline structure sheaf admits a natural surjection to the Zariski structure sheaf, whose kernel is an ideal sheaf $\mathcal{I}_{\mathrm{crys}}$ admitting divided powers. Concretely, given a PD thickening (U, T) with U a p-adic formal $\mathrm{Spf}(A)$-scheme with an $\mathrm{Spf}(A)$-map $U \to \mathrm{Spf}(R)$ and $U \hookrightarrow T$ a p-completely nilpotent PD thickening, we have $\mathcal{O}_{\mathrm{crys}}|_{(U,T)} = \mathcal{O}_T$ and $\mathcal{I}_{\mathrm{crys}}|_{(U,T)} = \ker(\mathcal{O}_T \twoheadrightarrow \mathcal{O}_U)$, which is a PD ideal sheaf inside $\mathcal{O}_{\mathrm{crys}}$. For any integer $r \geq 0$, we get a natural filtration on $\mathrm{R}\Gamma_{\mathrm{crys}}(R/A)$ given by $\mathrm{R}\Gamma_{\mathrm{crys}}(R/A, \mathcal{I}^{[r]}_{\mathrm{crys}})$. Results of Bhatt [3, Section 3.3] and Illusie [19, Section VIII.2] help us to understand this natural filtration in terms of the p-adic derived de Rham complex and its Hodge filtrations.

Theorem 4.1 (see [3, Proposition 3.25 and Theorem 3.27] and [19, Corollaire VIII.2.2.8]; see also [18, Theorem 3.4 (4)]). *Let R be a p-completely locally complete intersection A/I-algebra. Then there is a natural identification of filtered $\mathbb{E}_\infty$-$\mathcal{A}$-algebras*

$$(\mathrm{dR}^\wedge_{R/A}, \mathrm{Fil}^r_{\mathrm{H}}) \xrightarrow{\cong} (\mathrm{R}\Gamma_{\mathrm{crys}}(R/A), \mathrm{R}\Gamma_{\mathrm{crys}}(R/A, \mathcal{I}^{[r]}_{\mathrm{crys}})).$$

Here $\mathrm{Fil}^\bullet_{\mathrm{H}}$ denotes the (derived p-completed) Hodge filtration on $\mathrm{dR}^\wedge_{R/A}$, whose graded pieces are given by

$$\mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/A}) \cong \Gamma^*_R(\mathbb{L}^\wedge_{R/A}[-1]),$$

where Γ^* denotes the derived divided power algebra construction and $\mathbb{L}^\wedge_{R/A}$ denotes the derived p-completed cotangent complex of R over A. The triangle $A \to A/I \to R$ now gives us a triangle relating various p-completed cotangent complexes:

$$R \mathbin{\widehat{\otimes}}_{A/I} I/I^2[1] \cong R \mathbin{\widehat{\otimes}}_{A/I} \mathbb{L}^\wedge_{(A/I)/A} \to \mathbb{L}^\wedge_{R/A} \to \mathbb{L}^\wedge_{R/(A/I)},$$

where the (shifted) map $R \mathbin{\widehat{\otimes}}_{A/I} I/I^2 \to \mathbb{L}^\wedge_{R/A}[-1]$ comes from the $\mathcal{A}$-algebra structure on $\mathrm{dR}^\wedge_{R/A}$. Indeed, the multiplicativity of the Hodge filtrations and the fact that $I/I^2 \cong \mathcal{I}/\mathcal{I}^{[2]} \cong \mathrm{gr}^1_{\mathrm{H}}(\mathrm{dR}^\wedge_{(A/I)/A})$ naturally sits inside $\mathrm{gr}^1_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/A})$ give rise to

$$\mathrm{gr}^0_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/A}) \mathbin{\widehat{\otimes}}_{\mathrm{gr}^0_{\mathrm{H}}(\mathrm{dR}^\wedge_{(A/I)/A})} \mathrm{gr}^1_{\mathrm{H}}(\mathrm{dR}^\wedge_{(A/I)/A}) \to \mathrm{gr}^1_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/A}),$$

which is identified with the shifted map $R \mathbin{\widehat{\otimes}}_{A/I} I/I^2 \to \mathbb{L}^\wedge_{R/A}[-1]$.

The above discussion naturally extends to all A/I-algebras via left Kan extension. We restrict ourselves to those algebras that are quasisyntomic over A/I so that everything in sight is a sheaf with respect to the quasisyntomic topology. Recall that a basis of the quasisyntomic site is given by algebras that are large quasisyntomic over A/I (see [9, Definition 15.1]). Below we shall show that, on this basis, all these sheaves have values living in cohomological degree 0. The proof is inspired by [9, Section 12.5].

Lemma 4.2. *Let B be an $\mathbb{F}_p$-algebra and let S be a B-algebra which is relatively semiperfect with $\mathbb{L}_{S/B}[-1]$ given by a flat S-module. Then $\mathrm{dR}_{S/B}$ and its Hodge filtrations all live in cohomological degree* 0.

Proof. Using the conjugate filtration and the Cartier isomorphism, we see that $\mathrm{dR}_{S/B}$ (being its 0-th Hodge filtration) lives in degree 0. On the other hand, we also know that the graded pieces of the Hodge filtrations are given by divided powers $\Gamma^*_S(\mathbb{L}_{S/B}[-1])$, hence all the graded pieces live in degree 0 as well. In order to prove the statement about the Hodge filtrations, we need to show that the natural map $\mathrm{dR}_{S/B} \to \mathrm{dR}_{S/B}/\mathrm{Fil}^r_{\mathrm{H}}$ is surjective (note that both sides live in degree 0 by the last sentence).

To this end, we proceed by mimicking [9, proof of Theorem 12.2]. First we may replace B by the relative perfection of S, as the relevant cotangent complexes $\mathbb{L}_{S/B}$ and $\mathbb{L}_{S^{(1)}/B}$ are unchanged. Hence we may assume $B \to S$ is a surjection, as S/B is assumed to be relatively semiperfect. Next, by choosing the surjection $\mathbb{F}_p[X_b \mid b \in B] \twoheadrightarrow B$ and base change along the fully faithful map $\mathbb{F}_p[X_b \mid b \in B] \to \mathbb{F}_p[X_b^{1/p^\infty} \mid b \in B]$, we may further assume that B is semiperfect (as surjectiveness of a map can be tested after a fully faithful base change). In particular, any element in the kernel of $B \to S$ admits compatible p-power roots in B.

Now if the kernel is generated by a regular sequence, then the map $\mathrm{dR}_{S/B} \to \mathrm{dR}_{S/B}/\mathrm{Fil}^r_{\mathrm{H}}$ is identified as $D_B(S) \to D_B(S)/J^{[r]}$ where $D_B(S)$ denotes the PD envelope and $J^{[r]}$ is the r-th divided power ideal of $J = \ker(D_B(S) \twoheadrightarrow S)$. Therefore $\mathrm{dR}_{S/B} \to \mathrm{dR}_{S/B}/\mathrm{Fil}^r_{\mathrm{H}}$ is surjective by this concrete description.

Lastly, given any such surjection $B \twoheadrightarrow S$, denote by I the underlying set of its kernel. Then we look at the surjection of B-algebras

$$\widetilde{S} := B[X_i^{1/p^\infty} \mid i \in I]/(X_i \mid i \in I) \twoheadrightarrow S,$$

where X_i^{1/p^∞} is sent to (the image of) a compatible p-power root of the corresponding element $f_i \in I$ in S. The induced map $\mathbb{L}_{\widetilde{S}/B}[-1] \to \mathbb{L}_{S/B}[-1]$ sends X_i to f_i, hence is a surjection. Therefore the map $\mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}_{\widetilde{S}/B}) \to \mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}_{S/B})$ is also a surjection. Since $\widetilde{S}$ is a quotient of a relatively perfect algebra over B by an ind-regular sequence, applying (a filtered colimit of) what we proved in the previous paragraph we find that $\mathrm{dR}_{\widetilde{S}/B} \to \mathrm{dR}_{\widetilde{S}/B}/\mathrm{Fil}^r_{\mathrm{H}}$ is also a surjection. Looking at the commutative diagram

$$\begin{tikzcd}
\mathrm{dR}_{\widetilde{S}/B} \arrow[r] \arrow[d, two heads] & \mathrm{dR}_{S/B} \arrow[d] \\
\mathrm{dR}_{\widetilde{S}/B}/\mathrm{Fil}^r_{\mathrm{H}} \arrow[r, two heads] & \mathrm{dR}_{S/B}/\mathrm{Fil}^r_{\mathrm{H}}
\end{tikzcd}$$

we conclude that the right arrow must be surjective, which is what we need to show. ∎

Lemma 4.3. *Let S be a large quasisyntomic A/I-algebra. Then all of the Hodge filtrations on $\mathrm{dR}^{\wedge}_{S/A}$ and $\mathrm{dR}^{\wedge}_{S/(A/I)}$ are given by submodules, equivalently all the filtrations and their graded pieces are cohomologically supported in degree* 0*. Moreover, the Hodge filtrations of $\mathrm{dR}^{\wedge}_{S/(A/I)}$ are p-completely flat over A/I.*

Proof. Derived modulo p, we see that the first claim follows from Lemma 4.2. Also we see that $\mathrm{dR}^{\wedge}_{S/(A/I)} \to \mathrm{dR}^{\wedge}_{S/(A/I)}/\mathrm{Fil}^r_{\mathrm{H}}$ is surjective. So the statement about p-complete flatness of $\mathrm{dR}^{\wedge}_{S/(A/I)}$ and its Hodge filtrations now follows from p-completeness of $\mathrm{dR}^{\wedge}_{S/(A/I)}$ and of the graded pieces of its Hodge filtrations. Using the conjugate filtration and the Cartier isomorphism, the last two instances of p-complete flatness follow from the fact that $\mathbb{L}^{\wedge}_{S/(A/I)}[-1]$ is p-completely flat over S, and S is p-completely flat over A/I (as S is large quasisyntomic over A/I). ∎

Since $\mathrm{dR}^{\wedge}_{R/A}$ is naturally an $\mathcal{A}$-algebra for any A/I-algebra R, the filtration on $\mathcal{A}$ by the divided powers of $\mathcal{J}$ gives rise to another functorial decreasing filtration on $\mathrm{dR}^{\wedge}_{R/A}$:

$$\mathrm{Fil}^r_{\mathcal{J}}(\mathrm{dR}^{\wedge}_{R/A}) := \mathrm{dR}^{\wedge}_{R/A} \widehat{\otimes}_{\mathcal{A}} \mathcal{J}^{[r]}.$$

We caution the readers that this is *not* the $\mathcal{I}$-adic filtration, as we are using divided powers of $\mathcal{I}$ instead of symmetric powers. A basic understanding of these filtrations is provided by the following:

Lemma 4.4. *All of these* $\mathrm{Fil}^r_{\mathcal{I}}(\mathrm{dR}^\wedge_{R/A})$ *are quasisyntomic sheaves, whose values on large quasisyntomic* A/I*-algebras are supported in degree* 0. *The graded pieces are given by*

$$\mathrm{gr}^r_{\mathcal{I}} \cong \mathrm{dR}^\wedge_{R/(A/I)} \widehat{\otimes}_{A/I} \mathcal{I}^{[r]}/\mathcal{I}^{[r+1]}.$$

Proof. The statement about graded pieces follows from the following chain of identifications:

$$\begin{aligned}\mathrm{gr}^r_{\mathcal{I}} &\cong \mathrm{dR}^\wedge_{R/A} \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]}/\mathcal{I}^{[r+1]} \cong \mathrm{dR}^\wedge_{R/A} \widehat{\otimes}_{\mathcal{A}} \mathcal{A}/\mathcal{I} \widehat{\otimes}_{\mathcal{A}/\mathcal{I}} \mathcal{I}^{[r]}/\mathcal{I}^{[r+1]} \\ &\cong \mathrm{dR}^\wedge_{R/(A/I)} \widehat{\otimes}_{A/I} \mathcal{I}^{[r]}/\mathcal{I}^{[r+1]},\end{aligned}$$

where the last identification comes from $\mathrm{dR}^\wedge_{R/A} \widehat{\otimes}_{\mathcal{A}} A/I \cong \mathrm{dR}^\wedge_{R/(A/I)}$ (cf. [18, Proposition 3.11]) and $\mathcal{A}/\mathcal{I} \cong A/I$. In particular, these graded pieces are given by $\mathrm{dR}^\wedge_{R/(A/I)}$ twisted by a rank 1 locally free sheaf on $\mathrm{Spf}(A/I)$, hence are quasisyntomic sheaves themselves.

Since $\mathrm{dR}^\wedge_{R/A}$ and all these graded pieces are quasisyntomic sheaves, so is each $\mathrm{Fil}^r_{\mathcal{I}}$.

If S is large quasisyntomic over A/I, then $\mathrm{dR}^\wedge_{S/A}$ and all these graded pieces are supported in cohomological degree 0 by Lemma 4.3. By induction, in order to show the filtrations are in degree 0, it suffices to show $\mathrm{dR}^\wedge_{S/A} \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]} \to \mathrm{dR}^\wedge_{S/A} \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]}/\mathcal{I}^{[r+1]}$ is surjective for any r, which follows from the right exactness of p-complete tensoring. ∎

The filtration $\mathrm{Fil}^\bullet_{\mathcal{I}}(\mathrm{dR}^\wedge_{R/A})$ is a disguise of the Katz–Oda filtration $\mathrm{Fil}^\bullet_{\mathrm{KO}}(\mathrm{dR}_{C/A})$ discussed in [18], applied to the triple $(A \to B \to C) = (A \to A/I \to R)$. More precisely,

$$\mathrm{Fil}^i_{\mathcal{I}} \mathrm{dR}^\wedge_{R/A} \cong \mathrm{Fil}^\bullet_{\mathrm{KO}}(\mathrm{dR}_{R/A})^\wedge.$$

We refer the readers to Section 3.2 of *loc. cit.* for a general discussion of additional structures on the derived de Rham complex of $A \to C$ when it factorizes through $A \to B \to C$.

Let R be an A/I-algebra. By p-completing the double filtrations obtained in [18, Construction 3.12], we see that $\mathrm{dR}^\wedge_{R/A}$ can be naturally equipped with a decreasing filtration indexed by $\mathbb{N} \times \mathbb{N}$:

$$\mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{R/A}) := (\mathrm{Fil}^i_{\mathrm{KO}} \mathrm{Fil}^j_{\mathrm{H}}(\mathrm{dR}_{R/A}))^\wedge$$

The following proposition will describe $\mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{R/A})$ and declare its relation to the two systems of filtrations $\mathrm{Fil}^\bullet_{\mathrm{H}} \mathrm{dR}^\wedge_{R/A}$ and $\mathrm{Fil}^\bullet_{\mathcal{I}} \mathrm{dR}^\wedge_{R/A}$.

Proposition 4.5. *Let R be an A/I-algebra.*

(1) *For any j, we have an identification* $\mathrm{Fil}^{0,j}(\mathrm{dR}^\wedge_{R/A}) \cong \mathrm{Fil}^j_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/A})$.

(2) *For each pair* $0 \le j \le i$, *we have an identification*

$$\mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{R/A}) \cong \mathrm{Fil}^i_{\mathcal{I}}(\mathrm{dR}^\wedge_{R/A}).$$

(3) *For each pair* $0 \le i \le j$*, we have a natural identification*

$$\mathrm{Cone}\big(\mathrm{Fil}^{i+1,j}(\mathrm{dR}^\wedge_{R/A}) \to \mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{R/A})\big) \cong \mathrm{Fil}^{j-i}_{\mathrm{H}}\, \mathrm{dR}^\wedge_{R/(A/I)} \,\widehat{\otimes}_{A/I}\, \Gamma^i_{A/I}(I/I^2).$$

Moreover, this identification is compatible with

$$\begin{array}{ccc}
\mathrm{Cone}\big(\mathrm{Fil}^{i+1,j}(\mathrm{dR}^\wedge_{R/A}) \to \mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{R/A})\big) & \xrightarrow{\cong} & \mathrm{Fil}^{j-i}_{\mathrm{H}}\, \mathrm{dR}^\wedge_{R/(A/I)} \,\widehat{\otimes}_{A/I}\, \Gamma^i_{A/I}(I/I^2) \\
\downarrow & & \downarrow \\
\mathrm{Cone}\big(\mathrm{Fil}^{i+1,0}(\mathrm{dR}^\wedge_{R/A}) \to \mathrm{Fil}^{i,0}(\mathrm{dR}^\wedge_{R/A})\big) & \xrightarrow{\cong} & \mathrm{dR}^\wedge_{R/(A/I)} \,\widehat{\otimes}_{A/I}\, \Gamma^i_{A/I}(I/I^2).
\end{array}$$

(4) *The assignment* $R \mapsto \mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{R/A})$ *defines a sheaf on the quasisyntomic site of* A/I *for any* (i, j).

Proof. (1) follows from [18, Construction 3.12]. Indeed, $\mathrm{Fil}^{0,j}$ is the p-completed j-th filtration on $\mathrm{dR}_{R/A} \otimes_{\mathrm{dR}_A(I)} \mathrm{Fil}^0_{\mathrm{H}}(\mathrm{dR}_A(I)) \cong \mathrm{dR}_{R/A}$. Since this is a filtered isomorphism, we see that this is nothing other than the p-completed j-th Hodge filtration on $\mathrm{dR}_{R/A}$, hence it is $\mathrm{Fil}^j_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/A})$.

(2) follows from [18, Construction 3.9]. Indeed, the inequality $j \le i$ implies that Fil^j of each term appearing in [18, Construction 3.9] is the whole term. Hence the colimit just gives $\mathrm{dR}_{R/A} \otimes_{\mathrm{dR}_A(I)} \mathrm{Fil}^i_{\mathrm{H}}(\mathrm{dR}_A(I))$ back. After p-completing, we see that by definition we have $\mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{R/A}) \cong \mathrm{Fil}^i_{\mathcal{I}}(\mathrm{dR}^\wedge_{R/A})$.

(3) follows by p-completing [18, Proposition 3.13 (1)].

For (4), we first claim that the assignments $R \mapsto \mathrm{Fil}^m_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/A})$ and $R \mapsto \mathrm{Fil}^n_{\mathrm{H}}(\mathrm{dR}^\wedge_{R/(A/I)})$ define sheaves for all m and n. For $m = 0$ this is Proposition 3.13, and for $n = 0$ this is [8, Example 5.12]. Induction on m and n reduces the task to showing the sheaf property of graded pieces, which are given by $\bigwedge^i_R \mathbb{L}^\wedge_{R/A}[-i]$ and $\bigwedge^i_R \mathbb{L}^\wedge_{R/(A/I)}[-i]$. p-Completing [8, Theorem 3.1] gives the desired sheaf property of these graded pieces.

Fix a natural number j; then by (1) we see that $\mathrm{Fil}^{0,j}$ is a quasisyntomic sheaf. Each graded piece with respect to i, by (2) and (3), is also a sheaf. Therefore by induction on i, each $\mathrm{Fil}^{i,j}$ defines a sheaf. ■

To understand these sheaves more concretely, we look at their value on the basis of large quasisyntomic A/I-algebras.

Proposition 4.6. *Let S be a large quasisyntomic A/I-algebra.*

(1) *For any pair* $(i, j) \in \mathbb{N} \times \mathbb{N}$, *the* $\mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{S/A})$ *is concentrated in degree* 0, *and the natural map* $\mathrm{Fil}^{i,j}(\mathrm{dR}^\wedge_{S/A}) \to \mathrm{dR}^\wedge_{S/A}$ *is injective.*

(2) *For any* j, *the natural map*

$$\mathrm{Fil}^j_{\mathrm{H}}(\mathrm{dR}^\wedge_{S/A}) \to \mathrm{Fil}^j_{\mathrm{H}}(\mathrm{dR}^\wedge_{S/(A/I)})$$

is surjective.

(3) *For each pair* $0 \le i \le j$, *we have*

$$\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A}) = \sum_{r=i}^{j} (\mathrm{Fil}^{j-r}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A} \cdot \mathcal{I}^{[r]}),$$

where $\mathrm{Fil}^{j-r}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A} \cdot \mathcal{I}^{[r]}$ *denotes the image of* $\mathrm{Fil}^{j-r}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A} \widehat{\otimes}_{\mathcal{A}} \mathcal{I}^{[r]} \to \mathrm{dR}^{\wedge}_{S/A}$, *and the sum is taken in the algebra* $\mathrm{dR}^{\wedge}_{S/A}$.

(4) *We have another description:*

$$\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A}) = (\mathrm{Fil}^{j}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A}) \cap (\mathrm{Fil}^{i}_{\mathcal{I}} \mathrm{dR}^{\wedge}_{S/A}),$$

where the intersection is taken in the algebra $\mathrm{dR}^{\wedge}_{S/A}$.

Proof. (1) We argue by decreasing induction on i. When $j \le i$, by Proposition 4.5 (2) we see that $\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A}) \cong \mathrm{Fil}^{i}_{\mathcal{I}}(\mathrm{dR}^{\wedge}_{S/A})$, which is concentrated in degree 0 by Lemma 4.4. By Proposition 4.5 (3), the graded pieces with respect to i are all concentrated in degree 0 by Lemma 4.3. This in turn implies that

- all of $\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A})$ are in degree 0 for any (i, j);
- we have short exact sequences

$$0 \to \mathrm{Fil}^{i+1,j}(\mathrm{dR}^{\wedge}_{S/A}) \to \mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A}) \to \mathrm{Fil}^{j-i}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{R/(A/I)} \widehat{\otimes}_{A/I} \Gamma^{i}_{A/I}(I/I^2) \to 0.$$

In particular, $\mathrm{Fil}^{i+1,j}(\mathrm{dR}^{\wedge}_{S/A}) \to \mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A})$ is injective. Using Proposition 4.5 (1) and Lemma 4.3, we see that the map $\mathrm{Fil}^{0,j}(\mathrm{dR}^{\wedge}_{S/A}) \cong \mathrm{Fil}^{j}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{S/A}) \to \mathrm{dR}^{\wedge}_{S/A}$ is also injective. Therefore the composition $\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A}) \to \mathrm{dR}^{\wedge}_{S/A}$ is injective as well for any (i, j).

(2) follows from the short exact sequence obtained in the previous paragraph, specializing to $i = 0$.

(3) follows from the combination of (2), Proposition 4.5 (3), and the fact that p-completed tensoring is right exact.

For (4): first notice that this is true for $i = 0$, due to Proposition 4.5 (1). Next let us look at the commutative diagram in Proposition 4.5 (3). Since the right vertical map is an injection, we see that the map

$$\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A})/\mathrm{Fil}^{i+1,j}(\mathrm{dR}^{\wedge}_{S/A}) \to \mathrm{Fil}^{i,0}(\mathrm{dR}^{\wedge}_{S/A})/\mathrm{Fil}^{i+1,0}(\mathrm{dR}^{\wedge}_{S/A})$$

is injective. Therefore, by Proposition 4.5 (2), we know that

$$\mathrm{Fil}^{i+1,j}(\mathrm{dR}^{\wedge}_{S/A}) = (\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A})) \cap (\mathrm{Fil}^{i+1}_{\mathcal{I}} \mathrm{dR}^{\wedge}_{S/A}).$$

By increasing induction on i, we may assume

$$\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{S/A}) = (\mathrm{Fil}^{j}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A}) \cap (\mathrm{Fil}^{i}_{\mathcal{I}} \mathrm{dR}^{\wedge}_{S/A}).$$

Hence

$$\begin{aligned} \mathrm{Fil}^{i+1,j}(\mathrm{dR}^{\wedge}_{S/A}) &= (\mathrm{Fil}^{j}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A}) \cap (\mathrm{Fil}^{i}_{\mathcal{I}} \mathrm{dR}^{\wedge}_{S/A}) \cap (\mathrm{Fil}^{i+1}_{\mathcal{I}}(\mathrm{dR}^{\wedge}_{S/A})) \\ &= (\mathrm{Fil}^{j}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A}) \cap (\mathrm{Fil}^{i+1}_{\mathcal{I}} \mathrm{dR}^{\wedge}_{S/A}). \end{aligned}$$

■

Let us draw a table to summarize these filtrations on $\mathrm{dR}^{\wedge}_{R/A}$:

	R	$\mathbb{L}^{\wedge}_{R/(A/I)}[-1]$	$(\bigwedge^2_R \mathbb{L}_{R/(A/I)})^{\wedge}[-2]$	$\cdots$
A/I	$M_0 \widehat{\otimes}_{A/I} N_0$	$M_0 \widehat{\otimes}_{A/I} N_1$	$M_0 \widehat{\otimes}_{A/I} N_2$	$\cdots$
$\mathcal{J}/\mathcal{J}^{[2]}$	$M_1 \otimes_{A/I} N_0$	$M_1 \widehat{\otimes}_{A/I} N_1$	$M_1 \widehat{\otimes}_{A/I} N_2$	$\cdots$
$\mathcal{J}^{[2]}/\mathcal{J}^{[3]}$	$M_2 \widehat{\otimes}_{A/I} N_0$	$M_2 \widehat{\otimes}_{A/I} N_1$	$M_2 \widehat{\otimes}_{A/I} N_2$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

In the diagram above, $M_i = \mathcal{J}^{[i]}/\mathcal{J}^{[i+1]}$ and $N_j = (\bigwedge^j_R \mathbb{L}_{R/(A/I)})^{\wedge}[-j]$ for $i, j \in \mathbb{N}$. Here rows indicate graded pieces of the filtration $\mathrm{Fil}^r_{\mathcal{J}}$, and each term in the i-th row indicates the graded piece of the induced filtration on $\mathrm{dR}^{\wedge}_{R/(A/I)} \widehat{\otimes}_{A/I} \Gamma^i_{A/I}(I/I^2)$. The skewed dotted lines indicate the Hodge filtration on $\mathrm{dR}^{\wedge}_{R/A}$ (given by things below the dotted line). See also [18, p. 10].

As a consequence we get a structural result on the graded algebra associated with the Hodge filtration on $\mathrm{dR}^{\wedge}_{R/A}$.

