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Excision in entire cyclic cohomology

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Abstract. We prove that entire and periodic cyclic cohomology satisfy excision for extensions of bornological algebras with a bounded linear section. That is, for such an extension we obtain a six term exact sequence in cohomology.

1. Introduction

A convex vector *bornology* on a vector space is a collection of subsets satisfying some conditions [8]. A typical example is the collection of bounded subsets of a locally convex vector space. A *bornological algebra* is a (possibly non-unital) algebra with a bornology for which the multiplication is bounded. Following Cuntz and Quillen [5], we define the *entire cyclic cohomology* $\mathrm{HE}^*(A)$ and the *periodic cyclic cohomology* $\mathrm{HP}^*(A)$ of a bornological algebra A using the *X-complex* $X(\mathcal{T}A)$ of the *tensor algebra* $\mathcal{T}A$ of A. We furnish $X(\mathcal{T}A)$ with a certain bornology and define $\mathrm{HE}^*(A)$ as the homology of the complex of bounded linear maps $X(\mathcal{T}A) \to \mathbb{C}$. This definition generalizes Connes's original definition of entire cyclic cohomology for locally convex algebras [1]. Our main result is the following theorem:

Theorem 1. *Let*
$$0 \longrightarrow K \xrightarrow{i} E \xrightarrow{p} Q \longrightarrow 0$$
be an extension of bornological algebras with a bounded linear section $s\colon Q \to E$. *Thus* $E \cong K \oplus Q$ *as bornological vector spaces.*

Then there is a natural exact sequence

$$\begin{tikzcd}
\mathrm{HE}^0(Q) \arrow[r, "p^*"'] & \mathrm{HE}^0(E) \arrow[r, "i^*"'] & \mathrm{HE}^0(K) \arrow[d] \\
\mathrm{HE}^1(K) \arrow[u] & \mathrm{HE}^1(E) \arrow[l, "i^*"'] & \mathrm{HE}^1(Q) \arrow[l, "p^*"']
\end{tikzcd}$$

and a similar exact sequence for periodic cyclic cohomology $\mathrm{HP}^*(\sqcup)$.

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Cuntz and Quillen [6], [3] prove excision for the (bivariant) periodic cyclic cohomology of algebras without additional structure and certain topological algebras. In a critical step of their argument they use methods of Wodzicki [12] that are difficult to adapt to entire cyclic cohomology. Recently, Puschnigg has been able to do this [11]. However, his proof of excision works only for the special class of algebras that are inductive limits of Banach algebras. We will prove excision without resorting to Wodzicki's methods.

Let $X(\mathcal{T}E : \mathcal{T}Q)$ be the kernel of the map $X(\mathcal{T}p)\colon X(\mathcal{T}E) \to X(\mathcal{T}Q)$ induced by p. The excision theorem amounts to the assertion that the complexes of bornological vector spaces $X(\mathcal{T}K)$ and $X(\mathcal{T}E : \mathcal{T}Q)$ are homotopy equivalent. Our key idea is to use the left ideal $\mathcal{L}$ in $\mathcal{T}E$ generated by K. In the first part of the proof, we show that $X(\mathcal{T}E : \mathcal{T}Q)$ and $X(\mathcal{L})$ are homotopy equivalent. This is done by homological algebra: We have to verify that certain $\mathcal{L}$-bimodules are free and to compute their commutator quotients. In the second part of the proof, we show that $X(\mathcal{T}K)$ and $X(\mathcal{L})$ are homotopy equivalent. This follows from the homotopy invariance of the X-complex, once we have a bounded splitting homomorphism $\upsilon\colon \mathcal{L} \to \mathcal{T}\mathcal{L}$ for the natural projection $\mathcal{T}\mathcal{L} \to \mathcal{L}$. To construct υ, we write down a bilinear map $E \times \mathcal{T}\mathcal{L} \to \mathcal{T}\mathcal{L}$ and use the universal property of $\mathcal{T}E$. The main difficulty is to show that υ is bounded.

Periodic and entire cyclic cohomology are special cases of corresponding *bivariant* homology theories. We actually prove excision for these bivariant theories, in both variables. Furthermore, the argument below can be used to prove excision for HP* in various categories of algebras: topological algebras, bornological algebras, algebras over a commutative ground field of characteristic zero, etc. We do not pursue these generalizations because our presentation is optimized for the entire theory. The results about HP* that we prove require only little extra work.

2. Tensor algebras, filtrations, and periodic cyclic cohomology

We recall the Cuntz-Quillen approach to periodic cyclic cohomology [5].

Let A be an algebra, possibly without unit. Let A^+ be the algebra obtained by adjoining a unit to A. As a vector space, $A^+ = A \oplus \mathbb{C}$. Let $\Omega^0 A := A$, $\Omega^n A := A^+ \otimes A^{\otimes n}$ for $n \geq 1$, and $\Omega A := \sum_{n\geq 0} \Omega^n A$. We endow ΩA with the usual $\mathbb{Z}/2$-grading, multiplication, and differential d as defined in [2]. The multiplication of differential forms yields a natural A-bimodule structure on $\Omega^n A$ for all n.

Let $\Omega^{\mathrm{even}} A$ and $\Omega^{\mathrm{odd}} A$ be the even and odd part of ΩA. Let $\gamma = \mathrm{id}$ on $\Omega^{\mathrm{even}} A$ and $\gamma = -\mathrm{id}$ on $\Omega^{\mathrm{odd}} A$. The *Fedosov product* on ΩA is defined by

$$x \odot y := x \cdot y - d\gamma(x) \cdot dy.$$

That is, $x \odot y = xy - dxdy$ if x is even and $x \odot y = xy + dxdy$ if x is odd. The Fedosov product is associative. We call $\mathcal{T}A := (\Omega^{\mathrm{even}} A, \odot)$ the *tensor algebra* of A. It has the following universal property [4]. A linear map $l\colon A \to B$ into an algebra B can be extended uniquely to a homomorphism $f\colon \mathcal{T}A \to B$. In terms of the *curvature* $\omega_l(x_1, x_2) := l(x_1 \cdot x_2) - l(x_1) \cdot l(x_2)$ of l, we have

$$f(\langle x_0\rangle dx_1 \dots dx_{2n}) = l\langle x_0\rangle \cdot \omega_l(x_1, x_2) \cdots \omega_l(x_{2n-1}, x_{2n})$$

for all $\langle x_0\rangle \in A^+$, $x_1, \dots, x_{2n} \in A$, $n \geq 0$. The expression $\langle x_0\rangle$ should remind you of the notation for optional arguments in computer handbooks. Thus $\langle x_0\rangle$ is either $x_0 \in A$ or just missing. In the latter case, it behaves like $1 \in A^+$ and $l\langle x_0\rangle = 1 \in B^+$.

The *commutator quotient* $V/[,] = V/[V, A]$ of an A-bimodule V is defined as the cokernel of the map $A \otimes V \to V$, $a \otimes x \mapsto ax - xa$. Let $X_\beta(A)$ be the complex

$$0 \longrightarrow \Omega^1 A/[,] \xrightarrow{b_*} \Omega^0 A \longrightarrow 0$$

with A in degree 0, $\Omega^1 A/[,]$ in degree 1, and $b_*(\langle x_0\rangle dx_1 \bmod [,]) := \langle x_0\rangle x_1 - x_1\langle x_0\rangle$ for all $\langle x_0\rangle \in A^+$, $x_1 \in A$. The *X-complex* $X(A)$ of A is obtained by adding another boundary $d_*\colon A \to \Omega^1 A/[,]$, $x \mapsto dx \bmod [,]$ to the definition of $X_\beta(A)$.

The universal property implies that the tensor algebras $\mathcal{T}A$ and $\mathcal{T}B$ are isomorphic if A and B are isomorphic as vector spaces, that is, have the same dimension. Thus the naked complex $X(\mathcal{T}A)$ cannot contain any interesting information. We can, however, encode interesting homological information in $X(\mathcal{T}A)$ if we endow it with additional structure. We are going to define a filtration and a bornology on $X(\mathcal{T}A)$ that encode the periodic and the entire cyclic cohomology of A, respectively.

A *filtration* on a vector space V is a *decreasing* sequence of subspaces (V_n). Usually, we have $\bigcap V_n = \{0\}$ but it is not useful to require this. A *filtered vector space* is a vector space V furnished with a filtration (V_n). It gives rise to a projective system of vector spaces $(V/V_n)_{n\in\mathbb{N}}$. A linear map $f\colon V \to W$ between filtered vector spaces is called *filtered* iff it descends to a morphism of projective systems $(V/V_n) \to (W/W_n)$. That is, for all $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $f(V_m) \subset W_n$. A *filtered isomorphism* is a filtered linear map with a filtered two-sided inverse.

Let V and W be filtered vector spaces. We define canonical filtrations on the direct sum $V \oplus W$ and the tensor product $V \otimes W$ by $(V \oplus W)_n := V_n \oplus W_n$ and $(V \otimes W)_n := V_n \otimes W + V \otimes W_n$. Thus $V \oplus W/(V \oplus W)_n \cong (V/V_n) \oplus (W/W_n)$ and $V \otimes W/(V \otimes W)_n \cong (V/V_n) \otimes (W/W_n)$. A bilinear map $V \times W \to \sqcup$ is *filtered* iff the induced linear map $V \otimes W \to \sqcup$ is filtered. A *filtered algebra* is a filtered vector space with a filtered associative multiplication.

We filter a subspace $X \subset V$ by $X_n := V_n \cap X$ and the quotient V/X by $V_n \bmod X$. Each vector space V carries a *trivial* filtration defined by $V_n := \{0\}$ for all n. We always endow $\mathbb{C}$ with the trivial filtration.

