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# Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs

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**Abstract.** Using non-Archimedian integration over spaces of arcs of algebraic varieties, we define stringy Euler numbers associated with arbitrary Kawamata log-terminal pairs. There is a natural Kawamata log-terminal pair corresponding to an algebraic variety $V$ having a regular action of a finite group $G$. In this situation we show that the stringy Euler number of this pair coincides with the physicists' orbifold Euler number defined by the Dixon-Harvey-Vafa-Witten formula. As an application, we prove a conjecture of Miles Reid on the Euler numbers of crepant desingularizations of Gorenstein quotient singularities.

## 1. Introduction

Let $X$ be a normal irreducible algebraic variety of dimension $n$ over $\mathbb{C}$, $Z_{n-1}(X)$ the group of Weil divisors on $X$, $\mathrm{Div}(X) \subset Z_{n-1}(X)$ the subgroup of Cartier divisors on $X$, $Z_{n-1}(X) \otimes \mathbb{Q}$ the group of Weil divisors on $X$ with coefficients in $\mathbb{Q}$, $K_X \in Z_{n-1}(X)$ a canonical divisor of $X$.

Recall several definitions from the Minimal Model Program [14–16] (see also [17, 18]):

**Definition 1.1.** *Let $\Delta_X \in Z_{n-1}(X) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-divisor on a normal irreducible algebraic variety $X$. A resolution of singularities $\rho : Y \to X$ is called a* **log-resolution of** $(X, \Delta_X)$ *if the union of the $\rho$-birational transform $\rho^{-1}(\Delta_X)$ of $\Delta_X$ with the exceptional locus of $\rho$ is a divisor $D$ consisting of smooth irreducible components $D_1, \dots, D_m$ having only normal crossings.*

**Definition 1.2.** *Let $\rho : Y \to X$ be a log-resolution of a pair $(X, \Delta_X)$. We assume that $K_X + \Delta_X$ is a $\mathbb{Q}$-Cartier divisor and write*

$$K_Y = \rho^*(K_X + \Delta_X) + \sum_{i=1}^{m} a(D_i, \Delta_X) D_i,$$

*where $D_i$ runs through all irreducible components of $D$ and $a(D_i, \Delta_X) = -d_j$ if $D_i$ is a $\rho$-birational transform of an irreducible component $\Delta_j$ of*

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*$Supp\,\Delta_X$ of multiplicity $d_j$. Then the number rational number $a(D_i, \Delta_X)$ (resp. $a_l(D_i, \Delta_X) := a(D_i, \Delta_X) + 1$) is called the* **discrepancy** *(resp.* **log-discrepancy***) of $D_i$.*

**Definition 1.3.** *A pair $(X, \Delta_X)$ is called* **Kawamata log-terminal** *if the following conditions are satisfied:*

*(i) $\Delta_X = d_1\Delta_1 + \cdots + d_k\Delta_k$, where $\Delta_1, \ldots, \Delta_k$ are irreducible Weil divisors and $d_i < 1$ for all $i \in \{1, \ldots, k\}$;*

*(ii) $K_X + \Delta_X$ is a $\mathbb{Q}$-Cartier divisor;*

*(iii) for any log-resolution of singularities $\rho : Y \to X$, we have $a_l(D_i, \Delta_X) > 0$ for all $i \in \{1, \ldots, m\}$,*

Now we introduce a new invariant of Kawamata log-terminal pairs:

**Definition 1.4.** *Let $(X, \Delta_X)$ be a Kawamata log-terminal pair, $\rho : Y \to X$ a log-resolution of singularities as above. We put $I = \{1, \ldots, m\}$ and set for any subset $J \subset I$*

$$D_J := \begin{cases} \bigcap_{j\in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases}, \quad D_J^\circ := D_J \setminus \bigcup_{j\in(I\setminus J)} D_j,$$
$$e(D_J^\circ) := (\text{topological Euler number of } D_J^\circ).$$

*We call the rational number*

$$e_{\text{st}}(X, \Delta_X) := \sum_{J\subset I} e(D_J^\circ) \prod_{j\in J} a_l(D_j, \Delta_X)^{-1}$$

*the* **stringy Euler number** *of the Kawamata log-terminal pair $(X, \Delta_X)$ (in the above formula, we assume $\prod_{j\in J} = 1$ if $J = \emptyset$).*

Using non-Archimedian integrals, we show that the stringy Euler number $e_{\text{st}}(X, \Delta_X)$ is well-defined:

**Theorem 1.5.** *In the above definition, $e_{\text{st}}(X, \Delta_X)$ does not depend on the choice of a log-resolution $\rho : Y \to X$.*

We expect that the stringy Euler numbers have the following natural connections with log-flips in dimension 3 (see [21,22]):

*Conjecture 1.6.* Let $X$ be a normal 3-dimensional variety and $\Delta_X$ is an effective $\mathbb{Q}$-divisor such that $(X, \Delta)$ is Kawamata log-terminal, and $\varphi : (X, \Delta_X) \dashrightarrow (X^+, \Delta_{X^+})$ a log-flip with respect to $K_X + \Delta_X$. Then one has the following inequality:

$$e_{\text{st}}(X, \Delta_X) > e_{\text{st}}(X^+, \Delta_{X^+}).$$

*Remark 1.7.* In 4.11 we show that the above conjecture is true for toric log-flips in arbitrary dimension $n$.

Recall now a definition from the string theory [10] (see also [20]):

**Definition 1.8.** *Let $V$ be a smooth complex algebraic variety together with a regular action of a finite group $G$: $G \times V \to V$. For any element $g \in G$ we set*

$$V^g := \{x \in V \,:\, gx = x\}.$$

*Then the number*

$$e(V, G) := \frac{1}{|G|} \sum_{\substack{(g,h) \in G \times G \\ gh = hg}} e(V^g \cap V^h)$$

*is called the* **physicists' orbifold Euler number** *of $V$.*

Our main result of this paper is the following:

**Theorem 1.9.** *Let $V$ be as in 1.8, $X := V/G$ the geometric quotient, $\Delta_1, \ldots, \Delta_k \subset V/G$ the set of all irreducible components of codimension 1 in the ramification locus of the Galois covering $\phi : V \to X$. We denote by $\nu_i$ the order of a cyclic inertia subgroup $G_i \subset G$ corresponding to $\Delta_i$ and set*

$$\Delta_X := \sum_{i=1}^{k} \left( \frac{\nu_i - 1}{\nu_i} \right) \Delta_i.$$

*Then the pair $(X, \Delta_X)$ is Kawamata log-terminal and the following equality holds*

$$e_{\rm st}(X, \Delta_X) = e(V, G).$$

As corollary of 1.9, we obtain the following statement conjectured by Miles Reid in [20]:

**Theorem 1.10.** *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup acting on $V := \mathbb{C}^n$. Assume that there exists a crepant desingularization of $X := V/G$, i.e., a smooth variety $Y$ together with a projective birational morphism $\rho : Y \to X$ such that the canonical class $K_Y$ is trivial. Then the Euler number of $Y$ equals the number of conjugacy classes in $G$.*

The paper is organized as follows. In Sect. 2 we review a construction of a non-Archimedean measure on the space of arcs $J_\infty(X)$ of a smooth algebraic variety $X$ over $\mathbb{C}$. This measure associate to a measurable subset $C \subset J_\infty(X)$ an element $Vol_X(X)$ of a 2-dimensional noetherian ring $\widehat{A}_1$ which is complete with respect to a non-Archimedean topology defined by powers of a principal ideal $(\theta) \subset \widehat{A}_1$. In Sect. 3 we define exponentially integrable measurable functions and their exponential non-Archimedian integrals. Our main interest are measurable functions $F_D$ associated with $\mathbb{Q}$-divisors $D \in Div(X) \otimes \mathbb{Q}$. We prove Theorem 1.5 using a transformation formula for the exponential integral under a birational proper morphism.

In Sect. 4 we consider Kawamata log-terminal pairs $(X, \Delta_X)$, where $X$ is a toric variety and $\Delta_X$ is a torus invariant $\mathbb{Q}$-divisor. We give an explicit formula for $e_{\rm st}(X, \Delta_X)$ using a $\Sigma$-piecewise linear function $\varphi_{K,\Delta}$ corresponding to the torus invariant $\mathbb{Q}$-Cartier divisor $K_X + \Delta_X$. In Sect. 5 we investigate quotients of smooth algebraic varieties $V$ modulo regular actions of finite groups $G$. We define canonical sequences of blow ups of smooth $G$-invariant subvarieties in $V$ which allow us to construct in a canonical way a smooth $G$-variety $V'$ such that stabilizers of all points in $V'$ are abelian. This construction is used in Sect. 6 where we prove Theorem 1.9. In Sect. 7 we apply our results to a cohomological McKay correspondence in arbitrary dimension (this extends our $p$-adic ideas from [2]).

We note that Sects. 2 and 3 are strongy influenced by the idea of "motivic integral" proposed by Kontsevich [19]. Its different versions are containend in the papers of Denef and Loeser [6–9]. The case of divisors on surfaces was considered by Veys in [23, 24].

## 2. Non-Archimedian measure on spaces of arcs

Recall definitions of jets and spaces of arcs (see [11], Part A).

**Definition 2.1.** *Let $X$ be a smooth $n$-dimensional complex manifold, $x \in X$ an arbitrary point. A* **germ of a holomorphic curve at** *$x$ is a germ of a holomorphic map $\gamma$ of a small ball $\{|z| < \varepsilon\} \subset \mathbb{C}$ to $X$ such that $\gamma(0) = x$.*

*Let $l$ be a nonnegative integer. Two germs $\gamma_1, \gamma_2$ of holomorphic curves at $x$ are called $l$-***equivalent** *if the derivatives of $\gamma_1$ and $\gamma_2$ at $0$ coincide up to order $l$. The set of $l$-equivalent germs of holomorphic curves is denoted by $J_l(X, x)$ and called the* **jet space of order $l$ at $x$**. *The union*

$$J_l(X) = \bigcup_{x \in X} J_l(X, x)$$

*is a complex manifold of dimension $(l+1)n$ which is a holomorphic affine bundle over $X$. The complex manifold $J_l(X)$ is called the* **jet space of order $l$ of $X$**.

**Definition 2.2.** *Consider canonical mappings $j_l : J_{l+1}(X) \to J_l(X)$ $(l \geq 0)$ whose fibers are isomorphic to affine spaces $\mathbb{C}^n$. We denote by $J_\infty(X)$ the projective limit of $J_l(X)$ and by $\pi_l$ the canonical projection $J_\infty(X) \to J_l(X)$. The space $J_\infty(X)$ is called the* **space of arcs of $X$**.

*Remark 2.3.* Let $R$ be the formal power series ring $\mathbb{C}[[t]]$ considered as the inverse limit of finite dimensional $\mathbb{C}$-algebras $R_l := \mathbb{C}[t]/(t^{l+1})$. If $X$ is $n$-dimensional smooth quasi-projective algebraic variety over $\mathbb{C}$, then the set of points in $J_\infty(X)$ (resp. $J_l(X)$) coincides with the set of $R$-valued (resp. $R_l$-valued) points of $X$.

