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Effective Nullstellensatz for arbitrary ideals

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Abstract. Let f_i be polynomials in n variables without a common zero. Hilbert's Nullstellensatz says that there are polynomials g_i such that $\sum g_i f_i = 1$. The effective versions of this result bound the degrees of the g_i in terms of the degrees of the f_j. The aim of this paper is to generalize this to the case when the f_i are replaced by arbitrary ideals. Applications to the Bézout theorem, to Łojasiewicz–type inequalities and to deformation theory are also discussed.

1. Introduction

Let $X, Y \subset \mathbb{P}^n$ be closed irreducible subvarieties and Z_i the irreducible components of $X \cap Y$. One variant of the theorem of Bézout (cf. [Fulton84, 8.4.6]) says that

$$\sum_i \deg Z_i \leq \deg X \cdot \deg Y.$$

This result holds without any restriction on the dimensions of X, Y, Z_i and it can be easily generalized to the case when $X_1, \ldots, X_s$ are arbitrary subschemes of $\mathbb{P}^n$ and the Z_i are the reduced irreducible components of $X_1 \cap \cdots \cap X_s$.

It is frequently of interest to study finer algebraic or metric properties of intersections of varieties. In recent years considerable attention was paid to the case when the X_i are all hypersurfaces, in connection with the effective versions of Hilbert's *Nullstellensatz*. Assume that we have polynomials $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$ of degrees $d_i = \deg f_i$. There are three related questions one can ask about the intersection of the hypersurfaces $(f_i = 0)$, in each case attempting to minimize a bound $B(d_1, \ldots, d_s)$.

Algebraic Bézout version: [Brownawell89] Find prime ideals $P_j \supset (f_1, \ldots, f_s)$ and natural numbers a_j such that

$$\prod_j P_j^{a_j} \subset (f_1, \ldots, f_s) \quad \text{and} \quad \sum_j a_j \deg P_j \leq B(d_1, \ldots, d_s).$$

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Effective Nullstellensatz version: [Kollár88] If the f_i have no common zeros in $\mathbb{C}^n$, find polynomials g_i such that

$$\sum_i f_i g_i = 1 \quad \text{and} \quad \deg(f_i g_i) \le B(d_1, \dots, d_s).$$

Łojasiewicz inequality version: [JKS92] Fix a metric on $\mathbb{C}^n$ and let Z be the intersection of the hypersurfaces $(f_i = 0)$. Prove that if x varies in a bounded subset of $\mathbb{C}^n$ then

$$\operatorname{dist}(Z, x)^{B(d_1,\dots,d_s)} \le C \cdot \max_i |f_i(x)| \quad \text{for some } C > 0.$$

The optimal value of $B(d_1, \dots, d_s)$ is known in almost all cases. If we assume that $d_i \ge 3$ for every i, then

$$B(d_1, \dots, d_s) = d_1 \cdots d_s,$$

is best possible for $s \le n$. (See [Kollár88, 1.5] for the case $s > n$.)

The algebraic Bézout version is also called the *prime power product* variant of the Nullstellensatz.

The aim of this paper is to consider these problems in case the f_j are replaced by arbitrary ideals. The first step in this direction was taken in [Sombra97]. His methods can deal with special cases of the above problems if the ideals are Cohen–Macaulay. Some other cases are worked out in [Ploski-Tworzewski98]. Łojasiewicz–type inequalities for arbitrary analytic sets were studied in the works of Cygan, Krasiski and Tworzewski, see especially [Tworzewski95,Cygan98,CKT98]. Although they consider the related problem of separation exponents, their proof can easily be modified to give a general Łojasiewicz inequality for reduced subschemes.

My proofs grew out of an attempt to understand their work in algebraic terms. This leads to a general Łojasiewicz inequality in the optimal form and to an effective Nullstellensatz with a slightly worse bound. In the algebraic Bézout version my results are weaker. It should be noted, however, that the straightforward generalization of the algebraic Bézout version fails to hold (1.4).

All three of these results can be formulated for arbitrary ideals, but for simplicity here I state them for unmixed ideals. (I is called *unmixed* if all primary components of I have the same dimension.) These are the ideals that correspond to the usual setting of intersection theory. For such ideals the degree of I (cf. (2.1)) gives a good generalization of the degree of a hypersurface. The precise versions for arbitrary ideals are stated in (6.1), (6.2) and (7.6).

Theorem 1.1 (Algebraic Bézout theorem). *Let K be any field and $I_1, \dots, I_m$ unmixed ideals in $K[x_1, \dots, x_n]$. Then there are prime ideals $P_j \supset (I_1, \dots, I_m)$ and natural numbers a_j such that*

1. $\prod_j P_j^{a_j} \subset (I_1, \ldots, I_m)$*, and*
2. $\sum_j a_j \le n \cdot \prod_i \deg I_i$.

Theorem 1.2 (Effective Nullstellensatz). *Let K be any field and $I_1, \ldots, I_m$ unmixed ideals in $K[x_1, \ldots, x_n]$. The following are equivalent:*

1. $I_1, \ldots, I_m$ *have no common zero in* $\bar{K}^n$.
2. There are polynomials $f_j \in I_j$ *such that*

$$\sum_j f_j = 1 \quad \text{and} \quad \deg f_j \le (n+1) \cdot \prod_i \deg I_i.$$

Theorem 1.3 (Łojasiewicz inequality). *(cf.* [Cygan98,CKT98]*) Let $I_1, \ldots, I_m$ be unmixed ideals in $\mathbb{C}[x_1, \ldots, x_n]$ and $X_1, \ldots, X_m \subset \mathbb{C}^n$ the corresponding subschemes. Let f_{ij} be generators of I_i. Then for every bounded set $B \subset \mathbb{C}^n$ there is a $C > 0$ such that for every $x \in B$,*

$$\operatorname{dist}(X_1 \cap \cdots \cap X_m, x)^{\prod_i \deg I_i} \le C \cdot \max_{ij} |f_{ij}(x)|.$$

The difference between the geometric and algebraic versions of the Bézout theorem can be seen already in the case when an irreducible variety is intersected with a hyperplane.

Example 1.4. Pick coordinates u, v in $\mathbb{C}^2$ and x, y, z, s in $\mathbb{C}^4$. For odd $n \ge 3$ consider the morphism

$$F_n : \mathbb{C}^2 \to \mathbb{C}^4 \quad \text{given by} \quad F_n(u, v) = (u^n, u^2, uv, v).$$

Let $S_n \subset \mathbb{C}^4$ be the image of F_n. It is easy to see that $\deg S_n = n + 1$ and the ideal of S_n in $\mathbb{C}[x, y, z, s]$ is

$$I_n = \left(x^2 - y^n, z^2 - ys^2, xz - y^{\frac{n+1}{2}} s, xs - y^{\frac{n-1}{2}} z\right).$$

Let us intersect S_n with the hyperplane $(s = 0)$ to get a curve C_n. Set theoretically, the intersection is the image of $f_n : \mathbb{C} \to \mathbb{C}^4$ given by $f_n(u) = (u^n, u^2, 0, 0)$ and its ideal is

$$J_n = (x^2 - y^n, z, s).$$

On the other hand,

$$(I_n, s) = \left(x^2 - y^n, z^2, xz, y^{\frac{n-1}{2}} z, s\right),$$

and we see that, as a vectorspace,

$$J_n/(I_n, s) \cong \left\langle z, yz, \ldots, y^{\frac{n-3}{2}} z\right\rangle.$$

Let $m = (x, y, z, s)$ be the ideal of the origin. There are two minimal ways of writing an algebraic Bézout form of this example:

$$J_n^2 \subset (I_n, s) \quad \text{and} \quad m^{\frac{n-1}{2}} \cdot J_n \subset (I_n, s).$$

Taking degrees we get $2 \deg J_n = 2n > n + 1 = \deg I_n$ and $\frac{n-1}{2} \deg m + \deg J_n = \frac{n-1}{2} + n > n + 1 = \deg I_n$ for $n \geq 5$.

This example illustrates the nature of the difficulties, but it does not seem to give pointers as to the general shape of the theory. Unfortunately, I do not have any plausible conjectures about what happens in general. As in [Kollár88], the effect of embedded primes seems small, but the correct way of estimating it is still elusive.

