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Coxeter group actions on the complement of hyperplanes and special involutions

Abstract. We consider both standard and twisted actions of a (real) Coxeter group G on the complement $\mathcal{M}_G$ to the complexified reflection hyperplanes by combining the reflections with complex conjugation. We introduce a natural geometric class of special involutions in G and give explicit formulae which describe both actions on the total cohomology $H^*(\mathcal{M}_G, \mathbb{C})$ in terms of these involutions. As a corollary we prove that the corresponding twisted representation is regular only for the symmetric group S_n, the Weyl groups of type D_{2m+1}, E_6 and dihedral groups $I_2(2k+1)$. We also discuss the relations with the cohomology of Brieskorn's braid groups.

Keywords. Coxeter groups, hyperplane arrangements, Brieskorn's braid groups

Introduction

In 1969 V. I. Arnol'd [1] computed the cohomology algebra of the configuration space $\mathcal{M}_n$ of n distinct points of the complex plane. This remarkable short paper was the starting point of the active research in this area of mathematics at the crossroads of algebra, geometry and combinatorics.

In particular, Brieskorn [7] generalised Arnol'd's results to arbitrary irreducible Coxeter groups G and showed that the Poincaré polynomial of the complement $\mathcal{M}_G$ to the complexified reflection hyperplanes has the form

$$P(\mathcal{M}_G, t) = (1 + m_1 t) \cdots (1 + m_n t),$$

where $m_i = d_i - 1$ are the exponents of the Coxeter group G, d_i being the degrees of the generators of the algebra of G-invariants. Since the product $(1 + m_1) \cdots (1 + m_n) = d_1 \cdots d_n = |G|$ is known to be the order of the group G, it is tempting to suggest that the total cohomology space $H^*(\mathcal{M}_G) = H^*(\mathcal{M}_G, \mathbb{C})$ is the regular representation with respect to the natural action of G on $\mathcal{M}_G$. However this turns out not to be true already for the symmetric group $G = S_n$, as shown by Lehrer [18], although not far from being true.

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The starting point of this work was the following observation. Let us consider another action of G on $\mathcal{M}_G$ using the anti-holomorphic extensions of the reflections from the real to the complexified space. In other words we combine the usual action of the reflections with complex conjugation. The claim is that, for the symmetric group $G = S_n$ with this new action, $H^*(\mathcal{M}_G, \mathbb{C})$ is indeed a regular representation. We were not able to find this result in the literature, although the proof can be easily derived from known results. While looking for the simplest explanation of this fact, we have found a simple universal way to investigate this representation for any Coxeter group, both in the standard and twisted case.

The main result of this paper is the following two explicit formulae which show how far the corresponding representations are from the regular representation, and describe them as virtual representations.

Let G be a finite group generated by reflections in a real Euclidean space V of dimension n, and $\mathcal{M}_G$ be the complement to the complexified reflection hyperplanes in $V \otimes \mathbb{C}$. Let us denote the twisted representation of G on $H^*(\mathcal{M}_G, \mathbb{C})$ by $H_\epsilon^*(\mathcal{M}_G)$. Alternatively we can write

$$H_\epsilon^*(\mathcal{M}_G) = \bigoplus_{k=0}^{n} \epsilon^k \otimes H^k(\mathcal{M}_G, \mathbb{C}), \tag{1}$$

where ϵ is the alternating representation of G.

We claim that for all Coxeter groups G with the standard action on $\mathcal{M}_G$,

$$H^*(\mathcal{M}_G) = \sum_{\sigma \in X_G} (2 \operatorname{Ind}_{\langle \sigma \rangle}^{G}(1) - \rho), \tag{2}$$

and for the twisted action,

$$H_\epsilon^*(\mathcal{M}_G) = \sum_{\sigma \in X_G^\epsilon} (2 \operatorname{Ind}_{\langle \sigma \rangle}^{G}(1) - \rho). \tag{3}$$

Here ρ is the regular representation of G, X_G is a special set of involutions in G (more precisely, conjugacy classes of involutions), X_G^ϵ is the subset of X_G consisting of even involutions, and $\operatorname{Ind}_{\langle \sigma \rangle}^{G}(1)$ is the representation induced from the trivial representation of the subgroup generated by σ.

We should mention that for the standard action in most of the cases our result can be derived from the results of the papers [13, 14, 18, 19], so our main contribution here is the following universal geometric description of the set X_G.

Namely, consider any involution σ in the geometric realisation of G as a group of orthogonal transformations of a Euclidean space V and the corresponding splitting $V = V_1 \oplus V_2$ so that $V_1 = V^-(\sigma)$ and $V_2 = V^+(\sigma)$ are the eigenspaces of σ with eigenvalues -1 and 1, respectively. Let R_1 and R_2 be the intersections of the root system R of the group G with these subspaces, and G_1, G_2 be the corresponding Coxeter subgroups of G. We call the involution σ *special* if for any root $a \in R$ at least one of the projections of a onto V_1 and V_2 is proportional to a root from R_1 or R_2. In particular the identity and any simple reflection are always special involutions.