From now on we shall consider only the spaces $J_\infty(X)$, where $X$ is a smooth algebraic variety. In this case, $J_l(X)$ is a smooth algebraic variety for all $l \geq 0$.

**Definition 2.4.** *A set $C \subset J_\infty(X)$ is called* **cylinder set** *if there exists a positive integer $l$ such that $C = \pi_l^{-1}(B_l(C))$ for some constructible subset $B_l(C) \subset J_l(X)$. Such a constructible subset $B_l(C)$ will be called the $l$-***base of** *$C$. By definition, the empty set $\subset J_\infty(X)$ is a cylinder set and its $l$-base in $J_l(X)$ is assumed to be empty for all $l \geq 0$.*

*Remark 2.5.* Let $C \subset J_\infty(X)$ be a cylinder set with an $l$-base $B_l(X)$.

(i) It is clear that $B_{l+1}(C) := j_l^{-1}(B_l(C)) \subset J_{l+1}(X)$ is the $(l+1)$-base of $C$ and $B_{l+1}(X)$ is a Zariski locally trivial affine bundle over $B_l(C)$ whose fibers are isomorphic to $\mathbb{C}^n$.

(ii) Using (i), it is a standard exercise to show that finite unions, intersections and complements of cylinder sets are again cylinder sets.

The following property of cylinder sets will be important:

**Theorem 2.6.** *Assume that a cylinder set $C \subset J_\infty(X)$ is contained in a countable union $\bigcup_{i=1}^\infty C_i$ of cylinder sets $C_i$. Then there exists a positive integer $m$ such that $C \subset \bigcup_{i=1}^m C_i$.*

*Proof.* The proof of theorem 2.6 is based on a classical property of constructible sets (see [12], Cor. 7.2.6). For details see Theorem 6.6 in [3]. Another version of the same statement is contained in [8] (see Lemma 2.4). □

**Definition 2.7.** *Let $\mathbb{Z}[\tau^{\pm 1}]$ be the Laurent polynomial ring in variable $\tau$ with coefficients in $\mathbb{Z}$, $A$ the group algebra of $(\mathbb{Q}, +)$ with coefficients in $\mathbb{Z}[\tau^{\pm 1}]$. We denote by $\theta^s \in A$ the image of $s \in \mathbb{Q}$ under the natural homomorphism $(\mathbb{Q}, +) \to (A^*, \cdot)$, where $A^*$ is the multiplicative group of invertible elements in $A$ (the element $\theta \in A$ is transcendental over $\mathbb{Z}[\tau^{\pm 1}]$). For this reason, we write*

$$A := \mathbb{Z}[\tau^{\pm 1}][\theta^{\mathbb{Q}}]$$

*and identify $A$ with the direct limit of the subrings $A_N := \mathbb{Z}[\tau^{\pm 1}][\theta^{\frac{1}{N}\mathbb{Z}}] \subset A$, where $N$ runs over all positive integers.*

**Definition 2.8.** *We consider a topology on $A$ defined by the* **non-Archimedian norm**

$$\| \cdot \| \ : \ A \to \mathbb{R}_{\geq 0}$$

*which is uniquely characterised by the properties:*

*(i)* $\|ab\| = \|a\| \cdot \|b\|, \forall a, b \in A$*;*

*(ii)* $\|a + b\| = \max\{\|a\|, \|b\|\}, \forall a, b \in A$ *if* $\|a\| \neq \|b\|$*;*

*(ii)* $\|a\| = 1, \forall a \in \mathbb{Z}[\tau^{\pm 1}] \setminus \{0\}$*;*

*(iii)* $\|\theta^s\| = e^{-s}$ *if* $s \in \mathbb{Q}$.

*The* **completion** *of $A$ (resp. of $A_N$) with respect to the norm $\| \cdot \|$ will be denoted by $\widehat{A}$ (resp. by $\widehat{A}_N$). We set*

$$\widehat{A}_\infty := \bigcup_{N \in \mathbb{N}} \widehat{A}_N \subset \widehat{A}.$$

*Remark 2.9.* The noetherian ring $\widehat{A}_N$ consists of Laurent power series in variable $\theta^{1/N}$ with coefficients in $\mathbb{Z}[\tau^{\pm 1}]$. The ring $\widehat{A}$ consists consists of formal infinite sums

$$\sum_{i=1}^{\infty} a_i \theta^{s_i}, \quad a_i \in \mathbb{Z}[\tau^{\pm 1}],$$

where $s_1 < s_2 < \cdots$ is an ascending sequence of rational numbers having the property $\lim_{i \to \infty} s_i = +\infty$.

**Definition 2.10.** *Let $W$ be an arbitrary algebraic variety. Using a natural mixed Hodge structure in cohomology groups $H^i_c(W, \mathbb{C})$, $(0 \leq i \leq 2d)$, we define the number $h^{p,q}\left(H^i_c(W, \mathbb{C})\right)$ to be the dimension of the $(p, q)$-type Hodge component in $H^i_c(W, \mathbb{C})$. We set*

$$e^{p,q}(W) := \sum_{i \geq 0} (-1)^i h^{p,q}\left(H^i_c(W, \mathbb{C})\right)$$

*and call*

$$E(W; u, v) := \sum_{p,q} e^{p,q}(W) u^p v^q,$$

*the $E$-***polynomial** *of $W$. By the* **usual Euler number** *of $W$ we always mean $e(W) := E(W; 1, 1)$.*

*Remark 2.11.* For our purpose, it will be very important that $E$-polynomials have properties which are very similar to the ones of usual Euler numbers:

(i) if $W = W_1 \cup \cdots \cup W_k$ is a disjoint union of Zariski locally closed subsets $W_1, \ldots, W_k$, then

$$E(W; u, v) = \sum_{i=1}^{k} E(W_i; u, v);$$

(ii) if $W = W_1 \times W_2$ is a product of two algebraic varieties $W_1$ and $W_2$, then

$$E(W; u, v) = E(W_1; u, v) \cdot E(W_2; u, v);$$

(iii) if $W$ admits a fibering over $Z$ which is locally trivial in Zariski topology such that each fiber of the morphism $f : W \to Z$ is isomorphic to the affine space $\mathbb{C}^n$, then

$$E(W; u, v) = E(\mathbb{C}^n; u, v) \cdot E(Z; u, v) = (uv)^n E(Z; u, v).$$

**Definition 2.12.** *Let $V \subset W$ is a constructible subset in a complex algebraic variety $V$. We write $V$ as a union*

$$V = W_1 \cup \cdots \cup W_k$$

*of pairwise nonintersecting Zariski locally closed subsets $W_1, \ldots, W_k$. Then the $E$-***polynomial of** *$V$ is defined as follows:*

$$E(V; u, v) := \sum_{i=1}^{k} E(W_i; u, v).$$

*Remark 2.13.* Using 2.11(i), it is easy to check that the above definition does not depend on the choice of the decomposion of $V$ into a finite union of pairwise nonintersecting Zariski locally closed subsets.

Now we define a **non-Archimedian cylinder set measure** on $J_\infty(X)$.

**Definition 2.14.** *$C \subset J_\infty(X)$ be a cylinder set. We define the* **non-Archimedian volume** *$Vol_X(C) \in A_1$ of $C$ by the following formula:*

$$Vol_X(C) := E(B_l(C); \tau\theta^{-1}, \tau^{-1}\theta^{-1})\theta^{2(l+1)n} \in A_1,$$

*where $C = \pi_l^{-1}(B_l(C))$ and $E(B_l(C); u, v)$ is the $E$-polynomial of the $l$-base $B_l(C) \subset J_l(X)$. If $C = \emptyset$, we set $Vol_X(C) := 0$.*

*Remark 2.15.* Using 2.5(i) and 2.11, one immediately obtains that $Vol_X(C)$ does not depend on the choice of an $l$-base $B_l(C)$ and

$$\|Vol_X(C)\| = e^{2dim\ B_l(C)-2(l+1)n}.$$

In particular, one has the following properties

(i) If $C_1$ and $C_2$ are two cylinder sets such that $C_1 \subset C_2$, then

$$\|Vol_X(C_1)\| \leq \|Vol_X(C_2)\|.$$

(ii) If $C_1, \ldots, C_k$ are cylinder sets, then

$$\|Vol_X(C_1 \cup \cdots \cup C_k)\| = \max_{i=1}^{k} \|Vol_X(C_i)\|.$$

(iii) if a cylinder set $C$ is a finite disjoint union of cylinder sets $C_1, \ldots, C_k$, then

$$Vol_X(C) = Vol_X(C_1) + \cdots + Vol_X(C_k).$$

**Definition 2.16.** *We say that a subset $C \subset J_\infty(X)$ is* **measurable** *if for any positive real number $\varepsilon$ there exists a sequence of cylinder sets $C_0(\varepsilon)$, $C_1(\varepsilon)$, $C_2(\varepsilon), \cdots$ such that*

$$(C \cup C_0(\varepsilon)) \setminus (C \cap C_0(\varepsilon)) \subset \bigcup_{i \geq 1} C_i(\varepsilon)$$

*and $\|Vol_X(C_i(\varepsilon))\| < \varepsilon$ for all $i \geq 1$. If $C$ is measurable, then the element*

$$Vol_X(C) := \lim_{\varepsilon \to 0} C_0(\varepsilon) \in \widehat{A}_1$$

*will be called the* **non-Archimedean volume** *of $C$.*

**Theorem 2.17.** *If $C \subset J_\infty(X)$ is measurable, then $\lim_{\varepsilon \to 0} C_0(\varepsilon)$ exists and does not depend on the choice of sequences $C_0(\varepsilon), C_1(\varepsilon), C_2(\varepsilon), \cdots$.*

*Proof.* The property 2.6 plays a crucial role in the proof of this theorem. For details see [3], Theorem 6.18. □

The proof of the following statement is a standard exercise:

**Proposition 2.18.** *Measurable sets possess the following properties:*

(i) *Finite unions, finite intersections of measurable sets are measurable.*

(ii) *If $C$ is a disjoint union of nonintersecting measurable sets $C_1, \ldots, C_m$, then*

$$Vol_X(C) = Vol_X(C_1) + \cdots + Vol_X(C_m).$$

(iii) *If $C$ is measurable, then the complement $\overline{C} := J_\infty(X) \setminus C$ is measurable.*

(iv) *If $C_1, C_2, \dots, C_m, \dots$ is an infinite sequence of nonintersecting measurable sets having the property*

$$\lim_{i\to\infty} \|Vol_X(C_i)\| = 0,$$

*then*

$$C = \bigcup_{i=1}^{\infty} C_i$$

*is measurable and*

$$Vol_X(C) = \sum_{i=1}^{\infty} Vol_X(C_i).$$

The next example shows that our non-Archimedean measure does not have all properties of the standard Lebesgue measure:

*Example 2.19.* Let $C \subset R = \mathbb{C}[[t]]$ be the set consisting of all power series $\sum_{i\geq 0} a_i t^i$ such that $a_i \neq 0$ for all $i \geq 0$. For any $k \in \mathbb{Z}_{\geq 0}$, we define $C_k \subset R$ to be the set consisting of all power series $\sum_{i\geq 0} a_i t^i$ such that $a_i \neq 0$ for all $0 \leq i \leq k$. We identify $R$ with $J_\infty(\mathbb{C})$. Then every $C_k \subset J_\infty(\mathbb{C})$ is a cylinder set and $Vol_{\mathbb{C}}(C_k) = (1-\theta^2)^{k+1}$. Moreover, we have

$$C_0 \supset C_1 \supset C_2 \supset \cdots, \quad \text{and} \quad C = \bigcap_{k\geq 0} C_k.$$

However, the sequence

$$Vol_{\mathbb{C}}(C_0),\ Vol_{\mathbb{C}}(C_1),\ Vol_{\mathbb{C}}(C_2),\ \dots$$

does not converge in $\widehat{A}_1$.