Instead, I approach the question as follows. There are many different varieties $S_n^\lambda \subset \mathbb{C}^4$ whose intersection with the hyperplane $(s = 0)$ is C_n. (For instance, pick polynomials $f(u, v), g(u, v)$ with no common zero and let $S_n^{f,g}$ be the image of $(u, v) \mapsto (u^n, u^2, v f(u, v), vg(u, v))$.) Each S_n^λ gives an ideal I_n^λ and one can ask about all the quotients

$$J_n / (I_n^\lambda, s).$$

It turns out that their length is bounded independent of λ and it is not too big. The main lemma of [Kollár88, 3.4] is a formalization of this observation using local cohomology groups in some special cases.

This paper develops another approach to this problem, going back to [Cayley1860]. For any space curve $C \subset \mathbb{P}^3$ Cayley considered all cones defined by C with a variable point $p \in \mathbb{P}^3$ as vertex. These cones can be encoded as one equation on the Grassmannian of lines in $\mathbb{P}^3$. More generally, for any pure dimensional subscheme $Y^d \subset \mathbb{C}^n$ (or for any pure dimensional algebraic cycle on $\mathbb{C}^n$) consider the ideal $I^{ch}(Y)$ generated by all cones defined by Y with a variable $(n - d - 1)$-dimensional linear space as its vertex. Following [Dalbec-Sturmfels95], it is called the *ideal of Chow equations* (4.1). It turns out that this ideal controls the length of the embedded components of any intersection. With this observation at hand, the rest of the arguments turn out to be not very complicated.

Section 2 reviews some basic facts about algebraic cycles and their intersection theory on $\mathbb{C}^n$. Section 3 collects known results about integral closures of ideals.

The ideal of Chow equations is defined and studied in Sect. 4. The connection between the ideal of Chow equations and intersection theory is established in Sect. 5.

Finally the main results are proved in Sects. 6 and 7.

Another approach to such theorems is to reduce them to the hypersurface case. If $X \subset \mathbb{C}^n$ is a subscheme of degree m then, set theoretically, X can be defined by degree m equations. This gives reasonable bounds for each problem, roughly like $(\max_i\{\deg I_i\})^n$. For many ideals of about the same degree this is close to the optimal bound for the Nullstellensatz, but it is considerably worse in general. For the algebraic Bézout version this method and [Brownawell89] gives a bound in the original form taking into account the degrees of the P_j.

A modified version of this idea is to reduce everything to intersecting with the diagonal and then using the methods of [Kollár88] directly. This gives 3^n-times the optimal bounds. If, however, the quotients $K[x_1, \dots, x_n]/I_j$ (or more precisely, their homogenizations) are Cohen-Macaulay, then the methods of [Sombra97] give better bounds. The factor $(n+1)$ in (1.2) can be replaced by 2.

The above questions become much more difficult if the field K is replaced by a ring R which is equipped with a "size function". (For instance, if R is the ring of integers in a number field then the height is a suitable size function.) In this case one would like to find a solution of the effective Nullstellensatz where the size of the coefficients of the f_j is also controlled. The most general results in this direction are due to [Berenstein-Yger96,Berenstein-Yger97]. It is quite possible that there is a connection between the ideal of Chow equations and their residue calculus.

2. Intersection of cycles on $\mathbb{A}^n$

Definition 2.1. Let Y be a scheme. An *algebraic cycle* on Y is a formal linear combination of reduced and irreducible subschemes $Z = \sum a_i[Z_i]$, $a_i \in \mathbb{Z}$. In using this notation, it is tacitly assumed that the Z_i are different. I do not assume that the Z_i have the same dimension. The cycles form a free Abelian group $Z_*(Y)$. The subgroup generated by all reduced and irreducible subschemes of dimension d is denoted by $Z_d(Y)$.

If Y is proper and L is a line bundle on Y then one can define the L-degree of a cycle

$$\deg_L Z := \sum a_i (Z_i \cdot L^{\dim Z_i}),$$

where $(Z_i \cdot L^{\dim Z_i})$ denotes the top selfintersection number of the first Chern class of $L|_{Z_i}$. The function $Z \mapsto \deg_L Z$ is linear.

Let Y be a scheme with a compactification $Y \subset \bar{Y}$ and assume that L is the restriction of a line bundle $\bar{L}$ from $\bar{Y}$ to Y. For a cycle $Z = \sum a_i Z_i$ on Y set $\bar{Z} = \sum a_i \bar{Z}_i$ where $\bar{Z}_i$ is the closure of Z_i in $\bar{Y}$. Then one can define the degree of a cycle $Z = \sum a_i Z_i$ on Y by

$$\deg_L Z := \deg_{\bar{L}} \bar{Z}.$$

It is important to note that this depends on the choice of $\bar{Y}$ and $\bar{L}$. I use this version of the degree only for the pair $Y = \mathbb{A}^n$ and $\bar{Y} = \mathbb{P}^n$.

Definition 2.2. Let X be a scheme and D an effective Cartier divisior on X. Let $[Y] \in Z_d(X)$ be an irreducible d-cycle on X. Define $[Y] \pitchfork D \in Z_*(X)$ as follows.

1. If $Y \subset \operatorname{Supp} D$ then set $[Y] \pitchfork D := [Y] \in Z_d(X)$.
2. If $Y \not\subset \operatorname{Supp} D$ then $D|_Y$ makes sense as a Cartier divisor. Set $[Y] \pitchfork D := [D|_Y] \in Z_{d-1}(X)$.

This definition is extended to $Z_*(X)$ by linearity. Observe that if $Z \in Z_*(X)$ is effective then so is $Z \pitchfork D$.

If f is a defining equation of D then I also use $Z \pitchfork f$ to denote $Z \pitchfork D$.

It should be emphasized that this definition is not at all well behaved functorially. While it is well defined on cycles, it is not well defined on the Chow group. Furthermore, if D_1, D_2 are two Cartier divisors then in general

$$(Z \pitchfork D_1) \pitchfork D_2 \neq (Z \pitchfork D_2) \pitchfork D_1.$$

(For instance let $X = \mathbb{A}^2$, $Z = (y - x^2 = 0)$, $D_1 = (x = 0)$ and $D_2 = (y = 0)$.)

Lemma 2.3. *Let L be an ample line bundle on X, D a section of $L^{\otimes d}$ and Z an effective cycle on X. Then*

1. $\deg_L(Z \pitchfork D) \leq d \cdot \deg_L(Z)$.
2. *If X is proper, $d = 1$ and all the components of Z have positive dimension then* $\deg_L(Z \pitchfork D) = \deg_L(Z)$.

Proof. By linearity it is sufficient to check this when $Z = [Y]$ for an irreducible and reduced subvariety Y. If $Y \subset \operatorname{Supp} D$ then $\deg_L(Z \pitchfork D) = \deg_L(Y)$, and otherwise $\deg_L(Z \pitchfork D) \leq d \cdot \deg_L(Y)$ with equality holding if X is proper and $\dim Y \geq 1$ by the usual Bézout theorem. □

2.4. One would like to define $Z_1 \pitchfork Z_2$ for any two cycles Z_i on a scheme X. As usual, this is reduced to intersecting $Z_1 \times Z_2$ with the diagonal $\Delta \subset X \times X$. Traditional intersection theory works if X is smooth since in this case $\Delta \subset X \times X$ is a local complete intersection (cf. [Fulton84, Chap.8]). The usual intersection product $Z_1 \cdot Z_2$ is then a cycle of the expected dimension $d = \dim Z_1 + \dim Z_2 - \dim X$. If $\dim(Z_1 \cap Z_2) = d$ then $Z_1 \cdot Z_2$ is well defined as a cycle, but if $\dim(Z_1 \cap Z_2) > d$ then $Z_1 \cdot Z_2$ is defined only as a rational equivalence class inside $\operatorname{Supp}(Z_1 \cap Z_2)$.

Here I follow the path of [Stückrad-Vogel82,Vogel84] and try to define $Z_1 \pitchfork Z_2$ as a well defined cycle which may have components of different dimension. If $X = \mathbb{P}^n$, the Z_i are pure dimensional and $d \geq 0$ then $Z_1 \pitchfork Z_2$ is a cycle such that

$$\deg(Z_1 \pitchfork Z_2) = \deg Z_1 \cdot \deg Z_2.$$

The cases when $d < 0$ were not considered to have much meaning traditionally. [Tworzewski95] realized that the definition is meaningful and gives an interesting invariant.

