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An anti-classification theorem for ergodic measure preserving transformations

Received February 20, 2003 and in revised form September 26, 2003

Abstract. Despite many notable advances the general problem of classifying ergodic measure preserving transformations (MPT) has remained wide open. We show that the action of the whole group of MPT's on ergodic actions by conjugation is turbulent in the sense of G. Hjorth. The type of classifications ruled out by this property include countable algebraic objects such as those that occur in the Halmos–von Neumann theorem classifying ergodic MPT's with pure point spectrum. We treat both the classical case of \mathbb{Z} as well as the case of general countable amenable groups.

1. Introduction

Dynamical systems are often studied from a probabilistic or statistical point of view by finding an ergodic invariant probability measure on the space. This allows qualitative descriptions of the dynamical system such as being “truly random” (e.g. Bernoulli) or truly deterministic (e.g. 0-entropy).

The probabilistic behavior of many classical dynamical systems has been characterized precisely by determining the invariant measure up to measure theoretic isomorphism. This led to the general project of classifying measure preserving systems up to isomorphism. Despite dramatic successes, notably Ornstein's theorem classifying Bernoulli transformations in terms of entropy [11], and the Halmos–von Neumann theorem classifying ergodic translations on compact groups, this project largely remains unfulfilled.

Indeed, there has been some accumulation of evidence suggesting that such a classification is impossible. The main result of this note is a precise statement of such a theorem.

As a corollary of the main result, it is impossible to effectively associate to members of a generic collection of ergodic measure preserving transformations countable algebraic structures (e.g. groups) in such a way that two transformations are isomorphic just in case the associated algebraic objects are isomorphic. (In other words, there is no Borel functor from the category of ergodic measure preserving transformations to the category of (say) countable groups.)

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There have been some precursors to this result. For example Hjorth ([8]) showed the analogous result for measure distal transformations of height 2. We will use the notion of *turbulence* defined by Hjorth for our theorem.

We begin by giving some terminology.

Definition 1. *Let X, Y be Polish spaces, and E, F be equivalence relations on X and Y respectively. Then E is Borel reducible to F ($E \leq_B F$) iff there is a Borel measurable function $f : X \rightarrow Y$ such that for all $x_1, x_2 \in X$:*

$$x_1 E x_2 \quad \text{iff} \quad f(x_1) F f(x_2).$$

In other words, any question about E can be “reduced” via f to a question about F . There is an obvious generalization of this notion to equivalence relations on Borel subsets $A \subset X, B \subset Y$.

In this language, the classical result that any two perfect Polish spaces are Borel isomorphic can be interpreted as saying that for all perfect Polish X and Y , there is a Borel bijection $f : X \rightarrow Y$ such that f reduces the relation of equality on X to the relation of equality on Y and f^{-1} reduces equality on Y to equality on X . Thus the question of attaching complete invariants that come from an arbitrary Polish space is not more general than the issue of attaching complete numerical invariants.

If we let E_0 be the equivalence relation on $2^{\mathbb{N}}$ (infinite sequences of 0’s and 1’s) defined by setting $\vec{x} \sim \vec{y}$ in case there is an N such that $x_n = y_n$ for all $n > N$, then classical 0-1 laws show that there is no Borel reduction of E_0 to equality on a Polish space. (There are converses to this, see [5].)

As remarked by Feldman ([1]) the following result implies that there can be no numerical invariants attached to \mathcal{K} -automorphisms:

Theorem (Ornstein–Shields). *There is a Borel reduction of E_0 to the isomorphism equivalence relation on the \mathcal{K} -automorphisms.*

In particular, there is no numerical invariant of \mathcal{K} -automorphisms analogous to entropy.

On the other hand, the Halmos–von Neumann theorem is considered to give a suitable structure theorem for the discrete spectrum transformations in spite of the fact that the equivalence relation E_0 is also reducible to isomorphism of discrete spectrum transformations [2].

The Halmos–von Neumann theorem associates to each ergodic discrete spectrum transformation a countable subset of the unit circle in such a way that two transformations are isomorphic iff the associated countable sets are equal. By using predicates for rational intervals, one can effectively associate to each countable subset of the unit circle a countable structure in a manner that reduces equality of countable sets to isomorphism of the countable structures. Until this paper there was no clear result showing that this was not possible for a collection of generic measure preserving transformations (e.g. the weakly mixing transformations).

To establish such a result we use some notions of Hjorth [6]. Let S_∞ be the group of permutations of \mathbb{N} with the topology of pointwise convergence. We will consider continuous actions of S_∞ on Polish spaces.

This example is chosen specifically because it includes the isomorphism relations on countable algebraic objects. We illustrate this in a particular case:

Example 2. Let X be the space of functions $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow 2$ with the obvious product topology. Let S_∞ act on X by permuting all three copies of \mathbb{N} simultaneously. If G is a countable group (say with universe \mathbb{N} and group operation \odot) we can associate an element $f_G \in X$ by setting

$$f_G(l, m, n) = 1 \quad \text{iff} \quad l \odot m = n.$$

If we equip the space of functions $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow 2$ with the “usual” topology of pointwise convergence (i.e. the $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ product topology of the discrete space $2 = \{0, 1\}$) then the collection of f_G for G a countable group is a dense \mathcal{G}_δ set, and hence the usual topology restricted to the f_G 's is a Polish topology.

If G, H are two groups with universe \mathbb{N} , then $G \cong H$ iff there is a $g \in S_\infty$ such that

$$gf_G = f_H.$$

We leave it to the reader to see that isomorphism on countable rings, rings with an ordering, fields etc. are all S_∞ -actions. (Indeed, there is a universal such action, the *logic action*.)

To show that there is no way of computing isomorphism classes of countable structures as a complete invariant, it suffices to show that the associated equivalence relation is not reducible to an S_∞ -action. A mechanism for doing this was developed by Hjorth [6].

Definition 3. Let G be a Polish group acting continuously on a Polish space X . Then the action is turbulent iff:

- (1) Every orbit is dense.
- (2) Every orbit is meager.
- (3) For all $x, y \in X$, $V \subset X$, $U \subset G$ open, with $x \in V$, $1 \in U$, there exist $y_0 \in [y]_G$ and $\langle g_i \rangle_{i \in \mathbb{N}} \subset U$, $\langle x_i \rangle_{i \in \mathbb{N}} \subset V$ such that
 - (a) $x_0 = x$,
 - (b) $x_{i+1} = g_i x_i$,
 - (c) there is a subsequence i_n such that x_{i_n} converges to y_0 .