**Definition 2.20.** *We shall say that a subset $C \subset J_\infty(X)$ has* **measure zero** *if for any positive real number $\varepsilon$ there exists a sequence of cylinder sets $C_1(\varepsilon), C_2(\varepsilon), \cdots$ such that $C \subset \bigcup_{i\geq 1} C_i(\varepsilon)$ and $\|Vol_X(C_i(\varepsilon))\| < \varepsilon$ for all $i \geq 1$.*

**Definition 2.21.** *Let $Z \subset X$ be a Zariski closed subvariety. For any point $x \in Z$, we denote by $\mathcal{O}_{X,x}$ the ring of germs of holomorphic functions at $x$. Let $I_{Z,x} \subset \mathcal{O}_{X,x}$ be the ideal of germs of holomorphic functions vanishing on $Z$. We set*

$$J_l(Z,x) := \{y \in J_l(X,x) \,:\, g(y) = 0 \ \ \forall\, g \in I_{Z,x}\}, \quad l \geq 1,$$

$$J_\infty(Z,x) := \{y \in J_\infty(X,x) \,:\, g(y) = 0 \ \ \forall\, g \in I_{Z,x}\}$$

*and*

$$J_\infty(Z) := \bigcup_{x\in Z} J_\infty(Z,x).$$

*The space* $J_\infty(Z) \subset J_\infty(X)$ *will be called* **space of arcs with values in** $Z$.

**Proposition 2.22.** *Let* $Z$ *be an arbitrary Zariski closed subset in a smooth irredicible algebraic variety* $X$. *Then* $J_\infty(X,Z) \subset J_\infty(X)$ *is measurable. Moreover, one has*

$$Vol_X(J_\infty(Z)) = \begin{cases} 0 & \text{if } Z \neq X \\ Vol_X(J_\infty(X)) & \text{if } Z = X. \end{cases}$$

*Proof.* If $Z \neq X$, then the set $J_\infty(Z)$ can be obtained as an intersection of cylinder sets $C_k$ such that $\|Vol_X(C_k)\| \le e^{-2k}$ (see Theorem 6.22 in [3] and 3.2.2 in [8]). □

## 3. Non-Archimedean integrals

**Definition 3.1.** *By a* **measurable function** $F$ **on** $J_\infty(X)$ *we mean a function* $F : M \to \mathbb{Q}$, *where* $M \subset J_\infty(X)$ *is a subset such that* $J_\infty(X) \setminus M$ *has measure zero and* $F^{-1}(s)$ *is measurable for all* $s \in \mathbb{Q}$. *Two measurable functions* $F_i : M_i \to \mathbb{Q}$ $(i = 1, 2)$ *on* $J_\infty(X)$ *are called* **equal** *if* $F_1(\gamma) = F_2(\gamma)$ *for all* $\gamma \in M_1 \cap M_2$.

**Definition 3.2.** *A measurable function* $F : M \to \mathbb{Q}$ *is called* **exponentially integrable** *if the series*

$$\sum_{s\in\mathbb{Q}} \|Vol_X(F^{-1}(s))\| e^{-2s}$$

*converges. If* $F$ *is exponentially integrable, then the sum*

$$\int_{J_\infty(X)} e^{-F} := \sum_{s\in\mathbb{Q}} Vol_X(F^{-1}(s))\theta^{2s} \in \widehat{A}$$

*will be called the* **exponential integral of** $F$ **over** $J_\infty(X)$.

**Definition 3.3.** *Let $D \subset Div(X)$ be a subvariety of codimension* 1*, $x \in D$ a point, and $g \in \mathcal{O}_{X,x}$ the local equation for $D$ at $x$. We set $M(D) := J_\infty(X) \setminus J_\infty(Supp\, D)$. For any $\gamma \in M(D)$, we denote by $\langle D, \gamma\rangle_x$ the order of the holomorphic function $g(\gamma(t))$ at $t = 0$. The number $\langle D, \gamma\rangle_x$ will be called the* **intersection number** *of $D$ and $\gamma$ at $x \in X$. We define the function $F_D : M(D) \to \mathbb{Z}$ as follows:*

$$F_D(\gamma) = \begin{cases} 0 & \text{if } \pi_0(\gamma) = x \notin D \\ \langle D, \gamma\rangle_x & \text{if } \pi_0(\gamma) \in D \end{cases}$$

*Remark 3.4.* Using the property $\langle D' + D'', \gamma\rangle_x = \langle D', \gamma\rangle_x + \langle D'', \gamma\rangle_x$, we extend the definition of $F_D$ to an arbitrary $\mathbb{Q}$-Cartier divisor $D$: if $D = \sum_{i=1}^m a_i D_i \in Div(X) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-linear combination of irreducible subvarieties $D_1, \dots, D_m$, then we set

$$F_D := \sum_{i=1}^m a_i F_{D_i}.$$

It is easy to show that measurable functions form a $\mathbb{Q}$-vector space and $D \subset Div(X) \otimes \mathbb{Q}$ can be identified with its $\mathbb{Q}$-subspace, since $F_D : M(D) \to \mathbb{Q}$ is mesurable for all $D \subset Div(X) \otimes \mathbb{Q}$

The following theorem describes a transformation law for exponential integrals under proper birational morphisms:

**Theorem 3.5.** *Let $\rho : Y \to X$ be a proper birational morphism of smooth complex algebraic varieties, $D = \sum_{i=1}^r d_i D_i \in Div(Y)$ the Cartier divisor defined by the equality*

$$K_Y = \rho^* K_X + \sum_{i=1}^r d_i D_i.$$

*Denote by $\rho_\infty : J_\infty(Y) \to J_\infty(X)$ the mapping of spaces of arcs induced by $\rho$. Then a measurable function $F$ is exponentially integrable if an only if $F \circ \rho_\infty + F_D$ is exponentially integrable. Moreover, if the latter holds, then*

$$\int_{J_\infty(X)} e^{-F} = \int_{J_\infty(Y)} e^{-F\circ\rho_\infty - F_D}.$$

*Proof.* The proof of theorem 3.5 is based on the equality $Vol_Y(C) = Vol_X(\rho_\infty(C))\theta^{2a}$, where $C$ is a cylinder set in $J_\infty(Y)$ such that $F_D(\gamma) = a$ for all $\gamma \in C$ (see for details Theorem 6.27 in [3] and Lemma 3.3 in [8]). □

**Theorem 3.6.** *Let* $D := a_1D_1 + \cdots + a_mD_m \in Div(X) \otimes \mathbb{Q}$ *be a* $\mathbb{Q}$*-divisor. Assume* $Supp\, D$ *is a normal crossing divisor. Then* $F_D$ *is exponentially integrable if and only* $a_i > -1$ *for all* $i \in \{1, \dots, m\}$*. Moreover, if the latter holds, then*

$$\int_{J_\infty(X)} e^{-F_D} = \sum_{J \subset I} E(D_J^\circ; \tau\theta^{-1}, \tau^{-1}\theta^{-1})(\theta^{-2} - 1)^{|J|} \prod_{j \in J} \frac{\theta^{2(1+a_j)}}{1 - \theta^{2(1+a_j)}}$$

*Proof.* The set $M(D) \subset J_\infty(X)$ splits into a countable union of pairwise nonintersecting cylinder sets whose non-Archimedian volume can be computed via $E$-polynomials of the strata $D_J^\circ$ (see for details Theorem 6.28 in [3] and Theorem 5.1 in [8]).

□

**Definition 3.7.** *Let* $(X, \Delta_X)$ *be a Kawamata log-terminal pair. Consider a log-resolution* $\rho_1 : Y \to X$ *and write*

$$K_Y = \rho^*(K_X + \Delta_X) + \sum_{i=1}^{m} a(D_i, \Delta_X) D_i.$$

*Using the notations from 1.4, we define*

$$E_{\rm st}(X, \Delta_X; u, v) := \sum_{J \subset I} E(D_J^\circ; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_l(D_j, \Delta_X)} - 1}.$$

*The function* $E_{\rm st}(X, \Delta_X; u, v)$ *will be called* **stringy** $E$**-function of** $(X, \Delta_X)$.

**Theorem 3.8.** *Let* $(X, \Delta_X)$ *be a Kawamata log-terminal pair. Then the stringy* $E$*-function of* $(X, \Delta)$ *does not depend on the choice of a log-resolution.*

*Proof.* Let $\rho_1 : Y_1 \to X$ and $\rho_2 : Y_2 \to X$ be two log-resolutions of singularities such that

$$K_{Y_1} = \rho_1^*(K_X + \Delta_X) + D_1, \quad K_{Y_2} = \rho_2^*(K_X + \Delta_X) + D_2$$

where

$$D_1 = \sum_{i=1}^{r_1} a(D_i', \Delta_X) D_i' \quad \text{and} \quad D_2 = \sum_{i=1}^{r_2} a(D_i'', \Delta_X) D_i''$$

and all discrepancies $a(D_i', \Delta_X)$, $a(D_i'', \Delta_X)$ are $> -1$. Choosing a resolution of singularities $\rho_0 : Y_0 \to X$ which dominates both resolutions $\rho_1$ and $\rho_2$, we obtain two morphisms $\alpha_1 : Y_0 \to Y_1$ and $\alpha_2 : Y_0 \to Y_2$ such that

$\rho_0 = \rho_1 \circ \alpha_1 = \rho_2 \circ \alpha_2$. We set $F := F_{D_0}$, where $D_0 = K_{Y_0} - \rho_0^*(K_X + \Delta_X)$. Since

$$K_{Y_0} - \rho_0^*(K_X + \Delta_X) = (K_{Y_0} - \alpha_i^* K_{Y_i}) + \alpha_i^* D_i, \quad (i = 1, 2),$$

we obtain

$$\int_{J_\infty(Y_1)} e^{-F_{D_1}} = \int_{J_\infty(Y_0)} e^{-F_{D_0}} = \int_{J_\infty(Y_2)} e^{-F_{D_2}} \quad \text{(see 3.5)}.$$

It follows from 3.6 that

$$\int_{J_\infty(Y_i)} e^{-F_{D_i}} = E_{\mathrm{st}}(X, \Delta_X; \tau\theta^{-1}, \tau^{-1}\theta^{-1}), \quad i \in \{0, 1, 2\}.$$