We denote by X_G the set of all conjugacy classes of special involutions in G. By choosing a representative in each class we can realise X_G as a special set of involutions in G.

For the symmetric group S_n, Weyl groups of type D_{2m+1}, E_6 and dihedral groups $I_2(2k + 1)$ the set X_G consists of two elements: the identity and the class of a simple reflection s, so we have in that case

$$H^*(\mathcal{M}_G) = 2\operatorname{Ind}_{\langle s \rangle}^G(1) \quad \text{and} \quad H_\epsilon^*(\mathcal{M}_G) = \rho.$$

For Coxeter groups of type B_n (or C_n) the set X_G consists of $2n$ involutions which in the geometric realisation have the form $\sigma_k = P_{12} \oplus (-I_k) \oplus I_{n-k-2}$, $k = 0, 1, \ldots, n-2$, and $\tau_l = (-I_l) \oplus I_{n-l}$, $l = 0, 1, \ldots, n$. Here P_{12} denotes the permutation of the first two coordinates and I_k is the $k \times k$ identity matrix.

In the case of Weyl groups of type D_{2m}, E_7, E_8 and for the icosahedral groups H_3 and H_4, X_G consists of 4 involutions: $\pm \operatorname{Id}$ and $\pm s$, where s is a simple reflection.

For F_4 we have 8 involutions which in the standard geometric realisation [6] have the form $\pm s$ with s representing two different conjugacy classes of simple reflections, $\operatorname{diag}(-1, -1, 1, 1)$, $P_{12} \oplus \operatorname{diag}(-1, 1)$ and the centre $\pm \operatorname{Id}$.

Finally, for the dihedral groups $I_2(2k)$, X_G consists of 4 elements: two conjugacy classes of simple reflections and the central elements $\pm \operatorname{Id}$.

In the Appendix we list the graphs of all equivalence classes of nontrivial special involutions using Richardson's description [22].

As a corollary we find that the twisted action of an irreducible Coxeter group G on $H^*(\mathcal{M}_G)$ is a regular representation only for the symmetric group S_n, the Weyl groups of type D_{2m+1}, E_6 and dihedral groups $I_2(2k + 1)$. Note that besides the one-dimensional Coxeter group of type A_1 this is the list of Coxeter groups with trivial centre.

Our result can be reformulated in terms of the decomposition of the cohomology into irreducible G-modules: the multiplicity $m(W)$ of the irreducible G-module W in the decomposition of $H^*(\mathcal{M}_G)$ is

$$m(W) = \sum_{\sigma \in X_G} (\dim W^+(\sigma) - \dim W^-(\sigma)), \tag{4}$$

where $W^{\pm}(\sigma) = \{v \in W : \sigma v = \pm v\}$ are the eigenspaces of the involution σ. For the twisted action, X_G is replaced by X_G^ϵ.

In particular, the trivial representation has the multiplicity

$$m(1) = |X_G|,$$

where $|X|$ denotes the number of elements in the set X. This gives us a topological interpretation of $|X_G|$ as the total Betti number of the corresponding quotient space $\Sigma_G = \mathcal{M}_G/G$. More precisely we show that the Poincaré polynomial $P(\Sigma_G, t)$ has the form

$$P(\Sigma_G, t) = \sum_{\sigma \in X_G} t^{\dim V^-(\sigma)}, \tag{5}$$

where V is the geometric representation of G and $V^-(\sigma)$ is the (-1)-eigenspace of σ in this representation. Notice that, according to a classical result due to Brieskorn and Deligne, Σ_G is the Eilenberg–Mac Lane space $K(\pi, 1)$ for the corresponding generalised braid group $\pi = B_G$, so the last formula can also be interpreted in terms of the (rational) cohomology of B_G (see [7]).

Another interesting corollary of our results is that the multiplicity $m(\epsilon)$ of the alternating representation in $H^*(\mathcal{M}_G)$ is zero. Due to (4) this is equivalent to the fact that the numbers of even and odd classes in X_G are equal. For the group S_n this was conjectured (in different terms) by Stanley [23] and proved by Hanlon in [15]. For general Coxeter groups it was first proved by Lehrer [20].

Our approach is geometrical and based on the (generalised) Lefschetz fixed point formula which says that the Lefschetz numbers for actions of finite groups are equal to the Euler characteristics of the corresponding fixed sets (see [8]). We should mention that the idea to use the Lefschetz fixed point formula is not new in this area (see e.g. [9]), however in this form it does not seem to have been explored before.

