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Random walks on finite rank solvable groups

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Abstract. We establish the lower bound $p_{2t}(e,e) \gtrsim \exp(-t^{1/3})$, for the large times asymptotic behaviours of the probabilities $p_{2t}(e,e)$ of return to the origin at even times $2t$, for random walks associated with finite symmetric generating sets of solvable groups of finite Prüfer rank. (A group has finite Prüfer rank if there is an integer r, such that any of its finitely generated subgroup admits a generating set of cardinality less or equal to r.)

Key words. random walk – heat kernel decay – asymptotic invariants of infinite groups – Prüfer rank – solvable group

Contents

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1. Introduction

1.1. Notation and definitions

Let Γ be a finitely generated group. Let

$$p : \Gamma \times \Gamma \to [0, 1]$$

be a symmetric Markov kernel which is left-invariant, irreducible and whose support lies within bounded distance (with respect to a word metric on Γ) from the diagonal. We also denote by p the operator on the Hilbert space $l^2(\Gamma)$ defined on $f \in l^2(\Gamma)$ by the formula

$$pf(x) = \sum_y p(x, y)f(y)$$

and we denote by p^n the composition of p with itself n times. Let $\delta_x \in l^2(\Gamma)$ be the characteristic function of the point x. The scalar product

$$p_t(x, y) = \langle p^t \delta_y, \delta_x \rangle$$

can be interpreted as the probability for the random walk on Γ defined by p to go from x to y in time t. In particular

$$\| p^t \delta_e \|_2^2 = \langle p^t \delta_e, p^t \delta_e \rangle = \langle p^{2t} \delta_e, \delta_e \rangle = p_{2t}(e, e)$$

corresponds to the probability of return to the origin e after $2t$ steps.

If f, g are two non-negative functions defined on positive numbers, we use the notation $f \precsim g$ if there exist constants $a, b > 0$, such that for x large enough, $f(x) \leq ag(bx)$. If the symmetric relation also holds, we write $f \sim g$. When a function is defined only on the integers, we extend it to the positive real axis by linear interpolation. We will use the same name for the original function and its extension. When using this convention for the function $t \mapsto p_{2t}(e, e)$, the value of $p_t(e, e)$ at an odd integer t must be interpreted as $\frac{1}{2}(p_{t-1}(e, e) + p_{t+1}(e, e))$.

1.2. Stability

1. If Γ and H are two quasi-isometric (see [7, IVB] and [12, 02C]) finitely generated groups then $p_{2t}^{\Gamma}(e, e) \sim p_{2t}^{H}(e, e)$. (We consider even times $2t$ to avoid usual periodicity problems.) In particular, two Markov kernels, with the properties explained above, on the same group Γ have equivalent asymptotic behaviours in the sense of the relation $\sim$ and we call the equivalence class of $p_{2t}^{\Gamma}(e, e)$ the heat decay of the group Γ. Finite index subgroups and quotients by finite normal subgroups are quasi-isometric to the original group [7, IVB24].

2. If H is a finitely generated subgroup in a finitely generated group Γ then $p^{\Gamma}_{2t}(e,e) \precsim p^{H}_{2t}(e,e)$.
3. If Q is a quotient of Γ then $p^{\Gamma}_{2t}(e,e) \precsim p^{Q}_{2t}(e,e)$.
4. Let M be a closed Riemannian manifold and let $\tilde{M}$ be its universal cover (with the induced locally isometric Riemannian metric). Let $x_0 \in \tilde{M}$ be a base point. Let $\tilde{p}_t(x,y)$ be the heat kernel of the Riemannian Laplacian on $\tilde{M}$. Then

$$\tilde{p}_t(x_0,x_0) \sim p^{\pi_1(M)}_{2t}(e,e). \tag{1.1}$$

5. Let G and H be two connected Lie groups with quasi-isometric left-invariant Riemannian metrics. Then the heat kernels associated with the Riemannian Laplacians satisfy $p^{G}_{t}(e,e) \sim p^{H}_{t}(e,e)$.

See [20] for the proofs of these stability properties.

1.3. Lie groups

On a connected Lie group G, two left-invariant Riemannian metrics are bi-Lipschitz. They are three essentially different heat decays on Lie groups.

1. $p_t(e,e) \sim \exp(-t)$ if and only if G is non-amenable or non-unimodular.
2. $p_t(e,e) \sim t^{-d/2}$ if and only if the growth of G is polynomial of degree d.
3. $p_t(e,e) \sim \exp(-t^{1/3})$ if and only if G is amenable unimodular and of exponential growth.

See [17]. Notice that we do not exclude non-unimodular Lie groups. Whether the group is unimodular or not, we consider a left-invariant Riemannian metric on it, the corresponding left-invariant volume form and the corresponding left-invariant Riemannian Laplacian. In the case of non-unimodular Lie groups it is natural to introduce the modular function in order to get refined asymptotics (see [31]) but this is not our aim in this paper.

1.4. Finitely generated groups

A finitely generated group is non-amenable if and only if its heat decay is exponential [13].

Among solvable groups, there appears a great diversity of possible heat decays. For example, the wreath product $\mathbb{Z} \wr \mathbb{Z}$ of $\mathbb{Z}$ with itself (associated with the left-regular representation) satisfies

$$p_{2t}(e,e) \sim \exp(-t^{1/3}(\log t)^{2/3}),$$

and if the heat decay of a finitely generated group Γ is equivalent to

$$\exp(-t^{\alpha}(\log t)^{\beta})$$

with $\alpha \in (0, 1)$, then the heat decay on $G = \Gamma \wr \mathbb{Z}$ is equivalent to

$$\exp\big(-t^{(1+\alpha)/(3-\alpha)}(\log t)^{2\beta/(3-\alpha)}\big).$$

Let us give examples from among the subgroups of $GL(2, \mathbb{R})$. Let $\lambda_1, \ldots, \lambda_d > 1$ be real numbers which are algebraically independent over $\mathbb{Q}$. The subgroup of the affine transformations of the real line generated by the homotheties $x \mapsto \lambda_i x$ where $1 \leq i \leq d$ and by the the translation $x \mapsto x + 1$ is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}^d$. Its heat decay satisfies

$$p_{2t}(e, e) \sim \exp\big(-t^{d/(d+2)}(\log t)^{2/(d+2)}\big).$$

See [21] for the lower bound. For the upper bound, see [10] and apply the Nash inequality which involves the isoperimetric function as explained in [19] or apply [6, Prop. 4.1].

1.5. Relations with volume growth

Gromov's theorem on polynomial growth and some work by Varopoulos together imply that a finitely generated group is virtually nilpotent if and only if its heat decay satisfies

$$p_{2t}(e, e) \sim t^{-d/2} \tag{1.2}$$

where d is the growth degree of the group. See [32].

Let Γ be a finitely generated group with a word metric and let $|B_e(r)|$ be the cardinality of the ball of radius r. Let $0 < \alpha, \beta \leq 1$. If

$$\exp(r^\alpha) \precsim |B_e(r)| \precsim \exp(r^\beta)$$

then

$$\exp\big(-t^{\frac{\beta}{2-\beta}}\big) \precsim p_{2t}(e, e) \precsim \exp\big(-t^{\frac{\alpha}{2+\alpha}}\big). \tag{1.3}$$

(The implication between the inequalities involving α (resp. β) is true independently of β (resp. α).) See [6, Corollary 7.4], see also [30] for the upper bounds on the heat decay. The case when $\alpha = 1$ states that a group of exponential growth has a heat decay at least as fast as $\exp(-t^{1/3})$. Whether there exists a finitely generated group with a heat decay strictly slower than $\exp(-t^{1/3})$ but faster than $t^{-d/2}$ (i.e. not virtually nilpotent) is an open question. This would follow, by the above implication involving β, from the existence of a non-virtually nilpotent finitely generated group with subradical growth in a strong sense, that is with $|B_e(r)| \precsim \exp(r^\beta)$ where $0 < \beta < 1/2$. See [7] for the notion of subradical growth. According to [14, Theorem D] a residually nilpotent group of subradical growth is virtually nilpotent. In fact, it is not known if there is a finitely generated group with subexponential growth, not virtually nilpotent, with $p_{2t}(e, e) \succsim \exp(-t^{1/3})$.

1.6. Statement of the results

The first example of a finitely generated group with a heat decay equivalent to $\exp(-t^{1/3})$ was given by Varopoulos. He showed that the heat decay on the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is equivalent to the expectation $E[2^{-R_t}]$ where R_t is the random variable which counts the number of visited sites during time t for a random walk on $\mathbb{Z}$. See [29, Appendix II] and the correction in [28]. The heat decay of the standard wreath product $K \wr Q$ of two finitely generated groups K and Q with Q infinite (otherwise, the Cartesian product of $|Q|$ copies of K is a subgroup of index $|Q|$ in $K \wr Q$) behaves like $\exp(-t^{1/3})$ if and only if K is a finite non-trivial group and Q is a finite extension of $\mathbb{Z}$. This follows from technics developed in [21].

Alexopoulos established the lower bound $p_{2t}(e, e) \gtrsim \exp(-t^{1/3})$ for polycyclic groups [1]. The main result of this paper is the generalization of this lower bound to the class of finitely generated solvable groups of finite Prüfer rank. Recall that a finitely generated group has finite Prüfer rank if there is an integer r, such that any of its finitely generated subgroup admits a generating set of cardinality less or equal to r.

Theorem 1.1 *Let Γ be a finitely generated virtually solvable group of finite Prüfer rank. Then the heat decay in Γ satisfies*

$$p_{2t}(e, e) \gtrsim \exp(-t^{1/3}).$$

Corollary 1.2 *Let Γ be a finitely generated virtually solvable group of finite Prüfer rank. The heat decay of Γ satisfies $p_{2t}(e, e) \sim \exp(-t^{1/3})$ if and only if Γ is not virtually nilpotent.*

To deduce the corollary from the theorem, recall that a finitely generated solvable group of subexponential growth is virtually nilpotent [16], [33], [11]. If the growth is exponential, we apply (1.3) with $\alpha = 1$.