Hjorth proved the following result:

Theorem. Suppose that the action of G on X is turbulent. Then the orbit equivalence relation (xEy iff there is a $g \in G$ such that $gx = y$) cannot be reduced to an S_∞ -action.

Hjorth's result clearly implies that if the action of G is turbulent, there is no invariant comeager set on which the orbit equivalence relation can be reduced to any S_∞ -action. Hence the conclusion is “generic”.

The notion of turbulence almost exactly captures the combinatorial content of not being an S_∞ -action: Hjorth showed that for a Borel equivalence relation E , E is not reducible to an S_∞ -action just in case there is a turbulent action reducible to E . Rather embarrassingly, we do not know whether isomorphism for ergodic measure preserving transformations is a Borel equivalence relation.

We will use the following lemma (whose proof is left to the reader), which gives a sufficient condition that is easier to verify:

Lemma 4. *Suppose that G is a Polish group acting continuously on a Polish space X . Suppose further that:*

- (1) *Every orbit is dense.*
- (2) *Every orbit is meager.*
- (3) *For all $x \in X$ and all neighborhoods $V_0 \subset X$ of x and all neighborhoods $U \subset G$ with $1 \in U$ there is a $V \subset V_0$ with $x \in V$ such that for all $y \in V$ and all neighborhoods W of y , there are finite sequences $\langle g_i \rangle_{i \leq N-1} \subset U$, $\langle x_i \rangle_{i \leq N-1} \subset V$ with*
 - (a) $x_0 = x$,
 - (b) $x_{i+1} = g_i x_i$,
 - (c) $x_N \in W$.

Then the action is turbulent.

In order to apply this technology we must describe a Polish group acting continuously on a Polish space. (See e.g. [2] for a laborious development of this situation.)

Since every separable non-atomic probability measure space is isomorphic to the unit interval with Lebesgue measure, every ergodic, aperiodic measure preserving transformation on a probability space is isomorphic to an ergodic transformation on $[0, 1]$ and gives rise to a unitary operator on $L^2([0, 1])$. (We identify measure preserving transformations that agree on a set of full measure.) The space of unitary operators arising from measure preserving transformations is a Polish group when endowed with the weak operator topology. Halmos ([4]) showed that the ergodic measure preserving transformations are a dense \mathcal{G}_δ subset of the measure preserving transformations, and hence form a Polish space.

Moreover, the relation of isomorphism between measure preserving transformations is given by the action of conjugacy of the measure preserving transformations on the ergodic transformations.

This context can be generalized to show that if H is a fixed countable amenable group then there is a natural Polish topology on the space of H -actions on $[0, 1]$ by measure preserving transformations. (We give details in Section 4.) Moreover the action of conjugation by the group of measure preserving transformations on the space of H -actions again coincides with isomorphism.

We prove:

Theorem A. *Let H be a countable amenable group. Let X be the space of ergodic actions of H on $[0, 1]$. Let G be the group of measure preserving transformations acting on X by conjugation. Then this action is turbulent.*

An important special case of this is:

Theorem B. *Let G be the group of measure preserving transformations of $[0, 1]$ acting by conjugation on X , the space of ergodic measure preserving transformations. Then the action of G is turbulent.*

We note that Hjorth [7] has proved that it is possible to reduce a turbulent equivalence relation to the space of ergodic actions of any countable discrete group that is not “abelian-by-

finite". Hence it is not possible to classify the measure preserving actions of any countable group by using countable algebraic objects.

We begin by proving Theorem B, and indicate how to strengthen the proof in Section 4. The proof is somewhat different than the proof of the more general result, Theorem A. We present it because it gives slightly more information than the proof of Theorem A.

We need to concretely describe a neighborhood basis for the measure preserving transformations. The following description can be found in [4].

Definition 5. For each measure preserving transformation T and finite sequence of measurable sets A_0, \dots, A_{n-1} and $\epsilon > 0$, let $N(A_0, \dots, A_{n-1}, \epsilon)$ be the collection of all S such that for all $i < n$ the measure of $S(A_i) \Delta T(A_i)$ is less than ϵ .

Proposition 6. Let T be a measure preserving transformation of $[0, 1]$. Then the collection $\{N(A_0, \dots, A_{n-1}, \epsilon) : A_i \text{ is measurable, } \{A_i\} \text{ is a partition of } [0, 1] \text{ and } \epsilon > 0\}$ forms a neighborhood basis for T .

In [4], Halmos defines a metric on the space of measure preserving transformations by setting $d'(S, T)$ to be the measure of $\{x : S(x) \neq T(x)\}$. The topology of this metric is finer than the Polish topology described above. Some of our results can be restated using the d' metric.

The prerequisites from ergodic theory are very mild. Throughout we will be assuming our measure space is $[0, 1]$ and we will denote the Lebesgue measure by μ . We heavily use the following fact:

Rokhlin's Lemma I. Let T be an ergodic measure preserving transformation on $[0, 1]$, $N \in \mathbb{N}$, $\epsilon > 0$. Then there is a set $B \subset [0, 1]$ such that:

- (1) $B, TB, T^2B, \dots, T^{N-1}B$ are pairwise disjoint.
- (2) The measure of $\bigcup_{i < N} T^i B$ is at least $1 - \epsilon$.

As is standard, we will call $B, TB, T^2B, \dots, T^{N-1}B$ the levels of an (N, ϵ) Rokhlin tower with base B . We will call $X \setminus \bigcup_{i < N} T^i B$ the remainder of the tower.

This lemma can be refined using the ergodic theorem to yield:

Rokhlin's Lemma II. Let T be an ergodic measure preserving transformation on $[0, 1]$, $\epsilon > 0$ and $\{A_1, \dots, A_n\}$ be subsets of $[0, 1]$. Then there is an M and for all $N > M$ there is a set $B \subset [0, 1]$ such that:

- (1) $B, TB, T^2B, \dots, T^{N-1}B$ are pairwise disjoint.
- (2) The measure of $\bigcup_{i < N} T^i B$ is at least $1 - \epsilon$.
- (3) For all $y \in B$ and $j < n$,

$$|\#\{i < N : T^i y \in A_j\} / N - \mu(A_j)| < \epsilon.$$

Notation. For a point $y \in [0, 1]$ and a transformation T we will often write y^i for the point $T^i y$. We will confuse the reader by using absolute value symbol " $|C|$ " both for the cardinality of C if C is a set, and for the absolute value of C if C is a number.