Characters and Lefschetz fixed point formula

Consider first the standard action of G on $\mathcal{M}_G$. The character of an element $g \in G$ in the corresponding representation $H^*(\mathcal{M}_G)$ is

$$\chi(g) = \sum_{k=0}^{n} \operatorname{tr} g_k^*,$$

where g_k^* is the action of g on $H^k(\mathcal{M}_G)$. Now replace the action of g by its composition with the complex conjugation which we denote as $\bar{g}$. Then because complex conjugation is acting as $(-1)^k$ on the k-th cohomology of $\mathcal{M}_G$ we have

$$\chi(g) = \sum_{k=0}^{n} (-1)^k \operatorname{tr} \bar{g}_k^*,$$

which by definition is the *Lefschetz number* $L(\bar{g})$ of the map $\bar{g}$. Now we can apply the Lefschetz fixed point formula [8] which says that this number is equal to the Euler characteristic $\chi(F_{\bar{g}})$ of the fixed set $F_{\bar{g}} = \{z \in \mathcal{M}_G : \bar{g}(z) = z\}$.

In the twisted case all the even elements of G act in the standard way but the action of all odd elements are twisted and the corresponding character of an odd $g \in G$ in $H_\epsilon^*(\mathcal{M}_G)$ is

$$\chi_\epsilon(g) = \sum_{k=0}^{n} (-1)^k \operatorname{tr} g_k^* = L(g),$$

the Lefschetz number of the standard action of g on $\mathcal{M}_G$.

Proposition 1.   (i) *The character of an element $g \in G$ for the standard action of G on $H^*(\mathcal{M}_G)$ is equal to the Euler characteristic of the fixed set $F_{\bar{g}}$.*

(ii) *The character of an odd element $g \in G$ in $H_{\epsilon}^*(\mathcal{M}_G)$ for the twisted action of G on $\mathcal{M}_G$ is equal to the Euler characteristic of the fixed set F_g.*

(iii) *The fixed set F_g is empty unless g is the identity, so the character of any odd element in the twisted representation $H_{\epsilon}^*(\mathcal{M}_G)$ is zero. The fixed set $F_{\bar{g}}$ is empty unless g is an involution, so if the order of g is greater than 2 then $\chi(g) = 0$.*

Only the last part needs a proof. A fixed point of g corresponds to an eigenvector of g in the geometric realisation with eigenvalue 1, which can be chosen real (recall that G is a real Coxeter group). But on the reals G acts freely on the set of Coxeter chambers. Thus F_g is empty unless g is the identity. Now assume that $\bar{g}$ has a fixed point $z \in \mathcal{M}_G$; then obviously $h = \bar{g}^2 = g^2$ must also have a fixed point, which means that g^2 must be the identity.

Remark. The fact that only involutions may have nonzero characters is known (see Corollary 1.10 in [14], where a combinatorial explanation of this fact is given). Our proof is close to a similar proof from [9], where the case $G = S_n$ was considered.

Now we are ready to compute the characters for the classical series.

Let us start with A_{n-1}, corresponding to the symmetric group $G = S_n$ acting by permutations on the configuration space $\mathcal{M}_n$ of n distinct points of the complex plane. Any involution σ in that case is conjugate to one of the involutions of the form $\pi_k = (12)(34) \dots (2k-1, 2k)$, $k = 1, \dots, [n/2]$, where (ij) denotes the transposition of i and j. The fixed set F_k for the action $\bar{\pi}_k$ consists of the points $(z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, z_k, \bar{z}_k, x_1, \dots, x_{n-2k})$, where $z_i \neq \bar{z}_i$ (i.e. z_i is not real), $x_i \in \mathbb{R}$, $z_i \neq z_j, z_i \neq \bar{z}_j$ if $i \neq j$ and $x_p \neq x_q$ if $p \neq q$.

It is easy to see that topologically all connected components are the same and equivalent to the product $\mathcal{M}_k \times \mathbb{R}^{n-2k}$, where $\mathcal{M}_k$ is the standard configuration space of k different points in the complex plane. Because the Euler characteristic of $\mathcal{M}_k$ is zero for $k > 1$ we conclude that only the simple reflection $s = \pi_1 = (12)$ has nonzero character. In that case we have $2(n-2)!$ contractible connected components, so $\chi(s) = 2(n-2)!$. We can formulate this fact in the following form.

Proposition 2 (G. Lehrer [18]). *As S_n-module with respect to the standard action of S_n on $\mathcal{M}_n$,*

$$H^*(\mathcal{M}_n) = 2 \operatorname{Ind}_{\langle s \rangle}^{S_n}(1),$$

where $\operatorname{Ind}_{\langle s \rangle}^{S_n}(1)$ denotes the representation of S_n induced from the trivial representation of the subgroup $\mathbb{Z}_2$ generated by the simple reflection $s = (12)$.

Indeed, the characters for the representations $\operatorname{Ind}_H^G(1)$ induced from the trivial representation of the subgroup $H \subset G$ are given by the formula

$$\chi(g) = \frac{1}{|H|} \sum_{h \sim g, \, h \in H} |C(h)|, \tag{6}$$

where $h \sim g$ means that h is conjugate to g in G, and $C(h)$ is the centraliser of h: $C(h) = \{g \in G : gh = hg\}$. The centraliser of the simple reflection s is $C(s) =$

$\mathbb{Z}_2 \times S_{n-2}$, where $\mathbb{Z}_2$ is generated by $s = (12)$ and S_{n-2} is the subgroup of S_n permuting the numbers $3, 4, \ldots, n$. According to the formula (6) only the identity e and $g \sim s$ have nonzero characters in $2 \operatorname{Ind}_{\langle s \rangle}^{S_n}(1)$. Moreover $\chi(e) = |S_n| = n!$ and $\chi(s) = |C(s)| = 2(n-2)!$. Comparing this with our previous analysis we have the proposition.

