J. Eur. Math. Soc. 7, 51–68

JEMS

Karma Dajani · Martijn de Vries

Measures of maximal entropy for random β-expansions

Received September 11, 2003 and in revised form January 26, 2004

Abstract. Let $\beta > 1$ be a non-integer. We consider β-expansions of the form $\sum_{i=1}^{\infty} d_i/\beta^i$, where the digits $(d_i)_{i\geq 1}$ are generated by means of a Borel map K_β defined on $\{0,1\}^{\mathbb{N}} \times [0, \lfloor\beta\rfloor/(\beta-1)]$. We show that K_β has a unique mixing measure ν_β of maximal entropy with marginal measure an infinite convolution of Bernoulli measures. Furthermore, under the measure ν_β the digits $(d_i)_{i\geq 1}$ form a uniform Bernoulli process. In case 1 has a finite greedy expansion with positive coefficients, the measure of maximal entropy is Markov. We also discuss the uniqueness of β-expansions.

Keywords. Greedy expansions, lazy expansions, Markov chains, measures of maximal entropy

1. Introduction

Let $\beta > 1$ be a non-integer. There are two well-known expansions of numbers x in $[0, \lfloor\beta\rfloor/(\beta-1)]$ of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}$$

with $a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$. The largest in lexicographical order is the *greedy expansion* ([P], [R1], [R2]), and the smallest is the *lazy expansion* ([JS], [EJK], [DK1]). The greedy expansion is obtained by iterating the *greedy transformation* $T_\beta : [0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$, defined by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & 0 \leq x < 1, \\ \beta x - \lfloor\beta\rfloor, & 1 \leq x \leq \lfloor\beta\rfloor/(\beta-1). \end{cases}$$

The lazy expansion is obtained by iterating the *lazy transformation* $S_\beta : [0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$, defined by

$$S_\beta(x) = \beta x - d \quad \text{for } x \in \Delta(d),$$

K. Dajani: Universiteit Utrecht, Fac. Wiskunde en Informatica and MRI, Budapestlaan 6, P.O. Box 80.000, 3508 TA Utrecht, the Netherlands; e-mail: dajani@math.uu.nl

M. de Vries: Vrije Universiteit Amsterdam, Faculty of Exact Sciences, de Boelelaan 1081, 1081 HV Amsterdam, the Netherlands; e-mail: mdvries@cs.vu.nl

Mathematics Subject Classification (2000): 28D05

where

$$\Delta(0) = \left[0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right],$$

and

$$\begin{aligned}\Delta(d) &= \left(\frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - d + 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - d}{\beta}\right] \\ &= \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d}{\beta}\right], \quad d \in \{1, \dots, \lfloor\beta\rfloor\}.\end{aligned}$$

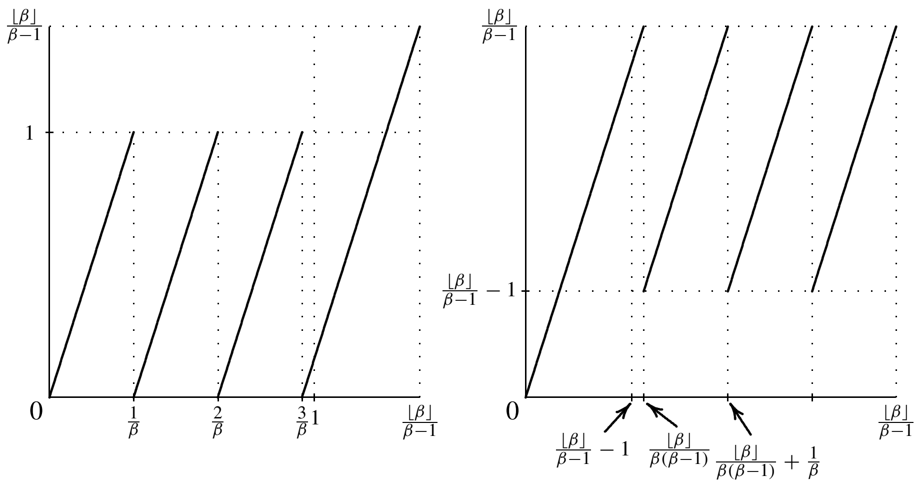

Fig. 1. The greedy map T_β (left), and the lazy map S_β (right). Here $\beta = \pi$.

