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Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents

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Abstract. There is an obvious topological obstruction for a finite energy unimodular harmonic extension of a S^1-valued function defined on the boundary of a bounded regular domain of R^n. When such extensions do not exist, we use the Ginzburg-Landau relaxation procedure. We prove that, up to a subsequence, a sequence of Ginzburg-Landau minimizers, as the coupling parameter tends to infinity, converges to a unimodular harmonic map away from a codimension-2 minimal current minimizing the area within the homology class induced from the S^1-valued boundary data. The union of this harmonic map and the minimal current is the natural generalization of the harmonic extension.

I. Introduction

I.1. Vortex equations

Complex Ginzburg-Landau equations originated in the theory of superconductivity [16]. When the Ginzburg-Landau parameter is chosen to be a special constant, the equations are called self-dual vortex equations which were carefully studied by Jaffe and Taubes [19].

For the vortex equation on a Riemannian surface Σ, one considers an open, smooth domain $\Omega \subseteq \Sigma$ with, possibly empty, smooth boundary $\partial\Omega$. Let L be a complex line bundle over Ω equipped with a Hermitian metric $< .,. >$. For a section u of L we write $|u(x)|^2 =< u(x), u(x) >$. Then the Ginzburg-Landau functionals are defined for a section u of L and a unitary connection A on L.

The self-dual case of this functional is given by

$$E(u, A, \Omega) = \int_\Omega \left[|dA|^2 + |\nabla_A u|^2 + \frac{1}{4}(1 - |u|^2)^2 \right] dx \quad , \qquad \text{(I.1)}$$

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where dx is the volume form of some fixed Kähler metric on Σ. As usual we adopt the following notations:

$$\nabla_A u = (d - iA)u \quad ;$$

d is the exterior derivative. Hence the unitary property simply means

$$d < u, v > = < \nabla_A u, v > + < u, \nabla_A v >$$

for sections u, v of L. The curvature of A is $F = dA$.
Thus (I.1) is the usual Yang-Mills-Higgs functional for this special case.

In local coordinates (x_1, x_2) on Σ, we write $\nabla_A^k = \nabla_A(\frac{\partial}{\partial x^k}) = \partial_k - iA^k$, $k = 1, 2$, and $F^{kj} = \partial_k A^j - \partial_j A^k = i(\nabla_A^k \nabla_A^j - \nabla_A^j \nabla_A^k)$. Then the Euler-Lagrange equations for E are

$$\begin{cases} \Delta_A u = -\dfrac{1}{2} u (1 - |u|^2) \\ \\ \partial_k F^{kj} = -\Im \left\langle (\partial_j - iA^j) u, u \right\rangle \quad , \end{cases} \tag{I.2}$$

where $\Delta_A = \nabla_A^k . \nabla_A^k$, and where we employ the usual summation convention.

E has two important properties. The first one is called the gauge invariance, i.e. the value of E is invariant under the gauge transformation $(u, A) \to (u \exp(i\psi), A + d\psi)$, for a real valued function ψ. The second important feature of E is the self duality. Namely, decomposing ∇_A into its $(1,0)$ and $(0,1)$ parts, $\nabla_A = \partial_A + \overline{\partial}_A$, in case $\Omega = \mathbb{R}^2$ and if $|u(x)| \to 1$, $\nabla_A u(x) \to 0$ sufficiently fast as $|x| \to \infty$, then E can be written as

$$\begin{aligned} E(u, A) = \int_{\mathbb{R}^2} \left[2 |\overline{\partial}_A u|^2 + \left| * F - \frac{1}{2}(1 - |u|^2) \right|^2 \right] dx \\ + 2\pi d \end{aligned} \tag{I.3}$$

for some integer d, the so-called vortex number (see [19], page 54). Thus we see that the infimum for E, namely $2\pi d$, is attained if and only if the vortex equations

$$\begin{cases} \overline{\partial}_A u = 0 \\ \\ *F = \dfrac{1}{2}(1 - |u|^2) \end{cases} \tag{I.4}$$

are satisfied.

Of course, since E is non negative, this is possible only if $d \geq 0$ (if $d < 0$, one should consider antiholomorphic sections instead of holomorphic ones). Taubes [33] showed that for any collection of N points $x_j \in \mathbb{R}^2$ with multiplicities N_j, there is a solution, unique up to gauge equivalence, of the vortex equations with $u(x_j) = 0$, $j = 1, ..., N$.

The situation for a compact Riemannian surface Σ is the same. One can rewrite E as

$$E(u, A, \Sigma) = \int_{\Sigma} \left[2|\overline{\partial}_A\, u|^2 + \left| *F - \frac{1}{2}(1 - |u|^2) \right|^2 \right] dx \\ + 2\pi\, deg\, L \quad , \tag{I.5}$$

where $deg\, L$ is the degree of L and $*$ denotes the contraction with the Kähler form of Σ. Thus the infimum $2\pi\, deg\, L$ is achieved by the solution of the vortex equations

$$\begin{cases} \overline{\partial}_A\, u = 0 \\ *F = \dfrac{1}{2}(1 - |u|^2) \quad . \end{cases} \tag{I.6}$$

We refer to the works by Bradlow and Garcia-Pradu for the detailed analysis on (I.6) ([6], [7], [13], [14]).

I.2. The scaling effect

On $\mathbb{R}^2$, for the functional

$$E(u, A) = \int_{\mathbb{R}^2} \left[|dA|^2 + |\nabla_A\, u|^2 + \frac{1}{4}(1 - |u|^2)^2 \right] dx \quad ,$$

one can easily introduce the scaling dimensions for u and A in such a way that the term $\int_{\mathbb{R}^2} |\nabla_A u|^2\, dx$ is scaling invariant. Thus we put u to be of dimension 0, A to be of dimension -1 and so $\nabla_A u$ is of dimension -1. The scaled functional is

$$E(u, A) = \int_{\mathbb{R}^2} \left[\varepsilon^2 |dA|^2 + |\nabla_A\, u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \right] dx \quad , \tag{I.7}$$

$0 < \varepsilon < \infty$. It still is self-dual and gauge invariant. The Euler-Lagrange equations for (I.7) are

$$\begin{cases} \Delta_A\, u = -\dfrac{1}{2\varepsilon^2} u(1 - |u|^2) \\ \varepsilon^2 \partial_k F^{kj} = -\Im \left\langle (\partial_j - iA^j)u, u \right\rangle \quad . \end{cases} \tag{I.8}$$

Again, the vortex equations on $\mathbb{R}^2$ are

$$\begin{cases} \overline{\partial}_A\, u = 0 \\ \\ \varepsilon^2 * F = \dfrac{1}{2}(1 - |u|^2) \quad . \end{cases} \tag{I.9}$$

On the general Riemannian surface Σ the second equation becomes

$$\varepsilon^2 * F = \frac{1}{2}(1 - |u|^2) \quad . \tag{I.10}$$

Note that a necessary condition for solving (I.10) on Σ is

$$2\pi\, \varepsilon^2\, deg\, L < \frac{1}{2}\, Vol\, \Sigma \quad . \tag{I.11}$$

The latter will obviously be true when ε is sufficiently small.

In [17], Hong-Jost-Struwe studied the asymptotic behavior of minimal solutions of

$$\begin{cases} \overline{\partial}_{A_\varepsilon} u_\varepsilon = 0 \\ \\ \varepsilon^2 * F_\varepsilon = \dfrac{1}{2}(1 - |u_\varepsilon|^2) \end{cases} \tag{I.12}$$

on a compact Riemannian surface Σ. They showed that, for a fixed $d = deg\, L \geq 0$, and for some sequence $\varepsilon_n \to 0$, there are points x_j, $j = 1, ..., l \leq d$, such that $|u_\varepsilon| \to 1$, $\nabla_{A_\varepsilon} u_\varepsilon \to 0$, $dA_\varepsilon \to 0$ uniformly on compact subsets of $\Sigma \setminus \{x_1, ..., x_l\}$. Moreover, for $h_\varepsilon = *dA_\varepsilon$, one has $h_\varepsilon \longrightarrow 2\pi \sum_{j=1}^{l} \delta_{x_j}$ in the sense of measures, where delta functions have to be counted with multiplicity. This yields a method for degenerating a line bundle L on Σ of degree d into a flat line bundle with $|\,\mathrm{d}\,|$ singularities (counted with multiplicity) and a covariantly constant section.

The above described result is a two-dimensional analogue of works by Taubes ([34], [35]) on the Seiberg-Witten equations. Taubes used them to relate the Seiberg-Witten and Gromov invariants in four dimensional geometry through a similar change of scales.

I.3. Superconductivity

In the theory of superconductivity, particularly for those high T_c superconductors, the coupling constant (or the Ginzburg-Landau parameter) is often very large. Hence instead of (I.7), one has to look at variational integrals:

$$E_\varepsilon(u, A, \Omega) = \int_\Omega \left[|dA|^2 + |\nabla_A\, u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \right] dx \quad , \tag{I.13}$$

for $0 < \varepsilon << 1$. The energy functional (I.13) is, though gauge invariant, no longer self-dual in the sense we discussed before. Thus the analysis has to be done on this variational integral and its corresponding second order Euler-Lagrange equations instead of the first order vortex equations.

When Ω is a two-dimensional domain, and if one ignores the effect of a magnetic field, i.e. the connection A, then it suffices to study the following model problem:

$$\min \int_\Omega \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(|u|^2 - 1)^2 \right] dx \quad . \tag{I.14}$$

The natural boundary condition for (I.14) is the standard Dirichlet boundary condition

$$u_{|\partial\Omega} = g \quad . \tag{I.15}$$

Here u is a complex-valued function and $g : \partial\Omega \longrightarrow S^1$ is a smooth unit vector field of degree d.

In [4], Bethuel-Brezis-Helein systematically analysed the problem (I.14)-(I.15). Then, by taking subsequences if necessary, one has

i)

$$u_{\varepsilon_n}(x) \to u_\star(x) = \prod_{j=1}^{d} \frac{x - a_j}{|x - a_j|} \exp(ih_a(x))$$

in

$$C^{1,\alpha}_{loc}(\overline{\Omega} \setminus \{a_1, ..., a_d\}) \quad ,$$

$$\begin{cases} \Delta h_a = 0 & \text{in} \quad \Omega \\ u_\star = g & \text{on } \partial\Omega \quad ; \end{cases}$$

ii)

$$\frac{1}{2}\int_\Omega \left[|\nabla u_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon^2}(|u_{\varepsilon_n}|^2 - 1)^2 \right] dx$$
$$= \pi d \log \frac{1}{\varepsilon_n} + \min_{b \in \Omega^d} W(b, g, \Omega) + o_{\varepsilon_n}(1) \quad .$$

Here $W(., g, \Omega)$ is a function defined on Ω^d which is called the renormalized energy;

iii) $a \equiv (a_1, ..., a_d)$ is a global minimum of $W(., g, \Omega)$;

iv)

$$\frac{(|u_{\varepsilon_n}|^2-1)^2}{\varepsilon_n^2} \rightharpoonup 2\pi \sum_{j=1}^{d} \delta_{a_j} \quad ,$$

$$\frac{|\nabla u_{\varepsilon_n}|^2}{2\pi \log \frac{1}{\varepsilon}} \rightharpoonup \sum_{j=1}^{d} \delta_{a_j}$$

in the sense of Radon measures.

We remark that the above statements were shown in [4] under the additional assumption that Ω is star-shaped. The key conclusion following from this assumption is the estimate

$$\frac{1}{\varepsilon^2}\int_\Omega (|u_\varepsilon|^2-1)^2 dx \leq C(g,\Omega) \quad . \tag{I.16}$$

Using the approach by Struwe [32], one can drop this additional assumption. Indeed the estimate (I.16) also follows from [32]. Later in [9] an elegant approach showed also this estimate without using the star-shaped property for Ω.

It turns out, from the point of view of analysis, the variational problem (I.13) is a small perturbation of the problem (I.14). Indeed, in [5], Bethuel-Rivière established corresponding results to the ones in [4] for the minimization problem associated with (I.13) with a suitable boundary condition by using similar analytical arguments. See also [10] and [29] for results under a more physical boundary condition and an external applied magnetic field.

I.4. Ginzburg-Landau equations in high dimensions

The purpose of this article is to study the asymptotic behavior of minimizers of the Ginzburg-Landau functionals (I.13) in high dimensions. In [25], the second author first studied the problem when the dimension of Ω is three. He proved among other results that the minimizers of (I.13) converge (by taking a subsequence if needed) away from a one-dimensional length minimizing current. Similar to the situations in the two dimensional case, the analysis in [25] suggest that the essential analytical difficulties in studying such problems lie in the following model problem:

$$\min \int_\Omega \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(|u|^2-1)^2 \right] dx \tag{I.17}$$

subject to the Dirichlet boundary condition

$$u = g_\varepsilon \quad : \partial\Omega \longrightarrow S^1 \quad . \tag{I.18}$$

For this reason we shall therefore discuss only the model problem (I.17)-(I.18).

I.4.1. The Dirichlet boundary condition. To further simplify the presentation we will make use of the following assumptions:

(A1) Ω is a smooth convex domain in $\mathbb{R}^n$, $n \geq 3$;
(A2) on $\partial\Omega$ we prescribed a family of boundary values $g_\varepsilon : \partial\Omega \longrightarrow \mathbb{C}$, for $\varepsilon \leq \varepsilon_0$, such that

(i) $d(\frac{g_\varepsilon}{|g_\varepsilon|}^* d\theta) = \mathbb{S}$, where $\mathbb{S}$ is a fixed smooth $(n-3)$-dimensional current with integer multiplicity (i.e. it can be represented by a $(n-3)$-dimensional smooth compact submanifold in $\partial\Omega$ with integer multiplicity);
(ii) $\|g_\varepsilon\|_\infty \leq 1$, $|g_\varepsilon|(x) \equiv 1$ if $r \geq c\varepsilon$, and $|\nabla^k g_\varepsilon|(x) \leq \frac{C}{\max(r,\varepsilon)^k}$ on $\partial\Omega$, where C and c are a positive constants independent of ε and $r = dist(x, \, spt\,\mathbb{S})$.

From $(A2)$ one deduces in particular

$$\int_{\partial\Omega} |\nabla g_\varepsilon|^2 \leq C \log \frac{1}{\varepsilon} \tag{I.19}$$

and

$$\int_{\partial\Omega} \frac{1}{\varepsilon^2}(|g_\varepsilon|^2 - 1)^2 \leq C \quad , \tag{I.20}$$

where C is a constant independent of ε.

In part IV of this paper we will need to strengthen a bit assumption $(A2)$ and prescribe a more precise shape of g close to its zero set: we will add a third hypothesis to i) and ii).

(iii) There exists $r_1 > 0$ such that for any x^0 in spt $\mathbb{S}$ there exists a diffeomorphism Φ_0 of $B_{r_1}(x^0)$ and a rotation R of $\mathbb{R}^n$ such that

$$g = f_\varepsilon\left(\frac{x}{\varepsilon}\right) \circ \Phi_0 \circ R \quad \text{ where}$$

$$\begin{cases} \Phi_0(x^0) = x^0 \text{ and } |\nabla(\Phi_0 - Id)|(x^0) = 0 \\ f_\varepsilon(x_1, ..., x_n) = h_\varepsilon\left(\frac{x_1 + ix_2}{|x_1 + ix_2|}\right)\chi(|x_1 + ix_2|) \quad , \end{cases}$$

where χ is an increasing function satifying $\chi(0) = 0$ and $\chi \equiv 1$ on $[1, +\infty)$, moreover h_ε is any function from S^1 into S^1 such that $\|\nabla^k h_\varepsilon\|_\infty$ are uniformly bounded, independently of ε.

$(A2)$ enforced by iii) is called $(A2')$.

I.4.2. The energy density concentration set. When $\dim \Omega = n = 3$, $\mathbb{S} = \sum_{j=1}^{k} d_j \delta_{a_j}$, for some $a_j \in \partial\Omega$, $j = 1, ..., k$. Since $\partial\Omega$ is compact, we have $\sum_{j=1}^{k} d_j = 0$. It was shown in [25] that, given ε_n tending to zero, from a subsequence u_{ε_n} of minimizers of (I.17) with $u_{\varepsilon_n} = g_{\varepsilon_n}$ on $\partial\Omega$ among $H^1_{g_{\varepsilon_n}}(\Omega, \mathbb{C})$ one can extract a subsequence which converges in $H^1_{loc}(\overline{\Omega} \setminus spt\,\mathbb{T})$ to a harmonic map u into S^1 where $\mathbb{T}$ is a length minimizing current supported in $\overline{\Omega}$ with $\partial\mathbb{T} = \sum_{j=1}^{k} d_j \delta_{a_j}$. Suppose such a length minimizing current is unique, then the whole family u_ε, $0 < \varepsilon < 1$, converges to u as $\varepsilon \to 0$ whenever g_ε does. In the beginning of Sect. III of this paper we shall give an alternative and much simpler proof of a part of the main result in [25]. For the general dimensions we have the following result which is the first part of the main result in this paper:

Theorem I.1 *Suppose the assumptions* $(A1)$, $(A2)$ *are valid and that* u_ε, $0 < \varepsilon < 1$ *are minimizers of (I.17)-(I.18). Let*

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)\,dx}{\pi \log \frac{1}{\varepsilon}} = \frac{\left[\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(|u_\varepsilon|^2 - 1)^2\right]\,dx}{\pi \log \frac{1}{\varepsilon}} \quad .$$

Then, for any sequence $\varepsilon_n \to 0$, *there is a subsequence of* $\{\mu_{\varepsilon_n}\}$ *that converges weakly (as Radon measures) to a Radon measure* μ *such that* $spt\,\mu = spt\,\mathbb{T}$, $\mu(\Omega) = M(\mathbb{T})$ *(mass of* $\mathbb{T}$*). Here* $\mathbb{T}$ *is an area minimizing codimension two current in* $\mathbb{R}^n$ *with* $\partial\mathbb{T} = \mathbb{S}$. *In the case that such* $\mathbb{T}$ *is unique, the whole family* μ_ε, $0 < \varepsilon < 1$, *converges to* μ *as* $\varepsilon \to 0^+$.

We should point out that the proof of the above theorem does not use the existence of area minimizing currents $\mathbb{T}$ with $\partial\mathbb{T} = \mathbb{S}$. The proof of the latter fact often needs the compactness theorem of Federer-Fleming for integral currents [12]. Thus the paper gives an alternative, though not necessary simple, proof of this useful fact.

At this point, it is interesting to point out that our arguments can be easily adopted to the problem studied in [34] to show the energy concentration set (the collapsing set for harmonic spinors) are two-dimensional area-minimizing surfaces. The holomorphic structure proved in [34] comes from the self-duality property of the Seiberg-Witten functionals considered there.

Though we have studied here a simple minded variational problem, we believe that we have developed here a very general analytical frame-work that can be used in various applications that latter may be more interesting than the main conclusion of the paper.

We should also point out that the infinite energy concentration sets have to be area minimizing is not particularly surprising from formal analysis. It is also naturally suggested by the results on its associated gradient flow [21].

I.4.3. The limiting map. We also give a description of the sequence of minimizers itself. This uses a somehow different approach as the one developed in part III to prove Theorem I.1. In this approach we assume hypothesis $(A2')$ above but we do not have to assume anymore that Ω is convex (hypothesis $(A1)$). Ω can be any regular bounded domain of $\mathbb{R}^n$. In particular $\partial\Omega$ can be topologically different from S^{n-1}. We are still interested in the situation where $g_\varepsilon/|g_\varepsilon|$ admits no extension in $W^{1,2}(\Omega, S^1)$. So either $d\left(\frac{g_\varepsilon}{|g_\varepsilon|}^* d\theta\right) = \mathbb{S} \neq 0$ or we can also have $\frac{g_\varepsilon}{|g_\varepsilon|} \in C^\infty(\Omega, S^1)$, $\pi_1(\partial\Omega) \neq \emptyset$ and there exists at least a generator γ of $\pi_1(\partial\Omega)$ which is contractible in Ω and such that $\deg(\frac{g_\varepsilon}{|g_\varepsilon|}; \gamma) \neq 0$. Of course one can also have both situations together.

We will use the following elementary lemma proved in the appendix:

Lemma A.7. *Let Ω be a bounded regular domain in $\mathbb{R}^n$, let g be a regular map from $\partial\Omega$ into $\mathbb{C}$ such that $g^{-1}(\{0\})$ is a submanifold of $\partial\Omega$. Denote by $\mathbb{S}$ the current $\mathbb{S} = d\left(\frac{g_\varepsilon}{|g_\varepsilon|}^* d\theta\right)$ and $S = spt\,\mathbb{S}$. Then, there exists a class $\mathcal{L}$ in $H_{n-2}(\overline{\Omega}, S, \mathbb{Z})$ such that, for any current $\mathbb{L}$ representing $\mathcal{L}$ one has*

i) g admits a regular extension from $\Omega \backslash spt\,\mathbb{L}$ into S^1.
ii) for any closed curve γ in $\partial\Omega \setminus g^{-1}(\{0\})$ such that $\gamma = \partial\sigma$ where σ is a 2-cycle in Ω we have

$$deg\ (g/|g|, \gamma) = \sigma \frown \mathcal{L} \quad .$$

This class $\mathcal{L}$ is uniquely determined by $\mathbb{S}$ and the degree of g on any closed curve in $\partial\Omega \setminus g^{-1}(\{0\})$. ■

In order to simplify the statement of our second main theorem we will make the following assumption on the boundary condition g_ε

(A3) The class $\mathcal{L} \in H_{n-2}(\Omega, S, \mathbb{Z})$ defined by g_ε is independent on ε, moreover $\mathcal{L} \neq 0$.

The following result generalizes to any dimension the result of F. Bethuel, H. Brezis and F. Hélein in [4] in dimension two and the result of the second author in [25] for the dimension three case.

Theorem I.2 *Let Ω be a bounded domain in $\mathbb{R}^n$, let ε_n be a sequence tending to zero and g_{ε_n} be a sequence of boundary conditions from $\partial\Omega$ into $\mathbb{C}$ verifying $(A2')$ and $(A3)$. If u_{ε_n} denotes a sequence of minimizers of E_{ε_n} then one can extract a subsequence (still denoted u_{ε_n}) which converges in $H^1_{loc}(\Omega \setminus spt\,\mathbb{T}, \mathbb{C})$ to an harmonic map $u_\star$ from $\Omega \setminus spt\,\mathbb{T}$ into S^1, where $\mathbb{T}$ minimizes the area in the class $\mathcal{T}$. Moreover $d(u_\star^* d\theta) = \mathbb{T}$.* ■

Remark I.1 In view of this result the union of the harmonic map $u_\star$ and the minimal current $\mathbb{T}$ is the right object which generalizes the harmonic extension of $g_* = \lim g_\varepsilon$ from Ω into S^1 when it does not exist. ■

I.5. Description of the paper

In Sect. II we shall establish two important ingredients of our proofs. The first is the energy monotonicity property. The second is the η-compactness lemma. The η-compactness lemma was first shown also in [25] for $\dim \Omega = n = 3$, here we generalized it to arbitrary dimension $n \geq 3$. It is the starting point of our analysis. The Sect. III is devoted to the proof of Theorem I.1. In the first part of Sect. III we restrict to the dimension 3 case and we give a relatively simplified proof of a part of the main result in [25]. It also presents the key idea we use in the second part of Sect. III to generalize it to high dimensions. Here we first analyze the defect measure μ and establish various properties concerning *spt* μ, such as its density with respect to $\mathcal{H}^{n-2}$, $(n-2)$-dimensional Hausdorff measure, its rectifiability and orientability. Then we use energy arguments to show $\mathrm{spt}\mu = \mathrm{spt}\,\mathbb{T}$, $M(\mathbb{T}) = \mu(\Omega)$ and $\mathbb{T}$ is an area-minimizing current in Ω with $\partial\mathbb{T} = \mathbb{S}$.

In the final section IV we prove Theorem I.2. This part is independent from part III, in particular we give an alternative proof of Theorem I.1, using the Federer-Fleming Theorem this time. The interest of this approach, which is the high-d version of the approach in [4] for $d = 2$, and [25] for $d = 3$, is that at the same time it gives the convergence of u_ε away from *spt* μ which could not be deduced directly from the approach in the previous section.

II. Fundamental lemmas

II.1. Bounding the energy density

II.1.1. Basic estimates. Suppose u_ε, $0 < \varepsilon < 1$, are minimizers of $E_\varepsilon(.)$ over $H^1_{g_\varepsilon}(\Omega, \mathbb{C})$, then using the maximum principle one has

Lemma II.1 *[3]*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq 1 \quad . \tag{II.1}$$

∎

Using a Gagliardo-Nirenberg type interpolation inequality, Lemma II.1 and the Euler-Lagrange equation one has

Lemma II.2 *[3] There exists $C > 0$ depending only on Ω and the constants in hypothesis $(A2)$ such that*

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} \quad . \tag{II.2}$$

∎

Finally a comparison construction yields the following (see also Lemma III.2):

Lemma II.3 *Let $\mathbb{T}$ be a current representing $\mathcal{T}$, then*

$$E_\varepsilon(u_\varepsilon) \leq \pi\ (M(\mathbb{T}) + \delta)\ \log \frac{1}{\varepsilon} \quad ,$$

for any $\delta > 0$ and for all sufficiently small $\varepsilon > 0$. ■

Note this estimate is particularly simple when $n = 3$.

II.1.2. The monotonicity formula

Lemma II.4 *(Monotonicity formula) The following identity holds*

$$\begin{aligned} &\frac{d}{dr}\left[\frac{1}{r^{n-2}}\int_{B_r}|\nabla u|^2 + \frac{n}{2(n-2)}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2\right] \\ &= \frac{1}{r}\left[\frac{1}{r^{n-3}}\int_{\partial B_r} 2\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{1}{n-2}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2\right] \quad . \end{aligned} \tag{II.3}$$

■

The Euler Lagrange equation for u_ε is

$$-\Delta u = \frac{u}{\varepsilon^2}\left(1-|u|^2\right) \tag{II.4}$$

We multiply $-\Delta u$ by the Pohozaev quantity $\sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}$ and we integrate by parts on B_r. We get

$$\begin{aligned} &\int_{B_r} -\Delta u\ \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i} \\ &= -r\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 + \int_{B_r}|\nabla u|^2 + \int_{B_r}\sum_{i,k=1}^n x_i \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_i} \quad . \end{aligned} \tag{II.5}$$

Integrating by parts on i the last integral of the right-hand side of (II.5) we obtain

$$\int_{B_r}\sum_{i,k=1}^n x_i \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_i} = \frac{1}{2}\int_{\partial B_r} r\ |\nabla u|^2 - \frac{n}{2}\int_{B_r}|\nabla u|^2 \quad . \tag{II.6}$$

Multiplying the right-hand side of the Euler equation (II.4) by the Pohozaev quantity and integrating by parts we obtain

$$\begin{aligned} \int_{B_r} \frac{u}{\varepsilon^2}\left(1-|u|^2\right) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = -\frac{1}{4}\int_{\partial B_r} r\, \frac{\left(1-|u|^2\right)^2}{\varepsilon^2} \\ +\frac{n}{4}\int_{B_r} \frac{\left(1-|u|^2\right)^2}{\varepsilon^2} \quad . \end{aligned} \tag{II.7}$$

Combining (II.4)...(II.7) we get

$$\begin{aligned} -\frac{1}{4}\int_{\partial B_r} r\, \frac{\left(1-|u|^2\right)^2}{\varepsilon^2} + \frac{n}{4}\int_{B_r} \frac{\left(1-|u|^2\right)^2}{\varepsilon^2} \\ = -r\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{r}{2}\int_{\partial B_r} |\nabla u|^2 - \frac{n-2}{2}\int_{B_r} |\nabla u|^2 \quad . \end{aligned} \tag{II.8}$$

We multiply this identity by $\frac{2}{r^{n-1}}$ and we get the desired result. ■

We also need a boundary version of the energy monotonicity formula. Consider a part of $\partial\Omega$ of the form $\partial\Omega \cap B_{r_1}(x^0)$ where $x^0 \in \Omega$. Assume that r_1 is sufficiently small. Assume $0 \in \partial\Omega \cap B_{r_1}(x^0)$. We can parameterize $\partial\Omega \cap B_{r_1}(x^0)$ in the following way

$$\partial\Omega \cap B_{r_1}(x^0) = \{x \in B_1(x^0) :\ x_n = \psi(x_1, ...x_{n-1})\}$$

such that $\psi(0) = |\nabla\psi(0)| = 0$ and let $\|\psi\|_{C^2} \leq \delta_0$. Denote by $d = \mathrm{dist}(x^0, \mathrm{spt}\,\mathbb{S})$ and $\Omega_r := \Omega \cap B_r(x^0)$ Let u_ε be a minimizer of $E_\varepsilon(.)$ on Ω_{r_1}. We have the following boundary version of the energy monotonicity formula.