Now let us consider the twisted action of the symmetric group S_n on $\mathcal{M}_n$. From the previous analysis of the fixed sets $F_{\bar{g}}$ and Proposition 1 we deduce the following

Proposition 3. *The twisted action of S_n on $\mathcal{M}_n$ induces the regular representation on the total cohomology $H^*(\mathcal{M}_n)$.*

Consider now the case B_n. In the standard geometric realisation any involution is conjugate either to the diagonal form $\tau_k = \operatorname{diag}(-1, \ldots, -1, 1, \ldots, 1) = (-I_k) \oplus I_{n-k}$ or to the form when we have m additional elementary transposition blocks P: $\sigma_{m,k} = P \oplus \ldots \oplus P \oplus (-I_k) \oplus I_{n-k-2m}$. A simple analysis similar to the previous case shows that in the last case the fixed set consists of several connected components each topologically equivalent to $\mathcal{M}_m \times \mathbb{R}^{n-2m}$. So if the number of transpositions m is greater than 1 the Euler characteristic of this set and thus the corresponding character $\chi(\sigma_{m,k})$ is zero. One can check that in the remaining cases, namely $\sigma_k = \sigma_{1,k}$ and τ_k, all the components are contractible and the action of the corresponding centralisers $C(\sigma)$ on them is effective. Thus if we define the set X_{B_n} as the union of the $2n$ involutions $\sigma_k, k = 0, \ldots, n-2$ and $\tau_l, l = 0, 1, \ldots, n$, we have the following

Proposition 4.
$$H^*(\mathcal{M}_{B_n}) = \sum_{\sigma \in X_{B_n}} (2 \operatorname{Ind}_{\langle \sigma \rangle}^{B_n}(1) - \rho), \tag{7}$$

where ρ is the regular representation of B_n. In the twisted case one should take the sum over the subset of X_{B_n} consisting of the involutions which are even elements of G.

For the standard action this result was first obtained by Lehrer in [19].

In the D_n case the results are slightly different for odd and even n.

For an odd $n = 2m + 1$ any involution is conjugate to one of the following involutions: $\sigma_{2k}, k = 0, 1, \ldots, m-1$ and $\tau_{2l}, l = 0, 1, \ldots, m$ (with the same notations as in the previous case).

The fixed set of σ_{2k} consists of the points $(z, \bar{z}, ix_1, \ldots, ix_{2k}, y_1, \ldots, y_{n-2k-2})$, where $z \neq \pm\bar{z}$ (i.e. z is not real or purely imaginary), $x_i, y_i \in \mathbb{R}$, $x_p \neq \pm x_q$, $y_p \neq \pm y_q$ if $p \neq q$. The number of connected components of the fixed set in that case is equal to the order of the group $D_2 \times D_{2k} \times D_{n-2k-2}$ which acts on this set. However if $k > 0$ we have to exclude from this set also the subspaces of codimension 2 given by the equations $x_j = y_l = 0$ for all $j = 1, \ldots, 2k$ and $l = 1, \ldots, n-2k-2$. One can check that as a result the Euler characteristic of all connected components (and thus of the whole fixed set) in that case is 0. Similar arguments show that the same is true for the fixed set of the involutions τ_{2l}.

In the only remaining case of the simple reflection σ_0 all the connected components of the corresponding fixed set are contractible. The centraliser $C(\sigma_0)$ coincides with the subgroup $D_2 \times D_{n-2}$, which acts freely and transitively on these components.

For the groups of type D_n with even $n = 2m$ the involutions σ_0, σ_{2m-2} and the central element $\tau_{2m} = -I_n$ have the centralisers whose order is equal to the number of connected components of the corresponding fixed sets, which are all contractible in this case. The Euler characteristic of the fixed set of any other involution can be shown to be zero in the same way as in the odd case.

Thus we have the following

Proposition 5. *For the Coxeter group G of type D_n with the standard action we have*

$$H^*(\mathcal{M}_G) = 2\operatorname{Ind}^G_{\langle\sigma_0\rangle}(1) \tag{8}$$

in the odd case $n = 2m + 1$ and

$$H^*(\mathcal{M}_G) = \rho + (2\operatorname{Ind}^G_{\langle\sigma_0\rangle}(1) - \rho) + (2\operatorname{Ind}^G_{\langle\sigma_{2m-2}\rangle}(1) - \rho) + (2\operatorname{Ind}^G_{\langle\tau_{2m}\rangle}(1) - \rho) \tag{9}$$

in the even case $n = 2m$. For the twisted action the formulas are respectively

$$H^*_\epsilon(\mathcal{M}_G) = \rho \tag{10}$$

in the odd case and

$$H^*_\epsilon(\mathcal{M}_G) = 2\operatorname{Ind}^G_{\langle\tau_{2m}\rangle}(1) \tag{11}$$

in the even case.