The *Hodge filtration* on ΩA is defined as follows. If A is an algebra without filtration or with the trivial filtration, we let $(\Omega A)_n$ be the linear span of monomials $\langle x_0\rangle dx_1 \dots dx_m$ with $m \geq n$. If the algebra A is itself a filtered algebra, we let $(\Omega A)_n$ be the linear span of monomials $\langle x_0\rangle dx_1 \dots dx_m$ with $m \geq n$ or $x_j \in A_n$ for some $j \in \{0, \dots, m\}$. The differential, grading, and multiplication are filtered maps. Hence the Fedosov product is filtered and $\mathcal{T}A$ is a filtered algebra. We endow $\Omega^1(\mathcal{T}A) = (\mathcal{T}A \oplus \mathbb{C}) \otimes \mathcal{T}A$ and $\Omega^1(\mathcal{T}A)/[,]$ with the induced filtrations to turn $X(\mathcal{T}A)$ into a complex of filtered vector spaces.

Let $C_\bullet$ be a complex of filtered vector spaces. The *filtered cohomology* of $C_\bullet$ is defined as the homology of the complex of filtered linear maps $C_\bullet \to \mathbb{C}$. Following Cuntz and Quillen [5], we define $\mathrm{HP}^*(A)$, $* = 0, 1$, as the filtered cohomology of $X(\mathcal{T}A)$. More generally, we define the *bivariant periodic cyclic homology* $\mathrm{HP}_*(A; B)$ as the homology of the complex of morphisms between the projective systems $\big(X(\mathcal{T}A)/X(\mathcal{T}A)_n\big)$ and $\big(X(\mathcal{T}B)/X(\mathcal{T}B)_n\big)$ associated to A and B.

3. Bornological vector spaces

We refer to [8] for the elementary theory of (convex) bornological vector spaces. We will only meet convex bornological vector spaces and therefore omit the qualifier "convex" from our notation. A *bornological vector space* is a vector space V together with a collection of subsets $\mathfrak{S}(V)$ called the *bornology* of V. The bornology $\mathfrak{S}(V)$ has to satisfy the following conditions. If $S_1 \subset S_2$ and $S_2 \in \mathfrak{S}(V)$, then $S_1 \in \mathfrak{S}(V)$. If $S_1, S_2 \in \mathfrak{S}(V)$, then $S_1 \cup S_2 \in \mathfrak{S}(V)$. If $S \in \mathfrak{S}(V)$, then $c \cdot S \in \mathfrak{S}(V)$ for all $c \in \mathbb{R}$. We have $\{x\} \in \mathfrak{S}(V)$ for all $x \in V$. If $S \in \mathfrak{S}(V)$, then the *disked hull* S° of S is in $\mathfrak{S}(V)$. The disked hull S° of S is defined as the smallest circled convex subset of V containing S. We call sets $S \in \mathfrak{S}(V)$ *small*.

If V is a locally convex topological vector space, then the collection of all bounded subsets of V is a bornology, the *bounded bornology* on V. If V is just a vector space, then there is a finest possible bornology on V, namely the collection of all bounded subsets of finite dimensional vector subspaces of V.

A linear map $l\colon V \to W$ between bornological vector spaces is *bounded* iff it maps small subsets of V to small subsets of W. That is, $l(S) \in \mathfrak{S}(W)$ for all $S \in \mathfrak{S}(V)$. A *bornological isomorphism* is a bounded linear map with a bounded two-sided inverse. A bilinear map $l\colon V_1 \times V_2 \to W$ is *bounded* iff $l(S_1 \times S_2) \in \mathfrak{S}(W)$ for all $S_1 \in \mathfrak{S}(V_1)$, $S_2 \in \mathfrak{S}(V_2)$. A *bornological algebra* is a bornological vector space A together with a bounded associative multiplication $m\colon A \times A \to A$.

There are canonical bornologies on subspaces, quotients, and direct sums of bornological vector spaces [8]. The canonical bornology on the tensor product $V \otimes W$ of two bornological vector spaces V and W is generated by the sets

$$S_V \otimes S_W := \{x \otimes y \mid x \in S_V,\ y \in S_W\}$$

with $S_V \in \mathfrak{S}(V)$, $S_W \in \mathfrak{S}(W)$. We write $S \Subset T$ iff $S \subset T^\circ$. A subset $T \subset V \otimes W$ is small iff $T \Subset S_V \otimes S_W$ for some $S_V \in \mathfrak{S}(V)$, $S_W \in \mathfrak{S}(W)$. Composition with the natural bilinear map $V \times W \to V \otimes W$ yields a bijection between bounded bilinear maps $V \times W \to \sqcup$ and bounded linear maps $V \otimes W \to \sqcup$. The tensor product bornology is associative and commutative in the usual sense.

Let $C_\bullet$ be a complex of bornological vector spaces. The *bounded cohomology* of $C_\bullet$ is defined as the homology of the complex of bounded linear maps $C_\bullet \to \mathbb{C}$.

Let A be a filtered bornological algebra and let V be a filtered bornological vector space. The *free A-bimodule* on V is $A^+ \otimes V \otimes A^+$ with the standard bimodule structure and the induced bornology and filtration. It can be characterized by the universal property that composition with the inclusion $V \cong 1 \otimes V \otimes 1 \subset$

$A^+ \otimes V \otimes A^+$ gives rise to a bijection between filtered bounded linear maps $V \to W$ and filtered bounded A-bimodule homomorphisms $A^+ \otimes V \otimes A^+ \to W$ for any A-bimodule W. We have $A^+ \otimes V \otimes A^+/[,] \cong A^+ \otimes V$ as filtered bornological vector spaces. The free left and right A-modules on V are $A^+ \otimes V$ and $V \otimes A^+$, respectively. An A-(bi)module is *projective* iff it is a direct summand of a free A-(bi)module.

3.1. Completeness

For a disk T in a vector space V, let V_T be the linear span of T in V furnished with the semi-norm whose unit ball is T. A disk $T \subset V$ is called *completant* iff it is the closed unit ball of V_T and V_T is a Banach space. If each $S \in \mathfrak{S}(V)$ is contained in a completant small disk $T \in \mathfrak{S}(V)$, then V is a *complete bornological vector space*. A locally convex vector space is called *quasi-complete* iff each bounded subset is contained in a completant bounded disk iff it is complete as a bornological vector space with respect to the bounded bornology. The following observation illustrates the usefulness of completeness in analysis.

Let V_1, V_2, and W be quasi-complete locally convex vector spaces furnished with the bounded bornology. Then a separately continuous bilinear map $b\colon V_1 \times V_2 \to W$ is bounded. Otherwise, there would be *completant* bounded disks $S_1 \subset V_1$, $S_2 \subset V_2$ such that $b(S_1 \times S_2)$ is unbounded. The map $(V_1)_{S_1} \times (V_2)_{S_2} \to V_1 \times V_2 \to W$ is separately continuous and hence continuous because $(V_1)_{S_1}$ and $(V_2)_{S_2}$ are Banach spaces. Thus $b(S_1 \times S_2) \subset W$ is bounded, a contradiction. Therefore, b is bounded.

Hence *a quasi-complete locally convex algebra with separately continuous multiplication is a complete bornological algebra with respect to the bounded bornology*.

Each bornological vector space V has a *completion* V^{c} that is characterized uniquely by a universal property [7]. The completion of a bornological algebra is a complete bornological algebra and has the same entire and periodic cyclic cohomology. Thus we could restrict attention to complete bornological algebras. However, it will be more convenient to avoid completions as much as possible because of the algebraic character of excision. We only need completions to define the *bivariant* versions of entire and periodic cyclic cohomology.

4. Entire cyclic cohomology

Let A be a bornological algebra. For $S \subset A$, let

$$\langle S\rangle(dS)^\infty := S \cup \{\langle x_0\rangle dx_1 \dots dx_n \mid n \geq 1,\ \langle x_0\rangle \in S \cup \{1\},\ x_1, \dots, x_n \in S\}.$$

We will frequently use $\langle S\rangle$ for $S \cup \{1\}$, where 1 behaves like a unit element. For instance, $\langle S\rangle(dS)^n := S(dS)^n \cup (dS)^n$. We furnish ΩA with the bornology generated by the sets $\langle S\rangle(dS)^\infty$ with $S \in \mathfrak{S}(A)$. Thus $T \in \mathfrak{S}(\Omega A)$ iff $T \Subset \langle S\rangle(dS)^\infty$ for some $S \in \mathfrak{S}(A)$. We endow $\mathcal{T}A \subset \Omega A$ with the subspace bornology.

Observe that $\sum_{j=1}^{n}\langle S\rangle(dS)^n \Subset \langle 2S\rangle(d2S)^n$. Thus we can convert sums into convex combinations by introducing a constant factor. This technique will be useful in many places. It yields that $\langle S\rangle(dS)^\infty \cdot \langle S\rangle(dS)^\infty \Subset \langle T\rangle(dT)^\infty$ for $T := 2S + 2S^2$. Hence the multiplication of differential forms is bounded. The differential d, the grading operator γ, and hence the Fedosov product are bounded as well. Consequently, $\mathcal{T}A$ is a bornological algebra. Its X-complex $X(\mathcal{T}A)$ is a complex of bornological vector spaces with respect to the induced bornology on $\Omega^1(\mathcal{T}A)/[,]$.

We define the *entire cyclic cohomology* $\mathrm{HE}^*(A)$ of A as the bounded cohomology of $X(\mathcal{T}A)$. In [10], we called this theory analytic cyclic cohomology. Here we stick to Connes's terminology. If $f\colon A \to B$ is a bounded homomorphism, then the induced maps $\mathcal{T}f\colon \mathcal{T}A \to \mathcal{T}B$ and $X(\mathcal{T}f)\colon X(\mathcal{T}A) \to X(\mathcal{T}B)$ are bounded as well. Composition with $X(\mathcal{T}f)$ makes HE^* a contravariant functor.

We define the *bivariant entire cyclic homology* $\mathrm{HE}_*(A;B)$ as the homology of the complex of bounded linear maps $X(\mathcal{T}A)^{\mathrm{c}} \to X(\mathcal{T}B)^{\mathrm{c}}$. This complex is equal to the complex of bounded linear maps $X(\mathcal{T}A) \to X(\mathcal{T}B)^{\mathrm{c}}$ by the universal property of completions. We have to complete the target in order to obtain a reasonable homology theory $\mathrm{HE}_*(\mathbb{C};B)$. The uncompleted complex $X(\mathcal{T}B)$ is acyclic.