Lemma II.5 *(Boundary energy monotonicity) With above notations one has, for $r \in (0, r_1)$ and any $0 < \alpha < 1$, that*

$$\begin{aligned} &\frac{d}{dr}\left\{e^{\Lambda r^\alpha} r^{-n+2}\int_{\Omega_r} \frac{1}{2}\left[|\nabla u_\varepsilon|^2 + \frac{n}{2(n-2)\varepsilon^2}\left(|u_\varepsilon|^2-1\right)^2\right] dx\right\} \\ &\geq \frac{r^{-n+2}}{2}\int_{\partial B_r(x^0)\cap\Omega} \left|\frac{\partial u_\varepsilon}{\partial \rho}\right|^2 + \frac{\left(1-|u|^2\right)^2}{\varepsilon^2} \\ &+ r^{-n+1}\int_{\partial\Omega\cap B_r(x^0)} |x.\nu| \left|\frac{\partial u}{\partial \nu}\right|^2 dx - C\frac{r}{d^2} e^{\Lambda r^\alpha} \quad . \end{aligned} \tag{II.9}$$

Here Λ and C are constants depending only on the upper bound of the ratio $r^{1-\alpha}/d$, on δ_0 and the constants in the hypothesis $(A2)$, but not on r or ε. ■

Proof. Assume first for the simplicity of the presentation that $\Omega_r = B_r(x^0)\cap\Omega$ is star-shaped around x^0, this implies in particular $(x-x^0).\nu \geq 0$ for any $x\in\partial\Omega\cap B_r(x^0)$. Multiply the equation $\Delta u+\frac{1}{\varepsilon^2}u(1-|u|^2)=0$ by the Pohozaev multiplier $\sum_i(x_i-x_i^0)\frac{\partial u}{\partial x_i}$. Integrating on Ω_r like in the proof of the interior monotonicity formula one gets

$$\begin{aligned}
&\frac{d}{dr}\left[\frac{1}{r^{n-2}}\int_{\Omega_r}|\nabla u|^2+\frac{n}{2(n-2)}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2\right]\\
&=\frac{1}{r}\left[\frac{1}{r^{n-3}}\int_{\partial B_r\cap\Omega}2\left|\frac{\partial u}{\partial\nu}\right|^2+\frac{1}{n-2}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2\right]\\
&+\frac{1}{r^{n-1}}\int_{\partial\Omega\cap B_r}(x-x^0).\nu\left|\frac{\partial u}{\partial\nu}\right|^2-\frac{1}{r^{n-1}}\int_{\partial\Omega\cap B_r}(x-x^0).\nu\,|\nabla g|^2\\
&+\frac{2}{r^{n-1}}\int_{\partial\Omega\cap B_r}(x-x^0).\tau\frac{\partial u}{\partial\nu}.\frac{\partial g}{\partial\tau}\quad ,
\end{aligned}\tag{II.10}$$

where B_r means $B_r(x^0)$ and where $\tau|x-x^0|$ is the orthogonal projection of $x-x^0$ on the tangent plane of $\partial\Omega$ at x. Of course, the worst term to deal with is the last one. In order to bound it one uses an idea from [8]. One can always find an extension $\overline{g}$ of g in Ω_r such that $|\nabla\overline{g}|\leq C/d$ and $|\nabla^2\overline{g}|\leq C/d^2$. The idea is to multiply the equation satisfied by u by $\sum_i(x_i-x_i^0)\frac{\partial\overline{g}}{\partial x_i}$ and to integrate it on Ω_r. This yields

$$\begin{aligned}
\int_{\partial\Omega\cap B_r}(x-x^0).\tau\frac{\partial u}{\partial\nu}.\frac{\partial g}{\partial\tau}&=-\int_{\partial\Omega_r}(x-x^0).\nu\frac{\partial\overline{g}}{\partial\nu}.\frac{\partial u}{\partial\nu}\\
&+\int_{\Omega_r}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_k}\left((x_i-x_i^0)\frac{\partial\overline{g}}{\partial x_i}\right)\\
&-\int_{\Omega_r}u\frac{(1-|u|^2)}{\varepsilon^2}(x_i-x_i^0)\frac{\partial\overline{g}}{\partial x_i}\quad .
\end{aligned}\tag{II.11}$$

The first term of the right-hand side of (II.11) is bounded by

$$\begin{aligned}
\int_{\partial\Omega_r}(x-x^0).\nu\frac{\partial\overline{g}}{\partial\nu}.\frac{\partial u}{\partial\nu}&\leq\frac{1}{d}\int_{\partial\Omega_r}(x-x^0).\nu\left|\frac{\partial u}{\partial\nu}\right|\\
&\leq\frac{C}{\delta}\frac{r^n}{d^2}+\delta\int_{\partial\Omega_r}(x-x_0).\nu\left|\frac{\partial u}{\partial\nu}\right|^2\quad .
\end{aligned}\tag{II.12}$$

The second term of the right-hand side of (II.11) can be bounded in the following way

$$\int_{\Omega_r} \frac{\partial u}{\partial x_k} \frac{\partial}{\partial x_k} \left((x_i - x_i^0) \frac{\partial \overline{g}}{\partial x_i} \right) \leq \frac{C}{d} \int_{\Omega_r} |\nabla u| \leq C \frac{r^{n-\alpha}}{d^2} + r^\alpha \int_{\Omega_r} |\nabla u|^2 \quad . \tag{II.13}$$

Finally for the last term of the right-hand side of (II.11) we write

$$\int_{\Omega_r} u \frac{(1-|u|^2)}{\varepsilon^2} (x_i - x_0) \frac{\partial \overline{g}}{\partial x_i} \leq \frac{Cr}{d} \int_{\Omega_r} \frac{|1-|u|^2|}{\varepsilon^2} \quad . \tag{II.14}$$

Now the difficulty is to handle the term $\int_{\Omega_r} \frac{1-|u_\varepsilon|^2}{\varepsilon^2} dx$. Here we should point out that $|u_\varepsilon| \leq 1$. To estimate $\int_{\Omega_r} \frac{1-|u_\varepsilon|^2}{\varepsilon^2} dx$, we use the same trick as in [8], we multiply the equation $\Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon \left(1 - |u_\varepsilon|^2\right) = 0$ by $\phi\, \left(1 - |u_\varepsilon|^2\right)\, u_\varepsilon$. Here $\phi(t)$ is a smooth positive function of $t \geq 0$ such that $\phi(0) = 0, \phi(t) = 1$ for $t \geq \varepsilon^2$, and $\phi'(t) \geq 0$. Recall that $|g|(x) \equiv 1$ if dist $(x,\, \mathrm{spt}\, \mathbb{S}) \geq \varepsilon$. After integration by parts, we obtain

$$\int_{\Omega_r} \frac{|u_\varepsilon|^2 \left(1 - |u_\varepsilon|^2\right)}{\varepsilon^2} dx \leq \int_{\Omega_r} |\nabla u_\varepsilon|^2 dx + \int_{\partial B_r(0) \cap \Omega} \left| u_\varepsilon . \frac{\partial u_\varepsilon}{\partial \rho} \right| + C \frac{r^n}{d^2} \quad .$$

Therefore

$$\begin{aligned} \int_{\Omega_r} \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} dx &\leq \int_{\Omega_r} \frac{(1-|u|^2)^2}{\varepsilon^2} + \int_{\Omega_r} \frac{|u_\varepsilon|^2 \left(1 - |u_\varepsilon|^2\right)}{\varepsilon^2} + C \frac{r^n}{d^2} \\ &\leq \int_{\Omega_r} e_\varepsilon(u) + \int_{\partial B_r(0) \cap \Omega} \left| \frac{\partial u_\varepsilon}{\partial \rho} \right| + C \frac{r^n}{d^2} \quad . \end{aligned} \tag{II.15}$$

combining (II.12), (II.14) and (II.15) we get

$$\int_{\Omega_r} u \frac{(1-|u|^2)}{\varepsilon^2} x_i \frac{\partial \overline{g}}{\partial x_i} \leq C \frac{r^{1-\alpha}}{d}\, r^\alpha E_\varepsilon(u) + \frac{C}{\delta} \frac{r^n}{d^2} + \delta \int_{\partial B_r \cap \Omega} x.\nu \left| \frac{\partial u}{\partial \nu} \right|^2 \quad . \tag{II.16}$$

Combining now (II.10), (II.11), (II.12), (II.13) and (II.16) we get, for $\delta = 1/8$

$$\begin{aligned} Y' + \frac{\alpha \Lambda}{r^{1-\alpha}} Y \geq \frac{1}{r} &\left[\frac{1}{2r^{n-3}} \int_{\partial B_r \cap \Omega} 2 \left| \frac{\partial u}{\partial \nu} \right|^2 + \frac{1}{n-2} \frac{1}{\varepsilon^2} \left(1 - |u|^2\right)^2 \right] \\ &+ \frac{1}{r^{n-2}} \int_{\partial \Omega \cap \Omega} |x.\nu| \left| \frac{\partial u}{\partial \nu} \right|^2 - C \frac{r}{d^2} \quad , \end{aligned} \tag{II.17}$$

where

$$Y = r^{-n+2} \int_{\Omega_r} \frac{1}{2} \left[|\nabla u_\varepsilon|^2 + \frac{n}{2(n-2)\varepsilon^2} \left(|u_\varepsilon|^2 - 1\right)^2 \right] dx \quad .$$

Multiplying (II.17) by $\exp(\Lambda r^\alpha)$ we get (II.9).

We do not assume anymore that Ω_r is star-shaped. We have to study the perturbation terms induced by omitting this assumption.

We claim that

$$\forall x \in \partial\Omega \cap B_r(x^0) \qquad (x - x^0).\nu \geq |(x - x^0).\nu| - cr^2 \quad , \tag{II.18}$$

where c is independent of r or x^0 in Ω. We have

$$\nu = \frac{1}{(1 + |\nabla \psi|^2)^{1/2}} \left(e_n - \sum_{i=1}^{n-1} \frac{\partial \psi}{\partial x_i} e_i \right) \quad .$$

Since $\nabla\psi(0) = 0$ we have $|\nabla\psi|(x) \leq Cr$ and in order to prove (II.18) it suffices to prove

$$\forall x \in \partial\Omega \cap B_r(x^0) \qquad (x - x^0).e_n \geq -cr^2 \quad . \tag{II.19}$$

One can notice that it suffices to prove the previous identity with y^0 instead of x^0, where y^0 is the projection of x^0 on $\partial\Omega$ along e_n (i.e. $y^0 = x^0 + \lambda e_n$, where $\lambda \geq 0$). Since $\nabla\psi(0) = 0$ we have $|\psi(x) - \psi(y^0)| \leq Cr^2$ and this implies

$$(x - y^0).e_n \geq -cr^2 \quad .$$

From this identity and the discussion above we deduce (II.18). So, without the star-shapedness assumption for Ω_r, instead of (II.17) we get

$$\begin{aligned} Y' + \frac{\alpha\Lambda}{r^{1-\alpha}} Y \geq \frac{1}{r} & \left[\frac{1}{2r^{n-3}} \int_{\partial B_r \cap \Omega} 2 \left| \frac{\partial u}{\partial \nu} \right|^2 + \frac{1}{n-2} \frac{1}{\varepsilon^2} \left(1 - |u|^2\right)^2 \right] \\ & + \frac{1}{r^{n-2}} \int_{\partial\Omega \cap B_r} |x.\nu| \left| \frac{\partial u}{\partial \nu} \right|^2 \\ & - \frac{C}{r^{n-3}} \int_{\partial\Omega \cap B_r} \left| \frac{\partial u}{\partial \nu} \right|^2 - C \frac{r}{d^2} \quad . \end{aligned} \tag{II.20}$$

So we have to bound the term $\frac{C}{r^{n-3}} \int_{\partial\Omega\cap B_r} \left|\frac{\partial u}{\partial\nu}\right|^2$. Observe that we have $\frac{C}{r^{n-3}}$ in front of the integral and not $\frac{C}{r^{n-2}}$, this is the reason why this term is not so bad. For r_1 sufficiently small compared to the C^2 -norm of $\partial\Omega$ one can ensure that Ω_r is always star-shaped for $r \leq r_1$ around some point z^0 for which one has $(x - z^0).\nu \geq cr$, where c is independent of r or x^0. One can apply all the previous Pohozaev arguments on Ω_r but around z^0. Using similar estimates as above one deduces that

$$\frac{1}{r^{n-3}}\int_{\partial\Omega\cap\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 \leq C\left[Y + r^{-n+3}(r^{n-2}Y)' + \frac{r^2}{d^2}\right] \quad . \tag{II.21}$$

Inserting this estimate in (II.20) one gets a similar estimate as (II.17) and we can conclude in the same way. ∎

II.1.3. A uniform bound of the energy density. In Part IV we will need the following bound for the density of energy

Lemma II.6 *For any x^0 in $\overline{\Omega}$ the following bound holds*

$$\frac{1}{r^{n-2}}\int_{B_r(x^0)\cap\Omega}|\nabla u|^2 + \frac{1}{\varepsilon^2}(1-|u|^2)^2 \leq C\log\frac{1}{\varepsilon} \quad , \tag{II.22}$$

where C is independent on ε, r and x^0. ∎

Proof. It is clear from the global upper bound of the energy given by Lemma II.3 and from the monotonicity formulas (Lemma II.4 and Lemma II.5) that (II.22) holds for any $x^0 \in K$, where K is a compact set included in $\overline{\Omega}\setminus \mathrm{spt}\,\mathbb{S}$ and for any $r > 0$ but the constant a priori could depend on K.

Let us take $x^0 \in \mathrm{spt}\,\mathbb{S}$ and prove that (II.22) holds for a C independent of r, $x^0 \in \mathrm{spt}\,\mathbb{S}$ and ε. We use the notations of the proof of Lemma II.5, for instance Y still denotes the density of energy on $\Omega_r = \Omega \cap B_r$: $Y = \frac{1}{r^{n-2}}E_\varepsilon(u)(B_r)$. (II.10) implies, using (II.19) (in fact since $x^0 \in \partial\Omega$ similar arguments as the ones used to prove (II.19) give also $|(x - x^0).\nu| \leq Cr^2$).

$$\begin{aligned} Y' \geq &- \frac{C}{r^{n-3}}\int_{\partial\Omega\cap B_r}\left|\frac{\partial u}{\partial\nu}\right|^2 - \frac{C}{r^{n-3}}\int_{\partial\Omega\cap B_r}|\nabla g|^2 \\ &+ \frac{2}{r^{n-1}}\int_{\partial\Omega\cap B_r}(x - x^0).\tau\frac{\partial u}{\partial\nu}.\frac{\partial g}{\partial\tau} \quad , \end{aligned} \tag{II.23}$$

where $\tau|x - x^0|$ is the orthogonal projection of $x - x^0$ on the tangent plane of $\partial\Omega$ at x. First of all we have

$$\frac{1}{r^{n-3}}\int_{\partial\Omega\cap B_r}|\nabla g|^2 \leq \frac{C}{r^{n-3}}\int_{\partial\Omega\cap B_r}\frac{1}{\max(\mathrm{dist}(x, \mathrm{spt}\,\mathbb{S}), \varepsilon)^2} \leq C\log\frac{1}{\varepsilon} \quad . \tag{II.24}$$

We have also the following a-priori bound

$$\frac{1}{r^{n-2}}\int_{\partial\Omega\cap B_r(x^0)}\left|\frac{\partial u}{\partial\nu}\right|^2 \leq C\left[Y' + \frac{Y}{r} + \frac{\log\frac{1}{\varepsilon}}{r}\right] \quad . \tag{II.25}$$

Indeed (II.25) is established in the following way: take a point z^0 in $\Omega \cap B_r(x^0)$ around which $\Omega \cap B_r(x^0)$ is star-shaped with $(x - z^0).\nu \geq cr^2$ where c is some universal constant. This is always possible for r small compared to the C^2 -norm of $\partial\Omega$. Let us apply the Pohozaev formula in $\Omega \cap B_r(x^0)$ around z^0. This easily implies

$$\frac{1}{r^{n-2}} \int_{\partial\Omega\cap B_r(x^0)} \left|\frac{\partial u}{\partial \nu}\right|^2 \leq \frac{C}{r^{n-1}} \int_{\Omega\cap B_r(x^0)} e_\varepsilon(u) + \frac{C}{r^{n-2}} \int_{\Omega\cap\partial B_r(x^0)} e_\varepsilon(u)$$
$$+ \frac{C}{r^{n-2}} \int_{\partial\Omega\cap B_r(x^0)} |\nabla g|^2 + |\nabla g| \left|\frac{\partial u}{\partial \nu}\right| \quad . \tag{II.26}$$

Observe that we have from hypothesis $(A2')$

$$\frac{1}{r^{n-2}} \int_{\partial\Omega\cap B_r(x^0)} |\nabla g|^2 \leq C \frac{\log \frac{1}{\varepsilon}}{r} \quad . \tag{II.27}$$

Thus combining (II.26) and (II.27) one gets (II.25). Observe now that from hypothesis $(A2')$ on g one deduces that

$$(x - x^0).\tau \frac{\partial g}{\partial \tau} \leq \left| \left(x_i \frac{\partial f_\varepsilon}{\partial x_i} \right) \circ \Phi_0 \right| + Cr^2 |\nabla g| \quad , \tag{II.28}$$

where $f_\varepsilon(x)$ denotes $f_\varepsilon(x) = h_\varepsilon \left(\frac{x_1 + ix_2}{|x_1 + ix_2|} \right) \chi \left(\frac{|x_1 + ix_2|}{\varepsilon} \right)$, (recall that from $(A2')$ we have $\operatorname{spt}(\chi - 1) \subset [0, 1]$). Thus $x_i \frac{\partial f_\varepsilon}{\partial x_i} \circ \Phi_0$ has a support in $S_\varepsilon = \{x \in \partial\Omega\ ;\ \operatorname{dist}(x; \operatorname{spt} \mathbb{S}) \leq \varepsilon\}$ and is bounded by Cr/ε. Combining this fact with (II.28) we get

$$\frac{1}{r^{n-1}} \int_{\partial\Omega\cap B_r} (x - x^0).\tau \frac{\partial u}{\partial \nu}.\frac{\partial g}{\partial \tau} \leq \frac{C}{r^{n-2}} \frac{1}{\varepsilon^2} |S_\varepsilon \cap B_r|$$
$$+ \frac{C}{r^{n-3}} \int_{\partial\Omega\cap B_r} |\nabla g| \left|\frac{\partial u}{\partial \nu}\right| \quad . \tag{II.29}$$

Using the fact that $|S_\varepsilon \cap B_r| \leq C\varepsilon^2 r^{n-3}$, (II.23), (II.24), (II.25) and (II.29) we finally obtain

$$Y' \geq -C \log \frac{1}{\varepsilon} - CY - \frac{C}{r} \quad . \tag{II.30}$$

This differential inequality integrated between 1 and any $r \geq \varepsilon$ gives the result for $x^0 \in \operatorname{spt} \mathbb{S}$ (for $r \leq \varepsilon$ (II.22) is a direct consequence of the L^∞ bounds $\|u\|_\infty \leq 1$ and $\|\nabla u\|_\infty \leq \frac{C}{\varepsilon}$, see Lemma II.1 and Lemma II.2).

Now take any point $x^0 \in \overline{\Omega}$ and any $r > 0$. Let $d = \text{dist}(x^0; \text{spt}\, \mathbb{S})$. If $r > d/2$, let $z^0 \in \text{spt}\, \mathbb{S}$ such that $d = |x^0 - z^0|$. Since we have proven the lemma for any point on spt $\mathbb{S}$ we have

$$\frac{1}{r^{n-2}} \int_{B_r(x^0)\cap\Omega} e_\varepsilon(u) \leq \frac{1}{r^{n-2}} \int_{B_{3r}(z^0)\cap\Omega} e_\varepsilon(u) \leq C \log \frac{1}{\varepsilon}$$

and the lemma is proven in this case. So we just have to consider the case where $r < d/2$ and we can also assume that $x^0 \in \partial\Omega$. Indeed, if the lemma is proven for the point on the boundary, for any point x^0 one has the estimate (II.22) for it's projection on $\partial\Omega$. We use it for $r \geq 2$ dist (x^0, Ω), this gives the estimate for x^0 and any $r >$ dist $(x^0, \Omega)/2$ and the estimate (II.22) between ε and dist $(x^0, \Omega)/2$ is just a consequence of the interior monotonicity formula. Thus we have $x^0 \in \partial\Omega$ and $r < d/2$. On $\partial\Omega \cap B_r(x^0)$ we have $|\nabla g| \leq \frac{C}{d}$. Thus the Pohozaev identity implies

$$Y' \geq -\frac{C}{r^{n-3}} \int_{\partial\Omega\cap B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 - C\frac{r^2}{d^2} - \frac{C}{r^{n-2}} \frac{1}{d} \int_{\partial\Omega\cap B_r} \left|\frac{\partial u}{\partial \nu}\right| \quad . \tag{II.31}$$

Using (II.25) we bound the last term of the right-hand side of (II.31) in the following way

$$\frac{C}{r^{n-2}} \frac{1}{d} \int_{\partial\Omega\cap B_r} \left|\frac{\partial u}{\partial \nu}\right| \leq C\frac{r^{\frac{1}{2}}}{d} \left(Y' + C\frac{Y}{r} + \frac{\log\frac{1}{\varepsilon}}{r}\right)^{\frac{1}{2}} \quad . \tag{II.32}$$

Combining (II.31) and (II.32) we get

$$Y' \geq -C\left[Y + \frac{1}{d}\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}} + \frac{r^{\frac{1}{2}}}{d}\left(Y' + C\frac{Y}{r}\right)^{\frac{1}{2}}\right] \quad . \tag{II.33}$$

So at any point $d/2 \geq s \geq r$ one of these 4 possibilities occur

$$\begin{cases} Y'(s) \geq -CY(s) \\ \\ Y'(s) \geq \dfrac{C}{d}\left(\log\dfrac{1}{\varepsilon}\right)^{\frac{1}{2}} \\ \\ Y'(s) \geq -C\frac{s}{d^2} \\ \\ Y'(s) \geq -\frac{C}{d}Y^{\frac{1}{2}} \quad . \end{cases} \tag{II.34}$$

Integrating all these possibilities between r and $d/2$ one get $Y(r) \leq CY(d) + C(\log\frac{1}{\varepsilon})^{\frac{1}{2}}$. Since (II.22) holds for $d/2$, this implies (II.22) for r and Lemma II.6 is proven. ∎

II.2. The eta-compactness Lemma

This part of our work is devoted to the proof of one of the main properties we use for solutions of the complex Ginzburg-Landau functional: the eta-compactness property. This roughly says that if the energy in a ball is sufficiently small then the density of the order parameter $|u|$ cannot approach 0 on the ball of half radius and if this remains true as the coupling constant tends to infinity we will have compactness on this ball. This property is reminiscent of the "ϵ-regularity" lemma proved by R. Schoen and K. Uhlenbeck for the minimizing harmonic map (see [28]). The eta-compactness Lemma was proved in the 3-dimensional case for minimizers in [25], it can also be used for the study of similar loss of compactness for the minimizing sequence of the gauge invariant Ginzburg-Landau functional in dimension 2 (see [26]). Here we give a proof of this eta-compactness property in any dimension and for critical points in general. This proof follows step by step the one in [25] except at the end, where the comparison argument using the minimality of the solution is replaced by a more refined one requiring only the fact that we have a critical point of the Ginzburg-Landau functional.

Let Ω be a domain in $\mathbb{R}^n$ for $n > 2$.

Lemma II.7 *(eta-compactness) Let u be a critical point of the Ginzburg-Landau functional satisfying $|u| \leq 1$ and $\|\nabla u\|_\infty \leq C/\varepsilon$ where C is independent of ε, then there exists η, λ and ε_o such that for any $\varepsilon < \varepsilon_o$ and for any ball $B_\rho(x^0) \subset \Omega$ where $\rho \geq \lambda\varepsilon$,*

$$\frac{1}{\rho^{n-2}} E_\varepsilon(u)\left(B_\rho(x^0)\right) \leq \eta\ \log\frac{\rho}{\varepsilon} \quad \Rightarrow \quad |u|(x^0) \geq \frac{1}{2} \quad .$$

■

Proof of the η-compactness lemma

We introduce the following notations

$$E_r = \int_{B_r} |\nabla u|^2 + \frac{n}{2(n-2)}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2 \quad ,$$

$$I_r = \int_{\partial B_r} |\nabla u|^2 + \frac{n}{2(n-2)}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2 = \frac{dE_r}{dr} \quad ,$$

$$F_r = \int_{B_r} 2\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{1}{n-2}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2 \quad ,$$

$$J_r = \int_{B_r} 2\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{1}{n-2}\frac{1}{\varepsilon^2}\left(1-|u|^2\right)^2 = \frac{dF_r}{dr} \quad .$$

Using these notations, Lemma II.4 becomes

$$\frac{d}{dr}\left[\frac{E_r}{r^{n-2}}\right] = \frac{1}{r^{n-2}}\frac{dF_r}{dr} \quad . \tag{II.35}$$

The hypothesis implies in particular

$$\int_\varepsilon^\rho \frac{1}{r}\left[\frac{1}{r^{n-3}}J_r\right] \leq \eta \ \log\frac{\rho}{\varepsilon} \quad .$$

Integrating by parts $\int_\varepsilon^\rho \frac{1}{r}\left[\frac{1}{r^{n-3}}J_r\right] = \int_\varepsilon^\rho \frac{1}{r^{n-2}}\frac{dF_r}{dr}$ we obtain

$$\frac{F_\rho}{\rho^{n-2}} + (n-2)\int_\varepsilon^\rho \frac{1}{r}\left[\frac{1}{r^{n-2}}F_r\right] \leq \eta \ \log\frac{\rho}{\varepsilon} + \frac{F_\varepsilon}{\varepsilon^{n-2}} \quad .$$

Using the fact that $\|\nabla u\|_\infty \leq C/\varepsilon$ and $|u| \leq 1$ we obtain

$$\frac{F_\varepsilon}{\varepsilon^{n-2}} = \int_{B_\varepsilon} \frac{2}{\varepsilon^{n-2}}\left|\frac{\partial u}{\partial \nu}\right|^2 + \frac{1}{n-2}\frac{1}{\varepsilon^n}\left(1-|u|^2\right)^2 \leq C \quad .$$

Thus, if $\eta \log\lambda \geq C$ (i.e. $\lambda \geq \exp(\frac{C}{\eta})$), since $\rho \geq \lambda\varepsilon$, we have $F_\varepsilon/\varepsilon^{n-2} \leq \eta \ \log \rho/\varepsilon$ and finally we get

$$\int_\varepsilon^\rho \frac{1}{r}\left[\frac{1}{r^{n-2}}F_r + \frac{1}{r^{n-3}}J_r\right] \leq C\eta \ \log\frac{\rho}{\varepsilon} \quad .$$

Using the mean value formula we deduce the existence of $r_1 \in [2\varepsilon, \rho]$ such that

$$\frac{1}{r_1^{n-3}}J_{r_1} + \frac{1}{\left(\frac{r_1}{2}\right)^{n-3}}J_{\frac{r_1}{2}} + \frac{F_{r_1}}{r_1^{n-2}} \leq C\,\eta \quad . \tag{II.36}$$

We make the following change of scale $r_1 \to 1$ and $u \to \tilde{u}$. Thus $\tilde{u}$ is a minimizer of

$$\int_{B_1} |\nabla\tilde{u}|^2 + \left(\frac{r_1}{\varepsilon}\right)^2\left(1-|\tilde{u}|^2\right)^2 \quad .$$

Using Lemma A.6 and the fact that $\Delta\tilde{u}$ is parallel to $\tilde{u}$ we have in $T_1 = B_1 \setminus B_{\frac{1}{2}}$

$$\begin{aligned}\Delta\left(\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial r}\right) &= -2\,(*)d_\top\left[\left(i\frac{\partial\tilde{u}}{\partial r};(*)d_\top\tilde{u}\right)\right]\\ &\quad + \frac{2}{r}\frac{\partial}{\partial r}\left(\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial r}\right) + \frac{n-1}{r^2}\left(\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial r}\right) \quad ,\end{aligned} \tag{II.37}$$

where $(*)$ and $d_\top$ respectively denote the Hodge operator and the external differentiation on ∂B_r and $(a;b)$ is the scalar product between two complex numbers a and b. Let Δ_0^{-1} be the inverse of the Laplace Beltrami operator on $n-2$-form Δ in T_1 for the Dirichlet boundary conditions ($v_{|\partial T_1}=0$ and $*v_{|\partial T_1}=0$) and let v be the following $n-2$-forms in T_1

$$v = -2\Delta_0^{-1}\left(i\frac{\partial\tilde{u}}{\partial r};(*)d_\top\tilde{u}\right) \quad . \tag{II.38}$$

Denote also by Δ_0^{-1} the inverse operator of the Laplace operator on functions for Dirichlet boundary conditions and by H the following function on T_1

$$H = \Delta_0^{-1}\left((*)d_\top\Delta v - \Delta(*)d_\top v\right) \quad . \tag{II.39}$$

We claim that $\forall 1<p<+\infty$ we have

$$\int_{T_1}|\nabla H|^p \le C_p\int_{T_1}|\nabla v|^p \quad . \tag{II.40}$$

Indeed, let $\omega_{S^{n-1}}$ be the volume form on $S_r=\partial B_r$ such that $*dr=\omega_{S^{n-1}}$. We have $(*)d_\top\Delta v = \langle d\Delta v;\omega_{S^{n-1}}\rangle$.
Write

$$v=\sum_{i<k}v_{ik}\,dx_1\wedge\ldots\wedge d\check{x}_i\ldots\wedge d\check{x}_k\ldots\wedge dx_n \quad .$$

We have $dv=\sum_{k=1}^n\sum_{i\neq n}\frac{\partial v_{ik}}{\partial i}(-1)^{i-1}dx_1\wedge\ldots\wedge d\check{x}_k\ldots\wedge dx_n$ (where $v_{ik}:=v_{ki}$ for $i>k$). Thus

$$\begin{aligned}\Delta(*)d_\top v &= \Delta\left(\langle dv;\omega\rangle\right)=\Delta\sum_{i\neq k}\frac{x_k}{r}\frac{\partial v_{ik}}{\partial x_i}\\ &=\sum_{i\neq k}\frac{x_k}{r}\frac{\partial\Delta v_{ik}}{\partial x_i}+(\text{ derivatives of } v \text{ of order } \le 2)\\ &=\langle d\Delta v;\omega\rangle+(\text{ derivatives of } v \text{ of order } \le 2)\quad .\end{aligned}$$