2. A useful lemma

We start with a definition:

Definition 7. We will say that a measure preserving transformation T' has the same orbits almost everywhere as T does if for almost all x the T' -orbit of x is equal to the T -orbit of x .

In this section we prove:

Lemma 8. Suppose that T is ergodic, $\epsilon > 0$ and $\{C_i\}_{i < k}, \{D_i\}_{i < k}$ are two measurable partitions of $[0, 1]$ such that for each $i < k$, C_i and D_i have the same measure. Then there is a T' that has the same orbits almost everywhere as T and is such that for all i the measure of $T'(C_i) \Delta D_i$ is less than ϵ .

From this lemma and the definition of the basic open neighborhoods, it is easy to see:

Lemma 9. Let S, T be ergodic measure preserving transformations and U a neighborhood of S . Then there is a $T' \in U$ that has the same orbits almost everywhere as T .

Proof of Lemma 8. Fix an $\epsilon > 0$ and partitions $\{C_l\}_{l < k}, \{D_l\}_{l < k}$, and let N be large enough and δ small enough that $(2k + 2)\delta + (2k + 4)/N < \epsilon$. Choose an (N, δ) Rokhlin tower with base B_0 and remainder R such that for all y in the base and $l < k$, the number of elements of $C_l \cap \{y^i\}_{i=1}^{N-2}$ differs from the number of elements of $D_l \cap \{y^i\}_{i=1}^{N-2}$ by less than δN .

We start by fixing an arbitrary y in the base of the tower and building a sequence of partial functions $t_0 \subset t_1 \subset \dots \subset t_k$ from $\{y^i\}_{i=1}^{N-2}$ to $\{y^i\}_{i=1}^{N-2}$ such that

- t_l is a one-to-one partial function from $\bigcup_{j < l} C_j$ to $\bigcup_{j < l} D_j$ sending the elements of C_j in its domain to elements of D_j .
- The graph of t_l contains no cycles.
- The domain of t_k contains more than $N - k(\delta + 1)N - 2$ points.

This is done inductively by setting $t_0 = \emptyset$ and obtaining t_l from t_{l-1} by taking t_l to be a maximal extension of t_{l-1} such that t_l maps C_{l-1} to D_{l-1} and has no cycles in its graph.

We note that there are at most $\max\{|C_{l-1} \cap \{y^i\}_{i=1}^{N-2}| - |D_{l-1} \cap \{y^i\}_{i=1}^{N-2}| + 1, 1\}$ points of $C_{l-1} \cap \{y^i\}_{i=1}^{N-2}$ left out of the domain of t_l . (If there were at least two points of D_i not in the range of t_l then it would be possible to extend t_l without creating a cycle.) Since the distributions of C_{l-1} and D_{l-1} are equal except for an error of proportion δ , t_l is defined on all of $C_{l-1} \cap \{y^i\}_{i=1}^{N-2}$ except perhaps $N\delta + 1$ points.

Let L_l be the set of points of $C_l \cap \{y^i\}_{i=1}^{N-2}$ not in the domain of t_l . Then $\bigcup_{l < k} L_l$ has cardinality less than $k(N\delta + 1)$. Thus we have constructed t_k as desired.

Now extend t_k to a total function t_y from $\{y^i\}_{i=0}^{N-2}$ to $\{y^i\}_{i=1}^{N-1}$ with no cycles in its graph such that $t_y^{N-1}(y) = y^{N-1}$. Define $s_y : \{0, \dots, N - 2\} \rightarrow \{1, \dots, N - 1\}$ by setting $s_y(i) = j$ iff $t_y(y^i) = y^j$.

Since there are only finitely many such s_y 's we can partition the base B_0 of the Rokhlin tower into measurable sets by setting $B_s = \{y \in B_0 : s_y = s\}$.

Define a measure preserving transformation T' as follows:

- (1) $T' \upharpoonright (R \cup T^{N-1} B_0) = T \upharpoonright (R \cup T^{N-1} B_0)$.
 (2) For $y \in B_s$ and $j < N - 1$ we let $T'(y^j) = y^{s(j)}$.

In other words, on each T -fiber above $y \in B_0$ we follow t_y , and outside the tower and on the top level of the tower we follow T . Since T' respects the columns of the Rokhlin tower and the entry and exit levels of the tower, it must have the same orbits as T .

To see that T' suffices: for each k , on each B_s the measure of the set of points $x \in C_i$ in the tower above B_s where $T'x \notin D_i$ is at most $\mu(B_s)(k(N\delta + 1) + 2)$. Hence the total measure of points $x \in C_i$ in the tower where $T'x \notin D_i$ is at most $k(\delta + 1/N) + 2/N$. Since the remainder R has measure at most δ the total measure of $T(C_i) \Delta D_i$ is at most $2\delta + 2k(\delta + 1/N) + 4/N = (2k + 2)\delta + (2k + 4)/N$. \square

It is well known, and easily established by an exhaustion argument, that if sets A and B have the same measure then there is a generalized power of T that maps A to B . Doing this for two partitions yields an element in the full group of T , but its orbits will, in general, only be finite parts of the T -orbits. The proof of Lemma 8 can be extended to prove the following:

Remark 10. Suppose that T is ergodic and $\{C_i\}_{i \in \mathbb{N}}, \{D_i\}_{i \in \mathbb{N}}$ are two measurable partitions of $[0, 1]$ such that for each $i \in \mathbb{N}$, C_i and D_i have the same measure. Then there is a T' such that for a.e. x the T' -orbit of x is infinite and included in the T -orbit of x and is such that for all i , $T'(C_i) = D_i$.