The construction of (2.2) needs Δ to be a connected component of a global complete intersection. Unfortunately this happens very rarely. The only such example that comes to mind is $X = \mathbb{A}^n$, or more generally, any scheme X which admits an étale map to $\mathbb{A}^n$. For simplicity of exposition, I work with $X = \mathbb{A}^n$. Homogenity considerations can then be used to define $\pitchfork$ for a few other interesting cases, most importantly for $X = \mathbb{P}^n$.

Definition 2.5 (Vogel–Tworzewski cycles). Let $X_i = \sum_j a_{ij} X_{ij}$ be effective cycles on $\mathbb{A}^n$ for $i = 1, \dots, s$. We would like to define a cycle which can reasonably be called the intersection of these cycles. This is done as follows.

Choose an identification $\mathbb{A}^{ns} = \mathbb{A}^n \times \cdots \times \mathbb{A}^n$. Using this identification define

$$\prod_{i=1}^{s} X_i := \sum_{j_1,\dots,j_s} \left(\prod_{i=1}^{s} a_{ij_i}\right)\left(\prod_{i=1}^{s} X_{ij_i}\right)$$

as a cycle in $Z_*(\mathbb{A}^{ns})$.

Let $\Delta \subset \mathbb{A}^n \times \cdots \times \mathbb{A}^n$ denote the diagonal. Each coordinate projection

$$\Pi_r : \mathbb{A}^n \times \cdots \times \mathbb{A}^n \to \mathbb{A}^n \quad \text{(onto the } r\text{th factor)}$$

gives an isomorphism $\Pi_r : \Delta \cong \mathbb{A}^n$ which is independent of r.

Let $\mathcal{L} := (L_1, \dots, L_{n(s-1)})$ be an ordered set of hyperplanes in $\mathbb{A}^{ns}$ such that their intersection is Δ. Set

$$(X_1 \pitchfork \cdots \pitchfork X_s, \mathcal{L}) := \left(\prod_{i=1}^{s} X_i\right) \pitchfork L_1 \pitchfork \cdots \pitchfork L_{n(s-1)},$$

where the right hand side means that we first intersect with L_1, then with L_2 and so on. To be precise, the right hand side is in $Z_*(\mathbb{A}^{ns})$, but every irreducible component of it is contained in Δ. Thus it can be viewed as a cycle in $Z_*(\Delta)$ and so it can be identified with a cycle in $Z_*(\mathbb{A}^n)$ using any of the projections Π_r.

$(X_1 \pitchfork \cdots \pitchfork X_s, \mathcal{L})$ is called an *intersection cycle* of $X_1, \dots, X_n$. Any of these cycles is denoted by $X_1 \pitchfork \cdots \pitchfork X_s$.

It should be emphasized that $X_1 \pitchfork \cdots \pitchfork X_s$ is not a well defined cycle since it depends on the choice of $\mathcal{L}$. In the papers [Vogel84,vanGastell91] the L_i are chosen generic and then $(X_1 \pitchfork \cdots \pitchfork X_s, \mathcal{L})$ is well defined as an element of a suitable Chow group. We would like to get a cycle which is defined over our field K. As long as K is infinite, a general choice of the L_i would work but there are some problems when K is finite. (It is for such reasons that [Brownawell89] does not work for all finite fields.) Furthermore, in our applications it is sometimes advantageous to make a special choice of the L_i. For these reasons I allow any choice of the L_i. The price we pay is that even the degree of $(X_1 \pitchfork \cdots \pitchfork X_s, \mathcal{L})$ depends on the L_i. This, however, does not seem to cause problems in the applications.

We obtain the following Bézout type inequality.

Theorem 2.6. *Let $X_1, \dots, X_s$ be effective cycles on $\mathbb{A}^n$. Then*

$$\deg(X_1 \pitchfork \cdots \pitchfork X_s) \le \prod_j \deg X_j.$$

Proof. $\deg \prod_{i=1}^s X_i = \prod_{i=1}^s \deg X_i$ and cutting with a hyperplane does not increase the degree by (2.3). □

Definition 2.7 (Refined intersection cyle). Let K be an infinite field. For a scheme Y let $B(Y)$ denote all subvarieties of Y which can be obtained by repeatedly taking irreducible components and their intersections. For a Zariski dense set of the $\mathcal{L}$ we can write

$$(X_1 \pitchfork \cdots \pitchfork X_s, \mathcal{L}) = \sum a_i [Z_i(\mathcal{L})],$$

where the $Z_i(\mathcal{L})$ depend algebraically on $\mathcal{L}$. For each $Z_i(\mathcal{L})$ there is a smallest $W \in B(X_1 \cap \cdots \cap X_s)$ such that $Z_i(\mathcal{L}) \subset W$ for general choice of $\mathcal{L}$. For each $W \in B(X_1 \cap \cdots \cap X_s)$, the sum of these cycles gives a well defined element of the Chow group $A_*(W)$. This cycle is denoted by $(X_1 \Cap \cdots \Cap X_s, W)$. Thus we obtain a refined intersection cycle

$$X_1 \Cap \cdots \Cap X_s := \sum_{W \in B(X_1 \cap \cdots \cap X_s)} (X_1 \Cap \cdots \Cap X_s, W)$$

If $Z \subset X_1 \cap \cdots \cap X_s$ is a connected component then

$$\deg(X_1 \Cap \cdots \Cap X_s, Z) := \sum_{W \subset Z} \deg(X_1 \Cap \cdots \Cap X_s, W)$$

is well defined. It is called the *equivalence* of Z in $X_1 \Cap \cdots \Cap X_s$ (cf. [Fulton84, 9.1]).

In analogy with [Tworzewski95], one can define a local variant of this number as follows. For every p,

$$\sum a_i \operatorname{mult}_p Z_i(\mathcal{L})$$

is constant on a Zariski open subset of the $\mathcal{L}$-s. I denote it by

$$\mathrm{mult}_p(X_1 \Cap \cdots \Cap X_s).$$

There is an inequality

$$\mathrm{mult}_p(X_1 \Cap \cdots \Cap X_s) \le \sum_{p \in W \in B(X_1 \cap \cdots \cap X_s)} \deg(X_1 \Cap \cdots \Cap X_s, W).$$

We need to set up a correspondence between ideal sheaves and algebraic cycles. This does not work as well as the usual correspondence between subschemes and ideal sheaves, but it is better suited for our purposes. Another way of going from cycles to ideal sheaves is studied in Sect. 4.

Definition 2.8. Let X be a scheme and $Z = \sum a_i[Z_i]$ an effective cycle. Let $I(Z_i) \subset \mathcal{O}_X$ denote the ideal sheaf of Z_i. Define the *ideal sheaf of* Z by

$$I(Z) := \prod_i I(Z_i)^{a_i} \subset \mathcal{O}_X.$$

It is clear that $I(Z_1 + Z_2) = I(Z_1)I(Z_2)$.

Definition 2.9. Let F be any coherent sheaf on X and $F_i \subset F$ the subsheaf of sections whose support has codimension at most i. Let x_{ij} be the generic points of the irreducible components $X_{ij} \subset \mathrm{Supp}(F_i/F_{i-1})$. Set

$$Z(F) := \sum_{ij} (\mathrm{length}_{x_{ij}} F_i) \cdot [X_{ij}].$$

$Z(F)$ is called the *cycle associated to* F.

Let $Q_{ij} \subset \mathcal{O}_X$ be the ideal sheaf of X_{ij} and $b_{ij} := \mathrm{length}_{x_{ij}} F_i$. Then $\prod_j Q_{ij}^{b_{ij}}$ maps F_i to F_{i-1}, thus $I(Z(F)) \subset \mathrm{Ann}(F)$.