According to the main theorem of [15] (see also [9, 6.13]), a group is finitely generated virtually solvable of finite Prüfer rank if and only if it is finitely generated residually finite and has polynomial subgroup growth. It is reasonable to expect a lower bound on the heat decay from a bound on the number of subgroups, but the proof we have does not follow this line. It would be interesting to establish such a direct relation for faster subgroup growth. Ascending HNN-groups over a polycyclic base are finitely generated solvable of finite Prüfer rank [3]. In particular, polycyclic groups are finitely generated solvable of finite Prüfer rank. The group of upper triangular $n \times n$ matrices with coefficients in the ring $\mathbb{Z}[1/d]$ where $d \in \mathbb{N}$ and with units on the diagonal is finitely generated solvable of finite Prüfer rank. See also [6] and [18] for examples and special cases of the theorem.

1.7. Main ideas from the proof

A Følner couple in a finitely generated group Γ is a finite set $\Omega \subset \Gamma$ together with a subset $\Omega' \subset \Omega$, such that a positive fraction of the "mass" of Ω (that is, its cardinality) is contained in the subset Ω' and such that Ω' "sits deep inside" Ω in the sense that the word distance $d(\Omega', \Gamma \setminus \Omega)$ is not small relative to the size of Ω. These two properties together imply that the time needed for the mass of Ω' to diffuse out of Ω is rather long. This geometric approach to on-diagonal heat kernel lower bounds has been formulated and formalized in [6].

A finitely generated virtually solvable group of finite Prüfer rank is quasi-isometric to an extension of a torsion-free countable nilpotent group M of finite Prüfer rank by a free abelian group of finite rank. We choose a finitely generated subgroup $M_0 \subset M$ whose normal closure in Γ is the group M. The main idea of the proof is, roughly speaking, to proceed "as if we could replace the group M with its finitely generated sugroup M_0 and construct Følner couples as explained in [6, Appendix 7.5] in the case of polycyclic groups". This strategy works because the rational structure associated with M in Theorem 4.8 below allows us to identify M_0 with a group of integral points. We can then exhaust M_0 with balls (with respect to a word metric) and we can add points of $M \setminus M_0$ in a controlled manner depending on the radius of the balls, by allowing points with progressively larger denominators. A crucial point is that an automorphism of a torsion-free countable nilpotent group M of finite Prüfer rank can be identified with a restriction of an automorphism of the rational Lie algebra associated with M (see the third point in Proposition 4.11 below).

1.8. Overview of the paper

The whole paper is devoted to the proof of Theorem 1.1. As explained above, the strategy consists in constructing a sequence of Følner couples and in translating the geometric information so obtained into an analytic one with the help of a result of [6] which is presented in Sect. 6. Følner exhaustions that serve to construct sequences of Følner couples are introduced in Subsect. 7.3.

Sections 2, 3, 5, 6 are independent. Section 4 depends on Sect. 3. Section 7 depends on the preceding ones.

In Sect. 2, we collect elementary facts about unipotent groups with coefficients in various subrings of $\mathbb{Q}$ and about regular maps between affine spaces over $\mathbb{Q}$. In Sect. 3, we recall the definitions and basic properties of groups of finite Prüfer rank. In Sect. 4, which is based on Mal'cev's theory, we characterize the countable torsion-free nilpotent groups of finite Prüfer rank in terms of nilpotent Lie algebras over $\mathbb{Q}$. In Sect. 5, we recall well-known facts from geometric group theory. In Sect. 6, we present in Theorem 6.2 a result from [6] and state in a corollary the special case we need. Section 7 consists in the proof of Theorem 1.1.

2. Unipotent groups with various coefficients

Let p be a prime number. Recall that the corresponding valuation on $\mathbb{Q}$ is defined as

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$$
$$v_p(0) = \infty, \ v_p(p^n a/b) = n$$

where $a, b \in \mathbb{Z}$ have no factor equal to p. Recall also that

$$v_p(x+y) \geq \min\{v_p(x), v_p(y)\},$$
$$v_p(xy) = v_p(x) + v_p(y).$$

We denote by $U(n, \mathbb{Q})$, or simply by $U(\mathbb{Q})$, the group of upper triangular unipotent $n \times n$ matrices with rational coefficients. Let $d \in \mathbb{N}$. Let $\mathbb{Z}[1/d]$ be the smallest subring of $\mathbb{Q}$ which contains $1/d$. For each $m \in \mathbb{N} \cup \{0\}$, there is an inclusion $\frac{1}{d^m}\mathbb{Z} \subset \frac{1}{d^{m+1}}\mathbb{Z}$ between cyclic subgroups. The group $\mathbb{Z}[1/d]$ is the direct limit

$$\mathbb{Z}[1/d] = \bigcup_{m \in \mathbb{N} \cup \{0\}} \frac{1}{d^m}\mathbb{Z}.$$

We agree that $\frac{1}{d^\infty}\mathbb{Z} = \mathbb{Z}[1/d]$. The matrices of $U(\mathbb{Q})$ with coefficients in $\mathbb{Z}[1/d]$ form a subgroup $U(\mathbb{Z}[1/d]) \subset U(\mathbb{Q})$. Let $v_p(d) = d_p \in \mathbb{N} \cup \{0\}$ be the power of p in the prime decomposition of d. That is:

$$d = \prod_p p^{d_p}.$$

If $d = 1$, then $d_p = 0$ for every p. In order to filter the elements of $U(\mathbb{Z}[1/d])$ with a sequence of finitely generated subgroups, we introduce the following definition.

Definition 2.1 *For each $m \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, let $U^m(\mathbb{Z}[1/d])$ be the set of elements $u \in U(\mathbb{Q})$, such that, for every prime p the coefficient u_{ij} of u satisfies:*

$$v_p(u_{ij}) \geq (i-j)md_p.$$

In the case $m = \infty$ and $d_p = 0$, we agree that $\infty \times 0 = 0$.

Lemma 2.2 *Let $d = \prod_p p^{d_p}$. Let $x, y \in \mathbb{Q}$ and $r, s \in \mathbb{N} \cup \{0\}$ such that $v_p(x) \geq -rd_p$ and $v_p(y) \geq -sd_p$. If $x \neq y$ then $|x-y| \geq 1/d^{r+s}$.*

Proof. Let $x = a/b$ where $a, b \in \mathbb{Z}$ are relatively prime.

$$v_p(a) - v_p(b) = v_p(x) \geq -rd_p.$$

Hence,

$$v_p(b) \leq v_p(a) + rd_p.$$

As a and b are relatively prime, we deduce that $v_p(b) \leq rd_p$. Applying this last inequality to each prime p in the decomposition of d, we deduce that $b \leq d^r$. Proceeding in the same way for the denominator b' of y, we obtain $b' \leq d^s$. Hence

$$|x-y| = \left|\frac{a}{b} - \frac{a'}{b'}\right| = \left|\frac{b'a - a'b}{b'b}\right| \geq \left|\frac{1}{b'b}\right| \geq 1/d^{r+s}.$$

□

Lemma 2.3 *Given a Euclidean structure on the space $M(n, \mathbb{R})$ of $n \times n$ matrices with real coefficients and given a ball $B(r) \subset M(n, \mathbb{R})$ of radius r, there exists $\lambda > 1$, depending on d, such that for all $m \in \mathbb{N} \cup \{0\}$,*

$$|B(r) \cap U^m(\mathbb{Z}[1/d])| < \lambda^m.$$

Proof. One easily deduces the lemma from the following uniform lower bound for the Euclidean distance between distinct elements u, v of $U^m(\mathbb{Z}[1/d])$.

$$\inf_{u,v} d(u, v) = \inf_{u,v} d(0, u - v) \geq \inf_{u_{ij} - v_{ij} \neq 0} |u_{ij} - v_{ij}| \geq 1/d^{2(1-n)m},$$

where the last inequality is deduced from Lemma 2.2. □

Lemma 2.4 *Let $a, b \in M(n, \mathbb{Q})$ and let $c = ab$. Let $r \in \mathbb{N} \cup \{0\}$. If $v_p(a_{ij}) \geq (i - j)r$ and $v_p(b_{ij}) \geq (i - j)r$. Then $v_p(c_{ij}) \geq (i - j)r$.*

Proof. We have $c_{ij} = \sum a_{ik} b_{kj}$ and

$$v_p(c_{ij}) \geq \min_k (v_p(a_{ik}) + v_p(b_{kj})) \geq \min_k ((i - k)r + (k - j)r) = (i - j)r.$$

□

Proposition 2.5 *There is a sequence of inclusions*

$$U^0(\mathbb{Z}[1/d]) \subset \cdots \subset U^m(\mathbb{Z}[1/d]) \subset U^{m+1}(\mathbb{Z}[1/d]) \subset \cdots \subset U^\infty(\mathbb{Z}[1/d])$$

beginning with $U(\mathbb{Z}) = U^0(\mathbb{Z}[1/d])$ and ending with $U(\mathbb{Z}[1/d]) = U^\infty(\mathbb{Z}[1/d])$. Each $U^m(\mathbb{Z}[1/d])$ is a group and if $m < \infty$ it is finitely generated.

Proof. Inclusions and equalities follow directly from the definitions. (The equality $U(\mathbb{Z}[1/d]) = U^\infty(\mathbb{Z}[1/d])$ is true because of the convention we decided on $\infty \times 0 = 0$.) Lemma 2.4 shows that if $a, b \in U^m(\mathbb{Z}[1/d])$ then $ab \in U^m(\mathbb{Z}[1/d])$. Let us show that $a \in U^m(\mathbb{Z}[1/d])$ implies $a^{-1} \in U^m(\mathbb{Z}[1/d])$. Let $x = 1 - a$. Hence x is nilpotent, strictly upper triangular (that is $x_{ij} = 0$ if $i \geq j$), in particular, $x^n = 0$. Recall the identity

$$(1 - x)(1 + x + \cdots + x^{n-1}) = 1 - x^n.$$

We deduce that $a^{-1} = 1 + x + \cdots + x^{n-1}$. Hence

$$v_p((a^{-1})_{ij}) \geq \min\{v_p(1_{ij}), v_p(x_{ij}), \ldots, v_p((x^{n-1})_{ij})\}.$$

By hypothesis $v_p(x_{ij}) \geq (i - j)md_p$. Lemma 2.4 implies that $v_p((x^k)_{ij}) \geq (i - j)md_p$ holds for all $k \in \mathbb{N}$. We deduce that $a^{-1} \in U^m(\mathbb{Z}[1/d])$. If m is finite, it follows from Lemma 2.3 that the group $U^m(\mathbb{Z}[1/d])$ is a discrete subgroup of $U(n, \mathbb{R})$. In particular, it is finitely generated [22, Theorem 2.10]. □

Let $\mathbb{A}^N$ be the N-dimensional affine space over $\mathbb{Q}$. That is

$$\mathbb{A}^N = \{(x_1, \dots, x_N) : x_i \in \mathbb{Q}\}.$$

We denote by $\mathbb{A}^N\left(\frac{1}{d^m}\mathbb{Z}\right)$ the $\mathbb{Z}$-submodule of $\mathbb{A}^N$ defined as

$$\mathbb{A}^N\left(\frac{1}{d^m}\mathbb{Z}\right) = \left\{(x_1, \dots, x_N) : x_i \in \frac{1}{d^m}\mathbb{Z}\right\}.$$

Let $\mathbb{Q}[X_1, \dots, X_N]$ be the ring of polynomials in N variables with coefficients in $\mathbb{Q}$.