Proof (sketch). The main point is a refinement lemma:

Lemma 11. Let T be an ergodic measure preserving transformation, $\epsilon > 0$, and $l \in \mathbb{N}$. Let $\{C_i : i \in \mathbb{N}\}$ and $\{D_i : i \in \mathbb{N}\}$ be two measurable partitions such that the measure of C_i is equal to the measure of D_i . There is a T' which has the same orbits almost everywhere as T does and such that:

- (1) For all i , if $x \in C_i$ and $T(x) \in D_i$ then $T'(x) = T(x)$.
 (2) For all $i < l$ the measure of $T(C_i) \Delta D_i$ is less than ϵ .

This refinement lemma is proved analogously to Lemma 8. We choose a Rokhlin tower to respect the distribution of the partitions $\{C_i : i < l\} \cup \{[0, 1] \setminus \bigcup_{i < l} C_i\}$ and $\{D_i : i < l\} \cup \{[0, 1] \setminus \bigcup_{i < l} D_i\}$. We then let $G = \{x : \text{for all } i, \text{ if } x \in C_i \text{ then } T(x) \in D_i\}$ and perform the same construction, except that we start with $t_0 \upharpoonright (G \cap \{y^i\}_{i=0}^{N-2}) = T \upharpoonright (G \cap \{y^i\}_{i=0}^{N-2})$. (If $y^0 \notin G$, then each t_i has domain contained in $\{y^i\}_{i=1}^{N-2}$, as before.)

To finish the proof of Remark 10, repeatedly use the refinement lemma to build a sequence of transformations $T_0, T_1, \dots, T_j, \dots$ with $\epsilon_j \rightarrow 0$ and $l_j \rightarrow \infty$, making sure that $T_{j+1} \upharpoonright G_j = T \upharpoonright G_j$, where $G_j = \{x : \text{for all } i, \text{ if } x \in C_i \text{ then } T_j(x) \in D_i\}$. Let T' be the measure preserving transformation such that for all j and $x \in G_j$, $T'(x) = T_j(x)$. \square

3. The main result

In this section we prove Theorem B.

Theorem 12. *Let G be the group of measure preserving transformations of $[0, 1]$ acting by conjugation on X , the space of ergodic measure preserving transformations. Then the action of G is turbulent.*

Corollary 13. *There is no dense \mathcal{G}_δ subset of the measure preserving transformations on which the equivalence relation of isomorphism can be reduced to an S_∞ -action.*

Proof of Theorem 12. In fact we will prove a stronger result about the metric d' :

Claim 14. *Suppose that $\epsilon, \delta > 0$, S, T are ergodic and S has the same orbits almost everywhere as T does. Then there is a sequence of elements $g_i \in G$ for $i < n$ such that:*

- (1) $\{x : g_i(x) \neq x\}$ has measure less than ϵ ,
- (2) if $T_0 = T, T_{i+1} = g_i T_i$ then for all i , $\{x : T_i(x) \notin \{T(x), S(x)\}\}$ has measure less than δ ,
- (3) $\{x : T_n(x) \neq S(x)\}$ has measure less than δ .

Using Lemma 4, and assuming the claim, we now verify that the action is turbulent.

We note first that Rokhlin's lemma implies that every transformation is d' -close to periodic transformations of sufficiently large period and small remainder. Since any two periodic transformations of the same period are conjugate, we see that every orbit is dense.

The fact that every orbit is meager follows easily from [9], where it is shown that for ergodic transformations T , the collection of ergodic transformations disjoint from T is a dense \mathcal{G}_δ . If there were a non-meager orbit, it would have to be comeager, since it is invariant. But then a transformation T in the orbit would have to be disjoint from itself, a contradiction. In the next section we reprove this fact in a more general context.

We now check the last clause of the definition of turbulence.

Let U be a neighborhood of the identity in G , and V_0 be a neighborhood of an ergodic transformation T . Let $V \subset V_0$ be a neighborhood of T of the form $N(A_0, \dots, A_l, \rho)$. Let $S \in V$ be ergodic and W be a neighborhood of S .

By Lemma 9, we can replace S by an $S' \in W$ that has the same orbits almost everywhere that T does and then replace W by a basic open neighborhood $W' \subset W$ of S' . Hence without loss of generality we can assume that S has the same orbits almost everywhere as T does.

Apply the claim. By choosing ϵ small enough, (1) ensures that each $g_i \in U$. If we choose δ small enough, (2) guarantees that each $T_i \in V$ and (3) guarantees that $T_n \in W$.

We now prove the claim. Choose now an M so large that the collection L_0 of x for which $\{Sx, S^{-1}x\}$ is not a subset of the partial orbit $\{T^{-M+1}x, T^{-M+2}x, \dots, T^{-1}x, x, Tx, \dots, T^{M-1}x\}$ has measure less than $\delta/3$.

Let N be so large that $2M/N < \delta/3$, and choose an $(N, \delta/3)$ Rokhlin tower with base B . Let L_1 be the union of the first and last M levels of the tower. Then L_1 has measure at most $\delta/3$. Let L_2 be the collection of x not in the levels of the Rokhlin tower. Then by assumption the measure of L_2 is less than $\delta/3$ and hence the measure of $L_0 \cup L_1 \cup L_2$ is less than δ .

Let $y \in B$. Then for all $x \in \{T^i y : M \leq i \leq N - M\} \setminus L_0$, both Sx and $S^{-1}x$ belong to $\{T^i y : 0 \leq i \leq N - 1\}$.

We can choose a cyclic permutation σ_y of $\{0, \dots, N - 1\}$ such that for all $i, j < N - 1$, if $S(T^i y) = T^j y$, then $\sigma_y(i) = j$. For each permutation σ of $\{0, \dots, N - 1\}$, the collection $B_\sigma = \{y : \sigma_y = \sigma\}$ is measurable and the sets B_σ partition B .

Partition each B_σ into sets $B_{\sigma,k}$, $k < k_\sigma$, such that the measure of each $B_{\sigma,k}$ is less than ϵ/N . Let τ_σ be a permutation of $\{0, \dots, N - 1\}$ such that $\tau_\sigma \sigma \tau_\sigma^{-1}$ is the cyclic permutation $(0, 1, \dots, N - 1)$ and define $u_{\sigma,k}$ on $\bigcup_{i < N} T^i B_{\sigma,k}$ by setting

$$u_{\sigma,k}(T^i y) = T^{\tau_\sigma(i)} y$$

for $y \in B_{\sigma,k}$ and $u_{\sigma,k}(x) = x$ for $x \notin \bigcup_{i < N} T^i B_{\sigma,k}$.