Lemma 4.7. *There is a functorial increasing exhaustive filtration* Fil^v_i *on the graded algebra* $\mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/A})$ *by graded* $(\mathrm{gr}^*_{\mathcal{J}} \mathcal{A} \cong \Gamma^*_{A/I}(I/I^2))$*-submodules with graded pieces given by*

$$\mathrm{gr}^v_i(\mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/A})) \cong (\bigwedge^i_R \mathbb{L}_{R/(A/I)})^{\wedge}[-i] \widehat{\otimes}_{A/I} \Gamma^*_{A/I}(I/I^2).$$

Here $(\bigwedge^i_R \mathbb{L}_{R/(A/I)})^{\wedge}[-i]$ *has degree* i *and the above is a graded isomorphism.*

We refer to this filtration Fil^v_i on $\mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/A})$ as the *vertical filtration* from now on (cf. [18, Construction 3.14]). This choice of name is because the Fil^v_i is literally the filtration given by the columns in the table before this lemma.

Proof of Lemma 4.7. Using the above table one can see this directly. Equivalently, we may use

$$\mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/A}) \cong (\Gamma^*_R(\mathbb{L}^{\wedge}_{R/A}[-1]))^{\wedge}$$

and the triangle

$$R\,\widehat{\otimes}_{A/I}\, I/I^2 \to \mathbb{L}^{\wedge}_{R/A}[-1] \to \mathbb{L}^{\wedge}_{R/(A/I)}[-1]. \qquad \blacksquare$$

Remark 4.8. Let (A, I) be a general bounded prism, and let S be a large quasisyntomic A/I-algebra. Combining Theorem 3.5, Construction 3.12 (4), and [9, Theorem 15.2 (1)], we can see that $\mathrm{dR}^{\wedge}_{S/A}$ is p-completely flat over $\mathrm{dR}_A(I)^{\wedge}$.

The remark below has been suggested to us by Bhatt. Using the conjugate filtration and the same argument of Lemma 4.7, we can give an alternative proof of this fact. Indeed, we can check this mod p, hence we shall assume A is p-torsion. Next we want to appeal to the conjugate filtrations on both algebras. We have the following pushout diagram:

$$\begin{tikzcd} A \arrow[r] & A/I^p \arrow[r] & R^{(1)} \\ A \arrow[u, "\varphi_A"] \arrow[r] & A/I \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

There is a similar functorial increasing exhaustive filtration on the graded algebra of the conjugate filtered $\mathrm{dR}_{S/A}$, with graded pieces given by

$$\Big(\bigwedge\nolimits^i_{R^{(1)}} \mathbb{L}_{R^{(1)}/(A/I^p)}\Big)[-i] \otimes_{A/I^p} \Gamma^*_{A/I^p}(I^p/I^{2p}).$$

It is flat over $\Gamma^*_{A/I^p}(I^p/I^{2p})$, which is the conjugate graded algebra of $\mathrm{dR}_A(I)$. Lastly, we conclude by recalling that an increasingly exhaustive filtered module of an increasingly exhaustive filtered algebra is flat if the graded counterpart is flat.

4.2. *Nygaard filtration*

Recall that in [9, Section 15], there is a natural decreasing filtration of quasisyntomic subsheaves on $\mathbb{\Delta}^{(1)}_{-/A}$, called the Nygaard filtration, with the following properties:

Theorem 4.9 (see [9, Theorems 15.2 and 15.3] and proofs therein). *Let S be a large quasisyntomic A/I-algebra.*

(1) *The Nygaard filtrations $\mathrm{Fil}^{\bullet}_{\mathrm{N}}$ on $\mathbb{\Delta}^{(1)}_{S/A}$ are given by p-completely flat A-submodules in $\mathbb{\Delta}^{(1)}_{S/A}$.*

(2) *We have an identification of algebras $\mathbb{\Delta}^{(1)}_{S/A}/I \cong \mathrm{dR}^{\wedge}_{S/(A/I)}$, under which the image of the Nygaard filtration becomes the Hodge filtration.*

(3) *For each $i \geq 0$, we have a short exact sequence*

$$0 \to \mathrm{Fil}^i_{\mathrm{N}}\, \mathbb{\Delta}^{(1)}_{S/A} \otimes_A I \to \mathrm{Fil}^{i+1}_{\mathrm{N}}\, \mathbb{\Delta}^{(1)}_{S/A} \to \mathrm{Fil}^{i+1}_{\mathrm{H}}\, \mathrm{dR}^{\wedge}_{S/(A/I)} \to 0.$$

Let R be a general quasisyntomic A/I-algebra. On $\mathbb{\Delta}^{(1)}_{R/A}$ there is also an I-adic filtration $\mathrm{Fil}^r_I\, \mathbb{\Delta}^{(1)}_{R/A} := \mathbb{\Delta}^{(1)}_{R/A} \otimes_A I^r$; by Theorem 4.9 (2), we identify the graded pieces as

$$\mathrm{gr}^r_I \cong \mathbb{\Delta}^{(1)}_{R/A}/I \otimes_{A/I} I^r/I^{r+1} \cong \mathrm{dR}^{\wedge}_{R/(A/I)} \otimes_{A/I} \mathrm{Sym}^r_{A/I}(I/I^2).$$

The I-adic filtration and the Nygaard filtration are related as follows. For any $(i, j) \in \mathbb{N} \times \mathbb{N}$, we define

$$\mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)}_{R/A} := \mathrm{Fil}^{j-i}_{\mathrm{N}} \mathbb{\Delta}^{(1)}_{R/A} \otimes_A I^i,$$

where we adopt the convention that $\mathrm{Fil}^{l}_{\mathrm{N}} \mathbb{\Delta}^{(1)}_{R/A} = \mathbb{\Delta}^{(1)}_{R/A}$ if $l \le 0$. One checks easily that this puts a decreasing filtration on $\mathbb{\Delta}^{(1)}_{R/A}$ indexed by $\mathbb{N} \times \mathbb{N}$. This filtration has very similar behavior to the $\mathrm{Fil}^{i,j}(\mathrm{dR}^{\wedge}_{R/A})$ studied in the previous subsection. The following is an analogue of Proposition 4.5.

Proposition 4.10. *Let R be an A/I-algebra.*

(1) *For any j, we have $\mathrm{Fil}^{0,j} \mathbb{\Delta}^{(1)}_{R/A} \cong \mathrm{Fil}^{j}_{\mathrm{N}} \mathbb{\Delta}^{(1)}_{R/A}$.*

(2) *For each pair $0 \le j \le i$, we have*

$$\mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)}_{R/A} \cong \mathrm{Fil}^{i}_{I} \mathbb{\Delta}^{(1)}_{R/A}.$$

(3) *For each pair $0 \le i \le j$, we have a natural identification*

$$\mathrm{Cone}(\mathrm{Fil}^{i+1,j} \mathbb{\Delta}^{(1)}_{R/A} \to \mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)}_{R/A}) \cong \mathrm{Fil}^{j-i}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/(A/I)}) \otimes_{A/I} \mathrm{Sym}^{i}_{A/I}(I/I^2).$$

Moreover, these identifications fit in the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Cone}(\mathrm{Fil}^{i+1,j} \mathbb{\Delta}^{(1)}_{R/A} \to \mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)}_{R/A}) & \xrightarrow{\cong} & \mathrm{Fil}^{j-i}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/(A/I)}) \otimes_{A/I} \mathrm{Sym}^{i}_{A/I}(I/I^2) \\
\downarrow & & \downarrow \\
\mathrm{Cone}(\mathrm{Fil}^{i+1,0} \mathbb{\Delta}^{(1)}_{R/A} \to \mathrm{Fil}^{i,0} \mathbb{\Delta}^{(1)}_{R/A}) & \xrightarrow{\cong} & \mathrm{dR}^{\wedge}_{R/(A/I)} \otimes_{A/I} \mathrm{Sym}^{i}_{A/I}(I/I^2)
\end{array}$$

(4) *The assignment $R \mapsto \mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)}_{R/A}$ defines a sheaf on $\mathrm{qSyn}_{A/I}$ for any (i, j).*

Proof. (1) and (2) follow from the definition; (3) follows from Theorem 4.9 (3); and (4) follows from (3). ∎

Proposition 4.11. *Let S be a large quasisyntomic A/I-algebra.*

(1) *We have*

$$\mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)}_{S/A} = \sum_{r=i}^{j} (\mathrm{Fil}^{j-r}_{\mathrm{N}}(\mathbb{\Delta}^{(1)}_{S/A}) \cdot I^r),$$

where the sum is taken in the algebra $\mathbb{\Delta}^{(1)}_{S/A}$.

(2) *We have*

$$\mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)}_{S/A} = (\mathrm{Fil}^{j}_{\mathrm{N}} \mathbb{\Delta}^{(1)}_{S/A}) \cap (\mathrm{Fil}^{i}_{I} \mathbb{\Delta}^{(1)}_{S/A}),$$

where the intersection is taken in the algebra $\mathbb{\Delta}^{(1)}_{S/A}$.

Proof. The proof is similar to that of Proposition 4.6 (3, 4). Notice that $\mathrm{Fil}^{j}_{\mathrm{N}} \mathbb{\Delta}^{(1)}_{S/A} \to \mathrm{Fil}^{j}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/(A/I)})$ is surjective by Theorem 4.9 (2). ∎

We can collect all these structures on $\Delta^{(1)}_{R/A}$ in the following graph similar to one in the previous subsection. One observes that the distinction is just that divided powers of I/I^2 get replaced by symmetric powers of I/I^2.

$$
\begin{array}{c|cccc}
\vdots & R & \mathbb{L}^{\wedge}_{R/(A/I)}[-1] & (\bigwedge^2_R \mathbb{L}_{R/(A/I)})^{\wedge}[-2] & \cdots \\
\hline
A/I & M_0 \mathbin{\widehat{\otimes}}_{A/I} N_0 & M_0 \mathbin{\widehat{\otimes}}_{A/I} N_1 & M_0 \mathbin{\widehat{\otimes}}_{A/I} N_2 & \cdots \\
\hline
I/I^2 & M_1 \otimes_{A/I} N_0 & M_1 \mathbin{\widehat{\otimes}}_{A/I} N_1 & M_1 \mathbin{\widehat{\otimes}}_{A/I} N_2 & \cdots \\
\hline
I^2/I^3 & M_2 \mathbin{\widehat{\otimes}}_{A/I} N_0 & M_2 \mathbin{\widehat{\otimes}}_{A/I} N_1 & M_2 \mathbin{\widehat{\otimes}}_{A/I} N_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Here rows indicate graded pieces of the filtration Fil^r_I, and each term in each row indicates the graded piece of the Hodge filtration on $\mathrm{dR}^{\wedge}_{R/(A/I)}$. The skewed dotted lines indicate the Nygaard filtration on $\Delta^{(1)}_{R/A}$ (given by things below the dotted line).

Also as a consequence we get a structural result on the graded algebra associated with the Nygaard filtration on $\Delta^{(1)}_{R/A}$.

Lemma 4.12. *There is a functorial increasing exhaustive filtration* Fil^v_i *on the graded algebra* $\mathrm{gr}^*_{\mathrm{N}}(\Delta^{(1)}_{R/A})$ *by graded* $(\mathrm{gr}^*_I A \cong \mathrm{Sym}^*_{A/I}(I/I^2))$*-submodules with graded pieces given by*

$$\mathrm{gr}^v_i(\mathrm{gr}^*_{\mathrm{N}}(\Delta^{(1)}_{R/A})) \cong (\textstyle\bigwedge^i_R \mathbb{L}_{R/(A/I)})^{\wedge}[-i] \mathbin{\widehat{\otimes}}_{A/I} \mathrm{Sym}^*_{A/I}(I/I^2).$$

Here $(\bigwedge^i_R \mathbb{L}_{R/(A/I)})^{\wedge}[-i]$ *has degree* i *and the above is a graded isomorphism.*

We also call the filtration Fil^v_i on $\mathrm{gr}^*_{\mathrm{N}}(\Delta^{(1)}_{R/A})$ the *vertical filtration* from now on.

Proof of Lemma 4.12. This follows from Theorem 4.9 (3); see also the proof of Lemma 4.7. ■

4.3. Promoting to a filtered map

Recall that we use (A, I) to denote a transversal prism. For the rest of this section, we shall use (B, J) to denote a general bounded prism. By Theorem 3.5, we have a map

$\Delta^{(1)}_{R/B} \to \mathrm{dR}^{\wedge}_{R/B}$ functorial in $B/J \to R$. The goal of this subsection is to show that this map can be promoted to a filtered map where the source is equipped with the Nygaard filtration and the target is equipped with the Hodge filtration. Our plan is to:

- show a certain rigidity of the map being filtered;
- show the map is filtered when the base prism (A, I) is transversal;
- show the map is filtered when the algebra R is large quasisyntomic over B/J of a particular type;
- show the map is functorially filtered when R is a p-completely smooth B/J-algebra, and hence finish the argument by left Kan extension.

Fix a natural number i. The main diagram that we shall look at in this subsection is

$$\begin{tikzcd} \mathrm{Fil}^i_{\mathrm{N}} \arrow[r] \arrow[d, dotted, "g_i"'] & \Delta^{(1)} \arrow[r] \arrow[d] & Q_{1,i} \arrow[d, dotted, "f_i"] \\ \mathrm{Fil}^i_{\mathrm{H}} \arrow[r] & \mathrm{dR} \arrow[r] & Q_{2,i} \end{tikzcd} \tag{⚂}$$

viewed as a commutative diagram of sheaves on $\mathrm{qSyn}_{B/J}$. Here $Q_{1,i}$ and $Q_{2,i}$ are the cones of the natural maps, so both rows are distinguished triangles of quasisyntomic sheaves. All the solid arrows are defined: for instance, the middle vertical arrow is given by Theorem 3.5. Our main task is to show that one can fill in the dotted arrows f_i and g_i making the diagram commute.

We first need a few lemmas to illustrate that the situation is pretty rigid and there is at most one choice of these dotted arrows.

Lemma 4.13. *Let S be a large quasisyntomic B/J-algebra. Then the values of*

$$\mathrm{Fil}^i_{\mathrm{N}},\ \Delta^{(1)},\ Q_{1,i},\ \textit{and}\ Q_{2,i}$$

at S are concentrated in cohomological degree 0.

Proof. The first three follow from how they are defined: see [9, Section 15.1]. The claim for $Q_{2,i}$ follows from the fact that $\mathbb{L}^{\wedge}_{S/B}[-1]$ lives in cohomological degree 0. ∎

Lemma 4.14. *Let S be a large quasisyntomic B/J-algebra.*

(1) *There is at most one choice of f_i making the right square of (⚂) commute.*

(2) *If $S \to T$ is a morphism of large quasisyntomic B/J-algebras, and the f_i are defined on both of them, then the diagram*

$$\begin{tikzcd} Q_{1,i}(S) \arrow[r] \arrow[d, "f_i(S)"'] & Q_{1,i}(T) \arrow[d, "f_i(T)"] \\ Q_{2,i}(S) \arrow[r] & Q_{2,i}(T) \end{tikzcd}$$

is commutative.

(3) *The existence of* $f_i(S)$ *is equivalent to the existence of* $g_i(S)$ *making the left square of* (⚂) *commute.*

(4) $g_i(S)$, *if it exists, must be unique.*

Proof. (1) Suppose there are two of them, and take their difference. Since precomposing this difference with $\mathbb{\Delta}^{(1)} \to Q_{1,i}$ is the zero map $\mathbb{\Delta}^{(1)} \xrightarrow{0} Q_{2,i}$ due to commutativity, the difference must factor through $\mathrm{Fil}^i_{\mathrm{N}}[1]$. But $\mathrm{Hom}(\mathrm{Fil}^i_{\mathrm{N}}[1], Q_{2,i}) = \{0\}$ by cohomological considerations in Lemma 4.13. Hence the difference must be zero.

(2) The argument is similar to (1). The difference of the two arrows from $Q_{1,i}(S)$ to $Q_{2,i}(T)$ will again factor through $\mathrm{Fil}^i_{\mathrm{N}}(S)[1]$, hence must again be zero.

(3) Just apply TR3, noticing that the two rows of (⚂) are exact triangles.

(4) This is similar to (1). The difference of two possible $g_i(S)$'s factors through an arrow $\mathrm{Fil}^i_{\mathrm{N}}(S) \to Q_{2,i}[-1](S)$, which is again zero by cohomological considerations. ■

Knowing the rigidity of our situation, we start proving the existence of f_i following the plan outlined above.

Proposition 4.15. *Let* (A, I) *be a transversal prism, and let*

$$S = A/I\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\rangle/(f_1, \ldots, f_r),$$

where (f_i) *is a* p*-completely regular sequence. Then* $\mathrm{Fil}^i_{\mathrm{N}}\, \mathbb{\Delta}^{(1)}_{S/A} \subset \mathrm{Fil}^i_{\mathrm{H}}\, \mathrm{dR}^{\wedge}_{S/A}$.

Proof. When $i = 0$, there is nothing to prove, and when $i = 1$, the triangle in Corollary 3.18 gives us a commutative diagram

$$\begin{tikzcd} & S & \\ \mathbb{\Delta}^{(1)}_{S/A} \arrow[ur, two heads] \arrow[rr] & & \mathrm{dR}^{\wedge}_{S/A} \arrow[ul, two heads] \end{tikzcd}$$

Since the kernels of these two surjections define the first Nygaard and Hodge filtrations respectively, we see the containment for $i = 1$. For general i, we argue by induction. Let us look at the induced map $g : \mathrm{Fil}^i_{\mathrm{N}}\, \mathbb{\Delta}^{(1)}_{S/A} \to \mathrm{dR}^{\wedge}_{S/A}/\mathrm{Fil}^i_{\mathrm{H}}$. We first notice that by induction and since $I \subset \mathcal{I}$, the submodule $I \cdot \mathrm{Fil}^{i-1}_{\mathrm{N}}\, \mathbb{\Delta}^{(1)}_{S/A}$ is sent to zero under g. By multiplicativity and the containment for $i = 1$, we see that $\mathrm{Sym}^i(\mathrm{Fil}^1_{\mathrm{N}}\, \mathbb{\Delta}^{(1)}_{S/A})$ is also sent to zero under g. Now we use Theorem 4.9 (3) to see that $\mathrm{Fil}^i_{\mathrm{N}}/I \cdot \mathrm{Fil}^{i-1}_{\mathrm{N}}$ is identified with $\mathrm{Fil}^i_{\mathrm{H}}\, \mathrm{dR}^{\wedge}_{S/(A/I)}$ and the image of $\mathrm{Sym}^i(\mathrm{Fil}^1_{\mathrm{N}}\, \mathbb{\Delta}^{(1)}_{S/A})$ becomes $\mathrm{Sym}^i(\mathrm{Fil}^1_{\mathrm{H}}\, \mathrm{dR}^{\wedge}_{S/(A/I)})$, so we get an induced map

$$\overline{g} : \mathrm{Fil}^i_{\mathrm{H}}\, \mathrm{dR}^{\wedge}_{S/(A/I)}/\mathrm{Sym}^i(\mathrm{Fil}^1_{\mathrm{H}}\, \mathrm{dR}^{\wedge}_{S/(A/I)}) \to \mathrm{dR}^{\wedge}_{S/A}/\mathrm{Fil}^i_{\mathrm{H}}.$$

But the source of this map has its p-power torsion submodule being p-adically dense and the target of this map is p-torsionfree and p-adically complete, so the map $\overline{g}$ must in fact be zero. This proves the containment $\mathrm{Fil}^i_{\mathrm{N}} \subset \mathrm{Fil}^i_{\mathrm{H}}$ as claimed. ■

The following is inspired by [9, Section 12.4].

Proposition 4.16. *Let (A, I) be a transversal prism. Then for any p-completely smooth A/I-algebra R, the map $\mathbb{\Delta}^{(1)}_{R/A} \to \mathrm{dR}^{\wedge}_{R/A}$ can be promoted to a map of filtered algebras. Moreover, this lift is functorial in the A/I-algebra R, hence left Kan extends to all animated A/I-algebras.*

Proof. For any surjection $A/I\langle X_1, \ldots, X_n\rangle \to R$, the ring

$$\widetilde{R} = A/I\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\rangle \otimes_{A/I\langle X_1,\ldots,X_n\rangle} R$$

is large quasisyntomic over A/I and Zariski locally of the form considered in the previous proposition. Therefore the map $\mathbb{\Delta}^{(1)}_{\widetilde{R}/A} \to \mathrm{dR}^{\wedge}_{\widetilde{R}/A}$ is canonically filtered: (Zariski locally) existence follows from the previous proposition, while Zariski glue as well as uniqueness are provided by Lemma 4.14. The same applies to all terms of the Čech nerve $\widetilde{R}^\bullet$ of $R \to \widetilde{R}$.

The filtered cosimplicial rings $\mathbb{\Delta}^{(1)}_{\widetilde{R}^\bullet/A}$ and $\mathrm{dR}^{\wedge}_{\widetilde{R}^\bullet/A}$ compute the filtered rings $\mathbb{\Delta}^{(1)}_{\widetilde{R}/A}$ and $\mathrm{dR}^{\wedge}_{\widetilde{R}/A}$ respectively. By Lemma 4.14, we get a map of filtered cosimplicial rings.

This construction is independent of the choice of the surjection $A/I\langle X_1, \ldots, X_n\rangle \to R$: adding extra variables to the X_i, one gets a square of maps between filtered cosimplicial algebras, and we use Lemma 4.14 to see the maps commute for each term associated with $[m] \in \Delta$. Note that the category of such surjections admits pairwise coproducts, it is therefore sifted, in particular, this category has weakly contractible nerve. Hence we have obtained a weakly contractible space worth of ways to promote the map $\mathbb{\Delta}^{(1)}_{R/A} \to \mathrm{dR}^{\wedge}_{R/A}$ to a filtered map. The naturality in R follows from exactly the same argument. This way we get the desired functorial map. ∎

Now we turn to the general situation where the base prism (B, J) is not necessarily transversal. We bootstrap the previous two propositions.

Proposition 4.17. *Let $S = B/J\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\rangle/(f_1, \ldots, f_r)$ where (f_i) is a p-completely regular sequence. Then there exist $f_i(S)$ and $g_i(S)$ making the diagram (⚂) commute.*

Proof. We shall utilize the knowledge of when the base prism is transversal. Without loss of generality, let us assume $(B, J) = (B, (d))$ is oriented: Zariski locally it is oriented, and the locally defined $f_i(S)$ and $g_i(S)$ will necessarily glue due to Lemma 4.14.

Following a private communication with Illusie, let us define a transversal prism $(A, (a))$ together with a surjection of prisms $(A, (a)) \twoheadrightarrow (B, (d))$ as follows. Let

$$A := \mathbb{Z}_p\{x_b \mid b \in B\}\{\delta(x_d)^{-1}\}^{\wedge}_{(x_d, p)}$$

be given by first adjoining a free δ-variable corresponding to each element in B to $\mathbb{Z}_p$ together with an inverse of the δ of the variable corresponding to $d \in B$, completed at the end with respect to (p, x_d). Denote $a := x_d$. Then $(A, (a))$ is a transversal prism, and there is an evident surjection of prisms $(A, (a)) \twoheadrightarrow (B, (d))$.

Consider the surjection $A/a\langle X_1^{1/p^\infty},\dots,X_n^{1/p^\infty}\rangle \to B/J\langle X_1^{1/p^\infty},\dots,X_n^{1/p^\infty}\rangle$ and lift the elements f_i to $\widetilde{f_i}$. Let

$$\widetilde{S} := \mathrm{Kos}(A/a\langle X_1^{1/p^\infty},\dots,X_n^{1/p^\infty}\rangle; \widetilde{f_1},\dots,\widetilde{f_r}).$$

We have $S = \widetilde{S} \otimes^{\mathbb{L}}_{(A/a)} B/d$. We know the analogous map for $\widetilde{S}/(A/a)$ exists, thanks to Proposition 4.16. Since both the Nygaard and Hodge filtrations satisfy base change, we may base change the maps for $\widetilde{S}/(A/a)$ to obtain the desired maps for $S/(B/b)$. ■

Following the same reasoning as in the proof of Proposition 4.16, one obtains the following:

Proposition 4.18. *For any p-completely smooth B/J-algebra R, the map $\mathbb{\Delta}^{(1)}_{R/B} \to \mathrm{dR}^\wedge_{R/B}$ can be promoted to a map of filtered algebras. Moreover, this lift is functorial in the B/J-algebra R, hence left Kan extends to all animated B/J-algebras.*

Proof. The argument is exactly the same as for Proposition 4.16; note that Lemma 4.14 applies to general bounded base prisms (B, J). ■

Remark 4.19. Fix a bounded base prism (B, J). Any such functorial filtered map is determined by its effect on p-complete polynomial algebras, by left Kan extension. Then by quasisyntomic descent, such a functorial filtered map is determined by its effect on a basis of $\mathrm{qSyn}_{B/J}$, such as the full subcategory generated by large quasisyntomic B/J-algebras. Therefore Lemma 4.14 implies that there is at most one such functorial filtered map. Combining this with the above proposition, we have both its existence and uniqueness.

Remark 4.20. Following the way these filtered maps are constructed, we have certain compatibility with base change: Let R be an animated B/J-algebra, let $(B, J)\to(C, JC)$ be a map of bounded prisms and denote $R' := R \otimes^{\mathbb{L}}_{B/J} C/JC$. Then the filtered map for R'/C arises as the filtered map for R/B base changed along $B \to C$. Indeed, it suffices to prove this when $R/(B/J)$ is p-completely smooth. Then one simply notices that a surjection $B/J\langle X_1,\dots,X_n\rangle \to R$ base changed along $B \to C$ will give rise to a surjection $C/JC\langle X_1,\dots,X_n\rangle \to R'$.

4.4. Comparing Hodge and Nygaard filtrations

We again use (A, I) to denote a transversal prism, and use (B, J) to denote a general bounded prism. All sheaves referred to in this subsection are viewed as objects in $\mathrm{Shv}(\mathrm{qSyn}_{A/I})$ or $\mathrm{Shv}(\mathrm{qSyn}_{B/J})$ depending on the context. Combining Theorem 3.5 and Proposition 4.18, we get a natural map of sheaves of filtered rings

$$(\mathbb{\Delta}^{(1)}_{-/B}, \mathrm{Fil}^\bullet_{\mathrm{N}}) \mathbin{\widehat{\otimes}}_{(B,J^\bullet)} (\mathrm{dR}^\wedge_{(B/J)/B}, \mathrm{Fil}^\bullet_{\mathrm{H}}) \to (\mathrm{dR}^\wedge_{-/A}, \mathrm{Fil}^\bullet_{\mathrm{H}}),$$

which is an isomorphism on the underlying sheaf of rings. Our objective in this subsection is to show that the above map is an isomorphism of sheaves of filtered rings.

Our plan is again to first understand the case of a transversal base prism, and then bootstrap to general base prisms.