Making the substitutions $u = \tau\theta^{-1}$, $v = \tau^{-1}\theta^{-1}$, we obtain that the definition of the stringy $E$-function $E_{\mathrm{st}}(X, \Delta_X; u, v)$ does not depend on the choice of log-resolutions $\rho_1$ and $\rho_2$. □

*Proof of Theorem 1.5.* The statement immediately follows from 3.8 using the equality

$$e_{\mathrm{st}}(X, \Delta_X) = \lim_{u,v \to 1} E_{\mathrm{st}}(X, \Delta_X; u, v).$$

□

## 4. Log-pairs on toric varieties

Let $X$ be a normal toric variety of dimension $n$ associated with a rational polyhedral fan $\Sigma \subset N_{\mathbb{R}} = N \otimes \mathbb{R}$, where $N$ is a free abelian group of rank $n$. Denote by $X(\sigma)$ the torus orbit in $X$ corresponding to a cone $\sigma \in \Sigma$ ($codim_X X_\sigma = dim\, \sigma$). Let $\overline{X}(\sigma)$ be the Zariski closure of $X(\sigma)$. Then the torus invariant $\mathbb{Q}$-divisors are $\mathbb{Q}$-linear combinations of the closed strata $\overline{X}(\sigma_1^{(1)}), \ldots, \overline{X}(\sigma_k^{(1)})$, where $\Sigma^{(1)} := \{\sigma_1^{(1)}, \ldots, \sigma_k^{(1)}\}$ is the set of all 1-dimensional cones in $\Sigma$. We denote by $e_1, \ldots, e_k$ the primitive lattice generators of the cones $\sigma_1^{(1)}, \ldots, \sigma_k^{(1)}$ and set $\Delta_i := \overline{X}(\sigma_i^{(1)})$ $i \in \{1, \ldots, k\}$.

**Definition 4.1.** *Let $\varphi_{K,\Delta} : N_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ be a continious function satisfying the conditions*

*(i) $\varphi_{K,\Delta}(N) \subset \mathbb{Q}$;*

*(ii) $\varphi_{K,\Delta}$ is linear on each cone $\sigma \in \Sigma$;*

*(iii) $\varphi_{K,\Delta}(p) > 0$ for all $p \in N \setminus \{0\}$.*

*Then we define a* **$\mathbb{Q}$-divisor** $\Delta_X \in Z_{n-1}(X)$ **associated with** $\varphi_{K,\Delta}$ *as follows:*

$$\Delta_X := \sum_{i=1}^{k} \left(1 - \varphi_{K,\Delta}(e_i)\right) \Delta_i.$$

*Remark 4.2.* It is well-known that the canonical class $K_X$ of a toric variety $X$ is equal to $-(\Delta_1 + \cdots + \Delta_k)$. The above definition of $\Delta_X$ implies that $K_X + \Delta_X$ is a $\mathbb{Q}$-Cartier divisor on $X$ corresponding to the $\Sigma$-piecewise linear function $-\varphi_{K,\Delta}$.

The following statement is well-known in toric geometry (see e.g. [16] §5-2):

**Proposition 4.3.** *Let $\rho : X' \to X$ be a toric desingularization of $X$, which is defined by a subdivision $\Sigma'$ of the fan $\Sigma$. Denote by $\{D_1, \ldots, D_m\}$ the set of all irreducible torus invariant strata on $Y$ corresponding to primitive lattice generators $e'_1, \ldots, e'_m$ of 1-dimensional cones $\sigma' \in \Sigma'$. Then $\sum_{i=1}^m D_i$ is a normal crossing divisor and one has*

$$K_{X'} = \rho^*(K_X + \Delta_X) + \sum_{i=1}^{m} a(D_i, \Delta_X) D_i,$$

*where $a(D_i, \Delta_X) = \varphi_{K,\Delta}(e'_i) - 1 \ \forall i \in \{1, \ldots, m\}$.*

**Corollary 4.4.** *Let $\varphi_{K,\Delta}$ be a $\Sigma$-piecewise linear function as in 4.1. Then the pair $(X, \Delta_X)$ is Kawamata log-termial.*

Denote by $\sigma^\circ$ the relative interior of $\sigma$ (we put $\sigma^\circ = 0$, if $\sigma = 0$). We give the following explicit formula for the function $E_{\rm st}(X, \Delta_X; u, v)$:

**Theorem 4.5.**

$$\begin{aligned} E_{\rm st}(X, \Delta_X; u, v) &= (uv-1)^n \sum_{\sigma \in \Sigma} \sum_{p \in \sigma^\circ \cap N} (uv)^{-\varphi_{K,\Delta}(p)} \\ &= (uv-1)^n \sum_{p \in N} (uv)^{-\varphi_{K,\Delta}(p)}. \end{aligned}$$

*Proof.* Let $T \subset X$ be an algebraic torus acting on $X$, $\partial X := X \backslash T$ its complement. Choose an isomorphism $N \cong \mathbb{Z}^n$ and write $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$. Denote by $K := \mathbb{C}((t))$ the field of Laurent power series and define a cylinder subset $C_p \subset J_\infty(X)$ as follows:

$$C_p := \{(x_1(t), \ldots, x_n(t)) \in K^n \ : \ Ord_{t=0} x_i(t) = p_i, \ 1 \leq i \leq n\}.$$

Consider the subset $M(\partial X) \subset J_\infty(X)$ consisting of all arcs which are not contained in $J_\infty(\partial X)$. Then $M(\partial X)$ splits into a disjoint union

$$M(\partial X) = \bigcup_{p \in N} C_p.$$

Let $\rho : X' \to X$ be a toric desingularization of $X$, and

$$K_{X'} = \rho^*(K_X + \Delta_X) + \sum_{i=1}^{m} a(D_i, \Delta_X) D_i.$$

By definition, we have

$$E_{\rm st}(X, \Delta_X; \tau\theta^{-1}, \tau^{-1}\theta^{-1}) = \int_{J_\infty(X')} e^{-F_D},$$

where

$$D = \sum_{i=1}^{m} a(D_i, \Delta_X) D_i.$$

Now we notice that $F_D$ is constant on each cylinder set $C_p$ $(p \in N)$ and

$$Vol(C_p)\theta^{2F(C_p)} = (\theta^{-2} - 1)^n \theta^{2\varphi_{K,\Delta}(p)}.$$

Summing over $p \in N$ and making the substitution $u = \tau\theta^{-1}$, $v = \tau^{-1}\theta^{-1}$, we come to the required formula. □

**Definition 4.6.** *Let $X$ be an arbitary $n$-dimensional normal toric variety defined by a fan $\Sigma$, and $X + \Delta_X$ a torus invariant $\mathbb{Q}$-Cartier divisor corresponding to a $\Sigma$-piecewise linear function $\varphi_{K,\Delta}$. Denote by $\Sigma^{(n)}$ the set of all $n$-dimensional cones in $\Sigma$. Let $\sigma \in \Sigma^{(n)}$ be a cone. Define* **$\Delta$-shed of** *$\sigma$ to be the pyramid*

$$\mathrm{shed}_\Delta \sigma = \sigma \cap \{y \in N \otimes \mathbb{R} : \varphi_{K,\Delta}(y) \le 1\}.$$

*Furthermore, define* **$\Delta$-shed of** $\Sigma$ *to be*

$$\mathrm{shed}_\Delta \Sigma = \bigcup_{\sigma \in \Sigma^{(n)}} \mathrm{shed}_\Delta \sigma.$$

**Definition 4.7.** *Let $\sigma \in \Sigma^{(n)}$ be an arbitrary cone. Define $vol_\Delta(\sigma)$ to be the volume of* $\mathrm{shed}_\Delta\sigma$ *with respect to the lattice $N \subset N_{\mathbb{R}}$ multiplied by $n!$. We set*

$$vol_\Delta(\Sigma) := \sum_{\sigma \in \Sigma^{(n)}} vol_\Delta(\sigma).$$

**Definition 4.8.** *Let $X_0$, $X$, $X^+$ be $n$-dimensional normal projective toric varieties. Denote by $\Sigma$ (resp. by $\Sigma^+$) the fan defining $X$ (resp. $X^+$). Let $(X, \Delta_X)$ (resp. $(X^+, \Delta_{X^+})$) be a torus invariant Kawamata log-terminal pair defined by a $\Sigma$-piecewise linear (resp. $\Sigma^+$-piecewise linear) function $\varphi_{K,\Delta}$ (resp. $\varphi^+_{K,\Delta}$). Assume that we are given two equivariant projective birational toric morphisms $\alpha : X \to X_0$ and $\beta : X^+ \to X_0$ such that*

$-(K_X + \Delta_X)$ *is* $\alpha$*-ample ,* $K_{X^+} + \Delta_{X^+}$ *is* $\beta$*-ample, and both* $\alpha$ *and* $\beta$ *are isomorphisms in codimension* 1*. Then the birational rational map* $\psi := \beta^{-1} \circ \alpha : (X, \Delta_X) \dashrightarrow (X^+, \Delta_{X^+})$ *is called a* **toric log-flip** *with respect to a* $\mathbb{Q}$*-Cartier divisor* $K_X + \Delta_X$*.*

**Proposition 4.9.** *Let* $\psi : (X, \Delta_X) \dashrightarrow (X^+, \Delta_{X^+})$ *be a toric log-flip with respect to* $K_X + \Delta_X$ *as above. Then*

$$vol_\Delta(\Sigma) > vol_\Delta(\Sigma^+).$$

*Proof.* Using a toric interpretation of ampleness via a combinatorial convexity, one obtains from the definition of toric log-flips that $\varphi_{K,\Delta}(p) \leq \varphi^+_{K,\Delta}(p)$ for all $p \in N$ and there exists a $n$-dimensional cone $\sigma \in \Sigma^{(n)}$ such that $\varphi_{K,\Delta}(p) < \varphi^+_{K,\Delta}(p)$ for all interior lattice points $p \in \sigma \cap N$. This implies the statement (cf. [3], Prop. 4.9). □

**Proposition 4.10.** *Let* $X$ *be an arbitary* $n$*-dimensional normal toric variety defined by a fan* $\Sigma$*, and* $K_X + \Delta_X$ *a torus invariant* $\mathbb{Q}$*-Cartier divisor corresponding to a* $\Sigma$*-piecewise linear function* $\varphi_{K,\Delta}$*. Then*

$$e_{\rm st}(X, \Delta_X) = vol_\Delta(\Sigma).$$

*Proof.* The statement follows from the formula in 4.5 using the same arguments as in the proof of Prop. 4.10 in [3]. □

**Corollary 4.11.** *Let* $(X, \Delta) \dashrightarrow (X^+, \Delta_{X^+})$ *be a toric log-flip. Then*

$$e_{\rm st}(X, \Delta_X) > e_{\rm st}(X^+, \Delta_{X^+}).$$

## 5. Canonical abelianization

Let $G$ be a finite group, $V$ a smooth $n$-dimensional algebraic variety over $\mathbb{C}$ having a regular effective action of $G$. If $x \in V$ is an arbitrary point, then by $St_G(x)$ we denote the stabilizer of $x$ in $G$. For any element $g \in G$ we set $V^g := \{x \in V : gx = x\}$.