Let g_1, \dots, g_{n-1} enumerate the $u_{\sigma,k}$'s (in any order). Note that it is clear that condition (1) is satisfied. Moreover, since the collections of x moved by the various $u_{\sigma,k}$'s are pairwise disjoint, the $u_{\sigma,k}$'s commute.

By the choice of τ_σ , for all $x \in \bigcup_{M \leq i \leq N-M} T^i x \setminus L_0$,

$$u_{\sigma,k}(x)T(x)u_{\sigma,k}^{-1}(x) = S(x),$$

hence for all $x \notin L_0 \cup L_1 \cup L_2$, $T_i(x) \in \{S(x), T(x)\}$ and hence (2) is satisfied.

Moreover, for all $x \notin L_0 \cup L_1 \cup L_2$ that are moved by any g_i , we have $T_{i+1}(x) = S(x)$. In particular $T_n(x) = S(x)$ except on a set of measure less than δ . \square

We remark that a fact analogous to Claim 14 is true for finitely generated amenable groups.

4. General discrete groups

In this section we discuss this theorem in the context of countable discrete group actions by measure preserving transformations. We will prove the result analogous to Theorem 12 for the case of monotilable amenable groups and discuss the situation for general groups.

We begin by fixing a countable group H . We put a topology on the space of H -actions on $[0, 1]$ by measure preserving transformations. (As usual we will identify actions that agree on a set of full measure.) For each H -action \mathcal{A} on $[0, 1]$ and $g \in H$ we will denote the measure preserving transformation determined by the action and g as $g^{\mathcal{A}}$. Fix such an action \mathcal{A} . We describe a basic open neighborhood of \mathcal{A} . For each $\epsilon > 0$, $g_1, \dots, g_n \in H$ and measurable partition $\{A_1, \dots, A_k\}$ we take as a basic open neighborhood $N(\mathcal{A}, g_1, \dots, g_n, A_1, \dots, A_k, \epsilon)$ the set

$$\{\mathcal{B} : \mu(g_i^{\mathcal{A}} A_j \Delta g_i^{\mathcal{B}} A_j) < \epsilon \text{ for all } i, j\}.$$

(We note that if H is finitely generated with generators $\{h_1, \dots, h_n\}$, then the collection of neighborhoods determined by taking the g_i 's to be the h_i 's forms a base for the topology.)

As before this is a Polish topology. The collection of free ergodic actions of a countable amenable group H form a dense \mathcal{G}_δ set in the collection of all actions and hence a Polish space (see [3]).

If T is a measure preserving transformation and \mathcal{A} is an H -action by measure preserving transformation on $[0, 1]$ then we let $\mathcal{A}' = T\mathcal{A}$ be the H -action defined by setting

$g^{\mathcal{A}'} = Tg^{\mathcal{A}}T^{-1}$. (We will sometimes write this as $\mathcal{A}' = T\mathcal{A}T^{-1}$.) This defines an action of the group of measure preserving transformations on the space of H -actions and the orbits coincide with isomorphism classes of H -actions.

Theorem 15. *Suppose that H is an amenable group. Then the action of the measure preserving transformations on the ergodic H -actions is turbulent.*

A closely related theorem is:

Theorem 16. *Suppose that H is an amenable group and \mathcal{A} is an ergodic action of H . Then the collection of actions \mathcal{B} that have the same orbits almost everywhere as \mathcal{A} does is dense in the space of H -actions.*

We now give the proof of this theorem in the case that the amenable group is monothilable. The general result follows tediously using the quasi-tiling techniques of [13]. We begin by discussing *monothilable* groups. For an exposition about monothilable groups we refer the reader to [14].

Definition 17. *An amenable group H is monothilable iff there is a Følner sequence $\langle F_n : n \in \mathbb{N} \rangle$ and a collection $\{c_n^i : i, n \in \mathbb{N}\}$ such that for each fixed n , $\{F_n c_n^i : i \in \mathbb{N}\}$ is a partition of H . The sequence $\langle F_n : n \in \mathbb{N} \rangle$ will be called a tiling sequence. A set B such that there are $\{c^i : i \in \mathbb{N}\}$ such that $\{Bc^i : i \in \mathbb{N}\}$ is a partition of H is called a tiling set and the $\{c^i : i \in \mathbb{N}\}$ are the centers of the tiling.*

All elementary groups are monothilable and it is an open question whether every amenable group is monothilable. For our purposes the main advantage of monothilability is an easily usable version of Rokhlin's Lemma.

If $\Gamma, F \subset H$, we will call the set $\{h \in F : gh \notin F \text{ for some } g \in \Gamma\}$ the *inner Γ -boundary* of F . The Γ -*interior* of F is the collection of elements of F not in the inner boundary.

Rokhlin's Lemma III ([13]). *Suppose that H is a monothilable group acting freely and ergodically on $[0, 1]$ by measure preserving transformations with tiling sequence $\langle F_n : n \in \mathbb{N} \rangle$. Then for all N large enough $\epsilon > 0$ and any measurable subsets $\{A_1, \dots, A_n\}$ of $[0, 1]$ there is a set $B \subset [0, 1]$ such that:*

- (1) *The sets $\{gB : g \in F_N\}$ are pairwise disjoint.*
- (2) *The measure of $\bigcup_{g \in F_N} gB$ is at least $1 - \epsilon$.*
- (3) *For all $y \in B$ and $j < n$,*

$$|\#\{g \in F_N : g(y) \in A_j\} / |F_N| - \mu(A_j)| < \epsilon.$$

We will call this an (F_N, ϵ) *Rokhlin tower* for $\{A_1, \dots, A_n\}$. We will refer to B as the *base* and $X \setminus \bigcup_{g \in F_N} gB$ as the *remainder* of the tower.

Given an (F, ϵ) Rokhlin tower with base B , we will call the set $\bigcup\{hB : h \text{ is in the inner } \Gamma\text{-boundary of } F\}$ the Γ -*boundary* of the tower, and the set $\bigcup\{hB : h \text{ is in the } \Gamma\text{-interior of } F\}$ the Γ -*interior* of the tower. Clearly if the boundary is a small fraction of F and ϵ is small, the vast majority of the unit interval is in the interior of the tower.