Theorem 4.21. *Let S be a large quasisyntomic A/I-algebra.*

(1) *The map $\Delta^{(1)}_{S/A} \to \mathrm{dR}^{\wedge}_{S/A}$ is injective.*

(2) *We have*
$$\mathrm{Fil}^r_I \Delta^{(1)}_{S/A} = (\mathrm{Fil}^r_{\mathcal{I}} \mathrm{dR}^{\wedge}_{S/A}) \cap (\Delta^{(1)}_{S/A}).$$

(3) *We have*
$$\mathrm{Fil}^i_{\mathrm{N}} \Delta^{(1)}_{S/A} = (\mathrm{Fil}^i_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A}) \cap (\Delta^{(1)}_{S/A}).$$

(4) *For any i, the natural map $\Delta^{(1)}_{S/A}/\mathrm{Fil}^i_{\mathrm{N}} \to \mathrm{dR}^{\wedge}_{S/A}/\mathrm{Fil}^i_{\mathrm{H}}$ is an injection of p-torsionfree modules, whose cokernel is $(i-1)!$-torsion. Hence multiplying by $(i-1)!$ gives a natural map backward and composing the two maps in either direction is the same as multiplying by $(i-1)!$. In particular, the natural map*
$$\Delta^{(1)}_{S/A}/\mathrm{Fil}^i_{\mathrm{N}} \to \mathrm{dR}^{\wedge}_{S/A}/\mathrm{Fil}^i_{\mathrm{H}}$$
is an isomorphism for any $i \leq p$.

(5) *The induced map $\mathrm{gr}^*_{\mathrm{N}} \Delta^{(1)}_{S/A} \to \mathrm{gr}^*_{\mathrm{H}} \mathrm{dR}^{\wedge}_{S/A}$ is compatible with the vertical filtrations on both sides, and the induced map on the graded pieces of the vertical filtrations $(\bigwedge^i_S \mathbb{L}_{S/(A/I)})^\wedge[-i] \mathbin{\widehat{\otimes}}_{A/I} \mathrm{Sym}^*_{A/I}(I/I^2) \to (\bigwedge^i_S \mathbb{L}_{S/(A/I)})^\wedge[-i] \mathbin{\widehat{\otimes}}_{A/I} \Gamma^*_{A/I}(I/I^2)$ is given by $\mathrm{id} \otimes_{A/I} (\mathrm{gr}^*_I A \to \mathrm{gr}^*_{\mathcal{I}} \mathcal{A})$.*

Taking $R = A/I$ suggests that our estimate in (4) is sharp.

Before the proof, let us remark that p-completed tensor product over A with an $\mathcal{A}$-module is the same as (p, I)-completed tensor product. This is because $I^p \mathcal{A} \subset p\mathcal{A}$.

Proof. (1) The map is given by (p, I)-completely tensoring the inclusion $A \hookrightarrow \mathcal{A}$ with $\Delta^{(1)}_{S/A}$ over A. Since $\Delta^{(1)}_{S/A}$ is (p, I)-completely flat over A (see Remark 4.8), we get the injectivity of $\Delta^{(1)}_{S/A} \to \mathrm{dR}^{\wedge}_{S/A}$.

(2) Clearly $I^r \Delta^{(1)}_{S/A}$ is contained in $\mathcal{I}^{[r]} \mathrm{dR}^{\wedge}_{S/A}$. To check the equality of intersection, it suffices to show the induced map $\Delta^{(1)}_{S/A}/I^r \to \mathrm{dR}^{\wedge}_{S/A}/\mathcal{I}^{[r]}$ is injective. But this map is given by (p, I)-completely tensoring $\Delta^{(1)}_{S/A}$ with the inclusion $A/I^r \hookrightarrow \mathcal{A}/\mathcal{I}^{[r]}$ over A, so we get the desired injectivity again by (p, I)-complete flatness of $\Delta^{(1)}_{S/A}$ over A.

(3) It suffices to show that the induced map $\widetilde{g} : \Delta^{(1)}_{S/A}/\mathrm{Fil}^i_{\mathrm{N}} \to \mathrm{dR}^{\wedge}_{S/A}/\mathrm{Fil}^i_{\mathrm{H}}$ is injective. The I-adic and $\mathcal{I}^{[\bullet]}$-filtrations on each side induce maps of graded pieces as
$$\mathrm{dR}^{\wedge}_{S/(A/I)}/\mathrm{Fil}^j_{\mathrm{H}} \mathbin{\widehat{\otimes}}_{A/I} I^{i-j}/I^{i-j+1} \to \mathrm{dR}^{\wedge}_{S/(A/I)}/\mathrm{Fil}^j_{\mathrm{H}} \mathbin{\widehat{\otimes}}_{A/I} \mathcal{I}^{[i-j]}/\mathcal{I}^{[i-j+1]}.$$
Here we have used Propositions 4.5 (3) and 4.10 (3). We conclude that the map $\widetilde{g}$ is injective as $\mathrm{dR}^{\wedge}_{S/(A/I)}/\mathrm{Fil}^j_{\mathrm{H}}$ is p-completely flat over A/I for any j and the natural map $I^{i-j}/I^{i-j+1} \to \mathcal{I}^{[i-j]}/\mathcal{I}^{[i-j+1]}$ is injective.

(4) Injectivity follows from the previous paragraph. Let $S = A/I\langle X_l^{1/p^\infty} \mid l \in L\rangle/M$, with each element $m \in M$ corresponding to a series f_m. Below we shall not distinguish m and f_m. Consider

$$S' = A/I\langle X_l^{1/p^\infty}, Y_m^{1/p^\infty} \mid l \in L, m \in M\rangle/(Y_m - f_m; m \in M) =: \widetilde{S}/(Y_m - f_m; m \in M).$$

There is a surjection $S' \twoheadrightarrow S$ of A/I-algebras, sending powers of Y_m to 0. This induces a surjection on $\mathbb{L}^{\wedge}_{-/A}$, hence also a surjection on $\mathrm{dR}^{\wedge}_{-/A}$. Therefore it suffices to prove the statement for S'.

Now we know $\mathrm{dR}^{\wedge}_{S'/A}$ is given by p-completely adjoining divided powers of I and $Y_m - f_m$ to $\widetilde{S}$, and the i-th Hodge filtration is given by the ideal p-completely generated by those degree-at-least-i divided monomials. Since the image of $\mathbb{\Delta}^{(1)}_{S'/A}$ already contains $\widetilde{S}$, it suffices to show that $(i-1)!$ times those degree-less-than-i divided monomials lie in $\widetilde{S}$, which follows from the definition.

(5) Since the generating factor $(\bigwedge^i_S \mathbb{L}_{S/(A/I)})^\wedge[-i]$ of both vertical filtrations comes from the i-th graded piece of the Hodge filtration on $\mathrm{dR}^{\wedge}_{S/(A/I)}$ (modulo I and $\mathcal{I}$ respectively), our statement follows from the commutative triangle in Corollary 3.18. ∎

The above statements can be immediately extended to the desired statement via several reduction steps.

Corollary 4.22. *Let R be a B/J-algebra. The natural map of filtered algebras*

$$(\mathbb{\Delta}^{(1)}_{R/B}, \mathrm{Fil}^\bullet_{\mathrm{N}}) \,\widehat{\otimes}_{(B,J^\bullet)}\, (\mathrm{dR}^{\wedge}_{(B/J)/B}, \mathrm{Fil}^\bullet_{\mathrm{H}}) \to (\mathrm{dR}^{\wedge}_{R/A}, \mathrm{Fil}^\bullet_{\mathrm{H}})$$

is a filtered isomorphism. In particular, the filtrations on the left hand side define quasisyntomic sheaves.

We refer the readers to [18, Sections 3.8–3.10] for a discussion of the filtration on a tensor product of filtered modules over a filtered algebra. Here we use $\widehat{\otimes}$ to mean that we take the derived p-completion of [18, Construction 3.9].

Proof of Corollary 4.22. We make a few reduction steps. First of all, both sides are left Kan extended from the case of p-complete polynomial algebras, so it suffices to show the map is a filtered isomorphism when $R = B/J\langle X_1, \dots, X_n\rangle$. Secondly, it suffices to prove the statement Zariski locally on $\mathrm{Spf}(B/J)$, hence we may assume (B, J) is oriented. Now we look at the universal map from the universal oriented prism (A^{univ}, I) to (B, J). Let R^{univ} be the corresponding p-complete polynomial algebra over the reduction of the universal oriented prism. By Remark 4.20, one sees that the filtered map for R is the base change of the analogous map for R^{univ}. Therefore we are finally reduced to the case where the base prism (A, I) is transversal and R is a p-completely smooth A/I-algebra.

Since the underlying algebra is an isomorphism by Theorem 3.5, it suffices to show the induced map of graded algebras is an isomorphism. By derived p-completing of [18, Lemma 3.10], we see that the graded algebra of the left hand side becomes

$$\mathrm{gr}^*_{\mathrm{N}}(\mathbb{\Delta}^{(1)}_{R/A}) \,\widehat{\otimes}_{\mathrm{Sym}^*_{A/I}(I/I^2)}\, \Gamma^*_{A/I}(I/I^2).$$

Now we invoke the vertical filtrations on graded algebras of both sides; see Lemma 4.7 and Lemma 4.12. The vertical filtration on $\mathrm{gr}^*_{\mathrm{N}}(\Delta^{(1)}_{R/A})$ induces an increasing filtration by $(-)\,\widehat{\otimes}_{\mathrm{Sym}^*_{A/I}(I/I^2)}\,\Gamma^*_{A/I}(I/I^2)$, and our morphism induces identifications

$$\mathrm{gr}^v_i\left(\mathrm{gr}^*_{\mathrm{N}}(\Delta^{(1)}_{R/A})\,\widehat{\otimes}_{\mathrm{Sym}^*_{A/I}(I/I^2)}\,\Gamma^*_{A/I}(I/I^2)\right) \cong \mathrm{gr}^v_i(\mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/A}))$$

for all i. Here we have used Theorem 4.21 (5). Since these vertical filtrations are increasing, exhaustive, and uniformly bounded below by 0, we conclude that the natural map

$$\mathrm{gr}^*_{\mathrm{N}}(\Delta^{(1)}_{R/A})\,\widehat{\otimes}_{\mathrm{Sym}^*_{A/I}(I/I^2)}\,\Gamma^*_{A/I}(I/I^2) \to \mathrm{gr}^*_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/A})$$

is also an isomorphism. ∎

In particular, we can specialize to the case of quasicompact quasiseparated smooth formal schemes over $\mathrm{Spf}(B/J)$.

Corollary 4.23 (cf. [20, Theorem 2.9]). *Let X be a quasicompact quasiseparated smooth formal scheme over $\mathrm{Spf}(B/J)$. Then we have a natural filtered isomorphism*

$$\left(\mathrm{R}\Gamma(X, \mathrm{Fil}^{\bullet}_{\mathrm{N}}(\Delta^{(1)}_{-/B}))\right)\widehat{\otimes}_{(B,J^{\bullet})}(\mathrm{dR}^{\wedge}_{(B/J)/B}, \mathrm{Fil}^{\bullet}_{\mathrm{H}}) \xrightarrow{\cong} \mathrm{R}\Gamma(X, \mathrm{Fil}^{\bullet}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{-/B}));$$

they are furthermore naturally filtered isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(X, \mathcal{J}^{[\bullet]}_{\mathrm{crys}})$ *if (B, J) is transversal. Similarly, whenever $i \leq p$, we also have natural isomorphisms*

$$\mathrm{R}\Gamma(X, \Delta^{(1)}_{-/A}/\mathrm{Fil}^i_{\mathrm{N}}) \xrightarrow{\cong} \mathrm{R}\Gamma(X, \mathrm{dR}^{\wedge}_{-/A}/\mathrm{Fil}^i_{\mathrm{H}});$$

these are furthermore naturally isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(X, \mathcal{O}_{\mathrm{crys}}/\mathcal{J}^i_{\mathrm{crys}})$ *if (B, J) is transversal. These isomorphisms are functorial in X, and satisfy the base change property as in Remark* 4.20.

Proof. These functorial isomorphisms are provided by Corollary 4.22 and Theorem 4.21 (4) respectively. The "furthermore" equality, when the base prism is transversal, follows from Theorem 4.1. ∎

Remark 4.24. Back to the transversal base prism case. A posteriori the filtration on the left hand side of Corollary 4.22 is a quasisyntomic sheaf, hence we may define it as the unfolding of its restriction to the basis of large quasisyntomic A/I-algebras. Also a posteriori, we know the value on such an algebra S must be concentrated in cohomological degree 0, so they have to be the image of the augmentation map

$$\mathrm{Fil}^i(\Delta^{(1)}_{S/A}\,\widehat{\otimes}_{\mathbb{Z}_p}\,\mathcal{A}) \to \mathrm{dR}^{\wedge}_{S/A},$$

where the filtration on the left hand side is given by the usual Day convolution. This implies

$$\mathrm{Fil}^n_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{S/A}) = \sum_{i=0}^{n}(\mathrm{Fil}^i_{\mathrm{N}}(\Delta^{(1)}_{S/A})\cdot \mathcal{J}^{[n-i]}),$$

which also follows from combining Proposition 4.5 (1), Proposition 4.6 (3), and Theorem 4.9 (2).

Therefore, for any $0 \le r \le p-1$, we see that the Frobenius on the derived de Rham complex when restricted to the r-th Hodge filtration,

$$\mathrm{Fil}^r_{\mathrm{H}}(\mathrm{dR}^\wedge_{S/A}) \xrightarrow{\varphi} \mathrm{dR}^\wedge_{S/A},$$

factors through multiplication by p^r. Since for large quasisyntomic A/I-algebras S, the $\mathrm{dR}^\wedge_{S/A}$ is p-completely flat over $\mathcal{A}$ (see Remark 4.8) which is p-torsionfree, we may uniquely divide the restriction φ by p^r. By unfolding, this gives rise to divided Frobenii as maps of sheaves on $\mathrm{qSyn}_{A/I}$:

$$\varphi_r : \mathrm{Fil}^r_{\mathrm{H}}(\mathrm{dR}^\wedge_{-/A}) \to \mathrm{dR}^\wedge_{-/A}.$$

By definition, they also satisfy $\varphi_r|_{\mathrm{Fil}^{r+1}_{\mathrm{H}}} = p\varphi_{r+1}$ when $r \le p-2$. Following the same argument of Theorem 3.14 (see also Remark 3.15), such a functorial divided Frobenius is unique for each $0 \le r \le p-1$.

When (A, I) is the Breuil–Kisin prism, this gives rise to an alternative definition of the divided Frobenii appearing in [10, p. 532].

5. Connection on $\mathrm{dR}^\wedge_{-/\mathfrak{S}}$ and structure of torsion crystalline cohomology

From this section onward, we focus on the Breuil–Kisin prism $A = (\mathfrak{S}, E)$ and crystalline cohomology over $S = \mathrm{dR}^\wedge_{\mathcal{O}_K/\mathfrak{S}}$. Let k be a perfect field with characteristic p, and let K be a finite totally ramified extension over $K_0 = W(k)[1/p]$ with a fixed uniformizer $\pi \in \mathcal{O}_K$. Fix an algebraic closure $\bar{K}$ of K and let $\mathbf{C}$ be the p-adic completion of $\bar{K}$. Write $G_K := \mathrm{Gal}(\bar{K}/K)$ and $e = [K : K_0]$. Let $E = E(u) \in W(k)[u]$ be the Eisenstein polynomial of π with constant term $a_0 p$; recall $\mathfrak{S} := W(k)[\![u]\!]$ is equipped with a Frobenius φ naturally extending that on $W(k)$ defined by $\varphi(u) = u^p$. Pick $\pi_n \in \mathcal{O}_{\bar{K}}$ so that $\pi_0 = \pi$ and $\pi^p_{n+1} = p$. Then $\underline{\pi} := (\pi_n)_{n\ge 0} \in \mathcal{O}^\flat_{\mathbf{C}}$. We embed $\mathfrak{S} \hookrightarrow A_{\mathrm{inf}}$ via $u \mapsto [\underline{\pi}]$ which is a map of prisms. Let $K_\infty := \bigcup_{n=0}^\infty K(\pi_n)$ and $G_\infty := \mathrm{Gal}(\bar{K}/K_\infty)$. It is clear that the embedding $\mathfrak{S} \subset A_{\mathrm{inf}}$ is compatible with the G_∞-actions. We extend φ from $\mathfrak{S}$ to S and let $\mathrm{Fil}^m S$ be the p-adic closure of the ideal generated by $\gamma_i(E) := E^i/i!, i \ge m$. We embed $S \hookrightarrow A_{\mathrm{crys}}$ also via $u \mapsto [\underline{\pi}]$. For $m \le p-1$, $\varphi(\mathrm{Fil}^m S) \subset p^m S$. We set $\varphi_m := \varphi/p^m : \mathrm{Fil}^m S \to S$. Similar notation also applies to A_{crys}. Write $c_1 := \frac{\varphi(E)}{a_0 p} \in S^\times$. Finally, there exists a $W(k)$-linear derivation $\nabla_S : S \to S$ defined by $\nabla_S(f(u)) = f'(u)$.

For $n \ge 1$, if M is a $\mathbb{Z}_p$-module then we always use M_n to denote $M/p^n M$. Similar notation applies to (p-adic formal) schemes: i.e., $X_n := X \times_{\mathrm{Spf}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}/p^n\mathbb{Z})$. Write $W = W(k)$ and reserve $\gamma_i(\cdot)$ for the i-th divided power.

5.1. *Connection on* $\mathrm{dR}^\wedge_{-/\mathfrak{S}}$

According to the philosophy that derived de Rham cohomology behaves a lot like crystalline cohomology, one expects there to be a connection on $\mathrm{dR}^\wedge_{-/\mathfrak{S}}$. We explain it in this section.

Lemma 5.1. *Let R be an $\mathcal{O}_K$-algebra. Then the natural morphism $\mathrm{dR}^{\wedge}_{R/W[u]} \to \mathrm{dR}^{\wedge}_{R/\mathfrak{S}}$ is an isomorphism, where R is regarded as an $\mathfrak{S}$- and $W[u]$-algebra via $W[u] \to \mathfrak{S} \to \mathcal{O}_K \to R$.*

Proof. Just notice the following p-complete pushout diagram:

$$\begin{tikzcd}
\mathfrak{S} \arrow[r] & \mathcal{O}_K \arrow[r] & R \\
W[u] \arrow[r] \arrow[u] & \mathcal{O}_K \arrow[r] \arrow[u] & R \arrow[u]
\end{tikzcd}$$

and appeal to the p-complete base change formula for derived de Rham complexes to get

$$\mathrm{dR}^{\wedge}_{R/W[u]} \widehat{\otimes}_{W[u]} \mathfrak{S} \xrightarrow{\cong} \mathrm{dR}^{\wedge}_{R/\mathfrak{S}}.$$

Next we observe that $\mathrm{dR}^{\wedge}_{R/W[u]}$ is an $S = \mathrm{dR}^{\wedge}_{\mathcal{O}_K/W[u]}$-complex and $S \widehat{\otimes}_{W[u]} \mathfrak{S} = S$, hence the base change on the left hand side gives $\mathrm{dR}^{\wedge}_{R/W[u]}$ back. ∎

Construction 5.2 (see also [22]). For any $W[u]$-algebra R, by (p-completely) applying [18, Lemma 3.13 (4)] to the triple $W \to W[u] \to R$, we see that there is a functorial triangle in the filtered derived ∞-category of W-modules,

$$\mathrm{dR}^{\wedge}_{R/W[u]} \widehat{\otimes}_{W[u]} \Omega^1_{W[u]/W}[-1] \to \mathrm{dR}^{\wedge}_{R/W} \to \mathrm{dR}^{\wedge}_{R/W[u]}.$$

Here $\Omega^1_{W[u]/W}[-1]$ is completely put in the first filtration. By choosing the generator $du \in \Omega^1_{W[u]/W}$, the above becomes

$$\mathrm{dR}^{\wedge}_{R/W} \to \mathrm{dR}^{\wedge}_{R/W[u]} \xrightarrow{\nabla} \mathrm{dR}^{\wedge}_{R/W[u]}(-1),$$

where (-1) indicates the shift of filtrations: $\mathrm{Fil}^i(\mathrm{dR}^{\wedge}_{R/W[u]}(-1)) = \mathrm{Fil}^{i-1}_{\mathrm{H}}(\mathrm{dR}^{\wedge}_{R/W[u]})$. If R is smooth over $W[u]$, then ∇ is given by the Lie derivative with respect to $\frac{\partial}{\partial u}$:

$$\nabla(\omega) = \mathcal{L}_{\frac{\partial}{\partial u}}(\omega).$$

Lemma 5.3. *Let R be an $\mathcal{O}_K$-algebra. Then we have a functorial triangle in the filtered derived ∞-category,*

$$\mathrm{dR}^{\wedge}_{R/W} \to \mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \xrightarrow{\nabla} \mathrm{dR}^{\wedge}_{R/\mathfrak{S}}(-1).$$

Moreover,

$$pu^{p-1} \cdot \varphi \circ \nabla = \nabla \circ \varphi,$$

where $\varphi : \mathrm{dR}^{\wedge}_{R/\mathfrak{S}} \to \mathrm{dR}^{\wedge}_{R/\mathfrak{S}}$ is the Frobenius defined in Section 2.3.

Proof. The first statement follows from Construction 5.2 and Lemma 5.1.

To check the equality, by left Kan extension it suffices to check it for polynomials. Then by quasisyntomic descent, it suffices to check the equality for large quasisyntomic

$\mathcal{O}_K$-algebras. Following the proof of Corollary 3.16, we reduce the problem to showing the equality for algebras of the form

$$R = \mathcal{O}_K\langle X_i^{1/p^\infty}, Y_j^{1/p^\infty} \mid i \in I,\ j \in J\rangle/(Y_j - f_j \mid j \in J) =: \widetilde{R}/(Y_j - f_j \mid j \in J).$$

Now the map $\widetilde{R} \to R$ induces a map between $\mathrm{dR}^\wedge_{-/\mathfrak{S}}$ given by

$$S\langle X_i^{1/p^\infty}, Y_j^{1/p^\infty} \mid i \in I,\ j \in J\rangle =: T \mapsto D_T(Y_j - f_j; j \in J)^\wedge.$$

Here S is the p-complete PD envelope of $\mathfrak{S}$ along (E) and the latter denotes p-completely adjoining divided powers of $(Y_j - f_j)$ in T. Since $D_T(Y_j - f_j; j \in J)^\wedge$ is p-complete and p-torsionfree, it suffices to check the identity on T. On T, the Frobenius φ acts by sending variables X, Y, u to their p-th power, and ∇ acts via $\frac{\partial}{\partial u}$. Finally, the task is reduced to checking the equality

$$pu^{p-1} \cdot \varphi\left(\frac{\partial}{\partial u}(F(u, \underline{X}, \underline{Y}))\right) = \frac{\partial}{\partial u}(\varphi(F(u, \underline{X}, \underline{Y})))$$

for any $F(u, \underline{X}, \underline{Y}) \in T$. ■

Consequently, for any $\mathcal{O}_K$-algebra R, we always have a long exact sequence

$$\cdots \to \mathrm{H}^i(\mathrm{dR}^\wedge_{R/W}) \to \mathrm{H}^i(\mathrm{dR}^\wedge_{R/\mathfrak{S}}) \xrightarrow{\nabla} \mathrm{H}^i(\mathrm{dR}^\wedge_{R/\mathfrak{S}}(-1)) \xrightarrow{+1} \cdots \qquad (⚃)$$

and its r-th filtration analogues for all $r \in \mathbb{N}$. In special situations, these will break into short exact sequences. Let us introduce some more notation. Let L be a perfectoid field extension of K containing all p-power roots of π. For instance, L could be the p-adic completion of K_∞ or it could be $\mathbf{C}$. Let $A_{\mathrm{inf}}(L) := W(\mathcal{O}_L^\flat)$ be Fontaine's A_{inf}-ring associated with L, and recall there is a natural map $\theta := A_{\mathrm{inf}}(L) \to \mathcal{O}_L$. Fixing a compatible system of p-power roots of π, we obtain a map $\mathfrak{S} \to A_{\mathrm{inf}}(L)$ with $u \mapsto [\underline{\pi}]$ compatible with θ and the inclusion $\mathcal{O}_K \to \mathcal{O}_L$.

Proposition 5.4. *With notation as above, let R be a quasisyntomic $\mathcal{O}_L$-algebra.*

(1) *The natural map $\mathrm{dR}^\wedge_{R/W} \to \mathrm{dR}^\wedge_{R/A_{\mathrm{inf}}(L)}$ is a filtered isomorphism.*

(2) *The sequence (⚃) and its r-th filtration analogues break into short exact sequences*

$$0 \to \mathrm{H}^i(\mathrm{Fil}^r_{\mathrm{H}}\, \mathrm{dR}^\wedge_{R/W}) \to \mathrm{H}^i(\mathrm{Fil}^r_{\mathrm{H}}\, \mathrm{dR}^\wedge_{R/\mathfrak{S}}) \xrightarrow{\nabla} \mathrm{H}^i(\mathrm{Fil}^{r-1}_{\mathrm{H}}\, \mathrm{dR}^\wedge_{R/\mathfrak{S}}) \to 0$$

for all i and r. In particular, $\mathrm{dR}^\wedge_{R/\mathfrak{S}} \xrightarrow{\nabla} \mathrm{dR}^\wedge_{R/\mathfrak{S}}(-1)$ is surjective on each H^i, and

$$\mathrm{H}^i(\mathrm{dR}^\wedge_{R/W}) = \mathrm{H}^i(\mathrm{dR}^\wedge_{R/\mathfrak{S}})^{\nabla=0}.$$

Proof. (1) is [18, Theorem 3.4 (2)].

As for (2), it suffices to show that the maps $\mathrm{H}^i(\mathrm{Fil}^r_{\mathrm{H}}\, \mathrm{dR}^\wedge_{R/W}) \to \mathrm{H}^i(\mathrm{Fil}^r_{\mathrm{H}}\, \mathrm{dR}^\wedge_{R/\mathfrak{S}})$ are injective for all i and r. By functoriality, we have maps of filtered algebras

$$\mathrm{dR}^\wedge_{R/W} \to \mathrm{dR}^\wedge_{R/\mathfrak{S}} \to \mathrm{dR}^\wedge_{R/A_{\mathrm{inf}}(L)}$$

whose composition is a filtered isomorphism by (1). Therefore the first morphism factorizing isomorphism induces the injection in cohomology. This explains why the long exact sequence (⚃) breaks into short exact sequences. The last statement follows easily by letting $r = 0$. ∎

5.2. Structures of torsion crystalline cohomology

Let X be a proper smooth formal scheme over $\mathcal{O}_K$. Let us summarize the structures on $\mathrm{H}^i_{\mathrm{crys}}(X/S) := \mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{O}_{\mathrm{crys}})$ constructed in the previous sections.