**Definition 5.1.** *Let* $D = \sum_{i=1}^m d_i D_i \in \mathrm{Div}(V)^G \otimes \mathbb{Q}$ *an effective* $G$*-invariant* $\mathbb{Q}$*-divisor on a* $G$*-manifold* $V$*. A pair* $(V, D)$ *will be called* $G$*-***normal** *if the following conditions are satisfied:*

*(i)* $Supp\, D$ *is a union of normal crossing divisors* $D_1, \ldots, D_m$*;*

*(ii) for any element* $g \in G$ *and any irredicible component* $D_i$ *of* $D$*, the divisor* $D_i$ *is* $St_G(x)$*-invariant for all* $x \in V^g \cap D_i$ *(i.e.,* $h(D_i) = D_i$ $\forall\, h \in St_G(x)$*, but the* $St_G(x)$*-action on* $D_i$ *itself may be nontrivial).*

**Theorem 5.2.** *Let $(V, D)$ be a $G$-normal pair. Then, using a canonically determined sequence of blow ups of $G$-invariant submanifolds, one obtains a $G$-normal pair $(V^{ab}, D^{ab})$ and a projective birational $G$-morphism $\psi : V^{ab} \to V$ having the properties:*

(i) $D^{ab} = (K_{V^{ab}} - \psi^* K_V) + \psi^* D$;

(ii) *for any point $x \in V^{ab}$ the stabilizer $St_G(x)$ is an abelian subgroup in $G$.*

*Proof.* Let $Z(V, G) \subset V$ be the set of all points $x \in V$ such that $St_G(x)$ is not abelian. If $Z(V, G)$ is empty, then we are done. Assume that $Z(V, G) \neq \emptyset$. We set

$$s(V, G) := \max_{x \in Z(V,G)} |St_G(x)|.$$

Consider a Zariski closed subset

$$Z_{\max}(V, G) := \{x \in Z(V, G) \ : \ |St_G(x)| = s(V, G)\} \subset Z(V, G).$$

We claim that the set $Z_{\max}(V, G) \subset V$ is a smooth $G$-invariant subvariety of codimension at least 2. By definition, $Z_{\max}(V, G)$ is a union of smooth subvarieties

$$F(H) := \{x \in V \ : \ gx = x \ \ \forall g \in H\},$$

where $H$ runs over all nonabelian subgroups of $G$ such that $|H| = s(V, G)$. This implies that $Z_{\max}(V, G)$ is $G$-invariant. Since the $G$-action is effective and $dim\, F(H) = n - 1$ is possible only for cyclic subgroups $H \subset G$, we obtain $dim\, Z_{\max}(V, G) \leq n - 2$. It remains to observe that any two subvarieties $F(H_1), F(H_2) \subset V$ must either coincide, or have empty intersection. Indeed, if $x \in F(H_1) \cap F(H_2)$, then $H_1, H_2 \subset St_G(x)$. Since $|H_1|, |H_2|$ are maximal, we obtain $H_1 = H_2 = St_G(x)$; i.e., $F(H_1) = F(H_2)$.

We set $V_0 := V$, $D_0 := D$ and define $V_1$ to be the $G$-equivariant blow-up of $V_0$ with center $Z_{\max}(V, G)$. Denote by $\varphi_1 : V_1 \to V_0$ the corresponding projective birational $G$-morphism. It is obvious that the support of $D_1 = K_{V_1} - \varphi_1^*(K_V - D)$ is a normal crossing divisor. If $x \in V_1^g \cap E$, where $E$ is a connected component of an $\varphi_1$-exceptional divisor, then $St_G(x) \subset St_G(\varphi(x))$. Since $\varphi(E)$ is a connected component of a smooth subvariety $Z_{\max}(V, G)$, $\varphi(E)$ must be $St_G(\varphi(x))$-invariant. Hence, we conclude that $(V_1, D_1)$ is a $G$-normal pair. If $Z(V_1, G) = \emptyset$, then we are done. Otherwise we apply the same procedure to the $G$-normal pair $(V_1, D_1)$, where $D_1 = \phi_1^* D_0$, and construct in the same way a next $G$-equivariant blow-up $\varphi_2 : V_2 \to V_1 \ldots$ etc.

It remains to show that the above procedure terminates. For this purpose, it suffices to show that $s(V_i, G) < s(V_0, G)$ for some $i > 0$. Assume that $s(V_0, G) = s(V_i, G)$ for all $i > 0$. Then there exist points $x_i \in V_i$ $(i \geq 0)$ such that $\varphi_i(x_i) = x_{i-1}$ and $St_G(x_i) = St_G(x_{i-1})$ $(i \geq 1)$. Let $S(x_i)$ be the set of those irreducible components of $Supp\, D_i$ which are $St_G(x_i)$-invariant and contain $x_i$. We denote by $n(x_i)$ the cardinality of $S(x_i)$ and denote by $D(x_i) \subset V_i$ the intersection of all divisors from $S(x_i)$. Then $F(St_G(x_i)) \subset D(x_i)$. If $F(St_G(x_i)) \neq D(x_i)$, then the point $n(x_{i+1}) = n(x_i)+1$ (we obtain one more component from the $\varphi_i$-exceptional divisor over $F(St_G(x_i))$). Since $n(x_i) \leq n$ for all $i \geq 0$, there exists a positive number $k$ such that $n(x_k) = n(x_{k+j})$ for all $j \geq 0$. So we obtain $F(St_G(x_{k+j})) = D(x_{k+j})$ for all $j \geq 0$. The latter means that the action of $St_G(x_k)$ on the tangent space to $x_k$ in $V_k$ splits into a direct sum of $n(x_k)$ 1-dimensional representations and a $(n - n(x_k))$-dimensional trivial representation. Since the action of $St_G(x_k)$ is effective, the group $St_G(x_k)$ must be abelian. Contradiction. □

**Definition 5.3.** *Let $(V, D)$ be a $G$-normal pair. Then the $G$-normal pair $(V^{ab}, D^{ab})$ obtained in 5.2 will be called* **canonical abelianization** *of a $G$-normal pair $(V, D)$.*

*Remark 5.4.* If the stabilisator $St_G(x) \subset G$ of every point $x \in V$ is already abelian, then one can't expect that $G$-equivariant blow ups of smooth subvarieties $Z \subset V$ could simplify singularities of the quotient-space $V/G$.

Here is the following simplest example: Let $V := \mathbb{C}^2$ and $G = \langle g \rangle$ is a cyclic group of order 5 whose generator $g$ acts by the diagonal matrix with the eigenvalues $e^{2\pi\sqrt{-1}/5}, e^{4\pi\sqrt{-1}/5}$. Let $V'$ be the blow up of $\mathbb{C}^2$ at 0. Then $V'$ has a natural covering by two open subsets $V_1'$ and $V_2'$ such that $V_1' \cong V_2' \cong \mathbb{C}^2$ and the $G$-action on one of these subsets coincides with the original $G$-action on $V$.

## 6. Orbifold $E$-functions

**Definition 6.1.** *Let $D = \sum_{j=1}^m d_j D_j$ be a $G$-invariant effective divisor on a smooth $G$-variety $V$ such that $(V, G)$ is a $G$-normal pair. Take an arbitrary element $g \in G$ and a connected component $W$ of $V^g$. Choose a point $x \in W$ and local $g$-invariant coordinates $z_1, \dots, z_n$ at $x$ so that irreducible components of $Supp\, D$ containing $x$ are defined by local equations $z_i = 0$ for some $i \in \{1, \dots, n\}$. Let $\delta_i (1 \leq i \leq n)$ be the multiplicity of $D$ along $\{z_i = 0\}$ $(\{\delta_1, \dots, \delta_n\} \subset \{0, d_1, \dots, d_m\})$, and $e^{2\pi\sqrt{-1}\alpha_i}$ $(1 \leq i \leq n)$ the eigenvalue of the $g$-action on $z_i$ $(\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{Q} \cap [0, 1))$. We define the $D$-***weight** of $g$ at $W$ as*

$$wt(g, W, D) := \sum_{i=1}^n \alpha_i(\delta_i + 1).$$

*If $D = 0$, then*

$$wt(g, W) := wt(g, W, 0) = \sum_{i=1}^n \alpha_i$$

*will be called simply the* **weight** *of $g$ at $W$. Let $I^g$ be the subset of $g$-fixed elements in $I := \{1, \dots, m\}$. For any subset $J \subset I^g$ we set*

$$F(g, W, D_J^\circ; u, v) := \prod_{j \in J} \frac{uv - 1}{(uv)^{d_j+1} - 1} E(W_J; u, v),$$

*where $W_J$ is the geometric quotient of $W \cap D_J^\circ$ modulo the subgroup $C(g, W, J) \subset C(g)$ consisting of those elements in the centralizer of $g$ which leave the component $W \subset V^g$ and the subset $J \subset I^g$ invariant.*

*Remark 6.2.* We note that $wt(g, W, D)$ does not depend on the choice of a point $x \in W$. Moreover, if $h \in C(g)$ is an element in the centralizer of $g$ and $W' = hW$ is another connected component of $V^g$, then $wt(g, W', D) = wt(g, W, D)$.

**Definition 6.3.** *We define the* **orbifold $E$-function of a $G$-normal pair** *$(V, D)$ by the formula:*

$$E_{\mathrm{orb}}(V, D, G; u, v) = \sum_{\{g\}} \sum_{\{W\}} (uv)^{wt(g,W,D)} \sum_{J \subset I^g} F(g, W, D_J^\circ; u, v),$$

*where $\{g\}$ runs over all conjugacy classes in $G$, and $\{W\}$ runs over the set of representatives of all $C(g)$-orbits in the set of connected components of $V^g$.*

*In the case $D = 0$, we call*

$$\begin{aligned} E_{\mathrm{orb}}(V, G; u, v) :&= E_{\mathrm{orb}}(V, 0, G; u, v) \\ &= \sum_{\{g\}} \sum_{\{W\}} (uv)^{wt(g,W_i)} E(W/C(g, W); u, v), \end{aligned}$$

*the* **orbifold $E$-function** *of a $G$-manifold $V$ (here $C(g, W)$ is the subgroup of all elements in $C(g)$ which leave the component $W \subset V^g$ invariant).*

*Remark 6.4.* Using the equalities

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} \sum_{h \in C(g)} e(V^g \cap V^h) &= \sum_{\{g\} \subset G} \frac{1}{|C(g)|} \sum_{h \in C(g)} e(V^g \cap V^h) \\ &= \sum_{\{g\} \subset G} e(V^g / C(g)), \end{aligned}$$

one immediately obtains that $E_{\mathrm{orb}}(V, G; 1, 1)$ equals the physicists' orbifold Euler number $e(V, G)$ (see 1.8).