If A is a bornological algebra, then we modify the definition of periodic cyclic (co)homology above to take into account the bornology of A. Let $\mathrm{HP}^*(A)$ be the *filtered bounded cohomology* of $X(\mathcal{T}A)$, that is, the homology of the complex of filtered bounded linear maps $X(\mathcal{T}A) \to \mathbb{C}$. In the bivariant theory, we replace the projective system $X(\mathcal{T}A/(\mathcal{T}A)_n)$ by the projective system of completions $X(\mathcal{T}A/(\mathcal{T}A)_n)^{\mathrm{c}}$ and restrict to bounded morphisms of projective systems.

In the following, we will meet many vector spaces that are constructed from $\mathcal{T}A$ and ΩA by taking quotients, direct sums and tensor products. These spaces are always furnished with the induced bornology and the induced filtration.

4.1. Comparison to Connes's definition of entire cyclic cohomology

We denote the standard derivations on ΩA and $\Omega(\mathcal{T}A)$ by d and D, respectively. Let $\sigma_A\colon A \to \mathcal{T}A$ be the natural linear map $A \cong \Omega^0 A \subset \Omega^{\mathrm{even}} A = \mathcal{T}A$.

Lemma 1. *Let A be a filtered bornological algebra. The map $D \circ \sigma_A\colon A \to \Omega^1(\mathcal{T}A)$ induces a natural isomorphism of filtered bornological $\mathcal{T}A$-bimodules*

$$\mu_1\colon (\mathcal{T}A)^+ \otimes A \otimes (\mathcal{T}A)^+ \to \Omega^1(\mathcal{T}A),$$
$$\langle x_0\rangle \otimes a \otimes \langle x_1\rangle \mapsto \langle x_0\rangle \odot \big(D\sigma_A(a)\big) \odot \langle x_1\rangle. \quad (1)$$

Thus $\Omega^1(\mathcal{T}A)$ is a free $\mathcal{T}A$-bimodule.

Proof. Cuntz and Quillen show in [4] that μ_1 is a vector space isomorphism. It is evident that μ_1 is filtered and bounded. We have to check that the inverse map μ_1^{-1} is also filtered and bounded. Since μ_1^{-1} is a bimodule map, it suffices to consider its restriction to the subspace $D(\mathcal{T}A)$. Since D is a derivation, we have

$$\mu_1^{-1}D(\langle a_0\rangle da_1\dots da_{2n}) = \big(\mu_1^{-1}D\langle a_0\rangle\big)\odot da_1\dots da_{2n} + \sum_{j=0}^{n-1}\langle a_0\rangle da_1\dots da_{2j}\odot\mu_1^{-1}D(da_{2j+1}da_{2j+2})\odot da_{2j+3}\dots da_{2n}.$$

The big sum can be converted into a convex combination of terms of the form

$$\langle a_0\rangle d2a_1\dots d2a_{2j}\odot\mu_1^{-1}D(d2a_{2j+1}d2a_{2j+2})\odot d2a_{2j+3}\dots d2a_{2n}.$$

The boundedness of μ_1^{-1} follows. It is not hard to show that μ_1^{-1} is filtered. □

Hence $\Omega^1(\mathcal{T}A)/[,] \cong (\mathcal{T}A)^+\otimes A \cong (\mathcal{T}A)^+\,dA = \Omega^{\mathrm{odd}}A$. Thus we obtain a natural isomorphism $X(\mathcal{T}A)\cong\Omega A$ of $\mathbb{Z}/2$-graded filtered bornological vector spaces.

Let A be a quasi-complete locally convex algebra with separately continuous multiplication. Let $\mathrm{HE}^*_C(A)$ be its entire cyclic cohomology as defined by Connes [1], [2]. Actually, Connes's original definition only works for unital algebras. Khalkhali has extended it to non-unital algebras in [9]. We rewrite Khalkhali's definition as follows. Endow A with the bounded bornology. Define $[k]=[k+1/2]=k$ for all $k\in\mathbb{Z}$. Let $\mathfrak{S}_{n!}$ be the convex bornology on ΩA generated by the sets

$$S\cup\bigcup_{n\geq 1}[n/2]!\langle S\rangle(dS)^n$$

with $S\in\mathfrak{S}(A)$. We furnish the $\mathbb{Z}/2$-graded vector space ΩA with the Hodge filtration, the bornology $\mathfrak{S}_{n!}$, and the boundary $B+b$ to obtain a filtered bornological complex $C(A)$. The bounded cohomology of $C(A)$ is equal to $\mathrm{HE}^*_C(A)$. Indeed, a linear map $l\colon C(A)\to\mathbb{C}$ can be described by a family of multi-linear maps $l_n\colon A^+\times A^n\to\mathbb{C}$, $n\geq 0$. The map l is bounded iff for all bounded subsets $S\subset A$, there is a constant C such that $|l_n(\langle a_0\rangle,a_1,\dots,a_n)|\leq C/[n/2]!$ for all $n\in\mathbb{N}$, $\langle a_0\rangle\in S\cup\{1\}$, $a_1,\dots,a_n\in S$.

The map l is filtered iff $l_n=0$ for all but finitely many n. If l is filtered, it is bounded iff l_n is bounded for all n. Hence the filtered bounded cohomology of $C(A)$ is the "usual" periodic cyclic cohomology of A.

Two chain maps between filtered bornological complexes are called *chain homotopic* iff they differ by the boundary of a *filtered bounded* linear map. Similarly, if two filtered bornological complexes are called *homotopy equivalent*, this means that all the four maps involved in the homotopy equivalence are filtered and bounded. Hence if C_1 and C_2 are homotopy equivalent, then they have the same bounded and filtered bounded cohomology.

Proposition 1. *Let A be a quasi-complete locally convex algebra furnished with the bounded bornology. The filtered bornological complexes $X(\mathcal{T}A)$ and $C(A)$ are naturally homotopy equivalent. Hence the definitions of entire and periodic cyclic cohomology given above agree with the standard definitions.*

Essentially, this is already proved by Cuntz and Quillen in [5]. They construct a homotopy equivalence between the complexes $X(\mathcal{T}A)$ and $(\Omega A, B+b)$. The maps implementing this homotopy equivalence are filtered and bounded. Boundedness is proved by writing down explicit formulas for these maps, see [10] for details. We do not go into these computations here but only observe that the homotopy equivalence involves multiplication by $(-1)^n n!$ in degrees $2n$ and $2n+1$. This is where the $[n/2]!$ in the definition of the bornology $\mathfrak{S}_{n!}$ enters.

4.2. Homotopy invariance of the X-complex

Let B be a filtered bornological algebra. Let $\mathbb{C}[t]$ be the polynomial ring in one variable. We turn $\mathbb{C}[t] \otimes B$ into a filtered bornological algebra as follows. The filtration consists of the subspaces $\mathbb{C}[t] \otimes B_n$. For $x \in \mathbb{C}$, let $\mathrm{ev}_x \colon \mathbb{C}[t] \otimes B \to B$ be evaluation at x. Let

$$S \in \mathfrak{S}(\mathbb{C}[t] \otimes B) \iff \{\mathrm{ev}_x(f),\ \mathrm{ev}_x(df/dt) \mid x \in [0,1],\ f \in S\} \in \mathfrak{S}(B).$$

Two homomorphisms $f_0, f_1 \colon A \to B$ are called *polynomially homotopic* iff there is a filtered bounded homomorphism $F \colon A \to \mathbb{C}[t] \otimes B$ with $\mathrm{ev}_t \circ F = f_t$ for $t = 0, 1$.

In general, the chain maps $X(f_0), X(f_1) \colon X(A) \to X(B)$ induced by polynomially homotopic homomorphisms f_0 and f_1 need not be chain homotopic. However, they are chain homotopic if the source A is *quasi-free* in the sense that the bimodule $\Omega^1 A$ is projective. Essentially, this is proved already by Cuntz and Quillen [5]. They write down a linear map $h \colon X(A) \to X(B)$ with $[\partial, h] = X(f_0) - X(f_1)$. It is straightforward to check that their map h is filtered and bounded. See also [10] for an explanation of the formula for h. We will only need homotopy invariance in the special case where A is a tensor algebra (and thus quasi-free by Lemma 1):

Proposition 2. *Let C, B be filtered bornological algebras and $A := \mathcal{T}C$. Let $f_0, f_1 \colon A \to B$ be polynomially homotopic homomorphisms. Then the induced chain maps $X(f_0), X(f_1) \colon X(A) \to X(B)$ are chain homotopic.*

5. The proof of the excision theorem

Consider an extension $K \to E \to Q$ as in Theorem 1. Identify $X(\mathcal{T}E) \cong \Omega E$ and $X(\mathcal{T}Q) \cong \Omega Q$ as above. The section s for p gives rise to a filtered bounded linear section $s_L \colon \Omega Q \to \Omega E$ for $X(\mathcal{T}p) \colon X(\mathcal{T}E) \to X(\mathcal{T}Q)$ that is defined by

$$s_L(\langle q_0 \rangle dq_1 \dots dq_n) := s\langle q_0 \rangle ds(q_1) \dots ds(q_n).$$

In addition, we will need the right-handed version

$$s_R(dq_1 \dots dq_n \cdot \langle q_{n+1} \rangle) := ds(q_1) \dots ds(q_n) \cdot s\langle q_{n+1} \rangle.$$

of s_L. Let

$$X(\mathcal{T}E : \mathcal{T}Q) := \ker\bigl(X(\mathcal{T}p)\colon X(\mathcal{T}E) \to X(\mathcal{T}Q)\bigr).$$

The chain map $X(\mathcal{T}i)\colon X(\mathcal{T}K) \to X(\mathcal{T}E)$ satisfies $X(\mathcal{T}p) \circ X(\mathcal{T}i) = 0$, so that we can view it as a chain map $\varrho\colon X(\mathcal{T}K) \to X(\mathcal{T}E : \mathcal{T}Q)$. We will show that ϱ is a homotopy equivalence of filtered bornological complexes. This implies excision in (bivariant) entire and periodic cyclic cohomology. We have an extension

$$X(\mathcal{T}E : \mathcal{T}Q) \xrightarrow{\subset} X(\mathcal{T}E) \xrightarrow{X(\mathcal{T}p)} X(\mathcal{T}Q) \tag{2}$$

of filtered bornological complexes with a filtered bounded linear section. Let C be another (filtered) bornological complex. Let $\mathrm{Lin}(V; W)$ be the space of (filtered) bounded linear maps $V \to W$. Since the extension (2) splits, it remains an extension of complexes of vector spaces after applying the functors $\mathrm{Lin}(C; \sqcup^{\mathrm{c}})$ or $\mathrm{Lin}(\sqcup; C^{\mathrm{c}})$. In the associated exact homology sequence, we can replace $X(\mathcal{T}E : \mathcal{T}Q)$ by $X(\mathcal{T}K)$ if ϱ is a homotopy equivalence. This yields the desired six term exact sequences.

