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Erratum

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Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents

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In our previous paper [2] "Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents", we studied the asymptotic behavior of energy minimizing solutions of the Ginzburg-Landau equations. But the η-compactness Lemma was for arbitrary solutions which may not be energy minimizing. We found there is a gap in this version of the proof of the η-compactness Lemma (Lemma II.7). This η-compactness Lemma as well as asymptotic behavior of arbitrary solutions are treated in our forthcoming paper [3]. Here we shall simply present our earlier proof of the η-compactness for energy minimizing solutions. All the statements in the rest of the paper [2] are not affected by this modification.

The arguments from (II.60) to (II.68) have to be modified in the following way. Starting from (II.60) we say:

In fact we will be mainly interested in p such that $W^{1,p}(\partial B_t) \hookrightarrow H^{\frac{1}{2}}(\partial B_t)$, that is $p \geqslant 2 - \frac{1}{n}$.

We will now extend the map $\tilde{u}$ into a map w in all of B_t by using as less energy as possible.

We will decompose ∂B_t into a union of disjoint "cubes" having edges of length δ, where δ will be taken very small (to be fixed later).

We will first extend $\tilde{u}$ between ∂B_t and $\partial B_{t-\delta}$. As it is proved in [1] it is possible to choose this union of cubes such that, if $\mathcal{C}_k$ denotes the corresponding k-skeleton for this union of cubes (for $2 \leqslant k \leqslant n-1$), we

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have in the same time

$$
\begin{cases}
\displaystyle\int_{\mathcal{C}_k} |\nabla\tilde{u}|^2 \leqslant K \left(\frac{1}{\delta}\right)^{n-1-k} \int_{\partial B_t} |\nabla\tilde{u}|^2 \qquad \forall\, 2 \leqslant k \leqslant n-1 \\[2ex]
\displaystyle\left(\frac{r_1}{\varepsilon}\right)^2 \int_{\mathcal{C}_k} \left(1-|\tilde{u}|^2\right)^2 \leqslant K \left(\frac{1}{\delta}\right)^{n-1-k} \left(\frac{r_1}{\varepsilon}\right)^2 \int_{\partial B_t} \left(1-|\tilde{u}|^2\right)^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \forall\, 2 \leqslant k \leqslant n-1 \\[2ex]
\displaystyle\int_{\mathcal{C}_k} |\tilde{u}\wedge d_\top \tilde{u}|^p \leqslant K \left(\frac{1}{\delta}\right)^{n-1-k} \int_{\partial B_t} |\tilde{u}\wedge d_\top \tilde{u}|^p \\
\qquad\qquad\qquad\qquad\qquad\qquad \forall\, 2 \leqslant k \leqslant n-1 \quad .
\end{cases}
\tag{1}
$$

Where K is a constant depending only on n not on ε nor on δ. First of all, δ will be chosen such that $|\tilde{u}| \geqslant 1/2$ in $\mathcal{C}_2$. Combining (II.47) of [2] and (1) we get

$$
\begin{cases}
\displaystyle\left(\frac{r_1}{\varepsilon}\right)^2 \int_{\mathcal{C}_2} \left(1-|\tilde{u}|^2\right)^2 \leqslant K \left(\frac{1}{\delta}\right)^{n-3} \eta \\[2ex]
\displaystyle\|\nabla\tilde{u}\|^2 \leqslant C\frac{r_1}{\varepsilon} \quad .
\end{cases}
$$

Thus, this is the case if η/δ^{n-3} is sufficiently small (independantly of ε). Denote by $\mathcal{C}_k^\delta$ the k-skeleton homothetic to $\mathcal{C}_k$, contained in $\partial B_{t-\delta}$, for the homothetie of rate $\frac{t-\delta}{t}$ and let $\mathcal{D}_{k+1}$ be the $k+1$-skeleton in the interior of $B_t \setminus B_{t-\delta}$ having $\mathcal{C}_k \cup \mathcal{C}_k^\delta$ as boundary. we will construct the extension w of $\tilde{u}$ in $B_t \setminus B_{t-\delta}$ on $\mathcal{D}_k$ by induction on k. For $k=3$ we take

$$
w(x) = \tilde{u}\left(t\frac{x}{|x|}\right) \qquad \text{in } \mathcal{D}_3 \quad .
$$

