J. Eur. Math. Soc. 1, 35–49

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On the ideal class groups of the maximal real subfields of number fields with all roots of unity

Received January 15, 1998 / final version received July 31, 1998

Abstract. In this paper, for a totally real number field k we show the ideal class group of $k\,(\cup_{n>0}\mu_n)^+$ is trivial. We also study the p-component of the ideal class group of the cyclotomic $\mathbf{Z}_p$-extension.

1. Introduction

For a positive integer $n > 0$, μ_n denotes the group of n-th roots of unity.

Iwasawa studied the ideal class groups of number fields containing $\mu_{p^\infty} = \cup_{n>0}\mu_{p^n}$ with a prime number p, and established a theory which clarifies their deep arithmetical meaning. One of the motivation lay in an analogy between a number field containing μ_{p^∞} and a function field over an algebraically closed field, so it is natural to ask how is the ideal class group of a field containing all roots of unity.

For an algebraic extension $K/\mathbf{Q}$, $C_K = \mathrm{Pic}(O_K)$ denotes the ideal class group of the integer ring O_K, namely the group of isomorphism classes of invertible O_K-submodules of K. So we have $C_K = \varinjlim C_k$ where k ranges over all intermediate fields with $[k : \mathbf{Q}] < \infty$.

For a totally real number field k, let $k_\infty = \cup_{n>0}k(\mu_n)$ be the field obtained from k by adjoining all the roots of unity. For example, $\mathbf{Q}_\infty$ is the maximal abelian extension $\mathbf{Q}^{ab}$ of $\mathbf{Q}$. The class group C_{k_∞} was studied by Brumer [1], and Horie [6] (cf. also [15]), but the following result does not seem to be known.

Theorem 1.1. *Let k be a totally real number field. We denote by $(k_\infty)^+$ the maximal real subfield of k_∞. Then, we have*

$$C_{(k_\infty)^+} = 0.$$

In particular, $C_{(\mathbf{Q}^{ab})^+} = 0$.

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Mathematics Subject Classification (1991): 11R23, 11R29,11R18

On the other hand, we know that the Pontrjagin dual of the minus part $C^-_{k_\infty} = \mathrm{Coker}(C_{(k_\infty)^+} \longrightarrow C_{k_\infty})$ is generated by infinitely many elements even as a $\hat{\mathbf{Z}}[[\mathrm{Gal}(k_\infty/k)]]$-module (for example, cf. [11]).

After some preparation in Sect. 2, we will prove this theorem in Sect. 3. For the proof, we only need class field theory. Using the same method, in Sect. 4 we will study the p-primary component of the ideal class group of the cyclotomic $\mathbf{Z}_p$-extension of a real abelian field. For a number field F, A_F denotes the p-primary component (p-Sylow subgroup) of the ideal class group C_F. Let φ be an even Dirichlet character of the first kind, and k_φ (resp. $k_{\varphi,\infty,p}$) be the real abelian field corresponding to the kernel of φ (resp. the cyclotomic $\mathbf{Z}_p$-extension of k_φ). Let k_i be the subfield of $k_{\varphi,\infty,p}$ such that $[k_i : k_\varphi] = p^i$, and consider $X_{k_{\varphi,\infty,p}} = \varprojlim A_{k_i}$ where the projective limit is taken with respect to the norm maps. We denote by X^φ the φ-component of $X_{k_{\varphi,\infty,p}}$. We will see in Sect. 4 that there are many φ's such that X^φ is finite (in other words, such that $A^\varphi_{k_{\varphi,\infty,p}} = 0$) (cf. Propositions 4.3, 4.4, 4.6, 4.1).

I would like to express my hearty thanks to Professor K. Iwasawa for his warm encouragement. I learned a lot from his papers [8,9] on the relation between A_K and A_L for a p-extension L/K.

Notation. For an abelian group A and an integer $n > 0$, the cokernel (resp. kernel) of the multiplication by n is denoted by A/n (resp. $A[n]$). (Even in the case A is multiplicative, we use A/n instead of A/A^n.) For a number field F, its integer ring is denoted by O_F. For an integer $n > 0$, μ_n denotes the group of n-th roots of unity.

2. Some lemmas

In this section, we assume that K is a totally real number field of finite degree over $\mathbf{Q}$, and p is a prime number.

Let ℓ be a prime number which is different from p, and $n > 0$ be a positive integer. In the following lemma, we consider a finite extension L/K of totally real fields such that L/K is cyclic of degree p^n, and that L/K is unramified outside ℓ, and totally ramified at all primes of K lying over ℓ.

For a place w of L, let L_w be the completion of L at w. We denote by E_L (resp. E_{L_w}) the unit group of the integer ring of L (resp. L_w). (If w is an infinite place, we define $E_{L_w} = L_w^\times$.) We have an exact sequence

$$0 \longrightarrow E_L \longrightarrow \prod_w E_{L_w} \longrightarrow \mathcal{C}_L \longrightarrow C_L \longrightarrow 0$$

where $\mathcal{C}_L$ (resp. C_L) is the idele class group (resp. ideal class group) of L. We fix a generator of $\mathrm{Gal}(L/K)$ and identify $\hat{H}^0(L/K, M)$ with $H^2(L/K, M)$ for any $\mathrm{Gal}(L/K)$-module M. Here, $\hat{H}^0$ is the Tate cohomology.

Lemma 2.1. *The above exact sequence yields an exact sequence*

$$\begin{array}{l} \longrightarrow \hat{H}^0(L/K, C_L) \longrightarrow \hat{H}^0(L/K, E_L) \longrightarrow (\bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}))^0 \\ \longrightarrow H^1(L/K, C_L) \longrightarrow H^1(L/K, E_L) \longrightarrow \bigoplus_{v|\ell} H^1(L_w/K_v, E_{L_w}) \\ \longrightarrow \quad \cdots \quad . \end{array}$$

The notation is as follows. v ranges over all primes of K lying over ℓ, and w is the unique prime of L lying over v. We define an isomorphism $\hat{H}^0(L_w/K_v, E_{L_w}) \simeq \mathbf{Z}/p^n$ by

$$\hat{H}^0(L_w/K_v, E_{L_w}) \simeq \hat{H}^0(L_w/K_v, L_w^\times) \simeq H^2(L_w/K_v, L_w^\times) \simeq \mathbf{Z}/p^n$$

where the last map is the invariant map of local class field theory. (The first two groups are isomorphic because L_w/K_v is totally ramified.) The group $(\bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}))^0$ denotes the kernel of

$$\bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}) \simeq \bigoplus_{v|\ell} \mathbf{Z}/p^n \xrightarrow{\Sigma} \mathbf{Z}/p^n$$

where Σ is the map defined by the sum.