In particular, if $J \subset \mathcal{O}_X$ is an ideal sheaf then

$$I(Z(\mathcal{O}_X/J)) \subset J.$$

Lemma 2.10. *Let $X_1, \ldots, X_m$ be schemes and Z_i an effective cycle on X_i for every i. Let $\pi_i : \prod_j X_j \to X_i$ be the i-th coordinate projection. Then*

$$I\Big(\prod_j Z_j\Big) \subset (\pi_1^* I(Z_1), \ldots, \pi_m^* I(Z_m)).$$

Proof. Using induction, it is sufficient to prove the case $m = 2$. Let $Z_k = \sum_j a_{kj} Z_{kj}$, then

$$Z_1 \times Z_2 = \sum_{ij} a_{1i} a_{2j} (Z_{1i} \times Z_{2j}).$$

If I, J are arbitrary ideals and $a, b \geq 1$, then

$$(I, J)^{ab} \subset (I, J)^{a+b-1} \subset (I^a, J^b).$$

Using this on $X_1 \times X_2$, we obtain that

$$\begin{aligned} I(Z_1 \times Z_2) &= \prod_{ij} I(Z_{1i} \times Z_{2j})^{a_{1i}a_{2j}} \\ &= \prod_{ij} (\pi_1^* I(Z_{1i}), \pi_2^* I(Z_{2j}))^{a_{1i}a_{2j}} \\ &\subset \prod_{ij} (\pi_1^* I(Z_{1i})^{a_{1i}}, \pi_2^* I(Z_{2j})^{a_{2j}}) \\ &\subset \prod_{j} (\pi_1^* \prod_{i} I(Z_{1i})^{a_{1i}}, \pi_2^* I(Z_{2j})^{a_{2j}}) \\ &= \prod_{j} (\pi_1^* I(Z_1), \pi_2^* I(Z_{2j})^{a_{2j}}) \\ &\subset (\pi_1^* I(Z_1), \pi_2^* I(Z_2)). \end{aligned}$$

□

Definition 2.11. Let X be proper and L a line bundle on X. Define the *L-arithmetic degree* of a sheaf F by

$$\text{arith-deg}_L F := \deg_L Z(F).$$

If $I \subset \mathcal{O}_X$ is an ideal sheaf then the arithmetic degree of $\mathcal{O}_X/I$ is also called the arithmetic degree of I and denoted by $\text{arith-deg}_L I$. Note that there is a possibility of confusion since I is also a sheaf.

As in (2.1), the arithmetic degree of an ideal sheaf on $\mathbb{A}^n$ is the arithmetic degree of its unique maximal extension to $\mathbb{P}^n$. (Note that one can not define the arithmetic degree of an arbitrary sheaf on $\mathbb{A}^n$.)

This definition is very natural and it appeared in several different places (see, for instance, [Hartshorne66,Kollár88,Bayer-Mumford91]). The concept was used extensively in many papers (cf. [STV95]).

3. Integral closure of ideals

In this section we recall some relevant facts concerning integral closure of ideals. [Teissier82, Chap.I] serves as a good general reference.

Definition 3.1. Let R be a ring and $I \subset R$ an ideal. $r \in R$ is called *integral over I* if r satisfies an equation

$$r^k + \sum_{j=1}^{k} i_j r^{k-j} = 0 \quad \text{where } i_j \in I^j.$$

All elements integral over I form an ideal $\overline{I}$, called the *integral closure* of I.

We use the following easy properties of the integral closure.

1. $(\overline{I}, \overline{J}) \subset \overline{(I, J)}$,
2. $\overline{I_1} \cdot \overline{I_2} \subset \overline{I_1 I_2}$, and so $(\overline{I})^m \subset \overline{I^m}$.

We also need the following special case of the Briançon–Skoda theorem. A short proof of it can be found in [Lipman-Teissier81, p.101].

Theorem 3.2. [Briancon-Skoda74] *If $R = K[x_1, \ldots, x_n]$ (or more generally, if R is regular of dimension n) then $\overline{I^n} \subset I$.* □

The following result gives the best way to compare integral closures (cf. [Teissier82, I.1.3.4]).

Theorem 3.3 (Valuative criterion of integral dependence). *Let R be a ring and $I, J \subset R$ two ideals. The following are equivalent.*

1. *$J \subset \overline{I}$.*
2. *If $p : R \to S$ is any homomorphism of R to a DVR S then $p(J) \subset \overline{p(I)}$.*

If K is an algebraically closed field and R a finitely generated K-algebra then in (2) it is sufficient to use homomorphisms to the power series ring $K[[t]]$. □

Integral closures usually do not commute with taking quotients, but this holds in some special cases.

Lemma 3.4. *Let $I \subset K[x_1, \ldots, x_n]$ be an ideal. Then*

$$\overline{(I, x_n)}/(x_n) = \overline{(I, x_n)/(x_n)}.$$

Proof. If $J_1 \subset J_2 \subset R$ are ideals then $\overline{J_2}/J_1 \subset \overline{J_2/J_1}$ always holds using (3.1). If $R \to R/J_1$ splits (as a ring homomorphism) then any equation over R/J_1 can be lifted to an equation over R, showing the other containment. □

We need two lemmas about ideals given by algebraic families of generators.

Lemma 3.5. *Let K be an infinite field, R a K-algebra and $L \subset R$ a finite dimensional K-vectorspace. Let U be a K-variety and*

$$F : U \to L \quad \text{given by} \quad u \mapsto r_u$$

a K-morphism. Let $V \subset U$ be Zariski dense. Then there is an equality of ideals

$$(r_u : u \in V) = (r_u : u \in U).$$

Proof. If J is any ideal in R then $L \cap J$ is a sub vector space in L. Thus $\{u \in U : r_u \in J\}$ is Zariski closed in U. Set $J = (r_u : u \in V)$. Since V is dense in U, we obtain that $r_u \in J$ for every $u \in U$. □

Lemma 3.6. *Notation as in (3.5). Assume in addition that U is irreducible. Let $u \mapsto r_u$ and $u \mapsto s_u$ be K-morphisms from U to L. Let $V \subset U$ be Zariski dense. Then we have an equality of ideals*

$$\overline{(r_u s_u : u \in V)} = \overline{(r_u : u \in U) \cdot (s_u : u \in U)}.$$

Proof. Let $p : R \to S$ be any homomorphism to a DVR. An ideal in S is characterized by the minimum order of vanishing of its elements. We need to prove that both ideals above give the same number.

The order of vanishing of each $p(r_u)$ in S is a lower semi continuous function of U, thus it achieves the minimum value on a dense open subset of U. Similarly for $p(s_u)$. Thus we can choose $u \in V$ where both $p(r_u)$ and $p(s_u)$ achieve their minimum. □

Example 3.7. Let $R = K[x, y]$, $L = \{ax + by\}$, $U = K$, $r_u = x - uy$, $s_u = x + uy$. Then $(r_u : u \in U) \cdot (s_u : u \in U) = (x, y)^2$ is different from $(r_u s_u : u \in U) = (x^2, y^2)$. This shows that (3.6) fails without integral closure.

Another such example is given in (4.11).

Remark 3.8. It is easy to check that the ideals (I_n, s) in (1.4) are integrally closed, hence integral closure alone cannot remove the embedded primes, even in a geometrically very simple situation.

4. The ideal of Chow equations

Let K be a field and Z any effective cycle in $\mathbb{A}^n$. In this section we define an ideal in $K[x_1, \ldots, x_n]$, called the ideal of Chow equations of Z. The main advantage of this notion is that it behaves well with respect to arbitrary hyperplane sections. This is the crucial property that one needs for the applications. On the other hand, the ideal of Chow equations is quite difficult to analyze and I leave several basic questions unresolved. (The explanation of the name and other variants are discussed in (4.2).)

Definition 4.1. Let $Z = \sum a_i[Z_i]$ be a purely d-dimensional cycle in $\mathbb{A}^n$. Let $\pi : \mathbb{A}^n \to \mathbb{A}^{d+1}$ be a linear projection such that $\pi : Z_i \to \mathbb{A}^{d+1}$ is finite for every i. We call such a projection *allowable*.

The center of the projection π is a linear space $L \subset \mathbb{P}^n \setminus \mathbb{A}^n$ of dimension $n - d - 2$ and π is allowable iff L is disjoint from $\cup_i \bar{Z}_i$. This shows that allowable projections can be parametrized by an irreducible quasiprojective variety.

If π is allowable then $\pi_*(Z)$ is a well defined codimension 1 cycle in $\mathbb{A}^{d+1}$, and so it corresponds to a hypersurface. Choose an equation of this hypersurface and pull it back by π to obtain a polynomial $f(\pi, Z)$.