This proves (II.40). Denote by K the following function in T_1

$$K=\Delta_0^{-1}\left(\frac{2}{r}\frac{\partial}{\partial r}\left(\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial r}\right)+\frac{n-1}{r^2}\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial r}\right) \quad . \tag{II.41}$$

Thus we have

$$\begin{cases}\Delta\left(\tilde{u}\wedge\dfrac{\partial\tilde{u}}{\partial r}-(*)d_\top v+H-K\right)=0 & \text{in } T_1\\[2ex] \tilde{u}\wedge\dfrac{\partial\tilde{u}}{\partial r}-(*)d_\top v+H-K=\tilde{u}\wedge\dfrac{\partial\tilde{u}}{\partial r} & \text{on } \partial T_1\quad .\end{cases}$$

Let $\xi = \tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r} - (*) d_\top v + H - K$ in T_1, using standard results on harmonic functions we have, for any domain $\omega \subset\subset T_1$,

$$\int_\omega |\nabla \xi|^2 \leq C(\omega) \int_{\partial T_1} |\xi|^2 \leq C(\omega) \int_{\partial T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2 \quad . \tag{II.42}$$

Choose $\omega = B_{7/8} \setminus B_{5/8}$. In ω we have

$$\begin{aligned} \frac{\partial}{\partial r}\left(\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r}\right) &= \frac{\partial \xi}{\partial r} + (*) \frac{\partial}{\partial r} d_\top v + \frac{\partial H}{\partial r} - \frac{\partial K}{\partial r} \\ &= \frac{\partial \xi}{\partial r} + (*) d_\top \left(\frac{\partial v}{\partial r}\right) - \frac{1}{r}(*) d_\top v + \frac{\partial H}{\partial r} - \frac{\partial K}{\partial r} \quad . \end{aligned} \tag{II.43}$$

Let $1 < q < \frac{n}{n-1}$, using standard elliptic estimates and the mean value formula, we deduce from (II.38), (II.40), (II.41), (II.42) and (II.43) that there exists $t \in (5/8, 7/8)$ such that

$$\begin{aligned} &\left(\int_{\partial B_t} |\nabla \xi|^2\right)^{\frac{1}{2}} \leq \left(\int_{\partial T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{\frac{1}{2}} \quad , \\ &\left(\int_{\partial B_t} \left|\frac{\partial v}{\partial r}\right|^q\right)^{\frac{1}{q}} \leq C \left(\int_{T_1} |\nabla \tilde{u}|^2\right)^{\frac{1}{2}} \times \left(\int_{T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{\frac{1}{2}} \quad , \\ &\left(\int_{\partial B_t} \left|\frac{\partial K}{\partial r}\right|^2\right)^{\frac{1}{2}} \leq C \left(\int_{T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{\frac{1}{2}} \quad , \\ &\left(\int_{\partial B_t} \left|\frac{\partial H}{\partial r}\right|^q\right)^{\frac{1}{q}} \leq C \left(\int_{T_1} |\nabla \tilde{u}|^2\right)^{\frac{1}{2}} \times \left(\int_{T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{\frac{1}{2}} \quad , \\ &\left(\int_{\partial B_t} |\nabla \tilde{u}|^2\right)^{\frac{1}{2}} \leq C \left(\int_{T_1} |\nabla \tilde{u}|^2\right)^{\frac{1}{2}} \quad , \\ &\left(\int_{\partial B_t} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2 + \left(\frac{r_1}{\varepsilon}\right)^2 \left(1 - |\tilde{u}|^2\right)^2\right)^{\frac{1}{2}} \leq C \left(\int_{T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2 + \left(\frac{r_1}{\varepsilon}\right)^2 \left(1 - |\tilde{u}|^2\right)^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{II.44}$$

Combining the previous inequalities, (II.42) and (II.43) we obtain

$$\begin{aligned} \left\|\frac{\partial}{\partial r}\left(\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r}\right)\right\|_{W^{-1,q}(\partial B_t)} \leq\ & C \left(\int_{T_1} |\nabla \tilde{u}|^2\right)^{\frac{1}{2}} \times \left(\int_{T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{\frac{1}{2}} \\ &+ C \left(\int_{T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{\frac{1}{2}} + C \left(\int_{\partial T_1} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{II.45}$$

Using the fact that $\tilde{u} \wedge \Delta \tilde{u} = 0$ we deduce that

$$d_\top^{(*)} (\tilde{u} \wedge d_\top \tilde{u}) = \tilde{u} \wedge \Delta_r \tilde{u} = + \frac{\partial}{\partial r}\left(\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r}\right) + \frac{n-1}{r} \tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r} \quad , \tag{II.46}$$

where Δ_r denotes the Laplace operator on ∂B_r and $d_\top^{(*)}$ the adjoint of the exterior differentiation $d_\top$ for the scalar product induced on ∂B_r.
Using (II.36) and (II.44) we have

$$\left\{\begin{aligned} &\int_{\partial B_t}\left(\frac{r_1}{\varepsilon}\right)^2\left(1-|\tilde{u}|^2\right)^2 \le C\,\eta \\ &\|\nabla\tilde{u}\|_\infty \le C\left(\frac{r_1}{\varepsilon}\right)\end{aligned}\right. \tag{II.47}$$

and we deduce that $\{x{:}|\tilde{u}(x)| < 1/2\}$ is contained in $C\eta(r_1/\varepsilon)^{n-3}$ balls of radius ε/r_1 in ∂B_t. Let ω_ε be this union of balls and let $2\,\omega_\varepsilon$ be the union of the balls having the same centers and radii $2\,\varepsilon/r_1$ in ∂B_t.
Let $a(x)$ be a positive function on ∂B_t satisfying

$$\left\{\begin{aligned} &a(x) = \frac{1}{|\tilde{u}|^2} \qquad \text{in } \partial B_t \setminus 2\omega_\varepsilon \\ &a(x) \equiv 1 \qquad \text{in } \omega_\varepsilon \\ &\text{and } \|\nabla a(x)\|_\infty \le C\frac{r_1}{\varepsilon} \qquad \text{in } \partial B_t \quad .\end{aligned}\right. \tag{II.48}$$

First observe that

$$\begin{aligned} d_\top\left(a(x)\,\tilde{u}\wedge d_\top\tilde{u}\right) &= d_\top\left(\frac{1}{|\tilde{u}|^2}\tilde{u}\wedge d_\top\tilde{u}\right) \\ &= d_\top\left(\frac{\tilde{u}}{|\tilde{u}|}\wedge d_\top\frac{\tilde{u}}{|\tilde{u}|}\right) = 0 \quad \text{ in } \partial B_t\setminus 2\omega_\varepsilon \quad .\end{aligned} \tag{II.49}$$

Let Δ denote the Hodge operator on forms on ∂B_t. Δ admits an inverse Δ^{-1} on 1- or 2-forms in ∂B_t (for $n \ge 4$). If $n = 3$ we restrict ourselves to exact 2-forms. We have

$$a(x)\tilde{u}\wedge d_\top\tilde{u} = d_\top^{(*)}\Delta^{-1}\left(d_\top\left(a(x)\tilde{u}\wedge d_\top\tilde{u}\right)\right) + d_\top\alpha \qquad \text{on } \partial B_t \quad , \tag{II.50}$$

where $\alpha(x)$ is the function equal to $d_\top^{(*)}\Delta^{-1}\left(a(x)\tilde{u}\wedge d_\top\tilde{u}\right)$.
Let $K(x,y) = \sum_{i=1}^{\infty}\frac{1}{\lambda_i}\,\psi_i(x)\otimes\psi_i^*(y)$ be the kernel of Δ on $d_\top(\wedge^1\partial B_t)$. $K(x,y)\in\pi_1^*\wedge^2\partial B_t\otimes\pi_2^*\left(\wedge^2\partial B_t\right)^*$, where $\pi_1(x,y)\to x$ and $\pi_2(x,y)\to y$, where λ_i are the eigenvalues of Δ, ψ_i the corresponding eigenforms and, for $v\in\wedge^2_y\partial B_t$ $\left\langle\psi_i^*(y); v\right\rangle := \psi_i(y).v$. Standard results on kernels imply

$$|K(x,y)| \le \frac{C}{|x-y|^{n-3}} \quad \text{and} \quad |\partial_x K(x,y)| \le \frac{C}{|x-y|^{n-2}} \tag{II.51}$$

(Recall dim $\partial B_t = n-1$). In view of (II.49) we have

$$\begin{aligned}\Delta^{-1}\left(d_\top\left(a(x)\tilde{u}\wedge d_\top\tilde{u}\right)\right) &= \int_{\partial B_t} K(x,y).d_\top\left(a(y)\tilde{u}\wedge d_\top\tilde{u}\right)(y) \\ &= \int_{2\omega_\varepsilon} K(x,y).d_\top\left(a(y)\tilde{u}\wedge d_\top\tilde{u}\right)(y) \quad .\end{aligned} \tag{II.52}$$

Using (II.48), (II.51) and the fact that $\|\nabla\tilde{u}\|_\infty \leq Cr_1/\varepsilon$, we get

$$\left|d_\top^{(*)}\Delta^{-1}\left(d_\top\left(a(x)\tilde{u}\wedge d_\top\tilde{u}\right)\right)\right|(x) \leq \int_{2\omega_\varepsilon}\frac{C}{|x-y|^{n-2}}\times\frac{r_1^2}{\varepsilon^2} \quad . \tag{II.53}$$

Letting $1<p<\frac{n-1}{n-2}$, we have

$$\begin{aligned}\int_{\partial B_t}\left|d_\top^{(*)}\Delta^{-1}\left(d_\top\left(a(x)\tilde{u}\wedge d_\top\tilde{u}\right)\right)\right|^p &\leq \int_{\partial B_t}\left[\int_{2\omega_\varepsilon}\frac{C}{|x-y|^{n-2}}\times\frac{r_1^2}{\varepsilon^2}\right]^p \\ &\leq C_p\left(\frac{r_1}{\varepsilon}\right)^{2p}|2\omega_\varepsilon|^p \quad ,\end{aligned} \tag{II.54}$$

where we have used Hölder + Fubini and the fact that $p(n-2) < n-1 = \dim\partial B_t$. Combining (II.47) and (II.54), we have, for any $1<p<\frac{n-1}{n-2}$,

$$\left\|d_\top^{(*)}\Delta^{-1}\left(d_\top\left(a(x)\tilde{u}\wedge d_\top\tilde{u}\right)\right)\right\|_{L^p} \leq C_p\,\eta \quad . \tag{II.55}$$

On the other hand, Combining (II.43), (II.46) and (II.50), we have

$$\begin{aligned}\Delta_t\alpha &= d_\top^{(*)}\left(\tilde{u}\wedge d_\top\tilde{u}\right) + d_\top^{(*)}\left((a-1)\tilde{u}\wedge d_\top\tilde{u}\right) \\ &= -(*)d_\top\left(\frac{\partial v}{\partial r}\right) - \frac{\partial\xi}{\partial r} + \frac{1}{r}(*)d_\top v - \frac{\partial H}{\partial r} + \frac{\partial K}{\partial r} \\ &\quad - \frac{n-1}{r}\tilde{u}\wedge\frac{\partial\tilde{u}}{\partial r} + d_\top^{(*)}\left((a-1)\tilde{u}\wedge d_\top\tilde{u}\right) \quad .\end{aligned} \tag{II.56}$$

Letting $1<q<\frac{n}{n-1}$, we have

$$\begin{aligned}\int_{\partial B_t}(a-1)^q\,|\tilde{u}\wedge d_\top\tilde{u}|^q &\leq \left(\int_{\partial B_t}|a-1|^{\frac{2q}{2-q}}\right)^{1-\frac{q}{2}}\times\left(\int_{\partial B_t}|d_\top\tilde{u}|^2\right)^{\frac{q}{2}} \\ &\leq C\left(\int_{\partial B_t}\left(1-|\tilde{u}|^2\right)^2+|2\omega_\varepsilon|\right)^{1-\frac{q}{2}}\times\left(\int_{T_1}|\nabla\tilde{u}|^2\right)^{\frac{q}{2}} \\ &\leq C\,\eta^{1-\frac{q}{2}}\left(\frac{\varepsilon}{r_1}\right)^{2-q}\left(\int_{T_1}|\nabla\tilde{u}|^2\right)^{\frac{q}{2}} \quad .\end{aligned} \tag{II.57}$$

Finally, combining (II.36), (II.44), (II.45), (II.56) and (II.57) we get, for any $1 < q < \frac{n}{n-1}$

$$||\nabla\alpha||_{L^q} \leq C_q\ \eta^{\frac{1}{2}}\ \left(\int_{T_1} |\nabla\tilde{u}|^2\right)^{\frac{1}{2}} + C_q\eta^{\frac{1}{2}} \quad . \tag{II.58}$$

Thus, (II.50), (II.55) and (II.58) imply, for any $1 < q < \frac{n}{n-1}$,

$$\left(\int_{\partial B_t} |\tilde{u} \wedge d_\top \tilde{u}|^q\right)^{\frac{1}{q}} \leq C_q\ \eta^{\frac{1}{2}} \left(\int_{T_1} |\nabla\tilde{u}|^2\right)^{1/2} + C_q\ \eta^{\frac{1}{2}} \quad . \tag{II.59}$$

If we take $\frac{n}{n-1} \leq p < 2$, choose any $q < \frac{n}{n-1}$, we have

$$\begin{aligned}\left(\int_{\partial B_t} |\tilde{u} \wedge d_\top \tilde{u}|^p\right)^{\frac{1}{p}} &\leq \left(\int_{\partial B_t} |\tilde{u} \wedge d_\top \tilde{u}|^q\right)^{\frac{\gamma}{q}} \times \left(\int_{\partial B_t} |\tilde{u} \wedge d_\top \tilde{u}|^2\right)^{\frac{1-\gamma}{2}} \\ &\leq C_q\ \eta^{\frac{\gamma}{2}}\ \left(\int_{T_1} |\nabla\tilde{u}|^2\right)^{\frac{1}{2}} + C_q\ \eta^{\frac{\gamma}{2}}\end{aligned} \tag{II.60}$$

where $\frac{\gamma}{q} + \frac{1-\gamma}{2} = \frac{1}{p}$.
Using the mean value formula simultaneously for 2 slices at the same time one can ensure inequality (II.60) holds for ∂B_t and $\partial B_{7t/8}$ in the same time and we have, denoting $T_t = B_t \setminus B_{7t/8}$

$$\left(\int_{\partial T_t} |\tilde{u} \wedge d\tilde{u}|^p\right)^{\frac{1}{p}} \leq C_q\ \eta^{\frac{\gamma}{2}}\ \left(\int_{T_1} |\nabla\tilde{u}|^2\right)^{\frac{1}{2}} + C_q\ \eta^{\frac{\gamma}{2}} \quad . \tag{II.61}$$

Let $a(x)$ be the function equal to $\frac{1}{|\tilde{u}|^2}$ in $\{x\ ;\ |u(x)| > 1/2\}$ and equal to 4 otherwise and let ω_ε be the set where $a(x) = 4$. The form $a(x)\tilde{u} \wedge d\tilde{u}$ is the solution of

$$\left\{\begin{array}{l}\Delta h = dd^*(a(x)\tilde{u} \wedge d\tilde{u}) + d^*d(a(x)\tilde{u} \wedge d\tilde{u}) \\ \\ h = a(x)\tilde{u} \wedge d\tilde{u} \quad \text{ on } \partial T_t \quad .\end{array}\right. \tag{II.62}$$

Observe that the boundary condition means that both normal and tangential components of the forms h and $\tilde{u} \wedge d\tilde{u}$ coincide on ∂T_t. $h = h_0 + h_1 + h_2$, where

$$\left\{\begin{array}{l}\Delta h_0 = 0 \qquad \text{ in } T_t \\ \\ h = a(x)\tilde{u} \wedge d\tilde{u} \quad \text{ on } \partial T_t\end{array}\right.$$

$$\begin{cases} \Delta h_1 = d^* d(a(x)\tilde{u} \wedge d\tilde{u}) & \text{in } T_t \\ h_1 = 0 \quad \text{on } \partial T_t \end{cases}$$

$$\begin{cases} \Delta h_2 = dd^*(a(x)\tilde{u} \wedge d\tilde{u}) & \text{in } T_t \\ h_2 = 0 \quad \text{on } \partial T_t \end{cases}$$

In view of (II.61) we have for $p \geq 2(n-1)/n$

$$\|h_0\|_{L^2} \leq C\|h_0\|_{W^{\frac{1}{p},p}(T_t)} \leq C\|a(x)\tilde{u} \wedge d\tilde{u}\|_{L^p(\partial T_t)} \quad . \tag{II.63}$$

Moreover we have $d(a(x)\tilde{u} \wedge d\tilde{u}) = 4d(\chi_\varepsilon \tilde{u} \wedge d\tilde{u})$ where χ_ε is the characteristic function of ω_ε. Thus

$$\begin{aligned} \|h_1\|^2_{L^2} &\leq C\|\chi_\varepsilon \tilde{u} \wedge d\tilde{u}\|^2_{L^2} \leq C|\omega_\varepsilon|^{\frac{1}{2}} \left(\int_{T_1} |\nabla \tilde{u}|^4\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{T_1} (1-|\tilde{u}|^2)^2 \left(\frac{r^1}{\varepsilon}\right)^2\right)^{\frac{1}{2}} \left(\int_{T_1} \left(\frac{\varepsilon}{r_1}\right)^2 |\nabla \tilde{u}|^4\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{T_1} (1-|\tilde{u}|^2)^2 \left(\frac{r^1}{\varepsilon}\right)^2\right)^{\frac{1}{2}} \left(\int_{T_1} |\nabla \tilde{u}|^2\right)^{\frac{1}{2}} \\ &\leq C\eta^{\frac{1}{2}} \left(\int_{T_1} |\nabla \tilde{u}|^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{II.64}$$

Using the fact that $\Delta\tilde{u} \wedge \tilde{u} = 0$, we write

$$d^*(a(x)\tilde{u} \wedge d\tilde{u}) = d^*((a(x)-1)\tilde{u} \wedge d\tilde{u})$$

and we get

$$\begin{aligned} \|h_2\|^2_{L^2} &\leq C\|(a(x)-1)\tilde{u} \wedge d\tilde{u}\|^2_{L^2} \\ &\leq C\left(\||u|^2 - 1\|_{L^2} + |\omega_\varepsilon|^{\frac{1}{2}}\right) \left(\int_{T_1} |\nabla \tilde{u}|^4\right)^{\frac{1}{2}} \end{aligned}$$

$$\leq C\left(\int_{T_1}(1-|\tilde{u}|^2)^2\left(\frac{r^1}{\varepsilon}\right)^2\right)^{\frac{1}{2}}\left(\int_{T_1}\left(\frac{\varepsilon}{r_1}\right)^2|\nabla\tilde{u}|^4\right)^{\frac{1}{2}}$$

$$\leq C\left(\int_{T_1}(1-|\tilde{u}|^2)^2\left(\frac{r^1}{\varepsilon}\right)^2\right)^{\frac{1}{2}}\left(\int_{T_1}|\nabla\tilde{u}|^2\right)^{\frac{1}{2}} \tag{II.65}$$

$$\leq C\eta^{\frac{1}{2}}\left(\int_{T_1}|\nabla\tilde{u}|^2\right)^{\frac{1}{2}} \quad .$$

Thus combining (II.63), (II.64) and (II.65) we get

$$\int_{T_t}|\tilde{u}\wedge d\tilde{u}|^2 \leq C\eta^\gamma\int_{T_1}|\nabla\tilde{u}|^2+\eta^{\frac{1}{2}}\left(\int_{T_1}|\nabla\tilde{u}|^2\right)^{\frac{1}{2}}+C\eta^\gamma \quad . \tag{II.66}$$

We can always find a good slice τ between t and $\frac{7t}{8}$ such that

$$\int_{\partial B_\tau}|\nabla\tilde{u}|^2+\left(\frac{r_1}{\varepsilon}\right)^2\left(1-|\tilde{u}|^2\right)^2 \leq C\,\eta^\gamma\int_{T_1}|\nabla\tilde{u}|^2+C\eta^\gamma \quad . \tag{II.67}$$

The monotonicity formula implies

$$\frac{1}{(\tau r_1)^{n-2}}\int_{B_{\tau r_1}}e_\varepsilon(u)\leq C_n\frac{1}{(\tau r_1)^{n-3}}\int_{\partial B_{\tau r_1}}e_\varepsilon(u) \quad .$$

Writing (II.67) in the usual scale, using (II.44) and the estimate above we have either $\frac{E_{r_1}}{r_1^{n-2}}\leq\eta^{1-2\gamma}$, which permits to conclude the lemma as it is explained below, (γ can be so close to zero as we want) or we have

$$\frac{E_{\tau r_1}}{(\tau r_1)^{n-2}}\leq C\eta^\gamma\,\frac{E_{r_1}}{r_1^{n-2}}+\eta^\gamma$$

where $\tau\geq 1/2$. Thus, because of the monotonicity formula we have

$$\frac{E_{\frac{r_1}{2}}}{\left(\frac{r_1}{2}\right)^{n-2}}\leq C\eta^\gamma\,\frac{E_{r_1}}{r_1^{n-2}}+\eta^\gamma \quad . \tag{II.68}$$

Using (II.35) we have

$$\frac{E_{r_1}}{r_1^{n-2}}-\frac{E_{\frac{r_1}{2}}}{\left(\frac{r_1}{2}\right)^{n-2}}\leq\int_{r_1/2}^{r_1}\frac{1}{r^{n-2}}J_r dr\leq\frac{C}{{r_1}^{n-2}}\int_{r_1/2}^{r_1}J_r dr\leq C\frac{F_{r_1}}{{r_1}^{n-2}}$$

and since we have chosen r_1 such that $F_{r_1}/r_1^{n-2}\leq C\eta$ (see (II.36)), we get

$$\frac{E_{r_1}}{r_1^{n-2}} \leq C\eta^\gamma \, \frac{E_{r_1}}{r_1^{n-2}} + \eta^\gamma + C\eta \quad .$$

Thus, for η sufficiently small, this implies $\frac{E_{r_1}}{r_1^{n-2}}$ is small, and in particular, because of the monotonicity formula, $E_\varepsilon/\varepsilon^{n-2}$ is small. Precisely we have

$$\frac{1}{\varepsilon^n}\int_{B_\varepsilon}\left(1-|u|^2\right)^2 \leq g(\eta) \quad ,$$

where $g(\eta) \to 0$ as $\eta \to 0$. Since $\|\nabla u\|_\infty \leq C/\varepsilon$, for η sufficiently small, we necessarily have $|u|(x_0) \geq \frac{1}{2}$. ∎

We need a version of the η compactness on the boundary. u_ε is a critical point of E_ε verifying $u_\varepsilon = g_\varepsilon$ on the boundary. Assume that g_ε verifies $(A2)$, thus we have

Lemma II.8 *(eta-compactness lemma at the boundary) For any* $0 < \alpha < 1$, *there are positive constants* η, λ, ρ_1, ε_0 *depending only on* $\partial\Omega$ *and the constants in condition* $(A2)$, *such that, for any* $\varepsilon < \varepsilon_0$, *for any* $x^0 \in \overline{\Omega}\setminus spt\,\mathbb{S}$ *and for any* ρ *verifying* $\min(\rho_1, d^{1/1-\alpha}) \geq \rho \geq \lambda\varepsilon$ *where* $d = dist(x^0, spt\,\mathbb{S})$ *one has*

$$\frac{1}{\rho^{n-2}}\int_{B_\rho(x^0)\cap\Omega}\left[|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\left(|u_\varepsilon|^2-1\right)^2\right] \leq \eta \log\frac{\rho}{\varepsilon}$$

$$\Longrightarrow |u_\varepsilon(x^0)| \geq \frac{1}{2} \quad .$$

∎

Proof of the eta-compactness lemma at the boundary
We adapt the proof of Lemma II.7 to our present situation. Denote by ξ the ratio

$$\xi = \frac{\rho}{\mathrm{dist}(x^0, \mathrm{spt}\,\mathbb{S})} \leq \rho_1^\alpha \frac{\rho^{1-\alpha}}{d} \leq \rho_1^\alpha \quad .$$

Observe that, in order to verify the hypothesis of the theorem, ξ has to tend to zero as ρ_1 is taken smaller and smaller. So one should think about ξ as something small since ρ_1 (like η) will be chosen to be sufficiently small at the end of the proof. From the hypothesis of the lemma $\frac{\rho^{1-\alpha}}{d}$ is bounded by 1 so all the constants in the monotonicity formula at the boundary (Lemma II.5) are bounded. Using this Lemma we deduce, like in the proof of (II.36), the existence of $r_1 \in [2\varepsilon, \rho]$ such that

$$\frac{1}{r_1^{n-3}}\int_{(\partial B_{r_1}\cup\partial B_{r_1/2})\cap\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 + \frac{1}{\varepsilon^2}(1-|u|^2)^2 \leq C\,(\eta+\xi) \quad ,$$

$$\frac{1}{r_1^{n-3}}\int_{\partial\Omega\cap B_{r_1}(x^0)}\left|\frac{x}{r}.\nu\right|\left|\frac{\partial u}{\partial\nu}\right|^2 dx \le C\,(\eta+\xi)\quad ,$$

$$\frac{1}{r_1^{n-2}}\int_{B_{r_1}\cap\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2+\frac{1}{\varepsilon^2}(1-|u|^2)^2\le C\,(\eta+\xi)\quad . \tag{II.69}$$

We have only to deal with the case where $\mathrm{dist}(x^0,\partial\Omega)$ is so small as we want compared to r_1. Indeed if $\mathrm{dist}(x^0,\partial\Omega)\ge Cr_1$ one can apply once again the mean value formula in order to get a possibly smaller r_1 satisfying (II.69) and such that $B_{r_1}(x^0)\subset\Omega$ and the remaining part of the proof is identical to the proof of Lemma II.7 in order to get $|u(x^0)|\ge 1/2$. Since we only consider the case $\mathrm{dist}(x^0,\partial\Omega) << r_1$ and since we can take ρ_1 so small as we want compared to the C^2-norm of $\partial\Omega$ we can be so close as we want to the situation where $\partial(\Omega/r_1)\cap B_1^n(x^0)$ is $B_1^{n-1}(0)$ the unit ball of $\mathbb{R}^{n-1}$ once we have made the change of scale $r_1\to 1$. This means that the constants (Sobolev constants, constants for the Dirichlet Problem for the Laplace Beltrami operator on $\partial B_1(x^0)\cap\Omega/r_1$...etc) can be bounded independently of x^0, r_1 and ε.