Remark 2.2. It is well known that the kernel of the map

$$H^1(L/K, E_L) \longrightarrow \bigoplus_{v|\ell} H^1(L_w/K_v, E_{L_w})$$

in Lemma 2.1 coincides with the kernel of the canonical map

$$i_{L/K} : C_K \longrightarrow C_L.$$

In fact, let D_L (resp. P_L) be the divisor group of L (resp. the group of principal divisors of L). From a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \uparrow & & \uparrow & & & & \\ & & H^1(L/K, E_L) & \longrightarrow & \bigoplus_{w|\ell} \mathbf{Z}/p^n & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & H^0(L/K, P_L) & \longrightarrow & H^0(L/K, D_L) & \longrightarrow & H^0(L/K, C_L) & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & H^0(L/K, L^\times) & \longrightarrow & D_K & \longrightarrow & C_K & \longrightarrow & 0 , \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

we have an exact sequence

$$0 \longrightarrow \mathrm{Ker}(i_{L/K}) \longrightarrow H^1(L/K, E_L) \longrightarrow \bigoplus_{w|\ell} \mathbf{Z}/p^n$$

(where w ranges over all primes of L lying over ℓ). Here, the last map coincides with $H^1(L/K, E_L) \longrightarrow \bigoplus_{v|\ell} H^1(L_w/K_v, E_{L_w})$, so its kernel coincides with $\mathrm{Ker}(i_{L/K})$.

Proof of Lemma 2.1. Let A be the kernel of $\mathcal{C}_L \longrightarrow C_L$. We will simply denote $\hat{H}^q(L/K, M)$ by $\hat{H}^q(M)$. First of all, since L/K is unramified outside ℓ, we have $\hat{H}^q(\Pi_w E_{L_w}) = \hat{H}^q(\Pi_{w|\ell} E_{L_w})$. Consider a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \mathbf{Z}/p^n & \xrightarrow{\simeq} & \hat{H}^0(\mathcal{C}_L) & & \\
 & & & & \uparrow{\scriptstyle\Sigma} & & \uparrow & & \\
\ldots \longrightarrow & \hat{H}^0(E_L) & \longrightarrow & & \bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}) & \longrightarrow & \hat{H}^0(A) & \longrightarrow H^1(E_L) & \longrightarrow \ldots \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & (\bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}))^0 & \longrightarrow & H^1(C_L) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Here, we used $H^1(\mathcal{C}_L) = 0$, and the canonical map

$$\bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}) \longrightarrow \hat{H}^0(\mathcal{C}_L)$$

coincides with the sum $\bigoplus_{v|\ell} \mathbf{Z}/p^n \xrightarrow{\Sigma} \mathbf{Z}/p^n$. Hence, we have an exact sequence

$$\begin{aligned}
\ldots \longrightarrow \hat{H}^0(E_L) \longrightarrow (\bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}))^0 \\
\longrightarrow H^1(C_L) \longrightarrow H^1(E_L) \longrightarrow \ldots
\end{aligned}$$

On the other hand, from the above commutative diagram, $\hat{H}^0(A) \longrightarrow \hat{H}^0(\mathcal{C}_L)$ is surjective. This fact together with $H^1(\mathcal{C}_L) = 0$ implies that $\hat{H}^0(C_L) \longrightarrow H^1(A)$ is bijective. Thus,

$$H^1(E_L) \longrightarrow \bigoplus_{v|\ell} H^1(L_w/K_v, E_{L_w}) \longrightarrow \hat{H}^0(C_L) \longrightarrow \hat{H}^0(E_L)$$

is exact. This completes the proof of Lemma 2.1. □

We fix a prime number p, and denote by A_F the p-primary component (p-Sylow subgroup) of the ideal class group C_F for a number field F. We also need the following lemma (cf. [1, Lemma 6]).

Lemma 2.3. *Let K be a totally real number field, and $n > 0$ a positive integer. We assume K contains $\mathbf{Q}(\mu_{2p^n})^+$. Let $\mathbf{c} \in A_K$ be any element of A_K. Then there exist infinitely many rational primes ℓ such that*

$$i(\mathbf{c}) \in p^n A_{K(\mu_\ell)^+}$$

where $i : A_K \longrightarrow A_{K(\mu_\ell)^+}$ is the canonical homomorphism.

In fact, let H be the Hilbert class field of K, namely H/K is the unramified extension such that $\mathrm{Gal}(H/K) \simeq C_K$. We consider an abelian extension $H(\mu_{2p^n})/K$ whose Galois group is $\mathrm{Gal}(H(\mu_{2p^n})/K) = C_K \times \{\pm 1\}$ where $\{\pm 1\} = \mathrm{Gal}(K(\mu_{2p^n})/K)$. By Tschebotareff density, there exist infinitely many primes λ of K, of degree 1, whose Frobenius coincides with $(\mathbf{c}, 1) \in C_K \times \{\pm 1\} = \mathrm{Gal}(H(\mu_{2p^n})/K)$. We denote by ℓ the prime number which is below λ. We may suppose ℓ is unramified in K. Our assumption implies that ℓ splits completely in $\mathbf{Q}(\mu_{2p^n})$, and that the class of λ in C_K is $\mathbf{c}$. Hence, $\ell \equiv 1 \pmod{2p^n}$, and $K(\mu_\ell)/K$ is totally ramified at λ. Let w be the prime of $K(\mu_\ell)^+$ lying over λ. Then, we have $\lambda = w^{(\ell-1)/2}$ in $K(\mu_\ell)^+$, hence the image $i(\mathbf{c})$ of the class of λ in $A_{K(\mu_\ell)^+}$ belongs to $p^n A_{K(\mu_\ell)^+}$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We will use the same notation as in Sect. 2. In particular, p denotes a prime number, and C_F (resp. A_F) denotes the ideal class group of O_F (resp. the p-Sylow subgroup of C_F) for a number field F. Let k be a totally real number field. In this section, we call an extension K/k an rcf extension (a real cyclotomic finite extension) if $K \subset (k_\infty)^+$ and $[K : k] < \infty$. We will show that for any rcf extension K/k, and any element $\mathbf{c} \in C_K$, there is an rcf extension L/k such that $K \subset L$ and that the image of $\mathbf{c}$ vanishes in C_L.