Assume first that K is infinite. Define the *ideal of Chow equations* of Z in the polynomial ring $K[\mathbb{A}^n] \cong K[x_1, \dots, x_n]$ as

$$I^{ch}(Z) := (f(\pi, Z) : \pi \text{ is allowable}) \subset K[\mathbb{A}^n].$$

For technical reasons we frequently work with the integral closure of this ideal, denoted by $\overline{I^{ch}}(Z)$.

We see in (4.5) that these are independent of the base field. Thus if K is finite, one can define $I^{ch}(Z)$ by taking any infinite field extension of K first. (By [Weil62, I.7.Lem.2] every ideal has a smallest field of definition. Since $I^{ch}(Z)$ is defined over $K(x) \subset K(x, y)$ and also over $K(y) \subset K(x, y)$, it is also defined over their intersection which is K.)

Finally, if $Z = \sum a_i[Z_i]$ is any effective cycle then write Z as a sum $Z = \sum Z^d$ where Z^d has pure dimension d and set

$$I^{ch}(Z) = \prod_d I^{ch}(Z^d).$$

Its integral closure is denoted by $\overline{I^{ch}}(Z)$. A product formula in terms of the Z_i is given in (4.10), but this only works for the integral closures.

Remark 4.2. The ideals $I^{ch}(Z)$ were first considered by [Cayley1860] and $I^{ch}(Z)$ is essentially equivalent to the Chow form of Z, as explained in [Catanese92,Dalbec-Sturmfels95]. This equivalence clarifies the definition of $I^{ch}(Z)$, but it obscures other versions of this concept.

In (4.1) we consider *linear* projections $\pi : \mathbb{A}^n \to \mathbb{A}^{d+1}$. It is, however, possible to use larger classes of morphisms. For instance we can allow π to be any algebraic automorphism of $\mathbb{A}^n$ followed by a projection or we can even allow π to be any smooth morphism. The latter case can be localized in various topologies.

More generally, if R is any smooth K-algebra and Z a d-cycle on Spec R then one can define the ideal of locally Chow equations (using étale or analytic topology or even working formally) and these ideals behave well with respect to intersections with smooth divisors. Here I concentrate on the simpler case of linear projections. I was unable to decide if the various definitions give the same ideals for a cycle in $\mathbb{A}^n$.

4.3. I do not know if it is essential to consider the integral closure or not in the definition above. The examples (4.8, 4.9) show that $I^{ch}(Z)$ is not integrally closed in general. More importantly, the crucial property (4.10) fails without integral closure as shown by (4.11).

The main question is whether (5.5) holds without integral closure on the right hand side. This would eliminate the extra factor $(n+1)$ in (1.2). I do not know the answer. This question is related to the degree bounds considered in [Sturmfels97, Sect. 4].

As a special case of (3.5) we obtain:

Lemma 4.4. *Let $\{\pi_\lambda : \lambda \in \Lambda\}$ be a Zariski dense set of allowable projections as in (4.1). Then*

$$I^{ch}(Z) = (f(\pi_\lambda, Z) | \lambda \in \Lambda).$$

□

Corollary 4.5. *$I^{ch}(Z)$ is independent of the base field K. That is, if $L \supset K$ is a field extension, then*

$$I^{ch}(Z) \otimes_K L = I^{ch}(Z_L).$$

Proof. If K is infinite, then the projections defined over K form a Zariski dense set of the projections defined over L. Thus by (4.4) we obtain the same ideals.

For finite K we defined $I^{ch}(Z)$ by forcing the above formula to hold. □

Example 4.6. Let $X \subset \mathbb{A}^n$ be a smooth subvariety with ideal sheaf $I(X)$. Then $I^{ch}(X) = I(X)$ and $I^{ch}(a \cdot X) = I(X)^a$. More generally, let $Z = \sum a_i Z_i$ be any cycle. Then the above relationship holds near any smooth point of Supp Z, cf. [Catanese92, 1.14.a].

Thus $I^{ch}(Z)$ is interesting only near the singular points of Supp Z.

Let $p \in \operatorname{Supp} Z$ be a point of multiplicity d and m_p the ideal of p. A general projection $\pi(Z)$ has multiplicity d at $\pi(p)$, thus each $f(\pi, Z)$ has multiplicity $\geq d$ at p. This shows that $I^{ch}(Z) \subset m_p^d \cap I(Z)$.

By [Catanese92, 1.14.b], if Z has codimension at least 2 then $I(Z) \neq I^{ch}(Z)$ along the singular locus of Z.

Example 4.7. Let $\mathbb{A}^{n-k} \subset \mathbb{A}^n$ be the subspace $(x_n = \cdots = x_{n-k+1} = 0)$. Let Z be a d-cycle on $\mathbb{A}^{n-k}$ and j_*Z the corresponding cycle on $\mathbb{A}^n$. We would like to compare $I^{ch}(Z)$ and $I^{ch}(j_*Z)$.

A general projection of j_*Z can be obtained as a projection $\rho : \mathbb{A}^n \to \mathbb{A}^{n-k}$ followed by a general projection $\pi : \mathbb{A}^{n-k} \to \mathbb{A}^{d+1}$. This shows that

$$f(\pi \circ \rho, j_*Z) = f(\pi, Z)(x_1 + L_1, \ldots, x_{n-k} + L_{n-k}),$$

where the L_i are linear forms in $x_{n-k+1}, \ldots, x_n$ defining ρ.

This shows that the restriction map

$$I^{ch}(j_*Z) \twoheadrightarrow I^{ch}(Z) \quad \text{is surjective.}$$

Example 4.8. Let $X \subset \mathbb{A}^n$ be defined by equations $g(x_1, \ldots, x_{n-1}) = x_n = 0$. A general projection of X is isomorphic to X and, at least in characteristic zero,

$$\begin{aligned} f(\pi, X) &= g(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1} x_n) \\ &= \sum_I c_I a^I x_n^{|I|} \frac{\partial^I g}{\partial x^I}, \end{aligned}$$

where the c_I are nonzero constants and $a_i \in K$. Since the a_i can vary independently, we see that the $f(\pi, X)$ generate the ideal

$$\left(x_n^{|I|} \frac{\partial^I g}{\partial x^I} : I = (i_1, \ldots, i_{n-1}) \right).$$

Consider for instance the case $n = 3$ and $g = x_1^3 + x_2^5$. Then

$$I^{ch}(X) = \left(x_1^3 + x_2^5, x_1^2 x_3, x_1 x_3^2, x_3^3, x_2^4 x_3, x_2^3 x_3^2, x_2^2 x_3^3, x_2 x_3^4 \right).$$

$x_2^2 x_3^2$ is integral over $I^{ch}(X)$ (since $(x_2^2 x_3^2)^2 - x_3^3 \cdot x_2^4 x_3 = 0$) but it is not in $I^{ch}(X)$. Hence $\overline{I^{ch}}(X) \neq I^{ch}(X)$.

Example 4.9. Assume that K has characteristic p and let $0 \in \mathbb{A}^2$ be the origin with ideal (x, y). Set $Z = p[0]$. Then $f(\pi, Z) = (ax + by)^p$ for some a, b, thus $I^{ch}(Z) = (x^p, y^p)$. Its integral closure is the much bigger ideal $(x, y)^p$.

The next lemma gives a product formula for $\overline{I^{ch}}(Z)$. This result is crucial for the applications and it fails if we do not take the integral closure, as the example after the lemma shows.

Lemma 4.10. *Let $Z = \sum a_i [Z_i]$ be an effective cycle. Then*

$$\overline{I^{ch}}(Z) = \overline{\prod_i I^{ch}(Z_i)^{a_i}}.$$

Proof. It is enough to check this for pure dimensional cycles.

Let π be any allowable projection for Z. Then π is allowable for every Z_i and $f(\pi, Z) = \prod_i f(\pi, Z_i)^{a_i}$ which proves the containment $\subset$. The converse follows by a repeated application of (3.6). □

Example 4.11. Choose $n \geq 3$ odd and in $\mathbb{A}^n$ consider the 1–cycle of the n coordinate axes $Z = \sum_{i=1}^n [Z_i]$. Then

$$\prod_i I^{ch}(Z_i) = \prod_i (x_1, \ldots, \widehat{x_i}, \ldots, x_n).$$

As a vectorspace this has a basis consisting of all monomials of degree at least n which involve at least 2 variables.