Definition 2.6 [26, I.2.2, I.2.3] *A map $f : \mathbb{A}^N \to \mathbb{A}^M$ is called regular if, for each coordinate function f_i, $1 \le i \le M$, there exists $P_i \in \mathbb{Q}[X_1, \dots, X_N]$ such that, for all $x \in \mathbb{A}^N$, $f_i(x) = P_i(x)$.*

Remark 2.7 We will only consider regular maps between affine spaces over $\mathbb{Q}$, hence f determines the P_i.

Definition 2.8 *The degree of f is defined as*

$$\deg(f) = \max_i \deg(P_i).$$

Lemma 2.9 *Let $f : \mathbb{A}^N \to \mathbb{Q}$ be regular. There exists a function with finite support $\mu : \mathcal{P} \to \mathbb{Q}$ on the set $\mathcal{P}$ of prime numbers, with the following property. Suppose $r \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{A}^N$ and a prime p are such that for $1 \le i \le N$, $v_p(x_i) \ge -r$. Then for $1 \le i \le N$, $v_p(f(x)) \ge \mu(p) - r\deg(f)$.*

Proof. Let $P \in \mathbb{Q}[X_1, \dots, X_N]$, such that for all $x \in \mathbb{A}^N$, $f(x) = P(x)$. We write P as

$$P(X_1, \dots, X_N) = \sum_{i_1, \dots, i_N} a_{i_1, \dots, i_N} X_1^{i_1} \cdots X_N^{i_N}$$

with $a_{i_1, \dots, i_N} \in \mathbb{Q}$. We define

$$\mu(p) = \min_{i_1, \dots, i_N} v_p(a_{i_1, \dots, i_N}).$$

Hence,

$$\begin{aligned} v_p(P(x_1, \dots, x_N)) &\ge \min_{i_1, \dots, i_N} v_p\left(a_{i_1, \dots, i_N} x_1^{i_1} \cdots x_N^{i_N}\right) \\ &\ge \min_{i_1, \dots, i_N} v_p(a_{i_1, \dots, i_N}) + \min_{i_1, \dots, i_N} (i_1 v_p(x_1) + \dots + i_N v_p(x_N)) \\ &\ge \mu(p) - r\deg(f). \end{aligned}$$

□

Proposition 2.10 *Let $f : \mathbb{A}^N \to \mathbb{A}^M$ be a regular map. There exist a constant $C > 0$ and an integer $d \in \mathbb{N}$ (both depending explicitly on f), such that for all $m \in \mathbb{N} \cup \{0\} \cup \{\infty\}$,*

$$f\left(\mathbb{A}^N\left(\frac{1}{d^m}\mathbb{Z}\right)\right) \subset \mathbb{A}^M\left(\frac{1}{d^{\deg(f)m+C}}\mathbb{Z}\right).$$

Proof. Let $P_j \in \mathbb{Q}[X_1, \dots, X_N]$ be the j-coordinate function of f. Let $\mu_j : \mathcal{P} \to \mathbb{Q}$ be defined as in Lemma 2.9. We choose d, such that for any prime p,

$$v_p(d) \geq \max_j |\mu_j(p)|.$$

We choose

$$C = \max_j \max_{p:\, d_p \neq 0} |\mu_j(p)|/d_p.$$

The proof follows easily from Lemma 2.9 by first considering the case when p is such that $d_p = 0$ (and therefore $\mu_j(p) = 0$). □

Corollary 2.11 *Let $f_i : \mathbb{A}^N \to \mathbb{A}^{M_i}$, $1 \leq i \leq k$ be a finite collection of regular maps. Let $D = \max_i \deg(f_i)$. There exist a constant $C > 0$ and an integer $d \in \mathbb{N}$ such that, for all $m \in \mathbb{N} \cup \{0\} \cup \{\infty\}$,*

$$f_i \left(\mathbb{A}^N \left(\frac{1}{d^m} \mathbb{Z} \right) \right) \subset \mathbb{A}^{M_i} \left(\frac{1}{d^{Dm+C}} \mathbb{Z} \right).$$

Proof. Let $M = \sum_i M_i$ and let

$$f : \mathbb{A}^N \to \mathbb{A}^M$$

be the regular map defined by $f = (f_1, \dots, f_k)$. We apply Proposition 2.10 to f. □

When we consider the property of being regular for maps with source and target spaces equal to the group $U(n, \mathbb{Q})$ or its Lie algebra $\mathfrak{u}(n, \mathbb{Q})$, we choose one identification (among the obvious $\frac{n(n-1)}{2}!$ possible ones) with the affine space $\mathbb{A}^{\frac{n(n-1)}{2}}$. Notice that for any of these identifications and for any $d \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\} \cup \{\infty\}$

$$\mathbb{A}^{\frac{n(n-1)}{2}} \left(\frac{1}{d^m} \mathbb{Z} \right) \subset U^m(\mathbb{Z}[1/d]). \tag{2.4}$$

3. Prüfer rank, torsion-free rank and Hirsch length

Definition 3.1 [24, Exercises 14.1, 3] *A group G has finite Prüfer rank r if every finitely generated subgroup of G contains a generating set of cardinality smaller or equal to r and r is the least such integer.*

In this paper, the rank of a group G, denoted by $rank(G)$, with no further specification, refers to the Prüfer rank of the group. The class of finite rank group is obviously closed with respect to forming subgroups, images and extensions.

For example, the additive group of the field of rational numbers, or the additive group $\mathbb{Z}[1/d]$ of rational numbers with denominators a power of $d \in \mathbb{N}$, have finite rank equal to one. The group $U(n, \mathbb{Q})$ has finite rank. A classical subsequence extraction procedure shows that the additive group of the ring $\mathbb{Z}_p$ of p-adic integers, although not countable, is of rank one.

Definition 3.2 [24, Exercises 14.1, 1] *A group G has finite torsion-free rank if it has a series of finite length whose factors are either torsion or infinite cyclic.*

Induction on the number of infinite cyclic factors shows that this number does not depend on the choice of the series. This number, denoted by $h(G)$ in this paper, is called the torsion-free rank or the Hirsch length of G.

The Prüfer rank of a non-trivial finite group is greater or equal to one, while its torsion-free rank is zero. The Prüfer rank and the torsion-free rank of $\mathbb{Q}$ or $\mathbb{Z}[1/d]$ are both equal to one.

Proposition 3.3 *The Prüfer rank of a strongly polycyclic group is bounded above by its torsion-free rank.*

Proof. If H is a subgroup of a strongly polycyclic group G and if G_i is a series for G with $G_i/G_{i+1} \simeq \mathbb{Z}$, then $H_i = H \cap G_i$ is a series for H and $H_i/H_{i+1} \subset G_i/G_{i+1}$. Hence H is generated by $h(G)$ elements. □

It seems that both ranks coincide within the class of strongly polycyclic groups.

Remark 3.4 When we refer to [22], note that in this reference the torsion-free rank of solvable groups is simply called the rank. See [22, 2.8, 2.9, 4.3] and recall, in order to check that the rank of a finitely generated nilpotent group as defined in [22, 2.8, 2.9] is equal to its torsion-free rank or Hirsch length, that a finitely generated nilpotent group has a finite (central) series whose factors are cyclic groups (with prime or infinite orders) [24, 5.2.18].

Proposition 3.5 *The Hirsch length does not increase when taking quotients.*

Proof. Let $f : G \to Q$ be an epimorphism between two groups. Suppose that G admits a series G_i of finite length with G_i/G_{i+1} cyclic or torsion. Then $Q_i = f(G_i)$ is a series of finite length for Q and f induces surjections $G_i/G_{i+1} \to Q_i/Q_{i+1}$. Hence, if G_i/G_{i+1} is cyclic or torsion, so is Q_i/Q_{i+1}. □

4. Nilpotent groups

The main result of this section is Theorem 4.8. It essentially follows from Mal'cev's theory. A celebrated theorem of Mal'cev states that a group is isomorphic to a lattice in a simply-connected nilpotent Lie group if and only if it is finitely generated torsion-free and nilpotent. Theorem 4.8 can be viewed as a generalization of this result to countable finite-rank torsion-free nilpotent groups.

4.1. Background and preliminaries

Proposition 4.1 *Let M be a nilpotent group. Then elements of finite order in M form a fully-invariant subgroup T, such that M/T is torsion-free and T is the internal direct product $Dr_p T_p$ where T_p is the unique maximal p-subgroup of M. In particular, if M is of finite rank, then T is finite.*

Proof. See [24, 5.2.7]. The fact that T is finite if M has finite rank follows from [24, 5.2.6, 4.2.1] because subgroups and quotient groups of finite rank groups have finite rank too. □

Lemma 4.2 *Let M be a group. Suppose we are given a sequence of subgroups M_i of M, $i \in \mathbb{N}$, such that $M_i \subset M_{i+1}$. Suppose that $M = \bigcup_i M_i$. Let G be a group and let*

$$f_i : M_i \to G$$

be a sequence of homomorphisms such that, for $i \leq j$, the restriction of f_j to M_i equals f_i. Then, the following holds.

