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Remark on some conformally invariant integral equations: the method of moving spheres

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1. Introduction

For $n \geq 3$, consider

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{R}^n. \tag{1}$$

It was proved by Gidas, Ni and Nirenberg [21] that any positive C^2 solution of (1) satisfying

$$\liminf_{|x|\to\infty} |x|^{n-2}u(x) < \infty, \tag{2}$$

must be of the form

$$u(x) \equiv \left(\frac{a}{1 + a^2|x - \bar{x}|^2}\right)^{(n-2)/2},$$

where $a > 0$ is some constant and $\bar{x} \in \mathbb{R}^n$.

Hypothesis (2) was removed by Caffarelli, Gidas and Spruck in [8]; this is important for applications. Such Liouville type theorems have been extended to general conformally invariant fully nonlinear equations by Li and Li ([24]–[27]); see also related works of Viaclovsky ([40]–[41]) and Chang, Gursky and Yang ([13]–[14]). The method used in [21], as well as in much of the above cited work, is the method of moving planes. The method of moving planes has become a very powerful tool in the study of nonlinear elliptic equations; see Aleksandrov [1], Serrin [38], Gidas, Ni and Nirenberg [21]–[22], Berestycki and Nirenberg [2], and others.

In [30], Li and Zhu gave a proof of the above mentioned theorem of Caffarelli, Gidas and Spruck using the method of moving spheres (i.e. the method of moving planes together with the conformal invariance), which fully exploits the conformal invariance of the problem and, as a result, captures the solutions directly rather than going through the usual procedure of proving radial symmetry of solutions and then classifying radial solutions. Significant simplifications to the proof in [30] have been made in Li and Zhang [29]. The method of moving spheres has been used in [24]–[27].

Liouville type theorems for various conformally invariant equations have received much attention; see, in addition to the above cited papers, [23], [17], [15], [33], [42] and [43].

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In this paper we study some conformally invariant integral equations. Lieb proved in [31], among other things, that there exist maximizing functions, f, for the Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^n$:

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|\cdot - y|^\lambda} \, dy \right\|_{L^q(\mathbb{R}^n)} \le N_{p,\lambda,n} \|f\|_{L^p(\mathbb{R}^n)},$$

with $N_{p,\lambda,n}$ being the sharp constant and $1/p + \lambda/n = 1 + 1/q$, $1 < p, q, n/\lambda < \infty$, $n \ge 1$. When $p = q' = q/(q-1)$ or $p = 2$ or $q = 2$, $N_{p,\lambda,n}$ and the maximizing f's are explicitly evaluated. When $p = q'$, i.e., $p = 2n/(2n - \lambda)$ and $q = 2n/\lambda$, the Euler–Lagrange equation for a maximizing f is, modulo a positive constant multiple,

$$f(x)^{p-1} = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^\lambda} \, dy. \tag{3}$$

Writing $\lambda = n - \alpha$ and $u = f^{p-1}$, we have $0 < \alpha < n$, and equation (3) becomes

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x - y|^{n-\alpha}} \, dy, \qquad \forall x \in \mathbb{R}^n. \tag{4}$$

As mentioned above, maximizing solutions f of (3) are classified in [31] and they are, in terms of u, of the form

$$u(x) \equiv \left(\frac{a}{d + |x - \bar{x}|^2} \right)^{(n-\alpha)/2}, \tag{5}$$

where $a, d > 0$ and $\bar{x} \in \mathbb{R}^n$. Of course, a is a fixed constant depending only on n and α, while d and $\bar{x}$ are free.

Equation (4), or (3), is conformally invariant in the following sense. Let v be a positive function on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $\lambda > 0$, we define

$$v_{x,\lambda}(\xi) = \left(\frac{\lambda}{|\xi - x|} \right)^{n-\alpha} v(\xi^{x,\lambda}), \qquad \xi \in \mathbb{R}^n, \tag{6}$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2 (\xi - x)}{|\xi - x|^2}. \tag{7}$$

Then, if u is a solution of (4), so is $u_{x,\lambda}$ for any $x \in \mathbb{R}^n$ and $\lambda > 0$. The conformal invariance of (4) was used in [31]. More studies on issues concerning the Hardy–Littlewood–Sobolev inequality, among other things, were made by Carlen and Loss in [9]–[12], where the conformal invariance of the problem was further exploited.

After classifying all maximizing solutions of (3), Lieb raised the beautiful question (page 361 of [31]) on the (essential) uniqueness of solutions of (3), or equivalently, of (4). He produced (page 363 of [31]) a nontrivial $2n$-parameter family of solutions of equation (3), or (4), which are not as regular as the maximizers. For instance, modulo a positive constant, $|x|^{(\alpha-n)/2}$ is a solution of (4).

In a recent paper, Chen, Li and Ou established the following result which answers the question of Lieb in the class of $L^\infty_{\text{loc}}(\mathbb{R}^n)$.

Theorem 1.1 ([18])**.** *Let $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ be a positive function satisfying* (4)*. Then u is given by* (5) *for some constants $a, d > 0$ and some $\bar{x} \in \mathbb{R}^n$.*

In an earlier version of the present paper ([28, version one]), we gave a simpler proof of Theorem 1.1. The proof, in the spirit of [30] and [29] and following Section 2 of [29], fully exploits the conformal invariance of the integral equation. It is different from the one in [18]. In particular, we do not follow the usual procedure of proving radial symmetry of solutions and then classifying radial solutions, and we do not need to distinguish $n \geq 2$ and $n = 1$. For the method of moving spheres or moving planes, there are, roughly speaking, three steps: one is to get started with the procedure, second is to prove that the function and the reflected one coincide if the procedure stops, and the third is to handle the case when the procedure never stops. Our arguments are also different for handling these steps. This proof is presented in Section 2.

Lieb pointed out to us that his question also concerns functions which are not in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, it is not known a priori that maximizers are in $L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n)$. This has led us to study the question further and to establish

Theorem 1.2. *For $n \geq 1$ and $0 < \alpha < n$, let $u \in L^{2n/(n-\alpha)}_{\mathrm{loc}}(\mathbb{R}^n)$ be a positive solution of* (4)*. Then $u \in C^{\infty}(\mathbb{R}^n)$.*

An answer to the question of Lieb is therefore known in the class $L^{2n/(n-\alpha)}_{\mathrm{loc}}(\mathbb{R}^n)$. The above mentioned solution $|x|^{(\alpha-n)/2}$ does not belong to $L^{2n/(n-\alpha)}_{\mathrm{loc}}(\mathbb{R}^n)$, though it belongs to $L^{t}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $t < 2n/(n-\alpha)$. The question remains unanswered for the class $L^{t}_{\mathrm{loc}}(\mathbb{R}^n)$ for $t < 2n/(n-\alpha)$. See [34] and the references therein for related results.

In the process of proving Theorem 1.2, we have established the following result which should be of independent interest.

For $n \geq 1$ and $0 < \alpha < n$, let $V \in L^{n/\alpha}(B_3)$ be a nonnegative function, and set

$$\delta(V) := \|V\|_{L^{n/\alpha}(B_3)}. \tag{8}$$

Theorem 1.3. *For $n \geq 1$, $0 < \alpha < n$ and $\nu > r > n/(n-\alpha)$, there exist positive constants $\bar{\delta} < 1$ and $C \geq 1$, depending only on n, α, r and ν, such that for any $0 \leq V \in L^{n/\alpha}(B_3)$ with $\delta(V) \leq \bar{\delta}$, $h \in L^{\nu}(B_2)$, and $0 \leq u \in L^{r}(B_3)$ satisfying*

$$u(x) \leq \int_{B_3} \frac{V(y)u(y)}{|x-y|^{n-\alpha}}\, dy + h(x), \qquad x \in B_2, \tag{9}$$

we have

$$\|u\|_{L^{\nu}(B_{1/2})} \leq C(\|u\|_{L^{r}(B_3)} + \|h\|_{L^{\nu}(B_2)}). \tag{10}$$

Corollary 1.1. *For $n \geq 1$, $0 < \alpha < n$, $\nu > r > n/(n-\alpha)$ and $R_2 > R_1 > 0$, let $0 \leq V \in L^{n/\alpha}(B_{R_2})$, $h \in L^{\nu}(B_{R_1})$ and $0 \leq u \in L^{r}(B_{R_2})$ satisfy*

$$u(x) \leq \int_{B_{R_2}} \frac{V(y)u(y)}{|x-y|^{n-\alpha}}\, dy + h(x), \qquad x \in B_{R_1}.$$

Then, for some $\epsilon > 0$, $u \in L^{\nu}(B_{\epsilon})$.

Remark 1.1. After we proved Theorems 1.2 and 1.3 in [28, version two], a revision of [18] was made which included another proof of Theorem 1.2.

For $\alpha = 2$ and $n \geq 3$, Theorem 1.3 is essentially equivalent to a result of Brezis and Kato (Theorem 2.3 in [6]), so it can be viewed as an integral equation analogue of their theorem. When informed of Theorem 1.3. Brezis kindly pointed out that it is similar to, though not the same as, Lemma A.1 in [7]. Indeed, our proof of the theorem makes use of special properties of the potential $|x|^{\alpha-n}$, and it is not clear to us at this point whether the conclusion of the theorem still holds if we replace $|x|^{\alpha-n}$ by any $Y \in L_w^{n/(n-\alpha)}$, the weak $L^{n/(n-\alpha)}$ space, as in Lemma A.1 of [7]. Theorem 1.2, Theorem 1.3 and Corollary 1.1 are established in Section 2.

We also study some equations similar to (4), though they do not have the same kind of conformal invariance property. For $n \geq 1$, $0 < \alpha < n$ and $\mu > 0$, let u be a positive Lebesgue measurable function in $\mathbb{R}^n$ satisfying

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\mu}}{|x-y|^{n-\alpha}}\, dy, \qquad \forall x \in \mathbb{R}^n. \tag{11}$$

Theorem 1.4. *Let $n \geq 1$ and $0 < \alpha < n$.*

(i) *For $0 < \mu < n/(n-\alpha)$, equation (11) does not have any positive Lebesgue measurable solution u, unless $u \equiv \infty$.*
(ii) *For $n/(n-\alpha) \leq \mu < (n+\alpha)/(n-\alpha)$, equation (11) does not have any positive solution $u \in L^{n(\mu-1)/\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$.*

For $\mu > (n+\alpha)/(n-\alpha)$, we know from Lemma 4.2 that if u is a positive solution in $L^{n(\mu-1)/\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, then u must be in $C^{\infty}(\mathbb{R}^n)$. Theorem 1.4 is proved in Section 4.

In [24]–[27], all conformally invariant second order fully nonlinear equations are classified and Liouville type theorems are established for the elliptic ones. It would be interesting to identify as many as possible conformally invariant integral equations for which (essential) uniqueness of solutions can be obtained. One class of such equations, similar to (4), is

$$u(x) = \int_{\mathbb{R}^n} |x-y|^p u(y)^{-(2n+p)/p}\, dy, \qquad \forall x \in \mathbb{R}^n,$$

where $n \geq 1$ and $p > 0$. We study more general equations, similar to (11), including those which are not conformally invariant.