By Corollary 4.22 and Theorem 4.1, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \mathbb{\Delta}^{(1)}_{-/\mathfrak{S}}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{crys}}(X/S, \mathcal{O}_{\mathrm{crys}}) \arrow[r, equal] & S \otimes^{\mathbb{L}}_{\varphi,\mathfrak{S}} \mathrm{R}\Gamma_{\mathbb{\Delta}}(X/\mathfrak{S}) \\
\mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \mathrm{Fil}^m_{\mathrm{N}}\, \mathbb{\Delta}^1_{-/\mathfrak{S}}) \arrow[r] \arrow[u] & \mathrm{R}\Gamma_{\mathrm{crys}}(X/S, \mathcal{I}^{[m]}_{\mathrm{crys}}) \arrow[u] &
\end{tikzcd} \tag{5.5}$$

Here the second isomorphism of the top row follows from the canonical isomorphism $\mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \mathbb{\Delta}_{-/\mathfrak{S}}) \simeq \mathrm{R}\Gamma_{\mathbb{\Delta}}(X/\mathfrak{S})$ and the fact that $\varphi : \mathfrak{S} \to \mathfrak{S}$ is flat.

For $m \leq p - 1$, Remark 4.24 allows us to define a φ-semilinear map $\varphi_m : \mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m]}_{\mathrm{crys}}) \to \mathrm{H}^i_{\mathrm{crys}}(X/S)$ so that the following diagram commutes for $m + 1 \leq p - 1$:

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m]}_{\mathrm{crys}}) \arrow[r, "\varphi_m"] & \mathrm{H}^i_{\mathrm{crys}}(X/S) \\
\mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m+1]}_{\mathrm{crys}}) \arrow[u] \arrow[ur, "p\varphi_{m+1}"] &
\end{tikzcd}$$

We abbreviate the above diagram by writing $\varphi_m|_{\mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m+1]}_{\mathrm{crys}})} = p\varphi_{m+1}$. It is also clear that for any $s \in \mathrm{Fil}^m S$ and $x \in \mathrm{H}^i_{\mathrm{crys}}(X/S)$ we have

$$\varphi_m(sx) = (c_1)^{-m}\varphi_m(s)\varphi_h(E(u)^m x).$$

Finally, the above subsection constructs a connection $\nabla : \mathrm{H}^i_{\mathrm{crys}}(X/S) \to \mathrm{H}^i_{\mathrm{crys}}(X/S)$. By Proposition 5.4 and Lemma 5.3, we conclude that

(1) $\nabla : \mathrm{H}^i_{\mathrm{crys}}(X/S) \to \mathrm{H}^i_{\mathrm{crys}}(X/S)$ is a $W(k)$-linear derivative satisfying

$$\nabla(f(u)x) = f'(u)x + f(u)\nabla(x).$$

(2) (Griffiths transversality) $\nabla(\mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m]}_{\mathrm{crys}}))$ factors through $\mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m-1]}_{\mathrm{crys}})$.

(3) The following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m]}_{\mathrm{crys}}) \arrow[r, "\varphi_m"] \arrow[d, "E(u)\nabla"'] & \mathrm{H}^i_{\mathrm{crys}}(X/S) \arrow[d, "c_1\nabla"] \\
\mathrm{H}^i_{\mathrm{crys}}(X/S, \mathcal{I}^{[m]}_{\mathrm{crys}}) \arrow[r, "u^{p-1}\varphi_m"] & \mathrm{H}^i_{\mathrm{crys}}(X/S)
\end{tikzcd}$$

The last diagram follows from the equality $pu^{p-1}\varphi \circ \nabla = \nabla \circ \varphi$ of Lemma 5.3 and from $\varphi(E) = pc_1$.

Now consider the p^n-torsion crystalline cohomology $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ together with the filtration $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[m]}_{\mathrm{crys}})$. We claim that $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ admits all the above structures $\varphi_m : \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[m]}_{\mathrm{crys}}) \to \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ for $m \le p-1$ and $\nabla : \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \to \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ satisfying all the above properties. To see this, note that $\mathrm{R}\Gamma_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[m]}_{\mathrm{crys}}) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/S, \mathcal{I}^{[m]}_{\mathrm{crys}}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ where $\mathcal{I}^{[0]}_{\mathrm{crys}} = \mathcal{O}_{\mathrm{crys}}$, thus all the above properties follow by taking $\otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$, except diagram (5.5) which requires torsion quasisyntomic cohomology. For this, we define the following torsion cohomologies: For $m \ge 0$, $\mathrm{R}\Gamma_{\mathrm{dR}}(X_n/\mathfrak{S}_n, \mathrm{Fil}^m_{\mathrm{H}}) := \mathrm{R}\Gamma_{\mathrm{dR}}(X/\mathfrak{S}, \mathrm{Fil}^m_{\mathrm{H}}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$, $\mathrm{R}\Gamma_{\mathrm{qSyn}}(X_n/\mathfrak{S}_n, \mathrm{Fil}^m_N \Delta^{(1)}) := \mathrm{R}\Gamma_{\mathrm{qSyn}}(X/\mathfrak{S}, \mathrm{Fil}^m_N \Delta^{(1)}_{-/\mathfrak{S}}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$, and finally $\mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) := \mathrm{R}\Gamma_{\Delta}(X/\mathfrak{S}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Then the derived modulo p^n version of diagram (5.5) still holds by taking the original diagram and taking the derived complexes modulo p^n.

5.3. *Galois action on torsion crystalline cohomology*

Keep the notations as above. Set $\mathcal{X}$ to be the base change of X to $\mathrm{Spf}\,\mathcal{O}_{\mathbf{C}}$ and $\mathcal{X}_n := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Then $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ has an S-linear G_K-action when we define the G_K-action on S to be trivial. Note that $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ also has an A_{crys}-semilinear G_K-action which is induced by the G_K-actions on $\mathcal{X}$ and A_{crys}. By Proposition 5.4 and its proof, we see that the natural map $W(k) \to \mathfrak{S} \to A_{\mathrm{inf}}$ induces the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/W_n(k)) \arrow[r, hook, "\alpha"] \arrow[dr, "\beta"', "\sim"] & \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n) \arrow[d, "\iota"] & \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \arrow[l, hook', "\tilde{\alpha}"'] \arrow[d, hook, "\tilde{\iota}"] \\
& \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) \arrow[r, equal] & \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \otimes_{S_n} A_{\mathrm{crys},n}
\end{tikzcd}$$

Note that the bottom row is an isomorphism because $\mathcal{X} = X \times_{\mathrm{Spec}(S)} \mathrm{Spec}(A_{\mathrm{crys}})$ and that $A_{\mathrm{crys},n}$ is flat over S_n. Thus $\tilde{\iota}$ is an injection. So is $\tilde{\alpha}$. Also we note that α and β are both compatible with the G_K-actions because both the maps $W(k) \to \mathfrak{S}$ and $W(k) \to A_{\mathrm{inf}}$ are G_K-compatible. But ι is not, because $\mathfrak{S} \subset A_{\mathrm{inf}}$ is only stable under the G_∞-action. It is also clear that $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \subset (\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n))^{G_K}$ via $\tilde{\alpha}$, and $\tilde{\alpha}$ is also compatible with connections on both sides.

Now we claim the G_K-action on $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$ is given by the following formula: For any $\sigma \in G_K$, any $x \otimes a \in \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \otimes_S A_{\mathrm{crys}} \simeq \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$,

$$\sigma(x \otimes a) = \sum_{i=0}^{\infty} \nabla^i(x) \otimes \gamma_i(\sigma([\underline{\pi}]) - [\underline{\pi}])\sigma(a). \tag{5.6}$$

To see this, for any $x \in \mathcal{M}^i := \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$, set

$$x^{\nabla} := \sum_{m=0}^{\infty} \nabla(x)\gamma_m([\underline{\pi}] - u) \in \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n).$$

Then we immediately see that $x^\nabla \in \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/W_n(k)) = \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)^{\nabla=0}$. Now we claim that $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/W_n(k))$ is generated by $\{x^\nabla \mid x \in \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)\}$ as an A_{crys}-module. If so then (5.6) follows from the fact that β is G_K-equivariant and the construction of x^∇ (note that both x and u are G_K-invariants).

To prove the claim, for any $y \in \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/W_n(k))$, suppose that $\beta(y) = \sum_j a_j \tilde{\iota}(x_j)$ with $a_j \in A_{\mathrm{crys}}$ and $x_j \in \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$. Then we see that $y^\nabla := \sum_j a_j x_j^\nabla \in \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/W_n(k))$. It suffices to that check $y = y^\nabla$. Since β is an isomorphism, it suffices to show that $\beta(y) = \beta(y^\nabla)$. This follows from $\beta(x^\nabla) = \iota(x)$ for $x \in \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ as $\iota([\underline{\pi}] - u) = [\underline{\pi}] - [\underline{\pi}] = 0$.

6. Torsion Kisin module, Breuil module and associated Galois representations

In this section, we set up the theory of generalized torsion Kisin modules which extends the theory of Kisin modules, which is discussed, for example, in [26, Section 2]. The key point for the generalized Kisin modules is that they may have u-torsion, and they are classical torsion Kisin modules when taken modulo u-torsion.

6.1. (Generalized) Kisin modules

Let $(\mathfrak{S}, E(u))$ be the Breuil–Kisin prism over $\mathcal{O}_K$ with $d = E(u) = E$ the Eisenstein polynomial of fixed uniformizer $\pi \in \mathcal{O}_K$. A *φ-module over $\mathfrak{S}$* is an $\mathfrak{S}$-module $\mathfrak{M}$ together with a $\varphi_{\mathfrak{S}}$-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$. Write $\varphi^*\mathfrak{M} = \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Note that $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is an $\mathfrak{S}$-linear map. A (*generalized*) *Kisin module of height h* is a φ-module $\mathfrak{M}$ of finite $\mathfrak{S}$-type such that there exists an $\mathfrak{S}$-linear map $\psi : \mathfrak{M} \to \varphi^*\mathfrak{M}$ such that $\psi \circ (1 \otimes \varphi) = E^h \,\mathrm{id}_{\varphi^*\mathfrak{M}}$ and $(1 \otimes \varphi) \circ \psi = E^h \,\mathrm{id}_{\mathfrak{M}}$. Maps between generalized Kisin modules are given by $\mathfrak{S}$-linear maps which are compatible with φ and ψ. We denote by $\mathrm{Mod}^{\varphi,h}_{\mathfrak{S}}$ the category of (generalized) Kisin modules of height h.

In [26], a *Kisin module of height h* is defined to be an *étale* φ-module $\mathfrak{M}$ of finite $\mathfrak{S}$-type such that $\mathrm{coker}(1 \otimes \varphi)$ is killed by E^h. Here étale φ-module means that the natural map $\mathfrak{M} \to \widehat{\mathfrak{S}[1/u]} \otimes_{\mathfrak{S}} \mathfrak{M}$ is injective. Since $E(u)$ is a unit in $\widehat{\mathfrak{S}[1/u]}$, we easily see that the étale assumption implies that $1 \otimes \varphi : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is injective. Then existence and uniqueness of $\psi : \mathfrak{M} \to \varphi^*\mathfrak{M}$, in the definition of (generalized) Kisin modules of height h, follows. That is, the Kisin module $\mathfrak{M}$ of height h defined classically is a (generalized) Kisin module of height h. So in the following, we drop "generalized" when we mention an object in $\mathrm{Mod}^{\varphi,h}_{\mathfrak{S}}$. If we need to emphasize that $\mathfrak{M}$ is also a Kisin module of height m classically defined, we will mention that it is étale.

Lemma 6.1. (1) $\mathrm{Mod}^{\varphi,h}_{\mathfrak{S}}$ *is an abelian category.*

(2) $\mathfrak{M}$ *is étale if and only if* $\mathfrak{M}$ *has no u-torsion.*

(3) $\mathfrak{M}[1/p]$ *is finite* $\mathfrak{S}[1/p]$*-free.*

Proof. (1) is easy to check because $\varphi : \mathfrak{S} \to \mathfrak{S}$ is faithfully flat.

(2) It is clear from the definition that if $\mathfrak{M}$ is étale then it has no u-torsion. Conversely, let $\mathfrak{M}[p^\infty] := \{x \in \mathfrak{M} \mid p^n x = 0 \text{ for some } n > 0\}$ and $\mathfrak{M}' := \mathfrak{M}/\mathfrak{M}[p^\infty]$. We get the short exact sequence $0 \to \mathfrak{M}[p^\infty] \to \mathfrak{M} \to \mathfrak{M}' \to 0$. It is clear that both $\mathfrak{M}[p^\infty]$ and $\mathfrak{M}'$ are objects in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$ and if $\mathfrak{M}$ has no u-torsion then neither $\mathfrak{M}[p^\infty]$ nor $\mathfrak{M}'$ has u-torsion. Since $\mathfrak{M}[p^\infty]$ is killed by some p-power, $\mathfrak{M} \otimes \widehat{\mathfrak{S}}[1/u] = \mathfrak{M}[1/u]$. So $\mathfrak{M}[p^\infty]$ has no u-torsion if and only if $\mathfrak{M}[p^\infty]$ is étale. Now since $\mathfrak{M}'$ has no p-torsion, we claim that $\mathfrak{M}'[1/p]$ is finite $\mathfrak{S}[1/p]$-free, which will imply (3) and étaleness of $\mathfrak{M}'$. By [16, Section 1.2.1], $\mathfrak{M}'[1/p] \simeq \bigoplus \mathfrak{S}[1/p]/P_i^{a_i}$ with $P_i \in W(k)[u]$ monic irreducible and $P_i \equiv u^{b_i} \bmod p$, or $P_i = 0$. Without loss of generality, we may assume that $P_i \neq 0$ and show such $\mathfrak{M}'$ does not exist when $\mathfrak{M}' \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$. Consider the wedge product $\mathfrak{N}$ of $\mathfrak{M}'[1/p]$. Then $\mathfrak{N} \simeq \mathfrak{S}[1/p]/f$ with $f = \prod P_i^{a_i}$ and write $\varphi^* := 1 \otimes \varphi$. We also obtain $\varphi^* : \varphi^*\mathfrak{N} \to \mathfrak{N}$ and $\psi : \mathfrak{N} \to \varphi^*\mathfrak{N}$ so that $\psi \circ \varphi^* = E(u)^h \,\mathrm{id}_{\varphi^*\mathfrak{N}}$ and $\varphi^* \circ \psi = E(u)^h \,\mathrm{id}_{\mathfrak{N}}$ for some h. Since $\varphi^*\mathfrak{N} \simeq \mathfrak{S}[1/p]/\varphi(f)$, we can write the above maps explicitly as

$$\mathfrak{S}[1/p]/\varphi(f) \xrightarrow{\varphi^*} \mathfrak{S}[1/p]/f \xrightarrow{\psi} \mathfrak{S}[1/p]/\varphi(f).$$

Write $x = \varphi^*(1)$ and $y = \psi(1)$. We have $\varphi(f)x = fz'$ and $fy = \varphi(f)w'$ for some $z', w' \in \mathfrak{S}[1/p]$. The condition $\psi \circ \varphi^* = E(u)^h \,\mathrm{id}_{\varphi^*\mathfrak{N}}$ and $\varphi^* \circ \psi = E(u)^h \,\mathrm{id}_{\mathfrak{N}}$ implies that $\varphi(f)E(u)^h = fz$ and $fE(u)^h = \varphi(f)w$ with $z, w \in \mathfrak{S}[1/p]$. So $E(u)^{2h} = zw$. Since $E(u)$ is an Eisenstein polynomial, $z = z_0 E(u)^l$ with z_0 a unit in $\mathfrak{S}[1/p]$. Then $\varphi(f) = z_0 f E(u)^{l-h}$. We easily see $z_0 \in \mathfrak{S}^\times$ as both f and $E(u)$ are monic. So $l - h > 0$ by reducing mod p on both sides. Let $a_0 = f(0)$ be the constant term of $f(u)$. Since $\varphi(f)(0) = \varphi(a_0) = z_0(0)a_0 p^{l-h}$, comparing the p-adic valuation on both sides, we see that $a_0 = 0$. Then we may write $f = u^m g$ with $g(0) \neq 0$. But then

$$u^{pm-m}\varphi(g) = z_0 g E(u)^{l-h},$$

which is impossible by comparing the constant terms on both sides. In summary, such an $\mathfrak{M}'$ cannot exist and $\mathfrak{M}'[1/p]$ is finite $\mathfrak{S}[1/p]$-free. ∎

Let $\mathfrak{M}$ be a Kisin module of height h and set $\mathfrak{M}[u^\infty] := \{x \in \mathfrak{M} \mid u^l x = 0 \text{ for some } l\}$. Then both $(1 \otimes \varphi_{\mathfrak{M}})(\varphi^*\mathfrak{M}[u^\infty]) \subset \mathfrak{M}[u^\infty]$ and $\psi(\mathfrak{M}[u^\infty]) \subset \varphi^*\mathfrak{M}[u^\infty]$. The above lemma shows that $\mathfrak{M}[u^\infty] \subset \mathfrak{M}[p^\infty]$ and $\mathfrak{M}/\mathfrak{M}[u^\infty]$ is étale.

Lemma 6.2. *We have the following short exact sequence in* $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$*:*

$$0 \to \mathfrak{M}[u^\infty] \to \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}[u^\infty] \to 0$$

with $\mathfrak{M}/\mathfrak{M}[u^\infty]$ *being étale.*

It turns out that an étale Kisin module enjoys many nice properties. Let $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h}$ denote the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$ whose objects $\mathfrak{M}$ are torsion, i.e., killed by p^n for some n. The following lemma is a part of [26, Proposition 2.3.2].

Lemma 6.3. *The following statements are equivalent for a torsion Kisin module* $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor}}^{\varphi,h}$*:*

(1) $\mathfrak{M}$ *is étale.*

(2) $\mathfrak{M}$ *can be written as a successive quotient of* $\mathfrak{M}_i$ *so that* $\mathfrak{M}_i \in \mathrm{Mod}^{\varphi,h}_{\mathfrak{S},\mathrm{tor}}$ *and* $\mathfrak{M}_i$ *is finite* $k[\![u]\!]$*-free.*

(3) $\mathfrak{M} = \mathfrak{N}/\mathfrak{N}'$ *where* $\mathfrak{N}' \subset \mathfrak{N}$ *are Kisin modules of height* h *and* $\mathfrak{N}'$ *and* $\mathfrak{N}$ *are finite free* $\mathfrak{S}$*-modules.*

Corollary 6.4. *Given an étale Kisin module* $\mathfrak{M} \in \mathrm{Mod}^{\varphi,h}_{\mathfrak{S}}$*, there exists an étale Kisin module* $\mathfrak{M}_n \in \mathrm{Mod}^{\varphi,h}_{\mathfrak{S}}$ *killed by* p^n *satisfying* $\mathfrak{M}/p^n\mathfrak{M}[1/u] = \mathfrak{M}_n[1/u]$ *and* $\mathfrak{M} = \varprojlim_n \mathfrak{M}_n$*.*

Proof. Let $M = \mathfrak{M} \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}[1/u]}$. Consider the exact sequence $0 \to p^n M \to M \xrightarrow{q} M/p^n M \to 0$. Since $\mathfrak{M}$ is étale, the natural map $\mathfrak{M} \to M$ is injective. Set $\mathfrak{M}_n = q(\mathfrak{M}) \subset M/p^n M$. It is easy to check that $\mathfrak{M}_n[1/u] = \mathfrak{M}/p^n\mathfrak{M}[1/u] = M/p^n M$, $\mathfrak{M}_n$ has no u-torsion and $\mathfrak{M} = \varprojlim_n \mathfrak{M}_n$ (since $\mathfrak{M}$ is p-adically closed in M). We just need to check that $\mathfrak{M}_n$ has height h. This was proved in [16, Proposition B 1.3.5]. ■

In general, the category of étale Kisin modules is not abelian but under some restrictions it could be abelian. Given $\mathfrak{M} \in \mathrm{Mod}^{\varphi,h}_{\mathfrak{S},\mathrm{tor}}$, let $M = \mathfrak{M}[u^\infty, p] := \{x \in \mathfrak{M}[u^\infty] \mid px = 0\}$.

Lemma 6.5. *If* $eh < p-1$ *then* $M = 0$ *and if* $eh < 2(p-1)$ *then* $M \simeq \bigoplus k$ *or* 0*.*

Proof. We have $\psi : M \to \varphi^* M$ such that $\psi \circ (1 \otimes \varphi) = d^h \,\mathrm{id}_{\varphi^* M}$. We can write $M = \bigoplus_{j=1}^m k[\![u]\!]/u^{a_j}$ with $a_j \geq 1$, and then $\varphi^* M \simeq \bigoplus_{j=1}^m k[\![u]\!]/u^{pa_j}$. Assume that $a = \max_j a_j$ and let $x \in \varphi^* M$ be such that $u^{pa}x = 0$ but $u^{pa-1}x \neq 0$. Since $\psi \circ (1 \otimes \varphi) = u^{eh}\,\mathrm{id}_{\varphi^* M}$, we conclude that $u^{eh}x \in \psi(M)$. Noting that $u^a M = \{0\}$ and ψ is $k[\![u]\!]$-linear, we have $u^{a+eh}x = 0$. This forces $a + eh \geq pa$, that is, $a \leq \frac{eh}{p-1}$. Hence such an a cannot exist if $eh < p-1$. If $eh < 2(p-1)$ then $a = 1$ or 0. This proves the lemma. ■

Proposition 6.6. *If* $eh < p-1$ *then* $\mathrm{Mod}^{\varphi,h}_{\mathfrak{S}}$ *is an abelian category.*

Proof. By Lemma 6.5, $\mathfrak{M}[u^\infty] = 0$. ■

Example 6.7. Let $E(u) = u - p$, $\mathfrak{M} = k \simeq k[\![u]\!]/u$ and $\varphi(1) = 1$. Let $\psi : k[\![u]\!]/u \to k[\![u]\!]/u^p$ by $\psi(1) = u^{p-1}$. Then $\mathfrak{M} \in \mathrm{Mod}^{\varphi,p-1}_{\mathfrak{S},\mathrm{tor}}$.

Let $\mathfrak{M} \in \mathrm{Mod}^{\varphi,h}_{\mathfrak{S}}$. Define the *Breuil–Kisin filtration* on $\varphi^*\mathfrak{M}$ by

$$\mathrm{Fil}^h_{\mathrm{BK}}\,\varphi^*\mathfrak{M} := \mathrm{Im}(\psi : \mathfrak{M} \to \varphi^*\mathfrak{M}).$$

If $\mathfrak{M}$ is étale then ψ is injective as explained above, and we have an identification

$$\mathrm{Fil}^h_{\mathrm{BK}}\,\varphi^*\mathfrak{M} \cong \{x \in \varphi^*\mathfrak{M} \mid (1 \otimes \varphi)(x) \in E(u)^h \mathfrak{M}\} \tag{6.8}$$

of submodules in $\varphi^*\mathfrak{M}$. Since there is only one filtration considered for Kisin modules in this section, we drop BK from the notation in this section. Finally, there is a $\varphi_{\mathfrak{S}}$-semi-linear map $\varphi := \varphi \otimes \varphi : \varphi^*\mathfrak{M} \to \varphi^*\mathfrak{M}$. It is clear that $\varphi(\mathrm{Fil}^i\,\varphi^*\mathfrak{M}) \subset \varphi(E(u))^i \varphi^*\mathfrak{M}$. If $\mathfrak{M}$ is étale, then we define $\varphi_i : \mathrm{Fil}^i\,\varphi^*\mathfrak{M} \to \varphi^*\mathfrak{M}$ by

$$\varphi_i(x) := \frac{\varphi(x)}{\varphi((a_0^{-1}E(u))^i)}, \quad \text{where} \quad a_0 p = E(0).$$

Lemma 6.9. *Suppose that* $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ *is an exact sequence in* $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$ *and all modules are étale. Then the following sequence is exact:*

$$0 \to \mathrm{Fil}^h \varphi^*\mathfrak{M}' \to \mathrm{Fil}^h \varphi^*\mathfrak{M} \to \mathrm{Fil}^h \varphi^*\mathfrak{M}'' \to 0.$$

Proof. This easily follows from the fact that $\varphi^* : \mathrm{Fil}^h \varphi^*\mathfrak{M} \to E^h\mathfrak{M}$ is bijective. ■

Remark 6.10. The above lemma fails in general if $i < h$ or if the modules are not étale.

6.2. *Galois representation attached to étale Kisin modules*

Recall that we fix $\pi_n \in \bar{K}$ so that $\underline{\pi} := (\pi_n) \in \mathcal{O}_{\mathbf{C}}^\flat$ and $\pi_0 = \pi$; moreover, we set $K_\infty := \bigcup_{n\geq 0} K(\pi_n)$ and $G_\infty := \mathrm{Gal}(\bar{K}/K_\infty)$. We embed $\mathfrak{S} \to A_{\mathrm{inf}}$ via $u \mapsto [\underline{\pi}]$. This embedding is compatible with φ, but not with the G_K-action. We have $\mathfrak{S} \subset A_{\mathrm{inf}}^{G_\infty}$.

To a Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$, we can associate a representation of G_∞ via

$$T_{\mathfrak{S}}(\mathfrak{M}) := (\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathbf{C}^\flat))^{\varphi=1} = (\mathfrak{M}/\mathfrak{M}[u^\infty] \otimes_{\mathfrak{S}} W(\mathbf{C}^\flat))^{\varphi=1}.$$

So the Galois representation attached to $\mathfrak{M}$ is insensible to u-torsion parts because $1/u \in W(\mathbf{C}^\flat)$. It is well-known that $T_{\mathfrak{S}}$ is exact and there exists a $W(\mathbf{C}^\flat)$-linear isomorphism

$$\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathbf{C}^\flat) \simeq T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} W(\mathbf{C}^\flat),$$

which is compatible with φ and with the G_∞-actions.