We denote by μ_β the extended *Parry measure* (see [P], [G]) on $[0, \lfloor\beta\rfloor/(\beta-1)]$ which is absolutely continuous with respect to Lebesgue measure, and with density

$$h_\beta(x) = \begin{cases} \dfrac{1}{F(\beta)} \displaystyle\sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T_\beta^n(1))}(x), & 0 \le x < 1, \\ 0, & 1 \le x \le \lfloor\beta\rfloor/(\beta-1), \end{cases}$$

where $F(\beta) = \int_0^1 (\sum_{x < T_\beta^n(1)} 1/\beta^n)\, dx$ is a normalizing constant.

Define $\ell : [0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$\ell(x) = \frac{\lfloor\beta\rfloor}{\beta-1} - x,$$

and consider the *lazy measure* ρ_β defined on $[0, \lfloor\beta\rfloor/(\beta-1)]$ by $\rho_\beta(A) = \mu_\beta(\ell(A))$ for every measurable set A. It is easy to see ([DK1]) that ℓ is a measurable isomorphism between $([0, \lfloor\beta\rfloor/(\beta-1)], \mu_\beta, T_\beta)$ and $([0, \lfloor\beta\rfloor/(\beta-1)], \rho_\beta, S_\beta)$.

In order to produce other expansions in a dynamical way, a new β-transformation K_β was introduced in [DK2]. The expansions generated by iterating this map are random mixtures of greedy and lazy expansions. This is done as follows. Superimpose the greedy map and the corresponding lazy map on $[0, \lfloor\beta\rfloor/(\beta-1)]$, one gets $\lfloor\beta\rfloor$ overlapping regions of the form

$$S_k = \left[\frac{k}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}\right], \qquad k = 1, \dots, \lfloor\beta\rfloor,$$

which one refers to as *switch regions*. On S_k, the greedy map assigns the digit k, while the lazy map assigns the digit $k-1$. Outside these switch regions both maps are identical, and hence they assign the same digits. Now, define a new random expansion in base β by randomizing the choice of the map used in the switch regions. So, whenever x belongs to a switch region flip a coin to decide which map will be applied to x, and hence which digit will be assigned. To be more precise, partition the interval $[0, \lfloor\beta\rfloor/(\beta-1)]$ into switch regions S_k and *equality regions* E_k, where

$$E_k = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}, \frac{k+1}{\beta}\right), \qquad k = 1, \dots, \lfloor\beta\rfloor - 1,$$
$$E_0 = \left[0, \frac{1}{\beta}\right), \qquad E_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right].$$

Let

$$S = \bigcup_{k=1}^{\lfloor\beta\rfloor} S_k, \qquad E = \bigcup_{k=0}^{\lfloor\beta\rfloor} E_k,$$

and consider $\Omega = \{0,1\}^{\mathbb{N}}$ with the product σ-algebra $\mathcal{A}$. Let $\sigma : \Omega \to \Omega$ be the left shift, and define $K_\beta : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)]$ by

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - k), & x \in E_k,\ k = 0, 1, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - k), & x \in S_k \text{ and } \omega_1 = 1,\ k = 1, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - k + 1), & x \in S_k \text{ and } \omega_1 = 0,\ k = 1, \dots, \lfloor\beta\rfloor. \end{cases}$$

The elements of Ω represent the coin tosses ("heads" $=1$ and "tails" $=0$) used every time the orbit hits a switch region. Let

$$d_1 = d_1(\omega, x) = \begin{cases} k & \text{if } x \in E_k,\ k = 0, 1, \dots, \lfloor\beta\rfloor, \\ & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_k,\ k = 1, \dots, \lfloor\beta\rfloor, \\ k-1 & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_k,\ k = 1, \dots, \lfloor\beta\rfloor. \end{cases}$$

Then

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - d_1) & \text{if } x \in E, \\ (\sigma(\omega), \beta x - d_1) & \text{if } x \in S. \end{cases}$$

Set $d_n = d_n(\omega, x) = d_1(K_\beta^{n-1}(\omega, x))$, and let $\pi_2 : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to [0, \lfloor\beta\rfloor/(\beta - 1)]$ be the canonical projection onto the second coordinate. Then

$$\pi_2(K_\beta^n(\omega, x)) = \beta^n x - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n,$$

and rewriting yields

$$x = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi_2(K_\beta^n(\omega, x))}{\beta^n}.$$