Let $\mathcal{L}$ be the left ideal in $\mathcal{T}E$ generated by $K \subset E \subset \mathcal{T}E$. Alternatively, we can define $\mathcal{L}$ as the linear span of the monomials $\langle x_0\rangle dx_1 \dots dx_{2n}$ with $x_{2n} \in K$. Our ultimate goals are to show that $X(\mathcal{L}) \sim X(\mathcal{T}E : \mathcal{T}Q)$ and $X(\mathcal{T}K) \sim X(\mathcal{L})$, where $\sim$ denotes homotopy equivalence of filtered bornological complexes.

Some conventions: We write x, x_j, $j \in \mathbb{N}$, for elements of E; y, y_j for elements of $\mathcal{T}E$; l, l_j for elements of $\mathcal{L}$; q, q_j for elements of Q and $\mathcal{T}Q$; k for elements of K; $\langle x\rangle$ for elements of E^+, and similarly $\langle y\rangle$, $\langle l\rangle$, $\langle q\rangle$. We write d and D for the standard derivations of ΩE and $\Omega(\mathcal{T}E)$. We declare $d1 = 0$ and $D1 = 0$. An expression $dx_1 \dots dx_{2n}$ or $Dy_1 \dots Dy_{2n}$ with $n = 0$ is 1. We will frequently use the grading # on $(\mathcal{T}E)^+ \supset \mathcal{L}^+$ defined by $\#1 = 0$ and $\#y = n$ iff $y \in \Omega^{2n}E$.

5.1. A free resolution

Let $\psi\colon X(\mathcal{L}) \to X(\mathcal{T}E : \mathcal{T}Q)$ be the chain map induced by the inclusion $\mathcal{L} \subset \mathcal{T}E$. Let $C_\bullet$ be the contractible complex $C_0 = C_1 = \mathcal{T}Q \otimes \mathcal{L}$ with boundary $\mathrm{id}\colon C_1 \to C_0$. We define a chain map $\psi'\colon C_\bullet \to X(\mathcal{T}E : \mathcal{T}Q)$ by $C_0 \ni q \otimes l \mapsto [l, s_L(q)] \in \mathcal{T}E$ and $C_1 \ni q \otimes l \mapsto l\, Ds_L(q) \bmod [,] \in X_1(\mathcal{T}E)$.

Lemma 2. *The map $(\psi, \psi')\colon X(\mathcal{L}) \oplus C_\bullet \to X(\mathcal{T}E : \mathcal{T}Q)$ is a filtered bornological isomorphism. Thus $\psi\colon X(\mathcal{L}) \to X(\mathcal{T}E : \mathcal{T}Q)$ is a homotopy equivalence.*

The map $(\psi, \psi')\colon X_\beta(\mathcal{L}) \oplus C_\bullet \to X_\beta(\mathcal{T}E : \mathcal{T}Q)$ is a filtered bornological isomorphism. Thus $\psi\colon X_\beta(\mathcal{L}) \to X_\beta(\mathcal{T}E : \mathcal{T}Q)$ is a homotopy equivalence.

Recall that the X-complex and the X_β-complex are equal as filtered bornological vector spaces. Hence the two paragraphs of Lemma 2 are equivalent. We will prove the second paragraph by homological algebra.

Let A be a filtered bornological algebra. Consider the following extension

$$\begin{aligned}
\Omega^1 A \xrightarrow{\alpha_1} A^+ \otimes A^+ \xrightarrow{\alpha_0} A^+,\\
\alpha_1(\langle x\rangle Dy) := \langle x\rangle \odot y \otimes 1 - \langle x\rangle \otimes y,\\
\alpha_0(\langle x\rangle \otimes \langle y\rangle) := \langle x\rangle \cdot \langle y\rangle.
\end{aligned}$$

A contracting homotopy $h_0\colon A^+ \to A^+ \otimes A^+$, $h_1\colon A^+ \otimes A^+ \to \Omega^1 A$ is defined by $h_0(\langle x\rangle) := 1 \otimes \langle x\rangle$ and $h_1(\langle x\rangle \otimes \langle y\rangle) := (D\langle x\rangle)\langle y\rangle$. It is straightforward to verify that the maps $\alpha_\bullet$ are A-bimodule homomorphisms and that $\alpha_\bullet^2 = 0$ and $\alpha_\bullet h_\bullet + h_\bullet \alpha_\bullet = 1$. We write $B_\bullet^A$ for the complex $\Omega^1 A \overset{\alpha}{\to} A^+ \otimes A^+$. Let $\mathbb{C}[0]$ be the complex with $\mathbb{C}$ in degree zero and 0 in all other degrees. We have

$$B_\bullet^A/[,] := B_\bullet^A/\big[B_\bullet^A, A\big] \cong X_\beta(A) \oplus \mathbb{C}[0].$$

We define another $\mathcal{L}$-bimodule resolution $P_1 \to P_0 \to \mathcal{L}^+$ of $\mathcal{L}^+$ by

$$\begin{aligned}
P_0 &:= \mathcal{L}^+ \otimes \mathcal{L}^+ + (\mathcal{T}E)^+ \otimes \mathcal{L} \subset (\mathcal{T}E)^+ \otimes (\mathcal{T}E)^+ = B_0^{\mathcal{T}E},\\
P_1 &:= (\mathcal{T}E)^+\, D\mathcal{L} \subset \Omega^1(\mathcal{T}E) = B_1^{\mathcal{T}E}.
\end{aligned}$$

The subspaces $P_0 \subset B_0^{\mathcal{T}E}$ and $P_1 \subset B_1^{\mathcal{T}E}$ are sub-$\mathcal{L}$-bimodules. We have

$$\alpha_1(P_1) \subset P_0, \quad \alpha_0(P_0) \subset \mathcal{L}^+, \quad h_0(\mathcal{L}^+) \subset P_0, \quad h_1(P_0) \subset P_1,$$

because $\mathcal{L}$ is a left ideal in $(\mathcal{T}E)^+$. Thus $(P_\bullet, \alpha_\bullet)$ is an $\mathcal{L}$-bimodule resolution of $\mathcal{L}^+$.

We are going to show that P_0 and P_1 are free $\mathcal{L}$-bimodules in order to compute their commutator quotients: $P_\bullet/[,] \cong X_\beta(\mathcal{T}E : \mathcal{T}Q) \oplus \mathbb{C}[0]$. Furthermore, we compare the $\mathcal{L}$-bimodule resolutions $P_\bullet$ and $B_\bullet^{\mathcal{L}}$ and obtain $P_\bullet \cong B_\bullet^{\mathcal{L}} \oplus (\mathcal{L}^+ \otimes C_\bullet)$. These facts together imply Lemma 2. First, we need some preparations.

Lemma 3. *Let A be a filtered bornological algebra. The natural maps*

$$\begin{aligned}
m_L\colon (\mathcal{T}A)^+ \otimes A \to \mathcal{T}A, \qquad \langle y\rangle \otimes x \mapsto \langle y\rangle \odot \sigma_A(x),\\
m_R\colon A \otimes (\mathcal{T}A)^+ \to \mathcal{T}A, \qquad x \otimes \langle y\rangle \mapsto \sigma_A(x) \odot \langle y\rangle
\end{aligned}$$

are filtered bornological isomorphisms.

Proof. We only prove that m_L is a filtered bornological isomorphism. The proof for m_R is similar. Let $m_L^{-1}(x) := 1 \otimes x$ for all $x \in A$. Monomials of higher degree are of the form $\langle y\rangle dx_1 dx_2$ with $\langle y\rangle \in (\mathcal{T}A)^+$, $x_1, x_2 \in A$. Let

$$m_L^{-1}(\langle y\rangle dx_1 dx_2) := \langle y\rangle \otimes (x_1 \cdot x_2) - (\langle y\rangle \odot x_1) \otimes x_2.$$

It is straightforward to verify that $m_L \circ m_L^{-1} = \mathrm{id}$ and $m_L^{-1} \circ m_L = \mathrm{id}$. Moreover, m_L^{-1} is filtered and bounded. Thus m_L is a filtered bornological isomorphism. □

If we let $A = E$, then m_L maps the direct summand $(\mathcal{T}E)^+ \otimes K \subset (\mathcal{T}E)^+ \otimes E$ onto the smallest left ideal that contains K, that is, $\mathcal{L}$. Hence we obtain a natural filtered bornological isomorphism

$$\mu_3 \colon (\mathcal{T}E)^+ \otimes K \to \mathcal{L}, \qquad \langle y\rangle \otimes k \mapsto \langle y\rangle \odot k. \tag{3}$$

Let $\mathcal{I}$ be the kernel of the homomorphism $\mathcal{T}p \colon \mathcal{T}E \to \mathcal{T}Q$.