For $n \geq 1$ and $p, q > 0$, let u be a nonnegative Lebesgue measurable function in $\mathbb{R}^n$ satisfying

$$u(x) = \int_{\mathbb{R}^n} |x-y|^p u(y)^{-q}\, dy, \qquad \forall x \in \mathbb{R}^n. \tag{12}$$

Theorem 1.5. *For $n \geq 1$, $p > 0$ and $0 < q \leq 1+2n/p$, let u be a nonnegative Lebesgue measurable function in $\mathbb{R}^n$ satisfying (12). Then $q = 1 + 2n/p$ and, for some constants $a, d > 0$ and some $\bar{x} \in \mathbb{R}^n$,*

$$u(x) \equiv \left(\frac{d + |x-\bar{x}|^2}{a} \right)^{p/2}. \tag{13}$$

Remark 1.2. For some $a = a(n,p) > 0$, (13) indeed solves (12) with $q = 1 + 2n/p$. This is proved in Appendix A. The argument also shows that, modulo a constant, (5) is a solution of (4), a known fact whose proofs can be found in [39, page 131], [31], and, for $n \geq 2$, in [36]. Our proof is different.

The proof of Theorem 1.5, similar to our proof of Theorem 1.1, is given in Section 5. It turns out that for $n = 3$, $p = 1$ and $q = 7$, integral equation (12) is associated with some fourth order conformal covariant operator on 3-dimensional compact Riemannian manifolds, arising from the study of conformal geometry. See, e.g., Paneitz [37], Fefferman and Graham [19], Branson [3] and Chang and Yang [16].

Question 1. *Is equation* (12)*, in the case $p > 0$ and $q = 1 + 2n/p$, associated with some kind of pseudo-differential conformal covariant operators on n-dimensional compact Riemannian manifolds, the same way the case $n = 3$, $p = 1$ and $q = 7$ is associated with the above mentioned fourth order conformal covariant operator?*

After posting [28, version one] on the Archive and essentially completing the proof of Theorem 1.5, we became aware of some recent work of Xu [44] where he proved Theorem 1.5 in the special case $n = 3$, $p = 1$ and $u \in C^4(\mathbb{R}^3)$. He also proved in the same paper that for $n = 3$, $p = 1$ and $q > 7$ $(= 1 + 2n/p)$, equation (12) does not admit any nonnegative solution u in $C^4(\mathbb{R}^3)$. Radial solutions of the biharmonic equations corresponding to (12) with $n = 3$ and $p = 1$ were studied by McKenna and Reichel in [35].

Question 2. *Is it true that for all $n \geq 1$, $p > 0$ and $q > 1 + 2n/p$ equation* (12) *does not admit any positive solutions?*

We point out that if we consider the integral equations of the form

$$u(x) = \int_{\mathbb{R}^n} G(|x - y|, u(y))\, dy, \qquad u > 0, \qquad \forall x \in \mathbb{R}^n, \tag{14}$$

and consider the transformation of the form

$$u_{x,\lambda}(\xi) = h\left(\left(\frac{\lambda}{|\xi - x|}\right)^{2n}\right) u(\xi^{x,\lambda}),$$

where $\xi^{x,\lambda}$ is given by (7), and wish that

$$h\left(\left(\frac{\lambda}{|\xi - x|}\right)^{2n}\right) \int_{\mathbb{R}^n} G(|\xi^{x,\lambda} - y|, u(y))\, dy \equiv \int_{\mathbb{R}^n} G(|\xi - z|, u_{x,\lambda}(z))\, dz \tag{15}$$

for all $x, \xi \in \mathbb{R}^n$, $\lambda > 0$ and all positive functions u, then we are only led to equation (4) and equation (12) with $q = 1 + 2n/p$ together with the transformations we use in the paper. Note that condition (15) guarantees that whenever u is a solution of (14) so is $u_{x,\lambda}$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$. The quantity $(\lambda/|\xi - x|)^{2n}$ is the Jacobian of the conformal transformation $\xi \mapsto \xi^{x,\lambda}$.

It looks worthwhile to study equation (12) on a bounded domain (existence of solutions, etc.). In this connection, we draw the reader's attention to some works of Brezis and Cabre [4] and Brezis, Dupaigne and Tesei [5].

2. Proof of Theorem 1.3, Corollary 1.1 and Theorem 1.2

In this section we prove Theorem 1.3. Let

$$\xi(x) := \int_{B_3} \frac{V(y)u(y)}{|x-y|^{n-\alpha}}\,dy + h(x) - u(x) \geq 0, \qquad x \in B_2.$$

Then

$$u(x) = (Lu)(x) + f(x) + h(x) - \xi(x), \qquad x \in B_2, \tag{16}$$

where

$$(Lu)(x) = \int_{B_2} \frac{V(y)u(y)}{|x-y|^{n-\alpha}}\,dy, \qquad x \in B_2,$$

and

$$f(x) = \int_{2<|y|<3} \frac{V(y)u(y)}{|x-y|^{n-\alpha}}\,dy.$$

Let p be determined by $1/r = 1/p - \alpha/n$. Then $p > 1$ and therefore, by the property of the Riesz potential (see, e.g., Theorem 1 on page 119 of [39]),

$$\begin{aligned}\|Lu\|_{L^r(B_2)} &\leq C\|Vu\|_{L^p(B_2)} = C(\|V^p u^p\|_{L^1(B_2)})^{1/p} \\ &\leq C(\|V^p\|_{L^{r/(r-p)}(B_2)}\|u^p\|_{L^{r/p}(B_2)})^{1/p} \leq C\|V\|_{L^{n/\alpha}(B_2)}\|u\|_{L^r(B_2)},\end{aligned} \tag{17}$$

where C depends on α, n and r. Similarly

$$\|f\|_{L^r(B_2)} \leq C\|V\|_{L^{n/\alpha}(B_3)}\|u\|_{L^r(B_3)}. \tag{18}$$

It follows, using also the fact $u, \xi \geq 0$, that

$$\|\xi\|_{L^r(B_2)} \leq C\|V\|_{L^{n/\alpha}(B_3)}\|u\|_{L^r(B_3)} + C\|h\|_{L^r(B_2)}. \tag{19}$$

For $i = 1, 2, \ldots,$ let

$$G_i(z) = \min\left(\frac{1}{|z|^{n-\alpha}}, i\right), \qquad u_i(z) = \min(u(z), i),$$

$$\xi_i(x) = \min\left(\xi(x), i\right), \qquad f_i(x) = \int_{2<|y|<3} G_i(x-y)V(y)u(y)\,dy.$$

We now give some preliminary estimates on $\{f_i\}$:

Lemma 2.1. *There exists some constant C, depending only on n and α, such that*

$$\|f_i\|_{L^\infty(B_1)} \leq C\|u\|_{L^r(B_3)}, \qquad \|f_i\|_{L^r(B_2)} \leq C\|u\|_{L^r(B_3)}. \tag{20}$$

Moreover, for any $p < r$,

$$\lim_{i\to\infty} \|f_i - f\|_{L^p(B_2)} = 0. \tag{21}$$

Proof. The first inequality in (20) follows easily:

$$\|f_i\|_{L^\infty(B_1)} \le \|f\|_{L^\infty(B_1)} \le C(n,\alpha)\int_{2<|y|<3} V(y)u(y)\,dy \le C(n,\alpha)\|u\|_{L^r(B_3)}.$$

Note that we have used the hypothesis $\|V\|_{L^{n/\alpha}(B_3)} \le \bar{\delta} < 1$. The second inequality in (20) follows from (18).

By the Fubini theorem,

$$\lim_{i\to\infty}\|f_i - f\|_{L^1(B_2)} \le \lim_{i\to\infty}\left\| G_i(\cdot) - \frac{1}{|\cdot|^{n-\alpha}}\right\|_{L^1(B_5)} \int_{2<|y|<3} V(y)u(y)\,dy = 0.$$

We deduce (21) from this and the second inequality in (20) using Hölder's inequality. □

Consider the following integral equation on w:

$$w(x) = (L_i w)(x) + f_i(x) + h(x) - \xi_i(x), \qquad x \in B_2, \tag{22}$$

where

$$(L_i w)(x) := \int_{|y|<2} G_i(x-y)V(y)w(y)\,dy.$$

Lemma 2.2. *For $r \le q \le \nu$, there exist some $0 < \bar{\delta} < 1$ and $C \ge 1$, depending only on α, n, r and q, such that if $0 < \delta(V) \le \bar{\delta}$, then, for all i, there exists $w_i \in L^q(B_2)$ solving* (22) *with $w = w_i$, satisfying*

$$\|w_i\|_{L^r(B_2)} + \|w_i^+\|_{L^q(B_{1/2})} \le C(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}), \tag{23}$$

where $w_i^+(x) = \max(w_i(x), 0)$.

Proof. Define, for $w \in L^q(B_2)$,

$$(T_i w)(x) = (L_i w)(x) + f_i(x) + h(x) - \xi_i(x), \qquad x \in B_2.$$

Clearly, $L_i, T_i : L^q(B_2) \to L^q(B_2)$.

Let p be determined by $1/q = 1/p - \alpha/n$. Then, using the property of the Riesz potential as in (17), we obtain

$$\|L_i w\|_{L^q(B_2)} \le \|L(|w|)\|_{L^q(B_2)} \le C\|V\|_{L^{n/\alpha}(B_2)}\|w\|_{L^q(B_2)} \le C\bar{\delta}\|w\|_{L^q(B_2)}.$$

Here and below (various) constants $C \ge 1$ depend only on r, q, α and n. Thus

$$\|T_i w\|_{L^q(B_2)} \le C\bar{\delta}\|w\|_{L^q(B_2)} + \|f_i\|_{L^q(B_2)} + \|h\|_{L^q(B_2)} + \|\xi_i\|_{L^q(B_2)}, \tag{24}$$

and

$$\|T_i(w-v)\|_{L^q(B_2)} \le \|L_i(w-v)\|_{L^q(B_2)} \le C\bar{\delta}\|w-v\|_{L^q(B_2)}.$$

Fix some positive $\bar{\delta}$ with $C\bar{\delta} \le 1/2$ and set

$$E_i = \{w \in L^q(B_2) \mid \|w\|_{L^q(B_2)} \le 2(\|f_i\|_{L^q(B_2)} + \|h\|_{L^q(B_2)} + \|\xi_i\|_{L^q(B_2)})\} \subset L^q(B_2).$$

Then T_i maps E_i to itself and is a contraction map. So there exists some $w_i \in E_i$ such that $T_i(w_i) = w_i$, i.e.,

$$w_i(x) = \int_{|y|<2} G_i(x-y)V(y)w_i(y)\,dy + f_i(x) + h(x) - \xi_i(x), \qquad x \in B_2. \tag{25}$$

Taking $q = r$ in (24), we deduce from (25) and (20) that

$$\|w_i\|_{L^r(B_2)} \le \frac{1}{2}\|w_i\|_{L^r(B_2)} + \|f_i\|_{L^r(B_2)} + \|h\|_{L^r(B_2)} + \|\xi\|_{L^r(B_2)}.$$

The estimate of $\|w_i\|_{L^r(B_2)}$ in (23) follows from this, in view of (19) and the second inequality in (20).