On the other hand, I claim that $I^{ch}(Z)$ does not contain the monomial $x_1 \cdots x_n$.

Let $\pi : (x_1, \dots, x_n) \mapsto (\sum a_i x_i, \sum b_i x_i)$ be a projection. This gives the equation

$$f(\pi, Z) = \prod_i \left(\sum_j m_{ji} x_j \right) \quad \text{where } m_{ji} = a_j b_i - a_i b_j.$$

Thus twice the coefficient of the $x_1 \cdots x_n$ term is

$$\begin{aligned} 2 \cdot \sum_{\sigma \in S_n} \prod_j m_{j\sigma(j)} &= \sum_{\sigma \in S_n} \left(\prod_j m_{j\sigma(j)} + \prod_j m_{j\sigma^{-1}(j)} \right) \\ &= \sum_{\sigma \in S_n} \left(\prod_j m_{j\sigma(j)} + \prod_j m_{\sigma(j)j} \right) \\ &= \sum_{\sigma \in S_n} \left(\prod_j m_{j\sigma(j)} + (-1)^n \prod_j m_{j\sigma(j)} \right) = 0. \end{aligned}$$

If $n = 3$ then it is easy to compute that $\prod_i I^{ch}(Z_i) = (I^{ch}(Z), x_1 x_2 x_3)$. I have not checked what happens for $n \geq 5$ or for even values of n.

The following result of [Amoroso94, Thm.B] shows that $\overline{I^{ch}}(Z)$ contains a fairly small power of $I(Z)$. (The statement in [Amoroso94] is slightly different since he is working with $I^{ch}(Z)$, but his proof actually gives this version.)

Theorem 4.12. [Amoroso94] *Let $Z = \sum a_i Z_i$ be a cycle in $\mathbb{A}^n$. Let $I(Z_i)$ denote the ideal of Z_i. Then*

$$\prod_i I(Z_i)^{a_i \deg Z_i} \subset \overline{I^{ch}}(Z).$$

More precisely, if $x \in \mathbb{A}^n$ is a point then

$$\prod_i I(Z_i)^{a_i \operatorname{mult}_x Z_i} \subset \overline{I^{ch}}(Z) \quad \text{in a neighborhood of } x.$$

□

Lemma 4.13. *Let Z be a cycle in $\mathbb{A}^n$. Then*

$$I^{ch}(Z) \subset I(Z).$$

Let $J \subset K[x_1, \dots, x_n]$ be an ideal. Then

$$I^{ch}(Z(K[x_1, \dots, x_n]/J)) \subset J.$$

Proof. Because of the multiplicative definitions of $I(Z)$ (2.8) and of $I^{ch}(Z)$ (4.1), it is sufficient to prove the first claim in case Z is pure dimensional.

Write $Z = \sum a_i[Z_i]$ and let $\pi : \mathbb{A}^n \to \mathbb{A}^{d+1}$ be an allowable projection. Then $\pi_*(Z) = \sum a_i \pi_*[Z_i]$, so $f(\pi, Z) = \prod f(\pi, Z_i)^{a_i}$. $f(\pi, Z_i) \in I(Z_i)$, so $f(\pi, Z) \in \prod I(Z_i)^{a_i} = I(Z)$.

The second part follows from the first and from (2.9). □

5. The ideal of Chow equations and intersection theory

The next result is the key property of the ideal of Chow equations.

Lemma 5.1. *Let $X \subset \mathbb{A}^n$ be an irreducible and reduced subvariety and $H = (x_n = 0)$ a hyperplane not containing X. Then*

$$I^{ch}(X \pitchfork H) \subset (I^{ch}(X), x_n).$$

Proof. By (4.5) we may assume that the base field is infinite. Choose a general linear subspace $L \subset \bar{H} \setminus H \setminus \bar{X}$ of dimension $n-d-2$. $\dim(\bar{H} \setminus H) = n-2$ and $\dim(\bar{X} \cap \bar{H}) \le d-1$. Since $(n-d-2)+(d-1) < n-2$, L is disjoint from $\bar{X}$. Let $\pi' : H \to H'$ and $\pi : \mathbb{A}^n \to \mathbb{A}^{d+1}$ be the projections with center L. Let $\rho : \mathbb{A}^n \to H$ be a projection and set $\pi'' := \pi \circ \rho : \mathbb{A}^n \to H'$. The 3 projections appear in the following diagram:

$$\begin{array}{ccc} H & \subset & \mathbb{A}^n \\ \pi' \downarrow & \swarrow \pi'' & \downarrow \pi \\ H' & \subset & \mathbb{A}^{d+1} \end{array}$$

$X \pitchfork H$ can be viewed as a cycle on H; in such a case I denote it by Z.

$\pi_*(X)$ is a hypersurface in $\mathbb{A}^{d+1}$ and $\pi'_*(Z)$ is a hypersurface in H' such that $\pi'_*(Z) = \pi_*(X) \cap H'$. Thus

$$f(\pi', Z)(x_1, \dots, x_{n-1}) = f(\pi, X)(x_1, \dots, x_{n-1}, 0).$$

As in (4.7), the generators of $I^{ch}(X \pitchfork H)$ are of the form

$$\begin{aligned} f(\pi'', X \pitchfork H) &= f(\pi', Z)(x_1 + a_1 x_n, \dots, x_{n-1} + a_{n-1} x_n) \\ &\equiv f(\pi', Z)(x_1, \dots, x_{n-1}) \quad \bmod (x_n) \\ &\equiv f(\pi, X)(x_1, \dots, x_{n-1}, x_n) \quad \bmod (x_n). \end{aligned}$$

□

Remark 5.2. More generally, (5.1) also holds if X is a pure dimensional cycle and H does not contain any of its irreducible components.

The generalization to intersecting with several linear equations is formal, but the induction seems to require the use of integral closure, as shown by the following example. The final result itself, however, may not need integral closure.

Example 5.3. The 3 coordinate axes C_x, C_y, C_z in $\mathbb{A}^3$ can be defined by the determinantal equations

$$\operatorname{rank}\begin{pmatrix} x & y & 0 \\ z & y & z \end{pmatrix} \le 1.$$

As we remarked at the end of (4.11),

$$I^{ch}(C_x)\cdot I^{ch}(C_y)\cdot I^{ch}(C_z) = (I^{ch}(C_x + C_y + C_z), xyz).$$

$C_x + C_y + C_z$ is a hyperplane section of the surface

$$Z_1 \quad \text{given by equations} \quad \operatorname{rank}\begin{pmatrix} x & y+as & bs \\ z & y+cs & z+ds \end{pmatrix} \le 1.$$

Z_1 is the cone over a rational normal curve for general a, b, c, d. By explicit computation, $xyz \in I^{ch}(Z_1)$ and using (5.1) this implies that

$$I^{ch}(C_x)\cdot I^{ch}(C_y)\cdot I^{ch}(C_z) \subset (I^{ch}(Z_1), s).$$

Next consider the surface

$$Z_2 \quad \text{with equations} \quad \operatorname{rank}\begin{pmatrix} x & y+as^2 & bs^2 \\ z & y+cs^2 & z+ds^2 \end{pmatrix} \le 1.$$

For general a, b, c, d this defines a rational triple point. By explicit computation, $xyz \notin (I^{ch}(Z_2), s)$, which implies that

$$I^{ch}(C_x)\cdot I^{ch}(C_y)\cdot I^{ch}(C_z) \not\subset (I^{ch}(Z_2), s).$$

Lemma 5.4. *Let Z be a cycle on $\mathbb{A}^n$ and $H_i = (\ell_i = 0)$ hyperplanes. Then*

$$I^{ch}(Z \pitchfork H_1 \pitchfork \cdots \pitchfork H_m) \subset \overline{(I^{ch}(Z), \ell_1, \ldots, \ell_m)}.$$

Proof. Consider first the case when Z is irreducible and $m = 1$. The claim is trivial if $Z \subset H_1$ and the $Z \not\subset H_1$ case is treated in (5.1).