*Example 6.5.* Let $G := \mu_d$ a cyclic group of order $d$ acting by roots of unity on $V := \mathbb{C}$. Then the corresponding orbifold $E$-function equals

$$\begin{aligned} E_{\mathrm{orb}}(V, G; u, v) &= uv + \sum_{k=1}^{d-1} (uv)^{k/d} \\ &= (uv)^{1/d} + (uv)^{2/d} + \cdots + (uv)^{d-1/d} + uv. \end{aligned}$$

**Lemma 6.6.** *Let $V := \mathbb{C}^r$ and $g \in \mathrm{GL}(r, \mathbb{C})$ a linear authomorphism of finite order. Denote by $V'$ the blow up of $V$ at 0. Let $D \cong \mathbb{P}^{r-1}$ be the exceptional divisor in $V'$ and $\{W_1, \ldots, W_s\}$ the set of connected components of $D^g$. Then*

$$\sum_{i=1}^{s} (uv)^{wt(g, W_i, D)} \frac{uv - 1}{(uv)^r - 1} E(W_i; u, v) = (uv)^{wt(g, V^g)}.$$

*Proof.* Let $\{e^{2\pi\sqrt{-1}\alpha_i}\}$ $(1 \leq i \leq n)$ be the set of the eigenvalues of $g$-action. Without loss of generality, we assume $0 \leq \alpha_1 \leq \cdots \leq \alpha_n < 1$. We write the number $r$ as a sum of $s$ positive integers $k_1 + \cdots + k_s$ where the numbers $k_1, \ldots, k_s$ are defined by the conditions

$$\alpha_i = \alpha_{i+1} \Leftrightarrow \exists j \in \{1, \ldots, s\} \,:\, k_1 + \cdots + k_j \leq i < k_1 + \cdots + k_j + k_{j+1}$$

and

$$\alpha_i < \alpha_{i+1} \Leftrightarrow \exists j \in \{1, \ldots, s\} \,:\, i + 1 = k_1 + \cdots + k_j.$$

Then $D^g$ is a union of $s$ projectives linear subspaces $W_1, \ldots, W_s$, where $W_j \cong \mathbb{P}^{k_i - 1}$ $(j \in \{1, \ldots, s\})$. By definition, we have $wt(g, V^g) = \sum_{i=1}^{n} \alpha_i$. By direct computations, one obtains $wt(g, W_j, D) = k_1 + \cdots + k_{j-1} + \sum_{i=1}^{r} \alpha_i$. Hence,

$$\begin{aligned} \sum_{W_i \subset D^g} (uv)^{wt(g, W_i, D)} E(W_i; u, v) &= (uv)^{wt(g, V^g)} \sum_{j=1}^{s} (uv)^{k_1 + \cdots + k_{j-1}} E(\mathbb{P}^{k_j - 1}; u, v) \\ &= (uv)^{wt(g, V^g)} \sum_{j=1}^{s} (uv)^{k_1 + \cdots + k_{j-1}} (1 + (uv) + \cdots + (uv)^{k_j - 1}) \\ &= (uv)^{wt(g, V^g)} \sum_{l=0}^{r-1} (uv)^l \\ &= (uv)^{wt(g, V^g)} \frac{(uv)^r - 1}{uv - 1}. \end{aligned}$$

This completes the proof. □

**Lemma 6.7.** *Let $V$ and $W$ be two smooth algebraic varieties having a regular action of a finite group $G$. Assume that $V$ is a Zariski locally trivial $\mathbb{P}^r$-bundle over $W$ such that the canonical projection $\pi : V \to W$ is $G$-equivariant. Then*

$$E(V/G; u, v) = \frac{(uv)^r - 1}{uv - 1} E(W/G; u, v).$$

*Proof.* Let $H \subset G$ be a subgroup and $W(H) := \{x \in W : St_G(x) = H\}$. Then $W \subset W$ is a locally closed subvariety, and $W$ admits a $G$-invariant stratification by locally closed strata

$$W = \bigcup_{\{H\}} W(\{H\}),$$

where $\{H\}$ runs over the conjugacy classes of all subgroups in $G$ and $W(\{H\}) := \bigcup_{H' \in \{H\}} W(H')$. Denote $V(\{H\}) := \pi^{-1}(W(\{H\}))$. Then $V(\{H\})$ is a $G$-equivariant $\mathbb{P}^r$-bundle over $W(\{H\})$ and we have isomorphisms $V(\{H\})/G \cong V(H)/N(H)$, $W(\{H\})/G \cong W(H)/N(H)$, where $W(H) := \pi^{-1}(W(H))$ and $N(H)$ is the normalizer of $H$ in $G$. Since $V(H)$ is a $N(H)$-equivariant $\mathbb{P}^r$-bundle over $W(H)$, it suffices to prove our statement for the case $G = N(H)$, $W = W(H)$, and $V = V(H)$. Furthermore, we can restrict ourselves to the case when $W$ is irreducible and $N(H)$ leaves $W$ invariant. The last conditions imply $N(H) = H$. Therefore, $W/G = W$ and the $H$-action on leaves each fiber of $\pi$ invariant. Hence, $E(V/G; u, v) = E(\mathbb{P}^r/H; u, v)E(W; u, v)$. Since all cohomology groups of $\mathbb{P}^r$ have rank 1 and they are generated by an effective algebraic cycle, we get $E(\mathbb{P}^r/H; u, v) = E(\mathbb{P}^r; u, v)$. Thus, we have obtained the required formula for $E(\mathbb{P}^r/H; u, v)$. □

**Theorem 6.8.** *Let $(V, G)$ be a $G$-normal pair, $Z \subset V$ a smooth $G$-invariant subvariety such that after the $G$-equivariant blow up $\psi : V' \to V$ with center in $Z$ one obtains a $G$-normal pair $(V', D')$, where $D'$ the effective divisor defined by the equality*

$$K_{V'} = \psi^*(K_V - D) + D'.$$

*Then*

$$E_{\text{orb}}(V, D, G; u, v) = E_{\text{orb}}(V', D', G; u, v).$$

*Proof.* Let $Z_1, \ldots, Z_k$ be the set of connected components of $Z$ and $D_1, \ldots, D_m$ the set of irreducible components of $Supp\ D$. Then $Supp\ D' = \psi^{-1}(Supp\ D) \cup D_{m+1} \cup \cdots \cup D_{m+k}$, where $D_{m+1}, \ldots, D_{m+k}$ are irreducible $\psi$-exceptional divisors over $Z_1, \ldots, Z_k$. It suffices to prove the equality

$$\forall g \in G: \quad \sum_{\{W\}} (uv)^{wt(g,W,D)} \sum_{J \subset I^g} F(g, W, D_J^\circ; u, v) =$$

$$\sum_{\{W'\}} (uv)^{wt(g,W',D')} \sum_{J' \subset (I')^g} F(g, W', (D')_{J'}^\circ; u, v),$$

where $I' = I \cup \{m+1, \ldots, m+k\}$. We note that the $G$-equivariant mapping $\psi\ (V')^g \to V^g$ is surjective. Therefore, it suffices to prove the equality

$$(uv)^{wt(g,W,D)} \sum_{J \subset I^g} F(g, W, D_J^\circ; u, v) =$$
$$\sum_{i=1}^{l} (uv)^{wt(g,W_i',D')} \sum_{J' \subset (I')^g} F(g, W_i', (D')_{J'}^\circ; u, v),$$

where $W$ is a given connected component of $V^g$ and $W_1', \ldots, W_l'$ are all connected components of $(V')^g$ such that $\psi(W_i') \subset W$ $(1 \leq i \leq k)$. Since $\psi$ is an isomorphism over $W \setminus W \cap Z$ and the $\psi$-exceptional divisors $D_{m+1}, \ldots, D_{m+k}$ are pairwise nonintersecting, it suffices to prove the equality

$$(uv)^{wt(g,W,D)} F(g, W \cap Z_j, D_J^\circ; u, v) =$$
$$\sum_{i=1}^{l} (uv)^{wt(g,W_i',D')} F(g, W_i' \cap D_{m+j}, (D')_{J'}^\circ; u, v),$$

where $j \in I$ and $J' = J \cup \{j + m\}$. The last equality follows from Lemmas 6.6 and 6.7 using the fact that each $W_i' \cap D_{m+j}$ is a locally trivial $\mathbb{P}^{k_i}$-bundle over $W \cap Z_j$. □

## 7. Main theorems

Let $V$ be a smooth $n$-dimensional algebraic variety, $G$ a finite group acting by regular authomorphism on $V$, $X := V/G$ it geometric quotient, and $\phi : V \to X$ the corresponding finite morphism. Then $G$ acts on the set of irreducible components of the ramification divisor $\Lambda$ on $V$. Denote by $\{\Lambda_1, \ldots, \Lambda_k\}$ the set of representatives of $G$-orbits in the set of irreducible components of $Supp\,\Lambda$. Let $\nu_1 - 1, \ldots, \nu_k - 1$ be the multiplicities of $\Lambda_1, \ldots, \Lambda_k$ in $\Lambda$ (the number $\nu_i$ equals the order of the cyclic intertia subgroup $St_G(\Lambda_i) \subset G$). Since $\phi : V \to X$ is a Galois covering, the multiplicity $\nu_i - 1$ of $\Lambda_i$ depends only on the $G$-orbit of $\Lambda_i$ in $Supp\,\Lambda$. We set $\Delta_i := \phi(\Lambda_i)$ $(1 \le i \le k)$ and consider the pair $(X, \Delta_X)$, where

$$\Delta_X := \sum_{i=1}^{k} \left( \frac{\nu_i - 1}{\nu_i} \right) \Delta_i \in Z_{n-1}(X) \otimes \mathbb{Q}.$$

By the ramification formula, we have

$$\phi^*(K_X + \Delta_X) = \phi^* K_X + \Lambda = K_V.$$

**Proposition 7.1.** *The pair $(X, \Delta_X)$ is Kawamata log-terminal.*

*Proof.* Let $\rho : Y \to X$ be a log-resolution of singularities of $(X, \Delta_X)$ and

$$K_Y = \rho^*(K_X + \Delta_X) + \sum_i a(D_i, \Delta_X) D_i.$$

We consider the fiber product $V_1 := V \times_X Y$. Then $V_1$ has a natural finite Galois morphism $\phi_1 : V_1 \to Y$ and a natural birational $G$-morphism $\rho_1 : V_1 \to V$. We write

$$K_{V_1} = \rho_1^* K_V + \sum_{j=1}^{m} a(E_j, 0) E_j,$$

where $E_j$ runs over irreducible exceptional divisors of $\rho_1$.

By definition, the multiplicity of any irreducible component $\Delta_i$ of $\Delta$ is equal to $(\nu_i - 1)/\nu_i < 1$. Therefore, $a(D_i, \Delta_X) = -(\nu_j - 1)/\nu_j > -1$ if $\rho(D_i)$ coincides with an irreducible component $\Delta_j$ of $Supp\,\Delta$. Now consider the case when $\rho(D_i)$ is not an irreducible component of $Supp\,\Delta$. Denote by $E_j$ an irreducible divisor on $V_1$ such that $\phi_1(E_j) = D_i \subset Y$. Let $r_j$ be the ramification index of $\phi_1$ along $E_j$. By the ramification formula, one has $a(E_j, 0) + 1 = r_j(a(D_i, \Delta_X) + 1)$. Since $V$ is smooth, we have $a(E_j, 0) \ge 1$ for all $j \in \{1, \ldots, m\}$. Therefore, $a(D_i, \Delta_X) = a(E_j, 0) + 1/r_j - 1 > -1$. □

**Definition 7.2.** *Let V be a smooth algebraic variety having a regular action of a finite group G, and $(X, \Delta_X)$ the pair constructed above. Then we call $(X, \Delta_X)$ the* **Kawamata log-terminal pair associated with** $(V, G)$.