Next we establish the second inequality in (23). For $0 < t < s < 1$, we have, by (25),

$$w_i^+(x) \le I_i(x) + II_i(x) + f_i(x) + h(x),$$

where

$$I_i(x) = \int_{|y|<s} \frac{V(y)w_i^+(y)}{|x-y|^{n-\alpha}}\,dy, \qquad II_i(x) = \int_{s<|y|<2} \frac{V(y)w_i^+(y)}{|x-y|^{n-\alpha}}\,dy.$$

By the property of the Riesz potential,

$$\begin{aligned}\|I_i\|_{L^q(B_t)} &\le C\|Vw_i^+\|_{L^p(B_s)} \le C\|V\|_{L^{n/\alpha}(B_s)}\|w_i^+\|_{L^q(B_s)}\\ &\le C\bar{\delta}\|w_i^+\|_{L^q(B_s)} \le \frac{1}{2}\|w_i^+\|_{L^q(B_s)}.\end{aligned}$$

Using the estimate of $\|w_i\|_{L^r(B_2)}$ in (23) yields

$$\begin{aligned}\|II_i\|_{L^q(B_t)} &\le C(s-t)^{\alpha-n}\int_{s<|y|<2} V(y)w_i^+(y)\,dy\\ &\le C(s-t)^{\alpha-n}\|w_i\|_{L^r(B_2)} \le C(s-t)^{\alpha-n}(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}).\end{aligned}$$

With (20) and the above estimates, we have, for all $0 < t < s < 1$,

$$\|w_i^+\|_{L^q(B_t)} \le \frac{1}{2}\|w_i^+\|_{L^q(B_s)} + C(s-t)^{\alpha-n}(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}).$$

By a calculus lemma (see, e.g., page 32 of [20]), we have, for a possibly larger C, still depending only on r, q, α and n,

$$\|w_i^+\|_{L^q(B_t)} \le C(s-t)^{\alpha-n}(\|u\|_{L^r(B_3)} + \|h\|_{L^r(B_2)}), \qquad \forall 0 < t < s < 1.$$

The estimate of $\|w_i^+\|_{L^q(B_{1/2})}$ in (23) follows from the above. Lemma 2.2 is proved. □

Proof of Theorem 1.3. For any $r < q \le \nu$, let $\bar{\delta} > 0$ and $\{w_i\} \in L^q(B_2)$ be given by Lemma 2.2. Since

$$\int_{|y|<2} V(y)w_i(y)\,dy \le C\|V\|_{L^{n/\alpha}(B_2)}\|w_i\|_{L^r(B_2)} \le C$$

for some C independent of i, we have

$$\lim_{|z|\to 0}\sup_i \|(L_iw_i)(\cdot + z) - (L_iw_i)(\cdot)\|_{L^1(B_2)} = 0.$$

Therefore $\{L_iw_i\}$ is precompact in $L^1(B_2)$.

We know from Lemma 2.1 that $\{f_i\}$ converges to f in $L^1(B_2)$. So $\{w_i\}$ is precompact in $L^1(B_2)$. After passing to a subsequence, $w_i \to w$ in $L^1(B_2)$. In view of (23), $w \in L^r(B_2)$, $w_i \to w$ in $L^p(B_2)$ for all $p < r$, $w^+ \in L^q(B_{1/2})$, and

$$\|w^+\|_{L^q(B_{1/2})} \leq C(\|u\|_{L^r(B_3)} + \|h\|_{L^\nu(B_2)}). \tag{26}$$

It follows that $L_i w_i \to Lw$ in $L^1(B_2)$. Thus,

$$w(x) = \int_{|y|<2} \frac{V(y)w(y)}{|x-y|^{n-\alpha}}\, dy + f(x) + h(x) - \xi(x), \qquad \text{a.e. } x \in B_2.$$

Taking the difference of this and (16), we obtain

$$(u-w)(x) = \int_{|y|<2} \frac{V(y)(u-w)(y)}{|x-y|^{n-\alpha}}\, dy, \quad \text{a.e. } x \in B_2.$$

By the usual estimates and using $0 < \delta(V) \leq \bar{\delta}$ and $C\bar{\delta} \leq 1/2$, we infer that

$$\|u-w\|_{L^r(B_2)} \leq C\bar{\delta}\|u-w\|_{L^r(B_2)} \leq \tfrac{1}{2}\|u-w\|_{L^r(B_2)}.$$

It follows that $u = w$ a.e. in B_2. Theorem 1.3 follows from (26). □

Proof of Corollary 1.1. For $\epsilon > 0$ small, let

$$u_\epsilon(x) = \epsilon^{(n-\alpha)/2} u(\epsilon x), \qquad V_\epsilon(x) = \epsilon^\alpha V(\epsilon x), \qquad x \in B_3,$$

and

$$h_\epsilon(x) = \epsilon^{(n-\alpha)/2} \int_{3\epsilon<|y|<R_2} \frac{V(y)u(y)}{|\epsilon x - y|^{n-\alpha}}\, dy + \epsilon^{(n-\alpha)/2} h(\epsilon x).$$

Then

$$u_\epsilon(x) \leq \int_{B_3} \frac{V_\epsilon(y)u_\epsilon(y)}{|x-y|^{n-\alpha}}\, dy + h_\epsilon(x), \qquad x \in B_2.$$

Clearly, $u_\epsilon \in L^r(B_3)$ and $h_\epsilon \in L^\nu(B_2)$. Let $\bar{\delta} > 0$ be the number in Theorem 1.3, and fix some small $\epsilon > 0$ so that

$$\|V_\epsilon\|_{L^{n/\alpha}(B_3)} = \|V\|_{L^{n/\alpha}(B_{3\epsilon})} < \bar{\delta}.$$

Applying Theorem 1.3 to u_ϵ, we have $u_\epsilon \in L^\nu(B_{1/2})$, i.e. $u \in L^\nu(B_{\epsilon/2})$. □

Proof of Theorem 1.2. Since $u \in L^{2n/(n-\alpha)}_{\text{loc}}(\mathbb{R}^n)$, we have, by (4), for some $|\bar{x}| < 1$,

$$\int_{|y|>2} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|y|^{n-\alpha}}\, dy \leq C \int_{|y|>2} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|\bar{x}-y|^{n-\alpha}}\, dy \leq \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|\bar{x}-y|^{n-\alpha}}\, dy = u(\bar{x}) < \infty. \tag{27}$$

For any $R > 0$, we write

$$u(x) = I_R(x) + II_R(x) := \left(\int_{|y|\leq 2R} + \int_{|y|>2R} \right) \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}}\, dy. \tag{28}$$

Take

$$V(x)=u(x)^{\frac{2\alpha}{n-\alpha}},\qquad h(x)=\int_{|y|>2R}\frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}}\,dy.$$

Since $u\in L^{2n/n-\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, we have $V\in L^{n/\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. By (27), $h\in L^\infty(B_R)$. For any $\nu>n/(n-\alpha)$, we have, by Corollary 1.1, $u\in L^\nu(B_{\epsilon(\nu)})$ for some $\epsilon(\nu)>0$. Since any point can be taken as the origin, we have proved that $u\in L^\nu_{\mathrm{loc}}(\mathbb{R}^n)$ for all $1<\nu<\infty$. By the Hölder inequality, $I_R\in L^\infty(B_R)$. By (27), we can differentiate $II_R(x)$ under the integral sign for $|x|<R$, so $II_R\in C^\infty(\mathbb{R}^n)$. Since R is arbitrary, $u\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Back to (28), I_R is at least Hölder continuous in B_R. Since $R>0$ is arbitrary, u is Hölder continuous in $\mathbb{R}^n$. Now $u^{(n+\alpha)/(n-\alpha)}$ is Hölder continuous in B_{2R}, the regularity of I_R further improves and, by bootstrap, we eventually have $u\in C^\infty(\mathbb{R}^n)$. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. As shown in the last paragraph of Section 2, $u\in C^\infty(\mathbb{R}^n)$. By (4) and the Fatou lemma,

$$\beta:=\liminf_{|x|\to\infty}|x|^{n-\alpha}u(x)=\liminf_{|x|\to\infty}\int_{\mathbb{R}^n}\frac{|x|^{n-\alpha}u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}}\,dy\ge\int_{\mathbb{R}^n}u(y)^{\frac{n+\alpha}{n-\alpha}}\,dy>0. \tag{29}$$

For $x\in\mathbb{R}^n$, $\lambda>0$, and a positive function v on $\mathbb{R}^n$, let $v_{x,\lambda}$ be given by (6). Making a change of variables

$$y=z^{x,\lambda}=x+\frac{\lambda^2(z-x)}{|z-x|^2},$$

we have

$$dy=\left(\frac{\lambda}{|z-x|}\right)^{2n}dz.$$

Thus

$$\begin{aligned}\int_{|y-x|\ge\lambda}\frac{v(y)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-y|^{n-\alpha}}\,dy&=\int_{|z-x|\le\lambda}\frac{v(z^{x,\lambda})^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-z^{x,\lambda}|^{n-\alpha}}\left(\frac{\lambda}{|z-x|}\right)^{2n}dz\\&=\int_{|z-x|\le\lambda}\frac{v_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-z^{x,\lambda}|^{n-\alpha}}\left(\frac{\lambda}{|z-x|}\right)^{n-\alpha}dz.\end{aligned}$$

Since

$$\frac{|z-x|}{\lambda}\,\frac{|\xi-x|}{\lambda}|\xi^{x,\lambda}-z^{x,\lambda}|=|\xi-z|, \tag{30}$$

we have

$$\left(\frac{\lambda}{|\xi-x|}\right)^{n-\alpha}\int_{|y-x|\ge\lambda}\frac{v(y)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-y|^{n-\alpha}}\,dy=\int_{|z-x|\le\lambda}\frac{v_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi-z|^{n-\alpha}}\,dz. \tag{31}$$

Similarly,

$$\left(\frac{\lambda}{|\xi-x|}\right)^{n-\alpha}\int_{|y-x|\le\lambda}\frac{v(y)^{\frac{n+\alpha}{n-\alpha}}}{|\xi^{x,\lambda}-y|^{n-\alpha}}\,dy=\int_{|z-x|\ge\lambda}\frac{v_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi-z|^{n-\alpha}}\,dz. \tag{32}$$

For a positive solution u of (4), applying (31) and (32) with $v = u$ and $v = u_{x,\lambda}$, and using the fact that $(\xi^{x,\lambda})^{x,\lambda} = \xi$ and $(v_{x,\lambda})_{x,\lambda} \equiv v$, we obtain

$$u_{x,\lambda}(\xi) = \int_{\mathbb{R}^n} \frac{u_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}}{|\xi - z|^{n-\alpha}}\, dz, \qquad \forall \xi \in \mathbb{R}^n, \tag{33}$$

and

$$u(\xi) - u_{x,\lambda}(\xi) = \int_{|z-x|\geq\lambda} K(x,\lambda;\xi,z)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_{x,\lambda}(z)^{\frac{n+\alpha}{n-\alpha}}]\, dz, \tag{34}$$

where

$$K(x,\lambda;\xi,z) = \frac{1}{|\xi - z|^{n-\alpha}} - \left(\frac{\lambda}{|\xi - x|}\right)^{n-\alpha} \frac{1}{|\xi^{x,\lambda} - z|^{n-\alpha}}.$$

It is elementary to check that

$$K(x,\lambda;\xi,z) > 0, \qquad \forall |\xi - x|, |z - x| > \lambda > 0.$$

Formula (33) is the conformal invariance of the integral equation (4) (see [31] and [32]).