3.1. It is clear that we may assume $[k : \mathbf{Q}]$ is finite. We will first see that we may assume $k/\mathbf{Q}$ is Galois. Let $\tilde{k}/\mathbf{Q}$ be the Galois closure of $k/\mathbf{Q}$, and put $m = [\tilde{k} : k]$. In order to show the above statement, we may assume the order of $\mathbf{c}$ is a power of p for some prime number p. By Lemma 2.3 there is an rcf extension K'/k such that $\mathbf{c}' \in m A_{K'}$ where $\mathbf{c}'$ is the image of $\mathbf{c}$ in $A_{K'}$. So we can take $\tilde{\mathbf{c}} \in A_{K'\tilde{k}}$ such that $N(\tilde{\mathbf{c}}) = \mathbf{c}'$ where $N : A_{K'\tilde{k}} \longrightarrow A_{K'}$ is the norm map. Assume that there is an rcf extension $\tilde{L}/\tilde{k}$ such that $\tilde{L} \supset K'\tilde{k}$ and the image of $\tilde{\mathbf{c}}$ is zero in $A_{\tilde{L}}$. We may suppose $\tilde{L} = L\tilde{k}$ where L/k is an rcf extension. Then, the image of $\mathbf{c}$ in A_L is zero. In the following, we assume $k/\mathbf{Q}$ is a finite Galois extension.

3.2. In this subsection, we will define two homomorphisms Φ and ϕ_ℓ. Let p be a prime number, and let K/k be an rcf extension. Note that $K/\mathbf{Q}$ is a finite Galois extension by our assumption.

For a ring R and a group G, we define $R[G]^0 = \mathrm{Ker}(R[G] \longrightarrow R)$ $(\Sigma a_\sigma \sigma \mapsto \Sigma a_\sigma)$ the augmentation ideal. Let E_K be the unit group of O_K. By Dirichlet's unit theorem $E_K \otimes \mathbf{R} \simeq \mathbf{R}[\mathrm{Gal}(K/\mathbf{Q})]^0$, $E_K \otimes \mathbf{Q}$ is isomorphic to $\mathbf{Q}[\mathrm{Gal}(K/\mathbf{Q})]^0$ (cf. for example [14, Cor to Th 30 in Sect. 13]), so we can take a $\mathrm{Gal}(K/\mathbf{Q})$-homomorphism

$$\Phi : E_K \otimes \mathbf{Z}_p \longrightarrow \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]^0$$

with finite cokernel. We take such a Φ and fix it.

For a finite place v of K, E_{K_v} denotes the unit group of the integer ring O_{K_v} of the completion K_v at v, and $\kappa(v)$ denotes the residue field of v. For a prime number ℓ, we have a canonical homomorphism

$$E_K \longrightarrow \bigoplus_{v|\ell} E_{K_v} \longrightarrow \bigoplus_{v|\ell} \kappa(v)^\times.$$

We assume that ℓ splits completely in K, and that $\ell \equiv 1 \pmod{2p^n}$. Fixing a prime v_0 of K over ℓ, we identify $\bigoplus_{v|\ell} \kappa(v)^\times/p^n$ with $\mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})] \otimes (\mathbf{F}_\ell^\times/p^n)$. Fixing also a generator of $\mathbf{F}_\ell^\times$ and an isomorphism $\mathbf{F}_\ell^\times/p^n \simeq \mathbf{Z}/p^n$, we have a $\mathrm{Gal}(K/\mathbf{Q})$-homomorphism

$$\phi_\ell : E_K \longrightarrow (\bigoplus_{v|\ell} \kappa(v)^\times)/p^n \simeq \mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})].$$

Our assumption on ℓ implies that the canonical map $E_\mathbf{Q} = \{\pm 1\} \longrightarrow \mathbf{F}_\ell^\times/p^n$ is zero, hence the image of ϕ_ℓ is in $(\mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})])^0$.

3.3. Let K/k be an rcf extension, and $\mathbf{c}$ be any element in C_K. We will show that there is an rcf extension L/k such that $K \subset L$ and that the image of $\mathbf{c}$ vanishes in C_L.

We may assume that the order of $\mathbf{c}$ is a power of p for some prime number p, namely $\mathbf{c}$ is in A_K. We may assume K is big enough, so we may assume $K(\mu_{p^\infty})^+/K$ is totally ramified at all primes of K lying over p. Further, by Lemma 2.3 we may assume $\mathbf{c} \in 2A_K$.

We fix $\Phi : E_K \otimes \mathbf{Z}_p \longrightarrow \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]^0$ as in 3.2. We take a positive integer n such that

$$p^n > (\#A_K)^2 \cdot \#(\mathrm{Coker}(4\Phi : E_K \otimes \mathbf{Z}_p \longrightarrow \mathbf{Z}_p[\mathrm{Gal}(K/\mathbf{Q})]^0)).$$

We choose a prime number ℓ such that

(i) ℓ splits completely in K
(ii) $\ell \equiv 1 \pmod{2p^n}$
(iii) There is a prime v_0 of K lying over ℓ such that the class of v_0 in A_K is $\mathbf{c}$.
(iv) $\mathrm{Image}(\phi_\ell : E_K \longrightarrow \bigoplus_{v|\ell} \kappa(v)^\times/p^n \simeq \mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})])$ coincides with $\mathrm{Image}(4\Phi) \bmod p^n$ in $\mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})]$.