In order to investigate the general case and to understand the nature of the exceptional set X_G we will need the general results about involutions in Coxeter groups, summarised in the next section.

Involutions in Coxeter groups and their centralisers

We start with the description of the conjugacy classes of involutions in a general Coxeter group due to Richardson [22]. We refer to Bourbaki [6] or Humphreys [17] for the standard definitions.

Let G be a Coxeter group, and Γ be its Coxeter graph. Recall that the vertices of Γ correspond to the generating reflections, which form the set S. We will identify G with its geometric realisation in the Euclidean vector space V, in which $s \in S$ acts as the reflection with respect to the hyperplane orthogonal to the corresponding root e_s.

Let J be a subset of S, and G_J be the subgroup G generated by J (such a subgroup is called *parabolic*). Let also $J^* = \{e_s : s \in J\}$ and V_J be the subspace of V generated by J^*. Following Richardson we say that J satisfies the (-1)-*condition* if G_J contains an element σ_J which acts on V_J as $-\operatorname{Id}$. This element, being in G_J, acts as the identity on the orthogonal complement of V_J. Thus σ_J is uniquely determined and is an involution.

Proposition 6 (Richardson [22]). *Let σ be any involution in G. Then*

(i) *there exists a subset $J \subset S$ satisfying the (-1)-condition such that σ is conjugate to σ_J;*

(ii) *the involutions σ_J and σ_K are conjugate in G if and only if $g(J^*) = K^*$ for some $g \in G$.*

Richardson also gave an algorithm for testing G-equivalence based on the results by Howlett [16] and Deodhar [11].

For an irreducible Coxeter group G the whole set $J = S$ satisfies the (-1)-condition only in the following cases:

$$A_1, B_n, D_{2n}, E_7, E_8, F_4, H_3, H_4, I_2(2n).$$

This means that each connected component of the Coxeter graph corresponding to the group G_J for any J satisfying the (-1)-condition must be of that form. In particular we see that the components of type A_n with $n > 1$ are forbidden, which imposes strong restrictions on such subsets J. In the Appendix we give a subset J for each conjugacy class of special involutions.

To describe the centralisers of the involutions σ_J we can use the results by Howlett [16] who described the normalisers of the parabolic subgroups in a general Coxeter group. Indeed, we have the following

Proposition 7. *The centraliser $C(\sigma_J)$ coincides with the normaliser $N(G_J)$ of the corresponding parabolic subgroup G_J.*

Recall that the *normaliser $N(H)$* of a subgroup $H \subset G$ consists of the elements $g \in G$ such that $gHg^{-1} = H$. Consider the orthogonal splitting $V = V_J \oplus V_J^\perp$, corresponding to the spectral decomposition of σ_J. Obviously the normaliser $N(G_J)$ preserves this splitting and thus $N(G_J)$ is a subgroup of the centraliser $C(\sigma_J)$. To show the opposite inclusion take any $g \in C(\sigma_J)$; then $g(V_J) = V_J$ and therefore $g(e_s)$ belongs to the intersection of the root system R with the subspace V_J. But according to [22, Proposition 1.10] this intersection coincides with the root system of the group G_J. This means that gsg^{-1} belongs to G_J for any generating reflection $s \in J$ and thus g belongs to the normaliser $N(G_J)$.

So the question now is only what are the special involutions which form the set X_G.

Special involutions in Coxeter groups

Let σ be any involution in the Coxeter group G, and $V = V_1 \oplus V_2$ be the corresponding spectral decomposition of its geometric realisation, where V_1 and V_2 are the eigenspaces with eigenvalues -1 and 1, respectively. According to the previous section one can assume that $V_1 = V_J$ for some subset $J \in S$.

Consider the intersections R_1 and R_2 of the root system R of G with the subspaces V_1 and V_2. According to [22], [24] they are the root systems of the Coxeter subgroups G_1 and G_2, where G_1 is the corresponding parabolic subgroup G_J and G_2 is the subgroup of G consisting of the elements fixing the subspace V_1.

Definition. *We will call the involution σ special if for any root from R at least one of its projections onto V_1 and V_2 is proportional to a root from R_1 or R_2.*

In particular, the identity and simple reflections are special involutions for all Coxeter groups.

The following proposition explains the importance of this notion for our problem.