Lemma 3.1. *For $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) \leq u(y), \qquad \forall 0 < \lambda < \lambda_0(x),\ |y - x| \geq \lambda. \tag{35}$$

Proof. The proof is essentially the same as that of Lemma 2.1 in [29]. For the reader's convenience, we include the details. Without loss of generality we may assume $x = 0$, and we use the notation $u_\lambda = u_{0,\lambda}$.

Since $\alpha < n$ and u is a positive C^1 function, there exists $r_0 > 0$ such that

$$\nabla_y (|y|^{(n-\alpha)/2} u(y)) \cdot y > 0, \qquad \forall 0 < |y| < r_0.$$

Consequently,

$$u_\lambda(y) < u(y), \qquad \forall 0 < \lambda < |y| < r_0. \tag{36}$$

By (29) and the positivity and continuity of u,

$$u(z) \geq \frac{1}{C(r_0)|z|^{n-\alpha}} \qquad \forall |z| \geq r_0. \tag{37}$$

For small $\lambda_0 \in (0, r_0)$ and for $0 < \lambda < \lambda_0$,

$$u_\lambda(y) = \left(\frac{\lambda}{|y|}\right)^{n-\alpha} u\left(\frac{\lambda^2 y}{|y|^2}\right) \leq \left(\frac{\lambda_0}{|y|}\right)^{n-\alpha} \sup_{B_{r_0}} u \leq u(y), \qquad \forall |y| \geq r_0.$$

Estimate (35), with $x = 0$ and $\lambda_0(x) = \lambda_0$, follows from (36) and the above. □

Define, for $x \in \mathbb{R}^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 \mid u_{x,\lambda}(y) \leq u(y)\ \forall 0 < \lambda < \mu,\ |y - x| \geq \lambda\}.$$

Lemma 3.2. *If $\bar{\lambda}(\bar{x}) < \infty$ for some $\bar{x} \in \mathbb{R}^n$, then*

$$u_{\bar{x},\bar{\lambda}(\bar{x})} \equiv u \quad \text{on } \mathbb{R}^n. \tag{38}$$

Proof. Without loss of generality, we may assume $\bar{x} = 0$, and we use notations $\bar{\lambda} = \bar{\lambda}(0)$, $u_\lambda = u_{0,\lambda}$. By the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}}(y) \le u(y), \qquad \forall |y| \ge \bar{\lambda}. \tag{39}$$

By (34), with $x = 0$ and $\lambda = \bar{\lambda}$, and the positivity of the kernel, either $u_{\bar{\lambda}}(y) = u(y)$ for all $|y| \ge \bar{\lambda}$—then we are done—or $u_{\bar{\lambda}}(y) < u(y)$ for all $|y| > \bar{\lambda}$, which we assume below. By the Fatou lemma,

$$\begin{aligned}\liminf_{|y|\to\infty} |y|^{n-\alpha}(u - u_{\bar{\lambda}})(y) &= \liminf_{|y|\to\infty} \int_{|z|\ge\bar{\lambda}} |y|^{n-\alpha} K(0,\bar{\lambda}; y, z)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_{\bar{\lambda}}(z)^{\frac{n+\alpha}{n-\alpha}}]\,dz \\ &\ge \int_{|z|\ge\bar{\lambda}} \left(1 - \left(\frac{\bar{\lambda}}{|z|}\right)^{n-\alpha}\right)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_{\bar{\lambda}}(z)^{\frac{n+\alpha}{n-\alpha}}]\,dz > 0.\end{aligned}$$

Consequently, there exists $\epsilon_1 \in (0, 1)$ such that

$$(u - u_{\bar{\lambda}})(y) \ge \frac{\epsilon_1}{|y|^{n-\alpha}} \qquad \forall |y| \ge \bar{\lambda} + 1.$$

By the above and the explicit formula for u_λ, there exists $0 < \epsilon_2 < \epsilon_1$ such that

$$(u - u_\lambda)(y) \ge \frac{\epsilon_1}{|y|^{n-\alpha}} + (u_{\bar{\lambda}} - u_\lambda)(y) \ge \frac{\epsilon_1}{2|y|^{n-\alpha}} \qquad \forall |y| \ge \bar{\lambda} + 1,\ \bar{\lambda} \le \lambda \le \bar{\lambda} + \epsilon_2. \tag{40}$$

Now, for $\epsilon \in (0, \epsilon_2)$ which we choose below, we have, for $\bar{\lambda} \le \lambda \le \bar{\lambda} + \epsilon$ and for $\lambda \le |y| \le \bar{\lambda} + 1$,

$$\begin{aligned}(u - u_\lambda)(y) &= \int_{|z|\ge\lambda} K(0,\lambda; y, z)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]\,dz \\ &\ge \int_{\lambda\le|z|\le\bar{\lambda}+1} K(0,\lambda; y, z)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]\,dz \\ &\quad + \int_{\bar{\lambda}+2\le|z|\le\bar{\lambda}+3} K(0,\lambda; y, z)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]\,dz \\ &\ge \int_{\lambda\le|z|\le\bar{\lambda}+1} K(0,\lambda; y, z)[u_{\bar{\lambda}}(z)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]\,dz \\ &\quad + \int_{\bar{\lambda}+2\le|z|\le\bar{\lambda}+3} K(0,\lambda; y, z)[u(z)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}]\,dz.\end{aligned}$$

Because of (40), there exists $\delta_1 > 0$ such that

$$u(z)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}} \ge \delta_1, \qquad \bar{\lambda} + 2 \le |z| \le \bar{\lambda} + 3.$$

Since

$$K(0,\lambda; y, z) = 0, \qquad \forall |y| = \lambda,$$

$$\nabla_y K(0,\lambda; y, z)\cdot y\big|_{|y|=\lambda} = (n-\alpha)|y-z|^{\alpha-n-2}(|z|^2 - |y|^2) > 0, \qquad \forall \bar{\lambda}+2 \le |z| \le \bar{\lambda}+3,$$

and the function is smooth in the relevant region, we have, using also the positivity of the kernel,

$$K(0,\lambda; y, z) \ge \delta_2(|y| - \lambda), \qquad \forall \bar{\lambda} \le \lambda \le |y| \le \bar{\lambda} + 1,\ \bar{\lambda} + 2 \le |z| \le \bar{\lambda} + 3,$$

where $\delta_2 > 0$ is some constant independent of ϵ. It is easy to see that for some constant $C > 0$ independent of ϵ, we have, for $\bar\lambda \le \lambda \le \bar\lambda + \epsilon$,

$$|u_{\bar\lambda}(z)^{\frac{n+\alpha}{n-\alpha}} - u_\lambda(z)^{\frac{n+\alpha}{n-\alpha}}| \le C(\lambda - \bar\lambda) \le C\epsilon, \quad \forall \bar\lambda \le \lambda \le |z| \le \bar\lambda + 1,$$

and (recall that $\lambda \le |y| \le \bar\lambda + 1$)

$$\begin{aligned}\int_{\lambda\le|z|\le\bar\lambda+1} K(0,\lambda;y,z)\,dz &\le \left|\int_{\lambda\le|z|\le\bar\lambda+1}\left(\frac{1}{|y-z|^{n-\alpha}} - \frac{1}{|y^\lambda - z|^{n-\alpha}}\right)dz\right| \\ &\quad + \int_{\lambda\le|z|\le\bar\lambda+1}\left|\left(\frac{\lambda}{|y|}\right)^{n-\alpha} - 1\right|\frac{1}{|y^\lambda - z|^{n-\alpha}}\,dz \\ &\le C|y^\lambda - y| + C(|y| - \lambda) \le C(|y| - \lambda).\end{aligned}$$

It follows from the above that for small $\epsilon > 0$ we have, for $\bar\lambda \le \lambda \le \bar\lambda + \epsilon$ and $\lambda \le |y| \le \bar\lambda + 1$,

$$\begin{aligned}(u - u_\lambda)(y) &\ge -C\epsilon\int_{\lambda\le|z|\le\bar\lambda+1} K(0,\lambda;y,z)\,dz + \delta_1\delta_2(|y| - \lambda)\int_{\bar\lambda+2\le|z|\le\bar\lambda+3} dz \\ &\ge \left(\delta_1\delta_2\int_{\bar\lambda+2\le|z|\le\bar\lambda+3} dz - C\epsilon\right)(|y| - \lambda) \ge 0.\end{aligned}$$

This and (40) violate the definition of $\bar\lambda$. Lemma 3.2 is established. □

By the definition of $\bar\lambda(x)$,

$$u_{x,\lambda}(y) \le u(y), \quad \forall 0 < \lambda < \bar\lambda(x),\ |y - x| \ge \lambda.$$

Multiplying the above by $|y|^{n-\alpha}$ and sending $|y|$ to infinity yields

$$\beta = \liminf_{|y|\to\infty} |y|^{n-\alpha}u(y) \ge \lambda^{n-\alpha}u(x), \quad \forall 0 < \lambda < \bar\lambda(x). \tag{41}$$

On the other hand, if $\bar\lambda(\bar x) < \infty$, we use Lemma 3.2 and multiply (38) by $|y|^{n-\alpha}$ and then send $|y|$ to infinity to obtain

$$\beta = \lim_{|y|\to\infty} |y|^{n-\alpha}u(y) = \bar\lambda(\bar x)^{n-\alpha}u(\bar x) < \infty. \tag{42}$$

Proof of Theorem 1.1. (i) If there exists some $\bar x \in \mathbb{R}^n$ such that $\bar\lambda(\bar x) < \infty$, then, by (42) and (41), $\bar\lambda(x) < \infty$ for all $x \in \mathbb{R}^n$. Applying Lemma 3.2, we have

$$u_{x,\bar\lambda(x)} \equiv u \quad \text{on } \mathbb{R}^n,\ \forall x \in \mathbb{R}^n.$$

By a calculus lemma (Lemma 11.1 in [29], see also Lemma 2.5 in [30] for $\alpha = 2$), any C^1 positive function u satisfying the above must be of the form (5).

(ii) If $\bar\lambda(x) = \infty$ for all $x \in \mathbb{R}^n$, then

$$u_{x,\lambda}(y) \le u(y), \quad \forall |y - x| \ge \lambda > 0,\ x \in \mathbb{R}^n.$$

By another calculus lemma (Lemma 11.2 in [29], see also Lemma 2.2 in [30] for $\alpha = 2$), $u \equiv$ const, violating (4). Theorem 1.1 is established. □

4. Proof of Theorem 1.4

In this section we establish Theorem 1.4.