Let $\tilde u(x)=u(r_1x)$, $\tilde g(x)=g(r_1x)$ on $\partial\Omega/r_1$ and $\tilde\Omega_s=B_s(x^0)\cap\Omega/r_1$. $\tilde u$ is a minimizer of

$$\int_{\tilde\Omega_1}|\nabla\tilde u|^2+\left(\frac{r_1}{\varepsilon}\right)^2(1-|\tilde u|^2)^2$$

with $\tilde u=\tilde g$ on $\partial\Omega/r_1\cap B_1(x^0)$. Observe that condition (ii) of $(A2)$ implies

$$\|\nabla^k\tilde g\|\le C\xi^k\qquad\text{on }\tilde\Omega_1\quad . \tag{II.70}$$

Denote $T_1^+=\tilde\Omega_1\setminus\tilde\Omega_{1/2}$. Working on T_1^+ instead of working on T_1, one can follow similar arguments as the ones used to pass from identity (II.37) to identity (II.45) in order to decompose $\frac{\partial}{\partial r}(\tilde u\wedge\frac{\partial\tilde u}{\partial r})$: Let Δ_0^{-1} be the inverse of the Laplace Beltrami Operator Δ on T_1^+ for $n-2$ or 0 forms. Denote also like in the proof of Lemma II.7

$$v=-\Delta_0^{-1}\left(i\frac{\partial\tilde u}{\partial r};(*)d_\top\tilde u\right)\quad ,$$

$$H=\Delta_0^{-1}\left((*)d_\top\Delta v-\Delta(*)d_\top v\right)\quad ,$$

$$K=\Delta_0^{-1}\left(\frac{2}{r}\frac{\partial}{\partial r}\left(\tilde u\wedge\frac{\partial\tilde u}{\partial r}\right)+\frac{n-1}{r^2}\tilde u\wedge\frac{\partial\tilde u}{\partial r}\right)\quad .$$

Let ζ be the solution of

$$\begin{cases} \Delta\zeta = 0 \quad \text{in } T_1^+ \\ \zeta = 0 \quad \text{on } (\partial B_1(x^0) \cup \partial B_{1/2}(x^0)) \cap \Omega/r_1 \\ \zeta = (*)d_\top v \quad \text{on } \partial\Omega/r_1 \cap (B_1(x^0) \cup B_{1/2}(x^0)) \quad . \end{cases}$$

From standard elliptic theory we have

$$\int_{T_1^+} |\zeta|^q \leq \|(*)d_\top v\|^q_{W^{-\frac{1}{q},q}(\partial\Omega/r_1 \cap (B_1(x^0)\cup B_{1/2}(x^0)))} \leq \int_{T_1^+} |\nabla v|^q \quad .$$

Let ξ be the harmonic extension of $\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r}$ in T_1^+. We have clearly on T_1^+.

$$\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r} = \xi + (*)d_\top v - \zeta - H + K \quad .$$

For $x \in \partial\Omega/r_1$ denote by $e_\tau(x)$ the orthogonal projection of $(x-x^0)/|x-x^0|$ on the tangent plane to $\partial\Omega/r_1$ at x. We have

$$\begin{aligned} &\int_{\partial\Omega/r_1 \cap B_1(x^0)} \left|\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r}\right|^2 \\ &\leq 2\int_{\partial\Omega/r_1 \cap B_1(x^0)} \left|\frac{\partial \tilde{u}}{\partial e_\tau}\right|^2 + 2\int_{\partial\Omega/r_1 \cap B_1(x^0)} \left|\frac{\partial \tilde{u}}{\partial \nu}\nu.\frac{(x-x^0)}{|x-x^0|}\right|^2 \\ &\leq \frac{2}{r_1^{n-3}}\int_{\partial\Omega \cap B_{r_1}(x^0)} |\nabla g|^2 + \frac{2}{r_1^{n-3}}\int_{\partial\Omega \cap B_{r_1}(x^0)} \left|\frac{\partial u}{\partial \nu}\right|^2 \left|\frac{x-x^0}{|x-x^0|}.\nu\right|^2 \\ &\leq C(\xi+\eta) \quad , \end{aligned} \tag{II.71}$$

where we have used (II.69). Since $\xi - \zeta$ is harmonic there exists a 2-form γ such that

$$d(\xi - \zeta) = d^*\gamma \quad \text{in } T_1^+ \quad \text{and } \gamma = d\sigma \text{ where } \sigma_{|_{\partial T_1^+}} = 0 \quad .$$

Moreover γ verifies

$$\|\gamma\|_{L^q} \leq \|d(\xi-\zeta)\|_{W^{-1,q}} \leq \|\xi - \zeta\|_{L^q} \leq C(\eta+\xi)^{1/2} \tag{II.72}$$

see for instance [18]. Thus we have

$$\frac{\partial}{\partial r}\left(\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r}\right) = \iota_{\frac{\partial}{\partial r}}.d^*\gamma + (*)d_\top\left(\frac{\partial v}{\partial r}\right) - \frac{1}{r}(*)d_\top v - \frac{\partial H}{\partial r} + \frac{\partial K}{\partial r} \quad ,$$

where $\iota_{\frac{\partial}{\partial r}}.d^*\gamma$ is the interior product between $\frac{\partial}{\partial r}$ and $d^*\gamma$ and we have

$$\iota_{\frac{\partial}{\partial r}}.d^*\gamma = (*)d_\top\gamma \quad . \tag{II.73}$$

Now, like in the proof of Lemma II.7, one can find a good slice, a $t \in (1/2, 1)$, such that for any $q \in (1, \frac{n}{n-1})$ (II.44) holds (where ∂B_t and T_1 are respectively replaced by $\partial B_t \cap \tilde{\Omega}_1$ and T_1^+) and such that also

$$\int_{\partial B_t \cap \tilde{\Omega}_1} |\gamma|^q \leq C \int_{T_1^+} |\gamma|^q \leq C(\eta + \xi)^{\frac{1}{2q}} \quad . \tag{II.74}$$

Since $(*)d_\top$ corresponds to tangential derivatives along ∂B_t, we have

$$\begin{aligned} \left\| \frac{\partial}{\partial r}\left(\tilde{u} \wedge \frac{\partial \tilde{u}}{\partial r}\right)\right\|_{W^{-1,q}(\partial B_t(x^0)\cap\Omega/r_1)} &\leq C(\eta+\xi)^{\frac{1}{2}} \left(\int_{T_1^+} |\nabla u|^2\right)^{1/2} \\ &\quad + C(\eta+\xi)^{\frac{1}{2}} \quad . \end{aligned} \tag{II.75}$$

Let $N = \partial B_t(x^0) \cap \Omega/r_1$. On N we define $a(x)$ like in (II.48). Let α be the function that is the solution of the following problem

$$\begin{cases} \Delta_t \alpha = d_\top^{(*)}(a(x)\tilde{u} \wedge d_\top \tilde{u}) & \text{in } N \\ \phi = 0 & \text{on } \partial N \quad , \end{cases} \tag{II.76}$$

where Δ_t denotes the Laplace Beltrami Operator on $\wedge^p N$ for any $0 \leq p \leq n$. From standard results on Hodge decomposition (see for instance [18]), there exists a unique 2-form β such that

$$\begin{cases} a(x)\tilde{u} \wedge d_\top \tilde{u} - d_\top \alpha = d_\top^{(*)}\beta & \text{on } N \\ \beta = d\sigma \quad \text{on } N \quad \text{for some } \sigma \in \wedge^1 N \text{ satisfying } \sigma_{|\partial N} = 0 \quad . \end{cases} \tag{II.77}$$

β is in fact the unique 2-form satisfying

$$\begin{cases} \Delta_t \beta = d_\top(a(x)\tilde{u} \wedge d_\top \tilde{u}) & \text{in } N \\ d_\top^{(*)}\beta_{|\partial N} = \dfrac{\tilde{g}}{|\tilde{g}|} \wedge d\dfrac{\tilde{g}}{|\tilde{g}|} & \text{on } \partial\Omega \\ \beta_{|\partial N} = 0 & \text{on } \partial N \quad . \end{cases} \tag{II.78}$$

β is in fact the minimizer of

$$\min \left\{ \int_N \left| d_\top^{(*)}\beta - a(x)\tilde{u} \wedge d_\top \tilde{u} + d_\top \alpha \right|^2 + |d_\top \beta|^2 \quad ; \quad \text{s. t. } \beta_{|\partial N} = 0 \right\} \quad . \tag{II.79}$$

For problems (II.76) and (II.78) the solutions are given by convolutions with Calderon-Zygmund Kernels and an analysis similar to the one developed for passing from (II.50) to (II.60). Using (II.75) instead of (II.45) yields

$$\left(\int_{\partial B_t(x^0)\cap\Omega/r_1} |\tilde{u}\wedge d\tilde{u}|^p\right)^{\frac{1}{p}} \leq C_q(\eta^{\frac{\gamma}{2}}+\xi^{\frac{\gamma}{2}}) \left(\int_{T_1^+} |\nabla\tilde{u}|^2\right)^{\frac{1}{2}} + C_q(\eta^{\frac{\gamma}{2}}+\xi^{\frac{\gamma}{2}}) \ , \tag{II.80}$$

where $1 < q < \frac{n}{n-1}$, $\frac{n}{n-1} \leq p < 2$ and γ is given by $\frac{\gamma}{q} + \frac{1-\gamma}{2} = \frac{1}{p}$. Now the remaining part of the proof can be established almost identically like the end of the proof of Lemma II.7 in order to obtain that

$$\frac{1}{\varepsilon^n}\int_{B_\varepsilon\cap\Omega}(1-|u|^2)^2 \leq h(\eta,\xi) \quad , \tag{II.81}$$

where $h(\eta,\xi) \longrightarrow 0$ as $\eta,\xi \to 0$. And since $\|\nabla u\|_\infty \leq C/\varepsilon$, for η and ρ_1 sufficiently small, $h(\eta,\xi)$ is sufficiently small in order to deduce from (II.81) that $|u(x^0)| \geq 1/2$. ∎

III. The energy concentration set as a minimal current

III.1. A short proof in dimension 3

We let Ω to be a bounded smooth convex domain in $\mathbb{R}^n$ and let $g : \partial\Omega \longrightarrow S^1$ be a smooth map. We consider

$$\min E_\varepsilon(u) \triangleq \min \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2\right] dx \quad 0<\varepsilon<<1 \tag{III.1}$$

for $u \in H^1_g(\Omega,\mathbb{C}) \equiv \{u\in H^1(\Omega,\mathbb{C}) \ : \ u_{|\partial\Omega} = g\}$.

Suppose $n = 2$ and degree of $g : \partial\Omega \longrightarrow S^1$ is $d \geq 0$. Then, one of the principal results proved in [4] can be stated as follows.

Let

$$\mu_\varepsilon = \frac{\frac{1}{2}\left[|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(|u|^2-1)^2\right] dx}{\pi\log\frac{1}{\varepsilon}}, \quad 0<\varepsilon<<1 \quad ,$$

here u_ε is a minimizer of (III.1). Then, for any sequence $\varepsilon_n \to 0$, there is a subsequence (still denoted by $\{\varepsilon_n\}$) such that $\mu_{\varepsilon_n} \rightharpoonup \mu$ weakly as Radon measures and that

$$\mu = \sum_{j=1}^d \delta_{a_j} \quad , \tag{III.2}$$

for some d distinct points $a_1, ..., a_d$ inside Ω. Moreover, the d-tuple point $a = (a_1, ..., a_d)$ is a global minimum of the so called renormalized energy $W(b, g, \Omega)$, $b \in \Omega^d$.

In the case where Ω is a ball there is no topological obstruction for extending a smooth map $g : \partial\Omega \longrightarrow S^1$ to a smooth map $\tilde{g} : \Omega \longrightarrow S^1$ when $n \geq 3$. In order to obtain a similar statement as that for $\dim\Omega = n = 2$ described above, we want to allow the boundary data g to have some topological non trivial singularities on $\partial\Omega$ (see part I.4.3 of the paper and Lemma A.7).

In [25] the second author considered the case $\dim\Omega = n = 3$, and a family of boundary data g_ε, $0 < \varepsilon < 1$, such that the assumption $(A2)$ is valid in particular:

$$d\left(\frac{g_\varepsilon}{|g_\varepsilon|}^* d\theta\right) = \mathbb{S} = \sum_{j=1}^{N} d_j \delta_{a_j} \quad , \tag{III.3}$$

for some $a_1, ..., a_N \in \partial\Omega$, and $d_1, ..., d_N \in \mathbb{Z}$. Here C_1, C_2 are positive constants independent of ε. Since Ω is compact, we must have $\sum_{j=1}^N d_j = 0$, $(\dim\Omega = 3)$.

We shall now give an alternative proof of the following main result of [25].

Theorem III.1 *[25] For any sequence $\varepsilon_n \to 0$, let u_{ε_n} be a sequence of minimizers of $E_{\varepsilon_n}(.)$. Let $\mu_{\varepsilon_n} = \frac{e_{\varepsilon_n}(u)\,dx}{\pi \log \frac{1}{\varepsilon_n}}$, $e_{\varepsilon_n}(u) = \frac{1}{2}\left[|\nabla u_{\varepsilon_n}|^2 + \frac{1}{2\varepsilon_n^2}(|u_{\varepsilon_n}|^2 - 1)^2\right]$. Then there is a subsequence of μ_{ε_n} (still denoted by μ_{ε_n}) such that $\mu_{\varepsilon_n} \rightharpoonup \mu$ as Radon measures. Moreover $\operatorname{spt}\mu = \operatorname{spt}\mathbb{T}$, $\mu(\Omega) = M(\mathbb{T})$. Here $\mathbb{T}$ is a length minimizing current in Ω such that $\partial\mathbb{T} = \mathbb{S}$, and $M(\mathbb{T})$ denote the mass of the current $\mathbb{T}$.* ∎

We note that the above formulation of the part of the main result of [25] which concerns the energy concentration set immediately unifies the statements of the results in both 2-D and 3-D cases. Unlike the 2-D case where the locations of the singularities (the support of μ) is determined by the finite part of the total energy (the so-called renormalized energy, the next term in the energy asymptotic expansions), in 3-D (or high dimensions) case the support of μ is essentially determined by the infinite energy concentrations (the first term in the energy asymptotic expansions). The proof of Theorem III.1 given below is rather different from [25].

Proof of Theorem III.1
Let

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)\,dx}{\pi \log \frac{1}{\varepsilon}} = \frac{\frac{1}{2}\left[|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(|u_\varepsilon|^2 - 1)^2\right] dx}{\pi \log \frac{1}{\varepsilon}} \quad .$$

Then $\mu_\varepsilon(\overline{\Omega}) \leq M(\mathbb{T})+\delta$ by Lemma II.3. Thus for any sequence $\varepsilon_n \to 0$, we may obtain a subsequence of $\{\mu_{\varepsilon_n}\}$ (still denoted by $\{\mu_{\varepsilon_n}\}$) such that $\mu_{\varepsilon_n} \rightharpoonup \mu$ weakly as Radon measures. As a consequence of the energy monotonicity Lemma II.4, $\frac{\mu(B_r(x^0))}{r}$ is a monotone nondecreasing function of r, for $r \in (0, r_{x^0})$, $x^0 \in \Omega$, $r_{x^0} = dist(x^0, \partial\Omega)$. Similarly, by Lemma II.5, we see $\exp(\Lambda r)\frac{\mu(B_r(x^0))}{r}$ is a monotone nondecreasing function of $r \in (0, r_0(\Omega))$, for any $x^0 \in \partial\Omega$. In particular,

$$\Theta^1(\mu, x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{2r}$$

exists for all $x \in \overline{\Omega}$. Moreover, $\Theta^1(\mu, x)$ is upper semi-continuous in $x \in \overline{\Omega}$. Let

$$\Sigma = \left\{x \in \overline{\Omega} \quad : \quad \Theta^1(\mu, x) > 0\right\} \quad .$$

Then the general results in the Sect. III.2 bellow show that Σ is, in particular, a $\mathcal{H}^1-$rectifiable set. But here we will not need this fact.

Next we assume $C_\varepsilon = C_\varepsilon^1 + iC_\varepsilon^2$ is a regular value of the map $u_\varepsilon : \Omega \longrightarrow \mathbb{C}$ such that $\frac{1}{16} < |C_\varepsilon^1|^2 + |C_\varepsilon^2|^2 < \frac{1}{4}$.

Let $\Gamma_\varepsilon = u_\varepsilon^{-1}\{C_\varepsilon\}$. Suppose, for the moment, that all $d_j = \pm 1$, $j = 1, ..., N$. There $N = 2l$, and from the properties of g_ε one may obtain k embedded C^1-curves Γ_ε^j, $j = 1, ..., k$, such that $\partial\left(\sum_{j=1}^k \Gamma_\varepsilon^j\right) = \sum_{j=1}^N d_j\delta_{a_j}$. Blaschke's theorem implies that (by taking subsequences if needed) $\Gamma_{\varepsilon_n}^j \to \Gamma^j$ as $\varepsilon_n \to 0$, for $j = 1, ..., k$, in the Hausdorff metric (see [12] page 183). Each Γ^j is a connected, compact subset of $\overline{\Omega}$.

Applying the eta-compactness lemmas (Lemma II.7 and Lemma II.8), we see that $\Theta^1(\mu, x) \geq \eta_0/2$ for all $x \in \Gamma^j \cap (\overline{\Omega}\setminus spt\,\mathbb{S})$, $j = 1, ..., k$. Indeed, if $\Theta^1(\mu, x) \leq \eta_0/2$ for some $x \in \Gamma^j$ and $x \notin spt\,\mathbb{S}$, then $\frac{\mu(B_r(x))}{r} \leq \eta_0/2$ for all sufficiently small r. We may assume r is so small that $B_{2r}(x) \cap spt\,\mathbb{S} = \emptyset$. Now for sufficiently small ε, and small r, we may let $\varepsilon_1 = \varepsilon/r$, and $v_{\varepsilon_1}(y) = u_\varepsilon(ry + x)$, then $\mu_{\varepsilon_1}(B_1(0)) \leq \eta_0/2$. Here $\mu_{\varepsilon_1} = \frac{e_{\varepsilon_1}(v_{\varepsilon_1})\,dx}{\pi \log\frac{1}{\varepsilon_1}}$. Note that $g_{\varepsilon_1}(y) = g_\varepsilon(ry + x)$ will satisfy the assumption of Lemma II.5. From the latter fact we would conclude that $|u_\varepsilon(x)| \geq \frac{1}{2}$. This contradicts to the fact $x \in \Gamma^j$, for some j. Therefore

$$\sum_{j=1}^k \mathcal{H}^1(\Gamma^j) \leq \frac{2}{\eta_0}\mu(\overline{\Omega}) \leq \frac{2}{\eta_0}M(\mathbb{T}) \quad .$$

the latter estimate implies, in particular, that each Γ^j is $\mathcal{H}^1$-rectifiable: indeed a connected 1-dimensional set of finite $\mathcal{H}^1$ measure is rectifiable (cf. [11], Theorem 2).

We want to show that $\Theta^1(\mu, x) \geq 1$ for $\mathcal{H}^1$-a.e. $x \in \cup_{j=1}^k \Gamma^j$. Then we can easily deduce that $\sum_{j=1}^k \mathcal{H}^1(\Gamma^j) \leq M(\mathbb{T})$.

Since $\mathbb{T}$ is a multiplicity 1 length minimizing current, and since $\partial(\sum_{j=1}^k \Gamma^j) = \partial\mathbb{T}$, we see each Γ^j must be a line segment since Ω is convex. Moreover, $\mu(\overline{\Omega}) = M(\mathbb{T})$, and $\mu = \mathcal{H}^1 \lfloor (\cup_{j=1}^k \Gamma^j)$. This is the conclusion of Theorem III.1.

Now we have to show $\Theta^1(\mu, x) \geq 1$, for $\mathcal{H}^1$-a.e. $x \in \Gamma^j$, $j = 1, ..., k$. Let us give a proof of this fact for x in the interior of Ω. If $x \in \Gamma^j \cap (\partial\Omega \setminus spt\,\mathbb{S})$, the arguments should be modified slightly.

Since $\Theta^1(\mu, x)$ is approximately continuous $\mathcal{H}^1$-a.e. $x \in \Gamma^j$, we get, for $\mathcal{H}^1$-a.e. x^0, $\Theta^1(\mu, x)$ is $\mathcal{H}^1$-approximate continuous (as a function defined on Γ^j) at x^0, and that Γ^j has a unique tangent line at x^0.

For simplicity we assume $x^0 = 0$ and the tangent line at x^0 is the z-axis. Let $\eta_\lambda : x \longrightarrow x/\lambda$, for $\lambda > 0$, and $\mu_\lambda(A) = \frac{1}{\lambda}\mu(\eta_\lambda^{-1}A)$, for any Borel measurable set $A \subseteq \mathbb{R}^3$.

By the monotonicity formula, we conclude that there is a sequence $\lambda_m \longrightarrow 0$, $m \longrightarrow \infty$ such that $\mu_{\lambda_m} \rightharpoonup \nu$ weakly as Radon measures. ν is a tangent measure of μ at 0 (cf. [30]). Moreover, $\frac{\nu(B_r(0))}{2r} \equiv \Theta^1(\nu, 0) \equiv \Theta^1(\mu, x^0)$. On the other hand the measure $\overline{\mu} = \Theta^1(\mu, x)\mathcal{H}^1 \lfloor \Gamma^j$ has a unique tangent measure $\overline{\nu}$ at x^0, $\overline{\nu} = \Theta^1(\mu, x)\mathcal{H}^1 \lfloor \{z - \text{axis}\}$. Note that $\mu \geq \overline{\mu}$ $\nu \geq \overline{\nu}$. We apply monotonicity formula again to obtain $\overline{\nu} \equiv \nu$. In other words, $\mu_\lambda \rightharpoonup \overline{\nu} = \Theta^1(\mu, x)\mathcal{H}^1 \lfloor \{z - \text{axis}\}$ as $\lambda \to 0$. Note that we have $\Theta^1(\mu, 0) \geq \eta > 0$.

Let $C_\delta = B_\delta^2(0) \times [-1, 1]$ for $1 > \delta > 0$. Then we may find a suitable $\delta \in (0, 1)$ such that

$$\nu\left(\partial B_\delta^2(0) \times [-1, 1]\right) = 0$$

and hence

$$\mu_\lambda(\partial B_\delta^2(0) \times [-1, 1]) \to 0 \text{ as } \lambda \to 0 \quad . \tag{III.4}$$

We note that

$$\mu_{\varepsilon_n, \lambda} = \frac{\frac{1}{2}\left[|\nabla v|^2 + \frac{\lambda^2}{2\varepsilon_n^2}(|v|^2 - 1)^2\right]}{\log \frac{1}{\varepsilon_n}} dx \quad ,$$

here $v(x) = u_{\varepsilon_n}(\lambda x)$. It is then clear from the eta-compactness lemma that for a.e. $z \in [-1, +1]$, $deg(v, \partial B_\delta^2(0) \times \{z\}) = d$ is well defined and is independent of $z \in [-1, 1]$. Indeed (III.4) and eta-compactness lemma imply that $|v| \geq \frac{1}{2}$ on $\partial B_\delta^2(0) \times [-1, 1]$.

We claim $d \neq 0$. Indeed, if $d = 0$, we first choose $t_1 \in [1-2\delta, 1-\delta]$, $t_2 \in [-1+\delta, -1+2\delta]$, such that

$$\frac{\int_{S_\delta(t_i)} \left[|\nabla v|^2 + \frac{\lambda^2}{2\varepsilon_n^2}(|v|^2-1)^2 \right]}{\log \frac{1}{\varepsilon_n}} << \eta \quad \text{for } j = 1, 2 \quad .$$

Where $S_\delta(t_j) = \partial B_\delta^2(0) \times \{t_j\}$. Since we are in dimension 2, we may construct a new map $\tilde{v}$ on $B_\delta^2(0) \times \{t_j\}$ such that

$$\int_{B_\delta^2(0)\times\{t_j\}} \frac{1}{2} \left[|\nabla_T \tilde{v}|^2 + \frac{\lambda^2}{2\varepsilon_n^2} \left(|\tilde{v}|^2 - 1\right)^2 \right] dx << \eta \, \log \frac{1}{\varepsilon_n} \quad .$$

We now define $\tilde{v}$ as follows:

- $\tilde{v} = v$ on ∂C_δ
- $\tilde{v}$ be as above on $B_\delta^2(0) \times \{t_j\}$, $j = 1, 2$,
- $\tilde{v}$ minimizes

$$\int_{B_\delta^2(0)\times[t_1,1]} \frac{1}{2} \left[|\nabla u|^2 + \frac{\lambda^2}{2\varepsilon_n^2} \left(|u|^2 - 1\right)^2 \right] dx$$

$$\int_{B_\delta^2(0)\times[-1,t_2]} \frac{1}{2} \left[|\nabla u|^2 + \frac{\lambda^2}{2\varepsilon_n^2} \left(|u|^2 - 1\right)^2 \right] dx$$

and

$$\int_{B_\delta^2(0)\times[t_2,t_1]} \frac{1}{2} \left[|\nabla u|^2 + \frac{\lambda^2}{2\varepsilon_n^2} \left(|u|^2 - 1\right)^2 \right] dx$$

subject to corresponding Dirichlet boundary conditions.

By the proof of energy-monotonicity formula, we see the first two integrals are bounded by $\eta C\delta \log \frac{1}{\varepsilon}$, the last integral is bounded by $o(\eta)C\delta \log \frac{1}{\varepsilon}$. Here $o(\eta) << \eta$. By choosing δ suitably small, we can be sure that

$$\int_{C_\delta} \frac{1}{2} \left[|\nabla \tilde{v}|^2 + \frac{\lambda^2}{2\varepsilon_n^2} \left(|\tilde{v}|^2 - 1\right)^2 \right] dx < \frac{1}{2}\eta \, \log \frac{1}{\varepsilon_n} \quad .$$

On the other hand $\mu_{\varepsilon_n,\lambda}(C_\delta) \geq \frac{3}{2}\eta$ for n large and λ suitably small. This contradicts the energy minimizing property of v. Thus we can assume $d \neq 0$. Then [4] implies that

$$\int_{B_\delta^2(0)\times\{z\}} \frac{1}{2} \left[|\nabla_T v|^2 + \frac{\lambda^2}{2\varepsilon_n^2} \left(|v|^2 - 1\right)^2 \right] dx \geq \pi \, |d| \log \frac{\lambda}{\varepsilon_n} - o\left(\log\left(\frac{1}{\varepsilon_n}\right)\right)$$

whenever $\int_{S_\delta(z)} \frac{1}{2} \left[|\nabla v|^2 + \frac{\lambda^2}{2\varepsilon_n^2} \left(|v|^2 - 1\right)^2 \right] dx = o(\log \frac{1}{\varepsilon_n})$, and $deg(v, S_\delta(z)) = d$. Note $\lambda > 0$ is fixed here.

Indeed, if we apply arguments in [22] or [27], then, since any (x, y) with $|v(x, y, z)| \leq 1/2$ is contained in $B^2_{\delta/2}(0)$, and $|\nabla_T v| \leq C\lambda/\varepsilon_n$, then $deg(v, S_\delta(z)) = d$ implies

$$\int_{B^2_\delta(0)\times\{z\}} \frac{1}{2}\left[|\nabla_T v|^2 + \frac{\lambda^2}{2\varepsilon_n^2}\left(|v|^2 - 1\right)^2\right] dx \geq \pi|d| \log\frac{\lambda}{\varepsilon_n} - K \quad .$$

We thus conclude that $\Theta^1(\mu, 0) \geq |d| \geq 1$.

Remark. Let $(\text{Supp}\mathbb{T})_\delta$ be the δ-neighborhood of the support of $\mathbb{T}$ in Proposition 1. Then arguments of [22] or [27] show that

$$E_\varepsilon(u_\varepsilon, (\text{Supp}\mathbb{T})_\delta) \geq \pi\ M(\mathbb{T})\ \log\frac{1}{\varepsilon} - K \quad .$$

Thus u_ε is locally uniformly bounded in $H^1_{loc}(\overline{\Omega} \setminus (\text{Supp}\ \mathbb{T}))$ and hence $u_{\varepsilon_n} \to u$ in $C^{1,\alpha}(\overline{\Omega} \setminus (\text{Supp}\mathbb{T}))$ as $\varepsilon_n \to 0$ (cf. [3] and [22]).

Now we allow d_j to be arbitrary integers such that $\sum_{j=1}^N d_j = 0$.
As before, we have $\mu_{\varepsilon_n} \rightharpoonup \mu$ as Radon measures. Let $\Gamma_\varepsilon = u_\varepsilon^{-1}\{C_\varepsilon\}$ be as before, and let Γ^j_ε, $j = 1, ..., m$ be m embedded C^1-curves such that $\partial\Gamma^j_\varepsilon = \delta_{a^j_\varepsilon} - \delta_{b^j_\varepsilon}$, here $a^j_\varepsilon, b^j_\varepsilon \in \partial\Omega$, with

$$|a^j_\varepsilon - a_k| \leq \varepsilon \quad |b^j_\varepsilon - a_l| \leq \varepsilon\ , \qquad \text{for some } a_k,\ a_l,\ k \neq l \quad .$$

Moreover, $\sum_{j=1}^m \partial\Gamma^j_\varepsilon = \sum_{j=1}^N d_j\delta_{a_j}$ and $m \leq \frac{1}{2}\sum_{j=1}^N |d_j|$.
Apply Blaschke's theorem to each $\{\Gamma^j_\varepsilon\}$ so that $\Gamma^j_{\varepsilon_n} \to \Gamma^j$ in the Hausdorff metric. Γ^j is compact, connected subset of $\overline{\Omega}$. Moreover, by the eta-compactness lemma $\mathcal{H}^1(\Gamma^j) \leq \frac{1}{\eta}\mu(\overline{\Omega}) < \infty$. Thus each Γ^j is rectifiable. We may find a lipschitz map f: $[0, \mathcal{H}^1(\Gamma^j)]$ into Γ^j such that $f(0) = \lim b^j_{\varepsilon_n} = a_l$, $f(\mathcal{H}^1(\Gamma^j)) = \lim a^j_{\varepsilon_n} = a_k$. Moreover, f gives an arc-parameterization of its image

$$\Gamma^j_\star = f([0, \mathcal{H}^1(\Gamma^j)]) \subset \Gamma^j \quad .$$

Note that $\cup_{j=1}^k \Gamma^j_\star \subseteqq \Sigma \subseteqq$ Supp(μ), $\partial\left(\sum_{j=1}^k \Gamma^\star_j\right) = \sum_{j=1}^N d_j\delta_{a_j}$. Moreover, the last arguments in the proof of Proposition 2 imply that for $\mathcal{H}^1$-a.e. x such that x belongs to d of curves in $\{\Gamma^1_\star, ..., \Gamma^k_\star\}$, we have $\Theta^1(\mu, x) \geq \pi d$. Thus

$$\mu(\overline{\Omega}) \geq \pi\ \sum_{j=1}^k \mathcal{H}^1(\Gamma^j_\star) \geq \pi\ M(\mathbb{T}) \quad .$$

The last inequality along with the fact $\mu(\overline{\Omega}) \leq \pi\ (M(\mathbb{T}) + \delta)$, for any $\delta > 0$, implies that $\mu = \sum_{j=1}^k \mathcal{H}^1 \lfloor \Gamma^j_\star$ and $\sum_{j=1}^k \Gamma^j_\star$ (with proper orientations) is a length minimizing current with boundary $\sum d_j\ \delta_{a_j}$.

This completes the proof of Theorem III.1. ■

III.2. General dimensions, $n = \dim \Omega \geq 4$

III.2.1. Basic energy estimates. Let Ω be a bounded, smooth convex domain in $\mathbb{R}^n$, and let u_ε be a minimizer of

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}\left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(|u|^2-1)^2\right] dx \quad , \quad 0 < \varepsilon < 1 \quad ,$$

subject to the boundary condition $u = g_\varepsilon$ on $\partial\Omega$. Here g_ε satisfies the assumption $(A2)$, in particular

$$d\left(\frac{g_\varepsilon}{|g_\varepsilon|}^* d\theta\right) = \mathbb{S} \quad , \tag{III.5}$$

where $\mathbb{S}$ is a fixed smooth $(n-3)$-dimensional current of integer multiplicity.