Proposition 8. *For any special involution σ all the connected components of the fixed set $F_{\bar{\sigma}}$ are contractible. Their number N_σ is equal to $|G_1||G_2|$, the product of the orders of the corresponding groups G_1 and G_2.*

Proof. The fixed set $F_{\bar{\sigma}}$ is the subspace $V_2 \oplus i V_1$ minus the intersections with the reflection hyperplanes. The hyperplanes corresponding to the roots from R_1 and R_2 split this subspace into $|G_1||G_2|$ contractible connected components. The meaning of the condition on the special involutions is to make all other intersections redundant. Indeed, the intersection with the hyperplane $(\alpha, z) = 0$ is equivalent to two conditions: $(\alpha_1, x) = 0$ and $(\alpha_2, y) = 0$, where α_1 and α_2 are the projections of the root α on V_1 and V_2 respectively, and $x \in V_1$, $y \in V_2$. If at least one of the vectors α_i is proportional to some root from R_i this intersection will be contained in the hyperplanes already considered.

Corollary 1. *The character of the action of any special involution σ on $H^*(\mathcal{M}_G)$ is nonzero and equal to $|G_1||G_2|$.*

It turns out that the converse statement is also true.

Proposition 9. *If the character $\chi(g)$ is not zero in $H^*(\mathcal{M}_G)$ then g is a special involution.*

Unfortunately the only proof we have is case by case check. For the classical series this follows from the fact that the special involutions in that case are exactly those described in the first section of this paper. For the exceptional Weyl groups one can also use the results of [14].

The list of special involutions (up to conjugation) for all irreducible Coxeter groups is given in the Introduction (see also the Appendix for the corresponding Richardson graphs). It is easy to check that they all have the following property which is very important for us.

Proposition 10. *The centraliser $C(\sigma)$ of any special involution σ coincides with the product of the corresponding Coxeter subgroups $G_1 \times G_2$.*

Obviously $G_1 \times G_2$ is a subgroup of $C(\sigma)$ but the equality $C(\sigma) = G_1 \times G_2$ is not true for a general involution (an example is any of the involutions σ_{2k} with $k > 0$ in the D_{2m+1}-case, for which $G_1 \times G_2$ is a subgroup of $C(\sigma)$ of index 2).

Summarising, we have the following main

Theorem 1. *Let X_G be the set of conjugacy classes of special involutions in the Coxeter group G. Then the total cohomology $H^*(\mathcal{M}_G)$ as G-module with respect to the standard action of G on $\mathcal{M}_G$ can be represented in the form*

$$H^*(\mathcal{M}_G) = \sum_{\sigma \in X_G} (2 \operatorname{Ind}_{\langle \sigma \rangle}^G (1) - \rho). \tag{12}$$

For the twisted action one should replace in this formula X_G by its subset consisting of the conjugacy classes of even special involutions.

In particular, we see that $H_\epsilon^*(\mathcal{M}_G)$ is the regular representation only for the symmetric group S_n, the Weyl groups of type D_{2m+1}, E_6 and dihedral groups $I_2(2k+1)$.

Applying the Frobenius reciprocity formula for the characters of induced representations we have the following corollary. Let W be an irreducible representation of G, and

$$W^\pm(\sigma) = \{v \in W : \sigma v = \pm v\}$$

be the eigenspaces of the involution σ with eigenvalues ± 1 respectively.

Corollary 2. *The multiplicity $m(W)$ in the decomposition of $H^*(\mathcal{M}_G)$ is equal to*

$$m(W) = \sum_{\sigma \in X_G} (\dim W^+(\sigma) - \dim W^-(\sigma)).$$

In particular, for the trivial and alternating representations we have respectively

$$m(1) = |X_G| \quad and \quad m(\epsilon) = |X_G^\epsilon| - |X_G^o|,$$

where X_G^o is the subset of X_G consisting of odd special involutions.

In other words the number of special involutions (up to conjugacy) is equal to the number of invariant cohomology classes in $H^*(\mathcal{M}_G)$. The corresponding numbers are given in the table below.

Numbers $|X_G|$ of conjugacy classes of special involutions

A_n	B_n	D_n, n odd	D_n, n even	E_6	E_7	E_8	F_4	H_3	H_4	$I_2(n), n$ odd	$I_2(n), n$ even
2	$2n$	2	4	2	4	4	8	4	4	2	4

A simple analysis of this list leads to the following

Proposition 11. *For any irreducible Coxeter group G the numbers of even and odd involutions in the special set X_G are equal:*

$$|X_G^o| = |X_G^\epsilon| = \frac{1}{2}|X_G|.$$

The multiplicity $m(\epsilon)$ of the alternating representation in $H^*(\mathcal{M}_G)$ is zero for all G.

In other words the alternating representation never appears in the cohomology of $\mathcal{M}_G$. For the symmetric group the fact that $m(\epsilon) = 0$ was conjectured by Stanley [23] (in different, combinatorial terms) and proved by Hanlon in [15]. The general case was settled by Lehrer in [20]. We give some explanations of this remarkable fact in the next section.

Special involutions and cohomology of Brieskorn's braid groups

Consider now the corresponding quotient space $\Sigma_G = \mathcal{M}_G/G$. The fundamental group of this space is called *Brieskorn's braid group* and denoted as B_G. The fundamental group of $\mathcal{M}_G$ is called *Brieskorn's pure braid group* and denoted as P_G.