We clearly have

$$
\begin{cases}
\displaystyle\int_{\mathcal{D}_3} |\nabla w|^2 \leqslant \delta \int_{\mathcal{C}_2} |\nabla\tilde{u}|^2 \leqslant K \left(\frac{1}{\delta}\right)^{n-4} \int_{\partial B_t} |\nabla\tilde{u}|^2 \\[2ex]
\displaystyle\left(\frac{r_1}{\varepsilon}\right)^2 \int_{\mathcal{D}_3} \left(1-|w|^2\right)^2 \leqslant K \left(\frac{1}{\delta}\right)^{n-4} \left(\frac{r_1}{\varepsilon}\right)^2 \int_{\partial B_t} \left(1-|\tilde{u}|^2\right)^2 \\[2ex]
\displaystyle\int_{\mathcal{D}_3} |w\wedge d_\top w|^p \leqslant K \left(\frac{1}{\delta}\right)^{n-4} \int_{\partial B_t} |\tilde{u}\wedge d_\top \tilde{u}|^p \quad .
\end{cases}
\tag{2}
$$

Before to construct the extension w of $\tilde{u}$ in the interior of $B_t \setminus B_{t-\delta}$ we construct the extension w of $\tilde{u}$ in ∂B_t, on the $\mathcal{C}_k^\delta$ by induction on k. On each cell of the 3-skeleton $\mathcal{C}_3^\delta$, extend w radially from the boundary of the cell, where $w(x) = \tilde{u}\left(t\frac{x}{|x|}\right)$, to the center of the cell. One verifies that we have

$$\begin{cases} \displaystyle\int_{\mathcal{C}_3^\delta} |\nabla w|^2 \leqslant C\delta \int_{\mathcal{C}_2^\delta} |\nabla w|^2 \leqslant C \left(\frac{1}{\delta}\right)^{n-4} \int_{\partial B_t} |\nabla \tilde{u}|^2 \\ \displaystyle\left(\frac{r_1}{\varepsilon}\right)^2 \int_{\mathcal{C}_3^\delta} \left(1-|w|^2\right)^2 \leqslant C \left(\frac{1}{\delta}\right)^{n-4} \left(\frac{r_1}{\varepsilon}\right)^2 \int_{\partial B_t} \left(1-|\tilde{u}|^2\right)^2 \\ \displaystyle\int_{\mathcal{C}_3^\delta} |w \wedge d_\top w|^p \leqslant C \left(\frac{1}{\delta}\right)^{n-4} \int_{\partial B_t} |\tilde{u} \wedge d_\top \tilde{u}|^p \\ \displaystyle|w| \geqslant \frac{1}{2} \qquad \text{a. e. in } \mathcal{C}_3^\delta \quad . \end{cases}$$

Repeating these extensions on $\mathcal{C}_k^\delta$ until $k = n-1$ we obtain that

$$\begin{cases} \displaystyle\int_{\partial B_{t-\delta}} |\nabla w|^2 \leqslant C \int_{\partial B_t} |\nabla \tilde{u}|^2 \\ \displaystyle\left(\frac{r_1}{\varepsilon}\right)^2 \int_{\partial B_{t-\delta}} \left(1-|w|^2\right)^2 \leqslant C \left(\frac{r_1}{\varepsilon}\right)^2 \int_{\partial B_t} \left(1-|\tilde{u}|^2\right)^2 \\ \displaystyle\int_{\partial B_{t-\delta}} |w \wedge d_\top w|^p \leqslant C \int_{\partial B_t} |\tilde{u} \wedge d_\top \tilde{u}|^p \\ \displaystyle|w| \geqslant \frac{1}{2} \qquad \text{a. e. in } \partial B_{t-\delta} \quad . \end{cases} \tag{3}$$

Now we construct w in $\mathcal{D}_k$ by induction on k. We have $\partial\mathcal{D}_4 = \mathcal{D}_3 \cup \mathcal{C}_3 \cup \mathcal{C}_3^\delta$. Here also, in each cell, we choose w to be the radial extension, relative to the center of the cell, from it's value on the boundary. We establish in the same way inequalities like (2), replacing $\mathcal{D}_3$ by $\mathcal{D}_4$ and $n-4$ by $n-5$. Repeating this procedure until $k = n$, we get in particular

$$\begin{cases} \displaystyle\int_{\mathcal{D}_n = B_t \setminus B_{t-\delta}} |\nabla w|^2 \leqslant C\,\delta \int_{\partial B_t} |\nabla \tilde{u}|^2 \\ \displaystyle\left(\frac{r_1}{\varepsilon}\right)^2 \int_{B_t \setminus B_{t-\delta}} \left(1-|w|^2\right) \leqslant C\,\delta \left(\frac{r_1}{\varepsilon}\right)^2 \int_{\partial B_t} \left(1-|\tilde{u}|^2\right)^2 \quad . \end{cases} \tag{4}$$