For many purposes, we define another variant $T_{\mathfrak{S}}^h$ of $T_{\mathfrak{S}}$: For an étale $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$, we can naturally extend $\varphi_h : \mathrm{Fil}^h \varphi^*\mathfrak{M} \to \varphi^*\mathfrak{M}$ to $\varphi_h : \mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \to \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$. We set

$$\begin{aligned} T_{\mathfrak{S}}^h(\mathfrak{M}) &:= (\mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}})^{\varphi_h=1} \\ &= \{x \in \mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \mid \varphi(x) = \varphi(a_0^{-1}E(u)^h)x\}. \end{aligned}$$

Lemma 6.11. *Assume that* $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,h}$ *is étale. Then:*

(1) $T_{\mathfrak{S}}^h(\mathfrak{M}) \simeq T_{\mathfrak{S}}(\mathfrak{M})(h)$.

(2) *The following sequence is exact:*

$$0 \to T_{\mathfrak{S}}^h(\mathfrak{M}) \to \mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \xrightarrow{\varphi_h - 1} \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \to 0.$$

Proof. First it is clear that $T_{\mathfrak{S}}(\mathfrak{M}) = (\varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathbf{C}^\flat))^{\varphi=1}$ because φ on $W(\mathbf{C}^\flat)$ is bijective. Recall that $pa_0 = E(0)$. Let $\underline{\varepsilon} = (\zeta_{p^n})_{n\geq 0} \in \mathcal{O}_{\mathbf{C}}^\flat$ with ζ_{p^n} satisfying $\zeta_1 = 1$, $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ and $\zeta_p \neq 1$. By [28, Example 3.2.3], there exists nonzero $\mathfrak{t} \in A_{\mathrm{inf}}$ such that $\mathfrak{t} \neq 0 \bmod p$, $\varphi(\mathfrak{t}) = a_0^{-1}E(u)\mathfrak{t}$ and $t := \log[\underline{\varepsilon}] = \mathfrak{c}\varphi(\mathfrak{t})$ with $\mathfrak{c} = \prod_{n=1}^\infty \varphi^n(a_0^{-1}E(u)/p) \in A_{\mathrm{crys}}^*$. Write $\beta = \varphi(\mathfrak{t})$. Consider the map $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \to T_{\mathfrak{S}}(\mathfrak{M})$ defined by $x \mapsto x/\beta^h$ for any $x \in \mathrm{Fil}^h \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$. Since $\varphi(\beta) = \varphi(a_0^{-1}E(u))\beta$, and $\beta \in W(\mathbf{C}^\flat)$ is invertible as $\mathfrak{t} \neq 0 \bmod p$, the definition of ι makes sense. Note that $\mathfrak{c} \in (A_{\mathrm{crys}})^{G_\infty}$. So $g(\beta)/\beta = g(t)/t$ is a cyclotomic character for any $g \in G_\infty$. So

$\iota : T^h_{\mathfrak{S}}(\mathfrak{M}) \to T_{\mathfrak{S}}(\mathfrak{M})(h)$ is a map compatible with the G_∞-actions. We claim that $T^h_{\mathfrak{S}}$ is an exact functor. If so then since $T_{\mathfrak{S}}$ is also exact, to show that ι is an isomorphism, we can reduce to the case that $\mathfrak{M}$ is killed by p by Corollary 6.4. In this case, $\mathfrak{M}$ is finite $k[\![u]\!]$-free. If $e_1, \ldots, e_d$ is a basis of $\mathfrak{M}$, then $\varphi_{\mathfrak{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$ with a $k[\![u]\!]$-matrix A so that there exists a $k[\![u]\!]$-matrix B satisfying $AB = BA = (a_0^{-1}E(u))^h I_d$. Let us still regard $\{e_i\}$ as a basis of $\varphi^*\mathfrak{M}$. Then it is easy to check that $(e_1, \ldots, e_d)B$ is a basis of $\mathrm{Fil}^h\, \varphi^*\mathfrak{M}$. Now for any $x = \sum_i e_i \otimes a_i \in \varphi^*\mathfrak{M} \otimes_{k[\![u]\!]} \mathbf{C}^\flat$, the equation $\varphi(x) = x$ is equivalent to $\varphi(X) = \varphi(A)^{-1}X$ where $X = (a_1, \ldots, a_d)^T$. The latter gives $\varphi(\beta^h X) = \varphi(a_0^{-h}E(u)^h A^{-1})(\beta^h X) = \varphi(B)(\beta^h X)$, which implies that $Y = \beta^h X$ is in $(\mathcal{O}^\flat_{\mathbf{C}})^d$. That is, $y = \beta^h x \in \varphi^*\mathfrak{M} \otimes_{k[\![u]\!]} \mathcal{O}^\flat_{\mathbf{C}}$. Furthermore, consider $Z = B^{-1}\beta^h X$. Since $\varphi(Z) = \varphi(a_0^{-h}B^{-1}A^{-1}E(u)^h)BZ = BZ$, we conclude that Z has all entries in $\mathcal{O}^\flat_{\mathbf{C}}$. Then $\beta^h x = (e_1, \ldots, e_d)BZ$ is in $\mathrm{Fil}^h\, \varphi^*\mathfrak{M} \otimes \mathcal{O}^\flat_{\mathbf{C}}$. This proves that ι is surjective. Since ι is clearly injective, it is an isomorphism.

Now we prove the claim that $T^h_{\mathfrak{S}}$ is exact. For this, it suffices to show that $\varphi_h - 1$ is surjective and we once again reduce to the case that $\mathfrak{M}$ is killed by p. By writing the $k[\![u]\!]$-basis of $\mathfrak{M}$ as above, we need to solve the equation $\varphi(X) - BX = Y$ for any $Y = (a_1, \ldots, a_d)^T$ for $a_i \in \mathcal{O}^\flat_{\mathbf{C}}$. Since $\mathbf{C}^\flat$ is algebraically closed, we see X exists with entries in $\mathbf{C}^\flat$. By comparing the valuation of each entry of both sides of the equation $\varphi(X) = BX + Y$, it is easy to show that all entries of X must be in $\mathcal{O}^\flat_{\mathbf{C}}$. ∎

6.3. *Torsion Breuil modules*

We fix $0 \le h \le p-2$ for this subsection. Recall that $S = \mathcal{A}$ is the p-adically completed PD envelope of $\theta : \mathfrak{S} \twoheadrightarrow \mathcal{O}_K, u \mapsto \pi$, and for $i \ge 1$ write $\mathrm{Fil}^i\, S \subseteq S$ for the (closure of the) ideal generated by $\{\gamma_n(E) = E^n/n!\}_{n \ge i}$. For $i \le p-1$, one has $\varphi(\mathrm{Fil}^i\, S) \subseteq p^i S$, so we may define $\varphi_i : \mathrm{Fil}^i\, S \to S$ by $\varphi_i := p^{-i}\varphi$. We have $c_1 := \varphi(E(u))/p \in S^\times$.

Let $'\mathrm{Mod}^{\varphi,h}_S$ denote the category whose objects are triples $(\mathcal{M}, \mathrm{Fil}^h\, \mathcal{M}, \varphi_h)$, consisting of

(1) an S-module $\mathcal{M}$;

(2) an S-submodule $\mathrm{Fil}^h\, \mathcal{M} \subset \mathcal{M}$ containing $\mathrm{Fil}^h\, S \cdot \mathcal{M}$;

(3) a φ-semilinear map $\varphi_h : \mathrm{Fil}^h\, \mathcal{M} \to \mathcal{M}$ such that for all $s \in \mathrm{Fil}^h\, S$ and $x \in \mathcal{M}$ we have
$$\varphi_h(sx) = (c_1)^{-h}\varphi_h(s)\varphi_h(E(u)^h x);$$
and such that

(4) $\varphi_h(\mathrm{Fil}^h\, \mathcal{M})$ generates $\mathcal{M}$ as an S-module.

Morphisms are given by S-linear maps preserving Fil^h's and commuting with φ_h. A sequence is defined to be *short exact* if it is short exact as a sequence of S-modules, and induces a short exact sequence on Fil^h's. Let $\mathrm{Mod}^{\varphi,h}_{S,\mathrm{tor}}$ denote the full subcategory of $'\mathrm{Mod}^{\varphi,h}_S$ so that $\mathcal{M}$ is killed by a p-power and $\mathcal{M}$ can be written as a successive quotient of $\mathcal{M}_i$ in $'\mathrm{Mod}^{\varphi}_S$ and each $\mathcal{M}_i \simeq \bigoplus S_1$ where $S_n := S/p^n S$.

For each object $\mathcal{M} \in \mathrm{Mod}_S^{\varphi,h}$, we can extend φ_h and Fil^h to $A_{\mathrm{crys}} \otimes_S \mathcal{M}$ in the following way: Since $A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}$ is faithfully flat over S/p^n by [12, Lemma 5.6], $A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M} \to A_{\mathrm{crys}} \otimes \mathcal{M}$ is injective and so we can define $\mathrm{Fil}^h(A_{\mathrm{crys}} \otimes_S \mathcal{M}) := A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M}$ and then φ_h extends to $A_{\mathrm{crys}} \otimes_S \mathcal{M}$. This allows us to define a representation of G_∞ via

$$T_S(\mathcal{M}) := (\mathrm{Fil}^h(A_{\mathrm{crys}} \otimes_S \mathcal{M}))^{\varphi_h=1}.$$

Now let us recall the relation of classical torsion Kisin modules to objects in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and their relationship to torsion Galois representations. Let $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor\,\acute{e}t}}^{\varphi,h}$ denote the category of étale torsion Kisin modules of height h. In this subsection, all torsion Kisin modules are étale torsion Kisin modules, i.e., $\mathfrak{M}$ is u-torsionfree. For each such $\mathfrak{M}$, we construct an object $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ as follow: $\mathcal{M} := S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and

$$\mathrm{Fil}^h \mathcal{M} := \{x \in \mathcal{M} \mid (1 \otimes \varphi)(x) \in \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M}\};$$

and $\varphi_h : \mathrm{Fil}^h \mathcal{M} \to \mathcal{M}$ is defined as the composite

$$\mathrm{Fil}^h \mathcal{M} \xrightarrow{1\otimes\varphi_{\mathfrak{M}}} \mathrm{Fil}^h S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_h \otimes 1} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathcal{M}.$$

We write $\underline{\mathcal{M}}(\mathfrak{M})$ for $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ built from the Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S},\mathrm{tor\,\acute{e}t}}^{\varphi,h}$ as above. Note that $A_{\mathrm{crys}} \otimes_S \underline{\mathcal{M}}(\mathfrak{M}) = A_{\mathrm{crys}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

Proposition 6.12. *The above functor induces an exact equivalence between* $\mathrm{Mod}_{\mathfrak{S},\mathrm{tor\,\acute{e}t}}^{\varphi,h}$ *and* $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$. *Furthermore, there exists a short exact sequence*

$$0 \to T_S(\mathcal{M}) \to A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M} \xrightarrow{\varphi_h - 1} A_{\mathrm{crys}} \otimes_S \mathcal{M} \to 0 \tag{6.13}$$

and an isomorphism of G_∞*-representations*

$$T_S(\underline{\mathcal{M}}(\mathfrak{M})) \simeq T_{\mathfrak{S}}(\mathfrak{M})(h).$$

Proof. The equivalence together with exactness is [14, Theorem 2.2.1], which builds on Breuil and Kisin's results (see [27, Proposition 3.3.1]). Consider an exact sequence in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$,

$$0 \to \mathcal{M}'' \to \mathcal{M} \to \mathcal{M}' \to 0.$$

Then we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & T_S(\mathcal{M}'') \arrow[r] \arrow[d] & T_S(\mathcal{M}) \arrow[r] \arrow[d] & T_S(\mathcal{M}') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M}'' \arrow[r] \arrow[d, "\varphi_h - 1"] & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M} \arrow[r] \arrow[d, "\varphi_h - 1"] & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^h \mathcal{M}' \arrow[r] \arrow[d, "\varphi_h - 1"] & 0 \\
0 \arrow[r] & A_{\mathrm{crys}} \otimes_S \mathcal{M}'' \arrow[r] & A_{\mathrm{crys}} \otimes_S \mathcal{M} \arrow[r] & A_{\mathrm{crys}} \otimes_S \mathcal{M}' \arrow[r] & 0
\end{tikzcd}$$

By the definition of exactness in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and since A_{crys}/p^n is flat over S/p^n, we see that the last two rows of the diagram are exact. So to show $\varphi_h - 1$ is surjective on $\mathcal{M}$, we reduce to the situation that $\mathcal{M}$ is killed by p. Also the surjectivity of $\varphi_h - 1$ implies that the functor T_S is exact from the above diagram. So let us first take for granted that $\varphi_h - 1$ is surjective and postpone the proof.

Now let us construct a natural map $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$. Write $\mathcal{M} := \underline{\mathcal{M}}(\mathfrak{M})$. It is clear that $\mathrm{Fil}^h \varphi^*\mathfrak{M} \subset \mathrm{Fil}^h \underline{\mathcal{M}}(\mathfrak{M})$ is compatible with the injection $\varphi^*\mathfrak{M} \hookrightarrow \mathcal{M}$. But φ_h defined on Kisin modules is slightly different from that on Breuil modules. By chasing definitions, we see that for any $x \in \mathrm{Fil}^h \varphi^*\mathfrak{M}$, $\varphi_{h,\mathcal{M}}(x) = (a_0^{-1}c_1)^h \varphi_{h,\varphi^*\mathfrak{M}}(x)$. Recall $\mathfrak{c} = \prod_{n=1}^{\infty} \varphi^n(a_0^{-1}E(u)/p) \in A_{\mathrm{crys}}^*$ in the proof Lemma 6.11. Since $\varphi(\mathfrak{c}) = a_0^{-1}c_1\mathfrak{c}$, the map $\iota : A_{\mathrm{inf}} \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} \to A_{\mathrm{crys}} \otimes_S \mathcal{M}$ given by $\iota(x) = \mathfrak{c}^h x$ induces a map $\iota : T_{\mathfrak{S}}^h(\mathfrak{M}) \to T_S(\mathcal{M})$.

To show that ι is an isomorphism, since $T_{\mathfrak{S}}$, T_S and $\underline{\mathcal{M}}$ are all exact, we reduce to the case that $\mathfrak{M}$ is killed by p when $\mathfrak{M}$ is finite $k[\![u]\!]$-free. By the same argument as in Lemma 6.11, there exists a basis $e_1, \ldots, e_d$ of $\varphi^*\mathfrak{M}$ such that $\mathrm{Fil}^h \varphi^*\mathfrak{M}$ has basis $(e_1, \ldots, e_d)B$, $\varphi(e_1, \ldots, e_d) = (e_1, \ldots, e_d)\varphi(A)$ and $AB = BA = (a_0^{-1}E(u))^h I_d$. So any $x \in T_{\mathfrak{S}}^h(\mathfrak{M})$ corresponds to a solution of $\varphi(X) = BX$. Since $\mathcal{M} = \underline{\mathcal{M}}(\mathfrak{M})$, it is straightforward to compute that $\mathcal{M}$ also has S_1-basis $e_1, \ldots, e_d$, and $\mathrm{Fil}^h \mathcal{M}$ is generated by $(e_1, \ldots, e_d)B$ and $\mathrm{Fil}^p S_1\mathcal{M}$. Note that $a_0^{-1}c_1 \equiv 1 \bmod (p, \mathrm{Fil}^p S)$. So $T_S(\mathcal{M})$ corresponds to solutions of $\varphi(X) = BX \bmod \mathrm{Fil}^p A_{\mathrm{crys},1}$ where $A_{\mathrm{crys},1} = A_{\mathrm{crys}}/pA_{\mathrm{crys}}$. Now it suffices to show the following map is bijective:

$$\{X \mid \varphi(X) = BX,\ x_i \in \mathcal{O}_{\mathbf{C}}^{\flat}\} \to \{X \mid \varphi(X) = BX \bmod \mathrm{Fil}^p A_{\mathrm{crys},1},\ x_i \in A_{\mathrm{crys},1}\}.$$

Let v denote the valuation on $\mathcal{O}_{\mathbf{C}}^{\flat}$ normalized by $v(u^e) = 1$. Suppose that X is in the kernel; then $X \in E(u)^p\mathcal{O}_{\mathbf{C}}^{\flat}$. So $v(x_j) \geq p$ for all j. Let x_i be the entry with least valuation. Note that $v(\varphi(x_j)) = pv(x_j)$ for any j and $A\varphi(X) = u^{eh}X$. The minimal possible left side valuation is $pv(x_i)$, while for the right side it is $h + v(x_i)$. This is impossible when $v(x_i) \geq p$ because $h \leq p-2$. So this implies that $X = 0$. Indeed, if $v(x_i) \geq 2$ then the same proof shows that $X = 0$. That is, if X_1, X_2 are two solutions in the left side and $X_1 \equiv X_2 \bmod E(u)^2$ then $X_1 = X_2$.

Conversely, let Z be the vector in $A_{\mathrm{crys},1}$ such that $\varphi(Z) = BZ \bmod \mathrm{Fil}^p A_{\mathrm{crys},1}$. Then there exists Z_0 with entries in $\mathcal{O}_{\mathbf{C}}^{\flat}$ such that $\varphi(Z_0) = BZ_0 + E(u)^pC$ where C is a vector with entries in $\mathcal{O}_{\mathbf{C}}^{\flat}$. Note that $E(u)^p = E(u)^{p-h}BA$. So we may write $\varphi(Z_0) = B(Z_0 + E(u)^{p-h}AC)$. Let $Z_1 = Z_0 + E(u)^{p-h}AC$. Then $\varphi(Z_1) = BZ_1 + u^{pe(p-h)}C_1$ with $C_1 = -\varphi(AC)$. Note that $pe(p-h) > pe > he$. We can write $BZ_1 + u^{pe(p-h)}C_1 = B(Z_1 + u^{\alpha}AC)$ with $\alpha = pe(p-h) - h$. Set $Z_2 = Z_1 + u^{\alpha}AC$. Then $\varphi(Z_2) = Z_2 + u^{p\alpha}C_2$. Continuing, we see that Z_n converges in $\mathcal{O}_{\mathbf{C}}^{\flat}$ to Z' such that $\varphi(Z') = BZ'$ with $Z' \equiv Z_0 \bmod E(u)^{p-h}$. This settles the bijection of these two sets and completes the proof.

It remains to show that $\varphi_h - 1 : \mathrm{Fil}^h \mathcal{M} \otimes_S A_{\mathrm{crys}} \to \mathcal{M} \otimes_S A_{\mathrm{crys}}$ is surjective and we may assume that $\mathcal{M} = \underline{\mathcal{M}}(\mathfrak{M})$ with $\mathfrak{M}$ killed by p. Note that $\mathcal{M} \otimes_S A_{\mathrm{crys}} = \varphi^*\mathfrak{M} \otimes_{k[\![u]\!]} \mathcal{O}_{\mathbf{C}}^{\flat} + \varphi^*\mathfrak{M} \otimes_{k[\![u]\!]} \mathrm{Fil}^p A_{\mathrm{crys},1}$. By Lemma 6.11 (2), it suffices to show that for

$y = m \otimes a$ with $m \in \varphi^*\mathfrak{M}$ and $a \in \mathrm{Fil}^p A_{\mathrm{crys},1}$ there exists an $x \in \mathrm{Fil}^h \mathcal{M} \otimes_S A_{\mathrm{crys}}$ such that $\varphi_h(x) - x = y$. Since $\varphi_h(a) = 0$ for $a \in \mathrm{Fil}^p A_{\mathrm{crys},1}$, we have $y = -x$ as required. ■

Remark 6.14. If we consider the isomorphism $\eta : T_{\mathfrak{S}}(\mathfrak{M})(h) \to T^h_{\mathfrak{S}}(\mathfrak{M}) \to T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ defined by $x \mapsto \beta^h x \mapsto (\beta \mathfrak{c})^h x = t^h x$, then $\eta : T_{\mathfrak{S}}(\mathfrak{M})(h) \simeq T_S(\underline{\mathcal{M}}(\mathfrak{M}))$ is natural in the following sense: Suppose that $\mathfrak{M} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ has a G_K-action that is semilinear in the G_K-action on A_{inf} and commutes with $\varphi_{\mathfrak{M}}$. Then this G_K-action induces a G_K-action on $\underline{\mathcal{M}}(\mathfrak{M}) \otimes_S A_{\mathrm{crys}}$ compatible with Fil^h and φ. Then both $T_{\mathfrak{S}}(\mathfrak{M})(h)$ and $T_S(\mathcal{M})$ have G_K-actions and η is a G_K-compatible isomorphism.

Regard S as a subring of $K_0[\![u]\!]$. Define $I^+S = S \cap uK_0[\![u]\!]$ and $I^+ = u\mathfrak{S}$. Clearly we have a natural map $q : \mathfrak{M}/I^+ \to \underline{\mathcal{M}}(\mathfrak{M})/I^+S$. By dévissage to the situation that $\mathfrak{M}$ is killed by p, we obtain

Corollary 6.15. *Let* $\mathfrak{M} \in \mathrm{Mod}^{\varphi,h}_{\mathfrak{S},\mathrm{tor\,ét}}$. *Then*

$$\begin{aligned}\mathrm{length}_{W(k)}(\underline{\mathcal{M}}(\mathfrak{M})/I^+S) &= \mathrm{length}_{W(k)}(\mathfrak{M}/u\mathfrak{M}) = \mathrm{length}_{\mathbb{Z}}\, T_S(\underline{\mathcal{M}}(\mathfrak{M}))\\ &= \mathrm{length}_{\mathbb{Z}}\, T_{\mathfrak{S}}(\mathfrak{M}).\end{aligned}$$

Now let us add an extra structure to $\mathrm{Mod}^{\varphi,h}_{S,\mathrm{tor}}$ to make $T_S(\mathcal{M})$ a G_K-representation. Let $\mathrm{Mod}^{\varphi,h,\nabla}_{S,\mathrm{tor}}$ denote the category of objects $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h, \nabla)$ where

(1) $(\mathcal{M}, \mathrm{Fil}^h \mathcal{M}, \varphi_h)$ is an object in $\mathrm{Mod}^{\varphi,h}_{S,\mathrm{tor}}$;

(2) $\nabla : \mathcal{M} \to \mathcal{M}$ is a connection satisfying

  (a) $E\nabla(\mathrm{Fil}^h \mathcal{M}) \subset \mathrm{Fil}^h \mathcal{M}$;

  (b) the following diagram commutes:

$$\begin{tikzcd} \mathrm{Fil}^h \mathcal{M} \arrow[r, "\varphi_h"] \arrow[d, "E(u)\nabla"'] & \mathcal{M} \arrow[d, "c_1\nabla"] \\ \mathrm{Fil}^h \mathcal{M} \arrow[r, "u^{p-1}\varphi_h"] & \mathcal{M} \end{tikzcd} \tag{6.16}$$

Let us explain the relationship between objects in $\mathrm{Mod}^{\varphi,h,\nabla}_{S,\mathrm{tor}}$ and Breuil modules studied by Breuil and Caruso. Let $N_S : S \to S$ be a $W(k)$-linear differentiation such that $N_S(u) = u$. An object $\mathcal{M}$ in $\mathrm{Mod}^{\varphi,h}_{S,\mathrm{tor}}$ is called a *Breuil module* if $\mathcal{M}$ admits a $W(k)$-linear morphism $N : \mathcal{M} \to \mathcal{M}$ such that

(1) for all $s \in S$ and $x \in \mathcal{M}$, $N(sx) = N_S(s)x + sN(x)$;

(2) $E(u)N(\mathrm{Fil}^h \mathcal{M}) \subset \mathrm{Fil}^h \mathcal{M}$;

(3) the following diagram commutes:

$$\begin{tikzcd} \mathrm{Fil}^h \mathcal{M} \arrow[r, "\varphi_h"] \arrow[d, "E(u)N"'] & \mathcal{M} \arrow[d, "c_1 N"] \\ \mathrm{Fil}^h \mathcal{M} \arrow[r, "\varphi_h"] & \mathcal{M} \end{tikzcd} \tag{6.17}$$

Remark 6.18. Breuil and Caruso use the convention $N_S(u) = -u$. In fact, there is almost no difference for the entire theory when using $N_S(u) = u$, except that in formula (6.20) one has to change sign compared to the similar formula in [27, (5.1.1)].

Let $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,N}$ denote the category of Breuil modules. There is a natural functor $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla} \to \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,N}$ defined by $N_{\mathcal{M}} = u\nabla$. It is easy to chase the diagram to see this functor makes sense. So we also call objects in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,\nabla}$ Breuil modules.

Now we can define a G_K-action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$: for any $\sigma \in G_K$ and any $x \otimes a \in \mathcal{M} \otimes_S A_{\mathrm{crys}}$, define

$$\sigma(x \otimes a) = \sum_{i=0}^{\infty} \nabla^i(x) \otimes \gamma_i(\sigma([\underline{\pi}]) - [\underline{\pi}])\sigma(a). \tag{6.19}$$

We can also define a G_K-action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ as in [28, Section 5.1]: for any $\sigma \in G_K$, recall $\underline{\varepsilon}(\sigma) = \frac{\sigma([\underline{\pi}])}{[\underline{\pi}]} \in A_{\mathrm{inf}}$. For any $x \otimes a \in \mathcal{M} \otimes_S A_{\mathrm{crys}}$, define

$$\sigma(x \otimes a) = \sum_{i=0}^{\infty} N^i(x) \otimes \gamma_i(\log(\underline{\varepsilon}(\sigma)))\sigma(a), \tag{6.20}$$

where $\gamma_i(x) = x^i/i!$ is the standard divided power. We claim that (6.19) and (6.20) are the same formula. Let us postpone the proof to Section 8.1 as it is just a long combinatorial calculation.

Note that if $\sigma \in G_\infty$, then $\log(\underline{\varepsilon}(\sigma)) = 0$ and $\sigma(x \otimes a) = x \otimes \sigma(a)$. Thus the G_K-action defined above (if well-defined) is compatible with the natural G_∞-action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$.