We can find infinitely many such ℓ's by Tschebotareff density theorem, by the same method as in [13, Theorem 3.1] (p odd), and [5, Theorem 3.7] ($p = 2$). In fact, put $F = K(\mu_{2p^n})$, and $F' = F(E_K^{1/p^n})$. Let H be the Hilbert p-class field of K. By our assumption that F^+/K is totally ramified at every prime lying over p, we have $F \cap H = K$, and we identify A_K with $\mathrm{Gal}(HF/F)$. Since $2\,\mathrm{Gal}(HF \cap F'/F) = 0$ and $\mathbf{c} \in 2A_K$, $\mathbf{c}$ can be regarded as an element of $\mathrm{Gal}(HF/HF \cap F')$. We choose a primitive p^n-th root ζ_{p^n} of unity, and define $\iota : \mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})] \longrightarrow \mu_{p^n}$ by $\iota(\sigma) = 1$ if $\sigma \neq 1_{\mathrm{Gal}(K/\mathbf{Q})}$ (for $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$), and $\iota(1_{\mathrm{Gal}(K/\mathbf{Q})}) = \zeta_{p^n}$. Then, $4\iota \circ \Phi : E_K \longrightarrow \mu_{p^n}$ corresponds to an element γ of $\mathrm{Gal}(F'/HF \cap F')$ by Kummer theory. We choose a prime λ of F, of degree 1, such that $HF'/\mathbf{Q}$ is unramified at ℓ which is the prime of $\mathbf{Q}$ below λ, and that the Frobenius of λ in $\mathrm{Gal}(HF/F)$ coincides with $\mathbf{c}$, and that the Frobenius of λ in $\mathrm{Gal}(F'/F)$ coincides with γ. If we define ℓ (resp. v_0) to be the prime of $\mathbf{Q}$ (resp. K) below λ, the properties (i)-(iv) are satisfied.

Let L be the totally real subfield of $K(\mu_\ell)$ such that L/K is cyclic of degree p^n. We denote by $i_{L/K} : C_K \longrightarrow C_L$ the canonical homomorphism. We will show that $i_{L/K}(\mathbf{c}) = 0$.

Suppose that $i_{L/K}(\mathbf{c}) \neq 0$. L/K is totally ramified at v_0, and we denote by w_0 the prime of L lying over v_0. Let $[w_0]$ (resp. $[v_0]$) be the class of w_0 in C_L (resp. v_0 in C_K). From $w_0^{p^n} = v_0$ in L, we have $[w_0] \in A_L^{\mathrm{Gal}(L/K)}$. Let p^r be the order of $[w_0]$ in

$$\hat{H}^0(L/K, C_L) = \hat{H}^0(L/K, A_L) = A_L^{\mathrm{Gal}(L/K)}/i_{L/K}A_K,$$

and α be a class in A_K such that $p^r[w_0] = i_{L/K}(\alpha)$. By our assumption, $i_{L/K}(p^{n-r}\alpha) = p^n[w_0] = i_{L/K}([v_0]) = i_{L/K}(\mathbf{c}) \neq 0$ in C_L, hence $p^{n-r}\alpha$ is not in the kernel of $i_{L/K}$. In particular, the order of A_K is greater than p^{n-r}. This implies that $p^r > p^n/\#A_K > \#A_K\#\,\mathrm{Coker}\,4\Phi$. Since $\#\hat{H}^0(L/K, C_L) \geq p^r$, we have

$$\#\hat{H}^0(L/K, C_L) = \#\hat{H}^1(L/K, C_L) > \#A_K\#\,\mathrm{Coker}\,4\Phi. \qquad (1)$$

Now we apply Lemma 2.1 to L/K.
First of all, $\bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w}) = \bigoplus_{v|\ell} \kappa(v)^\times/p^n \simeq \mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})]$, and $\hat{H}^0(L/K, E_L) \longrightarrow \bigoplus_{v|\ell} \hat{H}^0(L_w/K_v, E_{L_w})$ is induced from ϕ_ℓ. On the other hand, as we mentioned in Remark 2.2, the kernel of $i_{L/K} : C_K \longrightarrow C_L$ coincides with the kernel of $H^1(L/K, E_L) \longrightarrow \bigoplus_{v|\ell} H^1(L_w/K_v, E_{L_w})$. Hence by Lemma 2.1, we have an exact sequence

$$0 \longrightarrow \mathrm{Coker}(\phi_\ell : E_K \longrightarrow \mathbf{Z}/p^n[\mathrm{Gal}(K/\mathbf{Q})]^0) \longrightarrow$$
$$\rightarrow H^1(L/K, C_L) \longrightarrow \mathrm{Ker}(i_{L/K}) \longrightarrow 0. \qquad (2)$$

Since the image of ϕ_ℓ coincides with the image of 4Φ mod p^n, the inequality (1) implies that $\#\,\mathrm{Ker}(i_{L/K}) = \#\,\mathrm{Ker}(A_K \longrightarrow A_L) > \#A_K$, which is a contradiction. Hence $i_{L/K}(\mathbf{c}) = 0$. This completes the proof of Theorem 1.1.

4. Remarks on cyclotomic $\mathbf{Z}_p$-extensions of real abelian fields

In this section, we consider a simple situation, and assume p is an odd prime number. We also use the notation that A_K is the p-primary component (p-Sylow subgroup) of the ideal class group C_K for a number field K. $K_{\infty,p}/K$ denotes the cyclotomic $\mathbf{Z}_p$-extension, and for $i \geq 0$, K_i denotes the intermediate field of degree p^i. We define

$$X_{K_{\infty,p}} = \varprojlim A_{K_i}$$

where the projective limit is taken with respect to the norm maps. Note that

$$A_{K_{\infty,p}} = \varinjlim A_{K_i}.$$

For a Dirichlet character φ of the first kind whose values are in an algebraic closure of $\mathbf{Q}_p$, k_φ denotes the abelian field corresponding to the kernel of φ, and put $G_\varphi = \mathrm{Gal}(k_\varphi/\mathbf{Q})$ (so $\varphi : G_\varphi \hookrightarrow \overline{\mathbf{Q}}_p^\times$). For a $\mathbf{Z}_p[G_\varphi]$-module M, we define