By the energy monotonicity Lemmas (Lemma II.4 and Lemma II.5) along with (I.19) and (I.20), one deduces that

$$\mu_\varepsilon(\overline{\Omega}) \leq C \quad , \quad \text{for all } 0 < \varepsilon < 1 \quad . \tag{III.6}$$

Here again

$$\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)\, dx}{\pi \log \frac{1}{\varepsilon}} = \frac{\frac{1}{2}\left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(|u|^2-1)^2\right]}{\pi \log \frac{1}{\varepsilon}} \quad .$$

Hence, for any sequence $\varepsilon_n \to 0$, there is a subsequence of $\{\mu_{\varepsilon_n}\}$ that weakly converges, as Radon measures, to a Radon measure μ. From the Monotonicity lemmas (Lemma II.4 and Lemma II.5) we get that

$$\frac{\mu(B_r(a))}{r^{n-2}} \text{ is a mono. non decreas. function of } r \quad , \tag{III.7}$$

$r \in (0,\ \text{dist}\,(a, \partial\Omega))$ whenever $a \in \Omega$, and that

$$\frac{\exp(\Lambda r)\mu(B_r(a))}{r^{n-2}} \text{ is a mono. non decreas. function of } r \quad , \tag{III.8}$$

$r \in (0, r_0(\Omega))$ for all $a \in \partial\Omega$.
Hence $\Theta^{n-2}(\mu, a) = \lim_{r\to 0^+} \frac{\mu(B_r(a))}{r^{n-2}}$ exists for every $a \in \overline{\Omega}$. Moreover, $\Theta^{n-2}(\mu, a)$ is an upper-semi continuous function of $a \in \overline{\Omega}$.
We define

$$\Sigma = \{a \in \Omega \quad : \quad \Theta^{n-2}(\mu, a) > 0\}$$

and

$$\overline{\Sigma} = \{a \in \overline{\Omega} \quad : \quad \Theta^{n-2}(\mu, a) > 0\} \quad .$$

We should prove that $\overline{\Sigma}$ is a $\mathcal{H}^{n-2}$-rectifiable set. Here we first prove the following density lemma

Lemma III.1 *If $a \in \overline{\Sigma}$, then $\Theta^{n-2}(\mu, a) \geq \delta_0$ for some fixed positive constant δ_0.* ∎

As a consequence of Lemma III.1 and the usual covering arguments, we have

Corollary III.1 *$\overline{\Sigma}$ is relatively closed subset of Ω with $\mathcal{H}^{n-2}(\Sigma) \leq C(n)\,\mu(\Omega)/\delta_0$.* ∎

Proof of Lemma III.1
Suppose $a \in \Sigma$ and $\Theta^{n-2}(\mu, a) < \delta_0$. Then there is $r > 0$ such that $\frac{\mu(B_r(a))}{r^{n-2}} < \delta_0$. Hence for all sufficiently small ε_n,

$$\frac{\mu_{\varepsilon_n}(B_r(a))}{r^{n-2}} < \delta_0 \quad .$$

Via a simple scaling, one may replace r by 1, ε_n by $\overline{\varepsilon}_n = \frac{\varepsilon_n}{r}$ which one may assume to be very small, then we would be able to apply the η-compactness lemma for a suitable small δ_0 to obtain $|u_{\varepsilon_n}(x)| \geq \frac{1}{2}$ for all $|x-a| \leq \frac{r}{2}$.

Below we shall use a comparison map $\overline{u}_\varepsilon$ to deduce an energy growth estimate that would lead to

$$\frac{d}{d\rho}\frac{\mu_\varepsilon(B_\rho(a))}{\rho^{n-3/2}} \geq 0 \quad \text{for } 0 < \rho \leq \frac{r}{2} \quad . \tag{III.9}$$

This latter fact would imply $\mu(B_\rho(a)) \leq C\rho^{n-3/2}$, and therefore $\Theta^{n-2}(\mu, a) = 0$. The last fact contradicts that $a \in \Sigma$.

To show (III.9) we note first that the eta-compactness lemma (it's proof) implies actually that:

$$\forall\delta > 0,\ \exists\eta = \eta(\delta) > 0 \text{ s. t.} \quad \mu_\varepsilon(B_2) < \eta \Longrightarrow |u_\varepsilon| > 1-\delta \text{ in } B_1 \tag{III.10}$$

and for ε sufficiently small.
Next by a scaling it suffices to verify that

$$\int_{B_1} e_\varepsilon(u_\varepsilon)\,dx \leq \frac{1}{n-3/2}\int_{\partial B_1} e_\varepsilon(u_\varepsilon) \quad , \tag{III.11}$$

whenever $|u_\varepsilon|(x) \geq 1-\delta$ on B_1 (δ sufficiently small) and ε sufficiently small. Note that $\|u_\varepsilon\|_{L^\infty(B_1)} \leq 1$ by Lemma II.1. Note that $a = 0$ in above by a translation. Since u_ε can be written as $u_\varepsilon = \rho_\varepsilon \exp(i\theta_\varepsilon)$, $\rho_\varepsilon = |u_\varepsilon|$, we shall construct a comparison map of the form $\overline{u} = \overline{\rho}\exp(i\theta)$. In the polar coordinates system we simply define that

$$\overline{\rho}(r,\omega) = \begin{cases} 1 \quad \text{if } 0 \leq r \leq 1-\delta \quad , \\[2ex] \dfrac{1-r}{\delta} + \rho(1,\omega)\dfrac{r-(1-\delta)}{\delta} \quad , \quad 1-\delta \leq r \leq 1 \end{cases}$$

and that

$$\begin{cases} \Delta\overline{\theta} = 0 & \text{in } B_1 \quad , \\ \overline{\theta} = \theta_\varepsilon & \text{on } \partial B_1 \quad . \end{cases}$$

Then

$$\begin{aligned} \int_{B_1} e_\varepsilon(u_\varepsilon)\,dx &\leq \int_{B_1} e_\varepsilon(\overline{u})\,dx \\ &\leq \frac{1}{2}\int_{B_1} |\nabla\overline{\theta}|^2\,dx + C\delta\int_{\partial B_1} \frac{(1-|u|^2)^2}{\varepsilon^2} \qquad \text{(III.12)} \\ &+ \frac{C}{\delta}\int_{\partial B_1} (1-|u|^2)^2 + \frac{1}{2n}\left|\frac{\partial}{\partial\omega}\rho_\varepsilon\right|^2 \quad . \end{aligned}$$

Here C is a constant independent on δ and ε.

Since $\overline{\theta}$ is a harmonic function on B_1, we obtain by the monotonicity of the function $\dfrac{\int_{B_r} |\nabla\overline{\theta}|^2\,dx}{r^n}$ that

$$\int_{\partial B_1} |\nabla\overline{\theta}|^2 \geq n\int_{B_1} |\nabla\overline{\theta}|^2 \quad .$$

Also, an integration by parts yields

$$\int_{\partial B_1} \left|\frac{\partial}{\partial\omega}\overline{\theta}\right|^2 = \int_{\partial B_1} \left|\frac{\partial\overline{\theta}}{\partial r}\right|^2 + (n-2)\int_{B_1} |\nabla\overline{\theta}|^2 \quad .$$

Thus

$$\int_{\partial B_1} \left|\frac{\partial}{\partial\omega}\overline{\theta}\right|^2 \geq (n-2)\int_{B_1} |\nabla\overline{\theta}|^2 \quad .$$

Therefore, we obtain from the last part and (III.12) that, for $\varepsilon << \delta$,

$$\begin{aligned} &\int_{B_1} e_\varepsilon(u_\varepsilon)\,dx \\ &\leq \frac{1}{n-1}\int_{\partial B_1} \frac{1}{2}\left[\left|\frac{\partial}{\partial\omega}\overline{\theta}\right|^2 + \left|\frac{\partial}{\partial\omega}\rho_\varepsilon\right|^2 + \frac{1}{2\varepsilon^2}(1-|\rho_\varepsilon|^2)^2\right] \\ &\leq \frac{1+C\delta}{n-1}\int_{\partial B_1} e_\varepsilon(u_\varepsilon) \quad . \end{aligned}$$

If δ is suitable small, then $\frac{1+C\delta}{n-1} \leq \frac{1}{n-3/2}$, and thus (III.11) is valid.

Consider now $a \in \overline{\Sigma} \cap (\partial\Omega \setminus \operatorname{spt} \mathbb{S}, \Theta^{n-2}(\mu, a) < \delta_0$. We want to show

$$\frac{d}{d\rho} \frac{\mu_\varepsilon(B_\rho(a))}{\rho^{n-3/2}} \geq -C_a \rho^{\frac{1}{2}} \quad , \tag{III.13}$$

$0 < \rho \leq r/2$, for a constant C_a (depending on a) and a small positive number r (may also depend on a). From this differential inequality (III.13) we conclude

$$\mu(B_\rho(a)) \leq C(\rho^{n-3/2} + \rho^n)$$

and hence

$$\Theta^{n-2}(\mu, a) = 0 \quad .$$

The last fact will contradict $a \in \overline{\Sigma}$.

To show (III.13) we apply the boundary eta-compactness lemma. After a scaling, a translation and a suitable diffeomorphism it suffices to verify.

$$\int_{B_1^+} e_\varepsilon(u_\varepsilon)\, dx \leq \frac{1}{n - 3/2} \int_{\partial B_1^+ \cap \{x_n > 0\}} e_\varepsilon(u_\varepsilon)\, dx + C(\varepsilon, a, \rho) \quad . \tag{III.14}$$

Here

$$C(\varepsilon, a, \rho) \leq \sqrt{\rho}\, \|g_\varepsilon\|_{C^1(B_\rho(a))} \leq C_a \sqrt{\rho}$$

for all $\rho \leq r \leq r_a$ and ε sufficiently small. Note that $u_\varepsilon(x)$ has boundary value $g_\varepsilon(a + \rho x)$ after the above translation, scaling and suitable diffeomorphism.

We then follow the above proof for the case $a \in \Sigma$ to obtain

$$\int_{B_1^+} e_\varepsilon(u_\varepsilon)\, dx \leq \int_{B_1^+} e_\varepsilon(\overline{u})\, dx \quad .$$

We write $\overline{\theta}$ as $\overline{\theta}_1 + \overline{\theta}_2$ where

$$\begin{cases} \Delta\overline{\theta}_1 = 0 & \text{in } B_1^+ \\ \overline{\theta}_1 = \theta_\varepsilon & \text{on } \partial B_1^+ \cap \{x_n > 0\} \\ \overline{\theta}_1 = 0 & \text{on } \{x_n = 0\} \end{cases}$$

$$\begin{cases} \Delta\overline{\theta}_2 = 0 & \text{in } B_1^+ \\ \overline{\theta}_2 = \theta_\varepsilon & \text{on } \{x_n = 0\} \\ \overline{\theta}_2 = 0 & \text{on } \partial B_1^+ \cap \{x_n > 0\} \end{cases} \quad .$$

It is obvious that, one has again

$$\int_{B_1^+} |\nabla\overline{\theta}_1|^2\, dx \leq \frac{1}{n} \int_{\partial B_1^-} |\nabla\overline{\theta}_1|^2 \quad .$$

On the other hand

$$\int_{B_1^+} |\nabla\overline{\theta}_2|^2\, dx \leq C\rho\, \|g_\varepsilon\|_{C^1(B_\rho(a))} \quad .$$

Combining the last two estimates and the proof for the case $a \in \Sigma$, we easily deduce (III.14). This complete the proof of Lemma III.1. ■

Our next lemma is the precise upper-bound on the total mass of the Radon measures μ_ε and μ.
Let

$$A \equiv \inf\{M(\mathbb{T}) \ \ : \ \ \mathbb{T} \in \mathbb{I}_{n-2}(\mathbb{R}^n) \ \ , \ \ \partial\mathbb{T} = \mathbb{S}\} \quad ,$$

i.e. A is the infimum of masses of integral rectifiable currents $\mathbb{T}$ in $\mathbb{R}^n$ of dimension $(n-2)$ such that the boundary of $\mathbb{T}$, $\partial\mathbb{T}$, equals $\mathbb{S}$.

Lemma III.2

$$\mu(\overline{\Omega}) \leq A \quad .$$

■

Proof. It suffices to show, for any $\delta > 0 \ \Rightarrow \ \mu(\overline{\Omega}) \leq A + \delta$. By the definition of the value A, we may find a $\mathbb{T} \in \mathbb{I}_{n-2}(\mathbb{R}^n)$ such that $\partial\mathbb{T} = \mathbb{S}$ and $M(\mathbb{T}) \leq A + \delta/4$. Since Ω is convex and smooth, we assume the support of $\mathbb{T}$, $\operatorname{spt}\mathbb{T}$, is contained in $\overline{\Omega}$. Moreover, $\operatorname{spt}\mathbb{T}\backslash\operatorname{spt}\mathbb{S} \subset \Omega$ (cf. [12] for various definitions and notations). Without loss of generality, we assume $g_\varepsilon = |g_\varepsilon| \exp(i\theta_\varepsilon)$, for a multi-valued function θ_ε on $\partial\Omega$. Then we can find a multi-valued harmonic function h_ε in Ω as follows:

$$\begin{cases} \Delta h_\varepsilon = \delta_{\mathbb{T}} & \text{in } \Omega \\ \\ h_\varepsilon = \theta_\varepsilon & \text{on } \partial\Omega \quad . \end{cases} \tag{III.15}$$

Here $\delta_{\mathbb{T}}$ is the delta measure on $\operatorname{spt}\mathbb{T}$ with integer multiplicity exactly the same as the multiplicity of $\mathbb{T}$. That is $\delta_{\mathbb{T}} = \Theta(\mathbb{T}, x)\ \mathcal{H}^{n-2}\lfloor\operatorname{spt}\mathbb{T}$, here $\Theta(\mathbb{T}, x)$ is the multiplicity function of $\mathbb{T}$ at $x \in \operatorname{spt}\mathbb{T}$. Indeed dh_ε is simply the harmonic 1-form ω with given singularity $\mathbb{T}$ and has $d\theta_\varepsilon$ as it's tangential part on $\partial\Omega \setminus \operatorname{spt} S_0$ which was found in the Appendix V.2. Then a standard elliptic estimate yields

$$\|\nabla h_\varepsilon\|_{L^p(\Omega)} \leq C \quad , \qquad \text{for } 1 \leq p < \frac{n}{n-1} \quad . \tag{III.16}$$

Here C is a constant depending on A and various constants in the assumption $(A2)$ on the family g_ε of the boundary data (C is independent of ε). Indeed from the Appendix V.2 we see that (III.16) is valid for any $1 \leq p < 2$, where C may also depend on p.

Let $0 \in \Omega$, and let $\eta_\lambda(x) = \frac{x}{\lambda}$, for $x \in \mathbb{R}^n$, $0 < \lambda < \infty$ we choose a $\lambda \in (1, 1+\delta_1)$ and let $\Omega_\lambda = \eta_\lambda(\Omega)$.

We consider on Ω_λ a map v_ε such that, for $x \in \Omega$, one has

$$v_\varepsilon(x) = \begin{cases} \dfrac{\text{dist }(x, \text{spt } \mathbb{T})}{\varepsilon} \exp(ih_\varepsilon(x)) & , \text{ if dist } (x, \text{spt } \mathbb{T}) \leq \varepsilon \quad , \\ \exp(ih_\varepsilon(x)) \quad , & \text{if dist } (x, \text{spt } \mathbb{T}) \geq \varepsilon \end{cases} \tag{III.17}$$

and thus, for $x \in \Omega_\lambda \setminus \Omega$, $v_\varepsilon(x)$ minimizes $\int_{\Omega_\lambda \setminus \Omega} e_\varepsilon(v)\, dx$ among all maps v such that $v = v_\varepsilon$ on $\partial\Omega$ and $v = g_\varepsilon \circ \eta_\lambda^{-1}$ on $\partial\Omega_\lambda$. We need to estimate the total energy of v_ε on Ω_λ.

Let $\mathcal{O}_\lambda$ be the $(\lambda - 1)$-neighborhood of the support of S in $\partial\Omega$, and let C_λ be the infinite cone with vertex at 0 consisting of all rays emitted from 0 through $\mathcal{O}_\lambda$. We consider a torus like domain $D_\lambda = C_\lambda \cap (\Omega_\lambda \setminus \Omega)$ in $\mathbb{R}^n$. On $\partial D_\lambda \cap \partial\Omega_\lambda$, we have $v_\varepsilon = g_\varepsilon \circ \eta_\lambda^{-1}$, on $\partial D_\lambda \cap \partial\Omega$ v_ε is given by (III.17). First it is easy to find an extension v_ε^* of this map to the rest of the boundary of D_λ in such a way that

$$\int_{\partial D_\lambda} e_\varepsilon(v_\varepsilon^*) \leq C_\lambda \quad . \tag{III.18}$$

Then it is easy to see (cf. monotonicity Lemma II.4) that the energy minimizer on D_λ with boundary value described above has total energy not larger than $C(\lambda - 1) \log \frac{1}{\varepsilon}$.

On the other hand, one can easily find an extension of v_ε (and v_ε^*) on $(\Omega_\lambda \setminus \Omega) \setminus D_\lambda$ in such a way that the total energy is bounded by C_λ.

Next we want to estimate the energy of v_ε on Ω. It is easy to see

$$\begin{aligned} \int_\Omega \frac{(1 - |v_\varepsilon|^2)^2}{\varepsilon^2}\, dx &\leq 4 \int_{\{x \in \Omega \,:\, \text{dist}(x, \text{spt } \mathbb{T}) \leq \varepsilon\}} \frac{(1 - \frac{\text{dist}(x, \text{spt } \mathbb{T})}{\varepsilon})^2}{\varepsilon^2}\, dx \\ &\leq \frac{4}{\varepsilon^2} \mathcal{L}^n \{x \in \Omega \,:\, \text{dist}(x, \text{spt } \mathbb{T}) \leq \varepsilon\} \quad . \end{aligned}$$

Since $\mathbb{T}$ is an integral multiplicity rectifiable current with $\mathcal{H}^{n-2}(\text{spt } \mathbb{T}) \leq A + \delta$, one easily obtains

$$\mathcal{L}^n \{x \in \Omega \,:\, \text{dist}(x, \text{spt } \mathbb{T}) \leq \varepsilon\} \leq C(n)(A + \delta)\varepsilon^2$$

for all sufficiently small $\varepsilon > 0$. In other words,

$$\int_\Omega \frac{(1 - |v_\varepsilon|^2)^2}{\varepsilon^2}\, dx \leq C(n)\,(A + \delta) \quad . \tag{III.19}$$

To estimate $\int_\Omega |\nabla v_\varepsilon|^2\, dx$, we need the following approximation theorem (cf. [12] page 417): for any $\delta_1 > 0$, there are an integral polyhedron chain $\mathbb{P}$ in $\mathbb{R}^n$ with $\operatorname{spt}\mathbb{P} \subseteq \{x \;:\; \operatorname{dist}(x, \operatorname{spt}\mathbb{T}) \le \delta_1\}$ and a diffeomorphism f of class 1 mapping $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that

$$M(\mathbb{P} - f_\sharp\mathbb{T}) + M(\partial(\mathbb{P} - f_\sharp\mathbb{T})) \le \delta_1 \quad ,$$

$$\operatorname{lip} f \le 1 + \delta_1 \quad , \quad \operatorname{lip} f^{-1} \le 1 + \delta_1 \quad , \quad |f(x) - x| \le \delta_1 \quad x \in \mathbb{R}^n$$

and $f(x) = x$ whenever $\operatorname{dist}(x, \operatorname{spt}\mathbb{T}) \ge \delta_1$.

For this integral polyhedron chain $\mathbb{P}$, and for all sufficiently small $\delta_2 \in (0, \delta_1)$, one may find an open subset of $\operatorname{spt}\mathbb{P}$, $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \operatorname{spt}\mathbb{P}$ such that $\mathcal{O}_2$ is the δ_2-open neighborhood of $\mathcal{O}_1$ in $\operatorname{spt}\mathbb{P}$, and that on the δ_2-neighborhood of $\mathcal{O}_1$, say $\mathcal{N}_{\delta_2}(\mathcal{O}_1)$, the nearest point projection from $\mathcal{N}_{\delta_2}(\mathcal{O}_1)$ to $\mathcal{O}_2$ is well defined and smooth. Moreover,

$$\mathcal{H}^{n-2}(\operatorname{spt}\mathbb{P} \sim \mathcal{O}_1) \le C(\delta_2) \longrightarrow 0 \quad \text{as } \delta_2 \to 0 \quad .$$

We also observe that, as h_ε is harmonic in $\Omega\setminus \operatorname{spt}\mathbb{T}$ and because of (III.16), one gets

$$|\nabla h_\varepsilon(x)| \le \frac{C}{\operatorname{dist}\,(x,\ \operatorname{spt}\mathbb{T})} \quad . \tag{III.20}$$

Let $B(\delta_1, \delta_2)$ be the set defined by

$$B(\delta_1, \delta_2) = f^{-1}\left\{(f_\sharp\Omega \sim \mathcal{N}_{\delta_2}(\mathcal{O}_1)) \cup \delta_2 - \text{neighb. of } \operatorname{spt}(\mathbb{P} - f_\sharp\mathbb{T})\right\} \quad . \tag{III.21}$$

Then it is rather easy to calculate

$$\begin{aligned}\int_{B(\delta_1,\delta_2)} e_\varepsilon(v_\varepsilon)\, dx &\le C \log\frac{1}{\varepsilon}\Big[M(\mathbb{P} - f_\sharp\mathbb{T}) \\ &\quad + \mathcal{H}^{n-2}(\operatorname{spt}\mathbb{P} - \mathcal{O}_1)\Big] + C(\delta_1, \delta_2) \\ &\le (\delta_1 + C(\delta_2))\, C\ \log\tfrac{1}{\varepsilon} + C(\delta_1, \delta_2) \quad .\end{aligned} \tag{III.22}$$

Note $\delta_1 + C(\delta_2) \to 0$ as $\delta_1, \delta_2 \to 0^+$, and $C(\delta_1, \delta_2)$ is independent of ε. (III.22) gives an estimate of the energy away from a good set $G(\delta_1, \delta_2)$ for the current $\mathbb{T}$. Here $G(\delta_1, \delta_2) = \Omega \setminus B(\delta_1, \delta_2)$. Indeed, $\operatorname{spt}(\mathbb{T}) \cap G(\delta_1, \delta_2)$ is contained in $f^{-1}(\operatorname{spt}\mathbb{P} \cap \mathcal{N}_{\delta_2}(\mathcal{O}_1))$. By our construction, $\operatorname{spt}\mathbb{P} \cap \mathcal{N}_{\delta_2}(\mathcal{O}_1)$ is uniformly smooth in the sense that the nearest point projection from $\mathcal{N}_{\delta_2}(\mathcal{O}_1)$ to $\mathcal{O}_2$ is well defined and smooth.

To obtain an energy upper-bound for $\int_{G(\delta_1,\delta_2)} e_\varepsilon(v_\varepsilon)\, dx$, we first look at a special case.

Let $Q = B^{n-2}_{\delta_2}(0) \times B^2_{\delta_2}(0)$ be a cube in $\mathbb{R}^n$ and let h_ε be a (multi-valued) harmonic function in $Q \setminus B^{n-2}_{\delta_2}(0) \times \{0\}$, such that

$$\Delta h_\varepsilon = \mathcal{H}^{n-2} \lfloor B^{n-2}_{\delta_2}(0)$$

and that $\|\nabla h_\varepsilon\|_{L^p(Q)} \leq C$, for $1 \leq p < \frac{n}{n-1}$. Thus

$$|\nabla h_\varepsilon(x)| \leq \frac{C}{|x_{n-1}| + |x_n|} \quad .$$

Let

$$v_\varepsilon(x) = \begin{cases} \exp(ih_\varepsilon(x)) \quad , & \text{if } |(x_{n-1}, x_n)| \geq \varepsilon \\ \dfrac{|(x_{n-1}, x_n)|}{\varepsilon} \exp(ih_\varepsilon(x) \quad , & \text{otherwise.} \end{cases}$$

Then

$$\int_{Q_{1/2}} e_\varepsilon(v_\varepsilon)\, dx \leq \pi \log \frac{1}{\varepsilon} \mathcal{H}^{n-2}(B^{n-2}_{\frac{\delta_2}{2}}(0)) + M(C, \delta_2) \quad . \tag{III.23}$$

Here $Q_{1/2} = B^{n-2}_{\delta_2/2}(0) \times B^2_{\delta_2/2}(0)$. Indeed, let θ be the argument function on the (x_{n-1}, x_n) plane. Then $\Delta(h - \theta) = 0$ in Q, and $\|\nabla(h - \theta)\|_{L^p(Q)} \leq C$. Hence $\|\nabla(h - \theta)\|_{L^\infty(Q_{\frac{1}{2}})} \leq C$. Then estimate (III.23) follows from a direct computation.

It is clear from our construction that $G(\delta_1, \delta_2)$ can be covered by a bilipschitz image, under a bilipschitz mapping F, of cubes of the form $B^{n-2}_{\delta_2}(0) \times B^2_{\delta_2}(0)$ with

$$\text{lip } F \leq 1 + C(\delta_1 + \delta_2) \quad , \text{ lip } F^{-1} \leq 1 + C(\delta_1 + \delta_2) \quad .$$

Then we apply the change of variables formula to estimate the Dirichlet integral $\int |\nabla v_\varepsilon|^2(x)\, dx$ and also use (III.19) to obtain

$$\int_{G(\delta_1, \delta_2)} e_\varepsilon(v_\varepsilon)\, dx \leq \pi \log \frac{1}{\varepsilon}\, (1 + C(\delta_1 + \delta_2))^{n-2}\, M(\mathbb{T}) + M(\delta_1, \delta_2, C) \quad . \tag{III.24}$$

Let us summarize: We have constructed a comparison map v_ε on Ω_λ such that $v_\varepsilon = g_\varepsilon \circ \eta_\lambda^{-1}$ on Ω_λ and such that

$$\int_{\Omega_\lambda} e_\varepsilon(v_\varepsilon)\, dx \leq M(C, A, \delta, \delta_1, \delta_2, \lambda) + \pi \log \frac{1}{\varepsilon} \left[(1 + C(\delta_1 + \delta_2))^{n-2} + C(\lambda - 1)\right]\, M(\mathbb{T}) \quad .$$

Now for any given $\delta > 0$, we may choose δ_1, δ_2, $\lambda - 1$ sufficiently small such that

$$\lambda^n \left[C(\lambda - 1) + (1 + C(\delta_1 + \delta_2))^{n-2}\right] \, (A + \delta/4)$$

$$+ \, M(C, A, \delta, \delta_1, \delta_2, \lambda)/\pi \log \frac{1}{\varepsilon} < A + \delta$$

for all sufficiently small ε. Let $\tilde{v}_\varepsilon(x) = v_\varepsilon(\lambda x)$, $x \in \Omega$, then

$$\int_\Omega e_\varepsilon(\tilde{v}_\varepsilon) \, dx \leq (A + \delta) \, \pi \log \frac{1}{\varepsilon} \quad .$$

That is $\mu_\varepsilon(\overline{\Omega}) \leq A + \delta$, for all sufficiently small ε, and the conclusion of Lemma III.2 follows. ∎

Remark. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two integral rectifiable currents in Ω with $\partial\mathbb{T}_1 = \partial\mathbb{T}_2 = \mathbb{S}$ and $h_{\mathbb{T}_j}$, $j = 1, 2$ be such that (as for (III.15))

$$\begin{cases} \Delta h_{\mathbb{T}_j} = \delta_{\mathbb{T}_j} & \text{in } \Omega \quad , \\ h_{\mathbb{T}_j} = \theta_\varepsilon & \text{on } \partial\Omega \quad . \end{cases}$$

Let $v_j(x)$ be defined by (III.17) for $\mathbb{T} - \mathbb{T}_j$, $j = 1, 2$, respectively then

$$\int_\Omega e_\varepsilon(v_1) \, dx \leq \int_\Omega e_\varepsilon(v_2) \, dx + CM(\mathbb{T}_1 - \mathbb{T}_2) \, \log \frac{1}{\varepsilon} \quad .$$

This is an easy consequence of our proof of Lemma III.2 above.

III.2.2. Rectifiability of Σ *and* $\overline{\Sigma}$. We shall follow closely the argument in [20] to show that Σ is a $\mathcal{H}^{n-2}$-rectifiable set with $\Theta^{n-2}(\mu, x)$ is an integer for $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma$. Then we modify the proof slightly to show the same is true for $\overline{\Sigma}$. We note first that the function $\Theta^{n-2}(\mu, x)$, $x \in \overline{\Sigma}$ is Borel measurable (cf. [30]). In particular, $\Theta^{n-2}(\mu, x)$ is $\mathcal{H}^{n-2}$-approximate continuous $\mathcal{H}^{n-2}$-a.e. on $\overline{\Sigma}$. That is, for $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma$, and for every $\delta > 0$

$$\lim_{r \to 0} \frac{\mathcal{H}^{n-2}(\{y \in B_r(x) \cap \overline{\Sigma} \, : \, |\Theta^{n-2}(\mu, y) - \Theta^{n-2}(\mu, x)| > \delta\})}{r^{n-2}} = 0 \quad . \tag{III.25}$$

We have already verified in Lemma III.1 that $\Theta^{n-2}(\mu, a) \geq \delta_0$ for every $a \in \Sigma$. As in [20] the rectifiability of Σ follows from the following Lemma III.3, Lemma III.4 and Lemma III.5 (cf. [20]).

Lemma III.3 *(Existence of weak tangent planes) For $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma$, and for $\delta > 0$, there is a positive number $r_x > 0$ such that if $0 < r < r_x$ then there is a $(n-2)$-plane $V = V(x,r) \in GL(n, n-2)$ such that $\Sigma \cap B_r(x) \subseteq V_\delta$. Here V_δ is the δr-neighborhood of V in $\mathbb{R}^n$.* ∎

We note that Lemma III.3 immediately implies the following

Corollary III.2 *For any $\delta_1, \delta_2 \in (0,1)$ there is a positive number r_* and a subset E of Σ such that*

(a) $\mathcal{H}^{n-2}(\Sigma \setminus E) < \delta_1$,
(b) if $x \in E$, $0 < r < r_*$, then there is $V = V(x,r) \in GL(n, n-2)$ such that $B_r(x) \cap \Sigma \subseteq \delta_2 r$-neighborhood of V.

∎

Indeed, it is clear that if r_x in Lemma III.3 is the largest such number such that the conclusion of Lemma III.3 remains true for the given $x \in \Sigma$, then r_x is a $\mathcal{H}^{n-2}$-measurable function on Σ. The statement of Corollary III.2 follows from the standard facts in measure theory.