Fix an (F, ϵ) Rokhlin tower with base B . Suppose that $\mathbb{P} = \{P_\tau : \tau \in S\}$ is a partition of the base of B where S is a collection of permutations of a subset of F . For $y \in P_\tau$ and h in the domain of τ , let $T_{\mathbb{P}}(hy) = \tau(h)y$, and let $T_{\mathbb{P}}$ be the identity on the remainder of the tower. Then $T_{\mathbb{P}}$ is a measure preserving transformation that locally permutes the levels of the tower and preserves the F -orbits inside the tower. We will call $T_{\mathbb{P}}$ the transformation induced by the partition and S . We will let T_τ denote the restriction of T to the collection of F -orbits of P_τ .

Remark 18. For any \mathcal{A} the H -action $\mathcal{A}' = T_{\mathbb{P}}\mathcal{A}T_{\mathbb{P}}^{-1}$ has the same orbits almost everywhere as \mathcal{A} does.

We now prove Theorems 15 and 16 simultaneously, combining in one proof the two explicit steps from the case of \mathbb{Z} :

Proof of Theorems 15 and 16. Let H be an amenable group and let X be the space of free ergodic H -actions by measure preserving transformations on $[0, 1]$. As remarked above, Glasner and King showed in [3] that the free ergodic actions of H are a dense \mathcal{G}_δ set in the space of all actions of H . In particular, it is a Polish space with the induced topology. We let the group of measure preserving transformations act on $[0, 1]$ by conjugation.

Claim 19. *Every orbit of a free action of H is dense in X .*

Proof. This follows immediately from Rokhlin's Lemma: If \mathcal{A} and \mathcal{B} are H -actions, fix a neighborhood of \mathcal{B} , $N(\mathcal{B}, g_1, \dots, g_n, A_1, \dots, A_k, \epsilon)$. Let F_N be an element of the tiling sequence such that for all i , $|g_i F_N \Delta F_N|/|F_N| < \epsilon/2$. Apply Rokhlin's Lemma for $F_N, \epsilon/2$ and the actions \mathcal{A}, \mathcal{B} to find sets $B_{\mathcal{A}}, B_{\mathcal{B}}$. (For these purposes the last clause of Rokhlin's Lemma is irrelevant.) We can assume that $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$ have the same measure. Let $T_0 : B_{\mathcal{A}} \rightarrow B_{\mathcal{B}}$ be measure preserving and extend T_0 to $\bigcup_{g \in F_N} g^{-A} B_{\mathcal{A}}$ by setting $T_0(g^{-A}y) = g^B T_0(y)$ for $y \in B_{\mathcal{A}}$ and $g \in F_N$. Finally, extend T_0 to a measure preserving transformation T of $[0, 1]$ on the remainder of the Rokhlin tower arbitrarily. Then for each i , the measure of the set $\{x : T g_i^A T^{-1}(x) \neq g_i^B(x)\}$ is less than ϵ . In particular, for all i, j , $(T g_i^A T^{-1})(A_j) \Delta g_i^B(A_j)$ has measure less than ϵ . \square

Claim 20. *The orbit of every H -action is meager.*

Proof. (We note that this claim does not use monotilability for its proof.) Since every orbit of H is invariant under the group of measure preserving transformations, and dense, standard 0-1 laws show that a non-meager orbit must be comeager. (See e.g. [3] or [6] for a proof of this classical result.)

Fix an action \mathcal{A} of the group H . We recall the definition of entropy of the action. If $\langle F_n : n \in \omega \rangle$ is a Følner sequence for H , and \mathbb{P} is a partition of $[0, 1]$, we let $\bigvee_{F_n} \mathbb{P}$ be the partition generated by $\{h\mathbb{P} : h \in F_n\}$ and define

$$h_n(\mathcal{A}, \mathbb{P}) = - \sum \left\{ \mu(a) \log(\mu(a)) : a \in \bigvee_{F_n} \mathbb{P} \right\} / |F_n|.$$

Note that h_n is a continuous function in \mathcal{A} .

The entropy of \mathcal{A} with respect to \mathbb{P} is defined by $h(\mathcal{A}, \mathbb{P}) = \lim_{n \rightarrow \infty} h_n(\mathcal{A}, \mathbb{P})$ and the entropy of \mathcal{A} is given by $h(\mathcal{A}) = \sup\{h(\mathcal{A}, \mathbb{P}) : \mathbb{P} \text{ is a finite partition}\}$.

Subclaim. *For all $\gamma \geq 0$, $\{\mathcal{A} : h(\mathcal{A}) < \gamma\}$ is a dense \mathcal{G}_δ set. In particular, the set of zero entropy actions of an amenable group is a dense \mathcal{G}_δ set.*

Proof. (This proof is due to Rudolph.) It can be shown that $h(\mathcal{A}, \mathbb{P})$ is a continuous function of \mathbb{P} in the sense that for all $k \in \mathbb{N}$ and $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathbb{P} = \{P_1, \dots, P_k\}$ and $\mathbb{Q} = \{Q_1, \dots, Q_k\}$ are two partitions with $\sum \mu(P_i \Delta Q_i) < \delta$ then $|h(\mathcal{A}, \mathbb{P}) - h(\mathcal{A}, \mathbb{Q})| < \epsilon$. Moreover, entropy is monotone in the sense that if \mathbb{P} refines \mathbb{Q} then $h(\mathcal{A}, \mathbb{Q}) \leq h(\mathcal{A}, \mathbb{P})$. From this we can deduce that

$$h(\mathcal{A}) = \sup\{h(\mathcal{A}, \mathbb{P}) : \mathbb{P} \text{ is a finite partition with rational endpoints}\}.$$

From this it is routine to see that for all γ , $\{\mathcal{A} : h(\mathcal{A}) < \gamma\}$ is a \mathcal{G}_δ set.

For each positive γ , choose a sequence $\langle a_n : n \in \mathbb{N} \rangle$ of positive numbers such that $\sum a_n = 1$ and $-\sum_{n \in \mathbb{N}} a_n \log(a_n) < \gamma$. This makes \mathbb{N} into an atomic measure space and if we endow $X = \mathbb{N}^{\mathbb{H}}$ with the product measure we get a Lebesgue space. Let H act on $\mathbb{N}^{\mathbb{H}}$ by $hf(h) = f(h^{-1}g)$. This action is ergodic and has entropy less than γ . \square

We will call the action of H on $\mathbb{N}^{\mathbb{H}}$ with the measure given above the *Bernoulli action with coefficients $\{a_n\}$* .