Lemma 4. *The following two linear maps are filtered bornological isomorphisms:*

$$\mu_4 \colon (\mathcal{T}E)^+ \otimes K \otimes (\mathcal{T}Q)^+ \to \mathcal{I}, \qquad \langle y\rangle \otimes k \otimes \langle q\rangle \mapsto \langle y\rangle \odot k \odot s_L\langle q\rangle, \tag{4}$$
$$\mu_5 \colon (\mathcal{T}Q)^+ \otimes K \otimes (\mathcal{T}E)^+ \to \mathcal{I}, \qquad \langle q\rangle \otimes k \otimes \langle y\rangle \mapsto s_R\langle q\rangle \odot k \odot \langle y\rangle. \tag{5}$$

The restriction of μ_5 to $(\mathcal{T}Q)^+ \otimes K \otimes \mathcal{L}^+$ is a filtered bornological isomorphism onto $\mathcal{L}$. Thus $\mathcal{L}$ is free as a right $\mathcal{L}$-module.

Proof. We prove that (5) is an isomorphism by writing down the inverse map and checking that it is filtered and bounded. The proof for (4) is analogous.

We call $dx_1 \dots dx_{2n}\langle x_{2n+1}\rangle$ a *standard monomial* iff $x_j \in K \cup s(Q)$ for $j = 0, \dots, 2n$ and $\langle x_{2n+1}\rangle \in \{1\} \cup K \cup s(Q)$. Elements of $\mathcal{I}$ are linear combinations of standard monomials with at least one entry in K. We are going to write down a μ_5-preimage for such a standard monomial $dx_1 \dots dx_{2n}\langle x_{2n+1}\rangle \in \mathcal{I}$. We pick the first j with $x_j \in K$, so that $x_i = s(q_i)$ for $i < j$ with $q_i = p(x_i)$. We distinguish three cases: j is even; j is odd and $j \le 2n - 1$; or $j = 2n + 1$. If j is even, then

$$\begin{aligned}
&\mu_5^{-1}(dx_1 \dots dx_{2n}\langle x_{2n+1}\rangle)\\
&= \mu_5^{-1}(dsq_1 \dots dsq_{j-2} \odot (sq_{j-1} \cdot x_j - sq_{j-1} \odot x_j) \odot dx_{j+1} \dots dx_{2n}\langle x_{2n+1}\rangle)\\
&\qquad = dq_1 \dots dq_{j-2} \otimes sq_{j-1}x_j \otimes dx_{j+1} \dots dx_{2n}\langle x_{2n+1}\rangle\\
&\qquad\qquad - dq_1 \dots dq_{j-2} \cdot q_{j-1} \otimes x_j \otimes dx_{j+1} \dots dx_{2n}\langle x_{2n+1}\rangle.
\end{aligned}$$

If j is odd and $j \le 2n - 1$, then we obtain similarly

$$\begin{aligned}
\mu_5^{-1}(dx_1 \dots dx_{2n}\langle x_{2n+1}\rangle) &= dq_1 \dots dq_{j-1} \otimes x_j x_{j+1} \otimes dx_{j+2} \dots dx_{2n}\langle x_{2n+1}\rangle\\
&\quad - dq_1 \dots dq_{j-1} \otimes x_j \otimes x_{j+1} \odot dx_{j+2} \dots dx_{2n}\langle x_{2n+1}\rangle.
\end{aligned}$$

Finally, if $j = 2n + 1$, then $\langle x_{2n+1}\rangle = x_{2n+1} \in K$ and

$$\mu_5^{-1}(dx_1 \dots dx_{2n}\langle x_{2n+1}\rangle) = dq_1 \dots dq_{2n} \otimes x_{2n+1} \otimes 1.$$

These formulas yield a linear map $\mu_5^{-1} \colon \mathcal{I} \to (\mathcal{T}Q)^+ \otimes K \otimes (\mathcal{T}E)^+$ such that $\mu_5 \circ \mu_5^{-1} = \mathrm{id}$. It is left to the reader to check $\mu_5^{-1} \circ \mu_5 = \mathrm{id}$. Hence μ_5 is an isomorphism with inverse μ_5^{-1}. It is straightforward to check that μ_5 maps $(\mathcal{T}Q)^+ \otimes K \otimes \mathcal{L}^+$ into $\mathcal{L}$ and that μ_5^{-1} maps $\mathcal{L}$ into $(\mathcal{T}Q)^+ \otimes K \otimes \mathcal{L}^+$. Hence the restriction of μ_5 to $(\mathcal{T}Q)^+ \otimes K \otimes \mathcal{L}^+$ is an isomorphism onto $\mathcal{L}$.

It remains to show that μ_5^{-1} is filtered and bounded. The following estimates will be needed later. The above formulas for μ_5^{-1} show that $\mu_5^{-1}(y)$ is a sum of terms $\langle y_0\rangle \otimes k \otimes \langle y_1\rangle$ with $\#\langle y_0\rangle + \#\langle y_1\rangle \geq \#y - 1$. That is, μ_5^{-1} decreases # by at most 1. Thus μ_5^{-1} is filtering.

For $S \in \mathfrak{S}(E)$, let

$$(dS)^{\mathrm{even}}\langle S\rangle := \bigcup_{n\geq 0} (dS)^{2n}\langle S\rangle \subset (\mathcal{T}E)^+.$$

Thus $1 \in (dS)^{\mathrm{even}}\langle S\rangle$. If $T \subset \mathcal{I}$ is small, then $T \Subset (dS)^{\mathrm{even}}\langle S\rangle \cap \mathcal{I}$ for some $S \in \mathfrak{S}(E)$. We have $S \Subset S_E := S_K \cup s(S_Q)$ for suitable $S_K \in \mathfrak{S}(K)$, $S_Q \in \mathfrak{S}(Q)$ and therefore $T \Subset (dS_E)^{\mathrm{even}}\langle S_E\rangle \cap \mathcal{I}$. The above formulas for μ_5^{-1} show that

$$\mu_5^{-1}\big((dS_E)^{\mathrm{even}}\langle S_E\rangle \cap \mathcal{I}\big) \Subset (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \otimes \langle S_E\rangle \cdot S_K \cdot \langle S_E\rangle \otimes 2\langle S_E\rangle \odot (dS_E)^{\mathrm{even}}\langle S_E\rangle. \quad (6)$$

Since the right hand side in (6) is small, μ_5^{-1} is bounded. □

Using (3) and (4), we obtain a filtered bornological isomorphism $\mathcal{L} \otimes (\mathcal{T}Q)^+ \cong \mathcal{I}$. Since $(\mathcal{T}E)^+ \cong \mathcal{I} \oplus s_L(\mathcal{T}Q)^+$, the map

$$\mu_7 \colon \mathcal{L}^+ \otimes (\mathcal{T}Q)^+ \to (\mathcal{T}E)^+, \qquad \langle l\rangle \otimes \langle q\rangle \mapsto \langle l\rangle \odot s_L\langle q\rangle, \quad (7)$$

is a filtered bornological isomorphism. By construction, μ_7 is a left $\mathcal{L}$-module map. Thus $(\mathcal{T}E)^+$ *is a free left* $\mathcal{L}$*-module*.

Finally, we can prove that $P_0 := \mathcal{L}^+ \otimes \mathcal{L}^+ + (\mathcal{T}E)^+ \otimes \mathcal{L}$ and $P_1 := (\mathcal{T}E)^+ \, D\mathcal{L}$ are free $\mathcal{L}$-bimodules. Equation (7) yields an $\mathcal{L}$-bimodule isomorphism

$$P_0 \cong \mathcal{L}^+ \otimes \mathcal{L}^+ + \mathcal{L}^+ \otimes (\mathcal{T}Q)^+ \otimes \mathcal{L} \cong (\mathcal{L}^+ \otimes \mathcal{L}^+) \oplus (\mathcal{L}^+ \otimes \mathcal{T}Q \otimes \mathcal{L}). \quad (8)$$

Since $\mathcal{L}$ is a free right $\mathcal{L}$-module, it follows that P_0 is a free $\mathcal{L}$-bimodule.

We claim that $P_1 \subset \Omega^1(\mathcal{T}E)$ is equal to $\Omega^1(\mathcal{T}E) \odot K + (\mathcal{T}E)^+ \, DK \subset \Omega^1(\mathcal{T}E)$. Using $\langle y_0\rangle \, D(\langle y_1\rangle \odot k) = (\langle y_0\rangle \, D\langle y_1\rangle) \odot k + \langle y_0\rangle \odot \langle y_1\rangle \, Dk$ and (3), we conclude that $(\mathcal{T}E)^+ \, D\mathcal{L}$ and $\Omega^1(\mathcal{T}E) \odot K$ agree modulo $(\mathcal{T}E)^+ \, DK$. Since $(\mathcal{T}E)^+ \, DK \subset (\mathcal{T}E)^+ \, D\mathcal{L}$, we get $P_1 = \Omega^1(\mathcal{T}E) \odot K + (\mathcal{T}E)^+ \, DK$. The isomorphism

$$\mu_1^{-1} \colon \Omega^1(\mathcal{T}E) \to (\mathcal{T}E)^+ \otimes E \otimes (\mathcal{T}E)^+$$

of Lemma 1 maps $(\mathcal{T}E)^+ \, DK$ onto $(\mathcal{T}E)^+ \otimes K \otimes 1$ and maps $\Omega^1(\mathcal{T}E) \odot K$ onto $(\mathcal{T}E)^+ \otimes E \otimes (\mathcal{T}E)^+ \odot K = (\mathcal{T}E)^+ \otimes K \otimes \mathcal{L} + (\mathcal{T}E)^+ \otimes s(Q) \otimes \mathcal{L}$ because μ_1^{-1} is a $\mathcal{T}E$-bimodule homomorphism. Thus μ_1^{-1} restricts to an isomorphism

$$P_1 \cong (\mathcal{T}E)^+ \otimes K \otimes \mathcal{L}^+ \oplus (\mathcal{T}E)^+ \otimes Q \otimes \mathcal{L}$$

of filtered bornological $\mathcal{L}$-bimodules. It follows that P_1 is a free $\mathcal{L}$-bimodule.