Next we prove the $m = 1$ case by induction on the number of irreducible components of Z.

$$\begin{aligned} I^{ch}((Z_1 + Z_2) \pitchfork \ell_1) &\subset \overline{I^{ch}(Z_1 \pitchfork \ell_1)\cdot I^{ch}(Z_2 \pitchfork \ell_1)} \quad \text{(by (4.10))} \\ &\subset \overline{(I^{ch}(Z_1), \ell_1)\cdot(I^{ch}(Z_2), \ell_1)} \quad \text{(by induction)} \\ &\subset \overline{(I^{ch}(Z_1)I^{ch}(Z_2), \ell_1)} \\ &\subset \overline{(I^{ch}(Z_1 + Z_2), \ell_1)} \quad \text{(by (4.10))} \end{aligned}$$

Finally the case $m > 1$ is established by induction using the chain of inclusions

$$\begin{aligned} I^{ch}(Z \pitchfork \ell_1 \pitchfork \ell_2) &\subset \overline{(I^{ch}(Z \pitchfork \ell_1), \ell_2)} \\ &\subset \overline{(\overline{(I^{ch}(Z), \ell_1)}, \ell_2)} \\ &\subset \overline{(I^{ch}(Z), \ell_1, \ell_2)}. \end{aligned}$$

$\square$

We are ready to formulate our main technical theorem.

Theorem 5.5. *Let K be a field and $Z_1, \ldots, Z_m$ cycles in $\mathbb{A}^n$. Let $Z_1 \pitchfork \cdots \pitchfork Z_m$ be any of the intersection cycles. Then*

$$I^{ch}(Z_1 \pitchfork \cdots \pitchfork Z_m) \subset \overline{(I(Z_1), \ldots, I(Z_m))}.$$

Proof. Choose an identification $\mathbb{A}^{nm} = \mathbb{A}^n \times \cdots \times \mathbb{A}^n$ (m-times) and let $\Pi_r : \mathbb{A}^{nm} \to \mathbb{A}^n$ be the projection onto the r-th factor. Let $\Delta \subset \mathbb{A}^n \times \cdots \times \mathbb{A}^n$ denote the diagonal.

Choose an ordered set of hyperplanes $\mathcal{L} := (L_i = (\ell_i = 0) : i = 1, \ldots, n(m-1))$ in $\mathbb{A}^{nm}$ whose intersection is Δ. This gives us a cycle $(Z_1 \pitchfork \cdots \pitchfork Z_m, \mathcal{L})$ which we view as a cycle in $\mathbb{A}^{nm}$.

Applying (5.4) we obtain that

$$I^{ch}(Z_1 \pitchfork \cdots \pitchfork Z_m, \mathcal{L}) \subset \overline{(I^{ch}(Z_1 \times \cdots \times Z_m), \ell_1, \ldots, \ell_{n(m-1)})}.$$

$I^{ch}(Z_1 \times \cdots \times Z_m) \subset I(Z_1 \times \cdots \times Z_m)$ by (4.13) and using (2.10) this gives the inclusion

$$I^{ch}(Z_1 \pitchfork \cdots \pitchfork Z_m, \mathcal{L}) \subset \overline{(\Pi_1^* I(Z_1), \ldots, \Pi_m^* I(Z_m), \ell_1, \ldots, \ell_{n(m-1)})}.$$

Let us restrict to Δ. The left hand side becomes $I^{ch}(Z_1 \pitchfork \cdots \pitchfork Z_m)$ by (4.7), and the right hand side becomes $\overline{(I(Z_1), \ldots, I(Z_m))}$ by (3.4). □

Remark 5.6. It is possible that (5.5) can be considerably sharpened. The strongest and most natural statement would be

$$I^{ch}(Z_1 \pitchfork \cdots \pitchfork Z_m) \subset (I^{ch}(Z_1), \ldots, I^{ch}(Z_m)).$$

For the applications the main point would be to get rid of the integral closure since this would eliminate the extra factor $(n+1)$ in (1.2).

6. Effective Nullstellensatz

We are ready to formulate and prove the precise technical versions of our main theorems, using the notion of arithmetic degree as defined in (2.11).

Theorem 6.1 (Algebraic Bézout theorem). *Let K be any field and $I_1, \ldots, I_m$ ideals in $K[x_1, \ldots, x_n]$. Then there are prime ideals $P_j \supset (I_1, \ldots, I_m)$ and natural numbers a_j such that*

1. $\prod_j P_j^{a_j} \subset (I_1, \ldots, I_m)$, *and*
2. $\sum_j a_j \leq n \cdot \prod_i \text{arith-deg}\, I_i$.

Proof. As in (2.9), set $Z_i = Z(I_i)$ and let $\sum b_j X_i = (Z_1 \pitchfork \cdots \pitchfork Z_m, \mathcal{L})$ be any of the intersection cycles defined in (2.5). Set $d_j := \deg X_j$, then $\sum_j b_j d_j \le \prod_i \deg Z_i$ by (2.6). By (5.5),

$$\prod_j I^{ch}(X_j)^{b_j} \subset \overline{(I(Z_1), \ldots, I(Z_m))}.$$

$I(X_j)^{d_j} \subset \overline{I^{ch}}(X_j)$ by (4.12), and so we obtain that

$$\prod_j I(X_j)^{b_j d_j} \subset \overline{(I(Z_1), \ldots, I(Z_m))}.$$

$I(Z_s) \subset I_s$ by (2.9), hence

$$\overline{(I(Z_1), \ldots, I(Z_m))} \subset \overline{(I_1, \ldots, I_m)}.$$

$\overline{(I_1, \ldots, I_m)}^n \subset \overline{(I_1, \ldots, I_m)^n} \subset (I_1, \ldots, I_m)$ by (3.2). Putting these together we get that

$$\prod_j I(X_j)^{n b_j d_j} \subset (I_1, \ldots, I_m).$$

Setting $P_j := I(X_j)$ and $a_j := n b_j d_j$ gives (6.1). □

Theorem 6.2 (Effective Nullstellensatz). *Let K be any field and $I_1, \ldots, I_m$ ideals in $K[x_1, \ldots, x_n]$. The following are equivalent:*

1. *$I_1, \ldots, I_m$ have no common zero in $\bar{K}^n$.*
2. *There are polynomials $f_j \in I_j$ such that*

$$\sum_j f_j = 1 \quad \textit{and} \quad \deg f_j \le (n+1) \cdot \prod_i \text{arith-deg } I_i.$$

Proof. It is clear that (2) $\Rightarrow$ (1). To see the converse, introduce a new variable x_0 and let $\tilde{I}_s \subset K[x_0, \ldots, x_n]$ denote the homogenization of $I_s \subset K[x_1, \ldots, x_n]$. Then arith-deg $\tilde{I}_s =$ arith-deg I_s and x_0 is contained in the radical of $(\tilde{I}_1, \ldots, \tilde{I}_m)$, hence it is contained in any prime ideal containing $(\tilde{I}_1, \ldots, \tilde{I}_m)$. By (6.1) there are prime ideals P_j and natural numbers a_j such that

1. $\prod_j P_j^{a_j} \subset (\tilde{I}_1, \ldots, \tilde{I}_m)$, and
2. $\sum_j a_j \le (n+1) \cdot \prod_i$ arith-deg I_i.

Since $x_0 \in P_j$ for every j, we see that

$$x_0^{\sum a_j} \in (\tilde{I}_1, \ldots, \tilde{I}_m).$$

Thus there are $f_i \in I_i$ with homogenizations $\tilde{f}_i$ such that

$$x_0^{\sum a_j} = \sum_i \tilde{f}_i \quad \text{and} \quad \deg \tilde{f}_i = \sum_j a_j.$$

Setting $x_0 = 1$ we obtain (6.2). □

7. Łojasiewicz inequalities

Next we turn to applications of these results to the study of Łojasiewicz inequalities and separation exponents. These results are essentially reformulations of [Cygan98,CKT98].

Definition 7.1. Let f be a real analytic function on $\mathbb{R}^n$ and $Z := (f = 0)$. Fix a norm on $\mathbb{R}^n$ and set $\operatorname{dist}(Z, x) := \inf_{z \in Z} ||x - z||$. [Łojasiewicz59, p.124] proved that for every compact set K there are $m, C > 0$ such that

$$\operatorname{dist}(Z, x)^m \leq C \cdot |f(x)| \quad \text{for } x \in K.$$

Any inequality of this type is called a *Łojasiewicz inequality*.

In general it is rather difficult to obtain an upper bound for m in terms of other invariants of f. The problem becomes easier if $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$, but even in this case it is not straightforward to obtain sharp upper bounds for m. The question was investigated in [Brownawell88] and [JKS92]. Instead of $\mathbb{C}$, one can work over any algebraically closed field with an absolute value.