Since $\pi_2(K_\beta^n(\omega, x)) \in [0, \lfloor\beta\rfloor/(\beta - 1)]$, it follows that

$$x - \sum_{i=1}^{n} \frac{d_i}{\beta^i} = \frac{\pi_2(K_\beta^n(\omega, x))}{\beta^n} \to 0 \quad \text{as } n \to \infty.$$

This shows that for all $\omega \in \Omega$ and for all $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$ one has

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i}.$$

The random procedure just described shows that to each $\omega \in \Omega$ corresponds an algorithm that produces expansions in base β. Further, if we identify the point (ω, x) with $(\omega, (d_1(\omega, x), d_2(\omega, x), \ldots))$, then the action of K_β on the second coordinate corresponds to the left shift.

In [DK2], the dynamical properties of the map K_β were studied for β satisfying $\beta^2 = n\beta + k$ (with $1 \le k \le n$) and $\beta^n = \beta^{n-1} + \cdots + \beta + 1$. It was shown that for these values of β, the underlying random β-transformation is isomorphic to a mixing Markov chain. However, the invariant measure considered in [DK2] is not the measure of maximal entropy (see Section 4, Remarks 6(3)). In this paper, we study the dynamical properties of K_β for *any* non-integer $\beta > 1$. In Section 2, we show that the map K_β captures all possible expansions in base β which are lexicographically ordered by the natural lexicographical ordering on Ω. We also briefly discuss unique expansions. In Section 3, we prove that the maximal entropy of K_β is $\log(1 + \lfloor\beta\rfloor)$. Further, K_β has a unique measure ν_β of maximal entropy under which the random digits (d_i), generated by the map K_β, form a uniform Bernoulli process. Moreover, the projection of the measure ν_β on the second coordinate is an infinite convolution of Bernoulli measures. In Section 4, we show that if 1 has a finite greedy expansion of the form $1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n$ with $b_i \ge 1$ for $i = 1, \ldots, n$ and $n \ge 2$, then the measure ν_β is Markov, and the underlying Markov chain is explicitly given.

2. Basic properties of random β-transformations

Let $<_{\text{lex}}$ and $\le_{\text{lex}}$ denote the lexicographical ordering on both Ω and $\{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{N}}$. For each $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$, consider the set

$$D_x = \{(d_1(\omega, x), d_2(\omega, x), \ldots) : \omega \in \Omega\}.$$

We now show that the elements of D_x are ordered by the lexicographical ordering on Ω.

Theorem 1. *Suppose $\omega, \omega' \in \Omega$ are such that $\omega <_{\text{lex}} \omega'$. Then*

$$(d_1(\omega, x), d_2(\omega, x), \ldots) \leq_{\text{lex}} (d_1(\omega', x), d_2(\omega', x), \ldots).$$

Proof. Let i be the first index where ω and ω' differ. Since $\omega <_{\text{lex}} \omega'$, we have $\omega_i = 0$ and $\omega'_i = 1$. Notice that $\pi_2(K_\beta^j(\omega, x)) = \pi_2(K_\beta^j(\omega', x))$ for $j = 0, \ldots, t_i$, where $t_i \geq 0$ is the time of the i^{th} visit to the region $\Omega \times S$ of the orbit of (ω, x) under K_β. Then $d_j(\omega, x) = d_j(\omega', x)$ for all $j \leq t_i$.

If $t_i = \infty$, then $d_j(\omega, x) = d_j(\omega', x)$ for all j. If $t_i < \infty$, then $K_\beta^{t_i}(\omega, x) = K_\beta^{t_i}(\omega', x) \in \Omega \times S$. Since $\omega_i = 0$ and $\omega'_i = 1$, it follows that $d_{t_i+1}(\omega', x) = d_{t_i+1}(\omega, x) + 1$. Hence,

$$(d_1(\omega, x), d_2(\omega, x), \ldots) <_{\text{lex}} (d_1(\omega', x), d_2(\omega', x), \ldots). \qquad \square$$

The next theorem shows that for all $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$, any representation of x of the form $x = \sum_{i=1}^{\infty} a_i/\beta^i$ with $a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$ can be generated by means of the map K_β by choosing an appropriate $\omega \in \Omega$.