It is known after Brieskorn and Deligne that $\mathcal{M}_G$ and Σ_G are Eilenberg–Mac Lane spaces $K(\pi, 1)$ for π the pure and usual braid groups P_G and B_G respectively (see [7], [10]), so the cohomology of these spaces can be interpreted as cohomology of the corresponding braid groups. The investigation of this cohomology was initiated by Arnol'd in [1], [2] who considered the case of the symmetric group. The rational (or complex) cohomology of the braid groups related to an arbitrary Coxeter group was computed by Brieskorn [7]. Later Orlik and Solomon [21] found an elegant description of the de Rham cohomology as the algebra generated by the differential forms $\omega_\alpha = d \log \alpha, \alpha \in R$, with explicitly given relations (defining the so-called *Orlik–Solomon algebra $\mathcal{A}_G$*).

Since $H^*(\Sigma_G, \mathbb{C})$ is simply the G-invariant part of $H^*(\mathcal{M}_G, \mathbb{C})$, it follows from the previous section that the total Betti number of Σ_G, which is the dimension of $H^*(\Sigma_G, \mathbb{C})$, is equal to $|X_G|$. The relation between special involutions in G and the cohomology $H^*(\Sigma_G, \mathbb{C})$ can be described more precisely in the following way.

Proposition 12. *The Poincaré polynomial $P(\Sigma_G, t)$ of the cohomology of Brieskorn's braid group B_G has the form*

$$P(\Sigma_G, t) = \sum_{\sigma \in X_G} t^{\dim V^-(\sigma)}, \tag{13}$$

where V is the geometric representation of G and $V^-(\sigma)$ is the (-1)-eigenspace of σ in this representation.

Explicitly, these polynomials are:

A_n : $\qquad P = 1 + t$

B_n : $\qquad P = 1 + 2t + 2t^2 + \ldots + 2t^{n-1} + t^n = (1+t)(1+t+t^2+\ldots+t^{n-1})$

D_n, n odd: $\quad P = 1 + t$

D_n, n even: $\quad P = 1 + t + t^{n-1} + t^n = (1+t)(1+t^{n-1})$

E_6 : $\qquad P = 1 + t$

E_7 : $\qquad P = 1 + t + t^6 + t^7 = (1+t)(1+t^6)$

E_8 : $\qquad P = 1 + t + t^7 + t^8 = (1+t)(1+t^7)$

F_4 : $\qquad P = 1 + 2t + 2t^2 + 2t^3 + t^4 = (1+t)(1+t+t^2+t^3)$

H_3 : $\qquad P = 1 + t + t^2 + t^3 = (1+t)(1+t^2)$

H_4 : $\qquad P = 1 + t + t^3 + t^4 = (1+t)(1+t^3)$

$I_2(n), n$ odd: $\; P = 1 + t$

$I_2(n), n$ even: $P = 1 + 2t + t^2 = (1+t)^2$.

The simplest proof of this proposition is by comparison of the Brieskorn formulas from [7] and our list of simple involutions. A more satisfactory proof may be found in the following way in terms of the corresponding Orlik–Solomon algebra $\mathcal{A}_G \approx H^*(\mathcal{M}_G)$.

Let σ be a special involution and S_σ be the set of simple roots in the Coxeter subsystem R_1, which is the set of roots $\alpha \in R$ such that $\sigma\alpha = -\alpha$ (see the previous section).

Conjecture. *For any special involution σ the symmetrisation of the product $\Omega_\sigma = \prod_{\alpha \in S_\sigma} \omega_\alpha$ by the action of the group G is a nonzero element in $\mathcal{A}_G$. Any G-invariant element in this algebra is a linear combination of such elements.*

We have checked this for all Coxeter groups except E_7, E_8, F_4, H_3, H_4, but the proof is by straightforward calculation with the use of the Orlik–Solomon relations. It would be nice to find a more conceptual proof for all Coxeter groups.

Notice that in the S_n case our claim reduces to Arnold's result saying that the symmetrisation of any element from the cohomology of the pure braid group of degree more than 1 is equal to zero and the only G-invariant element of degree 1 is given by the differential form $\sum_{i \neq j} d \log(z_i - z_j)$ (see Corollary 6 in [1]).

Notice that as a corollary of Proposition 11 we see that the anti-symmetrisation of any element in the Orlik–Solomon algebra is zero. This is also related to the fact that all the polynomials $P(\Sigma_G, t)$ are divisible by $t + 1$. Indeed, from the previous section we know that $m(\epsilon) = |X_G^\epsilon| - |X_G^o|$, which due to Proposition 12 is equal to $P(\Sigma_G, -1)$ and thus is zero.

Although the fact that $P(\Sigma_G, t)$ is divisible by $t + 1$ is transparent from the list of all these polynomials given above we will give here an independent topological explanation.

Namely consider the *projectivisation* $P\mathcal{M}_G$ of the space $\mathcal{M}_G$. We have a diffeomorphism

$$\mathcal{M}_G \approx P\mathcal{M}_G \times \mathbb{C}^*.$$

Indeed, for any root α the map $x \mapsto ([x], (\alpha, x))$ establishes such a diffeomorphism. Notice that it is compatible with the action of the group G if we assume that the action on $\mathbb{C}^*$ is trivial.