The inclusion $P_\bullet \subset B_\bullet^{\mathcal{T}E}$ induces a chain map

$$\phi_\bullet \colon P_\bullet/[P_\bullet, \mathcal{L}] \to B_\bullet^{\mathcal{T}E}/\big[B_\bullet^{\mathcal{T}E}, \mathcal{T}E\big] \cong X_\beta(\mathcal{T}E) \oplus \mathbb{C}[0].$$

We claim that $\phi_\bullet$ is a filtered bornological isomorphism onto $X_\beta(\mathcal{T}E : \mathcal{T}Q) \oplus \mathbb{C}[0]$. This is verified by computing the left hand side $P_\bullet/[\,,\,]$. We have

$$P_0/[\,,\,] \cong \mathcal{L}^+ \oplus \mathcal{L} \otimes \mathcal{T}Q \cong \mathbb{C} \oplus \mathcal{L} \otimes (\mathcal{T}Q)^+ \cong \mathbb{C} \oplus \mathcal{I} = \mathbb{C} \oplus X_\beta(\mathcal{T}E : \mathcal{T}Q)_0$$

by (4) and (3). The isomorphism $X_\beta(\mathcal{T}E)_1 \to \Omega^{\mathrm{odd}} E$ maps $X_\beta(\mathcal{T}E : \mathcal{T}Q)_1$ onto $\mathcal{I}\, ds(Q) \oplus (\mathcal{T}E)^+ \, dK$. Using (4) and (7), we compute

$$P_1/[,] \cong \mathcal{L}^+ \otimes (\mathcal{T}Q)^+ \otimes K \oplus \mathcal{L} \otimes (\mathcal{T}Q)^+ \otimes Q \cong (\mathcal{T}E)^+ \otimes K \oplus \mathcal{I} \otimes Q$$
$$\cong (\mathcal{T}E)^+ \, dK \oplus \mathcal{I}\, ds(Q) = X_\beta(\mathcal{T}E : \mathcal{T}Q)_1.$$

It is not hard to check that these isomorphisms $P_0/[,] \to X_\beta(\mathcal{T}E : \mathcal{T}Q)_0 \oplus \mathbb{C}$ and $P_1/[,] \to X_\beta(\mathcal{T}E : \mathcal{T}Q)_1$ are equal to ϕ_0 and ϕ_1, respectively.

We construct an isomorphism $P_\bullet \cong B^{\mathcal{L}}_\bullet \oplus (\mathcal{L}^+ \otimes C_\bullet)$ of complexes of $\mathcal{L}$-bimodules. Recall that $C_0 = C_1 = \mathcal{T}Q \otimes \mathcal{L}$. There is a natural inclusion $B^{\mathcal{L}}_\bullet \subset P_\bullet$ because $\mathcal{L}^+ \otimes \mathcal{L}^+ \subset P_0$ and $\Omega^1 \mathcal{L} \subset P_1$. The inclusion $\mathcal{L}^+ \otimes C_\bullet \to P_\bullet$ is defined by

$$\begin{aligned}
&\mathcal{L}^+ \otimes C_1 \to P_1, \qquad \langle l_0 \rangle \otimes q \otimes l_1 \mapsto \langle l_0 \rangle \big(Ds_L(q)\big) l_1,\\
&\mathcal{L}^+ \otimes C_0 \to P_0, \qquad \langle l_0 \rangle \otimes q \otimes l_1 \mapsto \langle l_0 \rangle \odot s_L(q) \otimes l_1 - \langle l_0 \rangle \otimes s_L(q) \odot l_1.
\end{aligned}$$

The induced map $B^{\mathcal{L}}_0 \oplus (\mathcal{L}^+ \otimes C_0) \to P_0$ is a filtered bornological isomorphism by (8). Hence we obtain a filtered bornological isomorphism

$$B^{\mathcal{L}}_1 \oplus \mathcal{L}^+ \otimes C_1 \cong \ker\big(B^{\mathcal{L}}_0 \oplus \mathcal{L}^+ \otimes C_0 \to \mathcal{L}^+\big) \cong \ker(P_0 \to \mathcal{L}^+) \cong P_1.$$

Thus $B^{\mathcal{L}}_\bullet \oplus (\mathcal{L}^+ \otimes C_\bullet) \cong P_\bullet$ as desired. Taking commutator quotients, we get

$$X_\beta(\mathcal{L}) \oplus \mathbb{C}[0] \oplus C_\bullet \cong P_\bullet/[,].$$

The composition of the above isomorphisms

$$X_\beta(\mathcal{L}) \oplus C_\bullet \oplus \mathbb{C}[0] \to P_\bullet/[,] \to X_\beta(\mathcal{T}E : \mathcal{T}Q) \oplus \mathbb{C}[0]$$

is equal to the map $(\psi, \psi') \oplus \mathrm{id}_{\mathbb{C}}$, with ψ and ψ' as in Lemma 2. Consequently, (ψ, ψ') is a filtered bornological isomorphism. The proof of Lemma 2 is finished.

Moreover, we obtain that $\Omega^1 \mathcal{L} = B^{\mathcal{L}}_1$ is projective as a direct summand of the free $\mathcal{L}$-bimodule P_1. Thus $\mathcal{L}$ is quasi-free. We will not use this observation.

5.2. *A left $\mathcal{T}E$-action on $\mathcal{T}\mathcal{L}$*

We construct a homomorphism $\upsilon \colon \mathcal{L} \to \mathcal{T}\mathcal{L}$ that will be used to prove that $X(\mathcal{T}K) \sim X(\mathcal{L})$.

First we establish some conventions. For any algebra A, let $\tau_A \colon \mathcal{T}A \to A$ be the natural projection with $\tau_A|_{\Omega^0 A} = \mathrm{id}$ and $\tau_A|_{\Omega^{2n} A} = 0$ for all $n \neq 0$. We always consider $K \subset \mathcal{L} \subset \mathcal{T}\mathcal{L}$ via the natural linear maps $\sigma_E \colon E \to \mathcal{T}E$ and $\sigma_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{T}\mathcal{L}$. We write $\circledcirc$ for the Fedosov product in $\mathcal{T}\mathcal{L}$ and denote elements of $\mathcal{T}\mathcal{L}$ by z, z_j.

Let $G := s_R(\mathcal{T}Q)^+ \odot K \subset \mathcal{L}$. Let $\alpha \colon \mathcal{I} \to G\, D(\mathcal{T}E) \subset \Omega^1(\mathcal{T}E)$ be the composition of the isomorphism $\mu_5^{-1} \colon \mathcal{I} \to (\mathcal{T}Q)^+ \otimes K \otimes (\mathcal{T}E)^+$ with the linear map sending $\langle q \rangle \otimes k \otimes \langle y \rangle$ to $s_R \langle q \rangle \odot k\, D\langle y \rangle$. Thus $\alpha|_G = 0$ and $\alpha(g \odot y) := g\, Dy$ for all $g \in G$, $y \in \mathcal{T}E$. Restricting α to $\mathcal{L}$, we obtain a map $\alpha \colon \mathcal{L} \to \mathcal{L}\, D\mathcal{L} \subset \Omega^1 \mathcal{L}$.

Define a filtered bounded bilinear map $E \times \mathcal{TL} \to \mathcal{TL}$, $(e, z) \mapsto e \triangleright z$, by

$$x \triangleright (l \circledcirc \langle z \rangle) := (x \odot l) \circledcirc \langle z \rangle - D\alpha(x \odot l) \circledcirc \langle z \rangle$$

This is well-defined because $\mathcal{TL} \cong \mathcal{L} \otimes (\mathcal{TL})^+$ by Lemma 3.

By construction, $x \triangleright (z_0 \circledcirc z_1) = (x \triangleright z_0) \circledcirc z_1$. We have $\alpha(g \odot \langle y_0 \rangle \odot \langle y_1 \rangle) = g\, D(\langle y_0 \rangle \odot \langle y_1 \rangle) = \alpha(g \odot \langle y_0 \rangle) \odot \langle y_1 \rangle + g \odot \langle y_0 \rangle\, D\langle y_1 \rangle$. Thus

$$\alpha(y_0 \odot \langle y_1 \rangle) = \alpha(y_0) \odot \langle y_1 \rangle + y_0\, D\langle y_1 \rangle \qquad \forall y_0 \in \mathcal{I},\ \langle y_1 \rangle \in \mathcal{T}E.$$

Using also that D is a graded derivation, we get

$$\begin{aligned} x \triangleright Dl_1\, Dl_2 &= x \odot l_1 \odot l_2 - D\alpha(x \odot l_1 \odot l_2) - (x \odot l_1) \circledcirc l_2 + D\alpha(x \odot l_1) \circledcirc l_2 \\ &= -D\big(\alpha(x \odot l_1) \odot l_2\big) + D\alpha(x \odot l_1) \odot l_2 = \alpha(x \odot l_1)\, Dl_2. \end{aligned}$$

Hence we can alternatively define $\triangleright$ by

$$x \triangleright \langle l_0 \rangle\, Dl_1 \dots Dl_{2n} = x \odot \langle l_0 \rangle\, Dl_1 \dots Dl_{2n} - D\alpha(x \odot \langle l_0 \rangle)\, Dl_1 \dots Dl_{2n}, \tag{9}$$

$$x \triangleright Dl_1 \dots Dl_{2n} = \alpha(x \odot l_1)\, Dl_2 \dots Dl_{2n}. \tag{10}$$

We view $\triangleright$ as a linear map $E \to \mathrm{Lin}(\mathcal{TL}; \mathcal{TL})$ sending $x \in E$ to the endomorphism $z \mapsto x \triangleright z$ of $\mathcal{TL}$. We can extend this linear map uniquely to a unital homomorphism $\triangleright\colon (\mathcal{T}E)^+ \to \mathrm{Lin}(\mathcal{TL}; \mathcal{TL})$. Let $\upsilon(\langle y \rangle \odot k) := \langle y \rangle \triangleright k$ for all $\langle y \rangle \in (\mathcal{T}E)^+$, $k \in K$. That is, evaluate the endomorphism $\triangleright\langle y \rangle$ on $k \in K \subset \mathcal{TL}$. This well-defines a linear map $\upsilon\colon \mathcal{L} \to \mathcal{TL}$ because $\mathcal{L} \cong (\mathcal{T}E)^+ \otimes K$.