Lemma 4.1. *For $n \geq 1$, $0 < \alpha < n$ and $\mu > 0$, let u be a Lebesgue measurable positive solution of* (11) *which is not identically equal to ∞. Then, for any $t < n/(n-\alpha)$ and $u \in L^{\mu}_{loc}(\mathbb{R}^n) \cap L^t_{loc}(\mathbb{R}^n)$,*

$$\beta := \liminf_{|x|\to\infty} |x|^{n-\alpha} u(x) \geq \int_{\mathbb{R}^n} u(y)^{\mu}\, dy > 0, \tag{43}$$

and

$$\int_{|y|>2} \frac{u(y)^{\mu}}{|y|^{n-\alpha}}\, dy < \infty \tag{44}$$

Proof. Multiplying (11) by $|x|^{n-\alpha}$, we obtain (43) by applying the Fatou lemma. Since u is not identically equal to ∞, we see from (11) that u is finite almost everywhere. So, for some $x_1, x_2 \in B_1$, $x_1 \neq x_2$, we have

$$\sum_{i=1}^{2} \int_{\mathbb{R}^n} \frac{u(y)^{\mu}}{|x_i - y|^{n-\alpha}}\, dy \leq u(x_1) + u(x_2) < \infty.$$

It follows that $u \in L^{\mu}_{loc}(\mathbb{R}^n)$ and (44) holds. For $R > 0$, we write

$$u(x) = I_R(x) + II_R(x) := \left(\int_{|y|<2R} + \int_{|y|>2R} \right) \frac{u(y)^{\mu}}{|x-y|^{n-\alpha}}\, dy. \tag{45}$$

Since $u \in L^{\mu}_{loc}(\mathbb{R}^n)$ and (44) holds, $II_R \in L^{\infty}(B_R)$. On the other hand, for any $1 < t < n/(n-\alpha)$, we have, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \|I_R\|_{L^t(B_R)} &\leq \int_{|y|<2R} u(y)^{\mu} \big\| |\cdot - y|^{\alpha-n} \big\|_{L^t(B_R)}\, dy \\ &\leq \big\| |\cdot - y|^{\alpha-n} \big\|_{L^t(B_{3R})} \int_{|y|<2R} u(y)^{\mu}\, dy < \infty. \end{aligned}$$

Since $R > 0$ is arbitrary, $u \in L^t_{loc}(\mathbb{R}^n)$. □

Lemma 4.2. *Assume $n \geq 1$ and $0 < \alpha < n$.*

(i) *For $0 < \mu < n/(n-\alpha)$, let u be a positive Lebesgue measurable solution of* (11) *which is not identically infinity. Then $u \in C^{\infty}(\mathbb{R}^n)$.*

(ii) *For $\mu \geq n/(n-\alpha)$, let $u \in L^{n(\mu-1)/\alpha}_{loc}(\mathbb{R}^n)$ be a positive solution of* (11)*. Then $u \in C^{\infty}(\mathbb{R}^n)$.*

Proof. (i) For $0 < \mu < n/(n-2)$. We know from Lemma 4.1 that $u \in L^t_{loc}(\mathbb{R}^n)$ for all $t < n/(n-\alpha)$. For any $R > 0$, write u as in (45). As usual, $II_R \in C^{\infty}(B_R)$. For any $1 < p < n/\mu(n-\alpha)$, let $1/q = 1/p - \alpha/n$. Then $q > n/(n-\alpha)$. By the property of the Riesz potential,

$$\|I_R\|_{L^q(B_R)} \leq C\|u^{\mu}\|_{L^p(B_{2R})} = C\|u\|^{\mu}_{L^{p\mu}(B_{2R})} < \infty.$$

So $u \in L^q_{\rm loc}(\mathbb{R}^n)$. Let $\mu' = \max(1, \mu)$. Since $u^\mu \le C + Cu^{\mu'}$, we have

$$u(x) \le C \int_{|y|<2R} \frac{V(y)u(y)}{|x-y|^{n-\alpha}}\, dy + h(x), \qquad x \in B_R,$$

where

$$V(y) = u(y)^{\mu'-1}, \qquad h(x) = C + \int_{|y|>2R} \frac{u(y)^\mu}{|x-y|^{n-\alpha}}\, dy.$$

By (44), $h \in L^\infty(B_R)$. Since $n(\mu'-1)/\alpha < n/(n-\alpha)$, we obtain $V \in L^{n/\alpha}_{\rm loc}(\mathbb{R}^n)$. Since $u \in L^q_{\rm loc}(\mathbb{R}^n)$ with $q > n/(n-\alpha)$, we have, by applying Corollary 1.1, $u \in L^\nu(B_{\epsilon(\nu)})$ for any $\nu > 0$, where $\epsilon(\nu) > 0$. Now, back to (45), I_R is C^∞ near the origin by bootstrapping. By the translation invariance of the problem, $u \in C^\infty(\mathbb{R}^n)$.

(ii) For $\mu \ge n/(n-\alpha)$, let $V(y) = u(y)^{\mu-1}$. We know from Lemma 4.1 that $u \in L^t_{\rm loc}(\mathbb{R}^n)$ for all $t < n/(n-\alpha)$. Since $u \in L^{n(\mu-1)/\alpha}_{\rm loc}(\mathbb{R}^n)$ by assumption, we also have $V \in L^{n/\alpha}_{\rm loc}(\mathbb{R}^n)$. Now, for any $R_2 > R_1 > 0$, let

$$h(y) = \int_{|y|>R_2} \frac{u(y)^\mu}{|x-y|^{n-\mu}}\, dy.$$

Then $u \in L^r(B_{R_2})$ with $r = n(\mu-1)/\alpha$, $V \in L^{n/\alpha}(B_{R_2})$, $h \in L^\infty(B_{R_1}) \subset L^\nu(B_{R_1})$ for any $\nu > r$, and

$$u(x) = \int_{|y|>R_2} \frac{V(y)u(y)}{|x-y|^{n-\alpha}}\, dy + h(x), \qquad x \in B_{R_1}.$$

By Corollary 1.1, $u \in L^r(B_{R_1})$. Since $R_1 > 0$ is arbitrary, $u \in L^r_{\rm loc}(\mathbb{R}^n)$ for all $r > 1$. Bootstrap as usual to get $u \in C^\infty(\mathbb{R}^n)$. ◻

For $x \in \mathbb{R}^n$, $\lambda > 0$, and a positive function v on $\mathbb{R}^n$, let $v_{x,\lambda}$ be as in (6).

Lemma 4.3. *For $n \ge 1$, $0 < \alpha < n$ and $\mu > 0$, let u be a positive solution of* (11). *Then*

$$u_{x,\lambda}(\xi) = \int_{\mathbb{R}^n} \frac{u_{x,\lambda}(z)^\mu}{|\xi - z|^{n-\alpha}} \left(\frac{\lambda}{|z-x|}\right)^{n+\alpha-\mu(n-\alpha)} dz, \qquad \forall \xi \in \mathbb{R}^n, \tag{46}$$

and

$$u(\xi) - u_{x,\lambda}(\xi) = \int_{|z-x|\ge\lambda} K(x,\lambda;\xi,z)\left[u(z)^\mu - \left(\frac{\lambda}{|z-x|}\right)^{n+\alpha-\mu(n-\alpha)} u_{x,\lambda}(z)^\mu\right] dz, \tag{47}$$

where

$$K(x,\lambda;\xi,z) = \frac{1}{|\xi - z|^{n-\alpha}} - \left(\frac{\lambda}{|\xi - x|}\right)^{n-\alpha} \frac{1}{|\xi^{x,\lambda} - z|^{n-\alpha}}.$$

Moreover,

$$K(x,\lambda;\xi,z) > 0, \qquad \forall |\xi - x|, |z-x| > \lambda > 0.$$

Proof. For $\mu = (n+\alpha)/(n-\alpha)$ the assertion is established in Section 3. The proof works for all $\mu > 0$ with minor modifications. ◻

Lemma 4.4. *For $n \ge 1$, $0 < \alpha < n$ and $\mu > 0$, let $u \in C^1(\mathbb{R}^n)$ be a positive solution of* (11). *Then for any $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) \le u(y), \qquad \forall 0 < \lambda < \lambda_0(x),\ |y-x| \ge \lambda. \tag{48}$$

Proof. This has been proved in Section 3 for $\mu = (n+\alpha)/(n-\alpha)$. The same proof applies for all $\mu > 0$. □

Define, for $x \in \mathbb{R}^n$,

$$\bar\lambda(x) = \sup\{\mu' > 0 \mid u_{x,\lambda}(y) \le u(y),\ \forall 0 < \lambda < \mu',\ |y-x| \ge \lambda\}.$$

Lemma 4.5. *For $n \ge 1$, $0 < \alpha < n$ and $0 < \mu < (n+\alpha)/(n-\alpha)$, let $u \in C^1(\mathbb{R}^n)$ be a positive solution of* (11). *Then $\bar\lambda(x) = \infty$ for all $x \in \mathbb{R}^n$.*

Proof. We argue by contradiction. Suppose that $\bar\lambda(\bar x) < \infty$ for some $\bar x \in \mathbb{R}^n$. Without loss of generality, we may assume $\bar x = 0$, and we write $\bar\lambda = \bar\lambda(0)$, $u_\lambda = u_{0,\lambda}$. By the definition of $\bar\lambda$,

$$u_{\bar\lambda}(y) \le u(y), \qquad \forall |y| \ge \bar\lambda. \tag{49}$$

Since $n+\alpha-\mu(n-\alpha) > 0$, $(\bar\lambda/|z|)^{n+\alpha-\mu(n-\alpha)} < 1$ for $|z| > \bar\lambda$. So, by (49) and (47) with $x = 0$ and $\lambda = \bar\lambda$, and the positivity of the kernel, we have, for $|y| > \bar\lambda$,

$$\begin{aligned}(u-u_{\bar\lambda})(y) &= \int_{|z|\ge\bar\lambda} K(0,\bar\lambda;y,z)\left[u(z)^\mu - \left(\frac{\lambda}{|z|}\right)^{n+\alpha-\mu(n-\alpha)} u_{\bar\lambda}(z)^\mu\right]dz\\ &\ge \int_{|z|\ge\bar\lambda} K(0,\bar\lambda;y,z)\left[1 - \left(\frac{\lambda}{|z|}\right)^{n+\alpha-\mu(n-\alpha)}\right] u_{\bar\lambda}(z)^\mu\, dz > 0.\end{aligned}$$

Thus, by the Fatou lemma and the above,

$$\begin{aligned}\liminf_{|y|\to\infty} |y|^{n-\alpha}(u-u_{\bar\lambda})(y) &\ge \liminf_{|y|\to\infty} \int_{|z|\ge\bar\lambda} |y|^{n-\alpha} K(0,\bar\lambda;y,z)[u(z)^\mu - u_{\bar\lambda}(z)^\mu]\,dz\\ &\ge \int_{|z|\ge\bar\lambda}\left(1-\left(\frac{\bar\lambda}{|z|}\right)^{n-\alpha}\right)[u(z)^\mu - u_{\bar\lambda}(z)^\mu]\,dz > 0.\end{aligned}$$