Del Junco proves the following result for \mathbb{Z} -actions, but the proof carries over directly to amenable actions:

Theorem 21 (del Junco [9]). *Suppose that \mathcal{A} is an H -action. Then $\{\mathcal{B} : \mathcal{B} \text{ is disjoint from } \mathcal{A}\}$ is a \mathcal{G}_δ set.*

Moreover, classical methods show that if \mathcal{A} has zero entropy then \mathcal{A} is disjoint from any Bernoulli action.

We now finish the proof of Claim 20. If \mathcal{A} were an action with non-meager orbit then the orbit of \mathcal{A} would have to be comeager. In particular \mathcal{A} would have to have zero entropy. On the other hand, the transformations disjoint from \mathcal{A} are a non-empty \mathcal{G}_δ set (because they contain Bernoulli actions). Since any non-empty, conjugation invariant \mathcal{G}_δ set must be dense we see that \mathcal{A} must be disjoint from itself. This contradiction proves Claim 20. \square

We will prove the following:

Subclaim. *Suppose that $\epsilon > 0$ and let $W = N(\mathcal{B}, g_0, \dots, g_{n-1}, B_0, \dots, B_{k-1}, \eta)$ be a basic open neighborhood. Then there is a p and a sequence of measure preserving transformations $u_i, i < p$, such that:*

- (1) *The set $\{x : u_i(x) \neq x\}$ has measure less than ϵ .*
- (2) *If $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_{i+1} = u_i \mathcal{A}_i$ then $g_j^{\mathcal{A}_i}(x) \in \{g_j^{\mathcal{A}}(x), g_j^{\mathcal{A}_p}(x)\}$ for all i, j, x .*
- (3) *$\mathcal{A}_p(x) \in W$.*
- (4) *Each \mathcal{A}_i has the same orbits almost everywhere as \mathcal{A} does.*

To see the subclaim suffices for Theorem 16: take any ergodic \mathcal{A}, \mathcal{B} , and a neighborhood W of \mathcal{B} . Applying the subclaim, we see that \mathcal{A}_p has the same orbits almost everywhere as \mathcal{A} does and is in the neighborhood of \mathcal{B} .

For the proof of Theorem 15 we use the subclaim to check the last clause of the hypothesis of Lemma 4. Fix an action \mathcal{A} , a neighborhood $V = N(\mathcal{A}, g_0, \dots, g_m, A_0, \dots, A_{k-1}, \rho)$, an ergodic action $\mathcal{B} \in V$, a neighborhood $W = N(\mathcal{B}, h_0, \dots, h_n, B_0, \dots, B_{l-1}, \eta)$ and a neighborhood U of the identity in the group of measure preserving transformations. We note that without loss of generality we can assume that $m = n$, and $h_j = g_j$ for $j < n$. Moreover, the partition $\{B_0, \dots, B_{l-1}\}$ refines $\{A_0, \dots, A_{k-1}\}$ and $\eta < \rho$.

Applying the subclaim with ϵ small enough we get a sequence of u_i 's such that each u_i belongs to U . If we take δ small enough and use the fact that \mathcal{A}_p belongs to V , clause (2) of the subclaim shows that each \mathcal{A}_i belongs to V .

To prove the subclaim, we begin by constructing an action \mathcal{A}' which will eventually become \mathcal{A}_p . Having done this the proof follows the proof of Claim 14. The "prescription" for finding \mathcal{A}' goes as follows: we take a very large element F of a tiling Følner sequence, and two Rokhlin towers for F , one for \mathcal{A} and one for \mathcal{B} . We "purify" by breaking the bases of the towers into sets such that on each set P and for each $h \in F$, hP is either disjoint from or contained in the partition determined by the B_i 's and the $g_j^B B_i$'s. We then permute FP by some τ so that $\mathcal{A}' = T_\tau \mathcal{A} T_\tau^{-1}$ behaves exactly the same way as \mathcal{B} does with respect to the B_i 's and their images under g_j^B . Since the vast majority of mass lies in the interior of FP , \mathcal{A}' is in the neighborhood W .

We now follow the prescription more formally. Let F_M be an element of the tiling sequence for H such that for all j , $|g_j F_M \Delta F_M|/|F_M| < \eta/6$, i.e. the boundary of F_M has size less than $(\eta/6)|F_M|$.

Let \mathcal{Q} be the partition of $[0, 1]$ generated by $\{B_0, \dots, B_{l-1}\}$ and $\{g_j^B B_i : i < l \text{ and } j < n\}$. Let B^A and B^B be bases of $(F_M, \eta/6)$ Rokhlin towers for the partition \mathcal{Q} , and the systems \mathcal{A} and \mathcal{B} respectively. We can assume that the measures of B^A and of B^B are equal. We let L_0 be the remainder $[0, 1] \setminus \bigcup_{h \in F_M} h^A B^A$. Let L_1 be the boundary of the tower. Then the measures of both L_0 and L_1 are less than or equal to $\eta/6$ and $[0, 1] \setminus L_0 \cup L_1$ is the interior of the tower.

For each atom a of \mathcal{Q} and $y \in B^A$, $|\#\{h \in F_M : h^A(y) \in a\}/|F_M| - \mu(a)| < \eta/6$ and similarly for all $y \in B^B$. By shrinking the bases B^A and B^B , an arbitrarily small amount, we can preserve the fact that we have $(F_M, \eta/6)$ Rokhlin towers and find partitions P_0^A, \dots, P_{r-1}^A and P_0^B, \dots, P_{r-1}^B of the bases B^A and B^B such that:

- (1) For all $s < r$ the measure of P_s^A is equal to the measure of P_s^B .
- (2) For all $s < r$ and $h \in F_M$ there are atoms a, b of \mathcal{Q} such that $h^A P_s^A \subset a$ and $h^B P_s^B \subset b$.
- (3) For each s , there is a permutation $\sigma = \sigma(s)$ of F_M such that for all but $(2\eta/6)|F_M|$ many $h \in F_M$ and all atoms a of \mathcal{Q} ,

$$h^A P_s^A \subset a \quad \text{iff} \quad \sigma(h)^B P_s^B \subset a.$$

Let $S = \{\sigma(s) : s < r\}$. Let T be the measure preserving transformation induced by the partition $\{P_0^A, \dots, P_{r-1}^A\}$ and $S = \{\sigma(s) : s < r\}$ (as in the remarks before Remark 18). Define \mathcal{A}' to be $T\mathcal{A}T^{-1}$. Then by Remark 18, \mathcal{A}' has the same orbits almost everywhere as \mathcal{A} does.