Lemma 6.21. *The above action is a well-defined A_{crys}-semilinear G_K-action on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$, compatible with $\mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}})$ and φ_h.*

Proof. The proof of [28, Section 5.1] essentially applies here. It is standard to check that (6.19) is well-defined; it is A_{crys}-semilinear on $\mathcal{M} \otimes_S A_{\mathrm{crys}}$ and compatible with the G_K-action on A_{crys}; and G_∞ acts on $\mathcal{M} \otimes 1$ trivially. It is clear that $\log(\underline{\varepsilon}(\sigma)) \in \mathrm{Fil}^1 A_{\mathrm{crys}}$. Since $E(u)N(\mathrm{Fil}^h \mathcal{M}) \subset \mathrm{Fil}^h \mathcal{M}$, we see that

$$\sigma(\mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}})) \subset \mathrm{Fil}^h(\mathcal{M} \otimes_S A_{\mathrm{crys}}).$$

The only thing left to check is that φ_h commutes with the G_K-action, which can be reduced to checking the following: Write $a = -\log(\underline{\varepsilon}(\sigma))$ and pick $x \in \mathrm{Fil}^h \mathcal{M}$; then

$$\varphi_h(\gamma_i(a) \otimes N^i(x)) = \gamma_i(a) \otimes N^i(\varphi_h(x)).$$

It is clear that $\varphi(a) = pa$. So $\varphi(\gamma_i(a)) = \gamma_i(a)c_1^{-i}\varphi(E(u)^i)$. So the proof of the above equality is reduced to checking $c_1^{-i}\varphi_h(E(u)^i N^i(x)) = N^i(\varphi_h(x))$, which can be done by induction on i. ∎

Corollary 6.22. *If $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h,N}$ is a Breuil module, then $T_S(\mathcal{M})$ (as a G_∞-representation) extends to a G_K-representation.*

To summarize this section, we return to the situation of Section 5.2 where $\mathcal{M}^i := \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ is proved to admit structures $\mathrm{Fil}^i\,\mathcal{M}^i = \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})$, $\varphi_i : \mathrm{Fil}^i\,\mathcal{M}^i \to \mathcal{M}^i$ and $\nabla : \mathcal{M}^i \to \mathcal{M}^i$. Obviously, our axioms of $\mathrm{Mod}^{\varphi,h,\nabla}_{S,\mathrm{tor}}$ are aimed at describing these structures of $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$.

Definition 6.23. For $i \leq p-2$, we say that $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ is a *Breuil module* if the quadruple

$$\big(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n), \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}), \varphi_i, \nabla\big)$$

constructed in Section 5.2 is an object in $\mathrm{Mod}^{\varphi,i,\nabla}_{S,\mathrm{tor}}$, which is equivalent to the triple

$$\big(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n), \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}), \varphi_i\big)$$

being an object in $\mathrm{Mod}^{\varphi,i}_{S,\mathrm{tor}}$.

Our main theorem shows that $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ together with these structures is indeed a Breuil module when $ei < p-1$.

7. Torsion cohomology and comparison with étale cohomology

In this section, we collect our previous preparations to understand the structures of torsion crystalline cohomology and its relationship to étale cohomology via torsion prismatic cohomology. In the end, we show that if $ei < p-1$ then p^n-th torsion crystalline cohomology $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ has the structure of a torsion Breuil module to compare to $\mathrm{H}^i_{\mathrm{\acute{e}t}}(X_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ via T_S, where $X_{\bar{\eta}}$ is a geometric generic fiber of X.

7.1. Prismatic cohomology and (generalized) Kisin modules

Let (A, I) be any prism. As at the end of Section 5.2, for any $n \geq 1$, we define torsion prismatic cohomology $\mathrm{R}\Gamma_{\Delta}(X_n/A_n) := \mathrm{R}\Gamma_{\Delta}(X/A, \mathcal{O}_{\Delta}/p^n\mathcal{O}_{\Delta}) = \mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. We have $\mathrm{R}\Gamma_{\Delta}(X_n/A_n) \simeq \mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \Delta_{-/A}/p^n) \simeq \mathrm{R}\Gamma_{\mathrm{qSyn}}(X, \Delta_{-/A}) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$.

Warning 7.1. We warn the readers that the notation $\mathrm{R}\Gamma_{\Delta}(X_n/A_n)$ is misleading, as it might suggest that this cohomology theory only depends on the mod p^n reduction of X, which is not true. See [7, Remark 2.4] for a counterexample.

Proposition 7.2. *Assume that (A, I) is transversal and $\varphi : A \to A$ is flat. Then $\mathrm{H}^i_{\Delta}(X_n/A_n)$ has height i.*

Proof. We follow the idea of [9, Corollary 15.5] where it is proved that $\mathrm{H}^i_{\Delta}(X/A)$ has height i. Examining the proof, it suffices to show that $\varphi^*\mathrm{R}\Gamma_{\Delta}(X_n/A_n) \simeq L\eta_I \mathrm{R}\Gamma_{\Delta}(X_n/A_n)$ when $X = \mathrm{Spf}(R)$ is an affine smooth p-adic formal scheme over A/I. By Theorem 15.3 of *loc. cit.*, we have $\varphi^*\mathrm{R}\Gamma_{\Delta}(X/A) \simeq L\eta_I \mathrm{R}\Gamma_{\Delta}(X/A)$. Since $\varphi : A \to A$ is flat, it suffices to show that

$$(L\eta_I \mathrm{R}\Gamma_{\Delta}(X/A)) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} \simeq L\eta_I \big(\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}\big). \tag{7.3}$$

Now we may apply [4, Lemma 5.16] to the above by $g = p^n$ and $f = d$. So we need to check that $\mathrm{H}^*(\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_A A/d)$ has no p^n-torsion. This follows from the Hodge–Tate comparison

$$\mathrm{H}^i(\mathrm{R}\Gamma_{\Delta}(X/A) \otimes^{\mathbb{L}}_A A/I) \simeq \Omega^i_{X/(A/I)}\{i\}. \qquad \blacksquare$$

Corollary 7.4. *For $n \in \mathbb{N} \cup \{\infty\}$, the φ-module $\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ is an object of $\mathrm{Mod}^{\varphi,i}_{\mathfrak{S}}$, i.e., a (generalized) Kisin module of height i and $T_{\mathfrak{S}}(\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)) \simeq \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$.*

Proof. It suffices to prove that $T_{\mathfrak{S}}(\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)) \simeq \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$. Write $\mathfrak{M}^i_n := \mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ and $\mathcal{X} := \operatorname{Spf} \mathcal{O}_{\mathbf{C}} \times_{\operatorname{Spf} \mathcal{O}_K} X$. For $n \neq \infty$, by [9, Theorem 1.8 (4, 5)], we have

$$\begin{aligned}\mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) &\simeq \left(\mathrm{H}^i(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/A_{\mathrm{inf}})/p^n)\left[\frac{1}{E(u)}\right]\right)^{\varphi=1}\\ &= (\mathfrak{M}^i_n \otimes_{\mathfrak{S}} W_n(\mathcal{O}^\flat_{\mathbf{C}})[1/u])^{\varphi=1} = (\mathfrak{M}^i_n \otimes_{\mathfrak{S}} W_n(\mathbf{C}^\flat))^{\varphi=1},\end{aligned}$$

which is just $T_{\mathfrak{S}}(\mathfrak{M}^i_n)$. The case of $n = \infty$ easily follows by taking inverse limits. $\blacksquare$

Remark 7.5. The G_∞-action on $T_{\mathfrak{S}}(\mathfrak{M}^i_n)$ discussed in Section 6.2 naturally extends to a G_K-action by the isomorphism $\mathfrak{M}^i_n \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \simeq \mathrm{H}^i_{\Delta}(\mathcal{X}/A_{\mathrm{inf}})$, which admits a natural G_K-action that commutes with φ. In this way $T_{\mathfrak{S}}(\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)) \simeq \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism of G_K-actions.

Let $X_k := X \times_{\operatorname{Spf}(\mathcal{O}_K)} \operatorname{Spf}(k)$ be the closed fiber of X.

Lemma 7.6. *If $\mathrm{length}_{W(k)}\, \mathrm{H}^i_{\mathrm{crys}}(X_k/W_n(k)) = \mathrm{length}_{\mathbb{Z}}\, \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ then $\mathfrak{M}^j_n$ has no u-torsion for $j = i, i+1$.*

Proof. We claim that $\mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) \otimes^{\mathbb{L}}_{\mathfrak{S}} W(k) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_k/W_n(k))$. To see this, first note that $(\mathfrak{S}, E) \to (W(k), p)$ taken mod u is a map of prisms. So [9, Theorem 1.8 (5)] proves that $\mathrm{R}\Gamma_{\Delta}(X/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} W(k) \simeq \mathrm{R}\Gamma_{\Delta}(X_k/W(k))$. Then Theorem 1.8 (1) of *loc. cit.* shows that $\mathrm{R}\Gamma_{\Delta}(X/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} W(k) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_k/W(k))$, and the claim follows by taking $\otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ on both sides.

The claim immediately yields the exact sequence

$$0 \to \mathfrak{M}^i_n/u\mathfrak{M}^i_n \to \mathrm{H}^i_{\mathrm{crys}}(X_k/W_n(k)) \to \mathfrak{M}^{i+1}_n[u] \to 0. \tag{7.7}$$

So $\mathrm{length}_{W(k)}(\mathfrak{M}^i_n/u\mathfrak{M}^i_n) \le \mathrm{length}_{W(k)}\, \mathrm{H}^i_{\mathrm{crys}}(X_k/W_n(k))$. On the other hand, consider the exact sequence in Lemma 6.2 with $\mathfrak{M} := \mathfrak{M}^i_n$,

$$0 \to \mathfrak{M}[u^\infty] \to \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}[u^\infty] \to 0.$$

Write $\mathfrak{M}^{\text{ét}} := \mathfrak{M}/\mathfrak{M}[u^\infty]$. Since $\mathfrak{M}^{\text{ét}}$ has no u-torsion, the above exact sequence remains exact modulo u. So we have $\mathrm{length}_{W(k)}(\mathfrak{M}^{\text{ét}}/u\mathfrak{M}^{\text{ét}}) \le \mathrm{length}_{W(k)}(\mathfrak{M}/u\mathfrak{M})$ and equality holds only when $\mathfrak{M}[u^\infty] = \{0\}$. Since $T_{\mathfrak{S}}(\mathfrak{M}) = T_{\mathfrak{S}}(\mathfrak{M}^{\text{ét}})$, and $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ by Corollary 7.4, Corollary 6.15 proves the following inequalities:

$$\mathrm{length}_{\mathbb{Z}}\, \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) = \mathrm{length}_{W(k)}(\mathfrak{M}^{\text{ét}}/u\mathfrak{M}^{\text{ét}}) \le \mathrm{length}_{W(k)}(\mathfrak{M}/u\mathfrak{M}).$$

Combining this with the exact sequence (7.7), we conclude that

$$\mathrm{length}_{\mathbb{Z}}\, \mathrm{H}^i_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) \leq \mathrm{length}_{W(k)}\, \mathrm{H}^i_{\mathrm{crys}}(X_k/W_n(k)),$$

and equality holds only if all the above inequalities become equalities and $\mathfrak{M}^i_n$ and $\mathfrak{M}^{i+1}_n$ have no u-torsion. ■

7.2. Nygaard filtration and Breuil–Kisin filtration

By Corollary 7.4, $\mathfrak{M}^i_n := \mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ is a Kisin module of height i. Then $\varphi^*\mathfrak{M}^i_n \simeq \mathrm{H}^i_{\mathrm{qSyn}}(X, \Delta^{(1)}_{-/\mathfrak{S}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z})$ admits two filtrations: the Breuil–Kisin filtration defined in (6.8) and the Nygaard filtration $\mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}^i_{\mathrm{N}}\, \Delta^{(1)}_{-/\mathfrak{S}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z})$. The aim of this subsection is to compare these two filtrations.

This theme can be put in a more general setting for a bounded prism (A, I). Recall that in [9, Section 15] the authors studied $\Delta_{-/A}$ and $\Delta^{(1)}_{-/A} := A \mathbin{\widehat{\otimes}}^{\mathbb{L}}_{\varphi,A} \Delta_{-/A}$ as sheaves on $\mathrm{qSyn}_{A/I}$. Also constructed in *loc. cit.* is the so-called Nygaard filtration $\mathrm{Fil}^j_{\mathrm{N}}\, \Delta^{(1)}_{-/A}$, also discussed in Section 4.2. For any $n \in \mathbb{N} \cup \{\infty\}$, set $\Delta^{(1)}_n := \Delta^{(1)}_{-/A} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ and $\mathrm{Fil}^j_{\mathrm{N}}\, \Delta^{(1)}_n := \mathrm{Fil}^j_{\mathrm{N}}\, \Delta^{(1)}_{-/A} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. Here and below, we adopt the convention that $n = \infty$ means we do not perform any base change.

Lemma 7.8. *Let (A, I) be a bounded prism. Let X be a smooth (p-adic) formal scheme over $\mathrm{Spf}(A/I)$ of relative dimension n.*

(1) *The Nygaard filtration $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^\bullet_{\mathrm{N}})$ on $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \Delta^{(1)}_{-/A})$ is complete.*

(2) *The natural map*

$$\mathrm{Fil}^i_{\mathrm{N}} \otimes_A I^j \to \mathrm{Fil}^{i+j}_{\mathrm{N}}$$

of quasisyntomic sheaves induces a morphism

$$\mathrm{H}^l(X_{\mathrm{qSyn}}, \mathrm{Fil}^i_{\mathrm{N}}) \otimes_A I^j \to \mathrm{H}^l(X_{\mathrm{qSyn}}, \mathrm{Fil}^{i+j}_{\mathrm{N}})$$

which is an isomorphism when $l \leq i$ and injective when $l = i+1$. When $i \geq n$ this map induces an isomorphism

$$\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^n_{\mathrm{N}}) \otimes_A I^j \cong \mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^{n+j}_{\mathrm{N}}).$$

(3) *The natural map*

$$\varphi : \mathrm{Fil}^i_{\mathrm{N}} \to \Delta_{-/A} \otimes_A I^i$$

induces a map on cohomology

$$\mathrm{H}^l(X_{\mathrm{qSyn}}, \mathrm{Fil}^i_{\mathrm{N}}) \to \mathrm{H}^l(X_{\mathrm{qSyn}}, \Delta_{-/A}) \otimes_A I^i$$

which is an isomorphism when $l \leq i$ and injective when $l = i+1$.

Moreover, their derived mod p^m counterparts hold true as well.

We thank Bhargav for pointing out statement (3) above, which we did not realize can be proved so easily. This significantly simplifies an earlier draft.

Proof of Lemma 7.8. (1) follows from (2). Indeed, (2) implies the Nygaard filtration on $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^i_{\mathrm{N}})$ is simply the I-adic filtration, hence it is complete.

(2) follows from the following exact triangle of quasisyntomic sheaves:

$$\mathrm{Fil}^i_{\mathrm{N}} \otimes_A I \to \mathrm{Fil}^{i+1}_{\mathrm{N}} \to \mathrm{Fil}^{i+1}_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/(A/I)}.$$

Observe that

$$\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^l_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/(A/I)}) \cong \mathrm{R}\Gamma(X, \mathrm{Fil}^l_{\mathrm{H}} \mathrm{dR}^{\wedge}_{-/(A/I)})$$

lives in $D^{\geq l}(A/I)$, and vanishes when $l > n$. An easy induction gives what we want.

As for (3), we look at the map of filtered complexes

$$\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^i_{\mathrm{N}}) \xrightarrow{\varphi} \mathrm{R}\Gamma(X_{\mathrm{qSyn}}, {\mathbb{\Delta}}_{-/A} \otimes_A I^i)$$

where the former is equipped with the Nygaard filtration $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^{i+*}_{\mathrm{N}})$ and the latter is equipped with the I-adic filtration $\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, {\mathbb{\Delta}}_{-/A} \otimes_A I^{i+*})$. Notice that both filtrations are complete. Now [9, Theorem 15.2 (2)] implies that the cone of the $(i+*)$-th graded piece lives in $D^{>(i+*)}(A/I)$. Hence we conclude that the cone of φ lives in $D^{>i}(A)$. Therefore the induced maps of degree at most i cohomology groups are isomorphisms, and the induced map in degree $i+1$ is injective.

Their derived mod p^m counterparts are proved in exactly the same way. ■

Now let us return to the situation of the Breuil–Kisin prism $A = \mathfrak{S}$. Recall that

$${\mathbb{\Delta}}^{(1)}_n := {\mathbb{\Delta}}^{(1)}_{-/\mathfrak{S}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} \quad \text{and} \quad \mathrm{Fil}^i_{\mathrm{N}} {\mathbb{\Delta}}^{(1)}_n := \mathrm{Fil}^i_{\mathrm{N}} {\mathbb{\Delta}}^{(1)}_{-/\mathfrak{S}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}.$$

Also, recall that $\mathfrak{M}^i_n := \mathrm{H}^i_{\mathbb{\Delta}}(X_n/\mathfrak{S}_n)$ and that the Breuil–Kisin filtration on $\varphi^*\mathfrak{M}^i_n \cong \mathrm{H}^i_{\mathrm{qSyn}}(X, {\mathbb{\Delta}}^{(1)}_n)$ is defined as the image of $\psi : \mathfrak{M}^i_n \to \varphi^*\mathfrak{M}^i_n$.

Corollary 7.9. *For any $i \in \mathbb{N}$ and any $n \in \mathbb{N} \cup \{\infty\}$, there is a functorial commutative diagram*

$$\begin{tikzcd} \mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}^i_{\mathrm{N}} {\mathbb{\Delta}}^{(1)}_n) \arrow[rr, "\varphi_i"] \arrow[dr] & & \mathfrak{M}^i_n \arrow[dl, "\psi"] \\ & \varphi^*\mathfrak{M}^i_n & \end{tikzcd}$$

with φ_i an isomorphism.

Proof. First let us justify the existence of the functorial commutative diagram. We may work with affine formal schemes $Y = \mathrm{Spf}(R)$. In this case, by the proof of [9, Theorem 15.3 and Corollary 15.5], we see that ψ is constructed by the (right part of the) following diagram:

$$\begin{tikzcd}
\tau^{\leq i}\mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, \mathrm{Fil}^i_{\mathrm{N}} {\Delta}^{(1)}) \arrow[rr, dashed] \arrow[d] & & \tau^{\leq i}\mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, {\Delta}) \otimes (E)^i \arrow[dl, "\psi"] \arrow[d] \\
\tau^{\leq i}\mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, {\Delta}^{(1)}) \arrow[r, "\sim"] \arrow[rr, bend right, "\varphi"'] & \tau^{\leq i} L\eta_E \mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, {\Delta}) \arrow[r] & \tau^{\leq i}\mathrm{R}\Gamma_{\mathrm{qSyn}}(Y, {\Delta})
\end{tikzcd}$$

Here the top row is the ($\leq i$ truncation of the) morphism

$$\mathrm{R}\Gamma(X_{\mathrm{qSyn}}, \mathrm{Fil}^i_{\mathrm{N}}) \xrightarrow{\varphi} \mathrm{R}\Gamma(X_{\mathrm{qSyn}}, {\Delta}_{-/A} \otimes_A I^i)$$

appearing in Lemma 7.8. Taking the derived complex mod p^n gives the desired functorial commutative diagram. By Lemma 7.8 (3) we know that φ_i is an isomorphism. ■

Remark 7.10. In the context of filtered derived infinity categories, a filtration is nothing but an arrow. Hence one could define two "quasi-filtrations":[3] one being the Breuil–Kisin quasi-filtration $\mathfrak{M}^i_n \xrightarrow{\psi} \varphi^*\mathfrak{M}^i_n$; another being the i-th Nygaard quasi-filtration $\mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}^i_{\mathrm{N}} {\Delta}^{(1)}_n) \to \varphi^*\mathfrak{M}^i_n$. Then the above says that these two quasi-filtrations are canonically identified via φ_i.

Let us consider the map

$$\iota^{i,j}_n : \mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}^j_{\mathrm{N}} {\Delta}^{(1)}_n) \to \mathrm{Fil}^j_{\mathrm{BK}} \mathrm{H}^i_{\mathrm{qSyn}}(X, {\Delta}^{(1)}_n)$$

for any pair of natural numbers (i, j) and any $n \in \mathbb{N} \cup \{\infty\}$. We have the following information about the image of $\iota^{i,j}_n$ when $i \leq j$.

Corollary 7.11. *Let $i \leq j$. Then we have an identification*

$$\mathrm{Im}(\iota^{i,j}_n) \cong \mathrm{Im}(\psi : \mathfrak{M}^i_n \to \varphi^*\mathfrak{M}^i_n) \cdot E^{j-i}.$$

In particular, defining $\widetilde{\mathfrak{M}^i_n} := \mathfrak{M}^i_n/[u^\infty]$ and $\widetilde{\varphi^*\mathfrak{M}^i_n} := \varphi^*\mathfrak{M}^i_n/[u^\infty]$, we have an identification

$$\mathrm{Im}(\tilde{\iota}^{i,j}_n : \mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}^i_{\mathrm{N}} {\Delta}^{(1)}_n) \to \mathrm{Fil}^i_{\mathrm{BK}} \widetilde{\varphi^*\mathfrak{M}^i_n}) \cong \{x \in \widetilde{\varphi^*\mathfrak{M}} \mid (1 \otimes \varphi)(x) \in E(u)^j \widetilde{\mathfrak{M}^i_n}\}.$$

Proof. The first statement follows from combining Lemma 7.8 (2) and Corollary 7.9. The second statement follows from the first and the fact that $\mathfrak{M}^i_n$ has height i. ■

Below we make some initial investigations of what happens without assuming $i \leq j$.

Proposition 7.12. *Let $A = \mathfrak{S}$ be the Breuil–Kisin prism. For any triple (i, j, n), the kernel and cokernel of $\iota^{i,j}_n$ above are finite.*

[3] This terminology was suggested by S. Mondal.

Proof. Note that the kernel and cokernel of $\iota_n^{i,j}$ are finitely generated modules over $\mathfrak{S}/(p^n)$. We have the containment

$$E(u)^j \cdot \mathbb{\Delta}^{(1)} \subset \mathrm{Fil}_N^j \mathbb{\Delta}^{(1)} \subset \mathbb{\Delta}^{(1)}$$

of sheaves on $\mathrm{qSyn}_{A/I}$. This shows that the map $\iota_n^{i,j}$ admits a section up to multiplication by $E(u)^j$, therefore the kernel and cokernel of $\iota_n^{i,j}$ are annihilated by $E(u)^j$. If $n \in \mathbb{N}$, the kernel and cokernel of $\iota_n^{i,j}$ are finitely generated modules over $\mathfrak{S}/(p^n, E(u)^j)$, hence finite.

If $n = \infty$, denote the map by $\iota^{i,j}$; then we make the following

Claim 7.13. *The map* $\iota^{i,j} : \mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}_N^j \mathbb{\Delta}^{(1)})[1/p] \to \mathrm{Fil}^j_{\mathrm{BK}} \varphi^* \mathfrak{M}^i[1/p]$ *is an isomorphism.*

Granting this claim, the kernel and cokernel of $\iota^{i,j}$ are finitely generated modules over $\mathfrak{S}/(E(u)^j)$ annihilated by a power of p, hence finite. ■

Proof of Claim 7.13. First let us show injectivity, which is the same as the injectivity of

$$\mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}_N^j \mathbb{\Delta}^{(1)})[1/p] \to \mathrm{H}^i_{\mathrm{qSyn}}(X, \mathbb{\Delta}^{(1)})[1/p].$$

To this end, we use the filtration $\mathrm{Fil}^{i,j}$ discussed in Section 4.2. We will prove a slightly stronger statement: the maps

$$\mathrm{H}^m_{\mathrm{qSyn}}(X, \mathrm{Fil}^{i,j} \mathbb{\Delta}^{(1)})[1/p] \to \mathrm{H}^m_{\mathrm{qSyn}}(X, \mathrm{Fil}^{i,0} \mathbb{\Delta}^{(1)})[1/p]$$

are injective for all $i \geq 0$. The case of $i \geq j$ is trivial due to Proposition 4.10 (2). For the other i, we use descending induction on i. By the Five Lemma and Proposition 4.10 (3), it suffices to show that the maps

$$\mathrm{H}^m(X, \mathrm{Fil}_{\mathrm{H}}^{j-i} \mathrm{dR}_{X/\mathcal{O}_K})[1/p] \to \mathrm{H}^m(X, \mathrm{dR}_{X/\mathcal{O}_K})[1/p]$$

are injective. This injectivity is equivalent to the degeneration of the Hodge-to-de Rham spectral sequence for the rigid space X_K, which is a result due to Scholze [33, Theorem 1.8].

Next we show surjectivity by induction on j, the case of $j = 0$ being trivial. All we need to show is that the induced map

$$\mathrm{Coker}\big(\mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}_N^{j+1} \mathbb{\Delta}^{(1)})[1/p] \to \mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{Fil}_N^j \mathbb{\Delta}^{(1)})[1/p]\big) \xrightarrow{\overline{\varphi}} \frac{E(u)^j \mathfrak{M}^i}{E(u)^{j+1}\mathfrak{M}^i}[1/p]$$

is injective. By the injectivity of $\iota^{i,j}[1/p]$ proved in the previous paragraph, we can rewrite the left hand side as $\mathrm{H}^i_{\mathrm{qSyn}}(X, \mathrm{gr}_N^j \mathbb{\Delta}^{(1)})[1/p]$. Recall that $\mathfrak{M}^i[1/p]$ is finite free over $\mathfrak{S}[1/p]$ (see Lemma 6.1 (3)), so the right hand side can be rewritten as $\mathrm{H}^i_{\mathrm{qSyn}}(X, \overline{\mathbb{\Delta}})[1/p]\{j\}$, the j-th Breuil–Kisin twist of the i-th Hodge–Tate cohomology of X_K. By [9, Theorem 15.2], we can identify the left hand side further as the j-th conjugate filtration of the right hand side. Now it follows from the degeneration of the Hodge–Tate spectral sequence [7, Theorem 13.3] that $\overline{\varphi}$ is always injective. ■

Below we exhibit an example illustrating the necessity of the $i \le j$ assumption in Corollary 7.11.

Example 7.14 (see [24, Section 4]). Let K be a ramified quadratic extension of $\mathbb{Q}_p$ and let G be a lift of α_p over $\mathcal{O}_K$. Denote by BG the classifying stack of G. Below we summarize the previous study of various cohomologies of BG as documented in [24, 4.6–4.10], following the notation thereof.

(1) The Breuil–Kisin prismatic cohomology ring of BG is given by

$$\mathrm{H}^*_{\Delta}(BG/\mathfrak{S}) \cong \mathfrak{S}[\widetilde{u}]/(p \cdot \widetilde{u}),$$

where $\widetilde{u}$ has degree 2.