Lemma 5. *The map υ is a filtered bounded homomorphism satisfying $\tau_{\mathcal{L}} \circ \upsilon = \mathrm{id}_{\mathcal{L}}$. In addition, $\upsilon|_K$ is the standard inclusion $K \leq \mathcal{TL}$.*

Proof. Since $K \subset G$, we have $\alpha(k \odot l) = k\, Dl$ and hence $k \triangleright z = k \circledcirc z$ for all $k \in K$, $z \in \mathcal{TL}$. We have $\langle y \rangle \triangleright (z_0 \circledcirc z_1) = (\langle y \rangle \triangleright z_0) \circledcirc z_1$ for all $\langle y \rangle \in (\mathcal{T}E)^+$, $z_0, z_1 \in \mathcal{TL}$ because E generates $\mathcal{T}E$. Therefore,

$$\begin{aligned} \upsilon(\langle y_0 \rangle \odot k_0 \odot \langle y_1 \rangle \odot k_1) &= \langle y_0 \rangle \triangleright k_0 \triangleright \langle y_1 \rangle \triangleright k_1 = \langle y_0 \rangle \triangleright (k_0 \circledcirc \langle y_1 \rangle \triangleright k_1) \\ &= (\langle y_0 \rangle \triangleright k_0) \circledcirc (\langle y_1 \rangle \triangleright k_1) = \upsilon(\langle y_0 \rangle \odot k_0) \circledcirc \upsilon(\langle y_1 \rangle \odot k_1), \end{aligned}$$

so that υ is a homomorphism.

Equations (9) and (10) imply $\tau_{\mathcal{L}}(x \triangleright z) = x \odot \tau_{\mathcal{L}}(z)$ for all $x \in E$, $z \in \mathcal{TL}$. Consequently, $\tau_{\mathcal{L}}(\langle y \rangle \triangleright z) = \langle y \rangle \odot \tau_{\mathcal{L}}(z)$ for all $\langle y \rangle \in (\mathcal{T}E)^+$, $z \in \mathcal{TL}$. This implies $\tau_{\mathcal{L}} \circ \upsilon = \mathrm{id}$.

For the proof that υ is filtered and bounded, it suffices to consider the restriction of υ to $(dE)^{\mathrm{even}} \odot K$ because $\upsilon(\langle x_0 \rangle dx_1 \dots dx_{2n} \odot k) = \langle x_0 \rangle \triangleright (dx_1 \dots dx_{2n} \triangleright k)$. We let $\langle Dz \rangle := Dl_1 \dots Dl_{2n}$ and $Dz := Dl_2 \dots Dl_{2n}$ and extend α to a map $\alpha\colon \mathcal{L}(D\mathcal{L})^n \to \mathcal{L}(D\mathcal{L})^{n+1}$ by $\alpha(l_0\, Dl_1 \dots Dl_{2n}) := \alpha(l_0)\, Dl_1 \dots Dl_{2n}$. Straightforward computations show that

$$\begin{aligned} dx_1\, dx_2 \triangleright l_0\, \langle Dz \rangle &= (x_1 x_2) \triangleright l_0\, \langle Dz \rangle - x_1 \triangleright (x_2 \triangleright l_0\, \langle Dz \rangle) \\ &= dx_1\, dx_2 \odot l_0\, \langle Dz \rangle - D\alpha(dx_1\, dx_2 \odot l_0)\, \langle Dz \rangle \\ &\qquad + \alpha\big(x_1 \odot \alpha(x_2 \odot l_0)\big)\, \langle Dz \rangle. \end{aligned} \tag{11}$$

$$\begin{aligned} dx_1\, dx_2 \triangleright Dl_1\, Dz &= \alpha\big((x_1 x_2) \odot l_1\big)\, Dz - x_1 \odot \alpha(x_2 \odot l_1\big)\, Dz \\ &\qquad + D\alpha\big(x_1 \odot \alpha(x_2 \odot l_1)\big)\, Dz. \end{aligned} \tag{12}$$

To prove that υ is filtered we construct a filtration $(\mathcal{TL})'_n$ on $\mathcal{TL}$ that is equivalent to the standard filtration and for which $dEdE \triangleright (\mathcal{TL})'_n \subset (\mathcal{TL})'_{n+1}$ for all n. Hence $\upsilon\big((dE)^{2n} \odot K\big) \subset (\mathcal{TL})'_n$ for all n, so that υ is filtered. Define

$$\#_e(l_0\, Dl_1 \dots Dl_{2n}) := 3n, \qquad \#_e(Dl_1 \dots Dl_{2n}) := 3n - 2,$$

and

$$\#_i(\langle l_0\rangle\, Dl_1 \dots Dl_{2n}) := \#\langle l_0\rangle + \#l_1 + \cdots + \#l_{2n}$$

for homogeneous $\langle l_0\rangle, l_1, \dots, l_{2n}$. Define $\#_t = \#_e + \#_i$ and let $(\mathcal{TL})'_n$ be the linear span of all homogeneous monomials $z \in \mathcal{TL}$ with $\#_t z \geq n$. We have $\#_t\langle l_0\rangle\, Dl_1 \dots Dl_{2m} > n$ if $m > n$ or if $\#l_j > n$ for some j. Conversely, if neither $m > n$ nor $\#l_j > n$ for any j, then $\#_t\langle l_0\rangle\, Dl_1 \dots Dl_{2m} \leq 3n + (2n+1)n$. Therefore, the filtration $(\mathcal{TL})'_n$ is equivalent to the standard filtration.

During the proof of Lemma 4 we observed that μ_5^{-1} decreases # by at most 1. Therefore, $\alpha(l)$ is a sum of terms $l_0\, Dl_1$ with $\#l_0 + \#l_1 \geq \#l - 1$. Inspecting the summands in (11) and (12), we see that $dx_1\, dx_2 \triangleright z$ is a sum of homogeneous terms of degree $\#_t \sqcup \geq \#_t z + 1$. The possible loss in $\#_i$ through α is compensated by the gain in $\#_e$. Consequently, $dE\, dE \triangleright (\mathcal{T}E)'_n \subset (\mathcal{T}E)'_{n+1}$ as desired.

The boundedness of υ is more difficult. We are going to prove the following Lemma:

Lemma 6. *For all $T \in \mathfrak{S}(\mathcal{TL})$ and $S \in \mathfrak{S}(E)$, there is $F \in \mathfrak{S}(\mathcal{TL})$ with $T \Subset F$ and $dS\, dS \triangleright F \Subset F$.*

Let $T = S_K \subset K$ for some $S_K \in \mathfrak{S}(K)$. Replacing the set F of Lemma 6 by F°, we can achieve that $S_K \subset F$ and $dS\, dS \triangleright F \subset F$. By induction, $(dS)^{2n} \triangleright F \subset F$ for all n, that is, $(dS)^{\text{even}} \triangleright S_K \subset F$. Thus Lemma 6 implies that υ is bounded.

To prove Lemma 6, we make the Ansatz

$$F := F_0\, (DF_\infty)^{\text{even}} \cup DF_0\, DF_\infty\, (DF_\infty)^{\text{even}} \tag{13}$$

with certain $F_0, F_\infty \in \mathfrak{S}(\mathcal{L})$ and $(DF_\infty)^{\text{even}} := \bigcup_{n=0}^{\infty} (DF_\infty)^{2n} \subset (\mathcal{TL})^+$. Inspecting the summands in (11) and (12), we see that $dS\, dS \triangleright F \Subset F$ follows if

$$\begin{gathered} 3\, dS\, dS\, F_0 \Subset F_0, \\ 3\alpha(dS\, dS\, F_0) \Subset F_0\, DF_\infty, \\ 3S \odot \alpha(S \odot F_0) \Subset F_0\, DF_\infty, \end{gathered} \qquad \begin{gathered} 3\alpha\big(S \odot \alpha(S \odot F_0)\big) \Subset F_0\, DF_\infty\, DF_\infty, \\ 3\alpha(S^2 \odot F_0) \Subset F_0\, DF_\infty. \end{gathered} \tag{14}$$

The factors of 3 are needed to convert sums of three terms into convex combinations. Choose $S_2 \in \mathfrak{S}(E)$ with $2S \cup (2S)^2 \subset S_2$. The conditions (14) follow if

$$dS_2\, dS_2\, F_0 \Subset F_0, \qquad \langle S_2\rangle \odot \alpha(\langle S_2\rangle \odot F_0) \Subset F_0\, DF_\infty. \tag{15}$$

Let $\omega_s \colon Q \times Q \to K$ be the curvature of the linear section $s \colon Q \to E$. We choose $S_Q \in \mathfrak{S}(Q)$ and (afterwards) $S_K \in \mathfrak{S}(K)$ such that $2S_2 \cup 2S_2^2 \Subset S_E := S_K \cup s(S_Q)$

and $\omega_s(S_Q, S_Q) \subset S_K$. To avoid notational clutter, we will frequently suppress the map s and write S_Q instead of $s(S_Q)$. Let $S'_K := \langle S_E\rangle \cdot S_K \cdot \langle S_E\rangle \in \mathfrak{S}(K)$. Let

$$F_0 := (dS_2)^{\mathrm{even}}\langle S_2\rangle \odot (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K \in \mathfrak{S}(\mathcal{L}). \tag{16}$$