The next lemma needed in the proof of the rectifiability of Σ is a general fact (cf. [20] Lemma 2.5).

Lemma III.4 *(Null-projection Lemma) If $E \subset \Sigma$ is a purely $\mathcal{H}^{n-2}$-unrectifiable set, then $\mathcal{H}^{n-2}(\mathbb{P}_V(E)) = 0$, for any $V \in GL(n, n-2)$. Here $\mathbb{P}_V$ denotes the orthogonal projection of $\mathbb{R}^n$ onto V.* ∎

Proof. See [20] Lemma 2.5. ∎

The final key fact needed is the following

Lemma III.5 *(Positive projection density)*

$$\lim_{r \to 0} \sup_{V \in GL(n,n-2)} \frac{\mathcal{H}^{n-2}(\mathbb{P}_V(\Sigma \cap B_r(x)))}{\alpha(n-2) r^{n-2}} = 1$$

for $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma$. ∎

The proof of Lemma III.3 is identical to that of Lemma 2.1 of [20]. Indeed the same geometric lemma (Lemma 2.4 of [20]) and the monotonicity lemma are valid. We shall point out that when we apply the proof of Lemma 2.1 to our situation, we obtain the following fact:

For $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma$, and for any sequence of $r_i \to 0$, the sequence of scaled measures $\eta_{\frac{1}{r_i}\,\sharp}\,\mu$ (here $\eta_{\frac{1}{r_i}} : y \to \frac{y-x}{r}$, and $\eta_{\frac{1}{r_i}\,\sharp}\,\mu(A) = \frac{\mu(x + r_i A)}{r_i^{n-2}}$) contains a weakly converging subsequence. Moreover, any such weak limit ν (a tangent measure of μ at x) is of the form $\nu = \Theta^{n-2}(\mu, x)\ \mathcal{H}^{n-2} \lfloor V$ for some $V \in GL(n, n-2)$. In other words, though the total defect measure μ may have part other than $\xi = \Theta^{n-2}(\mu, x)\ \mathcal{H}^{n-2}\lfloor\Sigma$, any tangent measure μ at $\mathcal{H}^{n-2}$- a.e. $x \in \Sigma$ is simply the tangent measure of $\xi \leq \mu$ at x.

The proof of Lemma IV.5 is actually somewhat simpler than that of Lemma 2.6 in [20].

Proof of Lemma III.5.
We assume at x, Σ has the weak tangent planes property (that is true for $\mathcal{H}^{n-2}$-a.e. $x \in \Sigma$, by Lemma III.3). Thus for any sequence $r_i \to 0^+$, $\{\mu_i = \eta_{\frac{1}{r_i}\sharp}\,\mu\}$ contains a subsequence that converges weakly to ν.

$$\nu = \Theta^{n-2}(\mu, x)\ \mathcal{H}^{n-2}\lfloor V$$

for some $V \in GL(n, n-2)$. It suffices to verify

$$\frac{\mathcal{H}^{n-2}(\mathbb{P}_V(\Sigma \cap B_r(x)))}{\alpha(n-2)r^{n-2}} \to 1 \quad \text{as } r = r_i \to 0^+ \quad .$$

The conclusion of Lemma III.5 will follow.

Without loss of generality, we may assume $V = \{0\} \times \mathbb{R}^{n-2}$. Then for any $\delta > 0$

$$\mu_i\left((B_1^2(0) \setminus B_\delta^2(0)) \times B_1^{n-2}(0)\right) \longrightarrow 0 \quad \text{as } i \to \infty \quad . \tag{III.26}$$

Each μ_i is a weak limit of Radon measures of the form $e_\varepsilon(u_\varepsilon^i)\,dx$ for some minimizers of $E_\varepsilon(.)$, and for a sequence of ε's tending to zero. Now eta-compactness lemma implies that $|u_\varepsilon^i(x)| \geq \frac{1}{2}$ for all $x \in (B_1^2(0) \setminus B_\delta^2(0)) \times B_1^{n-2}(0)$, $\delta > 0$, whenever i is sufficiently large and ε is sufficiently small. Thus the degrees of the maps $\frac{u_\varepsilon^i}{|u_\varepsilon^i|} : \partial B_\delta^2(0) \times \{p\} \longrightarrow S^1$, $p \in B_1^{n-2}(0)$ are well defined and they are equal for every $p \in B_1^{n-2}(0)$, say $d \in \mathbb{Z}$.

If $d \neq 0$, then on each slice $B_\delta^2(0) \times p$, $p \in B_1^{n-2}(0)$, there must be a point $q_i^\varepsilon \in B_\delta^2(0) \times \{p\}$ such that $|u_\varepsilon^i(q_i^\varepsilon)| < \frac{1}{2}$. Hence the eta compactness Lemma implies that

$$r^{2-n} \int_{B_r(q_i^\varepsilon)} e_\varepsilon(u_\varepsilon^i)\,dx \geq \eta_0 \ \log\frac{1}{\varepsilon} \tag{III.27}$$

for all $r \in (\lambda\varepsilon, 1/4)$ and for some $\eta_0 > 0$. Suppose $q_i^\varepsilon \longrightarrow q_i$ as $\varepsilon \to 0^+$. Then $q_i \in \eta_{\frac{1}{r_i}\sharp}\,\Sigma$ and

$$\frac{\mu_i(B_r(q_i))}{r^{n-2}} \geq \eta_0 \quad \text{for every } r \in (0, 1/4) \quad . \tag{III.28}$$

Note that (III.26), (III.28) also implies that $q_i \longrightarrow (0, p) \in B_\delta^2(0) \times \{p\}$ as $i \to \infty$.

What we have just shown is that, if $d \neq 0$ and if (III.26) is true, then $\mathbb{P}_V(\Sigma \cap B_r(x)) \equiv B_r(x)$ for $r = r_i$. In particular the conclusion of Lemma III.5.

It remains to verify that $d \neq 0$. We shall construct comparison maps v_ε^i such that, if $d = 0$ then one may find v_ε^i with $v_\varepsilon^i = u_\varepsilon^i$ on $\partial\left[B_\delta^2(0) \times B_1^{n-2}(0)\right]$ and

$$\int_{B_\delta^2(0)\times B_1^{n-2}(0)} e_\varepsilon(v_\varepsilon^i)\, dx \leq C\,\delta\ \log\frac{1}{\varepsilon} \tag{III.29}$$

the latter will imply that $\Theta^{n-2}(\mu, x) \leq C\,\delta$. Since $\delta > 0$ can be made arbitrarily small, we obtain a contradiction as $x \in \Sigma$.

The construction below is done by an induction on the dimension n. If $n = 3$, then we can choose small $\delta > 0$ such that

(i) $\displaystyle\int_{\partial B_\delta^2(0)\times\{-1+\delta\}} e_\varepsilon(u_\varepsilon^i) = o(1)\ \log\frac{1}{\varepsilon}$

(ii) $\int_{\partial B_\delta^2(0)\times\{1-\delta\}} e_\varepsilon(u_\varepsilon^i) = o(1)\ \log\frac{1}{\varepsilon}$

(iii) The degrees of the maps $\frac{u_\varepsilon^i}{|u_\varepsilon^i|} : \partial B_\delta^2(0)\times\{1-\delta\} \longrightarrow S^1$ are well defined (by (i) and (ii)) and both are equal to zero since $d = 0$.

Here $o(1) \longrightarrow 0$ as $\varepsilon \to 0^+$, $i \to \infty$.

It is very easy to obtain extensions of u_ε^i onto $B_\delta^2(0) \times \{-1+\delta, 1-\delta\}$ in such a way that the resulting map u_ε^i has total energy on these two discs equals $o(1)\ \log\frac{1}{\varepsilon}$. Next we can easily extend the map on $B_\delta^2(0) \times [-1+\delta, 1-\delta]$ with the Dirichlet boundary condition given by u_ε^i on $\partial B_\delta^2(0) \times [-1+\delta, 1-\delta]$ and v_ε^i on $B_\delta^2(0) \times \{-1+\delta, 1-\delta\}$ such that the extended map still called v_ε^i has energy on $B_\delta^2(0) \times [-1+\delta, 1-\delta]$ equal to $o(1)\ \log\frac{1}{\varepsilon}$. The latter fact follows easily from the energy bound on the boundary.

Finally we extend boundary values defined on two cubes $B_\delta^2(0) \times [-1, -1+\delta]$, $B_\delta^2(0) \times [1-\delta, 1]$, into these two cubes. Since the boundary values have total energy less or equal to $C\ \log\frac{1}{\varepsilon}$, one may extend the map to have the energy bounded by $C\,\delta\ \log\frac{1}{\varepsilon}$. This completes the construction when $n = 3$.

Under the assumption that we can do a similar construction in dimension $n-1$, we show how to construct such a map in dimension n.

For a small $\delta > 0$, we consider the $(n-1)$-dimensional domain $B_\delta^2(0) \times \partial B_{1-\delta}^{n-2}(0)$. We may assume δ is chosen so that

$$\int_{\partial B_\delta^2(0)\times\partial B_{1-\delta}^{n-2}(0)} e_\varepsilon(u_\varepsilon^i) = o(1)\ \log\frac{1}{\varepsilon} \quad .$$

Suppose again the degree of the map $\frac{u^i_\varepsilon}{|u^i_\varepsilon|} : \partial B^2_\delta(0) \times \{p\} \longrightarrow S^1$ is zero, for $p \in \partial B^{n-2}_{1-\delta}(0)$. Then by induction assumption, one may extend u^i_ε from $\partial B^2_\delta(0) \times \partial B^{n-2}_{1-\delta}(0)$ into $B^2_\delta(0) \times \partial B^{n-2}_{1-\delta}(0)$ in such a way that the extended map denoted by v^i_ε satisfies

$$\int_{B^2_\delta(0)\times\partial B^{n-2}_{1-\delta}(0)} e_\varepsilon(v^i_\varepsilon) \leq C\,\delta\,\log\frac{1}{\varepsilon} \quad . \tag{III.30}$$

Now we may extend the map into $B^2_\delta(0) \times B^{n-2}_{1-\delta}(0)$ with the Dirichlet boundary conditions on $B^2_\delta(0) \times \partial B^{n-2}_{1-\delta}(0)$ given by v^i_ε and on $\partial B^2_\delta(0) \times B^{n-2}_{1-\delta}(0)$ given by u^i_ε such that the extended map on $B^2_\delta(0) \times B^{n-2}_{1-\delta}(0)$ has total energy bounded by $C\,\delta\,\log\frac{1}{\varepsilon}$.

Finally, in the torus like domain $B^2_\delta(0) \times B^{n-2}_1(0) \setminus B^{n-2}_{1-\delta}(0)$ we again extend the map by minimizing $E_\varepsilon(.)$ with given Dirichlet boundary condition on $\partial(B^2_\delta(0) \times (B^{n-2}_1(0) \setminus B^{n-2}_{1-\delta}(0)))$, since the domain $B^2_\delta(0) \times (B^{n-2}_1(0) \setminus B^{n-2}_{1-\delta}(0))$ can be covered by cubes of size δ with total number of such cubes bounded by C/δ^{n-3}. On each cube, the energy is bounded by $C\,\delta^{n-2}\,\log\frac{1}{\varepsilon}$. Hence the total energy of the map on the whole domain $B^2_\delta(0) \times (B^{n-2}_1(0) \setminus B^{n-2}_{1-\delta}(0))$ is bounded by $C\,\delta\,\log\frac{1}{\varepsilon}$. This completes the induction. ∎

We have therefore completed the proof of rectifiability of Σ.

Let us now prove the rectifiability of $\overline{\Sigma}$. For convenience we again assume that Ω is a convex domain. We extend our minimizers u_ε which are defined on $\overline{\Omega}$ outside Ω by a simply homogeneous degree zero extension. Here we may assume without loss of generality that $0 \in \Omega$. By the boundary energy monotonicity and by the boundary eta-compactness lemma, we can easily establish the similar statements as in Lemma III.3 and Lemma III.4 for the set $\overline{\Sigma} \setminus (\,\mathrm{spt}\,\mathbb{S})$. Thus it again suffices to verify Lemma III.5 for the set $\overline{\Sigma} \setminus (\,\mathrm{spt}\,\mathbb{S})$.

First, for $\mathcal{H}^{n-2}-$a.e. $x \in \overline{\Sigma} \setminus (\mathrm{spt}\,\mathbb{S})$ and $x \in \partial\Omega$, we have a tangent measure of μ at x of the form

$$\nu = \Theta^{n-2}(\mu, x)\mathcal{H}^{n-2}\lfloor V$$

for some $(n-2)$-dimensional plane V. Note that by Lemma III.1:

$$\Theta^{n-2}(\mu, x) \geq \delta_0 > 0 \quad .$$

Note also that V has to lie inside the tangent plane, $T_x\partial\Omega$ of $\partial\Omega$ at x. The latter is due to the fact that $\mu \equiv 0$ outside $\overline{\Omega}$ with exception on the cone over $\mathrm{spt}(\mathbb{S})$ with vertex at 0. Here, we have extended the maps u_ε in whole $\mathbb{R}^n$,

μ is a Radon measure on $\mathbb{R}^n$ with support of μ contained in $\overline{\Omega} \cap$ (cone over spt $\mathbb{S}$).

Then we follow the exact same arguments as in the proof of the Lemma III.5 for points belonging to Σ and using instead the boundary eta-compactness lemma to conclude that the conclusion of Lemma III.5 remains true for $\mathcal{H}^{n-2}$-a.e. points in $\overline{\Sigma} \setminus (\text{spt}\, \mathbb{S})$. This completes the proof of the rectifiability of $\overline{\Sigma}$ as spt ($\mathbb{S}$) is a smooth $n-3$-dimensional submanifold of $\partial\Omega$. ■

Important remark.
We note that the last part of the proof of the rectifiability of Σ (or $\overline{\Sigma}$) implies actually the following:

Suppose for $x \in \overline{\Sigma} \setminus (\text{spt}\, \mathbb{S})$ one has

$$\nu = \Theta^{n-2}(\mu, x)\mathcal{H}^{n-2} \llcorner V$$

as the tangent measure of μ at x, for some $(n-2)$-dimensional plane V. (This is true for $\mathcal{H}^{n-2}$-a.e. $x \in \overline{\Sigma} \setminus (\text{spt}\, \mathbb{S})$). Then $\Theta^{n-2}(\mu, x)$ is in fact equal to

$$N = \left| \deg \left(\frac{u_\varepsilon}{|u_\varepsilon|}, \gamma \right) \right| \quad .$$

Here γ is a sufficiently small circle linked with V and lies in a sufficiently small ball centered at x.

Indeed we look at the scaled measures μ_i with the properties as (III.26). Then the degrees of the map

$$\frac{u_{\varepsilon_i}}{|u_{\varepsilon_i}|} \ : \ \partial B_\delta(0) \times \{p\} \longrightarrow S^1$$

are well defined by eta-compactness lemmas, for $p \in B_1^{n-2}(0)$. As in the proof of (III.27) we see that this degree d is not zero. Moreover, by 2-dimensional estimates in [4], we have an improved form of (III.27) that is the following:

$$\begin{aligned} &\int_{B_\delta^2(0) \times B_1^{n-2}(0)} e_\varepsilon(u_\varepsilon^i)\, dx \ \geq \\ &\qquad \pi |d| |B_1^{n-2}(0)| \log \frac{1}{\varepsilon} - o_\varepsilon(1) \log \frac{1}{\varepsilon} \end{aligned} \tag{III.31}$$

with $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$.

This last estimate implies $\Theta^{n-2}(\mu, x) \geq |d|$. On the other hand we follow a similar construction as in the proof of estimate (III.29). But now

$d \neq 0$, we use the usual 2-dimensional construction again to obtain the following upper bound for the energy:

$$\begin{aligned}\int_{B_\delta^2(0)\times B_1^{n-2}(0)} e_\varepsilon(u_\varepsilon^i)\, dx \ &\leq C\delta \log\frac{1}{\varepsilon} \\ &+ |B_1^{n-2}(0)|\, \pi|d|\log\frac{1}{\varepsilon} + C\end{aligned} \tag{III.32}$$

(compare with (III.30)). Here C is a constant independent of δ and ε, and δ can be chosen arbitrarily small to start with. Thus we obtain

$$\Theta^{n-2}(\mu, x) \leq |d| \quad .$$

Note that γ here can be any circle $\partial B_\delta^2 \times \{p\}$, $p \in B_1^{n-2}(0)$ in this normalized situation.

To end this section, we want to show $\overline{\Sigma}$ can have an orientation in such a way that we will have an $(n-2)$-dimensional current that represents the exact same class in $H_{n-2}(\overline{\Omega}, \text{spt}\,\mathbb{S}, \mathbb{Z})$ as described in the Lemma A.7.

We already know that, for $\mathcal{H}^{n-2}$-a.e. $a \in \overline{\Sigma} \setminus \text{spt}\,\mathbb{S}$, μ has the tangent measure at a given by $\Theta^{n-2}(\mu, a)\, \mathcal{H}^{n-2} \lfloor V(a)$, for some $(n-2)$-dimensional plane $V(a)$ in $\mathbb{R}^n$ and for a positive integer $\Theta^{n-2}(\mu, a)$. Let γ be a circle in $\Omega_\lambda \setminus \Sigma_\lambda$, here $\Omega_\lambda = \Omega \cup$ $(\lambda - 1)$-neighborhood of Ω, $\Sigma_\lambda = \Sigma \cup ((\text{cone over spt } S) \cap \Omega_\lambda)$, for a $\lambda > 1$. Let $\mathbb{Z}_\gamma$ be the integral homology group of γ. Here $\mathbb{Z}_\gamma$ is isomorphic (but not canonically isomorphic) to the integers $\mathbb{Z}$. Any homeomorphism (in particular isotopy) between two such circles does induce a natural isomorphism between their corresponding cohomology groups. Note these are simply constant -integer length 1-forms on γ when γ, say, is lipschitz.

Let $\omega_\varepsilon(\gamma)$ be the pull-back by $\frac{u_\varepsilon}{|u_\varepsilon|} : \gamma \to S^1$ of the standard one-form on S^1, that is $d\theta$, then we have shown that if γ is linked with $V(a)$ and lies in a small ball centered at a, then $|\omega_\varepsilon(\gamma)| =$ the length of $\omega_\varepsilon(\gamma)$ simply coincides with the absolute degree of the map $\frac{u_\varepsilon}{|u_\varepsilon|} : \gamma \to S^1$, that is $\Theta^{n-2}(\mu, a)$.

Now we simply introduce an orientation on $V(a)$ so that the intersection number satisfies

$$\sigma \wedge V(a) = \deg\left(\frac{u_{\varepsilon_i}}{|u_{\varepsilon_i}|}, \gamma\right)$$

for ε_i sufficiently small, where $\mu_{\varepsilon_i} \rightharpoonup \mu$ and where σ is a regular 2-cycle in Ω_λ bounded by γ. In the notation [30] p. 146, the fact above and the fact that this being true for $\mathcal{H}^{n-2}$-a.e. $a \in \overline{\Sigma}$, we conclude that

$$\mathbb{T} = \tau\left(\Sigma, \Theta^{n-2}(\mu, .), \mathbf{T}(.)\right)$$

is an $(n-2)$-dimensional integer multiplicity rectifiable current.

Now let σ be a generic, regular 2-cycle in Ω_λ with $\partial\sigma = \gamma$ being a regular curve on $\partial\Omega_\lambda \setminus$ (cone over spt S). Suppose σ is a generic position so that

$$\sigma \cap \Sigma = \{a_1, ..., a_l\}$$

and at $a_1, ..., a_l$, μ has tangent measures

$$\Theta^{n-2}(\mu, a_j)\, \mathcal{H}^{n-2} \lfloor V(a_j) \quad , \quad j = 1, ..., l \quad .$$

Then

$$\deg \left(\frac{u_{\varepsilon_i}}{|u_{\varepsilon_i}|}, \gamma \right) = \sum_{j=1}^{l} \sigma \wedge V(a_j) = \sigma \wedge \mathbb{T}$$

by eta-compactness lemmas. Note $\sigma \wedge V(a_j) = \pm\Theta^{n-2}(\mu, a_j)$.

From this canonical property of intersection, we see, by the Poincaré-Lefschetz Duality theorem, that $\mathbb{T}$ represents the class in $H_{n-2}(\overline{\Omega}_\lambda,$ spt $S_\lambda, \mathbb{Z}) = H_{n-2}(\Omega,$ spt $S, \mathbb{Z})$ as given in Lemma A.7. In particular

$$\partial\mathbb{T} = \mathbb{S}_\lambda \quad .$$

Here $\mathbb{S}_\lambda = \eta_{\lambda\sharp}\mathbb{S}$, $\eta_\lambda(x) = \lambda x$ so that $\operatorname{spt}(\mathbb{S}_\lambda) \subseteq \partial\Omega_\lambda$. We should point out there is a more general lemma which implies such current $\mathbb{T}$ has boundary equal to $\partial\mathbb{S}_\lambda$ and which uses only the local structure of $\mathbb{T}$. We should leave it to a forthcoming work.

III.2.3. Proof of the minimality of $\mathbb{T}$

First proof.
We have, via energy comparison that

$$\mu(\overline{\Omega}) \leq A \quad ,$$

where

$$A = \inf \left\{ M(\mathbb{T}) \ : \ \mathbb{T} \in \mathbb{I}_{n-2}(\mathbb{R}^n) \ , \ \partial\mathbb{T} = \mathbb{S} \right\} \quad .$$

On the other hand, for the integral current $\mathbb{T}$ obtained at the end of the last subsection we have

$$A \leq M(\mathbb{T}) \leq \mu(\overline{\Omega}) \quad .$$

Thus

$$\mu = \Theta^{n-2}(\mu, .)\, \mathcal{H}^{n-2} \lfloor \overline{\Sigma}$$

and $\mathbb{T}$ is an area-minimizing current. ■

Second proof.
In this part we will use some results proved in the next section.

Step I. By energy comparisons, we have shown that

$$\mu(\overline{\Omega}) \leq A \quad , \quad A = \inf\left\{M(\mathbb{T}) \ : \ \mathbb{T} \in \mathbb{I}_{n-2}(\mathbb{R}^n) \ , \ \partial\mathbb{T} = \mathbb{S}\right\} \quad .$$

Here μ is the defect measure which is a weak limit of Radon measures $\mu_\varepsilon = \frac{e_\varepsilon(u_\varepsilon)\ dx}{\pi \log \frac{1}{\varepsilon}}$.

Step II. Let $\{\mathbb{T}^\varepsilon\}$, $0 < \varepsilon << 1$, be a family of ε^α-approximation of $d\left(\dfrac{u_\varepsilon}{|u_\varepsilon|}^* d\theta\right)$ given by Lemma V.2, such that $\mathbb{T}^\varepsilon \in \mathbb{I}_{n-2}(\mathbb{R}^n)$

(i) $\partial\mathbb{T}^\varepsilon = \mathbb{S}$,
(ii) $M(\mathbb{T}^\varepsilon) \leq C_0$,
(iii) $\forall\gamma \subset \Omega \setminus \cup_{i=1}^{N_\alpha} B_{\varepsilon^\alpha}(x_i)$, closed regular curve,

$$deg\left(\frac{u_\varepsilon}{|u_\varepsilon|}, \gamma\right) = D \wedge \mathbb{T}^\varepsilon \quad ,$$

where D is a two dimensional regular cycle bounding γ.

By the Federer-Fleming Compactness Theorem for integral currents, one may assume $\mathbb{T}^\varepsilon \rightharpoonup \mathbb{T}$ in flat norm, $\mathbb{T} \in \mathbb{I}_{n-2}(\mathbb{R}^n)$ such that $\partial\mathbb{T} = \mathbb{S}$, $M(\mathbb{T}) \leq C_0$.

Step III. Note that, by construction of the $\{\mathbb{T}^\varepsilon\}$'s, one also has $\mathrm{spt}(\mathbb{T}^\varepsilon) \subset \cup_{i=1}^{N_\alpha} B_{\varepsilon^\alpha}(x_i)$. Then $\mathrm{spt}(\mathbb{T}) \subseteq \overline{\Sigma} = \{x \in \overline{\Omega} \ : \ \Theta^{n-2}(\mu, x) > 0\}$. Indeed, $\forall x^0 \in \mathrm{spt}(\mathbb{T})$, there is a sequence $x_{\varepsilon_i} \longrightarrow x^0$ such that $|u_{\varepsilon_i}(x_{\varepsilon_i})| \leq \frac{1}{2}$. By eta-compactness, $\frac{\mu_{\varepsilon_i}(B_r(x_{\varepsilon_i}))}{r^{n-2}} \geq \eta_0 > 0$ for any $r > 0$ and sufficiently large i. Thus $\frac{\mu(B_r(x^0))}{r^{n-2}} \geq \eta_0/2 > 0$ for any $r > 0$, i.e. $x^0 \in \overline{\Sigma}$. We have already noted that for $\mathcal{H}^{n-2}$-a.e. $x^0 \in \mathrm{spt}(\mathbb{T}) \subseteq \overline{\Sigma}$, $\Theta^{n-2}(\mu, x^0) = d \in \mathbb{Z}^+$ and $d = deg\left(\frac{u_\varepsilon}{|u_\varepsilon|}, \gamma\right)$. Here γ is a standard circle in $\{0\} \times \mathbb{R}^2$ (with center at 0) where $\mathbb{R}^{n-2} \times \{0\}$ is the tangent of $\mathrm{spt}(\mathbb{T})$ at x^0.

Since by property (iii) and intersection theory for currents (see [12] Chap. 4), one sees that d is equal to $D \wedge \mathbb{T}$, the multiplicity of the current $\mathbb{T}$. Therefore

$$M(\mathbb{T}) \leq \mu(\mathrm{spt}\,\mathbb{T}) \leq \mu(\overline{\Omega}) \leq A \quad .$$

We conclude that $\mathbb{T}$ must be an area-minimizing current. Moreover $\mathrm{spt}\mu \subseteq \mathrm{spt}\,\mathbb{T}$, i.e.

$$\mu = \Theta^{n-2}(\mu, x)\ \mathcal{H}^{n-2} \lfloor \mathrm{spt}\,\mathbb{T} \quad .$$

∎

IV. Convergence of the minimizing maps

In this part we assume hypothesis $(A2')$ of Sect. I.4 but we do not assume anymore that Ω is convex, Ω can be any regular bounded domain of $\mathbb{R}^n$. This part is relatively independent from part III and follows closely the analysis in [25]. It also yields to a new proof of Theorem I.1 but this time uses the Federer-Fleming compactness theorem for integrable currents.

IV.1. The $W^{1,p}$ estimate

We first prove the following uniform bound for the minimizers.

Lemma IV.1 *Let Ω and g_ε satisfying $(A2')$ and $(A3)$, for $\varepsilon < \varepsilon_0$ for any minimizer u_ε of E_ε on Ω with $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$, we have the uniform bounds*

$$\forall 1 < p < \frac{n}{n-1} \qquad \|u_\varepsilon\|_{W^{1,p}(\Omega)} \leq C \tag{IV.1}$$

where C is a constant independent on ε, it only depends on p, Ω and g_ε. ■

Remark IV.1. In fact the $W^{1,p}$ bound above holds for critical points of the Ginzburg-Landau functional in general once one knows the a-priori bound $E_\varepsilon(u_\varepsilon) \leq C \log \frac{1}{\varepsilon}$ where C is independent on ε.