1. *There exists a unique homomorphism*

$$f : M \to G$$

such that the restriction of f to M_i equals f_i.
2. *$Im(f) = \bigcup_i Im(f_i)$.*
3. *If each f_i is injective then f is injective.*

Proof. Let $x \in M$. We choose $i \in \mathbb{N}$ large enough such that $x \in M_i$. We define $f(x) = f_i(x)$. This definition is independent of the choice of i because if $x \in M_j \cap M_k$ then $f_j(x) = f_k(x)$. The Lemma follows obviously. (A more formal way of handling the situation described in this lemma would be to identify M as the direct limit associated with the system of subgroup M_i, see for example [24, 1.4.9].) □

Proposition 4.3 *1. A simply-connected nilpotent Lie group is torsion-free.*
2. A closed connected subgroup of a simply-connected nilpotent Lie group is simply-connected.

Proof. The exponential mapping of a simply-connected nilpotent Lie group is a global diffeomorphism [27, Theorem 3.6.2]. The proposition follows. □

Definition 4.4 *A rational structure on a real Lie algebra $\mathfrak{n}$ is a Lie algebra $\mathfrak{m}$ over $\mathbb{Q}$ such that $\mathfrak{m} \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathfrak{n}$*

If $\mathfrak{n}$ has a rational structure $\mathfrak{m}$, we identify $\mathfrak{n}$ with $\mathfrak{m} \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathfrak{m}$ with the subalgebra $\mathfrak{m} \otimes_{\mathbb{Q}} 1$.

Proposition 4.5 *Let N be a simply-connected nilpotent Lie group and let $\mathfrak{n}$ be its Lie algebra.*

1. *Let Γ be a discrete cocompact subgroup of N. The $\mathbb{Z}$-span of* $\exp^{-1}(\Gamma)$ *is a lattice of maximal rank in the vector space underlying $\mathfrak{n}$. Let $\mathfrak{m}_\Gamma$ be the $\mathbb{Q}$-span of* $\exp^{-1}(\Gamma)$. *The rational subspace $\mathfrak{m}_\Gamma$ has the structure of a Lie algebra over $\mathbb{Q}$ and $\mathfrak{m}_\Gamma \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathfrak{n}$. More precisely, the structural constants of the Lie algebra $\mathfrak{n}$ associated with any basis contained in the $\mathbb{Z}$-span of* $\exp^{-1}(\Gamma)$ *are rational.*

2. *Let $A \subset B$ be two discrete subgroups of N. If A is cocompact in N then $\mathfrak{m}_A = \mathfrak{m}_B$.*

Proof. For the first part, we refer to [22, 2.12]. For the second part, as B is obviously also cocompact, we deduce from the first part that the $\mathbb{Z}$-spans $\mathbb{Z} - \exp^{-1}(A) \subset \mathbb{Z} - \exp^{-1}(B)$ are two lattices of maximal rank in $\mathfrak{n}$. Hence $\mathfrak{m}_A = \mathfrak{m}_B$. □

Proposition 4.6 *Let $B \subset U(n, \mathbb{Q})$ be a discrete subgroup (for the topology induced by the natural embedding $U(n, \mathbb{Q}) \subset U(n, \mathbb{R})$). There exists a diagonal matrix $g \in GL(n, \mathbb{Q})$ such that $gBg^{-1} \subset U(n, \mathbb{Z})$.*

Proof. Let us explain how this statement follows from the proof of [22, Theorem 2.12]. Let N be the unique minimal closed connected subgroup of $U(n, \mathbb{R})$ containing B as a uniform lattice [22, Propostion 2.5]. It follows from 2) in Proposition 4.3 that N is simply-connected. Let $\mathfrak{n}$ be its Lie algebra. We denote by $\mathfrak{u}(n, \mathbb{R})$ and $\mathfrak{u}(n, \mathbb{Q})$ the Lie algebras of $U(n, \mathbb{R})$ and $U(n, \mathbb{Q})$. The exponential

$$\exp : \mathfrak{u}(n, \mathbb{Q}) \to U(n, \mathbb{Q})$$

and its inverse are regular maps. Hence, the first point in Proposition 4.5 implies that $\mathbb{Z} - \exp^{-1}(B)$ is a lattice of maximal rank in the vector space $\mathfrak{u}(n, \mathbb{Q}) \cap \mathfrak{n}$. Therefore, there exists $\lambda \in \mathbb{Z}$, such that $\mathbb{Z} - \exp^{-1}(B) \subset \lambda^{-1} M(n, \mathbb{Z}) \cap \mathfrak{u}(n, \mathbb{R})$. Let $g \in GL(n, \mathbb{Q})$ be the diagonal matrix defined by $g_{ii} = (n!\,\lambda)^{n-i}$. An elementary argument (see [22, 2.12, p. 35]) shows that $g \exp(\lambda^{-1} M(n, \mathbb{Z}) \cap \mathfrak{u}(n, \mathbb{R}))g^{-1}$ is included in $U(n, \mathbb{Z})$. □

Lemma 4.7 *Let N and N' be simply-connected nilpotent Lie groups.*

1. *Let M be a subgroup of N. Suppose M contains an increasing sequence of subgroups $M_i \subset M_{i+1}$, $i \in \mathbb{N}$, such that $M = \bigcup_i M_i$ and such that each M_i is discrete and cocompact in N. Let $h : M \to N'$ be a homomorphism. Then there is a unique differentiable homomorphism $\tilde{h} : N \to N'$ which extends h.*
2. *Let $A \subset N$ and $A' \subset N'$ be discrete cocompact subgroups. Let $h : A \to A'$ be a homomorphism. Then $T_e\tilde{h}(\mathfrak{m}_A) \subset \mathfrak{m}_{A'}$.*

Proof. We denote by h_i the restriction of h to M_i. As M_i is discrete, h_i is obviously continuous. Let $\tilde{h_i} : N \to N'$ be the unique continuous extension of h_i [22, Theorem 2.11]. Recall that a continuous homomorphism between Lie groups is differentiable (see for example [5, I.3.12]). For $i, j \in \mathbb{N}$, the restrictions of $\tilde{h_i}$ and $\tilde{h_j}$ to M_1 coincide, we deduce that $\tilde{h_i} = \tilde{h_j}$. For the second part, by definition of the exponential map, the diagram

$$\begin{array}{ccc} N & \xrightarrow{\tilde{h}} & N' \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{n} & \xrightarrow{T_e\tilde{h}} & \mathfrak{n}' \end{array}$$

is commutative [5, I.3.2] or [27, 2.10.3]. Hence

$$T_e\tilde{h} \exp^{-1}(A) = \exp^{-1} \tilde{h}(A) \subset \exp^{-1}(A').$$

We conclude that $T_e\tilde{h}(\mathfrak{m}_A) \subset \mathfrak{m}_{A'}$. □

4.2. The rational structure of a countable nilpotent group of finite rank

Theorem 4.8 *Let M be a group. The following two conditions are equivalent.*

1. *M is countable, of finite rank, torsion-free, nilpotent.*
2. *There exists a triple $(N, \mathfrak{m}, f)$ where N is a simply-connected nilpotent Lie group, $\mathfrak{m}$ is a rational structure on the Lie algebra $\mathfrak{n}$ of N and*

$$f : M \to N$$

is an injective homomorphism such that

(a) the image of f is included in $\exp(\mathfrak{m})$*,*
(b) the image of any finitely generated subgroup of M is discrete in N,
(c) there exists a finitely generated subgroup of M whose image under f is cocompact.

If $(N', \mathfrak{m}', f')$ is another triple as above, then there exists an isomorphism

$$\phi : N \to N'$$

such that $\phi f = f'$. In particular the derivative $T_e\phi$ of ϕ at the identity induces an isomorphism between the rational Lie algebras $\mathfrak{m}$ and $\mathfrak{m}'$.

Proof. Suppose there exists a triple $(N, \mathfrak{m}, f)$ for the group M. As M is isomorphic to a subgroup of the simply-connected nilpotent group N, we deduce that M is nilpotent. According to 1) in Proposition 4.3, it is torsion-free. As the group $\exp(\mathfrak{m})$ is countable, hypothesis (a) implies that M is countable. Let H be a finitely generated subgroup of M. Hypothesis (b) implies that its image $f(H)$ is discrete in N. Let us denote this image also by H. Let $\tilde{H}$ be the unique minimal connected closed subgroup of N containing H as a uniform lattice [22, Proposition 2.5]. The dimension of $\tilde{H}$ is equal to the Hirsch length of H [22, Theorem 2.10 and Remark 2.6] (about the terminology, see also Remark 3.4 above). As H is strongly polycyclic [22, 3.10], we apply Proposition 3.3 and obtain

$$\operatorname{rank}(H) \leq h(H) = \dim(\tilde{H}) \leq \dim(N).$$

This proves that $\operatorname{rank}(M) \leq \dim(N)$.

Now we assume that M has the properties described in 1) and we construct a triple $(N, \mathfrak{m}, f)$. Let r be the rank of M and let c be the nilpotent class of M. Let F be the free nilpotent group on r generators and of class c. We enumerate the elements of M as $x_1, \ldots, x_i, \ldots$ and we define M_i as the subgroup generated by $x_1, \ldots, x_i$. As $M_i \subset M$, the nilpotent class of M_i is less or equal to c and at most r elements are needed in order to generate M_i. Hence, for each i, there exists an epimorphism h_i from F onto M_i. According to Proposition 3.5, the Hirsch lengths satisfy $h(M_i) \leq h(F)$. According to Mal'cev, there exists a (unique) simply-connected nilpotent Lie group N_i containing M_i as a uniform lattice [22, Theorem 2.18, Theorem 2.1, 2–3]. Let us explain how the choice of an embedding of M_{i+1} as a lattice in N_{i+1} enables us to define an embedding of N_i as a closed subgroup of N_{i+1}. In other words, we have

$$M_i \subset M_{i+1} \subset N_{i+1},$$

with M_{i+1} discrete in N_{i+1} and we want to deduce from this data an embedding $N_i \subset N_{i+1}$. We denote by $\tilde{M}_i$ the unique minimal connected closed subgroup of N_{i+1} containing M_i as a uniform lattice, see [22, Propostion 2.5]. It follows from 2) in Proposition 4.3 that $\tilde{M}_i$ is simply-connected. Hence, both Lie groups N_i and $\tilde{M}_i$ are simply-connected nilpotent and contain an isomorphic copy of the group M_i as a uniform lattice. It follows that N_i and $\tilde{M}_i$ are isomorphic (see [22, Theorem 2.11 Corollary 2] or apply 1) of Lemma 4.7). Choosing an isomorphism, we obtain the wanted embedding $N_i \subset N_{i+1}$. The dimension of N_i is equal to the Hirsch length of M_i [22, Theorem 2.10] which is bounded by $h(F)$. This implies that the chain of inclusions $N_i \subset N_{i+1}$ stabilizes. Let n be such that $N_n = N_{n+1}$. We define the Lie group N as $N = N_n$. In order to distinguish explicitly the subgroup $M_n \subset M$ from its discrete cocompact isomorphic copy in N, we denote by