*Example 7.3.* Let $G := \mu_d$ a cyclic group of order $d$ acting by roots of unity on $V := \mathbb{C}$. Then $X = V/G \cong \mathbb{C}$ and $\Delta_X = \frac{d-1}{d} x_0$, where $x_0 \in X$ is the zero point. The stringy $E$-function of $(X, \Delta_X)$ equals

$$\begin{aligned} E_{\rm st}(X, \Delta_X; u, v) &= (uv - 1) + \frac{uv - 1}{(uv)^{1/d} - 1} \\ &= (uv)^{1/d} + (uv)^{2/d} + \cdots + (uv)^{d-1/d} + uv. \end{aligned}$$

Thus, it coincides with the orbifold $E$-function from Example 6.5.

Our next statements show the last phenomenon in more general situations:

**Lemma 7.4.** *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite abelian subgroup acting by diagonal matrices on $V := \mathbb{C}^n$, and $(X, \Delta)$ the Kawamata log-terminal pair associated with $(V, G)$. Then*

$$E_{\mathrm{st}}(X, \Delta_X; u, v) = E_{\mathrm{orb}}(V, G; u, v).$$

*Proof.* First, we remark that the ramification locus $Supp\, \Lambda$ is contained in the union of the coordinate hyperplanes $\Lambda_i := \{z_i = 0\} \subset \mathbb{C}^n$ $(1 \leq i \leq n)$. Therefore, we can write $\Lambda = \sum_{i=1}^n \nu_i \Lambda_i$, where $\nu_i \geq 1$ $(1 \leq i \leq n)$. Second, we note that $X$ is a normal affine toric variety corresponding to the cone $\sigma := \mathbb{R}^n_{\geq 0}$ and the lattice

$$N := \mathbb{Z}^n + \sum_{g \in G} \mathbb{Z}(\alpha_1(g), \dots, \alpha_n(g)),$$

where $e^{2\pi\sqrt{-1}\alpha_1(g)}, \dots, e^{2\pi\sqrt{-1}\alpha_n(g)}$ are the eigenvalues of $g$, $\{\alpha_1(g), \dots, \alpha_n(g)\} \in \mathbb{Q} \cap [0, 1)$. Moreover, $\Delta_X$ is a torus invariant divisor on $X$. Let us denote by $\{e_1, \dots, e_n\}$ the standard basis of $\mathbb{Z}^n$. Then the $\mathbb{Q}$-divisor $K_X + \Delta_X$ corresponds to a linear function $\varphi_{K,\Delta}$ which has value 1 on each $e_i$ $(1 \leq i \leq n)$. By 4.4, $(X, \Delta_X)$ is a torus invariant Kawamata log-terminal pair. By 4.5, we obtain

$$E_{\mathrm{st}}(X, \Delta_X; u, v) = (uv - 1)^n \sum_{p \in N \cap \sigma} (uv)^{-\varphi_{K,\Delta}}.$$

We set $f_i := (1/\nu_i)e_i$ $(1 \leq i \leq n)$. Then the system of vectors $\{f_1, \dots, f_n\} \subset N$ generates a sublattice $N' \subset N$ containing $\mathbb{Z}^n$. Denote by $\mathcal{R} := \{v_1, \dots, v_r\} \subset N$ the set of representatives of $N/N'$, where each element $v \in \mathcal{R}$ has a form $v = \sum_{i=1}^n \lambda_i(v) f_i$ $(0 \leq \lambda_i < 1)$. Then, by summing a multidimensional geometric series, we obtain

$$\begin{aligned}(uv - 1)^n &\sum_{p \in (v + N') \cap \sigma} (uv)^{-\varphi_{K,\Delta}} \\ &= (uv - 1)^n \left( (uv)^{-\sum_{i=1}^n \lambda_i(v)/\nu_i} \right) \prod_{i=1}^n \frac{1}{1 - (uv)^{-1/\nu_i}}\end{aligned}$$

(we used the property $\varphi_{K,\Delta}(f_i) = 1/\nu_i$, $1 \leq i \leq n$). Thus, we have

$$E_{\mathrm{st}}(X, \Delta_X; u, v) = \left( \sum_{v \in \mathcal{R}} (uv)^{-\sum_{i=1}^n \lambda_i(v))/\nu_i} \right) \prod_{i=1}^n \frac{(uv - 1)}{1 - (uv)^{-1/\nu_i}} =$$

$$(uv)^n \left( \sum_{v \in \mathcal{R}} (uv)^{-\sum_{i=1}^n \lambda_i(v))/\nu_i} \right) \prod_{i=1}^n \left(1 + (uv)^{-1/\nu_i} + \cdots + (uv)^{-(\nu-1)/\nu_i}\right) =$$

$$(uv)^n \sum_{g \in G} (uv)^{-\sum_{i=1}^n \alpha_i(g)} = E_{\mathrm{orb}}(V, G; u, v).$$

□

Now we come to our main theorem:

**Theorem 7.5.** *Let $G$ be a finite group acting regularly on a smooth algebraic variety $V$ and $(X, \Delta)$ the Kawamata log-terminal pair associated with $(V, G)$. Then*

$$E_{\mathrm{st}}(X, \Delta_X; u, v) = E_{\mathrm{orb}}(V, G; u, v).$$

*Proof.* Let $(V^{ab}, D)$ be the canonical abelianization of the $G$-normal pair $(V, 0)$, $D = K_{V^{ab}} - \psi^* K_V = \sum_{i=1}^m d_i D_i$. Denote by $\phi^{ab}$ the finite morphism $V^{ab} \to Y := V^{ab}/G$. Then $\psi$ induces a birational proper morphism $\overline{\psi} : Y \to X$ which can be considered as a partial desingularization of $X$. Let $W_1, \dots, W_l$ be representatives of $G$-orbits in the set $\{D_1, \dots, D_m\}$ $(l \leq m)$, and $\overline{W}_1, \dots, \overline{W}_l$ their $\phi^{ab}$-images in $Y$. By the ramification formula, we have

$$K_Y = (\overline{\psi})^*(K_X + \Delta_X) + \sum_{j=1}^{l} \left( \frac{d_j + 1}{r_j} - 1 \right) \overline{W}_j + \sum_{i=l+1}^{l+k} \left( \frac{1}{\nu_i} - 1 \right) \overline{W_i},$$

where $\overline{W_{l+i}}$ is the $\phi^{ab}$-image of $\psi^{-1}(\Lambda_i) \subset V^{ab}$ in $Y$, and $r_j$ is the order of the ramification of $W_j$ over $\overline{W}_j$. We set $I_1 := \{1, \dots, \}$, $I_2 := \{l+1, \dots, l+k\}$ and $I := I_1 \cup I_2$. For any subset $J \subset I$ we set $J_1 = I_1 \cap J$ and $J_2 = I_2 \cap J$. Denote by $G(J)$ the $G$-stabilizer of a point $x \in V^{ab}$ such that $\phi^{ab}(x) \in \overline{W}_J^{\circ}$ and set

$$S(J; u, v) := \sum_{g \in G(J)} (uv)^{wt(g,x,D)}.$$

It is easy to see that if $x' \in V^{ab}$ is another point such that $\phi^{ab}(x') \in \overline{W}_J^{\circ}$, then $St_G(x')$ is conjugate to $St_G(x)$, i.e., $G(J)$ depends only on $J$, but not on the choice of a point $x \in (\phi^{ab})^{-1}(\overline{W}_J^{\circ})$. Let $G'(J)$ be the subgroup in $G(J)$ generated by the cyclic inertia subgroups $St_G(W_j)$ $(j \in J_1)$ and $St_G(\psi^{-1}(\Lambda_{j-l}))$ $(j \in J_2)$; i.e., we have $G'(J) \cong \prod_{j \in J_1} \mu_{r_j} \prod_{j \in J_2} \mu_{\nu_j}$, and

$$S'(J; u, v) := \sum_{g \in G'(J)} (uv)^{wt(g,x,D)} = \prod_{j \in J_1} \frac{(uv)^{d_j+1} - 1}{(uv)^{(d_j+1)/r_j} - 1} \prod_{j \in J_2} \frac{uv - 1}{(uv)^{1/\nu_j} - 1} \tag{1}$$

By 6.8, we have

$$E_{\mathrm{orb}}(V, G; u, v) = E_{\mathrm{orb}}(V^{ab}, D, G; u, v)$$
$$= \sum_{J \subset I} S(J; u, v) \prod_{j \in J_1} \frac{uv - 1}{(uv)^{d_j+1} - 1} E(\overline{W}^{\circ}_J; u, v).$$

Since the singularities along $\overline{W}^{\circ}_J$ are toroidal (cf. [5]), in follows from 7.4 that

$$E_{\mathrm{st}}(X, \Delta_X; u, v)$$
$$= \sum_{J \subset I} \overline{S}(J; u, v) \prod_{j \in J_1} \frac{uv - 1}{(uv)^{(d_j+1)/r_j} - 1} \prod_{j \in J_2} \frac{uv - 1}{(uv)^{1/\nu_j} - 1} E(\overline{W}^{\circ}_J; u, v),$$

where $\overline{S}(J; u, v) S'(J; u, v) = S(J; u, v)$. It remains to apply (1). □

*Proof of Theorem 1.9.* The statement immediately follows from 6.4 and 7.5 by taking limits:

$$e_{\mathrm{st}}(X, \Delta_X) = \lim_{u,v \to 1} E_{\mathrm{st}}(X, \Delta_X; u, v) = \lim_{u,v \to 1} E_{\mathrm{orb}}(V, G; u, v) = e(V, G).$$

□

**Corollary 7.6.** *Let $X$ be a normal complex algebraic surface with at worst log-terminal singularities. Then*

$$e_{\mathrm{st}}(X) = e(X \setminus X_{sing}) + \sum_{x \in X_{sing}} c_x,$$

*where $c_x$ is the number of conjugacy classes in the local fundamental group of $X \setminus \{x\}$. In particular, $e_{\mathrm{st}}(X)$ is always an integer.*

*Proof.* It is well-known that a germ of a singular point $x \in X_{sing}$ is isomorphic to a germ of 0 in $\mathbb{C}^2/G_x$ where $G_x \subset \mathrm{GL}(2, \mathbb{C})$ is a finite subgroup (the suggroup $G_x$ is the local fundamental group of $X \setminus \{x\}$). Therefore, we have $J_\infty(X, x) \cong J_\infty(\mathbb{C}^2/G_x, 0)$. Let $\rho : Y \to X$ be a log-resolution of singularities. Denote by $D_1(x), \ldots, D_m(x)$ the exceptional divisors over $x \in X_{sing}$ and by $\{a_1(x), \ldots, a_m(x)\}$ their discrepancies. By 1.5 and 1.9, we have

$$e_{\mathrm{st}}(X) := e(X \setminus X_{sing}) + \sum_{x \in X_{sing}} c_x,$$

where the number

$$c_x = \sum_{i=1}^{m} e(D_i^{\circ}(x)) \frac{1}{a_i(x) + 1} + \sum_{i<j} e(D_i(x) \cap D_j(x)) \frac{1}{(a_i(x) + 1)(a_j(x) + 1)}$$

does not depend on the choice of a resolution and equals the number of conjugacy classes in $G_x$. □