Theorem 2. *Let $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$, and let $x = \sum_{i=1}^{\infty} a_i/\beta^i$ with $a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$ be a representation of x in base β. Then there exists an $\omega \in \Omega$ such that $a_i = d_i(\omega, x)$.*

For the proof we need the following lemma.

Lemma 1. *For $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$, one has*

(i) *If $x \in E_j$ for some $j \in \{0, \ldots, \lfloor\beta\rfloor\}$, then $a_1 = j$.*
(ii) *If $x \in S_j$ for some $j \in \{1, \ldots, \lfloor\beta\rfloor\}$, then $a_1 \in \{j - 1, j\}$.*

Proof. The proof is by contradiction.

(i) Suppose $a_1 \neq j$. If $a_1 \leq j - 1$, then $j \geq 1$ and

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \leq \frac{j-1}{\beta} + \sum_{i=2}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^i} = \frac{j-1}{\beta} + \frac{\lfloor\beta\rfloor}{\beta(\beta - 1)}.$$

If $a_1 \geq j + 1$, then $j \leq \lfloor\beta\rfloor - 1$ and $x \geq (j + 1)/\beta$. In both cases $x \notin E_j$.

(ii) Suppose $a_1 \notin \{j - 1, j\}$. If $a_1 \leq j - 2$, then $j \geq 2$ and

$$x \leq \frac{j-2}{\beta} + \frac{\lfloor\beta\rfloor}{\beta(\beta - 1)}.$$

If $a_1 \geq j + 1$, then $j \leq \lfloor\beta\rfloor - 1$ and $x \geq (j + 1)/\beta$. In both cases $x \notin S_j$. $\square$

Proof of Theorem 2. Define the numbers $\{x_n : n \in \mathbb{N}\}$ by $x_n = \sum_{i=1}^{\infty} a_{i+n-1}/\beta^i$. Notice that $x_1 = x$. Furthermore, we define a set $\{\ell_n(x) : n \in \mathbb{N}\}$ that keeps track of the number of times we flip a coin. More precisely,

$$\ell_n(x) = \sum_{i=1}^{n} \mathbf{1}_S(x_i).$$

We use induction on the number of digits already determined.

- If $x \in E_j$, then $\ell_1(x) = 0$ and by Lemma 1, $a_1 = j$. We set $\Omega_1 = \Omega$.
- If $x \in S_j$, then $\ell_1(x) = 1$ and by Lemma 1, $a_1 \in \{j-1, j\}$.
  — If $a_1 = j - 1$, we set $\Omega_1 = \{\omega \in \Omega : \omega_1 = 0\}$.
  — If $a_1 = j$, we set $\Omega_1 = \{\omega \in \Omega : \omega_1 = 1\}$.

It follows that Ω_1 is a cylinder of length $\ell_1(x)$, and $d_1(\omega, x) = a_1$ for all $\omega \in \Omega_1$. By a cylinder of length 0 we mean of course the whole space Ω. Suppose we have obtained $\Omega_n \subseteq \cdots \subseteq \Omega_1$ so that Ω_n is a cylinder of length $\ell_n(x)$ and for all $\omega \in \Omega_n$, $d_1(\omega, x) = a_1, \ldots, d_n(\omega, x) = a_n$. Notice that $x_{n+1} = \pi_2(K_\beta^n(\omega, x))$ for all $\omega \in \Omega_n$.

- If $x_{n+1} \in E_j$, then $\ell_{n+1}(x) = \ell_n(x)$ and for all $\omega \in \Omega_n$, $d_{n+1}(\omega, x) = d_1(K_\beta^n(\omega, x)) = j = a_{n+1}$, by Lemma 1. We set $\Omega_{n+1} = \Omega_n$.
- If $x_{n+1} \in S_j$, then $\ell_{n+1}(x) = \ell_n(x) + 1$ and $a_{n+1} \in \{j-1, j\}$ by Lemma 1.
  — If $a_{n+1} = j - 1$, we set $\Omega_{n+1} = \{\omega \in \Omega_n : \omega_{\ell_{n+1}} = 0\}$. Then, for all $\omega \in \Omega_{n+1}$, $d_{n+1}(\omega, x) = d_1(K_\beta^n(\omega, x)) = j - 1 = a_{n+1}$.
  — If $a_{n+1} = j$, we set $\Omega_{n+1} = \{\omega \in \Omega_n : \omega_{\ell_{n+1}} = 1\}$. Then, for all $\omega \in \Omega_{n+1}$, $d_{n+1}(\omega, x) = d_1(K_\beta^n(\omega, x)) = j = a_{n+1}$.