$$f_n : M_n \to N$$

the chosen discrete cocompact embedding. As mentioned in 1) of Proposition 4.5, the lattice $f_n(M_n) \subset N$ defines a rational structure $\mathfrak{m}_{f_n(M_n)}$ on the Lie algebra $\mathfrak{n}$ of N. We define

$$\mathfrak{m} = \mathfrak{m}_{f_n(M_n)}.$$

Recall that

$$M = \bigcup_{i \geq n} M_i$$

and that for $i \leq j$, $M_i \subset M_j$. We will apply Lemma 4.2 to obtain an embedding

$$f : M \to N.$$

For $n \leq i \leq j$, we denote

$$\alpha_i^j : M_i \to M_j$$

the inclusion. As a first step, we have by construction an embedding $f_n : M_n \to N$. Given a discrete embedding $f_i : M_i \to N$ with $i \geq n$, we will define a discrete embedding $f_{i+1} : M_{i+1} \to N$, such that

$$f_{i+1}\alpha_i^{i+1} = f_i.$$

Let $g_{i+1} : M_{i+1} \to N$ be an embedding of M_{i+1} as a uniform lattice in N. As

$$g_{i+1}\alpha_i^{i+1}(M_i) \simeq f_i(M_i)$$

are isomorphic uniform subgroups in N, there is an isomorphism $h_i : N \to N$ so that the following diagram commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{f_i} & N \\ \alpha_i^{i+1}\downarrow & & \uparrow h_i \\ M_{i+1} & \xrightarrow{g_{i+1}} & N. \end{array}$$

See [22, Theorem 2.11 Corollary 2] or apply 1) of Lemma 4.7. We define f_{i+1} as

$$f_{i+1} = h_i g_{i+1}.$$

Applying Lemma 4.2, we obtain an injective homomorphism

$$f : M \to N$$

such that, for $i \geq n$, the restriction of f to M_i equals f_i. If $B \subset M$ is a finitely generated subgroup, there exists i, such that $B \subset M_i$. Hence $f(B) \subset f(M_i) = f_i(M_i)$ is discrete. This proves (b). Point (c) is true because $f(M_n) = f_n(M_n)$ is cocompact in N.

According to 2) of Lemma 4.2, in order to show that $f(M) \subset \exp(\mathfrak{m})$, it is enough to show that for $i \geq n$, the inclusion $f_i(M_i) \subset \exp(\mathfrak{m})$ holds true. For $i \geq n$, the discrete subgroups $f_n(M_n) \subset f_i(M_i) \subset N$ are cocompact. Applying 2) of Proposition 4.5, we deduce that

$$\mathfrak{m}_{f_n(M_n)} = \mathfrak{m}_{f_i(M_i)}.$$

We conclude that $f_i(M_i) \subset \exp(\mathfrak{m})$.

Now we prove that the triple $(N, \mathfrak{m}, f)$ associated with M is essentially unique. Let $(N', \mathfrak{m}', f')$ be another such triple. We choose finitely generated subgroups H and H' in M, such that $f(H)$ is cocompact in N and such that $f'(H')$ is cocompact in N'. Let B be the subgroup of M generated by H and H'. It is finitely generated. The images $f(B)$ and $f'(B)$ are both discrete and cocompact in the respective Lie groups N and N'. We deduce from Lemma 4.7 the existence of an isomorphism

$$\phi : N \to N',$$

such that

$$\forall x \in B,\ \phi f(x) = f'(x)$$

and such that

$$T_e\phi(\mathfrak{m}) = \mathfrak{m}'.$$

This completes the proof of the theorem. □

Remark 4.9 Let M be as in Theorem 4.8. Let us choose a triple $(N, \mathfrak{m}, f)$ associated with M. Let $(\overline{f(M)})^\circ$ be the connected component of the identity of the closure of $f(M)$. This is a closed connected Lie subgroup of N. We define the "dense" part $D(M)$ of M as $f^{-1}(f(M) \cap (\overline{f(M)})^\circ)$. Uniqueness properties of the triple $(N, \mathfrak{m}, f)$ imply that the definition of $D(M)$ does not depend on the choice of f and N. Let us show that $D(M)$ is a fully-invariant subgroup of M. Let h be an endomorphism of M. According to Lemma 4.7 there is a unique continuous endomorphism $\tilde{h} : N \to N$, such that $\tilde{h} f = fh$. Hence,

$$\tilde{h}((\overline{f(M)})^\circ) \subset (\overline{f(h(M))})^\circ \subset (\overline{f(M)})^\circ.$$

We deduce that $h(D(M)) \subset D(M)$.

We guess that there is a purely algebraic definition of the fully-invariant subgroup $D(M)$ in terms of divisibility.

Definition 4.10 *Let M be a countable group of finite rank which is torsion-free and nilpotent. A subset K of M is called relatively compact according to a triple $(N, \mathfrak{m}, f)$ for M if $f(K)$ is relatively compact in N.*

Uniqueness properties of the triples associated with M imply that this definition does not depend on the choice of the triple.

Proposition 4.11 *Let M be a countable group of finite rank which is torsion-free and nilpotent. Let $B \subset M$ be a finitely generated subgroup. There exist $n \in \mathbb{N}$, (not depending on B) and a faithful representation ρ of M in $U(n, \mathbb{Q})$ with the following properties.*

1. $\rho(B) \subset U(n, \mathbb{Z})$.
2. *If $K \subset M$ is relatively compact, then $\rho(K)$ is relatively compact in $U(n, \mathbb{R})$.*
3. *If $h : M \to M$ is a homomorphism, there exists a linear map $L : \mathfrak{u}(n, \mathbb{Q}) \to \mathfrak{u}(n, \mathbb{Q})$, such that the regular map H uniquely defined by the commutative diagram*

$$\begin{array}{ccc} U(n,\mathbb{Q}) & \xrightarrow{H} & U(n,\mathbb{Q}) \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{u}(n,\mathbb{Q}) & \xrightarrow{L} & \mathfrak{u}(n,\mathbb{Q}) \end{array}$$

extends h in the sense that $H\rho = \rho h$.

Proof. Let $(N, \mathfrak{m}, f)$ be a triple associated with M, as in Theorem 4.8. The Theorems of Ado and Engel together establish the existence of an injective morphism of Lie algebras

$$\rho_0 : \mathfrak{m} \to \mathfrak{u}(\mathbb{Q})$$

[27, 3.17.7], [25, I.V.3.2]. Here and below, we write $\mathfrak{u}(\mathbb{Q})$ instead of $\mathfrak{u}(n, \mathbb{Q})$. The integer n is fixed. We also write $U(\mathbb{Q})$ instead of $U(n, \mathbb{Q})$ and $GL(\mathbb{Q})$ instead of $GL(n, \mathbb{Q})$ etc.. As the exponential map of a simply-connected Lie group is bijective, there is a unique homomorphism ρ_1 which makes the diagram

$$\begin{array}{ccc} \exp(\mathfrak{m}) & \xrightarrow{\rho_1} & U(\mathbb{Q}) \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{m} & \xrightarrow{\rho_0} & \mathfrak{u}(\mathbb{Q}) \end{array}$$

commute. We define $\rho = \rho_1 f$. Notice that if $E \subset \exp(\mathfrak{m})$ is a discrete subset, then $\rho_1(E) \subset U(\mathbb{Q})$ is also discrete. If $B \subset M$ is finitely generated, Theorem 4.8 implies that $f(B) \subset \exp(\mathfrak{m})$ is discrete. Hence $\rho(B) \subset U(\mathbb{Q})$ is discrete. Let $g \in GL(\mathbb{Q})$ be a diagonal matrix. The conjugation by g defines a automorphism of the Lie algebra $\mathfrak{u}(\mathbb{Q})$ as well as an automorphism of the group $U(\mathbb{Q})$ and makes the diagram

$$\begin{array}{ccc} U(\mathbb{Q}) & \xrightarrow{c_g} & U(\mathbb{Q}) \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{u}(\mathbb{Q}) & \xrightarrow{c_g} & \mathfrak{u}(\mathbb{Q}) \end{array}$$

commute. Hence, applying Proposition 4.6 and replacing ρ_0 with $c_g\rho_0$ in the above construction if needed, we obtain our faithful representation ρ with $\rho(B) \subset U(\mathbb{Z})$. In what follows we assume that ρ_0 has been chosen so that $\rho(B) \subset U(\mathbb{Z})$.

Let K be relatively compact in M. By definition, this means that $f(K)$ is relatively compact in N. The continuous extension of ρ_1 to N defines an embedding into $U(\mathbb{R})$. Hence $\rho(K) = \rho_1 f(K)$ is relatively compact in $U(\mathbb{R})$.

Let $h : M \to M$ be a homomorphism. According to 1) of Lemma 4.7 there is a unique continuous homomorphism $\tilde{h} : N \to N$, such that $\tilde{h} f = fh$. Moreover, according to 2) of Lemma 4.7, the derivative at the identity $T_e\tilde{h}$ preserves $\mathfrak{m}$. Let $V = \rho_0(\mathfrak{m})$ and let W be a complement for V in $\mathfrak{u}(\mathbb{Q})$. We denote by $\pi_V : \mathfrak{u}(\mathbb{Q}) \to V$ and $\pi_W : \mathfrak{u}(\mathbb{Q}) \to W$ the projections onto V and W. The map $L : \mathfrak{u}(\mathbb{Q}) \to \mathfrak{u}(\mathbb{Q})$ defined by

$$L(x) = \rho_0 T_e\tilde{h}\rho_0^{-1}\pi_V(x) + \pi_W(x)$$

is a morphism of vector spaces over $\mathbb{Q}$. On $U(\mathbb{Q})$ the exponential and its inverse are regular maps, hence the unique map H which makes the diagram

$$\begin{array}{ccc} U(\mathbb{Q}) & \xrightarrow{H} & U(\mathbb{Q}) \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{u}(\mathbb{Q}) & \xrightarrow{L} & \mathfrak{u}(\mathbb{Q}) \end{array}$$

commute is also regular. We have

$$\begin{aligned} H\rho &= H\rho_1 f = H \exp \rho_0 \exp^{-1} f = \exp L\rho_0 \exp^{-1} f = \\ &= \exp \rho_0 T_e\tilde{h}\rho_0^{-1}\rho_0 \exp^{-1} f = \exp \rho_0 T_e\tilde{h} \exp^{-1} f = \\ &= \rho_1 \exp T_e\tilde{h} \exp^{-1} f = \rho_1\tilde{h} f = \rho_1 fh = \rho h. \end{aligned}$$

□

5. Left-invariant metrics on groups

Let G be a finitely generated group and let $S \subset G$ be a finite symmetric (i.e. $S = S^{-1}$) generating set of G. The norm $|x|_S$ of $x \in G$ associated with S is by definition the minimal number $n \in \mathbb{N} \cup \{0\}$ for which there exist $s_1, \dots, s_n \in S$ with $x = s_1 \cdots s_n$.