$$M^\varphi = M \otimes_{\mathbf{Z}_p[G_\varphi]} O_\varphi$$

where we regard $O_\varphi = \mathbf{Z}_p[\mathrm{Image}\,\varphi]$ as a $\mathbf{Z}_p[G_\varphi]$-module via φ, namely $\sigma x = \varphi(\sigma)x$ for any $\sigma \in G_\varphi$ and any $x \in O_\varphi$. We simply denote $A_{k_\varphi}^\varphi$ by A^φ. From $\mathrm{Gal}(k_{\varphi,\infty,p}/\mathbf{Q}) = G_\varphi \times \mathrm{Gal}(k_{\varphi,\infty,p}/k_\varphi)$, we regard G_φ as a subgroup of $\mathrm{Gal}(k_{\varphi,\infty,p}/\mathbf{Q})$, and consider $X_{k_{\varphi,\infty,p}}^\varphi = X_{k_{\varphi,\infty,p}} \otimes_{\mathbf{Z}_p[G_\varphi]} O_\varphi$ which becomes an $O_\varphi[[\mathrm{Gal}(k_{\varphi,\infty,p}/k_\varphi)]]$-module. We simply denote $X_{k_{\varphi,\infty,p}}^\varphi$ by X^φ. We will see in this section that there are many even φ's with $\# X^\varphi < \infty$.

We assume that χ is an even Dirichlet character of order prime to p such that $\chi(p) \neq 1$.

4.1. In this subsection, we assume that $A^\chi = 0$. Then we have $X^\chi = 0$ since $\chi(p) \neq 1$ (cf. [9, Lemma 3]). We begin with the following simple observation.

Proposition 4.1. *Let χ be an even character of order prime to p, satisfying $\chi(p) \neq 1$ and $A^\chi = 0$. For any $n > 0$, there exist infinitely many even Dirichlet characters ψ of order p^n with conductor a prime number such that $X^{\chi\psi} = 0$.*

Put $K = k_\chi$. In fact, by Tschebotareff density, we can take infinitely many ℓ such that ℓ splits completely in K, $\ell \equiv 1 \pmod{p^n}$, and that

$(E_K \otimes \mathbf{Z}/p^n)^\chi \longrightarrow (\bigoplus_{v|\ell} E_{K_v} \otimes \mathbf{Z}/p^n)^\chi$ is surjective. Let L be the subfield of $K(\mu_\ell)$ such that $[L : K] = p^n$.

For any $\mathbf{Z}_p[\mathrm{Gal}(L/\mathbf{Q})]$-module M, we regard M as a $\mathbf{Z}_p[G_\chi]$-module by regarding G_χ as a subgroup of $\mathrm{Gal}(L/\mathbf{Q})$ by the decomposition $\mathrm{Gal}(L/\mathbf{Q}) = G_\chi \times \mathrm{Gal}(L/K)$. We consider $M^\chi = M \otimes_{\mathbf{Z}_p[G_\chi]} O_\chi$ which is an $O_\chi[\mathrm{Gal}(L/K)]$-module.

By Lemma 2.1, the above conditions on ℓ imply $H^1(L/K, A_L)^\chi = H^1(L/K, A_L^\chi) = 0$ (cf. (2) in 3.3), hence $A_L^\chi = 0$. For $X_{L_{\infty,p}}^\chi = X_{L_{\infty,p}} \otimes_{\mathbf{Z}_p[G_\chi]} O_\chi$, by the standard argument (cf. [9, Lemma 3]), we have $(X_{L_{\infty,p}}^\chi)_{\mathrm{Gal}(L_{\infty,p}/L)} = A_L^\chi$. In fact, let γ be a generator of $\mathrm{Gal}(L_{\infty,p}/L)$ and $H = \mathrm{Ker}\left(X_{L_{\infty,p}} \longrightarrow A_L\right)$. Since $\chi(p) \neq 1$, $(\bigoplus_{w|p} \mathbf{Z}_p)^\chi = 0$ where w ranges over all primes of $L_{\infty,p}$ over p, so $(H/(\gamma-1)X_{L_{\infty,p}})^\chi = 0$, which implies $(X_{L_{\infty,p}}^\chi)_{\mathrm{Gal}(L_{\infty,p}/L)} = A_L^\chi$. Hence $X_{L_{\infty,p}}^\chi = 0$ by Nakayama's lemma, so $X^{\chi\psi} = 0$ for a character ψ of $\mathrm{Gal}(L/K)$. □

Concerning the finiteness of $X^{\chi\psi}$ for a character ψ of conductor ℓ and of order p under the assumption $A^\chi = 0$, by studying the group of cyclotomic units, we can easily see the following.

Proposition 4.2. *Let χ be an even character of order prime to p, satisfying $\chi(p) \neq 1$ and $A^\chi = 0$. For a character ψ of order p and of conductor ℓ, $X^{\chi\psi}$ is finite if and only if*

$$\#\hat{H}^0(L_i/K_i, E_{L_i})^\chi = \#(\mathbf{F}_p[[\mathrm{Gal}(K_{\infty,p}/\mathbf{Q})]]/(\mathit{Frob}_\ell - 1))^\chi$$

for some $i \geq 0$ where Frob_ℓ is the Frobenius at ℓ in $\mathrm{Gal}(K_{\infty,p}/\mathbf{Q})$.

Concerning this kind of criterion, more general case is treated by Fukuda et al. [3].

If $(E_K \otimes \mathbf{Z}_p)^\chi \longrightarrow (\bigoplus_{v|\ell} E_{K_v} \otimes \mathbf{Z}_p)^\chi$ is surjective as in the proof of Proposition 4.1, then the condition on $\hat{H}^0$ in Proposition 4.2 is satisfied for all $i \geq 0$. As an example, for $p = 3$, $K = \mathbf{Q}(\sqrt{2})$, nontrivial character χ of K, and a prime number ℓ which splits completely in K, and which satisfies $\ell \equiv 1 \pmod p$ with $\ell < 200$, $(E_K \otimes \mathbf{Z}_p)^\chi \longrightarrow (\bigoplus_{v|\ell} E_{K_v} \otimes \mathbf{Z}_p)^\chi$ is surjective (so $X^{\chi\psi} = 0$ by Proposition 4.1) except $\ell = 79$, 103 . For $\ell = 79$, and 103, by a table in [12] we know $\#\hat{H}^0(L/K, E_L)^\chi = 3$, hence by Proposition 4.2, $X^{\chi\psi}$ is finite (and nonzero).