(2) The Hodge–Tate spectral sequence does not degenerate on the E_2 page, but does degenerate on the E_3 page, giving rise to short exact sequences

$$0 \to \mathrm{H}^{i+1}(BG, \textstyle\bigwedge^{i-1} \mathbb{L}_{BG/\mathcal{O}_K}) \simeq \mathbb{F}_p \to \mathrm{H}^{2i}_{\mathrm{HT}}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K/(p) \\ \to \mathrm{H}^i(BG, \textstyle\bigwedge^i \mathbb{L}_{BG/\mathcal{O}_K}) \simeq \mathbb{F}_p \to 0$$

for all $i > 0$.

(3) The Hodge-to-de Rham spectral sequence does not degenerate on the E_1 page, but does degenerate on the E_2 page, giving rise to short exact sequences

$$0 \to \mathrm{H}^{2i-1}(BG, \mathbb{L}_{BG/\mathcal{O}_K}) \simeq \mathbb{F}_p \to \mathrm{H}^{2i}_{\mathrm{dR}}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K/(p) \\ \to \mathrm{H}^{2i}(BG, \mathcal{O}_{BG}) \simeq \mathbb{F}_p \to 0$$

for all $i > 0$.

By [9, Theorem 15.2], we have the commutative diagram

$$\begin{tikzcd}
\mathrm{R\Gamma}_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta^{(1)}) \arrow[rr, "\varphi"] \arrow[d] & & \mathrm{R\Gamma}_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta) \arrow[d] \\
\mathrm{R\Gamma}_{\mathrm{dR}}(BG/\mathcal{O}_K) \arrow[r] & \mathrm{R\Gamma}(BG, \mathcal{O}_{BG}) \arrow[r] & \mathrm{R\Gamma}_{\mathrm{HT}}(BG/\mathcal{O}_K)
\end{tikzcd}$$

where φ is the Frobenius on prismatic cohomology, the vertical maps are derived modulo $E(u)$ reductions, the two arrows on the bottom row are natural arrows appearing in the Hodge-to-de Rham and Hodge–Tate spectral sequences respectively. Looking at the degree 2 cohomology together with (2) and (3) above, we see that φ on $\mathrm{H}^2_{\Delta}(BG/\mathfrak{S})$ is given by, up to a unit in $\mathfrak{S}/p$, multiplication by $u \in \mathfrak{S}/p$. Since φ is a map of $\mathbb{E}_\infty$-algebras, using (1) we see that φ on $\mathrm{H}^4_{\Delta}(BG/\mathfrak{S})$ is given by, up to a unit in $\mathfrak{S}/p$, multiplication by $u^2 = E(u) \in \mathfrak{S}/p$. In particular, we see that $\mathrm{Fil}^1_{\mathrm{BK}} \mathrm{H}^4_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta^{(1)}) = \mathrm{H}^4_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta^{(1)})$ is the whole cohomology group.

On the other hand, we claim that the map

$$\mathrm{H}^4_{\mathrm{qSyn}}(BG/\mathfrak{S}, \mathrm{Fil}^1_{\mathrm{N}} \Delta^{(1)}) \to \mathrm{H}^4_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta^{(1)})$$

is not surjective. Indeed, we have a long exact sequence coming from the exact triangle $\mathrm{Fil}^1_{\mathrm{N}} \Delta^{(1)} \to \Delta^{(1)} \to \mathcal{O}_{BG}$ with the second arrow being the composition of taking the derived complex modulo $E(u)$ followed by projection modulo the first Hodge filtration. Hence (3) above shows that the cokernel is exactly of length 1. This shows that BG is a smooth proper stack counterexample for $(i, j, n) = (4, 1, \infty)$. Since all these cohomology groups are p-torsion, we see that this also provides a stacky counterexample for $(i, j, n) = (3, 1, 1)$.

Finally, let us use an approximation of BG to get a smooth proper scheme counterexample. By [24, Section 4.3] there is a smooth projective fourfold X over $\mathcal{O}_K$ together with a map $f : X \to BG$ such that the induced pullback map of Hodge cohomology is injective when the total degree is no larger than 4. By functoriality of the formation of Breuil–Kisin filtrations, we know that

$$\mathrm{Im}\big(f^* : \mathrm{H}^4_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta^{(1)}) \to \mathrm{H}^4_{\mathrm{qSyn}}(X/\mathfrak{S}, \Delta^{(1)})\big) \subset \mathrm{Fil}^1_{\mathrm{BK}} \mathrm{H}^4_{\mathrm{qSyn}}(X/\mathfrak{S}, \Delta^{(1)}).$$

Lastly, we claim $f^*(\widetilde{u}^2) \in \mathrm{H}^4_{\mathrm{qSyn}}(X/\mathfrak{S}, \Delta^{(1)})$ is not in the image of $\mathrm{H}^4_{\mathrm{qSyn}}(X/\mathfrak{S}, \mathrm{Fil}^1_{\mathrm{N}} \Delta^{(1)})$. To see this, it suffices to compare two exact sequences:

$$\begin{tikzcd}
\mathrm{H}^4_{\mathrm{qSyn}}(BG/\mathfrak{S}, \mathrm{Fil}^1_{\mathrm{N}} \Delta^{(1)}) \arrow[r] \arrow[d] & \mathrm{H}^4_{\mathrm{qSyn}}(BG/\mathfrak{S}, \Delta^{(1)}) \arrow[r] \arrow[d] & \mathrm{H}^4_{\mathrm{qSyn}}(BG, \mathcal{O}_{BG}) \arrow[d, "f^*"] \\
\mathrm{H}^4_{\mathrm{qSyn}}(X/\mathfrak{S}, \mathrm{Fil}^1_{\mathrm{N}} \Delta^{(1)}) \arrow[r] & \mathrm{H}^4_{\mathrm{qSyn}}(X/\mathfrak{S}, \Delta^{(1)}) \arrow[r] & \mathrm{H}^4_{\mathrm{qSyn}}(X, \mathcal{O}_X)
\end{tikzcd}$$

and invoke the fact that f^* is injective by our choice of X. This gives us a smooth projective fourfold over $\mathcal{O}_K$, violating the conclusion of Corollary 7.11 for $(i, j, n) = (4, 1, \infty)$ or $(i, j, n) = (3, 1, 1)$.

7.3. Torsion crystalline cohomology

Now we are ready to discuss the structure of $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ via prismatic cohomology. First, we provide an application of the comparison $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} S \cong \mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{X}/S)$, which concerns the module structure of the cohomology of the latter. We need some preparations.

Lemma 7.15. *The rings S/p^n are coherent for all $n \in \mathbb{N}$.*

We do not know if the ring S itself is coherent.

Proof of Lemma 7.15. We use induction on n. The base case $n = 1$: since S is given by p-completely adjoining the divided powers of the Eisenstein polynomial $E(u)$ to $\mathfrak{S}$, we see that S/p is obtained by adjoining the divided powers of $E(u) \equiv u^e$ to $\mathfrak{S}/p = k[\![u]\!]$. It is well-known that the result is $S/p \cong k[u]/u^{pe} \otimes_k k[u_1, u_2, \ldots]/(u_i^p)$ where u_i is the image of the p^i-th divided powers of $E(u)$. One checks that this explicit algebra is coherent by noting that any finitely generated ideal is generated by polynomials involving only finitely many variables.

Now we do induction, which largely relies on [7, Lemma 3.26]. Indeed, the cited lemma reduces our task to showing that the ideal $(p^n)/(p^{n+1})$ in S/p^{n+1}, when viewed as an S/p-module, is finitely presented. But in fact S is p-torsionfree, hence the ideal $(p^n)/(p^{n+1})$ is free when viewed as an S/p-module with generator p^n. ∎

Lemma 7.16. *Suppose that $C^\bullet$ is a perfect $\mathfrak{S}_n$-complex. Then there exists an exact sequence of S-modules*

$$0 \to \mathrm{H}^i(C^\bullet) \otimes_{\mathfrak{S}} S \to \mathrm{H}^i(C^\bullet \otimes^{\mathbb{L}}_{\mathfrak{S}} S) \to \mathrm{Tor}_1^{\mathfrak{S}}(\mathrm{H}^{i+1}(C^\bullet), S) \to 0.$$

In particular, S has Tor-amplitude 1 over $\mathfrak{S}$ and the functor $M \mapsto \mathrm{Tor}_1^{\mathfrak{S}}(M, S)$ is left exact.

Proof. For the first claim, see [12, before the proof of Theorem 5.4] and replace A_{inf} (resp. A_{crys}) there by $\mathfrak{S}$ (resp. S). The fact that S has Tor-amplitude 1 over $\mathfrak{S}$ follows from the Auslander–Buchsbaum formula and torsionfreeness of S. ∎

Proposition 7.17. *Let M be a finitely generated Kisin module. Then $\mathrm{Tor}_1^{\mathfrak{S}}(M, \varphi_* S)$ is a finitely presented S-module.*

Proof. Denote $N := M[u^\infty]$, which is the maximal finite length $\mathfrak{S}$-submodule inside M.

We first show $\mathrm{Tor}_1^{\mathfrak{S}}(N, \varphi_* S) \to \mathrm{Tor}_1^{\mathfrak{S}}(M, \varphi_* S)$ is an isomorphism. Since S has Tor-amplitude 1 over $\mathfrak{S}$ by Lemma 7.16, it suffices to show the vanishing of $\mathrm{Tor}_1^{\mathfrak{S}}(M/N, \varphi_* S)$. Noting that M/N is an étale Kisin module, we have a sequence

$$0 \to (M/N)_{\mathrm{tor}} \to M/N \to (M/N)_{\mathrm{tf}} \to 0,$$

where $(M/N)_{\mathrm{tor}}$ is a successive extension of $k[\![u]\!] = \mathfrak{S}/p$ as M/N is étale, and $(M/N)_{\mathrm{tf}}$ is torsionfree. Next observe that both these structures are preserved under base change along the Frobenius on $\mathfrak{S}$. Therefore it suffices to show $\mathrm{Tor}_1^{\mathfrak{S}}(-, S) = 0$ whenever the input $\mathfrak{S}$-module is $\mathfrak{S}/p$ or torsionfree. In the former case, we apply the fact that S has no p-torsion. In the latter, consider the reflexive hull $M'^{\vee\vee}$ of the input module $M' \subset M'^{\vee\vee}$, which is finite free as $\mathfrak{S}$ is regular Noetherian of dimension 2. Finally, the desired vanishing of $\mathrm{Tor}_1^{\mathfrak{S}}(M', S)$ follows from the left exactness of Tor_1 against S over $\mathfrak{S}$: see Lemma 7.16.

It suffices to show $\mathrm{Tor}_1^{\mathfrak{S}}(N', S)$ is finitely presented for any finite length $\mathfrak{S}$-module, which is the content of the next lemma. ∎

Lemma 7.18. *Let N be a finite length $\mathfrak{S}$-module. Then $N \otimes_{\mathfrak{S}} S$ and $\mathrm{Tor}_1^{\mathfrak{S}}(N, S)$ are finitely presented S-modules.*

Proof. When $N = k \cong \mathfrak{S}/(p, u)$, the statement for $k \otimes_{\mathfrak{S}} S = S/(p, u) = (S/p)/u$ and $\mathrm{Tor}_1^{\mathfrak{S}}(k, S) \cong S/p[u]$ follows from the fact that S/p is coherent (Lemma 7.15) and [34, Tag 05CW (3)].

Next we use induction on the length of N. By considering $N \twoheadrightarrow N/(p,u) \simeq k^{\oplus r}$, we have a short exact sequence $0 \to N' \to N \to k \to 0$, which induces a long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathfrak{S}}(N', S) \to \mathrm{Tor}_1^{\mathfrak{S}}(N, S) \to \mathrm{Tor}_1^{\mathfrak{S}}(k, S) \to N' \otimes_{\mathfrak{S}} S \to N \otimes_{\mathfrak{S}} S \to k \otimes_{\mathfrak{S}} S \to 0.$$

Induction hypotheses imply that all terms except $N \otimes_{\mathfrak{S}} S$ and $\mathrm{Tor}_1^{\mathfrak{S}}(N, S)$ are finitely presented S-modules. Note that the finite length assumption implies all modules are S/p^N-modules for some sufficiently large N. The coherence of S/p^N (Lemma 7.15) and [34, Tag 05CW (3)] show the boundary map $\mathrm{Tor}_1^{\mathfrak{S}}(k, S) \to N' \otimes_{\mathfrak{S}} S$ has finitely presented kernel and cokernel. Now we use [34, Tag 0519] to finish the proof. ■

Proposition 7.19. *Let $\mathcal{X}$ be a smooth proper p-adic formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$*. The S/p^n-module $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/S_n)$ is finitely presented for any integer i and any $n \in \mathbb{N} \cup \{\infty\}$.*

Let us stress again that this already follows from [9, Theorem 5.2].

Proof of Proposition 7.19. The case of finite n follows from Lemma 7.15: the prismatic cohomology complex is a perfect complex, hence the comparison [9, Theorem 5.2] or Theorem 3.5 shows the crystalline cohomology complex is also perfect over the coherent ring S/p^n. Therefore all of its cohomology modules are finitely presented as S/p^n-modules.

Now we turn to the case $n = \infty$. By Lemma 7.16 there is a short exact sequence

$$0 \to \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} S \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}/S) \to \mathrm{Tor}_1^{\mathfrak{S}}(\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S}), \varphi_* S) \to 0.$$

Since the prismatic cohomology complex is perfect and the ring $\mathfrak{S}$ is Noetherian, the term $\mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} S$ is finitely presented. Using [34, Tag 0519] we are reduced to showing that $\mathrm{Tor}_1^{\mathfrak{S}}(\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S}), \varphi_* S)$ is finitely presented. This in turn follows from Proposition 7.17 and the fact that $\mathrm{H}^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})$ is a Kisin module: see Corollary 7.4. ■

Now we turn to the main result of our paper, which concerns the Breuil-module structure of crystalline cohomology. Write $\mathfrak{M}^j_n := \mathrm{H}^j_{\Delta}(X_n/\mathfrak{S}_n)$ and $\mathcal{M}^j_n := \mathrm{H}^j_{\mathrm{crys}}(X_n/S_n)$.

Lemma 7.20. *The sequence*

$$0 \to \mathfrak{M}^i_n/u\mathfrak{M}^i_n \to \mathcal{M}^i_n/I^+S \to \mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}^{i+1}_n, \varphi_* S)/I^+S \to 0$$

is exact.

Proof. From the derived mod p^n version of Theorem 3.11, we deduce that $S \otimes^{\mathbb{L}}_{\varphi,\mathfrak{S}} \mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_n/S_n)$. So Lemma 7.16 yields an exact sequence

$$0 \to S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^i_n \to \mathcal{M}^i_n \to \mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}^{i+1}_n, \varphi_* S) \to 0 \tag{7.21}$$

as φ on $\mathfrak{S}$ is finite flat.

We only need to show that the above exact sequence remains exact after reduction modulo I^+S. To see this, note that $\mathrm{R}\Gamma_{\mathrm{crys}}(X_k/W_n(k)) \simeq \mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) \otimes^{\mathbb{L}}_{\mathfrak{S}} W(k) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_n/S_n) \otimes^{\mathbb{L}}_{S} W(k)$, where in the last identification we use the fact that the Frobenius on $W(k)$ is an isomorphism. Using the exact sequence (7.7), we have the commutative diagram

$$\begin{tikzcd} & S/I^+S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^i_n \arrow[r] \arrow[d, "\wr"] & \mathcal{M}^i_n/I^+S \arrow[r] \arrow[d] & \mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}^{i+1}_n, \varphi_* S_n)/I^+S \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{M}^i_n/u\mathfrak{M}^i_n \arrow[r] & \mathrm{H}^i(X_k/W_n(k)) \arrow[r] & \mathfrak{M}^{i+1}_n[u] \arrow[r] & 0 \end{tikzcd}$$

Since the left column is an isomorphism, we conclude that the top row is left exact as desired. ■

Recall in Definition 6.23, $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ is defined to be a Breuil module if the quadruple

$$\left(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n), \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}), \varphi_i, \nabla\right)$$

constructed in Section 5.2 is an object of $\mathrm{Mod}^{\varphi,i,\nabla}_{S,\mathrm{tor}}$. This condition is equivalent to the triple

$$\left(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n), \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}), \varphi_i\right)$$

being an object of $\mathrm{Mod}^{\varphi,i}_{S,\mathrm{tor}}$.

Theorem 7.22. *Let $n \in \mathbb{N}$ and assume $i \leq p-2$. Then $\mathrm{H}^j_{\Delta}(X_n/\mathfrak{S}_n)$ has no u-torsion for $j = i, i+1$ if and only if $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ is a Breuil module. In that case we have $\underline{\mathcal{M}}(\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)) \simeq \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ inside $\mathrm{Mod}^{\varphi,i}_{S,\mathrm{tor}}$.*

Proof. Write $\mathfrak{M}^j_n := \mathrm{H}^j_{\Delta}(X_n/\mathfrak{S}_n)$. Suppose that it has no u-torsion for $j = i, i+1$. So $\mathfrak{M}^i_n$ is an étale Kisin module of height i by Proposition 7.2. By the discussion of Section 6.3, we know $\mathcal{M}^i_n := \underline{\mathcal{M}}(\mathfrak{M}^i_n)$ is an object of $\mathrm{Mod}^{\varphi,i}_{S,\mathrm{tor}}$. By the derived mod p^n version of Theorem 3.11, we have $S \otimes^{\mathbb{L}}_{\varphi,\mathfrak{S}} \mathrm{R}\Gamma_{\Delta}(X_n/\mathfrak{S}_n) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_n/S_n)$. So Lemma 7.16 yields

$$0 \to S \otimes_{\varphi,\mathfrak{S}} \mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n) \to \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \to \mathrm{Tor}_1^{\mathfrak{S}}(\mathrm{H}^{i+1}_{\Delta}(X_n/\mathfrak{S}_n), \varphi_* S_n) \to 0. \quad (7.23)$$

Our assumption that $\mathfrak{M}^{i+1}_n$ has no u-torsion gives an isomorphism

$$\iota : S \otimes_{\varphi,\mathfrak{S}} \mathrm{H}^j_{\Delta}(X_n/\mathfrak{S}_n) \simeq \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n).$$

Now we claim that ι induces a natural map $\iota^i : \mathrm{Fil}^i\, \underline{\mathcal{M}}(\mathfrak{M}^i_n) \to \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})$ and both the source and target are natural submodules of $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$. In particular, ι^i is an injection. To see this, we note that ι is induced by the natural map $\varphi^*\mathfrak{M}_n^i \to \mathcal{M}_n^i$, which we still denote by ι. By Theorem 4.21, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{i-1}_{\mathrm{qSyn}}(X_n, \Delta^{(1)}_{-/\mathfrak{S}}/\operatorname{Fil}^i_{\mathrm{N}}\Delta^{(1)}_{-/\mathfrak{S}}) \arrow[r, "\alpha"] \arrow[d, "\wr"] & \mathrm{H}^i_{\mathrm{qSyn}}(X_n, \operatorname{Fil}^i_{\mathrm{N}}\Delta^{(1)}_{-/\mathfrak{S}}) \arrow[r, "\beta"] \arrow[d] & \mathrm{H}^i_{\mathrm{qSyn}}(X_n, \Delta^{(1)}_{-/\mathfrak{S}}) \arrow[r] \arrow[d, "\iota"] & \cdots \\
\mathrm{H}^{i-1}_{\mathrm{qSyn}}(X_n, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}/\operatorname{Fil}^i_{\mathrm{H}}\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}) \arrow[r, "\alpha'"] & \mathrm{H}^i_{\mathrm{qSyn}}(X_n, \operatorname{Fil}^i_{\mathrm{H}}\mathrm{dR}^{\wedge}_{-/\mathfrak{S}}) \arrow[r, "\beta'"] & \mathrm{H}^i_{\mathrm{qSyn}}(X_n, \mathrm{dR}^{\wedge}_{-/\mathfrak{S}}) \arrow[r] & \cdots
\end{tikzcd}$$

with both rows being exact. By Theorem 4.21 (4), the left column is an isomorphism. As $\mathfrak{M}_n^i$ is assumed to have no u-torsion, Corollary 7.11 shows that β is an injection. Thus α and hence α' are zero maps. So β' is an injection. Therefore, Theorem 4.1 gives the commutative diagram

$$\begin{tikzcd}
\operatorname{Fil}^i_{\mathrm{BK}}\varphi^*\mathfrak{M}_n^i \arrow[r, hook] \arrow[d] & \varphi^*\mathfrak{M}_n^i \arrow[d, "\iota"] \\
\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \arrow[r, hook] & \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)
\end{tikzcd}$$

Since $\iota : \underline{\mathcal{M}}(\mathfrak{M}_n^i) \xrightarrow{\simeq} \mathcal{M}_n^i = \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ is an isomorphism and $\operatorname{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i)$ is the S-submodule of $\mathcal{M}_n^i$ generated by the image of $\operatorname{Fil}^i \varphi^*\mathfrak{M}_n^i$ and $\operatorname{Fil}^i S \cdot \mathcal{M}_n^i$, we see that $\operatorname{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i) \subset \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})$ via ι. This shows that ι induces an injection $\iota : \operatorname{Fil}^i \underline{\mathcal{M}}(\mathfrak{M}_n^i) \hookrightarrow \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})$.

Next we claim that ι^i is an isomorphism. After a faithfully flat base change along $S_n \to A_{\mathrm{crys},n} := A_{\mathrm{crys}}/p^n$, we are now working with $\mathcal{X} := X_{\mathcal{O}_{\mathbf{C}}}$. We need some facts about the sheaf $\mathbb{Z}_p(h)$ on $\mathcal{X}_{\mathrm{qSyn}}$ defined in [8, Section 7.4]. First by [8, Theorem 10.1], we have $\mathbb{Z}/p^n\mathbb{Z}(h) \simeq \tau^{\leq h}\mathrm{R}\psi_*(\mathbb{Z}/p^n\mathbb{Z}(h))$, where $\psi : (\mathcal{X}_{\mathbf{C}})_{\text{ét}} \to \mathcal{X}_{\text{ét}}$ is the natural map of étale sites. By [1, Theorem F], when $h \leq p-2$ we have

$$\mathbb{Z}_p(h) \simeq \operatorname{fib}(\varphi_h - 1 : \operatorname{Fil}^h_{\mathrm{H}}\mathrm{dR}^{\wedge}_{-/\mathbb{Z}_p} \to \mathrm{dR}^{\wedge}_{-/\mathbb{Z}_p}).$$

Now Proposition 5.4 implies

$$\mathbb{Z}_p(h) \simeq \operatorname{fib}(\varphi_h - 1 : \operatorname{Fil}^h_{\mathrm{H}}\mathrm{dR}^{\wedge}_{-/A_{\mathrm{inf}}} \to \mathrm{dR}^{\wedge}_{-/A_{\mathrm{inf}}}).$$

Since fib commutes with taking the derived mod p^n version, we may apply $\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z}$ to this equation. Finally, by Theorem 4.1, for $i \leq h \leq p-2$, we get the exact sequence

$$\begin{aligned}\cdots \to \mathrm{H}^{i-1}_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) \to \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z}(h)) \to \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[h]}_{\mathrm{crys}}) \\ \xrightarrow{\varphi_h - 1} \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}). \qquad (7.24)\end{aligned}$$

By (7.24) and Proposition 6.12, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_S(\underline{\mathcal{M}}(\mathfrak{M}_n^i)) \arrow[r] \arrow[d, "\alpha"] & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^i\, \underline{\mathcal{M}}(\mathfrak{M}_n^i) \arrow[r] \arrow[d, "1\otimes\iota^i"] & A_{\mathrm{crys}} \otimes_S \underline{\mathcal{M}}(\mathfrak{M}_n^i) \arrow[r] & 0 \\
& \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(i) \arrow[r, "s"] & \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}}) \arrow[r] & \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) &
\end{tikzcd}$$

with both rows being exact. Since $1 \otimes \iota^i$ is an injection, so is α. Then α must be an isomorphism because $T_S(\underline{\mathcal{M}}(\mathfrak{M}_n^i)) \simeq T_{\mathfrak{S}}(\mathfrak{M}_n^i)(i) \simeq \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(i)$ due to Proposition 6.12 and [9, Theorem 1.8 (4)]. Therefore s is also injective. Now by the snake lemma, $\mathrm{coker}(1 \otimes \iota) = 0$ as required.

Conversely, assume that $\mathcal{M}_n^i := \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ is an object in $\mathrm{Mod}^{\varphi,i}_{S,\mathrm{tor}}$ with $\mathrm{Fil}^i\, \mathcal{M}_n^i = \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})$. As before, we consider the base change $\mathcal{X} := X_{\mathcal{O}_{\mathbf{C}}}$ and we still have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_S(\mathcal{M}_n^i) \arrow[r] \arrow[d, "\alpha"] & A_{\mathrm{crys}} \otimes_S \mathrm{Fil}^i\, \mathcal{M}_n^i \arrow[r] \arrow[d, "\wr"] & A_{\mathrm{crys}} \otimes_S \mathcal{M}_n^i \arrow[r] & 0 \\
& \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z})(i) \arrow[r, "s"] & \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}, \mathcal{I}^{[i]}_{\mathrm{crys}}) \arrow[r] & \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n}) &
\end{tikzcd}$$

The difference here is that the middle column is now an isomorphism, whereas the first column α is not known to be an isomorphism.

First it is easy to see that α is an injection by chasing the diagram. Now by Corollary 6.15, we have

$$\mathrm{length}_{W(k)}(\mathcal{M}_n^i/I^+S) = \mathrm{length}_{\mathbb{Z}}\, T_S(\mathcal{M}_n^i) \leq \mathrm{length}_{\mathbb{Z}}\, \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z}).$$

On the other hand, by the proof of Lemma 7.6 and Lemma 7.20, we see that

$$\mathrm{length}_{\mathbb{Z}}\, \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z}) \leq \mathrm{length}_{W(k)}(\mathfrak{M}_n^i/u\mathfrak{M}_n^i) \leq \mathrm{length}_{W(k)}(\mathcal{M}_n^i/I^+S).$$

Combining the above two inequalities, we see that

$$\mathrm{length}_{\mathbb{Z}}\, \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\mathbf{C}}, \mathbb{Z}/p^n\mathbb{Z}) = \mathrm{length}_{W(k)}(\mathfrak{M}_n^i/u\mathfrak{M}_n^i) = \mathrm{length}_{W(k)}(\mathcal{M}_n^i/I^+S).$$

Now the proof of Lemma 7.6 implies that $\mathfrak{M}_n^i$ has no u-torsion. By the length equality, the injection $\mathfrak{M}_n^i/u\mathfrak{M}_n^i \hookrightarrow \mathcal{M}_n^i/I^+S$ in Lemma 7.20 is in fact an isomorphism. and hence $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_* S)/I^+S = 0$. It is easy to see that $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_* S)$ is a finitely generated S-module, and applying Nakayama's lemma yields $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}_n^{i+1}, \varphi_* S) = 0$. Therefore $\mathfrak{M}_n^{i+1}$ has no u-torsion by the following claim.