Definition 5.1 *The word metric on G associated with S is the left-invariant distance function on $G \times G$ defined as $d_S(x, y) = |x^{-1}y|_S$.*

Proposition 5.2 [7, IVB.23] *Let (X, x_0, d) be a pointed metric space which is geodesic and proper. Let G be a group acting on X by isometries. Assume the action is proper and the quotient $G\backslash X$ is compact. Then G is finitely generated and for any generating set S as above, there exists a constant C, such that*

$$\forall g, h \in G,\ d(gx_0, hx_0)/C \le d_S(g, h) \le Cd((gx_0, hx_0) + C.$$

Proposition 5.3 *Let G and H be connected Lie groups with left-invariant Riemannian metrics. Let $h : G \to H$ be a continuous homomorphism. Then there is a constant C such that*

$$\forall x, y \in G,\ d(h(x), h(y)) \leq C d(x, y).$$

Proof. We denote by l_g the left-translation by g. The diagram

$$\begin{array}{ccc} G & \stackrel{h}{\to} & H \\ {\scriptstyle l_g}\uparrow & & \uparrow {\scriptstyle l_{h(g)}} \\ G & \stackrel{h}{\to} & H \end{array}$$

commutes. Taking derivatives at the identity, we obtain the corresponding commutative diagram between tangent spaces

$$\begin{array}{ccc} T_e G & \stackrel{T_g h}{\to} & T_{h(g)} H \\ {\scriptstyle T_e l_g}\uparrow & & \uparrow {\scriptstyle T_e l_{h(g)}} \\ T_e G & \stackrel{T_e h}{\to} & T_e H. \end{array}$$

As the Riemannian metrics are left-invariant, the vertical arrows denote isometries. We conclude that h is Lipschitz with constant

$$C(h) = \sup_{|v|_e = 1} |T_e h(v)|_e = \sup_{|v|_g = 1} |T_g h(v)|_{h(g)}.$$

□

Corollary 5.4 *Let F be the free group on the letters $a_1, \ldots, a_n$. We denote by $|w|$ the length of a reduced word $w \in F$. Let G be connected Lie group with a left-invariant Riemannian metric. Let $\rho : F \to Aut(G)$ be a homomorphism. Let $C = \max_i\{C(\rho(a_i)), C(\rho(a_i^{-1}))\}$. Then*

$$\forall x \in G,\ d(e, \rho(w)(x)) \leq C^{|w|} d(e, x).$$

6. Følner couples and lower bounds for the heat decay

We recall from [6] the minimal machinery needed for our purpose.

Let $\mathcal{V}$ be a positive continuous increasing function on $[1, +\infty)$ whose inverse $\mathcal{V}^{-1}$ is defined on $[\mathcal{V}(1), +\infty)$.

Definition 6.1 [6, Definition 4.7] *We say that a finitely generated group Γ with word metric d_S admits a sequence of Følner couples adapted to $\mathcal{V}$ if there exist two constants $C > 1$ and $c > 0$ and a sequence $\{(\Omega'_n, \Omega_n)\}_{n \in \mathbb{N}}$ of pairs of non-empty finite sets $\Omega'_n \subset \Omega_n$ in Γ with the following properties.*

1. *The cardinals $v_n = |\Omega_n|$ satisfy $\lim_{n\to\infty} v_n = \infty$.*
2. *$c v_n \leq |\Omega'_n|$.*
3. *$v_n \leq \mathcal{V}(n)$.*
4. *$\mathcal{V}^{-1}(v_{n+1}) \leq C \mathcal{V}^{-1}(v_n)$.*
5. *$d_S(\Omega'_n, \Gamma \setminus \Omega_n) \geq cn$.*

Any individual pair (Ω_n, Ω_n') *from this sequence is called a Følner couple. The function* $\mathcal{V}$ *is called a Følner volume function.*

We denote by $\mathcal{L}f = \frac{f'}{f}$ the logarithmic derivative. Hence $\mathcal{L}^2 f = \frac{f''}{f'} - \frac{f'}{f}$.

Theorem 6.2 [6, Theorem 4.8] *Assume that* Γ *admits a sequence of Følner couples adapted to a function* $\mathcal{V}$*. Assume also that* $\mathcal{V} \in C^2(1, +\infty)$*,* $\mathcal{V}' > 0$*, and*

$$-\frac{1}{s} \leq \mathcal{L}^2\mathcal{V}(s) \leq \frac{C}{s},$$

for all large enough positive s*. Let* γ *be the function defined by the equality*

$$t = \int_{\mathcal{V}(1)}^{\gamma(t)} \left[\mathcal{V}^{-1}(v)\right]^2 \frac{dv}{v}.$$

Then, there exist constants $C > 1$ *and* $c > 0$*, such that for all large enough even integers* $2t$*,*

$$p_{2t}(e, e) \geq \frac{c}{\gamma(Ct)}.$$

Corollary 6.3 [6, Example 4.1] *If a finitely generated group admits a sequence of Følner couples adapted to the function* $\mathcal{V}(s) = \exp(Cs)$ *for some constant* $C > 0$*, then, up to multiplicative constants, its heat decay on the diagonal is bounded below, for large enough even times, by* $\exp(-t^{1/3})$.

Proof. For the function $\mathcal{V}(s) = \exp(Cs)$, whose logarithmic derivative is C, the integral in the theorem is easily computed because

$$\int \frac{\log(x)^2}{x} dx = \frac{1}{3}\log(x)^3.$$

□

7. Proof of the main theorem

7.1. Reduction steps

As the heat decay is stable when taking a finite index subgroup and when taking the quotient by a finite normal subgroup (see Subsect. 1.2 above), the first step in proving Theorem 1.1 is to apply these transformations in order to simplify the structure of the group in consideration. Let Γ be a finitely generated solvable group of finite rank. According to a theorem of Mal'cev, see [23, Theorem 3.5 p.79], there is an exact sequence

$$1 \to M \to \Gamma \to Q \to 1$$

with M nilpotent and Q abelian-by-finite. As Q is finitely generated, we can choose a finite index subgroup A in Q which is free abelian of finite rank. Let $p : \Gamma \to Q$

be the projection. We obtain an exact sequence

$$1 \to M \cap p^{-1}(A) \to p^{-1}(A) \to A \to 1$$

with $p^{-1}(A)$ of finite index in Γ. According to Proposition 4.1, the torsion subgroup T of $M \cap p^{-1}(A)$ is fully-invariant and finite. We obtain the exact sequence

$$1 \to (M \cap p^{-1}(A))/T \to p^{-1}(A)/T \to A \to 1.$$

We conclude that, up to taking a finite index subgroup and up to taking a quotient by a finite subgroup, we can assume that our finitely generated solvable group of finite rank Γ fits into an exact sequence

$$1 \to M \to \Gamma \to A \to 1$$

with M nilpotent torsion-free and A free abelian of rank d.

7.2. *A direct limit of nilpotent lattices*

The torsion-free nilpotent group M is a subgroup of a finitely generated group of finite rank, hence it is also countable and of finite rank. Let $(N, \mathfrak{m}, f)$ be a triple associated with M as in Theorem 4.8. We choose $a_1, \dots, a_d \in \Gamma$, such that $p(a_1), \dots, p(a_d)$ form a basis in $A \cong \mathbb{Z}^d$. According to (c) in Theorem 4.8, we can choose a finite symmetric set X_1 in M, such that the the image under f of the smallest subgroup containing X_1 is cocompact in N. Let X_2 be a symmetric finite subset of M, such that the smallest subgroup of Γ which contains X_2 and $a_1, \dots, a_d$ is the whole group Γ. Let

$$X_3 = \left\{\left[a_i^{\epsilon_i}, a_j^{\epsilon_j}\right],\ 1 \le i, j \le d,\ \epsilon_i, \epsilon_j \in \{1, -1\}\right\}.$$

Notice that $X_3 \subset M$. We define M_0 as the subgroup of M generated by $X = X_1 \cup X_2 \cup X_3$.

Let F be the free group on the letters $A_1, \dots, A_d$. Let $c : F \to Aut(M)$ be the unique homomorphism which extends the map $A_i \mapsto c_{a_i}$ where

$$\forall x \in M,\ c_{a_i}(x) = a_i x a_i^{-1}. \tag{7.5}$$

If $w \in F$ is a reduced word, we denote its image under c by c_w. Let M_n be the subgroup of M generated by the set

$$\bigcup_{0 \le |w| \le n} c_w(M_0).$$

As $X_2 \subset M_0$ we have

$$M = \bigcup_{n \in \mathbb{N} \cup \{0\}} M_n.$$

Lemma 7.1 *Let* $w \in F$, $n \in \mathbb{N} \cup \{0\}$, $1 \le i \le d$, $\epsilon \in \{1; -1\}$. *We have*

1. $c_w(M_n) \subset M_{n+|w|}$.
2. $c_w(a_i^{\epsilon})a_i^{-\epsilon} \in M_{|w|}$.