In all cases we see that Ω_{n+1} is a cylinder of length $\ell_{n+1}(x)$, and for all $\omega \in \Omega_{n+1}$, $d_1(\omega, x) = a_1, \ldots, d_{n+1}(\omega, x) = a_{n+1}$.

If the map K_β hits the switch regions infinitely many times, then $\ell_n(x) \to \infty$ and, as is well known, $\bigcap \Omega_n$ consists of a single point. If this happens only finitely many times, then the set $\{\ell_n(x) : n \in \mathbb{N}\}$ is finite and $\bigcap \Omega_n$ is exactly a cylinder set. In both cases $\bigcap \Omega_n$ is non-empty and $\omega \in \bigcap \Omega_n$ satisfies $d_j(\omega, x) = a_j$ for all $j \geq 1$. □

Remark 1. Theorems 1 and 2 give another proof of the fact that among all possible β-expansions of a point $x \in [0, \lfloor\beta\rfloor/(\beta - 1)]$, the greedy expansion is the largest in lexicographical order (it corresponds to the largest element $(1, 1, \ldots)$ of Ω), and the lazy one is the smallest (it corresponds to the smallest element $(0, 0, \ldots)$ of Ω). Furthermore, from Theorem 2, one sees that x has a unique representation in base β of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}$$

with $a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$ if and only if $a_i = d_i(\omega, x)$ for all $i \geq 1$ and *all* $\omega \in \Omega$. Equivalently, the greedy expansion of x is the only representation of x in base β if and only if $x_n \in E$ for all $n \geq 1$. In this case, we have $x_n = T_\beta^{n-1}x = S_\beta^{n-1}x$ for all $n \geq 1$.

Remark 1 gives in fact a characterization of unique expansion in terms of the greedy expansion. Namely, if x has an expansion of the form $x = a_1/\beta + a_2/\beta^2 + \cdots$, then x has a unique expansion in base β if and only if $T_\beta^n x \in E_{a_{n+1}}$ for all $n \geq 0$. We would like to give other characterizations. Although some of the results are already known (see [KL]), we give simple proofs for completeness. We first observe that $1 \in S_{\lfloor\beta\rfloor} \cup E_{\lfloor\beta\rfloor}$, and $1 \in E_{\lfloor\beta\rfloor}$ if and only if $\lfloor\beta\rfloor/(\beta - 1) - 1 \in E_0$. The following proposition gives a characterization of the case $1 \in E_{\lfloor\beta\rfloor}$ using the greedy expansion of 1.

Proposition 1. *Suppose* 1 *has a greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \cdots$.

(i) *If* $b_i = 0$ *for all* $i \geq 3$, *then* $1 \in E_{b_1}$ *if and only if* $b_2 \geq 2$. *Moreover, if* $b_2 = 1$, *then* $1 = \lfloor\beta\rfloor/(\beta-1) - 1/\beta$.

(ii) *If* $b_i \geq 1$ *for some* $i \geq 3$, *then* $1 \in E_{b_1}$ *if and only if* $b_2 \geq 1$.

Proof. First observe that $\lfloor\beta\rfloor = b_1$, and that

$$1 = \frac{b_1}{\beta} + \frac{b_2}{\beta^2} + \frac{1}{\beta^2}T_\beta^2 1.$$

This implies that $\beta^2 - b_1\beta = b_2 + T_\beta^2 1$. Now, by definition $1 \in E_{b_1}$ if and only if $1 > b_1/(\beta-1) - 1/\beta$, or equivalently $\beta^2 - b_1\beta > 1$.

In case (i), we have $T_\beta^2 1 = 0$, which implies that $\beta^2 - b_1\beta = b_2$. Hence, $1 \in E_{b_1}$ if and only if $b_2 \geq 2$. If $b_2 = 1$, then $\beta^2 - b_1\beta = 1$; equivalently, $1 = \lfloor\beta\rfloor/(\beta-1) - 1/\beta$.

In case (ii), we have $0 < T_\beta^2 1 < 1$. Hence, $\beta^2 - b_1\beta = b_2 + T_\beta^2 1 > 1$ if and only if $b_2 \geq 1$. □

Before we proceed to the characterization of the uniqueness of the β-expansion of x, we need the following simple lemma.