4.2. Next we consider the case $A^\chi \simeq O_\chi/p$.

Proposition 4.3. *Let χ be an even character of order prime to p such that $\chi(p) \neq 1$, and $A^\chi \simeq O_\chi/p$. If $pX^\chi \neq 0$, then for any $n > 0$ there exist infinitely many even Dirichlet characters ψ of order p^n with a prime conductor such that $X^{\chi\psi}$ is finite.*

This proposition says that if $A^\chi \simeq O_\chi/p$, there is at least one character ψ (ψ is of order 1 or p^n) such that $X^{\chi\psi}$ is finite.

We will prove Proposition 4.3. We use the same notation as in the proof of Proposition 4.1. Let $K = k_\chi$, and $\mathbf{c}$ be a generator of A_K^χ. We take ℓ which satisfies the conditions that ℓ splits completely in K, $\ell \equiv 1 \pmod{p^n}$, and that $(E_K \otimes \mathbf{Z}/p^n)^\chi \longrightarrow (\bigoplus_{v|\ell} E_{K_v} \otimes \mathbf{Z}/p^n)^\chi$ is surjective, and that $[v_0] = \mathbf{c}$ where v_0 is a prime lying over ℓ. Tschebotareff density asserts the existence of infinitely many ℓ satisfying these conditions.

Let L be the subfield of $K(\mu_\ell)$ such that $[L : K] = p^n$. By the surjectivity of $(E_K \otimes \mathbf{Z}/p^n)^\chi \longrightarrow (\bigoplus_{v|\ell} E_{K_v} \otimes \mathbf{Z}/p^n)^\chi$, Lemma 2.1 implies that

$$H^1(L/K, A_L)^\chi = \mathrm{Ker}(i_{L/K} : A_K^\chi \longrightarrow A_L^\chi) \tag{3}$$

(cf. (2) in 3.3). On the other hand, the prime w_0 of L lying over v_0 yields a nontrivial class in $\hat{H}^0(L/K, A_L)^\chi$. So $\#\hat{H}^0(L/K, A_L)^\chi = \#H^1(L/K, A_L)^\chi \neq 1$, hence by (3) we get $i_{L/K}(\mathbf{c}) = 0$.

The isomorphism (3) also implies that $(A_L^\chi)_{\mathrm{Gal}(L/K)} \simeq A_K^\chi$. (Note that $A_L^\chi = A_L \otimes_{\mathbf{Z}_p[G_\chi]} O_\chi$, and not $A_L \otimes_{\mathbf{Z}_p[\mathrm{Gal}(L/\mathbf{Q})]} O_\chi$.) So A_L^χ is generated by $[w_0]$. We have $A_L^\chi \simeq O_\chi/p$ by induction on n. In fact, let K' be the subfield of L such that $[K' : K] = p^{n-1}$. Since the inclusion induces an isomorphism $(\bigoplus_{v|\ell} E_{K_v} \otimes \mathbf{Z}/p)^\chi \simeq (\bigoplus_{v'|\ell} E_{K'_{v'}} \otimes \mathbf{Z}/p)^\chi$, the map $(E_{K'} \otimes \mathbf{Z}/p)^\chi \longrightarrow (\bigoplus_{v'|\ell} E_{K'_{v'}} \otimes \mathbf{Z}/p)^\chi$ is also surjective, hence the same argument as above implies that $i_{L/K'}([v'_0]) = 0$ where v'_0 is the prime of K' lying above v_0. From $p[w_0] = i_{L/K'}([v'_0]) = 0$, we have $A_L^\chi \simeq O_\chi/p$.

Put $G = \mathrm{Gal}(L/K)$. Fixing a generator γ of $\mathrm{Gal}(K_{\infty,p}/K)$ and identifying γ with $1 + T$, we write $\Lambda = O_\chi[[\mathrm{Gal}(K_{\infty,p}/K)]] \simeq O_\chi[[T]]$, and

$$\Lambda_L = O_\chi[[\mathrm{Gal}(L_{\infty,p}/K)]] \simeq O_\chi[[T]][G] = \Lambda[G].$$

Let σ be a generator of $G = \mathrm{Gal}(L/K)$. We take a character ψ of G such that $\psi(\sigma) = \zeta_{p^n}$ where ζ_{p^n} is a primitive p^n-th root of unity. We define $\alpha : \Lambda_L \longrightarrow \Lambda_K = O_\chi[[T]]$ (resp. $\psi : \Lambda_L \longrightarrow \Lambda_L^\psi = O_{\chi\psi}[[T]]$) to be a ring homomorphism defined by $\sigma \mapsto 1$ (resp. $\sigma \mapsto \psi(\sigma)$).

As in the proof of Proposition 4.1, we have $(X_{L_{\infty,p}}^\chi)_{\mathrm{Gal}(L_{\infty,p}/L)} \simeq A_L^\chi$ from our assumption $\chi(p) \neq 1$. So $X_{L_{\infty,p}}^\chi$ is cyclic as a Λ_L-module by Nakayama's lemma. We write $X_{L_{\infty,p}}^\chi = \Lambda_L/I$. From $(X_{L_{\infty,p}}^\chi)_{\mathrm{Gal}(L_{\infty,p}/L)} \simeq \Lambda_L/(I, T) = O_\chi/p$, I contains elements $G(T) = \sigma - 1 + Tg$ and $H(T) = p + Th$ for some $g, h \in \Lambda_L$.

We assume $X^{\chi\psi}$ is infinite. Since $\psi(G(0)) = \zeta_{p^n} - 1$ is a prime element of $O_{\chi\psi}$ and $\mu = 0$ [2], $\psi(G(T))^*$ is an Eisenstein polynomial, so irreducible where $\psi(G(T))^*$ is the polynomial obtained from $\psi(G(T))$ by Weierstrass preparation theorem. From the assumption $\#X^{\chi\psi} = \infty$, this implies that $\psi(G(T))$ generates $\psi(I)$. Hence, $\psi(G(T))$ divides $\psi(H(T))$.