IV.1.1. An ε^α-approximation of $d\left[\frac{u_\varepsilon}{|u_\varepsilon|}^* d\theta\right]$ by a $(n-2)$-current having a uniformly bounded mass. In this section we prove the following lemma

Lemma IV.2 *Let $0 < \nu < \alpha < 1$, for $\varepsilon \leq \varepsilon_0$ there exists N_α balls of radius ε^α, $(B_{\varepsilon^\alpha}(x_i))_{i=1\ldots N_\alpha}$, n_ν ball of radius ε^ν $(B_{\varepsilon^\nu}(z_j))_{j=1\ldots n_\nu}$ and a Lipschitz $(n-2)$-current $\mathbb{T}^\varepsilon$, having integer multiplicity such that*

i) $|u| \geq \frac{1}{2}$ *on* $\Omega \setminus \cup_{i=1}^{N_\alpha} B_{\varepsilon^\alpha}(x_i) \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$.
ii) $N_\alpha \leq \frac{C}{(\varepsilon^\alpha)^{n-2}}$ *and* $n_\nu \leq \frac{C}{(\varepsilon^\nu)^{n-3}}$ *(C independent on ε).*
iii) *supp*$(\mathbb{T}^\varepsilon) \subset \cup_{i=1}^{N_\alpha} B_{\varepsilon^\alpha}(x_i) \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$.
iv) $\partial\mathbb{T}^\varepsilon = \mathbb{S}$.
v) $M(\mathbb{T}^\varepsilon) \leq C$ *(C independent on ε).*
vi) $\forall\gamma \subset \Omega \setminus \cup_{i=1}^{N_\alpha} B_{\varepsilon^\alpha}(x_i) \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$, *closed regular curve,*

$$deg\left(\frac{u}{|u|}, \gamma\right) = \Sigma \wedge \mathbb{T}^\varepsilon$$

where Σ is a two-dimensional regular cycle bounding γ which is transversal to $\mathbb{T}^\varepsilon$ and $\Sigma \wedge \mathbb{T}^\varepsilon$ denotes the intersection number between Σ and $\mathbb{T}^\varepsilon$. ■

Remark IV.2. The last condition can also be written like this: Let Δ^{-1} be the inverse of the Laplace-Hodge Operator Δ on $\{\omega \in \wedge^2\Omega$ s.t. $\omega_{|\partial\Omega} = 0$ and $*\omega_{|\partial\Omega} = 0\}$, we have

$$\frac{u}{|u|} \wedge d\frac{u}{|u|} - d^*\Delta^{-1}(\mathbb{T}^\varepsilon)$$

is exact in $\Omega \setminus \cup_{i=1}^{N_\alpha} B_{\varepsilon^\alpha}(x_i)$. ∎

IV.1.2. Preliminaries: rectangular coverings. We will start by introducing the notion of a perfect rectangular covering of size ε^α. We first give an example: Consider all the cubes of $\mathbb{R}^n$, having faces parallel to the canonical hyperplanes $\{x_k = 0\}$, with edges of length $2\varepsilon^\alpha$ and with center in the lattice $\varepsilon^\alpha\mathbb{Z}^n$. We will translate now a bit the centers of this covering, keeping the faces of the cube parallel to the canonical hyperplanes $\{x_k = 0\}$ and without changing their size. We will do it in a periodic way such that the following holds: two parallel faces of two different cubes are at the distance at least $\mu\varepsilon^\alpha$, where μ will be some fixed constant independent of ε. Let $\delta_{(l_1,\ldots,l_n)}$ be a sequence of vectors of $\mathbb{R}^n$ indexed by $\mathbb{Z}_5^n = \{0, \ldots, 4\}^n$ such that all the coordinates of all these vectors are different and denote by μ the infimum, in absolute value, of all these differences:

$$0 < \mu = \min\{|\delta_l^i - \delta_{l'}^i| \quad \forall l \neq l' \in \mathbb{Z}_5^n \quad \forall i = 1\ldots n\} \quad .$$

Suppose also that the maximum of $|\delta_l|$ is small enough. If we translate each of our cubes of center say $\varepsilon^\alpha(k_1\ldots k_n)$ by the vector $\varepsilon^\alpha\delta_{(\overline{k}_1\ldots\overline{k}_n)}$ where $\overline{k}_i \equiv k_i$ in $\mathbb{Z}_5$, then we are done. Indeed, first, since the maximum of the $|\delta_l|$ is supposed to be small compare to one, two parallel sides of cubes which should be at a distance less than $\varepsilon^\alpha/2$ after this translation necessarily had to touch before the translation. Secondly, if two sides of cubes, say perpendicular to e_1, were touching before this translation, if $\varepsilon^\alpha(k_1\ldots k_n)$ and $\varepsilon^\alpha(k_1'\ldots k_n')$ are their centers, we know that they will be at a distance of at least $\varepsilon^\alpha \times |\delta^1_{\overline{k}} - \delta^1_{\overline{k'}}| \geq \mu\varepsilon^\alpha$ after the translation (necessarily $\overline{k} \neq \overline{k'}$ otherwise they would not touch before). Let (C_i) be the covering that we obtain. We will need the following definitions

Definition IV.1 *A rectangular set is a polyhedral set in $\mathbb{R}^n$ whose faces are parallel to one of the hyperplane $\{x_k = 0\}$*

Definition IV.2 *Let R be a union of disjoint rectangular sets in $\mathbb{R}^n$. We call the union of the n-rectangular sets forming R the n-skeleton of R, denoted by R_n, and more generally it's k-skeleton, denoted R_k, for $k \leq n-1$, is the union of the rectangular sets boundaries of the rectangular sets in R_{k+1}.*

R is now a union of disjoint rectangular sets such that $\overline{R} = \mathbb{R}^n$. We call such a union a rectangular covering. For instance the union of the connected components of $\mathbb{R}^n \setminus \cup_i \partial C_i$ is a rectangular covering. We will use the following definitions

Definition IV.3 *The external size (possibly infinite) of a rectangular covering R is the infimum among the length of the edges of the cubes which contain any component of R.*

Definition IV.4 *Let R be a rectangular covering. The maximum of the length of edges of the cubes (possibly 0) such that for any component of R and any point in this component there exists a cube of the same size included in this component and containing the point, is called the internal size of the covering.*

Definition IV.5 *We say that a rectangular covering R is μ-perfect, if μ is the ratio between the internal and the external size of R. μ will also be called the perfection coefficient.*

Observe that the perfection coefficient of $R = \mathbb{R}^n \setminus \cup_i \partial C_i$ is positive and independent of ε.

Definition IV.6 *We say that a rectangular covering R has no topology if for any $k \leq n$ all the rectangular sets of its k-skeleton are homeomorphic to B^k.*

We claim that $R = \mathbb{R}^n \setminus \cup_i \partial C_i$ has no topology. Indeed, for any K_k rectangular component in it's k-skeleton we have the following property: the intersection of K_k with any line parallel to one of the canonical direction has at most one component. We can prove it in the following way, any point of K_k is contained in a fixed family of cubes among the C_i and this family does not depend on the point in K_k. Take a line parallel to the l-th canonical direction which intersects K_k. When this line starts to leave K_k that means in particular that this line either leaves definitively one of the cubes in the family mentioned above or start to enter in a new cube and it will stay in this new cube during a length exactly equal to $2\varepsilon^\alpha$ (since this direction is parallel to one of the edges of the cube) and after it is too late for entering again in K_k since the maximal distance along a canonical direction in K_k is bounded by $2\varepsilon^\alpha$.

IV.1.3. Proof of Lemma IV.2. Let $0 < \nu < \alpha < \beta$. We consider first a good covering of $S = \mathrm{spt}\,\mathbb{S}$ by balls of radius ε^ν (each point of S is covered by at most a finite number of balls B_{ε^ν} depending only on the dimension n). Denote by S^ν_ε the union of these balls: $S^\nu_\varepsilon = \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$. Of course, since we have a good covering the condition $n_\nu \leq \frac{C}{(\varepsilon^\nu)^{n-3}}$ is automatically ensured.

Denote by $\Omega_\varepsilon^\nu = \Omega \setminus S_\varepsilon^\nu$. We will work now mainly on Ω_ε^ν in order to have the hypothesis of the η-compactness Lemma II.8 fulfilled for sets of size ε^α or ε^β.

We consider a good covering of $\overline{\Omega_\varepsilon^\nu}$ by ball of radius ε^β whose centers are contained in $\overline{\Omega_\varepsilon^\nu}$. Among these balls there are the one for which $|u| > \frac{1}{2}$ everywhere inside, we will call them the good balls, and the other ones, the bad balls. We will use the following notation for any subset K of Ω:

$$\mathcal{N}_\beta(K) \text{ is the number of bad balls intersecting } K \quad .$$

Now take the union of cubes C_i of size ε^α constructed in the preliminaries and it's associated covering. In fact we only consider the ones which intersect $\overline{\Omega_\varepsilon^\nu}$ but we will not mention it explicitly anymore. We will change these union of cubes a little bit, still keeping the perfection coefficient bounded from below by a positive number independent of ε and still keeping the "no topology" condition but in order to ensure the two following key conditions for any K_k rectangular component of the k-skeleton for any $2 \le k \le n$:

$$(H1)\ \frac{1}{(\varepsilon^\alpha)^{k-2}} \int_{K_k} e_\varepsilon(u) \le C \log \frac{1}{\varepsilon} \quad ,$$

$$(H2)\ \mathcal{N}_\beta(K_k) \le C \left(\frac{\varepsilon^\alpha}{\varepsilon^\beta} \right)^{k-2} \quad ,$$

where C in both $(H1)$ and $(H2)$ is independent of K, R, u or ε. First observe that these two conditions $(H1)$ and $(H2)$ are satisfied for the n-skeleton: $(H1)$ follows from Lemma II.7 and $(H2)$ from $(H1)$, the η-compactness Lemma II.8 and the following argument that we will often use and that was previously developed in [25]:

Let K_n be one of the n-dimensional rectangular sets of R. Let $B_{2\varepsilon^\beta}(y_i)$ for $i = 1...\mathcal{N}_\beta(K_n)$ be the bad balls intersecting K_n. By the η-compactness Lemma we have

$$\frac{1}{(\varepsilon^\beta)^{n-2}} \int_{B_{2\varepsilon^\beta}(y_i)} e_\varepsilon(u) \ge \eta\ (1-\beta)\ \log \frac{1}{\varepsilon} \quad .$$

Summing over all the $i = 1, .., \mathcal{N}_\beta(K_n)$ and since, by Lemma II.6, the total energy in K_n is bounded by $\frac{1}{(\varepsilon^\alpha)^{n-2}} \log \frac{1}{\varepsilon}$ we get $(H2)$ for K_n. So for this first step we do not need to change C_i in order to obtain $(H1)$ and $(H2)$ for the n-skeleton. We will proceed inductively in order to get $(H1)$ and $(H2)$ for the k-skeleton starting from n to 2.

Suppose now we have a covering of Ω by cubes of size $2\varepsilon^\alpha$, say (C_i^p), such that the induced rectangular covering has a perfection coefficient $\mu > 0$ independent of ε and such that it has no topology and suppose also that $(H1)$

and $(H2)$ are satisfied for $k \geq p$. We will deduce a new covering by cubes (C_i^{p-1}) satisfying $(H1)$ and $(H2)$ for $k \geq p-1$.

Let $C(2\varepsilon^\alpha, x_i)$ be any of the cubes of (C_i^p). Let $n-1 \geq l \geq p-1$ and K_l be any rectangular component of the l-skeleton which is contained in $\partial C(2\varepsilon^\alpha, x_i)$. Let m be the number of cubes of (C_i^p) whose boundary contain K_l (we can have $m = 0, ..., n-l-1$). Let H_{n-m} be the $n-m$-rectangular set realized by the intersection of all the faces of these cubes which contain K_l. Let's move the center x_i of our cube $C(2\varepsilon^\alpha, x_i)$ and call x the new center which becomes a variable. As x moves in $B_{\mu\varepsilon^\alpha/4}(x_i)$ (recall that μ is the perfection coefficient), K_l moves continuously in H_{n-m}. Denote by $K_l(x)$ the family of rectangular sets in H_{n-m} that we get. We claim that

$$\int_{x \in B_{\mu\varepsilon^\alpha/4}(x_i)} dx \int_{K_l(x)} e_\varepsilon(u) \leq C\, (\varepsilon^\alpha)^{n+l-2} \log \frac{1}{\varepsilon} \tag{IV.2}$$

and

$$\int_{x \in B_{\mu\varepsilon^\alpha/4}(x_i)} \mathcal{N}_\beta(K_l(x))\, dx \leq C \frac{(\varepsilon^\alpha)^{n+l-2}}{(\varepsilon^\beta)^{l-2}} \quad . \tag{IV.3}$$

Let $B^{n-m-l}_{\mu\varepsilon^\alpha/4}(x_i)$ be the ball of radius $\mu\varepsilon^\alpha/4$, center x_i and dimension $n-m-l$ for the directions perpendicular to K_l and included in H_{n-m}. We clearly have

$$\int_{x \in B^{n-m-l}_{\mu\varepsilon^\alpha/4}(x_i)} dx \int_{K_l(x)} e_\varepsilon(u) \leq C \int_{H_{n-m}} e_\varepsilon(u) \quad .$$

Since (C_i^p) is a covering in particular satisfying $(H1)$ for $k \geq p$ and since $n-m \geq l+1 \geq p$,

$$\int_{x \in B^{n-m-l}_{\mu\varepsilon^\alpha/4}(x_i)} dx \int_{K_l(x)} e_\varepsilon(u) \leq C \int_{H_{n-m}} e_\varepsilon(u) \leq C(\varepsilon^\alpha)^{n-m-2} \log \frac{1}{\varepsilon} \quad . \tag{IV.4}$$

Integrating both sides of the inequality (IV.4) along the $m+l$ remaining directions we get (IV.2). Similarly we have

$$\int_{x \in B^{n-m-l}_{\mu\varepsilon^\alpha/4}(x_i)} \mathcal{N}_\beta(K_l(x)) \leq C(\varepsilon^\beta)^{n-m-l}\, \mathcal{N}_\beta(H_{n-m}) \quad .$$

Since $(H2)$ is satisfied for (C_i^p) for $k \geq p$ and since $n-m \geq l+1 \geq p$ we have $\mathcal{N}_\beta(H_{n-m}) \leq C\left(\frac{\varepsilon^\alpha}{\varepsilon^\beta}\right)^{n-m-2}$. Thus

$$\int_{x \in B^{n-m-l}_{\mu\varepsilon^\alpha/4}(x_i)} \mathcal{N}_\beta(K_l(x)) \leq C(\varepsilon^\alpha)^{n-m-2}(\varepsilon^\beta)^{-l+2} \quad . \tag{IV.5}$$

Integrating both sides of the inequality (IV.5) along the $m+l$ remaining directions we get (IV.3). Considering inequalities (IV.2) and (IV.3) for all the K_l contained in $\partial C(2\varepsilon^\alpha, x_i)$ for $l \geq p-1$, we can simultaneously apply the mean value formula for deducing the existence of $x = \overline{x}_i$ such that for any of these K_l we have $(H1)$ and $(H2)$. We do that one by one for all the cubes of (C_i^p) and we call (C_i^{p-1}) the new family of cubes that we obtain. By construction, this family gives a new rectangular covering which has a perfection coefficient larger than $\mu/2$,which satisfies $(H1)$ and $(H2)$ for $k \geq 2$, which is homotopic to the (C_i) we start with and thus of course has also no topology.

Denote by R the rectangular covering deduced from (C_i^2). Denote by R' the union of rectangular sets of R which contain a point x such that $|u|(x) < 1/2$ and denote it's k-skeleton by R'_k. We know, by a similar argument used above to bound the number of bad balls, that the number of rectangular components in R' is bounded by $C/(\varepsilon^\alpha)^{n-2}$.

Let K_2 be any rectangular component of R_2. We have

$$\int_{K_2} e_\varepsilon(u) \leq C \log \frac{1}{\varepsilon} \quad \text{and} \quad \mathcal{N}_\beta(K_2) \leq C \quad . \tag{IV.6}$$

Moreover K_2 is Lipschitz diffeomorphic to the ball $B^2_{\varepsilon^\alpha}$ (for a diffeo. ψ verifying $\|\nabla\psi\|_\infty + \|\nabla\psi^{-1}\|_\infty \leq C$ indep. on ε). For all this reason we can deduce like in [5] or [25] that there exist $\delta \in (\alpha, \beta)$ and $z_1, \ldots, z_N$ in K_2 such that

i) $\forall i \neq j \quad B_{\varepsilon^\delta}(z_i) \cap B_{\varepsilon^\delta}(z_j) = \emptyset$
ii) $|u| \geq \frac{1}{2}$ in $K_2 \setminus \cup_i B_{\varepsilon^\delta}(z_i)$
iii) $N \leq C$ and deg $(\frac{u}{|u|}, \partial(K_2 \cap B_{\varepsilon^\delta}(z_i))) \leq C$ where C is independent of K_2 and ε.

Denote by T_2 the sum, among all the K_2 in R'_2, of the Dirac masses at the z_j with the multiplicity given by the degree of $u/|u|$ at z_j. Denote by v_2 a given regular map from $R_2 \setminus \text{supp} T_2$ into S^1 equal to $u/|u|$ outside of $B_{\varepsilon^\delta}(z_j)$.

We will construct by induction from $k = 3$ to $k = n$ both a $k-2$-Lipschitz current T_k in R_k and a map $v_k : R_k \setminus \text{Supp} T_k \to S^1$ such that

i) $\text{supp}(T_k) \subset R'_k$
ii) $\forall$ rectangular components K_k of R'_k

$$\partial(T_k \lfloor \overline{K}_k) = T_{k-1} \lfloor \partial K_k$$

iii) $\forall$ rectangular component of K_k R'_k

$$M(T_k \lfloor K_k) \leq C(\varepsilon^\alpha)^{k-2}$$

Where C is independent of ε.

iv) $v_{k|_{\partial(R_k\setminus \mathrm{supp} T_k)}} = v_{k-1}$
v) $\deg(v_k, \partial\Sigma) = \Sigma \wedge T_k$
for any Lipschitz oriented 2-surface Σ in R_k such that $\partial\Sigma \subset R_k \setminus \mathrm{supp} T_k$

Suppose T_k and v_k are constructed as above. Let K_{k+1} be one of the components of R_{k+1}. We claim that

$$\partial(T_k \lfloor \partial K_{k+1}) = 0$$

This is a direct consequence of the existence of v_k on ∂K_{k+1} satisfying iv). There exists a Lipschitz-diffeomorphism ψ which sends K_{k+1} to the unit ball of radius ε^α and which satisfies $\|\nabla\psi\|_\infty + \|\nabla\psi^{-1}\|_\infty \leq C$, where C is independent of ε. The choice of $T_{k+1}\lfloor K_{k+1}$ is as follows: the image by ψ^{-1} of the radial extension of $\psi_*(T_k\lfloor\partial K_{k+1})$. So it is clear that it satisfies conditions i), ii) and iii). The fact that there exists an extension v_{k+1} of v_k as a map from $K_{k+1}\setminus \mathrm{supp} T_{k+1}$ satisfying condition v) is explained in Appendix A.7.

Using arguments above we can change the good covering $B_{2\varepsilon^\gamma}(z_j)$ of $\mathbb{S}$ a bit into $B_{(2-\delta_j)\varepsilon^\gamma}(z_j)$, where $\delta_j < 1/4$ in order to ensure that each $\partial B_{(2-\delta_j)\varepsilon^\gamma}(z_j)$ intersects at most $C\varepsilon^{(\nu-\alpha)(n-3)}$ rectangular components containing a point where $|u| < 1/2$. Denote $\Omega^\nu_\varepsilon := \Omega \setminus \cup_j B_{(2-\delta_j)\varepsilon^\gamma}(z_j)$ instead of $\Omega \setminus \cup_j B_{\varepsilon^\gamma}(z_j)$. Let $T^\gamma_\varepsilon = T_n \lfloor \Omega^\nu_\varepsilon$. Because of conditions ii), iii) and the remark just above we have $\mathrm{spt}\partial T^\gamma_\varepsilon \subset \partial\Omega^\nu_\varepsilon$ and since $n_\nu \leq \frac{C}{\varepsilon^{\gamma(n-3)}}$ we can deduce that $M(\partial T^\nu_\varepsilon) \leq C$. Because of v) we easily deduce that $\partial T^\nu_\varepsilon - \mathbb{S} = \partial N_\varepsilon$ for some $N_\varepsilon \subset \cup_j B_{(2-\delta_j)\varepsilon^\nu}(z_j)$, moreover we can ensure $M(N_\varepsilon) \leq \varepsilon^\nu$. Now we claim that $\mathbb{T}^\varepsilon = T_n + N_\varepsilon$ is a solution of our problem. The fact that the conditions i),..,v) of Lemma V.1 are satisfied for $\mathbb{T}^\varepsilon = T_n$ comes from the construction of T_n. Let's look at the last condition. Let γ be a closed curve in $\Omega \setminus R'$. It is clear that γ can be continuously deformed into a curve lying in $\partial R'$, staying completely in the set where $|u| \geq \frac{1}{2}$, which ensures the fact that the degree along it does not change. The curve is now in R'_{n-1} but in the components of R'_{n-1} where $|u| \geq \frac{1}{2}$. So we can deform it again in R'_{n-2} and so on until reaching the 2-skeleton R'_2 and by construction of T_k we know that condition v) of Lemma V.2 is satisfied for such a curve and since, the degree of $u/|u|$ did not change during this transformation we have proved Lemma V.2.

IV.1.4. Proof of Lemma IV.1. Take $0 < \nu < \mu < \alpha < 1$ (to be fixed later) and let $\cup_{i=1}^{N_\alpha} B_{\varepsilon^\alpha}(x_i) \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$ be the union of balls of radius ε^α and ε^ν given by Lemma IV.2. Like in the proof of Lemma V.2, we can find a covering of $\Omega \setminus \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$ by $2\varepsilon^\mu$-cubes such that $(H1)$ and $(H2)$ are satisfied (for $\beta := \alpha$) at least for the n and $n-1$-skeletons. Denote by Ω_μ the complements in $\Omega \setminus \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$ of the union of the rectangular sets $(R'_i)_1, \ldots, N_\mu$ of this rectangular covering which contain at least one point where $|u| < 1/2$. We will use the notation $S_\varepsilon = \cup_{j=1}^{n_\nu} B_{\varepsilon^\nu}(z_j)$. We have chosen in fact the good covering $B_{\varepsilon^\nu}(z_j)$ of $\mathbb{S}$ such that

$$\int_{\partial S_\varepsilon} e_\varepsilon(u) \leq \log \frac{1}{\varepsilon} \quad . \tag{IV.7}$$

This is always possible, using Lemma II.7 + the mean value formula and the bound $n_\nu \leq \frac{C}{\varepsilon^{\nu(n-3)}}$ (the covering $B_{\varepsilon^\nu}(z_j)$ is eventually changed into a covering $B_{(1-\delta_j)\varepsilon^\nu}(z_j)$ where $0 < \delta_j < 1/4$). Recall that $N_\mu \leq C/(\varepsilon^\mu)^{n-2}$.

Let $1 > s > 0$, to be fixed later. We proceed with the following Hodge decomposition

$$\frac{u \wedge du}{|u \wedge du|^s} = d^* k + dL \qquad \text{in } \Omega \quad , \tag{IV.8}$$

where L is a function in Ω such that $L \equiv 0$ on $\partial\Omega$. We know (see [18]) that such a decomposition exists, since Ω is diffeo. to B^n, with the following bound for any $1 < q < +\infty$

$$\|k\|_{W^{1,q}(\Omega)} \leq C_q \left[\int_\Omega |u \wedge du|^{q(1-s)} \right]^{\frac{1}{q}} \quad . \tag{IV.9}$$

In the remaining part q will be chosen such that $q > n$ in order to ensure, by Sobolev injection, that

$$\|k\|_\infty + \|k\|_{C^{0,\delta}(\Omega)} \leq C_q \left[\int_\Omega |u \wedge du|^{q(1-s)} \right]^{\frac{1}{q}} \quad , \tag{IV.10}$$

for $\delta = 1 - \frac{n}{q}$. We will chose q and s such that

$$q = \frac{2-s}{1-s} \quad . \tag{IV.11}$$

Since $q > n$ the constraint on s becomes $\frac{n-2}{n-1} < s < 1$. Multiply (IV.8) by $u \wedge du$ and integrate on Ω. We get

$$\int_\Omega |u \wedge du|^{2-s} = \int_\Omega d * k \wedge (u \wedge du) + \int_\Omega dL \wedge *(u \wedge du) \quad . \tag{IV.12}$$

Using the fact that $L \equiv 0$ on $\partial\Omega$ and the fact that $d(*(u \wedge du)) = 0$ we get

$$\int_\Omega |u \wedge du|^{2-s} = \int_\Omega d * k \wedge (u \wedge du) \quad . \tag{IV.13}$$

Decompose $\int_\Omega d * k \wedge (u \wedge du) = \int_{\Omega_\mu} ... + \int_{\Omega\setminus\Omega_\mu} ...$. We bound the second integral like this

$$\begin{aligned}\left|\int_{\Omega\setminus\Omega_\mu} d * k \wedge (u \wedge du)\right| &\leq C\left(\int_{\Omega\setminus\Omega_\mu} |d^*k|^2\right)^{\frac{1}{2}} \left(\int_\Omega |du|^2\right)^{\frac{1}{2}} \\ &\leq (\varepsilon^{2\mu})^{1-\frac{2}{q}} \left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}} \left[\int_\Omega |u \wedge du|^{2-s}\right]^{\frac{1}{q}} \quad .\end{aligned} \tag{IV.14}$$

In Ω_μ we write $u \wedge du = \frac{u}{|u|} \wedge d\frac{u}{|u|} + (1 - \frac{1}{|u|^2})u \wedge du$. We have

$$\begin{aligned}\left|\int_{\Omega_\mu}\left(1 - \frac{1}{|u|^2}\right) d * k \wedge (u \wedge du)\right| &\leq C\|dk\|_{L^q}\left(\int_\Omega (1 - |u|^2)^{q'}\ |u \wedge du|^{q'}\right)^{\frac{1}{q'}} \\ &\leq C\varepsilon^{1-\frac{q'}{2}}\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}} \left[\int_\Omega |u \wedge du|^{2-s}\right]^{\frac{1}{q}} \quad .\end{aligned} \tag{IV.15}$$

We integrate the remaining term by parts and since $d(\frac{u}{|u|} \wedge d\frac{u}{|u|}) = 0$ on Ω_μ we get

$$\int_{\Omega_\mu} d * k \wedge \left(\frac{u}{|u|} \wedge d\frac{u}{|u|}\right) = \int_{\partial\Omega_\mu} *k \wedge \left(\frac{u}{|u|} \wedge d\frac{u}{|u|}\right) \quad . \tag{IV.16}$$

We write $\partial\Omega_\mu = (\partial\Omega_\mu \cap \Omega) \cup (\partial\Omega \cap \overline{\Omega_\mu})$ and we establish the following bound

$$\begin{aligned}\int_{\partial\Omega\cap\overline{\Omega_\mu}} *k \wedge \left(\frac{u}{|u|} \wedge d\frac{u}{|u|}\right) &\leq C\|k\|_\infty \int_{\partial\Omega\cap\overline{\Omega_\mu}} |g \wedge dg| \\ &\leq C\left[\int_\Omega |u \wedge du|^{2-s}\right]^{\frac{1}{q}} \quad .\end{aligned} \tag{IV.17}$$

Once again we write $\frac{u}{|u|} \wedge d\frac{u}{|u|} = (\frac{1}{|u|^2} - 1)u \wedge du + u \wedge du$, and using (IV.7) and the fact that $\int_{\partial\Omega_\mu\cap\Omega} e_\varepsilon(u) \leq \frac{1}{\varepsilon^\mu}\log\frac{1}{\varepsilon}$ we establish the bound

$$\begin{aligned}\left|\int_{\partial\Omega_\mu\cap\Omega}\left(\frac{1}{|u|^2} - 1\right) * k \wedge (u \wedge du)\right| &\leq \|k\|_\infty\varepsilon\int_{\partial\Omega_\mu\cap\Omega} e_\varepsilon(u) \\ &\leq \|k\|_\infty\varepsilon^{1-\mu}\log\frac{1}{\varepsilon} \quad .\end{aligned} \tag{IV.18}$$

Now decompose $\Omega\setminus\Omega_\mu$ as a union of S_ε and the rectangular sets $(R'_i)_{i=1,\dots,N_\mu}$. We have $\int_{\partial\Omega_\mu\cap\Omega} *k\wedge(u\wedge du) = \int_{\partial S_\varepsilon\cap\Omega}\dots + \sum_{i=1}^{N_\mu}\int_{\partial R'_i}\dots - \int_{\partial\Omega\cap\overline{\Omega\setminus\Omega_\mu}}\dots$. The following bounds hold, using the hypothesis on g_ε and (IV.7):

$$\int_{\partial S_\varepsilon\cap\Omega} *k\wedge(u\wedge du) \le C\|k\|_\infty\varepsilon^\nu\left(\int_{S_\varepsilon} e_\varepsilon(u)\right)^{\frac12} \tag{IV.19}$$

$$\le C\varepsilon^\nu\left(\log\frac1\varepsilon\right)^{\frac12}\left[\int_\Omega |u\wedge du|^{2-s}\right]^{\frac1q} \quad .$$

$$\int_{\partial\Omega\cap\overline{\Omega\setminus\Omega_\mu}} *k\wedge(u\wedge du) = \int_{\partial\Omega\cap\overline{\Omega\setminus\Omega_\mu}} *k\wedge(g\wedge dg) \le C\,\|k\|_\infty \quad . \tag{IV.20}$$

Let $*\overline{k}_i$ be the mean value of $*k$ on R'_i, in particular

$$\|*k - *\overline{k}_i\|_\infty \le C\varepsilon^{\mu\delta}\left[\int_\Omega |u\wedge du|^{2-s}\right]^{\frac1q} \quad .$$

Thus, using the fact that $\int_{\partial R'_i}|du|^2 \le \varepsilon^{\mu(n-3)}\log\frac1\varepsilon$, we get

$$\left|\int_{\partial R'_i}(*k - *\overline{k}_i)\wedge(u\wedge du)\right| \le C\varepsilon^{\mu(\delta+n-2)}\left(\log\frac1\varepsilon\right)^{\frac12}\left[\int_\Omega |u\wedge du|^{2-s}\right]^{\frac1q} \quad . \tag{IV.21}$$

In order to find a bound for $\int_{\partial R'_i} *\overline{k}_i\wedge(u\wedge du)$, we look for a bound of $\int_{\partial R'_i} *(dx_{n-1}\wedge dx_n)\wedge d(u\wedge du)$ since $\overline{k}_i$ is a linear combination of the $dx_k\wedge dx_l$ for $k\neq l$ with constant coefficients.
$dx_1\wedge\dots\wedge dx_{n-2} = *(dx_{n-1}\wedge dx_n)$ does not cancel on the faces where either x_n is constant or x_{n-1} is constant. For a given $(t_1,\dots,t_{n-2})$ the intersection with the two-plane $x_1 = t_1,\dots,x_{n-2} = t_{n-2}$ and $\partial R'_i$ is a 1-dim. closed Lipschitz line that we will denote by $\Gamma_{t_1,\dots,t_{n-2}}$. We denote by U_i the set of coordinates $(t_1,\dots,t_{n-2})$ such that $\Gamma_{t_1\dots t_{n-2}}$ is non empty. Among U_i we only want to keep the coordinates such that $\Gamma_{t_1\dots t_{n-2}}$ does not intersect the balls $(B_{\varepsilon^\alpha}(x_i))$. Denote this set by V_i. There exist at most $C\varepsilon^{(n-3)(\mu-\alpha)}$ of such balls, thus the $(n-2)$-measure of V_i is bounded by $C\varepsilon^{\alpha+\mu(n-3)}$. Let

$$\mathcal{S}_i = \{x\in R'_i \text{ s. t. } \exists(t_1\dots t_{n-2})\in V_i,\ \text{s. t. } x\in\Gamma_{t_1\dots t_{n-2}}\} \quad .$$

It is clear that $|\mathcal{S}_i|_{n-1}\le C\varepsilon^{\alpha+\mu(n-2)}$. Thus

$$\int_{\mathcal{S}_i}|u\wedge du| \le C\varepsilon^{\frac\alpha2+\mu(n-2)} \quad . \tag{IV.22}$$

On $\partial R_i' \setminus \mathcal{S}_i$ let us write $u \wedge du = \frac{u}{|u|} \wedge d\frac{u}{|u|} + (1 - \frac{1}{|u|^2}) u \wedge du$. We have the bound

$$\int_{\partial R_i' \setminus \mathcal{S}_i} (1 - |u|^2)|u \wedge du| \leq C\varepsilon^{1+\mu(n-3)} \log \frac{1}{\varepsilon} \quad . \tag{IV.23}$$

Now using Lemma V.1 we deduce that

$$\begin{aligned} \int_{\partial R_i' \setminus \mathcal{S}_i} dx_1 ... dx_{n-2} \wedge \left(\frac{u}{|u|} \wedge d\frac{u}{|u|} \right) &= \int_{U_i \setminus V_i} \int_{\Gamma_{t_1 ... t_{n-2}}} \frac{u}{|u|} \wedge d\frac{u}{|u|} \\ &= \int_{U_i \setminus V_i} \Sigma_{t_1 ... t_{n-2}} \wedge \mathbb{T}^\varepsilon \quad , \end{aligned} \tag{IV.24}$$

where $\Sigma_{t_1 ... t_{n-2}} = \{x \in R_i' \text{ s. t. } x_1 = t_1 ... x_{n-2} = t_{n-2}\}$. Summing (IV.19), (IV.21)...(IV.24) over i we obtain

$$\left| \int_{\partial \Omega_\mu \cap \Omega} *k \wedge (u \wedge du) \right| \leq C(M(\mathbb{T}^\varepsilon) + o(1)) \left[\int_\Omega |u \wedge du|^{2-s} \right]^{\frac{1}{q}} \quad . \tag{IV.25}$$

Combining now (IV.13)...(IV.20) and (IV.25) we obtain that $\int_\Omega |u \wedge du|^{2-s}$ is bounded independently of ε, moreover recall that the constraint on s is $\frac{n-2}{n-1} < s < 1$. That means that the constraint on $2 - s$ is $1 < 2 - s < \frac{n}{n-1}$. Thus we prove a uniform L^p bound for $u \wedge du$ for $1 < p < \frac{n}{n-1}$ and the passage from this L^p bound to the result of Lemma V.1 can be done like in [4] or [25].