Lemma 2. *Suppose* x *has a greedy expansion of the form* $x = a_1/\beta + a_2/\beta^2 + \cdots$. *If* $a_{n+1} \geq 1$, *then* $T_\beta^n x \in E_{a_{n+1}}$ *if and only if* $T_\beta^{n+1} x > \lfloor\beta\rfloor/(\beta-1) - 1$.

Proof. Notice that

$$T_\beta^n x = \frac{a_{n+1}}{\beta} + \frac{1}{\beta}T_\beta^{n+1} x \in S_{a_{n+1}} \cup E_{a_{n+1}}.$$

Thus, $T_\beta^n x \in E_{a_{n+1}}$ if and only if

$$T_\beta^n x > \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{a_{n+1}-1}{\beta}.$$

Rewriting one finds that $T_\beta^n x \in E_{a_{n+1}}$ if and only if $T_\beta^{n+1} x > \lfloor\beta\rfloor/(\beta-1) - 1$. □

Note that if $a_{n+1} = 0$, then $T_\beta^n x \in E_0$.

The following theorem is an immediate consequence of the above lemma. We remark that a lexicographical version of this theorem was obtained independently for the case $x = 1$, and via other methods in [KL, Theorem 3.1].

Theorem 3. *Suppose* x *has a greedy expansion of the form* $x = a_1/\beta + a_2/\beta^2 + \cdots$. *Then* x *has a unique expansion in base* β *if and only if* $T_\beta^{n+1} x > \lfloor\beta\rfloor/(\beta-1) - 1$ *for all* $n \geq 0$ *with* $a_{n+1} \geq 1$.

Corollary 1. *Suppose* x *has a greedy expansion of the form* $x = a_1/\beta + a_2/\beta^2 + \cdots$ *with* $a_i \geq 1$ *for all* $i \geq 1$. *Then* x *has a unique* β*-expansion.*

Proof. Observe that $T_\beta^n x \geq 1/(\beta-1)$ for all $n \geq 0$, and $1/(\beta-1) > \lfloor\beta\rfloor/(\beta-1) - 1$. The result follows from Theorem 3. □

Corollary 2. *If* 1 *has a unique* β*-expansion, then there exists a* $k \geq 1$ *such that in the greedy expansion of* 1*, every block of consecutive zeros consists of at most* k *terms.*

Proof. Let $1 = b_1/\beta + b_2/\beta^2 + \cdots$ be the greedy expansion. By uniqueness $1 \in E_{b_1}$, so $b_1/(\beta - 1) - 1 < 1/\beta$. Hence, there exists a $k \geq 1$ such that

$$\frac{1}{\beta^{k+1}} \leq \frac{b_1}{\beta - 1} - 1 < \frac{1}{\beta^k}.$$

If $b_{i-1}b_i \dots b_j$ is a block with $b_{i-1} \geq 1$, $b_i = \cdots = b_j = 0$ and $j - i + 1 \geq k + 1$, then

$$T_\beta^{i-1} 1 < \frac{1}{\beta^{k+1}} \leq \frac{b_1}{\beta - 1} - 1,$$

contradicting Theorem 3. □

Another immediate corollary of Theorem 3 and Proposition 1 is the following.

Corollary 3. *Suppose* 1 *has an infinite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \cdots$ *with* $b_2 \geq 1$*. Let* $k \geq 1$ *be the unique integer such that*

$$\frac{1}{\beta^{k+1}} \leq \frac{b_1}{\beta - 1} - 1 < \frac{1}{\beta^k}.$$

If in the greedy expansion of 1 *every block of consecutive zeros contains at most* $k - 1$ *terms, then* 1 *has a unique* β*-expansion.*

3. Measures of maximal entropy for random β-expansions

In this section we show that the map K_β on $\Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)]$ can be essentially identified with the left shift on $\{0, \dots, \lfloor\beta\rfloor\}^{\mathbb{N}}$. This will enable us to prove that K_β has a unique measure of maximal entropy.