Proof. Point 1) follows directly from the definition. For 2), we proceed by induction on $|w|$. If $|w| = 1$ the statement is true because

$$\left[a_i^{\epsilon_i}, a_j^{\epsilon_j}\right] \in M_0 \subset M_1.$$

Suppose the statement is true for w. Let $s \in F$, such that $|s| = 1$. Let us write a instead of a_i^{ϵ}. We have

$$c_{ws}(a)a^{-1} = c_w(c_s(a)a^{-1})c_w(a)a^{-1}.$$

As $c_s(a)a^{-1} \in M_0$, the first point in the lemma implies that $c_w(c_s(a)a^{-1}) \in M_{|w|}$. The induction hypothesis implies that $c_w(a)a^{-1} \in M_{|w|}$. We conclude that

$$c_{ws}(a)a^{-1} \in M_{|w|} \subset M_{|w|+1}.$$

□

From now on, in order to simplify notation, *we identify M with its image f(M) in N*. With this convention, M_0 is a lattice in N. Let us choose a left-invariant Riemannian metric on N. We denote by

$$B_e(R) = \{x \in N : \ d(e, x) \le R\}$$

the Riemannian ball in N with center the identity element e and with radius R. As the set X is finite, we can choose the Riemannian metric such that $X \subset B_e(1)$.

We arrive at a geometric version of the previous lemma.

Lemma 7.2 *There exists* $\lambda \ge 2$, *such that for* $w \in F$, $n \in \mathbb{N} \cup \{0\}$, $1 \le i \le d$, $\epsilon \in \{1; -1\}$ *the following hold.*

1. $c_w(B_e(\lambda^n)) \subset B_e(\lambda^{n+|w|})$, *in particular if* $s \in X$ *then* $c_w(s) \in B_e(\lambda^{|w|})$.
2. $c_w(a_i^{\epsilon})a_i^{-\epsilon} \in B_e(\lambda^{|w|})$.

Proof. Point 1) follows from Corollary 5.4. For 2), we proceed by induction on $|w|$. If $|w| = 1$ the statement is true because

$$\left[a_i^{\epsilon_i}, a_j^{\epsilon_j}\right] \in X_3 \subset B_e(1).$$

Suppose the statement is true for w. Let $s \in F$ such that $|s| = 1$. Again, we write a instead of a_i^{ϵ} and we will use the identity

$$c_{ws}(a)a^{-1} = c_w(c_s(a)a^{-1})c_w(a)a^{-1}.$$

As $c_s(a)a^{-1} \in B_e(1)$, the first point in the lemma implies that $c_w(c_s(a)a^{-1}) \in B_e(\lambda^{|w|})$. The induction hypothesis implies that $c_w(a)a^{-1} \in B_e(\lambda^{|w|})$. We conclude that

$$c_{ws}(a)a^{-1} \in B_e(2\lambda^{|w|}) \subset B_e(\lambda^{|w|+1}).$$

□

We denote by $|g|$ or $d_X(e, g)$ the word norm of the element $g \in M_0$ with respect to the generating set X. Let $D \subset N$ be a bounded fundamental domain for M_0 which contains the identity. Hence,

$$N = \bigsqcup_{g \in M_0} gD.$$

We define for each radius $R \geq 0$ the set

$$E(R) = \bigcup_{g \in M_0,\ |g| \leq R} gD.$$

Lemma 7.3 *There exists a constant $C_0 > 1$, such that the following inclusions hold.*

1. $\forall R \geq 1,\ B_e(R) \subset E(C_0 R)$.
2. $\forall R \geq 0,\ \forall R' \geq 1,\ E(R)B_e(R') \subset E(R + C_0 R')$.

Proof. Let $\delta = \sup_{x,y \in D} d(x, y)$. We claim that

$$B_e(R) \subset \bigcup_{g \in M_0,\ d(e,g) \leq R+\delta} gD.$$

Indeed, if $x \in B_e(R)$ let g be the unique element of M_0, such that $x \in gD$. As $e \in D$, $d(g, x) \leq \delta$ hence $d(e, g) \leq R + \delta$. This proves the claim. As the action of M_0 on N is proper and cocompact by isometries, we can apply Proposition 5.2 to deduce the existence of a constant $C_0 > 1$, such that

$$\forall R \geq 1,\ \forall g \in M_0,\ d(e, g) \leq R + \delta \Rightarrow |g| \leq C_0 R.$$

To prove 2), let $x \in E(R)$ and $x' \in B_e(R')$ and let $y = xx'$. Let g_x and g_y be the unique elements of M_0, such that $x \in g_x D$ and $y \in g_y D$. We have

$$d(g_x, g_y) \leq d(g_x, x) + d(x, y) + d(y, g_y) \leq 2\delta + R'.$$

Hence, applying Proposition 5.2, we deduce the existence of a constant $C_0 > 1$, not depending on $R' \geq 1$, such that $d_X(g_x, g_y) \leq C_0 R'$. This proves that $y \in E(R + C_0 R')$ because

$$d_X(e, g_y) \leq d_X(e, g_x) + d_X(g_x, g_y) \leq R + C_0 R'.$$

□

For $n \in \mathbb{N} \cup \{0\}$, we define

$$D_n = D \cap M_n.$$

Lemma 7.4 *For any subset $A \subset M_0$ and any $n \in \mathbb{N} \cup \{0\}$,*

$$\bigcup_{g \in A} gD_n = \Big(\bigcup_{g \in A} gD\Big) \cap M_n.$$

Proof. As M_n is a group and as $g \in M_0 \subset M_n$, we obtain $gM_n = M_n$. The lemma follows from obvious set-theoretical equalities. □

Lemma 7.5 *There exists a constant* $\mu > 1$*, such that*

$$\forall n \in \mathbb{N} \cup \{0\},\ |D_n| \leq \mu^n.$$

Proof. Let ρ be a faithful representation of M into $U(\mathbb{Q})$ as in Proposition 4.11 chosen such that $\rho(M_0) \subset U(\mathbb{Z})$. Let A^* be the free semi-group on the set

$$S = \left\{A_1, A_1^{-1}, \dots, A_d, A_d^{-1}\right\}.$$

Let $End(\mathfrak{u}(\mathbb{Q}))$ be the algebra of linear maps from $\mathfrak{u}(\mathbb{Q})$ into itself and let $End(U(\mathbb{Q}))$, be the algebra of rational maps from $U(\mathbb{Q})$ into itself. (Here an endomorphism $f : U(\mathbb{Q}) \to U(\mathbb{Q})$ is a regular map from the affine space $U(\mathbb{Q}) = U(n, \mathbb{Q}) \simeq \mathbb{Q}^{\frac{n(n-1)}{2}}$ into itself. It is not a group homomorphism from the unipotent group $U(\mathbb{Q})$ into itself.) In particular, $End(\mathfrak{u}(\mathbb{Q}))$ and $End(U(\mathbb{Q}))$ are semi-groups for the composition of maps. For each $w \in S$, we apply Proposition 4.11 in order to obtain $L_w \in End(\mathfrak{u}(\mathbb{Q}))$ and the corresponding extension $H_w \in End(U(\mathbb{Q}))$ of the conjugation automorphism c_w of M defined as in (7.5) (to save notation, when we write c_w, we make no explicit distinction between a word w in the semi-group A^* and its canonical image in F; hence, according to the context, we regard $w \mapsto c_w$ as a morphism of semi-groups from A^* to $Aut(M)$ or as a homomorphism from F to $Aut(M)$). Let

$$L : A^* \to End(\mathfrak{u}(\mathbb{Q}))$$

and

$$H : A^* \to End(U(\mathbb{Q}))$$

be the morphisms of semi-groups which extend the maps $w \mapsto L_w$ and $w \mapsto H_w$, with source equals to the generating set S. If $w \in A^*$ we denote by L_w its image under L and H_w its image under H. We denote by $|w|$ the number of letters of a word $w \in A^*$. By definition, if $w \in S$, i.e. if $|w| = 1$, the diagrams

$$\begin{array}{ccc} U(\mathbb{Q}) & \overset{H_w}{\to} & U(\mathbb{Q}) \\ \exp\uparrow & & \uparrow\exp \\ \mathfrak{u}(\mathbb{Q}) & \overset{L_w}{\to} & \mathfrak{u}(\mathbb{Q}) \end{array}$$

and

$$\begin{array}{ccc} U(\mathbb{Q}) & \overset{H_w}{\to} & U(\mathbb{Q}) \\ \rho\uparrow & & \uparrow\rho \\ \mathfrak{u}(\mathbb{Q}) & \overset{c_w}{\to} & \mathfrak{u}(\mathbb{Q}) \end{array}$$

commute. As L, H and c are morphisms of semi-groups, induction on $|w|$ shows that the diagrams commute for all $w \in A^*$. If E is a subset of a group G we denote by $\langle E \rangle$ the smallest subgroup of G containing E. For $n \in \mathbb{N} \cup \{0\}$ we have

$$\rho(M_n) = \rho\Big\langle \bigcup_{0\le|w|\le n} c_w(M_0)\Big\rangle = \Big\langle \bigcup_{0\le|w|\le n} \rho c_w(M_0)\Big\rangle =$$
$$\Big\langle \bigcup_{0\le|w|\le n} H_w \rho(M_0)\Big\rangle \subset \Big\langle \bigcup_{0\le|w|\le n} H_w(U(\mathbb{Z}))\Big\rangle.$$

At this point, our aim is to apply Corollary 2.11 to the family $\mathcal{F}$ of regular maps which consists in the linear maps $L_w : \mathfrak{u}(\mathbb{Q}) \to \mathfrak{u}(\mathbb{Q})$ with $w \in S$ as well as in $\exp : \mathfrak{u}(\mathbb{Q}) \to U(\mathbb{Q})$ and its inverse $\exp^{-1}$. With the same notation as in Corollary 2.11, let $d \in \mathbb{N}$ and $C > 0$, such that $\forall m \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, if $f \in \mathcal{F}$ then

$$f\left(\mathbb{A}^N\left(\frac{1}{d^m}\mathbb{Z}\right)\right) \subset \mathbb{A}^N\left(\frac{1}{d^{Dm+C}}\mathbb{Z}\right).$$

(Here $N = \dim(U(\mathbb{Q})) = \dim(\mathfrak{u}(\mathbb{Q}))$ and we keep n for the index in M_n). As already mentioned, for any $w \in A^*$, $H_w = \exp L_w \exp^{-1}$. As L_w is linear, we come to the obvious but crucial conclusion that

$$\deg(L_w) \le 1.$$

Also, as the group is nilpotent, $\deg(\exp) < \infty$. It implies that if $|w| \le n$ then

$$H_w(\mathbb{A}^N(\mathbb{Z})) \subset \mathbb{A}^N\left(\frac{1}{d^{C\deg(\exp)n+C}}\mathbb{Z}\right).$$