Finally we extend w in $B_{t-\delta}$ in the following way. We have $|w| \geqslant 1/2$ in $\partial B_{t-\delta}$ a. e. and $w \in W^{1,2}(\partial B_{t-\delta})$. Thus $\frac{w}{|w|} \wedge d\frac{w}{|w|} \in L^2(\partial B_{t-\delta})$ and since

$$d\left[\frac{w}{|w|} \wedge d\frac{w}{|w|}\right] = 0 \quad ,$$

and $H^1_{dR}(\partial B_{t-\delta}) = 0$, there exists $\phi \in W^{1,2}(\partial B_{t-\delta})$ such that

$$\begin{cases} \dfrac{w}{|w|} \wedge d\dfrac{w}{|w|} = d\phi \\ \\ \text{and} \qquad \displaystyle\int_{\partial B_{t-\delta}} \phi = 0 \quad . \end{cases}$$

Thus $w = |w| \ \exp i\phi + i\phi_0$, where $\phi_0 \in [0, 2\pi)$, and from (II.68) of [2] and (3) we deduce that

$$\|\phi\|_{W^{1,p}(\partial B_{t-\delta})} \leqslant C_q\ \eta^{\frac{\gamma}{2}} \ \left(\int_{T_1} |\nabla\tilde{u}|^2\right)^{\frac{1}{2}} + C_q\ \eta^{\frac{\gamma}{2}}$$

for some fixed $0 < \gamma < 1$, where $p = 2 - \frac{1}{n}$. By Sobolev embedding we have

$$\|\phi\|_{H^{\frac{1}{2}}(\partial B_{t-\delta})} \leqslant C\ \eta^{\frac{\gamma}{2}} \ \left(\int_{T_1} |\nabla\tilde{u}|^2\right)^{\frac{1}{2}} + C\ \eta^{\frac{\gamma}{2}} \quad .$$

Let $\overline{\phi}$ be the harmonic extension of ϕ in $B_{t-\delta}$, we have

$$\int_{B_{t-\delta}} |\nabla\overline{\phi}|^2 \leqslant C\|\phi\|_{H^{\frac{1}{2}}(\partial B_{t-\delta})} \leqslant C\ \eta^{\gamma} \int_{T_1} |\nabla\tilde{u}|^2 + C\ \eta^{\gamma} \quad . \tag{5}$$

We take $w/|w| = \exp i\overline{\phi} + i\phi_0$ in $B_{t-\delta}$. For the modulus $|w|$ of w in $B_{t-\delta}$ we choose $w = \omega$, where ω is the solution of the following problem

$$\begin{cases} -\left(\dfrac{\varepsilon}{r_1}\right)\Delta\omega + \omega = 1 \qquad \text{in } B_{t-\delta} \\ \\ \omega = |w| \qquad \text{in } \partial B_{t-\delta} \end{cases} \tag{6}$$

In [4] we proved that $\min_{B_{t-\delta}} \omega \geqslant \min_{\partial B_{t-\delta}} |w| \geqslant 1/2$ and

$$\begin{aligned} &\int_{B_{t-\delta}} |\nabla\omega|^2 + \left(\frac{r_1}{\varepsilon}\right)^2 \left(1-\omega^2\right)^2 \\ &\quad \leqslant C\left(\int_{\partial B_t} \left(1-|\tilde{u}|^2\right)^2\right)^{\frac{1}{2}} \times \left(\int_{\partial B_t} |\nabla\tilde{u}|^2 + \left(\frac{r_1}{\varepsilon}\right)^2 \left(1-|\tilde{u}|^2\right)^2\right)^{\frac{1}{2}} \quad . \end{aligned} \tag{7}$$

Combining (4), (5), (7) and the minimality of $\tilde{u}$ in B_t we get

$$\int_{B_t} |\nabla\tilde{u}|^2 + \left(\frac{r_1}{\varepsilon}\right)^2 \left(1 - |\tilde{u}|^2\right)^2 \leqslant C\,\eta^\gamma \int_{T_1} |\nabla\tilde{u}|^2 + $$

$$+ C\delta \int_{\partial B_t} |\nabla\tilde{u}|^2 + C\eta^\gamma \quad .$$

Choose $\delta = C\eta^{\frac{1}{n-3}}$ (recall that we had already chosen $\delta \geqslant C\eta^{\frac{1}{n-3}}$).
Using (II.44) of [2] and going back to the usual scale we get, for $\beta = \inf\left(\frac{1}{n-3}, \gamma\right)$,

$$\frac{E_{tr_1}}{(tr_1)^{n-2}} \leqslant C\eta^\beta \, \frac{E_{r_1}}{r_1^{n-2}} + \eta^\beta$$

where $t \geqslant 1/2$. Thus, because of the monotonicity formula we have (II.68) of [2] and the proof can be ended like in [2].

References

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