Put $\Phi_{p^n}(S) = (S^{p^n} - 1)/(S^{p^{n-1}} - 1)$ the p^n-th cyclotomic polynomial. Then, we can write $H(T) = G(T)A(T) + \Phi_{p^n}(\sigma)B(T)$ for some $A(T), B(T) \in \Lambda_L$. Since $p = \alpha(H(0)) = p\alpha(B(0))$, $\alpha(B(0)) = 1$ and $B(T)$ is a unit in Λ_L. So $\Phi_{p^n}(\sigma)$ is in I, which implies $p \in \alpha(I)$. Since we have a surjective homomorphism $(X^{\chi}_{L_{\infty,p}})_G = \Lambda/\alpha(I) \longrightarrow X^{\chi}_{K_{\infty,p}} = X^{\chi}$, this implies $pX^{\chi} = 0$, which contradicts our assumption. □

For example, by the same method, we have

Corollary 4.4. *Let K be a real quadratic field, and p be an odd prime number. We assume that p does not decompose in K, and $\#A_K = p$. If $pX_{K_{\infty,p}} \neq 0$, for any $n > 0$, there exist infinitely many real cyclic fields L' of degree p^n with a prime conductor such that $X_{L_{\infty,p}}/i(X_{K_{\infty,p}})$ is finite where $L = L'K$ and $i : X_{K_{\infty,p}} \longrightarrow X_{L_{\infty,p}}$ is the canonical map.*

Proof. Let χ_0 be the trivial character ($\chi_0 = 1$), and χ be the nontrivial character of K. Let K_1 be the subfield of $K_{\infty,p}$ such that $[K_1 : K] = p$. We take a prime number ℓ such that ℓ splits completely in K_1, $\ell \equiv 1 \pmod{p^n}$, and that both $(E_{K_1} \otimes \mathbf{Z}/p^n)^{\chi_0} \longrightarrow [(\bigoplus_{v|\ell} E_{K_{1,v}} \otimes \mathbf{Z}/p^n)^0]^{\chi_0}$ and $(E_K \otimes \mathbf{Z}/p^n)^{\chi} \longrightarrow (\bigoplus_{v|\ell} E_{K_v} \otimes \mathbf{Z}/p^n)^{\chi}$ are surjective, and that $[v_0]$ is a generator of A_K where v_0 is a prime of K lying over ℓ. Tschebotareff density asserts the existence of infinitely many such ℓ. Then, by the method in the proof of Proposition 4.3 we get the conclusion. □

Remark 4.5. For $p = 3, 5, 7$, there are a lot of examples of quadratic fields K such that $A_K \simeq \mathbf{Z}/p$, $pX_{K_{\infty,p}} \neq 0$, and that $X_{K_{\infty,p}}$ is finite (for example, $K = \mathbf{Q}(\sqrt{254})$, $\mathbf{Q}(\sqrt{473})$ for $p = 3$, cf. [7,10]). For these K, by Corollary 4.4 we can find infinitely many real cyclic fields L ($\subset K(\mu_\ell)$ for some ℓ) of degree $2p^n$ such that

$$A_L \neq 0 \quad \text{and} \quad \#X_{L_{\infty,p}} < \infty.$$

We remark that in [9] Iwasawa constructed infinitely many K such that $[K : \mathbf{Q}] = p$, $A_K \neq 0$, and $\#X_{K_{\infty,p}} < \infty$ for arbitrary odd prime number p.

4.3. We will give one more Proposition which has the same nature as Proposition 4.3.

Proposition 4.6. *Let χ be an even character of order prime to p, such that $\chi(p) \neq 1$. We assume that X^{χ} is isomorphic to a quotient of $O_{\chi}[[T]]/(f(T))$ as an $O_{\chi}[[\mathrm{Gal}(k_{\chi,\infty,p}/k_{\chi})]] = O_{\chi}[[T]]$- module where $f(T) \in O_{\chi}[[T]]$ is an irreducible polynomial. Then at least either X^{χ} is finite, or for any $n > 0$ there exist infinitely many even Dirichlet characters ψ of order p^n with a prime conductor such that $X^{\chi\psi}$ is finite.*

We use the same notation as in the proof of Proposition 4.3. Put $K = k_\chi$, and suppose $A_K^\chi \simeq O_\chi/p^m \neq 0$. We take ℓ as in the proof of Proposition 4.3 and take $L \subset K(\mu_\ell)$ such that $[L : K] = p^n$. By the same argument, we have $A_L^\chi \simeq O_\chi/p^m$.

Let ψ be a faithful character of $G = \mathrm{Gal}(L/K)$. Assume both X^χ and $X^{\chi\psi}$ are infinite. Our assumption implies that $X^\chi \simeq O_\chi[[T]]/(f(T))$. We write $X^\chi_{L_{\infty,p}} \simeq \Lambda_L/I$ as in the proof of Proposition 4.3. Since $A_L^\chi \simeq O_\chi/p^m$, I contains $G(T) = \sigma - 1 + Tg$ and $H(T) = p^m + Th$ for some $g, h \in \Lambda_L$. Let α and ψ be the homomorphisms defined in the proof of Proposition 4.3. Since $\alpha(H(0)) = f(0) = p^m$, $\alpha(H(T))$ generates $(f(T))$. So $\alpha(H(T))$ divides $\alpha(G(T))$. On the other hand, $\psi(G(T))$ divides $\psi(H(T))$ as in the proof of Proposition 4.3.