## 8. Cohomological McKay correspondence

**Definition 8.1.** *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup acting linearly on $V := \mathbb{C}^n$ and $X := V/G$. A resolution of singularities $\rho : Y \to X$ is called* **crepant** *if the canonical class $K_Y$ is trivial.*

**Proposition 8.2.** *Let $\mathbb{C}^* \times X \to X$ be the regular $\mathbb{C}^*$-action on $X$ induced by the action of scalar matrices on $\mathbb{C}^n$. Assume that there exists a crepant resolution of singularities $\rho : Y \to X$. Then the $\mathbb{C}^*$-action on $X$ extends uniquely to a regular $\mathbb{C}^*$-action on $Y$.*

*Proof.* Since $Y$ is birational to $X$, the $\mathbb{C}^*$-action on $X$ extends uniquely to a rational $\mathbb{C}^*$-action $\mathbb{C}^* \times Y \dashrightarrow Y$. It remains to show that it is regular. Let $\{D_1, \dots, D_m\}$ be the set of all irreducible divisors on $Y$ in the exceptional locus of $\rho$. It was shown in [13] that the corresponding discrete valuations $\mathcal{V}_{D_1}, \dots, \mathcal{V}_{D_m}$ of the field of rational functions on $Y$ are determined uniquely. Since the algebraic group $\mathbb{C}^*$ is connected, every such a valuation $\mathcal{V}_{D_1}, \dots, \mathcal{V}_{D_m}$ must be invariant under the rational $\mathbb{C}^*$-action on $Y$. Therefore, the rational $\mathbb{C}^*$-action on $Y$ can be extended to a regular action on some Zariski dense open subsets $U_j \subset D_j$ $(j = 1, \dots, m)$, i.e., the rational $\mathbb{C}^*$-action on $Y$ is regular outside some Zariski closed subset

$$Z := \bigcup_{j=1}^{m} (D_j \setminus U_j) \subset Y, \quad codim_Y\, Z \geq 2.$$

Let $TY$ be the tangent vector bundle over $Y$. By the extension theorem of Hartogs, the restriction mapping on global sections $\Gamma(Y, TY) \to \Gamma(Y \setminus Z, TY)$ is bijective. Hence, the regular vector field $\eta \in \Gamma(Y \setminus Z, TY)$ corresponding to the regular $\mathbb{C}^*$-action on $Y \setminus Z$ extends to a regular vector field on the whole variety $Y$. The latter shows that the $\mathbb{C}^*$-action on $Y \setminus Z$ extends to a regular action on the whole $Y$. □

**Lemma 8.3.** *Let $V$ be a smooth algebraic variety, and $W = \bigcup_j W_j$ a stratification of $W$ by locally closed irreducible subvarieties. Assume that the Hodge structure in the cohomology with compact supports $H_c^i(W_j, \mathbb{Q})$ is pure for all $i, j$. Then the Hodge structure in $H_c^i(W, \mathbb{Q})$ is pure for all $i$.*

*Proof.* The statement follows by induction using tha fact that for any closed subvariety $W' \subset W$ the long exact cohomology sequence

$$\to H_c^{i-1}(W') \to H_c^i(W \setminus W') \to H_c^i(W) \to H_c^i(W') \to H_c^{i+1}(W \setminus W') \to$$

respects the Hodge structure. □

The following statement was conjectured in [1] (see also [13]):

**Theorem 8.4.** *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup. Assume that there exists a crepant desingularization $\rho : Y \to X := \mathbb{C}^n/G$. Then the Hodge structure in the cohomology $H^*(Y, \mathbb{C})$ is pure. Moreover, $H^{2i+1}(Y, \mathbb{C}) = 0$, $H^{2i}(Y, \mathbb{C})$ has the Hodge type $(i, i)$ for all $i$, and the dimension of $H^{2i}(Y, \mathbb{C})$ is equal to the number of conjugacy classes $\{g\} \subset G$ having the weight $wt(g) = i$.*

*Proof.* Let $Y^{\mathbb{C}^*}$ be the fixed point set of the $\mathbb{C}^*$-action on $Y$, $Y^{\mathbb{C}^*} = \bigcup_{j=1}^{l} Y_j$ a decomposition of $Y^{\mathbb{C}^*}$ in its connected components, $Y_0 := \rho^{-1}(x_0) \subset X$, where $x_0$ is the image of $0 \in \mathbb{C}^n$ modulo $G$. Since $Y_0$ is the fiber over the unique $\mathbb{C}^*$-fixed point $x_0 \in X$, we have $Y^{\mathbb{C}^*} \subset Y_0$. Therefore $Y^{\mathbb{C}^*}$ is compact. Since the fixed point subvariety $Y^{\mathbb{C}^*}$ is smooth and compact, the cohomology of every connected component $Y_1, \dots, Y_k$ of $Y^{\mathbb{C}^*}$ have pure Hodge structure. Consider the Bialynicki-Birula cellular decomposition [4]: $Y = \bigcup_{j=1}^{l} W_j$, where $W_j = \{y \in Y : \lim_{z \to 0} z(y) \in Y_j, \ z \in \mathbb{C}^*\}$. Since every $W_j$ is a vector bundle over $Y_j$, the groups $H^i_c(W_j, \mathbb{C})$ have pure Hodge structures for all $i, j$. By 8.3, the Hodge structure in $H^i_c(Y, \mathbb{C})$ is pure for all $i$.

Denote by $C_i(G)$ the number of conjugacy classes $\{g\} \subset G$ having the weight $wt(g) = i$. Since $G$ is contained in $\mathrm{SL}(n, \mathbb{C})$, the ramification divisors $\Lambda$ and $\Delta_X$ are zero. By 3.6 and 7.5, we have

$$E(Y; u, v) = E_{\mathrm{st}}(X, 0; u, v) = E_{\mathrm{orb}}(\mathbb{C}^n, G; u, v).$$

Using the purity of $H^i_c(Y, \mathbb{C})$ and the fact that the Poincaré duality

$$H_c^{2n-i}(Y, \mathbb{C}) \otimes H^i(Y, \mathbb{C}) \to H_c^{2n}(Y, \mathbb{C}) \cong \mathbb{C}(n)$$

respects the Hodge structure, it remains to show that

$$E_{\mathrm{orb}}(\mathbb{C}^n, G; u, v) = \sum_{\{g\}} C_i(G)(uv)^{n-i}. \tag{2}$$

Indeed, we have $E_{\mathrm{orb}}(V, G; u, v) = \sum_{\{g\}} (uv)^{wt(g, V^g)} E(V^g/C(g); u, v)$, where $V := \mathbb{C}^n$. Since $V^g$ is a linear subspace of dimension $k(g) := dim\ Ker(g - id)$, we obtain $E(V^g/C(g); u, v) = (uv)^{k(g)}$. Hence,

$$(uv)^{wt(g, V^g)} E(V^g/C(g); u, v) = (uv)^{n - wt(g^{-1}, V^g)}.$$

The summing over all conjugacy classes $\{g^{-1}\}$ implies (2). □

*Proof of Theorem 1.10.* Now it follows immediatelly from 8.4. □

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## References


1. V.V. Batyrev, D. Dais: Strong McKay Correspondence, String-Theoretic Hodge Numbers and Mirror Symmetry. Topology **35**, 901–929 (1996)
2. V.V. Batyrev: Birational Calabi-Yau *n*-folds have equal Betti numbers, to appear in Proc. European Algebraic Geometry Conference (Warwick, 1996), alg-geom/9710020
3. V.V. Batyrev: Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In: Proc. Taniguchi Symposium 1997, "Integrable Systems and Algebraic Geometry, Kobe/Kyoto", 1–32 (1998)
4. A. Bialynicki-Birula: Some theorems on action of algebraic groups. Ann. Math. **98**, 480–497 (1973)
5. V.I. Danilov: Geometry of toric varieties. Russ. Math. Surv. **33**(2), 97–154 (1978)
6. J. Denef: Local zeta functions and Euler characteristics. Duke Math. J. **63**, 713–721 (1991)
7. J. Denef, F. Loeser: Charactéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques. J. Am. Math. Soc. **5**, 705–720 (1992)
8. J. Denef, F. Loeser: Germs of arcs on singular algebraic varieties and motivic integration. Invent. Math. **135**, 201–232 (1999)
9. J. Denef, F. Loeser: Motivic Igusa zeta functions. J. Algebraic Geometry **7**, 505–537 (1998)
10. L. Dixon, J. Harvey, C. Vafa, E. Witten: Strings on Orbifolds I,II. Nucl. Phys. B **261**, 678–686 (1985); **274**, 285–314 (1986)
11. M. Green, Ph. Griffiths: Two Applications of Algebraic Geometry to Entire Holomorphic Mappings. In: The Chern Symposium, Springer 1980, pp. 41–74
12. A. Grothendieck, J.A. Dieudonné: Eléments de Géométrie Algébrique I, Die Grundlehren der mathematischen Wissenschaften **166**. Berlin Heidelberg New York, Springer IX 1971
13. Y. Ito, M. Reid: The McKay correspondence for finite subgroups of $SL(3, \mathbb{C})$. In: Higher Dimensional Complex Varieties (Trento, June 1994), M. Andreatta, Th. Peternell (eds.). De Gruyter 1996, pp. 221–246
14. Y. Kawamata: The cone of curves of algebraic varieties. Ann. Math. **119**, 603–633 (1984)
15. Y. Kawamata: Pluricanonical systems on minimal algebraic varieties. Invent. Math. **79**(3), 567–588 (1985)
16. Y. Kawamata, K. Matsuda, K. Matsuki: Introduction to the Minimal Model Program. Adv. Studies Pure Math. **10**, 283–360 (1987)
17. J. Kollár (with 14 coauthors): Flips and Abundance for Algebraic Threefolds, Vol. **211**. Astérisque 1992
18. J. Kollár: Singularities of Pairs. Proc. Symp. Pure Math. **62.1**, 221–287 (1997)
19. M. Kontsevich: Lecture at Orsay (December 7, 1995)
20. M. Reid: The McKay correspondence and the physicists' Euler number. Lect. Notes given at Univ. of Utah (1992) and MSRI (1992)
21. V.V. Shokurov: 3-fold log flips (with Appendix by Yujiro Kawamata: The minimal discrepancy coefficients of terminal singularities in dimension three). Russ. Acad. Sci., Izv., Math. **40**(1), 95–202 (1993)
22. V.V. Shokurov: An addendum to the paper "3-fold log flips". Russ. Acad. Sci., Izv., Math. **43**(3), 527–558 (1994)
23. W. Veys: Zeta functions for curves and log canonical models. Proc. London Math. Soc. **19**, 360–378 (1997)
24. W. Veys: The topological zeta function associated to a function on a normal surface germ (Preprint 1997)