We now argue that \mathcal{A}' is in the neighborhood W . For each s , we consider the levels of the Rokhlin tower above P_s^A . We will call a level $h^A P_s^A$ *bad* if for the atom $a \in \mathcal{Q}$ with $h^A P_s^A \subset a$ we have $\sigma_s(h)^B P_s^B \not\subset a$.

For each x in the interior of the tower above P_s^A , if $x \in g_j^{A'} B_i \Delta g_j^B B_i$ then either x or $(g_j^{A'})^{-1}x$ belongs to a bad level of the tower over P_s^A , or else $(g_j^{A'})^{-1}x$ belongs to L_0 . Hence the measure of the intersection of the tower over P_s^A with $g_j^{A'} B_i \Delta g_j^B B_i$ is at most twice the measure of the union of the bad levels of the tower above P_s^A plus the measure of $\{x \in L_0 : g_j^{A'}(x) \in P_s^A\}$.

Let L_3^s be the union of the bad levels of the tower over P_s^A . Then the measure of L_3^s is less than $2(\eta/6)|F_M|\mu(P_s^A)$. Since $\bigcup_s P_s^A = B^A$, $\bigcup_s L_3^s$ has measure less than $1/|F_M|$. Thus if $L_3 = \bigcup_{s < r} L_3^s$ then the measure of L_3 is less than $2(\eta/6)$.

Thus for each j and i , the intersection of $g_j^{A'} B_i \Delta g_j^B B_i$ with the interior of the tower $\bigcup_{h \in F_M} hB^A$ has measure at most $2\mu(L_3) + \mu(L_0) < 5(\eta/6)$. Hence the total measure of $g_j^{A'} B_i \Delta g_j^B B_i$ is less than $\mu(L_0) + \mu(L_1) + 2\mu(L_3) = \eta$.

Divide each P_s into finite partitions $\{P_{s,t}\}_t$ such that the measure of each $P_{s,t}$ is less than $\epsilon/|F_M|$. Define $u_{s,t}$ by setting it equal to T on the orbits of $P_{s,t}$ and the identity otherwise. Then:

- The measure of $\{x : u_{s,t}(x) \neq x\}$ is less than ϵ .
- The $u_{s,t}$'s commute.
- T is equal to the product of all of the $u_{s,t}$.

In particular, if we enumerate the $u_{s,t}$'s as $\langle u_0, u_1, \dots, u_{p-1} \rangle$, then $\mathcal{A}_p = \mathcal{A}'$ and we have proven the subclaim. □

4.1. General amenable groups

If the amenable group H is not known to be monotilable the proof is carried out by using the quasi-tiling machinery of [13]. Roughly speaking, for any free action of an amenable group H on $[0, 1]$ by measure preserving transformations, this machinery provides a finite collection of disjoint Rokhlin towers parametrized by different Følner sets that fill up most of the space. The proof that we have given can then be carried out, *mutatis mutandis*, with these towers replacing the single tower of the monotilable case. We shall content ourselves with formulating the generalization of the Rokhlin Lemma for the general case and leave the details of the proof to the reader.

We will first formulate the basic Rokhlin lemma and then the version that is needed which incorporates an application of the ergodic theorem.

If we are given finite subsets K, F of H and a $\delta > 0$, then we will say that F is (K, δ) -invariant if $|\bigcup_{k \in K} kF \Delta F| < \delta|F|$. For K with $K = K^{-1}$ and $e \in K$, this definition is equivalent to the one given in [13].

Rokhlin's Lemma IV ([13]). *Suppose that H is an amenable group, and a finite subset K of H and $\delta, \epsilon > 0$ are given. Then there are sets $\{F_1, \dots, F_k\}$ that are (K, δ) -invariant and numbers $b_1, \dots, b_k \in [0, 1]$ with $\sum b_i > 1 - \epsilon$ so that for any free ergodic action of H on $[0, 1]$ by measure preserving transformations there are sets $\{B_1, \dots, B_k\}$ in the unit interval that satisfy:*

- (1) *For each i the sets $\{f B_i : f \in F_i\}$ are disjoint.*
- (2) *The sets $\{F_i B_i : 1 \leq i \leq k\}$ are pairwise disjoint.*
- (3) *The measure of $F_i B_i$ is b_i .*

It is important to note that the sets F_i depend only on K, δ and ϵ and not on the specific action. Also the measure of each of the $F_i B_i$ depends only on i and not on the action. For these addenda see the proof of Theorem 5 in II.§2 of [13] and Remark 3 in II.§3 there.

For the final version of the lemma assume that in addition to the action of H we are also given a finite number of measurable subsets A_j in $[0, 1]$. Then we have the following:

Rokhlin's Lemma V ([13]). *Suppose that H is an amenable group, and a finite subset K of H and $\delta, \epsilon > 0$ are given. Then there are sets $\{F_1, \dots, F_k\}$ that are (K, δ) -invariant and numbers $b_1, \dots, b_k \in [0, 1]$ with $\sum b_i > 1 - \epsilon$ so that for any free ergodic action of H on $[0, 1]$ by measure preserving transformations there are sets $\{B_1, \dots, B_k\}$ in the unit interval that satisfy:*

- (1) *For each i the sets $\{f B_i : f \in F_i\}$ are disjoint.*
- (2) *The sets $\{F_i B_i : 1 \leq i \leq k\}$ are pairwise disjoint.*
- (3) *The measure of $F_i B_i$ is b_i .*
- (4) *For all $y \in \bigcup B_i$ and $j < n$,*

$$|\#\{g \in F_N : g(y) \in A_j\}/|F_N| - \mu(A_j)| < \epsilon.$$

As before we note that the sets F_i depend only on K, δ and ϵ and not on the specific action as do the measures of the $F_i B_i$.

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