Consequently, there exists $\epsilon_1 \in (0,1)$ such that

$$(u-u_{\bar\lambda})(y) \ge \frac{\epsilon_1}{|y|^{n-\alpha}}, \qquad \forall |y| \ge \bar\lambda + 1.$$

By the above and the explicit formula for u_λ, there exists $0 < \epsilon_2 < \epsilon_1$ such that

$$(u-u_\lambda)(y) \ge \frac{\epsilon_1}{|y|^{n-\alpha}} + (u_{\bar\lambda}-u_\lambda)(y) \ge \frac{\epsilon_1}{2|y|^{n-\alpha}} \qquad \forall |y| \ge \bar\lambda+1,\ \bar\lambda \le \lambda \le \bar\lambda+\epsilon_2. \tag{50}$$

Now, using (49) and (50) as in Section 3, for $\epsilon \in (0,\epsilon_2)$ which we choose below, we have, for $\bar\lambda \le \lambda \le \bar\lambda+\epsilon$ and for $\lambda \le |y| \le \bar\lambda+1$,

$$\begin{aligned}(u-u_\lambda)(y) \ge &\int_{\lambda\le|z|\le\bar\lambda+1} K(0,\lambda;y,z)[u_{\bar\lambda}(z)^\mu - u_\lambda(z)^\mu]\,dz\\ &+ \int_{\bar\lambda+2\le|z|\le\bar\lambda+3} K(0,\lambda;y,z)[u(z)^\mu - u_\lambda(z)^\mu]\,dz.\end{aligned}$$

Because of (50), there exists $\delta_1 > 0$ such that

$$u(z)^\mu - u_\lambda(z)^\mu \ge \delta_1, \quad \bar\lambda + 2 \le |z| \le \bar\lambda + 3.$$

It was shown in Section 3 that

$$K(0,\lambda; y, z) \ge \delta_2(|y| - \lambda), \quad \forall \bar\lambda \le \lambda \le |y| \le \bar\lambda + 1,\ \bar\lambda + 2 \le |z| \le \bar\lambda + 3,$$

where $\delta_2 > 0$ is some constant independent of ϵ. It is easy to see that for some constant $C > 0$ independent of ϵ, we have, for $\bar\lambda \le \lambda \le \bar\lambda + \epsilon$,

$$|u_{\bar\lambda}(z)^\mu - u_\lambda(z)^\mu| \le C(\lambda - \bar\lambda) \le C\epsilon, \quad \forall \bar\lambda \le \lambda \le |z| \le \bar\lambda + 1,$$

and (recall that $\lambda \le |y| \le \bar\lambda + 1$), as in Section 3,

$$\int_{\lambda \le |z| \le \bar\lambda + 1} K(0,\lambda; y, z)\, dz \le C(|y| - \lambda).$$

It follows from the above that for small $\epsilon > 0$ we have, for $\bar\lambda \le \lambda \le \bar\lambda + \epsilon$ and $\lambda \le |y| \le \bar\lambda + 1$,

$$\begin{aligned}(u - u_\lambda)(y) &\ge -C\epsilon \int_{\lambda \le |z| \le \bar\lambda + 1} K(0,\lambda; y, z)\, dz + \delta_1\delta_2(|y| - \lambda) \int_{\bar\lambda + 2 \le |z| \le \bar\lambda + 3} dz \\ &\ge \left(\delta_1\delta_2 \int_{\bar\lambda + 2 \le |z| \le \bar\lambda + 3} dz - C\epsilon\right)(|y| - \lambda) \ge 0.\end{aligned}$$

This and (64) violate the definition of $\bar\lambda$. Lemma 4.5 is established. □

Proof of Theorem 1.4. According to Lemma 4.5, $\bar\lambda(x) = \infty$ for all $x \in \mathbb{R}^n$, i.e.,

$$u_{x,\lambda}(y) \le u(y), \quad \forall |y - x| \ge \lambda > 0, x \in \mathbb{R}^n.$$

By a calculus lemma (Lemma 11.2 in [29], see also Lemma 2.2 in [30] for $\alpha = 2$), $u \equiv$ const, violating (11). Theorem 1.4 is established. □

5. Proof of Theorem 1.5

In this section we establish Theorem 1.5.

Lemma 5.1. *For $n \ge 1$ and $p, q > 0$, let u be a nonnegative Lebesgue measurable function in $\mathbb{R}^n$ satisfying* (12). *Then*

$$\int_{\mathbb{R}^n} (1 + |y|^p) u(y)^{-q}\, dy < \infty, \tag{51}$$

$$\begin{aligned}\gamma &:= \lim_{|x|\to\infty} |x|^{-p} u(x) \\ &= \lim_{|x|\to\infty} \int_{\mathbb{R}^n} \frac{|x - y|^p}{|x|^p} u(y)^{-q}\, dy \int_{\mathbb{R}^n} u(y)^{-q}\, dy \in (0, \infty),\end{aligned} \tag{52}$$

and, for some constant $C \ge 1$,

$$\frac{1 + |x|^p}{C} \le u(x) \le C(1 + |x|^p), \quad \forall x \in \mathbb{R}^n. \tag{53}$$

Proof. We see from (12) that u must be positive everywhere and

$$|\{y \in \mathbb{R}^n \mid u(y) < \infty\}| > 0,$$

where $|\cdot|$ denotes the Lebesgue measure of the set. So there exist $R > 1$ and some measurable set E such that

$$E \subset \{y \mid u(y) < R\} \cap B_R, \quad |E| \geq 1/R.$$

By (12),

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} |x-y|^p u(y)^{-q}\, dy \geq \int_E |x-y|^p u(y)^{-q}\, dy \\ &\geq R^{-q} \int_E |x-y|^p\, dy, \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

The first inequality in (53) follows from the above.

For some $1 \leq |\bar{x}| \leq 2$,

$$\int_{\mathbb{R}^n} |\bar{x}-y|^p u(y)^{-q}\, dy = u(\bar{x}) < \infty.$$

We deduce (51) from the first inequality in (53) and the above.

For $|x| \geq 1$,

$$\left| \frac{|x-y|^p}{|x|^p} u(y)^{-q} \right| \leq (1+|y|^p) u(y)^{-q},$$

so, in view of (51), (52) follows from the Lebesgue dominated convergence theorem. The second inequality in (53) follows from (12), (51) and (52). □

Lemma 5.2. *For $n \geq 1$ and $p, q > 0$, let u be a nonnegative Lebesgue measurable function in $\mathbb{R}^n$ satisfying* (12). *Then $u \in C^\infty(\mathbb{R}^n)$.*

Proof. For $R > 0$, write (12) as

$$u(x) = I_R(x) + II_R(x) := \left(\int_{|y| \leq 2R} + \int_{|y| > 2R} \right) |x-y|^p u(y)^{-q}\, dy.$$

Because of (51), we can differentiate $II_R(x)$ under the integral sign for $|x| < R$, and therefore $II_R \in C^\infty(B_R)$. On the other hand, since $u^{-q} \in L^\infty(B_{2R})$, clearly I_R is at least Hölder continuous in B_R. Since $R > 0$ is arbitrary, u is Hölder continuous in $\mathbb{R}^n$. Now u^{-q} is Hölder continuous in B_{2R}, the regularity of I_R further improves and, by bootstrap, we eventually have $u \in C^\infty(\mathbb{R}^n)$. □

Let v be a positive function on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $\lambda > 0$, consider

$$v_{x,\lambda}(\xi) = \left(\frac{|\xi - x|}{\lambda} \right)^p v(\xi^{x,\lambda}), \quad \xi \in \mathbb{R}^n,$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2(\xi - x)}{|\xi - x|^2}.$$

Note that the notation $v_{x,\lambda}$ in this section is different from that in Sections 1–4.

Making a change of variables

$$y = z^{x,\lambda} = x + \frac{\lambda^2(z-x)}{|z-x|^2},$$

we have

$$dy = \left(\frac{\lambda}{|z-x|}\right)^{2n} dz.$$

Thus

$$\begin{aligned}\int_{|y-x|\geq\lambda} |\xi^{x,\lambda} - y|^p v(y)^{-q}\, dy &= \int_{|z-x|\leq\lambda} |\xi^{x,\lambda} - z^{x,\lambda}|^p v(z^{x,\lambda})^{-q} \left(\frac{\lambda}{|z-x|}\right)^{2n} dz \\ &= \int_{|z-x|\leq\lambda} |\xi^{x,\lambda} - z^{x,\lambda}|^p \left(\frac{\lambda}{|z-x|}\right)^{2n-pq} v_{x,\lambda}(z)^{-q}\, dz.\end{aligned}$$

Using (30), we have

$$\begin{aligned}&\left(\frac{\lambda}{|\xi-x|}\right)^{-p} \int_{|y-x|\geq\lambda} |\xi^{x,\lambda} - y|^p v(y)^{-q}\, dy \\ &\qquad = \int_{|z-x|\leq\lambda} |\xi - z|^p \left(\frac{\lambda}{|z-x|}\right)^{2n-pq+p} v_{x,\lambda}(z)^{-q}\, dz. \qquad (54)\end{aligned}$$

Similarly,

$$\begin{aligned}&\left(\frac{\lambda}{|\xi-x|}\right)^{-p} \int_{|y-x|\leq\lambda} |\xi^{x,\lambda} - y|^p v(y)^{-q}\, dy \\ &\qquad = \int_{|z-x|\geq\lambda} |\xi - z|^p \left(\frac{\lambda}{|z-x|}\right)^{2n-pq+p} v_{x,\lambda}(z)^{-q}\, dz. \qquad (55)\end{aligned}$$

Lemma 5.3. *Let u be a positive solution of* (12). *Then*

$$u_{x,\lambda}(\xi) = \int_{\mathbb{R}^n} |\xi - z|^p \left(\frac{\lambda}{|z-x|}\right)^{2n-pq+p} u_{x,\lambda}(z)^{-q}\, dz, \qquad \forall \xi \in \mathbb{R}^n, \qquad (56)$$

and

$$u_{x,\lambda}(\xi) - u(\xi) = \int_{|z-x|\geq\lambda} k(x,\lambda;\xi,z)\left[u(z)^{-q} - \left(\frac{\lambda}{|z-x|}\right)^{2n-pq+p} u_{x,\lambda}(z)^{-q}\right] dz, \qquad (57)$$

where

$$k(x,\lambda;\xi,z) = \left(\frac{|\xi-x|}{\lambda}\right)^p |\xi^{x,\lambda} - z|^p - |\xi - z|^p.$$