Condition (15) follows if $\alpha(\langle S_2\rangle \odot F_0) \Subset (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K\, DF_\infty$. Since $2S_2 \cup 2S_2^2 \Subset S_E$, we have $\langle S_2\rangle \odot (dS_2)^{\mathrm{even}}\langle S_2\rangle \Subset (dS_E)^{\mathrm{even}}\langle S_E\rangle$. Hence

$$\alpha((dS_E)^{\mathrm{even}}\langle S_E\rangle \odot (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K) \Subset (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K\, DF_\infty \tag{17}$$

implies $dS\,dS \triangleright F \Subset F$. Let $T' := (dS_E)^{\mathrm{even}}\langle S_E\rangle \cap \mathcal{I}$. We are going to show that the left hand side of (17) is contained in $\alpha(T' \odot T'')^\circ$ for some small set $T'' \in \mathfrak{S}(\mathcal{L})$. Since $(dS_E)^{\mathrm{even}}\langle S_E\rangle = (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \cup T'$, the only problematic part is $\alpha\big((dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K\big)$. Let

$$y = dq_1 \dots dq_{2n}\langle q_{2n+1}\rangle \odot dq_{2n+2} \dots dq_{2n+2m-1}\langle q_{2n+2m}\rangle \odot k$$

with $q_j \in S_Q = s(S_Q)$, $k \in S'_K$. We omit the section s whenever this does not create confusion. When bringing y into right handed standard form, we get a sum of terms of the form $\pm dq_1 \dots dq_{j-1} d\big(s(q_j)s(q_{j+1})\big) dq_{j+2} \dots$. We can replace $s(q_j)s(q_{j+1}) = s(q_jq_{j+1}) - \omega_s(q_j, q_{j+1})$ by $-\omega_s(q_j, q_{j+1})$ because α annihilates $dq_1 \dots dq_{j-1} ds(q_jq_{j+1}) dq_{j+2} \dots dq_{2n+2m-1}\langle q_{2n+2m}\rangle \odot k \in G$. By assumption, $\omega_s(q_j, q_{j+1}) \in \omega_s(S_Q, S_Q) \subset S_K$. Thus we can replace y by a sum of terms in

$$(dS_Q)^{j-1}\, dS_K\, (dS_Q)^{2n+2m-j-2}\langle S_Q\rangle \odot S'_K$$

and a last term (for $j = 2n + 2m - 1$) in $(dS_Q)^{2n+2m-2} S_K \odot S'_K$. Since the geometric series $\sum 2^{-j}$ converges, we can write this sum as a convex combination of terms in

$$\big((dS_E)^{\mathrm{even}} \cap \mathcal{I}\big)(d2S_Q)^{\mathrm{even}}\langle S_Q\rangle \odot CS'_K \subset T' \odot (d2S_Q)^{\mathrm{even}}\langle S_Q\rangle \odot CS'_K$$

for a suitable constant $C \geq 1$. Consequently,

$$\alpha\big((dS_E)^{\mathrm{even}}\langle S_E\rangle \odot (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K\big) \Subset \alpha(T' \odot T'')$$

with $T'' := (d2S_Q)^{\mathrm{even}}\langle S_Q\rangle \odot CS'_K$ as desired.

The map μ_5^{-1} is a right $\mathcal{T}E$-module homomorphism. Therefore and because $T' \subset \mathcal{I}$, we have $\mu_5^{-1}(T' \odot T'') = \mu_5^{-1}(T') \odot T''$. Equation (6) implies

$$\mu_5^{-1}(T' \odot T'') \Subset (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \otimes S'_K \otimes 2\langle S_E\rangle \odot (dS_E)^{\mathrm{even}}\langle S_E\rangle \odot T''.$$

As a result, we get $\alpha(T' \odot T'') \Subset (dS_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K\, DF_\infty$ with

$$\begin{aligned} F_\infty &:= 2\langle S_E\rangle \odot (dS_E)^{\mathrm{even}}\langle S_E\rangle \odot T'' \\ &= 2C\langle S_E\rangle \odot (dS_E)^{\mathrm{even}}\langle S_E\rangle \odot (d2S_Q)^{\mathrm{even}}\langle S_Q\rangle \odot S'_K. \end{aligned} \tag{18}$$

Consequently, sets F of the form (13) with F_0 as in (16) and F_∞ as in (18) satisfy $dS\,dS \triangleright F \Subset F$. Any small subset $T \subset \mathcal{T}\mathcal{L}$ is contained in the disked hull of F for suitably big $S_2 \in \mathfrak{S}(E)$. Therefore, Lemma 6 is true, so that υ is bounded. □

5.3. Homotopy equivalence of $\mathcal{L}$ and $\mathcal{T}K$

Let $\kappa\colon \mathcal{L} \to K$ be the restriction of $\tau_E\colon \mathcal{T}E \to E$ to $\mathcal{L}$. We claim that

$$X(\mathcal{T}\kappa) \circ X(\upsilon)\colon X(\mathcal{L}) \to X(\mathcal{T}\mathcal{L}) \to X(\mathcal{T}K)$$

is a homotopy inverse for $X(j)\colon X(\mathcal{T}K) \to X(\mathcal{L})$. We have $(\mathcal{T}\kappa) \circ \upsilon \circ j = \mathrm{id}_{\mathcal{T}K}$ because this holds on the subspace $\sigma(K) \subset \mathcal{T}K$ generating $\mathcal{T}K$. Hence $X(\mathcal{T}\kappa) \circ X(\upsilon)$ is a section for $X(j)$. We are going to show that the homomorphisms $j \circ \mathcal{T}\kappa\colon \mathcal{T}\mathcal{L} \to \mathcal{L}$ and $\tau_{\mathcal{L}}\colon \mathcal{T}\mathcal{L} \to \mathcal{L}$ are polynomially homotopic.

Define a linear map $h\colon \mathcal{L} \to \mathbb{C}[t] \otimes \mathcal{L}$ by $h(l) := t^{\# l} \otimes l$ for homogeneous l. Its curvature $\omega_h(l_1, l_2) := h(l_1 \odot l_2) - h(l_1) \odot h(l_2)$ is given by

$$\begin{aligned}\omega_h(l_1, l_2) &= h(l_1 \cdot l_2 - dl_1 dl_2) - h(l_1) \cdot h(l_2) + dh(l_1) \cdot dh(l_2)\\ &= t^{\# l_1 + \# l_2}(l_1 \cdot l_2 - t dl_1 dl_2 - l_1 \cdot l_2 + dl_1 dl_2) = t^{\# l_1 + \# l_2}(1-t) \otimes dl_1 dl_2\end{aligned}$$

for homogeneous $l_1, l_2 \in \mathcal{L}$. The universal property of the tensor algebra $\mathcal{T}\mathcal{L}$ allows us to extend h to a homomorphism $H\colon \mathcal{T}\mathcal{L} \to \mathbb{C}[t] \otimes \mathcal{L}$ by

$$\begin{aligned}H(\langle l_0 \rangle\, Dl_1 \dots Dl_{2n}) &= h\langle l_0 \rangle \odot \omega_h(l_1, l_2) \odot \cdots \odot \omega_h(l_{2n-1}, l_{2n})\\ &= t^{\#\langle l_0 \rangle + \# l_1 + \cdots + \# l_{2n}}(1-t)^n \otimes \langle l_0 \rangle dl_1 \dots dl_{2n}.\end{aligned}$$

The map $H_0 = \mathrm{ev}_0 \circ H$ annihilates $\langle l_0 \rangle\, Dl_1 \dots Dl_{2n}$ if $\# l_j > 0$ for some j, whereas $H_1 = \mathrm{ev}_1 \circ H$ annihilates $\langle l_0 \rangle\, Dl_1 \dots Dl_{2n}$ if $n > 0$. Consequently, $H_0 = j \circ \mathcal{T}\kappa$ and $H_1 = \tau_{\mathcal{L}}$. The map H is filtered and bounded with respect to the bornology and filtration on $\mathbb{C}[t] \otimes \mathcal{L}$ defined in Sect. 4.2. Hence $j \circ \mathcal{T}\kappa$ and $\tau_{\mathcal{L}}$ are polynomially homotopic. It follows that $j \circ (\mathcal{T}\kappa) \circ \upsilon$ is polynomially homotopic to $\mathrm{id}_{\mathcal{L}}$. Therefore, the algebras $\mathcal{L}$ and $\mathcal{T}K$ are homotopy equivalent.

Prop. 2 yields that $X(j \circ \mathcal{T}\kappa)$ and $X(\tau_{\mathcal{L}})$ are chain homotopic. As a result, $X(j) \circ X(\mathcal{T}\kappa) \circ X(\upsilon) \sim X(\tau_{\mathcal{L}}) \circ X(\upsilon) = \mathrm{id}$ because $\tau_{\mathcal{L}} \circ \upsilon = \mathrm{id}_{\mathcal{L}}$. Thus $X(j)$ is a homotopy equivalence. The map $\psi\colon X(\mathcal{L}) \to X(\mathcal{T}E : \mathcal{T}Q)$ is a homotopy equivalence as well by Lemma 2. Consequently, $\varrho = \psi \circ X(j)$ is a homotopy equivalence. The proof of the excision theorem is complete.

6. Excision results in cyclic cohomology

We can describe $\mathrm{HP}^*(A)$ as the inductive limit of the cyclic cohomology groups $\mathrm{HC}^{*+2n}(A)$. We may ask how the connecting map $\partial\colon \mathrm{HP}^*(K) \to \mathrm{HP}^{*-1}(Q)$ shifts the dimensions in cyclic cohomology. Puschnigg was the first to show that ∂ maps $\mathrm{HC}^n(K)$ to $\mathrm{HC}^{3n+3}(Q)$. He also proves that this estimate is optimal for certain extensions [11]. The proof of excision above yields, in principle, an explicit formula for a degree 1 chain map $\delta\colon X(\mathcal{T}Q) \to X(\mathcal{T}K)$ that induces the boundary map in the six term exact sequences. Messy bookkeeping in [10] shows that the map on cohomology induced by δ maps $\mathrm{HC}^n(K)$ to $\mathrm{HC}^{3n+3}(Q)$ and thus realizes the optimal bound. [10] also contains estimates about the dimension shifts that occur when switching between $\mathrm{HC}^*(K)$ and the relative cyclic cohomology $\mathrm{HC}^*(E : Q)$.

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