Notation 7.2. Let K be a field with an absolute value $|\ |$. (The case when $K = \mathbb{C}$ and $|\ |$ is the usual absolute value is the most interesting, but the cases when K is of positive characteristic or $|\ |$ is nonarchimedian are also of interest.) $|\ |$ induces a norm on K^n by $||\mathbf{x}|| := (|x_1| + \cdots + |x_n|)^{1/2}$. This defines a distance on K^n as in (7.1).

Definition 7.3. Let X be any topological space and F, G two sets of K-valued functions on X. We say that F is *integral over* G, denoted by $F \ll G$, if the following condition holds:

(∗) For every $f \in F$ and $x \in X$ there are $g_1, \dots, g_m \in G$ and a constant C such that $|f(x')| \leq C \max_i |g_i(x')|$ for every x' in a neighborhood of x.

If F and G are continuous (which will always be the case for us) then (∗) is automatic if $g(x) \neq 0$ for some $g \in G$. Thus (∗) is a local growth condition near the common zeros of G.

The two notions of integral dependence are closely related by the following result of [Teissier82, 1.3.1]. (The proof given there assumes $K = \mathbb{C}$ but it is not hard to modify it to work in general.)

Lemma 7.4. *Let K be an algebraically closed field with an absolute value $|\ |$. Let X be an affine variety over K (with the metric topology) and $I \subset \mathcal{O}_X$ an ideal sheaf. A polynomial function is integral over I in the sense of (3.1) iff it is integral over I in the sense of (7.3).* □

The relationship between the distance function and the ideal of Chow equations was established in earlier papers.

Lemma 7.5. *(cf.* [JKS92, 8], [Cygan98, 3.7]*) Let K be an algebraically closed field with an absolute value $|\ |$. Let $Z \subset K^n$ be an irreducible subvariety, $z \in Z$ a point and $m = \operatorname{mult}_z Z$. Then, in a neighborhood of z,*

$$\operatorname{dist}(Z, x)^m \ll I_Z^{ch} \ll I_Z \ll \operatorname{dist}(Z, x).$$

□

The main result about Łojasiewicz inequalities is the following.

Theorem 7.6 (Łojasiewicz inequality). *(cf.* [CKT98]*) Let K be an algebraically closed field with an absolute value $|\ |$. Let $I_1, \dots, I_m$ be ideals in $K[x_1, \dots, x_n]$ and $X_1, \dots, X_m \subset K^n$ the corresponding subschemes. Set $D :=$ arith-deg $I_1 \cdots$ arith-deg I_m. Then*

$$\operatorname{dist}(X_1 \cap \cdots \cap X_m, z)^D \ll (I_1, \dots, I_m) \ll \max_i\{\operatorname{dist}(X_i, z)\}.$$

Proof. Let $(X_1 \pitchfork \cdots \pitchfork X_m) = \sum a_i Z_i$ be one of the intersection cycles. $\sum a_i \deg Z_i \leq D$ by (2.6) and $Z_i \subset X_1 \cap \cdots \cap X_m$ by construction. Thus

$$\begin{aligned}
\operatorname{dist}(X_1 \cap \cdots \cap X_m, z)^D &\ll \operatorname{dist}(X_1 \cap \cdots \cap X_m, z)^{\sum a_i \deg Z_i} \\
&\leq \prod_i \operatorname{dist}(Z_i, z)^{a_i \deg Z_i} \\
&\ll \prod_i I^{ch}(Z_i)^{a_i} \quad \text{(by (7.5))} \\
&\subset \overline{(I_1, \dots, I_m)} \quad \text{(by (6.1))} \\
&\ll (I_1, \dots, I_m) \quad \text{(by (7.4))} \\
&\ll \max_i\{\operatorname{dist}(X_i, z)\} \quad \text{(by (7.5))}.
\end{aligned}$$

□

With a similar proof we obtain the following local version.

Corollary 7.7. *(cf.* [Cygan98, 4.5]*) Let K be an algebraically closed field with an absolute value $|\ |$. Let $I_1, \dots, I_m$ be ideals in $K[x_1, \dots, x_n]$ and set $D_p := \operatorname{mult}_p(Z(I_1) \Cap \cdots \Cap Z(I_m))$. Then, in a neighborhood of p,*

$$\operatorname{dist}(X_1 \cap \cdots \cap X_m, z)^{D_p} \ll (I_1, \dots, I_m) \ll \max_i\{\operatorname{dist}(X_i, z)\}.$$

□

8. Application to deformation theory

In usual deformation theory we are given a scheme X_0 and we would like to understand all flat families $\{X_t : t \in \Delta\}$ where Δ is the unit disc. There are, however, some deformation problems where we are interested in flat families $\{Y_t : t \in \Delta\}$ where $X_0 = Y_0 \setminus$ (embedded points), or, more generally, when X_0 and Y_0 have the same fundamental cycles. This question arises for instance in studying the Chow varieties. (See [Hodge-Pedoe52] or [Kollár96, Chap.I] for definitions and properties of the Chow varieties.) A point in the Chow variety of $\mathbb{P}^n$ is not a subscheme but a pure dimensional cycle $W \in Z_d(\mathbb{P}^n)$. Thus if we want to study the Chow variety near W then we need to understand the deformations of all subschemes $X \subset \mathbb{P}^n$ whose fundamental cycle is W. If $d \geq 1$ then there are infinitely many such subschemes X since adding embedded points does not change the fundamental cycle.

Assume that we find a subscheme $X_0 \subset \mathbb{P}^n$ whose fundamental cycle is W and a deformation $\{X_t : t \in \Delta\}$. From the point of view of the Chow variety we are interested only in the fundamental cycle of X_t and not in X_t itself. Hence the only case we need to study is when X_t has no embedded points for general t. The affine version of this problem can be stated as follows.

Question 8.1. *Let $W \in Z_d(\mathbb{A}^n)$ be a d-cycle. Let $Y \subset \mathbb{A}^n \times \mathbb{A}$ be a subscheme of pure dimension $(d+1)$ without embedded points such that the second projection $\pi : Y \to \mathbb{A}^1$ is flat. Let $Y_0 = \pi^{-1}(0)$ be the central fiber and assume that $Z(Y_0) = W$.*

What can we say about Y_0 in terms of W?

This question is related to the problems considered in [Kollár95].

As an application of (5.2) we obtain the following partial answer. This is a place where it would be more natural to use the ideal of locally Chow equations (4.2).

Proposition 8.2. *With the above notation, $I^{ch}(W) \subset I(Y_0)$.* □

Example 8.3. Consider the case when $W = [z = x^2 - y^n = 0] \in Z_1(\mathbb{A}^3)$. As in (4.8) we obtain that

$$I^{ch}(W) = \left(x^2 - y^n, z^2, xz, y^{n-1}z\right).$$

On the other hand, in (1.4) we found an example of a deformation S such that

$$I(S_0) = \left(x^2 - y^n, z^2, xz, y^{\frac{n-1}{2}}z\right).$$

Using [Teissier80] we obtain that the length of $I(W)/I(Y_0)$ is at most the arithmetic genus of W which is $\frac{n-1}{2}$. Comparing these two results we conclude that

$$(x^2 - y^n, z) \supset I(Y_0) \supset \left(x^2 - y^n, z^2, xz, y^{\frac{n-1}{2}} z\right) \qquad (**)$$

for every deformation Y. It is not hard to see that for every ideal satisfying $(**)$ there is a corresponding deformation.

Using (4.8) one can compute I^{ch} for all monomial plane curves in $\mathbb{A}^3$. The results give strong restrictions on Y_0 but I do not see how to get a complete answer as in the above example. P. Roberts computed several examples of monomial space curves and in each case I^{ch} turned out to be quite close to the ideal of the curve.

In higher dimensions I^{ch} gives a very unsatisfactory answer when $W = [X_0]$ and X_0 is normal. By [Hartshorne77, III.9.12] in this case $Y_0 = X_0$ but $I^{ch}(X_0) \neq I(X_0)$ if X_0 is singular.

On the other hand, (8.2) gives information about Y_0 even if W has multiple components. I do not see how to get any information about Y_0 by other methods in this case.

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