Moreover

$$k(x,\lambda;\xi,z) > 0, \qquad \forall |\xi - x|,\ |z-x| > \lambda > 0.$$

Proof. Since $(\xi^{x,\lambda})^{x,\lambda} = \xi$ and $(v_{x,\lambda})_{x,\lambda} \equiv v$, identity (56) follows from (12) and (54) and (55) with $v = u$. Similarly, using also (56), we obtain

$$\begin{aligned} u(\xi) &= \int_{|z-x|\geq\lambda} |\xi - z|^p u(z)^{-q}\, dz + \int_{|y-x|<\lambda} |\xi - y|^p u(y)^{-q}\, dy \\ &= \int_{|z-x|\geq\lambda} |\xi - z|^p u(z)^{-q}\, dz \\ &\quad + \left(\frac{|\xi - x|}{\lambda}\right)^p \int_{|z-x|\geq\lambda} |\xi^{x,\lambda} - z|^p \left(\frac{\lambda}{|z-x|}\right)^{2n-pq+p} u_{x,\lambda}(z)^{-q}\, dz, \\ u_{x,\lambda}(\xi) &= \int_{\mathbb{R}^n} |\xi - z|^p \left(\frac{\lambda}{|z-x|}\right)^{2n-pq+p} u_{x,\lambda}(z)^{-q}\, dz \\ &= \int_{|z-x|\geq\lambda} |\xi - z|^p \left(\frac{\lambda}{|z-x|}\right)^{2n-pq+p} u_{x,\lambda}(z)^{-q}\, dz \\ &\quad + \left(\frac{|\xi - x|}{\lambda}\right)^p \int_{|z-x|\geq\lambda} |\xi^{x,\lambda} - z|^p u(z)^{-q}\, dz. \end{aligned}$$

Identity (57) follows from the above. The positivity of the kernel k is elementary. □

Lemma 5.4. *For $n \geq 1$ and $p, q > 0$, let u be a solution of* (12). *Then for any $x \in \mathbb{R}^n$, there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(y) \geq u(y), \quad \forall 0 < \lambda < \lambda_0(x),\ |y - x| \geq \lambda. \tag{58}$$

Proof. The proof is similar to that of Lemma 2.1 in [29] and Lemma 3.1 in Section 3. Without loss of generality we may assume $x = 0$, and we write $u_\lambda = u_{0,\lambda}$.

Since $p > 0$ and u is a positive C^1 function, there exists $r_0 > 0$ such that

$$\nabla_y (|y|^{-p/2} u(y)) \cdot y < 0, \quad \forall 0 < |y| < r_0.$$

Consequently,

$$u_\lambda(y) > u(y), \quad \forall 0 < \lambda < |y| < r_0. \tag{59}$$

By (53),

$$u(z) \leq C(r_0)|z|^p \quad \forall |z| \geq r_0. \tag{60}$$

For small $\lambda_0 \in (0, r_0)$ and for $0 < \lambda < \lambda_0$, we have, using (53) and (59),

$$u_\lambda(y) = \left(\frac{|y|}{\lambda}\right)^p u\left(\frac{\lambda^2 y}{|y|^2}\right) \geq \left(\frac{|y|}{\lambda_0}\right)^p \inf_{B_{r_0}} u \geq u(y), \quad \forall |y| \geq r_0.$$

Estimate (58), with $x = 0$ and $\lambda_0(x) = \lambda_0$, follows from (59) and the above. □

Define, for $x \in \mathbb{R}^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 \mid u_{x,\lambda}(y) \geq u(y),\ \forall 0 < \lambda < \mu,\ |y - x| \geq \lambda\}.$$

Lemma 5.5. *For $n \geq 1$, $p > 0$ and $0 < q \leq 1 + 2n/p$, let u be a solution of* (12). *Then*

$$\bar{\lambda}(x) < \infty, \qquad \forall x \in \mathbb{R}^n,$$

and

$$u_{x,\lambda(x)} \equiv u \quad \text{on } \mathbb{R}^n, \ \forall x \in \mathbb{R}^n. \tag{61}$$

Consequently, $q = 1 + 2n/p$.

Proof. By the definition of $\bar{\lambda}(x)$,

$$u_{x,\lambda}(y) \geq u(y), \qquad \forall 0 < \lambda < \bar{\lambda}(x), |y - x| \geq \lambda.$$

Multiplying the above by $|y|^{-p}$ and sending $|y|$ to infinity yields, using (52),

$$0 < \gamma = \lim_{|y|\to\infty} |y|^{-p} u(y) \leq \lambda^{-p} u(x), \qquad \forall 0 < \lambda < \bar{\lambda}(x). \tag{62}$$

Thus $\bar{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$.

Now we prove (61). Without loss of generality, we may assume $x = 0$, and we write $\bar{\lambda} = \bar{\lambda}(0)$, $u_\lambda = u_{0,\lambda}$, and $y^\lambda = y^{0,\lambda}$. By the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}}(y) \geq u(y), \qquad \forall |y| \geq \bar{\lambda}. \tag{63}$$

Since $2n - pq + p \geq 0$, we have $(\bar{\lambda}/|z|)^{2n-pq+p} \leq 1$ for $|z| \geq \bar{\lambda}$. So, by (63), (57), with $x = 0$ and $\lambda = \bar{\lambda}$, and the positivity of the kernel, either $u_{\bar{\lambda}}(y) = u(y)$ for all $|y| \geq \bar{\lambda}$—then we are done (using (57) to see that $2n - pq + p = 0$)—or $u_{\bar{\lambda}}(y) > u(y)$ for all $|y| > \bar{\lambda}$, which we assume below.

By (57), with $x = 0$ and $\lambda = \bar{\lambda}$, and the Fatou lemma,

$$\begin{aligned}
&\liminf_{|y|\to\infty} |y|^{-p}(u_{\bar{\lambda}} - u)(y) \\
&\quad = \liminf_{|y|\to\infty} \int_{|z|\geq\bar{\lambda}} |y|^{-p} k(0, \bar{\lambda}; y, z) \left[u(z)^{-q} - \left(\frac{\bar{\lambda}}{|z|}\right)^{2n-pq+p} u_{\bar{\lambda}}(z)^{-q} \right] dz \\
&\quad \geq \int_{|z|\geq\bar{\lambda}} \left(\left(\frac{|z|}{\bar{\lambda}}\right)^p - 1 \right) [u(z)^{-q} - u_{\bar{\lambda}}(z)^{-q}]\, dz > 0.
\end{aligned}$$

Consequently, using also the positivity of $u_{\bar{\lambda}} - u$, there exists $\epsilon_1 \in (0, 1)$ such that

$$(u_{\bar{\lambda}} - u)(y) \geq \epsilon_1 |y|^p, \qquad \forall |y| \geq \bar{\lambda} + 1.$$

By the above and the explicit formula of u_λ, there exists $0 < \epsilon_2 < \epsilon_1$ such that

$$(u_\lambda - u)(y) \geq \epsilon_1 |y|^p + (u_\lambda - u_{\bar{\lambda}})(y) \geq \frac{\epsilon_1}{2} |y|^p, \qquad \forall |y| \geq \bar{\lambda} + 1, \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \epsilon_2. \tag{64}$$

Recall that $2n-pq+p \geq 0$ and therefore $(\lambda/|z|)^{2n-pq+p} \leq 1$ for $|z| \geq \lambda$. For $\epsilon \in (0, \epsilon_2)$ which we choose below, we have, for $\bar{\lambda} \leq \lambda \leq \bar{\lambda}+\epsilon$ and for $\lambda \leq |y| \leq \bar{\lambda}+1$,

$$\begin{aligned}(u_\lambda - u)(y) &\geq \int_{|z|\geq\lambda} k(0,\lambda; y,z)[u(z)^{-q} - u_\lambda(z)^{-q}]\,dz \\ &\geq \int_{\lambda\leq|z|\leq\bar{\lambda}+1} k(0,\lambda; y,z)[u(z)^{-q} - u_\lambda(z)^{-q}]\,dz \\ &\quad + \int_{\bar{\lambda}+2\leq|z|\leq\bar{\lambda}+3} k(0,\lambda; y,z)[u(z)^{-q} - u_\lambda(z)^{-q}]\,dz \\ &\geq \int_{\lambda\leq|z|\leq\bar{\lambda}+1} k(0,\lambda; y,z)[u_{\bar{\lambda}}(z)^{-q} - u_\lambda(z)^{-q}]\,dz \\ &\quad + \int_{\bar{\lambda}+2\leq|z|\leq\bar{\lambda}+3} k(0,\lambda; y,z)[u(z)^{-q} - u_\lambda(z)^{-q}]\,dz.\end{aligned}$$

Because of (64), there exists $\delta_1 > 0$ such that

$$u(z)^{-q} - u_\lambda(z)^{-q} \geq \delta_1, \quad \bar{\lambda}+2 \leq |z| \leq \bar{\lambda}+3.$$

Since

$$k(0,\lambda; y,z) = 0, \quad \forall |y| = \lambda,$$

$$\nabla_y k(0,\lambda; y,z)\cdot y\Big|_{|y|=\lambda} = p|y-z|^{p-2}(|z|^2 - |y|^2) > 0, \quad \forall \bar{\lambda}+2 \leq |z| \leq \bar{\lambda}+3,$$

and the function is smooth in the relevant region, we have, using also the positivity of the kernel,

$$k(0,\lambda; y,z) \geq \delta_2(|y|-\lambda), \quad \forall \bar{\lambda} \leq \lambda \leq |y| \leq \bar{\lambda}+1,\ \bar{\lambda}+2 \leq |z| \leq \bar{\lambda}+3,$$

where $\delta_2 > 0$ is some constant independent of ϵ. It is easy to see that for some constant $C > 0$ independent of ϵ, we have, for $\bar{\lambda} \leq \lambda \leq \bar{\lambda}+\epsilon$,

$$|u_{\bar{\lambda}}(z)^{-q} - u_\lambda(z)^{-q}| \leq C(\lambda - \bar{\lambda}) \leq C\epsilon, \quad \forall \bar{\lambda} \leq \lambda \leq |z| \leq \bar{\lambda}+1,$$

and (recall that $\lambda \leq |y| \leq \bar{\lambda}+1$)

$$\begin{aligned}\int_{\lambda\leq|z|\leq\bar{\lambda}+1} k(0,\lambda; y,z)\,dz &\leq C(|y|-\lambda) + \int_{\lambda\leq|z|\leq\bar{\lambda}+1} (|y^\lambda - z|^p - |y-z|^p)\,dz \\ &\leq C(|y|-\lambda) + C|y^\lambda - y| \leq C(|y|-\lambda).\end{aligned}$$

It follows that for small $\epsilon > 0$ we have, for $\bar{\lambda} \leq \lambda \leq \bar{\lambda}+\epsilon$ and $\lambda \leq |y| \leq \bar{\lambda}+1$,

$$\begin{aligned}(u_\lambda - u)(y) &\geq -C\epsilon \int_{\lambda\leq|z|\leq\bar{\lambda}+1} k(0,\lambda; y,z)\,dz + \delta_1\delta_2(|y|-\lambda)\int_{\bar{\lambda}+2\leq|z|\leq\bar{\lambda}+3} dz \\ &\geq \left(\delta_1\delta_2 \int_{\bar{\lambda}+2\leq|z|\leq\bar{\lambda}+3} dz - C\epsilon\right)(|y|-\lambda) \geq 0.\end{aligned}$$

This and (64) violate the definition of $\bar{\lambda}$. Lemma 5.5 is established. □

Proof of Theorem 1.5. According to Lemma 5.5, $q = 1 + 2n/p$ and

$$u_{x,\bar{\lambda}(x)} \equiv u \quad \text{on } \mathbb{R}^n,\ \forall x \in \mathbb{R}^n.$$

By a calculus lemma (Lemma 11.1 in [29], see also Lemma 2.5 in [30] for $\alpha = 2$), any C^1 positive function u satisfying the above must be of the form (13). □

Appendix A

In this appendix, we show, as pointed out in Remark 1.2, that for some $a = a(n, p) > 0$, (13) solves (12) with $q = 1 + 2n/p$. Our proof works equally well for equation (4).