On the cohomology level we have an isomorphism

$$H^*(\mathcal{M}_G) = H^*(P\mathcal{M}_G) \times H^*(\mathbb{C}^*),$$

which immediately implies the following result.

Let W be any irreducible representation of G and $P_W(t)$ be the Poincaré polynomials of the multiplicities of W in $H^k(\mathcal{M}_G)$. An explicit description of these polynomials is one of the most interesting open problems in this area. For the trivial representation the corresponding polynomial coincides with the Poincaré polynomial $P(\Sigma_G, t)$ according to Proposition 12.

Proposition 13. *All irreducible representations of the Coxeter group G appear in $H^*(\mathcal{M}_G)$ in pairs in degrees differing by 1. The corresponding polynomials $P_W(t)$ are divisible by $t + 1$.*

In particular, this implies that $P(\Sigma_G, -1) = 0$ and (modulo Propositions 11 and 12) the fact that the alternating representation never appears in the cohomology of generalised pure braid groups.

Concluding remarks

We have shown that the action of the Coxeter groups G on the total cohomology of the corresponding complement space $\mathcal{M}_G$ can be described in a very simple way (2) in terms of the special involutions. Although one might expect a formula like that knowing that only involutions may have nonzero characters in this representation, there are two facts which we found surprising:

1) the involutions with nonzero characters admit a very simple geometric characterisation;
2) the character of the action of such an involution σ on $H^*(\mathcal{M}_G)$ is exactly the order of the corresponding centraliser $C(\sigma)$.

Special involutions also appear to play a key role in the description of the multiplicative structure of the cohomology of the Brieskorn braid group: if the conjecture formulated in the last section is true, a basis of representatives of the cohomology in the Orlik–Solomon algebra can be given in terms of special involutions.

The nature of the set X_G also needs a better understanding. In particular, it may be worth looking at the geometry of the subgraphs Γ_J in the Coxeter graph Γ corresponding to the special involutions (see the Appendix for a complete list of these subgraphs). We would like to note that in all the cases except D_4, Γ_J consists of at most two connected components, which together with the (-1)-condition implies very strong restrictions on J. The only exception is the graph of the involution $-s$ in D_4, which consists of three components.

It would be interesting to generalise our results to the space $\mathcal{M}_G(\mathbb{R}^N)$ which is $V \otimes \mathbb{R}^N$ without the corresponding root subspaces $\Pi_\alpha \otimes \mathbb{R}^N, \alpha \in R$ (our case corresponds to $N = 2$). For the symmetric group $G = S_n$ this is the configuration space of n distinct points in $\mathbb{R}^N$; this case was investigated by Cohen and Taylor in [9]. In this relation we would like to mention the recent very interesting papers [3, 4, 5, 12]. In particular, the idea of Atiyah [4] to use equivariant cohomology may be the clue to understanding the graded structure of the G-module $H^*(\mathcal{M}_G)$.

Another obvious generalisation to look at is the case of complex reflection groups.

Appendix: Richardson graphs of special involutions

The following table lists the equivalence classes of nontrivial special involutions for all Coxeter groups. For each class we give a representative σ_J associated with a subset J of the set of nodes of the Coxeter graph by the Richardson correspondence (see Proposition 6): the nodes in J, whose simple roots span the (-1)-eigenspace of σ, are coloured in black. The notations in the third column follow the description in the Introduction. Besides the special involutions listed here there is the identity, which is special for all groups and corresponds to a Coxeter graph with white nodes only. The conventions for Coxeter graphs are those of [6, Ch. IV, §1, nᵒ 9] except that we draw a double edge instead of an edge with label 4.

G	(Γ, J)	σ_J
A_n		s
B_n		σ_k
		τ_l
D_n, n odd		s
D_n, n even		s
		$-s$
		$-\operatorname{Id}$
E_6		s
E_7		s
		$-s$
		$-\operatorname{Id}$
E_8		s
		$-s$
		$-\operatorname{Id}$

F_4	●———○═══○———○	s_1
	○———○═══○———●	s_2
	○———●═══●———○	$\mathrm{diag}(-1,-1,1,1)$
	●———○═══●———○	$P_{12} \oplus \mathrm{diag}(-1,1)$
	○———●═══●———●	$-s_1$
	●———●═══●———○	$-s_2$
	●———●═══●———●	$-\,\mathrm{Id}$
H_3	●—5—○———○	s
	●—5—○———●	$-s$
	●—5—●———●	$-\,\mathrm{Id}$
H_4	●—5—○———○———○	s
	●—5—●———●———○	$-s$
	●—5—●———●———●	$-\,\mathrm{Id}$
$I_2(m)$, m odd	●—m—○	s
$I_2(m)$, m even	●—m—○	s_1
	○—m—●	s_2
	●—m—●	$-\,\mathrm{Id}$

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