IV.2. Proof of Theorem I.2

Let $\mathbb{T}$ be the limit (modulo a subsequence) of the currents $\mathbb{T}_\varepsilon$ given by Lemma IV.2. Let $\overline{\Omega}_\delta = \overline{\Omega} \setminus \{\delta -\text{neighborhood of spt}\, \mathbb{T}\}$, $\delta > 0$. Then the eta-compactness Lemma implies, in particular, that $|u_\varepsilon|(x) \geq 1/2, \forall x \in \overline{\Omega}_\delta$.

We write $u_\varepsilon(x) = \rho_\varepsilon(x) \ \exp(i\theta_\varepsilon)$ locally near any point of $\overline{\Omega}_\delta$. Then $div(\rho_\varepsilon^2 \nabla \Theta_\varepsilon) = 0$. Using the fact that $\rho_\varepsilon \to 1^-$ (as $\varepsilon \to 0$) pointwise on $\overline{\Omega}_\delta$ and the $W^{1,p}$ estimate for u_ε, one has $\nabla \theta_\varepsilon \in L^2(\overline{\Omega}_\delta)$. At this stage of the proof one easily deduces that $\nabla \rho_\varepsilon \in L^2(\Omega_\delta)$. The minimality of $\mathbb{T}$ is obtained by a comparison argument and proving a lower bound of the form

$$\forall \alpha > 0 \quad \exists C_\alpha \geq 0 \qquad \text{s.t.} \qquad \pi\,(M(\mathbb{T}) - \alpha) \log \frac{1}{\varepsilon} - C_\alpha \leq E_\varepsilon(u_\varepsilon) \quad . \tag{IV.26}$$

This can be done working perpendicularly to the integral current $\mathbb{T}$ as in [25] Sect. 7 and using Lemma A.6 of [25] at each perpendicular 2-plane of $\mathbb{T}$. ∎

V. Appendix

V.1. Few technical lemmas

Lemma A.1. *Let φ be a p-form in $\mathbb{R}^n$, we have*

$$\frac{\partial}{\partial r}d\varphi = d\frac{\partial\varphi}{\partial r} - \frac{1}{r}d\varphi + d\log r \wedge \frac{\partial\varphi}{\partial r} \quad .$$

■

Proof of Lemma A.1.

$$\varphi = \sum_I \phi_I dx_I \quad , \quad d\varphi = \sum_{k,I} \frac{\partial\varphi_I}{\partial x_k} dx_k \wedge dx_I$$

$$\begin{aligned}
\frac{\partial}{\partial r}d\varphi &= \sum_{k,I} \frac{\partial}{\partial r}\left[\frac{\partial\varphi}{\partial x_k}\right] dx_k \wedge dx_I \\
&= \sum_{k,I,l} \frac{\partial}{\partial x_l}\left[\frac{\partial\varphi_I}{\partial x_k}\right] \frac{x_l}{|x|} dx_k \wedge dx_I \\
&= \sum_{k,I,l} \frac{\partial}{\partial x_k}\left[\frac{\partial\varphi_I}{\partial x_l}\right] \frac{x_l}{|x|} dx_k \wedge dx_I - \sum_{k,l,I} \frac{\partial\varphi_I}{\partial x_l}\frac{\delta_{kl}}{|x|} dx_k \wedge dx_I + \\
&+ \sum_{k,I,l} \frac{\partial\varphi_I}{\partial x_l}\frac{x_l x_k}{|x|} dx_k \wedge dx_I \\
&= d\frac{\partial\varphi}{\partial r} - \frac{1}{r}d\varphi + d\log r \wedge \frac{\partial\varphi}{\partial r} \quad .
\end{aligned}$$

■

Denote by $d_\top$ and $(*)$ respectively the exterior differentiation and the Hodge operator on $S_r = \partial B_r$. For $\omega \in \wedge^*\mathbb{R}^n$, $d_\top\omega$ not only denotes, the exterior differentiation of the restriction ω to S_r, but also the form in $\wedge^{*+1}\mathbb{R}^n$ which coincides with this restriction at each point of $\mathbb{R}^n$: $d_\top\omega = d\omega - dr \wedge \frac{\partial\omega}{\partial r}$.

Lemma A.2. *We have*

$$d_\top\frac{\partial\varphi}{\partial r} = \frac{\partial}{\partial r}\left(d_\top\varphi\right) + \frac{1}{r}d_\top\varphi \quad .$$

■

Proof of Lemma A.2.

$$
\begin{aligned}
d_\top \frac{\partial\varphi}{\partial r} &= d\frac{\partial\varphi}{\partial r} - dr \wedge \frac{\partial^2\varphi}{\partial r^2} \\
&= \frac{\partial}{\partial r} d\varphi + \frac{1}{r} d\varphi - \frac{1}{r} dr \wedge \frac{\partial\varphi}{\partial r} - dr \wedge \frac{\partial^2\varphi}{\partial r^2} \\
&= \frac{\partial}{\partial r}\left[d\varphi - dr \wedge \frac{\partial\varphi}{\partial r}\right] + \frac{1}{r}\left[d\varphi - dr \wedge \frac{\partial\varphi}{\partial r}\right] \\
&= \frac{\partial}{\partial r}\left[d_\top\varphi\right] + \frac{1}{r} d_\top\varphi \quad .
\end{aligned}
$$

■

Lemma A.3. *We have*

$$
\frac{\partial}{\partial r}\left[(*)\iota_r^* \eta\right] = (*)\iota_r^* \frac{\partial\eta}{\partial r} \quad ,
$$

where ι_r is the isometric embedding of ∂B_r into $\mathbb{R}^n$. ■

Proof of Lemma A.3. We have

$$
(*)\iota_r^*\eta = - * \eta \wedge dr \quad .
$$

Thus

$$
\frac{\partial}{\partial r}(*)\iota_r^*\eta = - * \frac{\partial\eta}{\partial r} \wedge dr = (*)\iota_r^* \frac{\partial\eta}{\partial r} \quad .
$$

■

Lemma A.4. *Let ϕ be a 0-form we have*

$$
\frac{\partial}{\partial r}\Delta_r\varphi = \Delta_r \frac{\partial\varphi}{\partial r} - \frac{2}{r}\Delta_r.\varphi \quad .
$$

■

Proof of Lemma A.4.

$$
\begin{aligned}
\frac{\partial}{\partial r}\Delta_r\varphi &= -\frac{\partial}{\partial r}(*)d_\top(*)d_\top\varphi = -(*)\frac{\partial}{\partial r}d_\top(*)d_\top\varphi \\
&= -(*)d_\top \frac{\partial}{\partial r}\left[(*)d_\top\varphi\right] + (*)\left[\frac{1}{r}d_\top(*)d_\top\varphi\right]
\end{aligned}
$$

$$= -(*)d_\top(*)\frac{\partial}{\partial r}d_\top\varphi - \frac{1}{r}\Delta_r\varphi$$

$$= \Delta_r\frac{\partial\phi}{\partial r} + (*)d_\top(*)\frac{1}{r}d_\top\varphi - \frac{\Delta_r\varphi}{r} = \Delta_r\frac{\partial\phi}{\partial r} - 2\frac{\Delta_r\varphi}{r} \quad .$$

■

Lemma A.5.

$$\Delta_r\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right) - \frac{\partial}{\partial r}\left[\varphi\wedge\Delta_r\varphi\right] = 2(*)d_\top\left[\left(i\frac{\partial\varphi}{\partial r};(*)d_\top\varphi\right)\right]$$
$$+\frac{2}{r}\left(\varphi; i\Delta_r\varphi\right) \quad .$$

■

Proof of Lemma A.5.

$$\Delta_r\left(\varphi; i\frac{\partial\varphi}{\partial r}\right) = -(*)d_\top(*)d_\top\left[\left(\varphi; i\frac{\partial\varphi}{\partial r}\right)\right]$$
$$= -(*)d_\top\left[\left((*)d_\top\varphi; i\frac{\partial\varphi}{\partial r}\right)\right] - (*)d_\top(*)\left[\left(\varphi; id_\top\frac{\partial\varphi}{\partial r}\right)\right] \quad . \tag{V.27}$$

Using Lemma A.2. we compute

$$(*)d_\top(*)\left[\left(\varphi; id_\top\frac{\partial\varphi}{\partial r}\right)\right] = \frac{\partial}{\partial r}\left[(*)d_\top\left[(\varphi; i(*)d_\top\varphi)\right]\right]$$
$$+\tfrac{2}{r}(*)d_\top\left[(\varphi; i(*)d_\top\varphi)\right] - (*)d_\top\left[\left(\tfrac{\partial\varphi}{\partial r}; i(*)d_\top\varphi\right)\right] \quad . \tag{V.28}$$

A short computation shows that

$$-(*)d_\top\left[(\varphi; i(*)d_\top\varphi)\right] = (\varphi; i\Delta_r\varphi) \quad .$$

Combining the previous identity and (V.28) we get the desired result. ■

Lemma A.6.

$$\Delta\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\varphi\wedge\Delta\varphi\right) - 2(*)d_\top\left[\left(i\frac{\partial\varphi}{\partial r};(*)d_\top\varphi\right)\right]$$
$$+\frac{2}{r}\left(\varphi\wedge\Delta\varphi\right) + \frac{2}{r}\frac{\partial}{\partial r}\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right) + \frac{n-1}{r^2}\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right) \quad .$$

■

Proof of Lemma A.6.

$$\Delta\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right)=\Delta_r\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right)-\frac{\partial^2}{\partial r^2}\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right)$$
$$-\frac{n-1}{r}\frac{\partial}{\partial r}\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right)\quad .$$

Using Lemma A.5 we get

$$\Delta\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right)=\frac{\partial}{\partial r}\left(\varphi\wedge\Delta_r\varphi\right)-2(*)d_\top\left[\left(i\frac{\partial\varphi}{\partial r};(*)d_\top\varphi\right)\right]$$
$$+\frac{2}{r}\varphi\wedge\Delta_r\varphi-\frac{\partial^2}{\partial r^2}\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right)-\frac{n-1}{r}\frac{\partial}{\partial r}\left(\varphi\wedge\frac{\partial\varphi}{\partial r}\right)\quad .$$

Replacing $\Delta_r\varphi$ by $\Delta\varphi-\frac{\partial^2\varphi}{\partial r^2}-\frac{n-1}{r}\frac{\partial\varphi}{\partial r}$ we get the desired result. ■

Proof of Lemma A.7.
By a continuity argument we can always assume that 0 is a regular value for g. Let $\overline{g}$ be a regular extension of g from Ω into $\mathbb{C}$ such that 0 is also a regular point for $\overline{g}$. Denote by $\mathbb{L}$ the integral current associated to the regular oriented submanifold realized by $\overline{g}^{-1}(\{0\})$. $\mathbb{L}$ defines a homology class $\mathcal{L}$ in $H_{n-2}(\overline{\Omega},\ \mathrm{spt}\,\mathbb{L})$ which satisfies of course i). It also satisfies ii) by of the following fact: let $\sigma=\sum n_i\sigma_i$ be a cellular decomposition of σ, $n_i\in\mathbb{Z}$ and $\sigma_i=f_i(D^2)$. The degree of $\overline{g}/|\overline{g}|$ on the one-chain realized by $\partial\sigma_i$ is $\int_{\partial D^2}\left(\frac{\overline{g}}{|\overline{g}|}\circ f_i\right)^* d\theta$. This is equal to

$$\int_{D^2}d\left(\left(\frac{\overline{g}}{|\overline{g}|}\circ f_i\right)^* d\theta\right)=\sum_{a_j\in\sigma_i\cap\mathrm{spt}\,\mathbb{L}}d_j\delta_{a_j}=\sigma_i\wedge\mathbb{L}\quad .$$

Multiplying this identity by n_i and summing over i one gets ii).

The uniqueness of $\mathcal{L}$ is a direct consequence of the Poincaré-Lefschetz duality theorem (see [31] page 296). We have

$$H_{n-2}(\overline{\Omega};\mathrm{spt}\,\mathbb{L})\simeq H^2(\overline{\Omega}\setminus\mathrm{spt}\,\mathbb{S};\partial\Omega\setminus\mathrm{spt}\,\mathbb{S})$$

and this isomorphism is exactly given by the intersection number. ii) means that we define uniquely a class in $H^2(\overline{\Omega}\setminus\mathrm{spt}\,\mathbb{S};\partial\Omega\setminus\mathrm{spt}\,\mathbb{S})$ and its image by the isomorphism above is $\mathbb{L}$. ■

V.2. A harmonic form with given singularity

First let us fix the notations.

Euclidean space $\mathbb{R}^n$, with $n \geq 3$. For $x_0 \in \mathbb{R}^n$, $r > 0$,
$U_r(x_0) = \{x \backslash x \in \mathbb{R}^n, \mid x - x_0 \mid < r\}$,
$B_r(x_0) = \{x \backslash x \in \mathbb{R}^n, \mid x - x_0 \mid \leq r\}$,
$U_r = U_r(0)$, $B_r = B_r(0)$.
$\omega_n =$ the volume of $B_1 \subset \mathbb{R}^n$.
$\Gamma(x) = \frac{1}{n(n-2)\omega_n |x|^{n-2}}$ the fundamental solution of the Laplacian operator.

For any open set $\Omega \subset \mathbb{R}^n$, any integer $k \in [0, n]$,
$D^k(\Omega) = \{$all smooth differential k forms on Ω with compact support$\}$.
For a locally integrable k-form ω on Ω, we have a $(n-k)$-current T_ω given by $T_\omega(\tau) = \int_\Omega \omega \wedge \tau$, for any $\tau \in D^{n-k}(\Omega)$. For any k-current T on Ω, σT is a $(k+1)$-current defined by $(\sigma T)(\tau) = T(\delta\tau)$, for any $\tau \in D^{k+1}(\Omega)$.
$\triangle T = \sigma\partial T + \partial\sigma T$.

V.2.1. The question. Suppose $S_0 = (M_0, \theta_0, \xi_0)$ is an integer multiplicity $(n-2)$-current, where M_0 is a $(n-2)$-rectifiable subset of B_1 with $H^{n-2}(M_0) < \infty$, $\theta_0 : M_0 \to \mathbb{Z}$ is a measurable function s.t. for any $x \in M_0$ there exists a positive number c_0 with $\mid \theta_0(x) \mid \leq c_0$ ξ_0 is the orientation $n-2$ vector function. We assume $\partial S_0 = 0$ in U_1 (not in $\mathbb{R}^n$!).

ζ is a given smooth closed tangential differential 1-form on $\partial B_1 \backslash \overline{M_0}$ s.t. $\mid \zeta(x) \mid \leq c \cdot d(x, \overline{M_0})^{-1}$ for $x \in \partial B_1$, c being a positive constant.

Compatibility condition. For any oriented closed smooth curve C lying in $\partial B_1 \backslash \overline{M_0}$, $\int_C \zeta = 2\pi$(the winding number of C around S_0).

Aim. We are looking for a canonical smooth harmonic 1-form in $B_1 \backslash \overline{M_0}$, which has ζ as its tangential part on $\partial B_1 \backslash \overline{M_0}$ and satisfies $\int_C \omega = 2\pi$ (the winding number of C around S_0) for any smooth closed curve C in $B_1 \backslash \overline{M_0}$.

We want to pick up a canonical one because there are many solutions of the problem which just look like higher order poles in complex analysis.

V.2.2. The solution to the problem. Suppose $n = 3$ and S_0 is the part of the z-axis in B_1. Let α be the angle on the horizontal plane, then as the standard picture we have $d\alpha$ in our mind. Since $d\alpha$ is integrable in B_1, we may consider $T_{d\alpha}$ and

$$\sigma(T_{d\alpha}) = 0,\ \partial(T_{d\alpha}) = 2\pi((z \text{ axis}) \cap B_1) \text{ in } U_1 \quad .$$

This motivates the following formulation of the question. Suppose we have S_0, ζ as in the former section. We hope to find an integrable 1 form ω on B_1 which is harmonic in $U_1 \backslash \overline{M_0}$, smooth in $B_1 \backslash \overline{M_0}$ with tangential part ζ on $\partial B_1 \backslash \overline{M_0}$. Besides this ω needs to satisfy

$$\sigma(T_\omega) = 0,\ \partial(T_\omega) = 2\pi S_0 \text{ in } U_1 \quad .$$

Uniqueness of solutions.

Suppose we have two solutions, say ω_1 and ω_2. Set $\omega = \omega_1 - \omega_2$, then

$$\sigma(T_\omega) = 0,\ \partial(T_\omega) = 0 \text{ in } U_1 \quad .$$

From elliptic regularity we know ω is smooth and harmonic in U_1. $\omega = df$, where f is a harmonic function in U_1. By the boundary condition we have $d(f \mid_{\partial B_1 \backslash \overline{M_0}}) = 0$, which implies $f \mid_{\partial B_1 \backslash \overline{M_0}} \equiv$ const. So $f \equiv$ const in B_1. $\omega = df = 0$, which gives the uniqueness.

Existence of a solution.

Suppose we have an integrable 1 form ω_0 in B_1 which satisfies

$$\sigma(T_{\omega_0}) = 0,\ \partial(T_{\omega_0}) = 2\pi S_0 \text{ in } U_1 \quad ,$$

but doesn't satisfy the boundary condition, then we may get the ω by solving a Dirichlet problem for a harmonic function. So the main step is to get a ω_0. We first choose another integer multiplicity $(n-2)$-current namely $S_1 = (M_1, \theta_1, \xi_1)$, where M_1 is bounded and lies in $\mathbb{R}^n \backslash B_1$ with $H^{n-2}(M_1) < \infty$, $\mid \theta_1(x) \mid \leq c_1$ c_1 being a positive constant. We choose S_1 suitably such that

$$S = S_0 + S_1 = (M_0 \cup M_1, \theta_0 \cup \theta_1, \xi_0 \cup \xi_1) = (M, \theta, \xi)$$

satisfies $\partial S = 0$ in $\mathbb{R}^n$. Notice that M is bounded and $\mid \theta(x) \mid \leq \max\{c_0, c_1\}$. Let $\Lambda = 2\pi\sigma(\Gamma * S)$, then

$$\sigma\Lambda = 2\pi\sigma(\sigma(\Gamma * S)) = 0 \quad .$$

$$\partial(\Gamma * S) = \Gamma * \partial S = 0 \quad .$$

$$\begin{aligned} \partial\Lambda &= 2\pi\partial\sigma(\Gamma * S) = 2\pi(\partial\sigma + \sigma\partial)(\Gamma * S) \\ &= 2\pi\triangle(\Gamma * S) = 2\pi(\triangle\Gamma * S) = 2\pi S \text{ in } \mathbb{R}^n \quad . \end{aligned}$$

Now for $x \in M$ suppose that $e_1(x), \cdots, e_n(x)$ is a positive orthonormal base of $\mathbb{R}^n$ s.t. $e_1(x), \cdots, e_{n-2}(x)$ is a positive base for $T_x M$. Let $e^1(x), \cdots, e^n(x)$ be the dual base, then we call $\chi(x) = e^1(x) \wedge \cdots \wedge e^{n-2}(x)$ the orientation form of M. Let

$$\Theta(x) = \int_M \Gamma(x-y)\theta(y)\chi(y)dH^{n-2}(y) \ \text{ for } x \in \mathbb{R}^n \quad .$$

It follows from Hölder's inequality and Fubini's theorem that $\Theta \in W^{1,\frac{n}{n-1}-\varepsilon}_{loc}$ for arbitrary small $\varepsilon > 0$. We have

$$T_{*\Theta} = \Gamma * S,\ \ \Lambda = 2\pi\sigma(T_{*\Theta}) = 2\pi T_{\delta(*\Theta)} \quad .$$

Put $\omega_0 = 2\pi\delta(*\Theta)$. We have $\omega_0 \in L^{\frac{n}{n-1}-\varepsilon}_{loc}$ for arbitrary small $\varepsilon > 0$.

$$\omega_0(x) = \frac{2\pi}{n\omega_n}\int_M i_{\frac{x-y}{|x-y|^n}}(*\chi(y))\theta(y)dH^{n-2}(y) \ \text{ for } x \in \mathbb{R}^n \quad ,$$

where i means the contraction of forms by a vector. Hence

$$\sigma(T_{\omega_0}) = 0,\ \ \partial(T_{\omega_0}) = 2\pi S \text{ in } \mathbb{R}^n \quad .$$

$$\sigma(T_{(\omega_0|_{B_1})}) = 0,\ \ \partial(T_{(\omega_0|_{B_1})}) = 2\pi S_0 \text{ in } U_1 \quad .$$

We know$|\ \omega_0(x)\ |\leq c \cdot d(x,\overline{M})^{-1}$ for x close to $\overline{M}$. ($\zeta-$tangential part of ω_0) is exact in $\partial B_1\backslash\overline{M}$, say it is equal to df. It follows from the growth of ζ and ω_0 that f is integrable (in fact it is in $L^p(\partial B_1)$ for any $p \in [1,\infty)$). Solve the Dirichlet problem $\triangle u = 0$ in U_1, $u\ |_{\partial B_1} = f$. Then $\omega = \omega_0\ |_{U_1} + du$ is a solution to the problem.

Remark A.1. We may do similar problems where the unit ball is replaced by an arbitrary smooth domain Ω in $\mathbb{R}^n$, which doesn't need to be simply connected, or by a smooth domain Ω in a compact oriented Riemannian manifold. For these cases, the necessary condition for a $(n-2)$-current to be the singular set of some harmonic form is $S_0(\tau) = 0$, for τ any closed form in $D^{n-2}(\Omega)$. We may use a similar idea of finding a complement current but replace Γ by the Green's operator in the case of a Riemannian manifold. Now because the topology is nontrivial, the solution space is a finite dimensional affine space. One may find more information in Chapter V of [24] and Chapter 7 of [23]. ■

V.2.3. Higher integrability of the solution. We know if $U_1 \subset \mathbb{R}^3$ and α is the horizontal angle, then the standard model $d\alpha$ lies in $L^{2-\varepsilon}$ for arbitrary small $\varepsilon > 0$. Now we will get this integrability for the solution in the former section under a local growth condition of the singular set. Because $\omega = \omega_0\ |_{U_1} + du$ and u is a nice harmonic function, it suffices to study the form ω_0.

Proposition A.1. *Suppose $S = (M,\theta,\xi)$ is an integer multiplicity $n-2$ current in $\mathbb{R}^n$. $H^{n-2}(M) < \infty$. There exists $r_0 > 0$ s.t. $M \subset B_{r_0}$. There exists $c_M > 0$ s.t.*

$$H^{n-2}(M \cap B_r(x)) \leq c_M \cdot r^{n-2},\ \ \textit{for } r \in (0,1], x \in \mathbb{R}^n \quad .$$

There exists $c_0 > 0$ s.t. $|\theta(x)| \leq c_0$ for $x \in M$. Let

$$\omega_0(x) = \frac{2\pi}{n\omega_n} \int_M i_{\frac{x-y}{|x-y|^n}} (*\chi(y))\theta(y) dH^{n-2}(y) \ \ \text{for } x \in \mathbb{R}^n \quad ,$$

where χ is the orientation form. Then

$$\omega_0 \in L^{2-\varepsilon}_{loc} \text{ for arbitary small } \varepsilon > 0 \quad .$$

∎

Proof. Observing that $\omega_0(x) \leq c(n, c_0) I_M(x)$, where

$$I_M(x) = \int_M \frac{1}{|x-y|^{n-1}} dH^{n-2}(y) \ \ \text{for } x \in \mathbb{R}^n \quad ,$$

we only need to show $I_M \in L^{2-\varepsilon}_{loc}$ for arbitrary small $\varepsilon > 0$. For any $r_1 > 0$,

$$\int_{B_{r_1}(x)\cap M} \frac{1}{|x-y|^{n-1}} dH^{n-2}(y)$$

$$= \int_{B_{r_1}(x)\cap M} \left(\frac{1}{|x-y|^{n-1}} - \frac{1}{r_1^{n-1}} \right) dH^{n-2}(y) + \frac{H^{n-2}(B_{r_1}(x)\cap M)}{r_1^{n-1}}$$

$$= (n-1) \int_{B_{r_1}(x)\cap M} dH^{n-2}(y) \int_{|x-y|}^{r_1} r^{-n} dr + \frac{H^{n-2}(B_{r_1}(x)\cap M)}{r_1^{n-1}}$$

$$= (n-1) \int_0^{r_1} dr \int_{B_r(x)\cap M} r^{1-n} dH^{n-2}(y) + \frac{H^{n-2}(B_{r_1}(x)\cap M)}{r_1^{n-1}}$$

$$= (n-1) \int_0^{r_1} \frac{H^{n-2}(B_r(x)\cap M)}{r^n} dr + \frac{H^{n-2}(B_{r_1}(x)\cap M)}{r_1^{n-1}} \quad .$$

Let r_1 go to ∞. This implies

$$I_M(x) = (n-1) \int_0^{\infty} \frac{H^{n-2}(B_r(x)\cap M)}{r^n} dr \quad .$$

For any $p \in [1, 2)$, any $r_1 > 0$,

$$\left(\int_{B_{r_1}} I_M(x)^p dx\right)^{\frac{1}{p}} \leq (n-1) \int_0^{\infty} \left[\int_{B_{r_1}} (H^{n-2}(B_r(x)\cap M))^p dx\right]^{\frac{1}{p}} r^{-n} dr \quad .$$

Now we have for $r \in (0, 1]$,

$$\int_{B_{r_1}} [H^{n-2}(B_r(x)\cap M)]^p dx \leq c \cdot r^{(p-1)(n-2)} \int_{B_{r_1}} H^{n-2}(B_r(x)\cap M) dx$$

$$\leq c \cdot r^{(p-1)(n-2)} \int_{\mathbb{R}^n} dx \int_{B_r(x)\cap M} dH^{n-2}(y)$$

$$= c \cdot r^{(p-1)(n-2)} \int_M r^n dH^{n-2}(y) = c \cdot r^{pn-2(p-1)} \quad .$$

Hence

$$(\int_{B_{r_1}} [H^{n-2}(B_r(x) \cap M)]^p dx)^{\frac{1}{p}} r^{-n} \leq c \cdot r^{-2(1-\frac{1}{p})} \quad .$$

For $r \in [1, \infty)$, we have

$$(\int_{B_{r_1}} [H^{n-2}(B_r(x) \cap M)]^p dx)^{\frac{1}{p}} r^{-n} \leq c(n, p, r_1) H^{n-2}(M) r^{-n} \quad .$$

The above two inequalities imply $\int_{B_{r_1}} I_M(x)^p dx < \infty$.

Remark. The function I_M in the proof is closely related to the trace problem because of the following,

$$f(x) = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{(x^i - y^i)\partial_i f(y)}{| x - y |^n} dy \ \text{ for any } f \in C_c^\infty(\mathbb{R}^n), x \in \mathbb{R}^n \quad .$$

$$\Rightarrow | f(x) | \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{| \nabla f(y) |}{| x - y |^{n-1}} dy \quad .$$

So

$$\int_M | f(x) | dH^{n-2}(x) \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} | \nabla f(x) | I_M(x) dx \quad .$$

For this aspect we may refer to Sect. 7.8 of [15] and Chapter 7 of [1] (especially P197 Theorem 7.2.2. and P211, 7.6.6.).

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