Claim. *If $\mathfrak{M}$ is a p^n-torsion $\mathfrak{S}$-module and $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, \varphi_* S) = 0$ then $\mathfrak{M}$ has no u-torsion.*

To prove this, we first note that $\mathrm{Tor}_1^{\mathfrak{S}}(-, \varphi_* S)$ is a left exact functor by Lemma 7.16. Secondly, note that $\mathfrak{M}$ has no u-torsion if and only if it has no (u, p)-torsion. Let $\mathfrak{M}' \subset \mathfrak{M}$ be the (u, p)-torsion submodule in $\mathfrak{M}$. The above discussion implies that

$\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}', \varphi_* S) = 0$. Now by definition, we have $\mathfrak{M}' \cong \bigoplus_\Lambda k$ as an $\mathfrak{S}$-module, where Λ is an indexing set. One computes directly that

$$\begin{aligned}\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}', \varphi_* S) &= \bigoplus_\Lambda \mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{S}/(p,u), \varphi_* S) = \bigoplus_\Lambda \mathrm{Tor}^1_{\mathfrak{S}}(\mathfrak{S}/(p,u^p), S) \\ &= \bigoplus_\Lambda \ker(S/p \xrightarrow{\cdot u^p} S/p).\end{aligned}$$

Since $\ker(S/p \xrightarrow{\cdot u^p} S/p)$ is nonzero ($u^{pe} = 0$ in S/p), the above computation implies $\Lambda = \emptyset$, as claimed. ∎

Corollary 7.25. *If $ei < p-1$ then $\mathrm{H}^j_{\Delta}(X_n/\mathfrak{S}_n)$ has no u-torsion for $j = i, i+1$, and $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$ is a Breuil module.*

Proof. By Lemma 6.5 and Proposition 7.2, we know that $\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ has no u-torsion. To show that $\mathrm{H}^{i+1}_{\Delta}(X_n/\mathfrak{S}_n)$ has no u-torsion, we first consider the case $n = 1$. The main theorem of [13] shows that $\mathrm{H}^i_{\mathrm{crys}}(X_1/S_1)$ is a Breuil module when $n = 1$ and $ei < p-1$. Then Theorem 7.22 shows that $\mathrm{H}^{i+1}_{\Delta}(X_1/\mathfrak{S}_1)$ has no u-torsion.

Let us prove by induction that $\mathfrak{M}_n^{i+1} := \mathrm{H}^{i+1}_{\Delta}(X_n/\mathfrak{S}_n)$ has no u-torsion. We use the long exact sequence relating various $\mathfrak{M}_n^{i+1} := \mathrm{H}^{i+1}_{\Delta}(X_n/\mathfrak{S}_n)$:

$$\cdots \to \mathfrak{M}^i_{n-1} \xrightarrow{f} \mathfrak{M}^{i+1}_1 \to \mathfrak{M}^{i+1}_n \to \mathfrak{M}^{i+1}_{n-1} \to \cdots.$$

By induction, we may assume that $\mathfrak{M}^{i+1}_{n-1}$ has no u-torsion. It suffices to prove that $\mathfrak{M}^{i+1}_1/f(\mathfrak{M}^i_{n-1})$ has no u-torsion. To that end, write $\mathfrak{N} := f(\mathfrak{M}^i_{n-1})$ which has height i, $\mathfrak{M} := \mathfrak{M}^{i+1}_1$ which has height $i+1$, and $\mathfrak{L} := \mathfrak{M}^{i+1}_1/\mathfrak{N}$. By construction we have the exact sequence

$$0 \to \mathfrak{N} \xrightarrow{f} \mathfrak{M} \xrightarrow{g} \mathfrak{L} \to 0.$$

Let $\mathfrak{M}' = g^{-1}(\mathfrak{L}[u^\infty])$. Then we obtain two exact sequences

$$0 \to \mathfrak{N} \to \mathfrak{M}' \to \mathfrak{L}[u^\infty] \to 0 \quad \text{and} \quad 0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{L}/\mathfrak{L}[u^\infty] \to 0.$$

The second sequence has all terms being étale Kisin modules. Since $\mathfrak{M}$ have height $i+1$, we conclude that both $\mathfrak{M}'$ and $\mathfrak{L}/\mathfrak{L}[u^\infty]$ have height $i+1$. Since both $\mathfrak{N}$ and $\mathfrak{M}'$ are étale, they are finite free over $k[\![u]\!]$. This allows us to choose a basis $e_1, \ldots, e_d$ of $\mathfrak{N}$ and a basis $e'_1, \ldots, e'_d$ of $\mathfrak{M}'$ so that $(e_1, \ldots, e_d) = (e'_1, \ldots, e'_d)\Lambda$, where $\Lambda = \mathrm{diag}(u^{a_1}, \ldots, u^{a_d})$ is a diagonal matrix such that $a_1 \leq \cdots \leq a_d$. Let A and A' be the matrices of the Frobenius for the corresponding basis. We easily see that

$$\Lambda A = A' \varphi(\Lambda).$$

Hence the last column of $A'\varphi(\Lambda)$ is divisible by u^{pa_d}. Consequently, the last column of A is divisible by $u^{(p-1)a_d}$. But $\mathfrak{N}$ has height i, which means that there exists a matrix B with entries in $k[\![u]\!]$ such that $AB = BA = u^{ei} I_d$. But this is impossible as $ei < p-1$ unless $a_d = 0$. This forces that $\Lambda = I_d$ and hence a posteriori $\mathfrak{L}$ has no u-torsion as desired. ∎

Remark 7.26. Let T be the largest integer satisfying $Te < p-1$, and let $n \in \mathbb{N}$. It is a result of Min [30, Lemma 5.1] that $\mathrm{H}^i_{\Delta}(X/\mathfrak{S})$ has no u-torsion when $0 \le i \le T+1$. By a similar argument, one can also show that $\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n)$ has no u-torsion for $0 \le i \le T$. The slight improvement along this direction in Corollary 7.25 is the statement that $\mathrm{H}^{T+1}_{\Delta}(X_n/\mathfrak{S}_n)$ is also u-torsionfree. This would imply Min's result. As far as we can tell, Min's strategy does not give u-torsionfreeness of $\mathrm{H}^{T+1}_{\Delta}(X_n/\mathfrak{S}_n)$.

Proposition 7.27. *Let $i \le p-2$ be an integer. Suppose that $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}) \to \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) =: \mathcal{M}^i_n$ is injective, and denote its image by $\mathrm{Fil}^i\,\mathcal{M}^i_n$. Assume furthermore that $\mathcal{M}^i_n$ together with*

$$(\mathrm{Fil}^i\,\mathcal{M}^i_n = \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}}), \varphi_i, \nabla)$$

is an object of $\mathrm{Mod}^{\varphi,i,\nabla}_{S,\mathrm{tor}}$. Then $T_S(\mathcal{M}^i_n) \simeq \mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i)$ as G_K-representations.

Proof. Theorem 7.22 together with Proposition 6.12 already yield the isomorphisms

$$T_S(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)) \overset{\iota_1}{\simeq} T_{\mathfrak{S}}(\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n))(i) \overset{\iota_2}{\simeq} \mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i).$$

The main point here is to check that ι_1, ι_2 are compatible with G_K-actions. Let $\mathcal{X} := X_{\mathcal{O}_{\mathrm{C}}}$.

First, $A_{\inf} \otimes_{\mathfrak{S}} \mathfrak{M}^i_n \simeq \mathrm{H}^i_{\Delta}(\mathcal{X}_n/A_{\inf,n})$, which admits a natural G_K-action. Since $A_{\inf}$ is a perfect prism, [9, Theorem 1.8 (4)] proves that

$$T_{\mathfrak{S}}(\mathfrak{M}^i_n) = (\mathrm{H}^i_{\Delta}(\mathcal{X}_n/A_{\inf,n}))^{\varphi=1} \simeq \mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$$

is compatible with the G_K-action, as explained in Remark 7.5. This implies that ι_2 is compatible with the G_K-actions.

Now Theorem 3.11 shows that the comparison isomorphism

$$\overline{\iota} : \mathrm{H}^i_{\Delta}(\mathcal{X}_n/A_{\inf,n}) \otimes_{A_{\inf},\varphi} A_{\mathrm{crys}} \simeq \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$$

is functorial. So $\overline{\iota}$ is compatible with the natural G_K-actions on both sides. Also $\overline{\iota}$ is compatible with the isomorphism $\iota : \underline{\mathcal{M}}(\mathfrak{M}^i_n) \simeq \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$. Applying Remark 6.14 then implies that

$$\iota_1 : T_S(\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)) \simeq T_{\mathfrak{S}}(\mathrm{H}^i_{\Delta}(X_n/\mathfrak{S}_n))(i)$$

is compatible with the G_K-actions *if* we define the G_K-action on $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \otimes_S A_{\mathrm{crys}}$ via the identification $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \otimes_S A_{\mathrm{crys}} = \mathrm{H}^i_{\mathrm{crys}}(\mathcal{X}_n/A_{\mathrm{crys},n})$. Recall the G_K-action on $\mathrm{H}^i_{\mathrm{crys}}(X_n/S_n) \otimes_S A_{\mathrm{crys}}$ is defined by (6.19), and we have shown in Section 5.3 that these two G_K-actions are the same. This proves that ι_1 is also compatible with the G_K-actions. ∎

To end this subsection, we explain how our results are related to Fontaine–Messing theory [17] (see also [21]) for a proper smooth *formal* scheme X over $W(k)$. For any $n \ge 1$, the scheme X_n is smooth proper over $\mathrm{Spec}(W_n(k))$. So when $0 \le j \le i \le p-1$, the triple $M^i := (\mathrm{H}^i_{\mathrm{crys}}(X_n/W_n(k)), \mathrm{H}^i_{\mathrm{crys}}(X_n/W_n(k), \mathcal{I}^{[j]}_{\mathrm{crys}}), \varphi_j)$ is known to be a Fontaine–Laffaille data.

Now let $i \le p-2$; one wants to show that $T_{\mathrm{crys}}(M^i) \simeq \mathrm{H}^i_{\mathrm{et}}(X_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})(i)$. We recall the construction of $T_{\mathrm{crys}}(M^i)$: Write $\mathrm{Fil}^j\, M^i := \mathrm{H}^i_{\mathrm{crys}}(X_n/W_n(k), \mathcal{I}^{[j]}_{\mathrm{crys}})$ and let

$$\mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_{W(k)} M^i) = \sum_{j=0}^{i} \mathrm{Fil}^j\, A_{\mathrm{crys}} \otimes_{W(k)} \mathrm{Fil}^{i-j}\, M^i \subset A_{\mathrm{crys}} \otimes_{W(k)} M^i.$$

Then one can define $\varphi_i : \mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_{W(k)} M^i) \to A_{\mathrm{crys}} \otimes_{W(k)} M^i$ by

$$\varphi_i := \sum_{j=0}^{i} \varphi_j|_{\mathrm{Fil}^j\, A_{\mathrm{crys}}} \otimes \varphi_{i-j}|_{\mathrm{Fil}^{i-j}\, M^i},$$

and $T_{\mathrm{crys}}(M^i) := (\mathrm{Fil}^i(A_{\mathrm{crys}} \otimes M^i))^{\varphi_i=1}$.

Let $\mathcal{M}^i := \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n)$, which is an object of $\mathrm{Mod}^{\varphi,i,\nabla}_{S,\mathrm{tor}}$ by Corollary 7.25. It is clear that the base change map $\iota : S \otimes_{W(k)} M^i \to \mathcal{M}^i$ is an isomorphism as $W(k) \to S$ is flat. Define

$$\mathrm{Fil}^i(S \otimes_{W(k)} M^i) := \sum_{j=0}^{i} \mathrm{Fil}^j\, S \otimes_{W(k)} \mathrm{Fil}^{i-j}\, M^i \subset S \otimes_{W(k)} M^i.$$

Since $\mathrm{Fil}^j\, M^i$ is a direct summand of $\mathrm{Fil}^{j-1}\, M^i$, the natural map $\mathrm{Fil}^i(S \otimes_{W(k)} M^i) \to \mathrm{H}^i_{\mathrm{crys}}(X_n/S_n, \mathcal{I}^{[i]}_{\mathrm{crys}})$ induced by ι is injective. Therefore, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{\mathrm{crys}}(M^i) \arrow[r] \arrow[d, hook] & \mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_{W(k)} M^i) \arrow[r, "\varphi_i - 1"] \arrow[d, hook] & A_{\mathrm{crys}} \otimes_{W(k)} M^i \arrow[d, "\wr"] \\
0 \arrow[r] & T_S(\mathcal{M}^i) \arrow[r] & \mathrm{Fil}^i(A_{\mathrm{crys}} \otimes_S \mathcal{M}^i) \arrow[r, "\varphi_i - 1"] & A_{\mathrm{crys}} \otimes_S \mathcal{M}^i
\end{tikzcd}$$

It is well-known from Fontaine–Laffaille theory that $\mathrm{length}_{\mathbb{Z}}\, T_{\mathrm{crys}}(M^i) = \mathrm{length}_{W(k)}\, M^i$. By Corollary 6.15, $\mathrm{length}_{\mathbb{Z}}\, T_S(\mathcal{M}^i) = \mathrm{length}_{W(k)}(\mathcal{M}^i/I^+S) = \mathrm{length}_{W(k)}\, M^i$. Therefore, the left column must be bijective. By Proposition 7.27, it remains to check that the isomorphism $T_{\mathrm{crys}}(M^i) \to T_S(\mathcal{M}^i)$ is compatible with the G_K-actions. Since the G_K-action on $T_S(\mathcal{M}^i)$ is the G_K-action on $A_{\mathrm{crys}} \otimes_S \mathcal{M}^i$ via (6.19), it suffices to show that $M^i \subset (\mathcal{M}^i)^{\nabla=0}$, which follows from Proposition 5.4 (1).

Corollary 7.28. *Fontaine–Messing theory* [17] *and* [21] *accommodate X being a proper smooth formal scheme over $W(k)$.*

8. Some calculations on T_S

8.1. *Identification of* (6.20) *and* (6.19)

In this section, we show that (6.20) and (6.19) are the same.

Lemma 8.1. *If we write $N^n = \sum_{i=1}^n A_{i,n} u^i \nabla^i$ then $A_{i,n+1} = A_{i-1,n} + iA_{i,n}$ and $A_{1,n} = A_{n,n} = 1$.*

Proof. Easy induction on n using $N = u\nabla$. ∎

Recall that $\gamma_i(t)$ denotes the i-th divided power of t.

Lemma 8.2. $\sum_{n\geq i} A_{i,n}\gamma_n(t) = \gamma_i(e^t - 1)$.

Proof. It suffices to show that Taylor's t-expansions of both sides are equal. It is clear that the coefficients of t^n, the first nonzero term, coincide on both sides. If we write $\gamma_i(e^t - 1) = \sum_{n\geq i} B_{i,n}\gamma_n(t)$ then it suffices to show that $B_{i,n}$ satisfies the recursive formula $B_{i,n+1} = B_{i-1,n} + iB_{i,n}$ for $n \geq i$. Note that

$$\gamma_i(e^t - 1) = \frac{1}{i!}\Big(\sum_{m=0}^{i}\binom{i}{m}(-1)^{i-m}e^{mt}\Big).$$

Therefore,

$$B_{i,n} = \frac{1}{i!}\Big(\sum_{m=0}^{i}\binom{i}{m}(-1)^{i-m}m^n\Big).$$

So $B_{i-1,n} + iB_{i,n} = B_{i,n+1}$ is equivalent to

$$\begin{aligned}\frac{1}{(i-1)!}\Big(\sum_{m=0}^{i-1}\binom{i-1}{m}(-1)^{i-1-m}m^n + \sum_{m=0}^{i}\binom{i}{m}(-1)^{i-m}m^n\Big)\\ = \frac{1}{i!}\sum_{m=0}^{i}\binom{i}{m}(-1)^{i-m}m^{n+1},\end{aligned}$$

which follows from $i\left(\binom{i}{m} - \binom{i-1}{m}\right) = \binom{i}{m}m$. ∎

Now by the above lemmas,

$$\begin{aligned}\sum_{n=0}^{\infty} N^n(x)\gamma_n(\log(\underline{\varepsilon}(\sigma))) &= \sum_{n=0}^{\infty}\nabla^n(x)u^n\gamma_n(e^{\log(\underline{\varepsilon}(\sigma))} - 1) = \sum_{n=0}^{\infty}\nabla^n(x)\gamma_n(u(\underline{\varepsilon}(\sigma) - 1))\\ &= \sum_{n=0}^{\infty}\nabla^n(x)\gamma_n(\sigma(u) - u).\end{aligned}$$

This proves that (6.20) and (6.19) are the same.

8.2. T_S and $T_{\mathrm{st},\star}$

In this subsection, we explain that our functor T_S and the functor $T_{\mathrm{st},\star}$ used in [13] are the same. For this purpose, we have to review the period ring $\widehat{A}_{\mathrm{st}}$ from [13]. Let $\widehat{A}_{\mathrm{st}} = A_{\mathrm{crys}}\langle X\rangle$ be the p-adic completion of the PD algebra of A_{crys}. We extend the Frobenius φ

and the filtration of A_{crys} to $\widehat{A}_{\mathrm{st}}$ as follows: Let $\varphi(X) = (1+X)^p - 1$ and

$$\mathrm{Fil}^i \widehat{A}_{\mathrm{st}} := \Big\{ \sum_{j=0}^{\infty} a_j \gamma_j(X) \;\Big|\; a_j \in \mathrm{Fil}^{\max\{i-j,0\}} A_{\mathrm{crys}},\ \lim_{j\to\infty} a_j = 0\ p\text{-adically} \Big\}.$$

It is easy to see that we can define $\varphi_r : \mathrm{Fil}^r \widehat{A}_{\mathrm{st}} \to \widehat{A}_{\mathrm{st}}$ similar to that for A_{crys}. To extend the G_K-action to $\widehat{A}_{\mathrm{st}}$, for any $g \in G_K$, recall that $\underline{\varepsilon}(g) = \frac{g([\underline{\pi}])}{[\underline{\pi}]} \in A_{\mathrm{inf}}$ defined before (6.20). Set $g(X) = \underline{\varepsilon}(g)X + \underline{\varepsilon}(g) - 1$. Finally, define an A_{crys}-linear monodromy by setting $N(X) = -(1+X)$. We embed S in $\widehat{A}_{\mathrm{st}}$ via $u \mapsto [\underline{\pi}](1+X)^{-1}$. At this point, we have two embeddings $S \to \widehat{A}_{\mathrm{st}}$: the embedding $\iota_1 : S \hookrightarrow A_{\mathrm{crys}} \subset \widehat{A}_{\mathrm{st}}$ via $u \mapsto [\underline{\pi}] \in A_{\mathrm{inf}}$, and $\iota_2 : S \hookrightarrow \widehat{A}_{\mathrm{st}}$ via $u \mapsto [\underline{\pi}](1+X)^{-1}$. We will use both. Notice that there is an A_{crys}-linear projection $q : \widehat{A}_{\mathrm{st}} \to A_{\mathrm{crys}}$ sending $\gamma_i(X)$ to 0. It is easy to check that q is compatible with the filtration, Frobenius, G_K-actions, and *both* embeddings $\iota_i : S \hookrightarrow \widehat{A}_{\mathrm{st}}$. Set $\beta := \log(1+X) \in \widehat{A}_{\mathrm{st}}$.

Remark 8.3. Breuil–Caruso's theory has $N(1+X) = 1+X$. Our setting has a minus sign to fit $\nabla(u) = 1$. There is no difference between these two settings apart from some signs.

Given a Breuil module $\mathcal{M} \in \mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,N,h}$, we extend filtration, φ_h, monodromy and G_K-actions to $\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ as follows:

$$\mathrm{Fil}^h \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M} = \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathrm{Fil}^h \mathcal{M} + \mathrm{Fil}^h \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}.$$

For $a \otimes m \in \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathrm{Fil}^h \mathcal{M}$, set $\varphi_h(a \otimes m) = \varphi(a) \otimes \varphi_h(m)$, and for $a \otimes m \in \mathrm{Fil}^h \widehat{A}_{\mathrm{st}} \otimes_{\iota_2 S} \mathcal{M}$, set $\varphi_h(a \otimes m) = \varphi_h(a) \otimes \varphi_h(E^h m)$. It is easy to check that these φ_h are compatible with intersection so that φ_h extends to $\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$. We extend the G_K-action from $\widehat{A}_{\mathrm{st}}$ to $\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ by acting on $\mathcal{M}$ trivially, and $N(a \otimes m) = N(a) \otimes m + a \otimes N(m)$ for all $a \in \widehat{A}_{\mathrm{st}}$ and $m \in \mathcal{M}$. Now set

$$T_{\mathrm{st}}(\mathcal{M}) := (\mathrm{Fil}^h(\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}))^{\varphi_h=1,N=0}.$$

Proposition 8.4. *There is an isomorphism $T_S(\mathcal{M}) \simeq T_{\mathrm{st},\star}(\mathcal{M})$ as G_K-representations.*

Proof. For $m \in \mathcal{M} \subset \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$, set $m^\nabla := \sum_{i=0}^{\infty} N^i(m)\gamma_i(\beta)$ and $\mathcal{M}^\nabla = \{m^\nabla \mid m \in \mathcal{M}\} \subset \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$. To understand the map $\alpha : \mathcal{M} \to \mathcal{M}^\nabla$, consider the following diagram induced by $q : \widehat{A}_{\mathrm{st}} \twoheadrightarrow A_{\mathrm{crys}}$:

$$\begin{tikzcd}
 & \mathcal{M}^\nabla \arrow[dr, hook, "j"] \arrow[ddl, "\alpha'"'] & \\
\mathcal{M} \arrow[ur, two heads, "\alpha"] \arrow[rr, hook] \arrow[d, equal] & & \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M} \arrow[d, two heads, "q"] \\
\mathcal{M} \arrow[rr, hook] & & A_{\mathrm{crys}} \otimes_S \mathcal{M}
\end{tikzcd}$$

where $\alpha' : \mathcal{M}^\nabla \to q(\mathcal{M}) = \mathcal{M}$ is induced by q. By definition of α, it is easy to show that α and α' are bijective. Also α is an isomorphism of S-modules in the sense that $\alpha(\iota_2(s)m) = \iota_1(s)\alpha(m)$ for $s \in S$ and $m \in \mathcal{M}$. Using the fact that N satisfies Griffiths transversality and diagram (6.17) together with the facts that $\beta \in \mathrm{Fil}^1 \widehat{A}_{\mathrm{st}}$ and $\varphi(\beta) = p\beta$, a similar argument to that in Lemma 6.21 (replacing a with β) shows that for any $m \in \mathrm{Fil}^h \mathcal{M}$ we have $m^\nabla \in \mathrm{Fil}^h(\widehat{A}_{\mathrm{st}} \otimes_{\iota_2, S} \mathcal{M})$ and $\varphi_h(m^\nabla) = (\varphi_h(m))^\nabla$. In summary, $\alpha : \mathcal{M} \to \mathcal{M}^\nabla$ is an isomorphism in $\mathrm{Mod}_{S,\mathrm{tor}}^{\varphi,h}$ and the injections $\mathcal{M} \overset{\alpha}{\simeq} \mathcal{M}^\nabla \subset \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ are compatible with the filtrations and φ_h.

Now consider the natural map $A_{\mathrm{crys}} \otimes_S \mathcal{M}^\nabla \in \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ induced by inclusion $j : \mathcal{M}^\nabla \subset \widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ which is still denoted by $\widetilde{j}$. Since $q \circ \widetilde{j}$ is an isomorphism (because $A_{\mathrm{crys}} \otimes_S (q \circ \alpha)$ is), we conclude that $A_{\mathrm{crys}} \otimes_S \mathcal{M}^\nabla$ is an A_{crys}-submodule of $\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ which is compatible with the filtration and φ_h.

As $N(\beta) = -1$, we easily see that $\mathcal{M}^\nabla \subset (\widehat{A}_{\mathrm{st}} \otimes_S \mathcal{M})^{N=0}$. In particular, we have an injection $\widetilde{j} : A_{\mathrm{crys}} \otimes_S M^\nabla \to (\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M})^{N=0}$ compatible with the filtration and φ_h. Therefore $\widetilde{j}$ induces an injection

$$\begin{aligned} T_S(\mathcal{M}) &= (\mathrm{Fil}^h(A_{\mathrm{crys}} \otimes_S \mathcal{M}))^{\varphi_h=1} \\ &\overset{\alpha}{\simeq} (\mathrm{Fil}^h(A_{\mathrm{crys}} \otimes_S \mathcal{M}^\nabla))^{\varphi_h=1} \subset (\mathrm{Fil}^h(\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}))^{\varphi_h=1,N=0} = T_{\mathrm{st},\star}(\mathcal{M}). \end{aligned}$$

To see that this injection is an isomorphism, by dévissage we can reduce to the case that $\mathcal{M}$ is killed by p because both T_S and $T_{\mathrm{st},\star}$ are exact functors [13, Corollary 2.3.10]. In this case, it is also well-known that $\dim_{\mathbb{F}_p} T_{\mathrm{st},\star} = \mathrm{rank}_{S_1} \mathcal{M} = \dim_{\mathbb{F}_p} T_S(\mathcal{M})$. This establishes the isomorphism $T_S(\mathcal{M}) \simeq T_{\mathrm{st},\star}(\mathcal{M})$. Finally, we check this isomorphism is compatible with G_K-actions. Note that $T_S(\mathcal{M})$ has G_K-action via (6.20), while $T_{\mathrm{st},\star}$ has G_K-action from that on $\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ with trivial G_K-action on $\mathcal{M}$. We have to show that $A_{\mathrm{crys}} \otimes_S \mathcal{M}^\nabla$ has G_K-action as the subspace of $\widehat{A}_{\mathrm{st}} \otimes_{\iota_2,S} \mathcal{M}$ is the same as that defined in (6.20). But this easily follows from the formulas $m^\nabla := \sum_{i=0}^\infty N^i(m)\gamma_i(\beta)$ and $g(\beta) = \log(\underline{\varepsilon}(g)) + \beta$. ∎

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