In general, for a discrete valuation ring R with maximal ideal m_R, and a power series $f \in R[[T]]$, we define $\lambda(f) = \mathrm{ord}_T(f \mod m_R)$ where ord_T is the normalized additive valuation of $(R/m_R)[[T]]$ defined by T. The above divisibility and $\mu = 0$ [2] imply that

$$\begin{aligned}\lambda(\alpha(H(T))) &\leq \lambda(\alpha(G(T))) = \lambda(\psi(G(T)))\\ &\leq \lambda(\psi(H(T))) = \lambda(\alpha(H(T))) < \infty.\end{aligned}$$

This implies both $\alpha(H(T))$ and $\alpha(G(T))$ generate the same ideal in $O_\chi[[T]]$. This is a contradiction because $\alpha(H(0)) \neq \alpha(G(0)) = 0$. □

Appendix

After this paper was accepted to be published, I was suggested by John Coates to study about $C_{k^{ab}}$ by the same method where k is an imaginary quadratic field and k^{ab} is the maximal abelian extension of k in an algebraic closure of k. Further I was informed by Nguyen Quang Do and R. Schoof of a paper by G. Gras [4]. I would like to express my hearty thanks to John Coates for his suggestion, and to Nguyen Quang Do and R. Schoof for telling me about [4].

By the same method as in Sect. 3, we can prove

Theorem A.1. *Let k be an imaginary quadratic field, and k^{ab} its maximal abelian extension in an algebraic closure. For any algebraic extension K/k, we have*

$$C_{Kk^{ab}} = 0$$

where Kk^{ab} is the compositum of K and k^{ab}.

For the maximal abelian extension K^{ab} of a number field K, Gras [4, Conjecture 0.5] conjectures $C_{K^{ab}} = 0$ for K which is not totally real. From Theorem A.1, we obtain

Corollary A.2. *For a number field K which contains an imaginary quadratic field, we have*

$$C_{K^{ab}} = 0.$$

Proof of Theorem A.1. In principle, instead of μ_n, we can use division points of an elliptic curve with complex multiplication. For an ideal $\mathbf{a}$ of O_k, $k(\mathbf{a})$ denotes the ray class field modulo $\mathbf{a}$ of k. Note that $[k(\mathbf{a}) : k(1)] = \#(O_k/\mathbf{a})^\times w_{\mathbf{a}}/w_k$ where w_k (resp. $w_{\mathbf{a}}$) denotes the number of roots of unity (resp. the number of roots of unity congruent to 1 modulo $\mathbf{a}$). We will apply the method in Sect. 3. Let K/k be a finite Galois extension. For any ideal class $\mathbf{c} \in A_K$, we will show the existence of an abelian extension k'/k such that the image of $\mathbf{c}$ vanishes in $A_{Kk'}$.

Suppose $\mu_{p^\infty}(K) = \mu_{p^\infty}^{\mathrm{Gal}(K(\mu_{p^\infty})/K)} = \mu_{p^m}$. We may assume m is big enough. By the method of Lemma 2.3, for any $n > 0$ we can show that

$$i(\mathbf{c}) \in p^n A_{Kk(\lambda_1 \cdot \ldots \cdot \lambda_r)}$$

for some primes $\lambda_1,...,\lambda_r$ of degree 1, so we may assume $\mathbf{c} \in p^m A_K$. Further, we may assume that K contains the Hilbert class field of k.

We take $\Phi : E_K \otimes \mathbf{Z}_p \longrightarrow \mathbf{Z}_p[\mathrm{Gal}(K/k)]^0$ as in Sect. 3, and take $n > 0$ such that

$$p^n > (\#A_K)^2 \cdot \#(\mathrm{Coker}(2p^m\Phi)).$$

As in 3.3, by Tschebotareff density theorem, we can choose a prime number ℓ such that ℓ splits completely in K, $\ell \equiv 1 \pmod{12p^n}$, $\mathbf{c} = [v_0]$ where v_0 is a prime of K lying over ℓ, and that if λ is a prime of k below v_0,

$$\begin{aligned}\mathrm{Image}(\phi_\lambda : E_K \longrightarrow \bigoplus_{v|\lambda} \kappa(\lambda)^\times/p^n \simeq \mathbf{Z}/p^n[\mathrm{Gal}(K/k)])\\ = \mathrm{Image}(2p^m\Phi \bmod p^n).\end{aligned}$$

In the notation of 3.3 we have $p^m\,\mathrm{Gal}(HF \cap F'/F) = 0$, so this is possible.

We define L to be the subfield of $Kk(\lambda)$ such that $[L : K] = p^n$, and apply an exact sequence

$$\begin{aligned}\ldots \longrightarrow \hat{H}^0(L/K, E_L) \longrightarrow (\bigoplus_{v|\lambda} \hat{H}^0(L_w/K_v, E_{L_w}))^0 \longrightarrow\\ \rightarrow H^1(L/K, C_L) \longrightarrow \ldots,\end{aligned}$$

which is obtained by the same method as Lemma 2.1. Then we have $i_{L/K}(\mathbf{c}) = 0$ by the same method as in 3.3.

Remark A.3. In [4], Gras studied the capitulation of ideals of number fields, and gave some examples. For $K = \mathbf{Q}(\sqrt{79})$, he showed the elements of C_K ($\simeq \mathbf{Z}/3$) become trivial in C_M where M is the compositum of K and the cubic field of conductor 97. Our method in this paper shows that we can take a field of smaller conductor for the capitulation. For example, the ideals of K become principal in the compositum of K and the cubic field of conductor 7, namely

$$C_K \longrightarrow C_{K(\cos(2\pi/7))}$$

is zero. (This can be also checked from a numerical calculation $C_{K(\cos(2\pi/7))} \simeq \mathbf{Z}/3$. Our method in this paper shows that the third and the fifth conditions in [4, page 421] can be replaced by a simpler condition that a prime over ℓ is not principal.)

In general, we can show the following by the method in the proof of Proposition 4.3. We use the notation in Sect. 4. Let χ be a Dirichlet character with order prime to p, and $K = k_\chi$ be the field corresponding to the kernel of χ. Assume that $A_K^\chi \simeq O_\chi/p^n$. If ℓ is a prime number which splits completely in K, and satisfies $\ell \equiv 1 \pmod{2p^n}$, and if A_K^χ is generated by a prime above ℓ, and if $(E_K \otimes \mathbf{Z}/p^n)^\chi \longrightarrow (\bigoplus_{v|\ell} \kappa(v)^\times \otimes \mathbf{Z}/p^n)^\chi$ is surjective, then the canonical map $A_K^\chi \longrightarrow A_L^\chi$ is zero where L is the subfield of $K(\mu_\ell)^+$ such that $[L : K] = p^n$.

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