For $n \geq 1$ and $p \in (-n, 0) \cup (0, \infty)$, we consider the integral equation

$$u(x) = \int_{\mathbb{R}^n} |x - y|^p u(y)^{-(2n+p)/p}\, dy, \qquad \forall x \in \mathbb{R}^n. \tag{65}$$

Lemma 5.6. *For $n \geq 1$ and $p \in (-n, 0) \cup (0, \infty)$, there exists a unique $a = a(n, p) > 0$ such that for any $\bar{x} \in \mathbb{R}^n$ and $d > 0$,*

$$u(x) = \left(\frac{d + |x - \bar{x}|^2}{a}\right)^{p/2}$$

satisfies (65).

Proof. Let $q = 1 + 2n/p$. For a positive function v, and for $x \in \mathbb{R}^n$ and $\lambda > 0$, we set

$$v_{x,\lambda}(\xi) = \left(\frac{|\xi - x|}{\lambda}\right)^p v(\xi^{x,\lambda}), \qquad \xi \in \mathbb{R}^n,$$

where $\xi^{x,\lambda}$ is given by (7). By conformal invariance, we only need to prove that modulo a positive constant multiple,

$$u(x) := (1 + |x|^2)^{p/2}$$

satisfies (65). Set

$$\tilde{u}(x) = \int_{\mathbb{R}^n} |x - y|^p u(y)^{-q}\, dy, \qquad x \in \mathbb{R}^n.$$

We only need to show that $\tilde{u}$ is a constant multiple of u.

For any $x \in \mathbb{R}^n$, let $\lambda(x) := \sqrt{1 + |x|^2}$. Observe that

$$u_{x,\lambda(x)} \equiv u, \qquad \text{on } \mathbb{R}^n,\ \forall x \in \mathbb{R}^n. \tag{66}$$

Making a change of variables

$$y = z^{x,\lambda(x)} = x + \frac{\lambda(x)^2 (z - x)}{|z - x|^2},$$

we have, using (66) and the conformal invariance of the equation ((54), (55), (31) and (32)),

$$\begin{aligned}
\tilde{u}_{x,\lambda(x)}(\xi) &= \left(\frac{|\xi - x|}{\lambda(x)}\right)^p \int_{\mathbb{R}^n} |\xi^{x,\lambda(x)} - y|^p u(y)^{-q}\, dy \\
&= \int_{\mathbb{R}^n} |\xi - z|^p u_{x,\lambda(x)}(z)^{-q}\, dz = \tilde{u}(\xi).
\end{aligned}$$

Multiplying this by $|\xi|^{-p}$ and sending $|\xi|$ to ∞ leads to

$$\lim_{|\xi|\to\infty} |\xi|^{-p}\tilde{u}(\xi) = \lambda(x)^{-p}\tilde{u}(x) = (1+|x|^2)^{-p/2}\tilde{u}(x) = u(x)^{-1}\tilde{u}(x), \qquad \forall x \in \mathbb{R}^n.$$

So $\tilde{u}$ is a constant multiple of u, and we are done. □

Appendix B

In this appendix, we present some calculus lemmas obtained jointly with L. Nirenberg. These lemmas under the stronger assumption $f \in C^1(\mathbb{R}^n)$ have been used repeatedly in some works on Liouville type theorems for conformally invariant equations (see, e.g., [29], [30], [24]–[27] and the present paper).

Lemma 5.7. *For $n \geq 1$ and $\nu \in \mathbb{R}$, let f be a function defined on $\mathbb{R}^n$ with values in $[-\infty, \infty]$ satisfying*

$$\left(\frac{\lambda}{|y-x|}\right)^{\nu} f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y), \qquad \forall |x-y| > \lambda > 0. \tag{67}$$

Then $f \equiv$ const or $\pm\infty$.

Remark 5.1. If the first inequality in (67) is reversed, the conclusion still holds, since we can replace f by $-f$.

Proof. For all $b > 1$ and $y, z \in \mathbb{R}^n$ with $y \neq z$, let

$$x = x(b) = y + b(z-y), \qquad \lambda = \lambda(b) = \sqrt{|z-x|\,|y-x|}.$$

Then

$$z = x + \frac{\lambda^2(y-x)}{|y-x|^2},$$

and, by (67),

$$\left(\frac{\lambda}{|y-x|}\right)^{\nu} f(z) \leq f(y).$$

Since

$$\lim_{b\to\infty} \frac{\lambda}{|y-x|} = \lim_{b\to\infty} \sqrt{\frac{|z-x|}{|y-x|}} = 1,$$

we have $f(z) \leq f(y)$. Lemma 5.7 follows since $y \neq z$ are arbitrary. □

Lemma 5.8. *Let $n \geq 1$, $\nu \in \mathbb{R}$ and $f \in C^0(\mathbb{R}^n)$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda(x) > 0$ such that*

$$\left(\frac{\lambda(x)}{|y-x|}\right)^{\nu} f\left(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2}\right) = f(y), \qquad \forall y \in \mathbb{R}^n \setminus \{x\}. \tag{68}$$

Then, for some $a \geq 0$, $d > 0$ and $\bar{x} \in \mathbb{R}^n$,

$$f(x) \equiv \pm a\left(\frac{1}{d + |x-\bar{x}|^2}\right)^{\nu/2}.$$

Proof. By (68) and the continuity of f,

$$\alpha := \lim_{|y|\to\infty} |y|^{\nu} f(y) = \lambda(x)^{\nu} f(x), \qquad \forall x \in \mathbb{R}^n. \tag{69}$$

If $\nu = 0$, then $f \equiv \alpha$, and we are done. On the other hand, the case $\nu < 0$ can easily be reduced to the case of $\nu > 0$ if we let $z = x + \lambda(x)^2(y-x)/|y-x|^2$ in (68). So we will assume that $\nu > 0$.

If $\alpha = 0$, then $f \equiv 0$, and we are done. Otherwise, replacing f by a nonzero multiple of f, we may assume that $\alpha = 1$. Since $f(y) \to 0$ as $|y|$ tends to ∞, and since f is continuous and positive, f has a maximum point, and we may assume that f has a maximum point at the origin.

For any $x \in \mathbb{R}^n$, we have, for large $|y|$,

$$\begin{aligned}|y|^{\nu} f(y) &= \lambda(x)^{\nu}\left(\frac{|y|}{|y-x|}\right)^{\nu} f\left(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2}\right)\\ &= \lambda(x)^{\nu}\left[1 + \frac{\nu x\cdot y}{|y|^2} + O(|y|^{-2})\right] f\left(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2}\right),\end{aligned}$$

and, by (69) and $\alpha = 1$,

$$\begin{aligned}|y|[|y|^{\nu} f(y) - 1] = |y|\lambda(x)^{\nu}&\left[f\left(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2}\right) - f(x)\right]\\ &+ \left[\frac{\lambda(x)^{\nu}\nu x\cdot y}{|y|} + O(|y|^{-1})\right] f\left(x + \frac{\lambda(x)^2(y-x)}{|y-x|^2}\right).\end{aligned} \tag{70}$$

Taking $x = 0$ in the above and using the fact that f has a maximum point at the origin, we obtain

$$\limsup_{|y|\to\infty} |y|[|y|^{\nu} f(y) - 1] \le 0. \tag{71}$$

Claim. *For any $\epsilon > 0$, there exists M_ϵ such that for any $|y| \ge M_\epsilon$, there exists $\tilde{x} = \tilde{x}(y)$ satisfying*

$$\tilde{x} + \frac{\lambda(\tilde{x})^2(y-\tilde{x})}{|y-\tilde{x}|^2} = 0 \quad \textit{and} \quad |\tilde{x}| \le \epsilon. \tag{72}$$

Indeed, we know from (69) and $\alpha = 1$ that $\lambda(x) = f(x)^{-1/\nu}$ for all $x \in \mathbb{R}^n$. For any $\epsilon \in (0,1)$, pick $M_\epsilon > 1$ so that

$$(M_\epsilon - \epsilon)^{-1} \max_{|x|\le\epsilon} f(x)^{-2/\nu} < \frac{\epsilon}{2}.$$

Then, for all $|y| \ge M_\epsilon$,

$$\max_{|x|\le\epsilon}\left|\frac{\lambda(x)^2(y-x)}{|y-x|^2}\right| = \max_{|x|\le\epsilon} |y-x|^{-1} f(x)^{-2/\nu} \le (M_\epsilon - \epsilon)^{-1}\max_{|x|\le\epsilon} f(x)^{-2/\nu} < \frac{\epsilon}{2}.$$

Thus, by a degree argument using the continuity of f, there exists $\tilde{x} = \tilde{x}(y)$ satisfying (72).

With $x = \tilde{x}(y)$ in (70), we obtain

$$\liminf_{|y|\to\infty} |y|[|y|^{\nu} f(y) - 1] \geq -\epsilon \nu f(0) \max_{|z|\leq\epsilon} f(z)^{-1/\nu}.$$

Sending ϵ to 0, we have

$$\liminf_{|y|\to\infty} |y|[|y|^{\nu} f(y) - 1] \geq 0.$$

Thus, in view of (71),

$$\lim_{|y|\to\infty} |y|[|y|^{\nu} f(y) - 1] = 0. \tag{73}$$

Let $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$. For any $x \in \mathbb{R}^n$, $1 \leq i \leq n$, and $t \in \mathbb{R}$, let $y = y(x, t, i) \in \mathbb{R}^n$ be defined by

$$te_i = \frac{\lambda(x)^2 (y - x)}{|y - x|^2}.$$

Taking this y in (70) and sending t to 0, we obtain, in view of (73),

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t\to 0} \frac{f(x + te_i) - f(x)}{t} = -\frac{\nu x \cdot e_i f(x)}{\lambda(x)^2} = -\nu x_i f(x)^{1+2/\nu}.$$

By the continuity of f, we know that f is in $C^1(\mathbb{R}^n)$, and we complete the proof of Lemma 5.8 by writing the above system of PDEs as $\frac{\partial}{\partial x_i}[f(x)^{-2/\nu} - |x|^2] = 0$ and solving it. □

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Added in proof. Question 2 is answered in the affirmative by Xingwang Xu in: *A theorem on integral equations*, preprint.

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