Remembering the identification we made between $\mathbb{A}^N$ and $U(\mathbb{Q})$ and (2.4), we deduce that for $|w| \le n$,

$$H_w(U(\mathbb{Z})) \subset U^{C\deg(\exp)n+C}(\mathbb{Z}[1/d]).$$

According to Proposition 2.5, the set $U^{C\deg(\exp)n+C}(\mathbb{Z}[1/d])$ is a group. We deduce that

$$\rho(D_n) = \rho(D \cap M_n) \subset \rho(M_n) \subset$$
$$U^{C\deg(\exp)n+C}(\mathbb{Z}[1/d]).$$

On the other hand, according to Definition 4.10, the subset $M \cap D$ of M is relatively compact because D is bounded in N (the embedding f does not appear here because of our convention of identifying M with its image in N). Proposition 4.11 states that $\rho(M \cap D)$ is relatively compact in $U(\mathbb{R})$. Lemma 2.3 enables us to conclude the proof. $\square$

7.3. Construction of Følner exhaustions

Let $\beta > 1$, such that

$$\beta \geq C_0 \lambda \tag{7.6}$$

where C_0 is the constant of Lemma 7.3 and λ is the constant of Lemma 7.2. Let $\alpha > 1$, such that for all $n \in \mathbb{N}$,

$$\alpha^n \geq 2n\beta^n. \tag{7.7}$$

Each element of Γ can be written in a unique way xa where $x \in M$ and $a = a_1^{k_1} \cdots a_d^{k_d}$ with $k_1, \dots, k_d \in \mathbb{Z}$. We denote by $|a| = \sum |k_i|$ the l^1-norm of the projection of a onto $\mathbb{Z}^d$. Let $n \in \mathbb{N}$ and $m \in \mathbb{R}$, such that $n > m \geq 0$. We are ready for the definition of the Følner exhaustions (which will enable us to obtain Følner couples). We define

$$\Omega_{n,m} = \{xa : x \in E(\alpha^n - m\beta^n) \cap M_n,\ |a| \leq n - m\}.$$

We have the following estimates for the cardinality of $\Omega_{n,m}$.

Lemma 7.6 *There exist $r \in \mathbb{N} \cup \{0\}$ and two constants $C > 1,\ c > 0$, such that*

$$c(n-m)^d(\alpha^n - m\beta^n)^r |D_n| \leq |\Omega_{n,m}| \leq C(n-m)^d(\alpha^n - m\beta^n)^r |D_n|.$$

Moreover, if $0 < \epsilon \leq 1/2$ and m, n satisfy $m \leq \epsilon n$ then

$$cn^d\alpha^{nr}|D_n| \leq |\Omega_{n,m}| \leq Cn^d\alpha^{nr}|D_n|.$$

Proof. The finitely generated group M_0 is nilpotent. Hence $|\{g \in M_0 : |g| \leq k\}|$ behaves, up to multiplicative constants, like a polynomial in k of a certain degree r. See [2] and [8]. The first point in the lemma follows from this fact and from Lemma 7.4. The second statement is true for the same reasons and because (7.7) implies

$$\alpha^n - m\beta^n \geq \alpha^n - n\beta^n \geq \alpha^n/2.$$

□

Recall that we have chosen X so that

$$S = X \cup \left\{a_1, a_1^{-1}, \dots, a_d, a_d^{-1}\right\}$$

generates Γ.

Lemma 7.7 *Let $s \in S$. Let $1 \leq m < n$. If $\gamma \in \Omega_{n,m}$ then $\gamma s \in \Omega_{n,m-1}$.*

Proof. Let $x \in M$ and $a = a_1^{k_1} \cdots a_d^{k_d}$ with $k_1, \dots, k_d \in \mathbb{Z}$ uniquely defined by the equation

$$\gamma = xa.$$

We consider two cases.

The first case is when $s \in X$. We have $\gamma s = xas = x(asa^{-1})a$. Hence to show that $\gamma s \in \Omega_{n,m-1}$, we have to show that

$$xasa^{-1} \in E(\alpha^n - (m-1)\beta^n) \cap M_n.$$

Applying 1) of Lemma 7.1, we deduce that $asa^{-1} \in M_n$. By hypothesis, x is also in the subgroup M_n. This shows that $xasa^{-1} \in M_n$. According to 1) of Lemma 7.2, $asa^{-1} \in B_e(\lambda^n)$. By hypothesis, $x \in E(\alpha^n - m\beta^n)$. According to 2) of Lemma 7.3 and according to (7.6),

$$E(\alpha^n - m\beta^n)B_e(\lambda^n) \subset E(\alpha^n - m\beta^n + C_0\lambda^n) \subset E(\alpha^n - (m-1)\beta^n).$$

This concludes the proof in the first case.

For the second case, let $s = a_i^{\epsilon}$, where $1 \leq i \leq d$ and $\epsilon \in \{1; -1\}$. If $d = 1$ or if $i = d$, the proof is trivial. (Notice that if the exact sequence splits, that is if Γ is a semi-direct product of M and A, then we can choose the a_i so that they commute together. Hence in this case the proof is also trivial.) We assume $1 \leq i < d$ and we write

$$a = uv$$

where $u = a_1^{k_1} \cdots a_i^{k_i}$ and $v = a_{i+1}^{k_{i+1}} \cdots a_d^{k_d}$. We have

$$\gamma s = xas = xuvs = xu[v,s]sv =$$
$$xu[v,s]u^{-1}a_1^{k_1} \cdots a_i^{k_i+\epsilon} a_{i+1}^{k_{i+1}} \cdots a_d^{k_d}.$$

Hence we have to show that

$$xu[v,s]u^{-1} \in E(\alpha^n - (m-1)\beta^n) \cap M_n.$$

We proceed almost as in the first case. Applying 2) of Lemma 7.1, we deduce that $[v,s] \in M_{|v|}$. Applying 1) of the same lemma we deduce that $u[v,s]u^{-1} \in M_{|u|+|v|} \subset M_n$. For the geometric part, applying 2) of Lemma 7.2, we deduce that $[v,s] \in B_e(\lambda^{|v|})$. Applying 1) of Lemma 7.2, we deduce that

$$u[v,s]u^{-1} \in B_e(\lambda^{|v|+|u|}) \subset B_e(\lambda^n).$$

We conclude as in the first case. □

7.4. The Følner couples

Proposition 7.8 *There is a constant $K > 1$, such that the sets*

$$(\Omega_n, \Omega'_n)_{n\in\mathbb{N}} = (\Omega_{n,0}, \Omega_{n,n/2})_{n\in\mathbb{N}}$$

form a sequence of Følner couples in Γ with Følner volume function $\mathcal{V}(s) =$ $\exp(Ks)$.

Proof. Conditions 1), 2) and 4) of Definition 6.1 follow from Lemma 7.6. Condition 3) follows from Lemmas 7.6 and 7.5. Condition 5) follows from Lemma 7.7. □

Theorem 1.1 follows from the above proposition and Corollary 6.3.

8. Questions and speculations

1. Let Γ be a finitely generated torsion-free solvable group which is not virtually nilpotent. Is it true that Γ has finite rank if and only if its heat decay satisfies $p_{2t} \sim \exp(-t^{1/3})$? If the derived length is two, the following holds.

   Theorem 8.1 *Let G be a finitely generated metabelian group without torsion and of exponential growth. The following conditions are equivalent.*
   (a) The group G has finite rank.
   (b) The heat decay of G satisfies $p_{2t} \sim \exp(-t^{1/3})$.
   (c) The group G does not contain a subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.
   (d) The heat decay of G is strictly slower than $\exp(-t^{1/3}(\log t)^{2/3})$.

   In the case $G_{ab} = G/[G, G]$ is (infinite) cyclic, the above statement follows from [18]. If $G_{ab} \simeq \mathbb{Z}^k$ with $k > 1$, the exact sequence

   $$1 \to [G, G] \to G \to \mathbb{Z}^k \to 1$$

   does not split in general but, as $[G, G]$ is abelian, $[G, G] \otimes \mathbb{Q}$ is nevertheless a finitely generated module over $\mathbb{Q}[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$ and the technics of [18] still apply.
2. (Alexopoulos) Does the (asymptotic) entropy of the density associated with the Markov kernel p vanishes on a finitely generated solvable group of finite rank? See [1, 2].
3. Does the isoperimetric profile of a finitely generated solvable group of finite rank satisfy $I(n) \precsim n/\log(n)$? See [17].
4. A polycyclic group has a finite index subgroup which is a uniform lattice in a simply-connected solvable Lie group. Hence, the possible heat decays for polycyclic groups can be deduced from the knowledge of the heat decays for Lie groups (see Sect. 1.3) and from (1.1). Does this strategy generalize if one replaces polycyclic groups with finitely generated solvable groups of finite rank and Lie groups with analytic groups? This leads to the following considerations.
   (a) Let Γ be a finitely generated solvable group of finite rank. There is a finite index subgroup H in Γ and a locally compact compactly generated unimodular solvable analytic group G, such that H embeds as a discrete subgroup in G.
   (b) Let Γ be a finitely generated discrete subgroup of a locally compact compactly generated analytic group G. Is it true that $p_{2t}^{\Gamma}(e, e) \succsim p_t^G(e, e)$?
   (c) What are the possible heat decays of locally compact compactly generated analytic groups? One of the simplest interesting cases is the following. Let p be a prime. The formula

   $$\lambda(x, y) = (\lambda x, \lambda^{-1} y),$$

   for $\lambda \in \mathbb{Q}_p^*$ and $x, y \in \mathbb{Q}_p$, defines an action of the multiplicative group $\mathbb{Q}_p^*$ of the field of p-adic numbers on the additive group $\mathbb{Q}_p^2$. Let $Sol(\mathbb{Q}_p)$ be the corresponding semi-direct product. This is a locally compact compactly generated solvable unimodular analytic group over $\mathbb{Q}_p$. We guess that the heat decay on $Sol(\mathbb{Q}_p)$ behaves like $\exp(-t^{1/3})$.

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