Let $D = \{0, \dots, \lfloor\beta\rfloor\}^{\mathbb{N}}$ be equipped with the product σ-algebra $\mathcal{D}$ and the uniform product measure $\mathbb{P}$. Let σ' be the left shift on D. On the set $\Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)]$ we consider the product σ-algebra $\mathcal{A} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $[0, \lfloor\beta\rfloor/(\beta - 1)]$, and $\mathcal{A}$ the product σ-algebra on Ω. Define the function $\varphi : \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] \to D$ by

$$\varphi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \dots).$$

It is easily seen that φ is measurable, and $\varphi \circ K_\beta = \sigma' \circ \varphi$. Furthermore, Theorem 2 implies that φ is surjective. We will now show that φ restricted to an appropriate K_β-invariant subset is in fact invertible. Let

$$Z = \{(\omega, x) \in \Omega \times [0, \lfloor\beta\rfloor/(\beta - 1)] : K_\beta^n(\omega, x) \in \Omega \times S \text{ infinitely often}\},$$

$$D' = \left\{(a_1, a_2, \dots) \in D : \sum_{i=1}^{\infty} \frac{a_{j+i-1}}{\beta^i} \in S \text{ for infinitely many } j\text{'s}\right\}.$$

Then $\varphi(Z) = D'$, $K_\beta^{-1}(Z) = Z$ and $(\sigma')^{-1}(D') = D'$. Let φ' be the restriction of the map φ to Z.

Lemma 3. *The map $\varphi' : Z \to D'$ is a bimeasurable bijection.*

Proof. For any sequence $(a_1, a_2, \ldots) \in D'$, define recursively

$$r_1 = \min\left\{ j \geq 1 : \sum_{l=1}^{\infty} \frac{a_{j+l-1}}{\beta^l} \in S \right\}, \qquad r_i = \min\left\{ j > r_{i-1} : \sum_{l=1}^{\infty} \frac{a_{j+l-1}}{\beta^l} \in S \right\}.$$

If $\sum_{l=1}^{\infty} a_{r_i+l-1}/\beta^l \in S_j$ then, according to Lemma 1, $a_{r_i} \in \{j-1, j\}$. If $a_{r_i} = j-1$, let $\omega_i = 0$, otherwise let $\omega_i = 1$. Define $(\varphi')^{-1} : D' \to Z$ by

$$(\varphi')^{-1}((a_1, a_2, \ldots)) = \left(\omega, \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}\right).$$

It is easily checked that $(\varphi')^{-1}$ is measurable, and is the inverse of φ'. □

Lemma 4. $\mathbb{P}(D') = 1$.

Proof. For any sequence $(a_1, a_2, \ldots) \in D$ and $m \geq 1$, define

$$x_m = \frac{1}{\beta} + \frac{a_1}{\beta^{m+1}} + \frac{a_2}{\beta^{m+2}} + \cdots.$$

Clearly $x_m \geq 1/\beta$. On the other hand,

$$x_m \leq \frac{1}{\beta} + \sum_{i=1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^{m+i}} = \frac{1}{\beta}\left(1 + \frac{\lfloor\beta\rfloor}{\beta^{m-1}(\beta-1)}\right).$$

Since $1 + \frac{\lfloor\beta\rfloor}{\beta^{m-1}(\beta-1)} \downarrow 1$ as $m \to \infty$, there exists an integer $N > 0$ such that for all $m \geq N$,

$$\frac{1}{\beta} \leq x_m \leq \frac{\lfloor\beta\rfloor}{\beta(\beta-1)},$$

i.e. $x_m \in S_1$ for all $m \geq N$. Let

$$D'' = \{(a_1, a_2, \ldots) \in D : a_j a_{j+1} \ldots a_{j+N-1} = 1\underbrace{00\ldots0}_{N-1 \text{ zeros}} \text{ for infinitely many } j\}.$$

From the above, we conclude that $D'' \subseteq D'$. Clearly $\mathbb{P}(D'') = 1$, hence $\mathbb{P}(D') = 1$. □

Now, consider the K_β-invariant measure ν_β defined on $\mathcal{A} \times \mathcal{B}$ by $\nu_\beta(A) = \mathbb{P}\left(\varphi(Z \cap A)\right)$. The following theorem is a simple consequence of Lemmas 3 and 4.

Theorem 4. *Let $\beta > 1$ be a non-integer. The dynamical systems $(\Omega \times [0, \lfloor\beta\rfloor/(\beta-1)], \mathcal{A} \times \mathcal{B}, \nu_\beta, K_\beta)$ and $(D, \mathcal{D}, \mathbb{P}, \sigma')$ are measurably isomorphic.*

Remark 2. The above theorem implies that $h_{\nu_\beta}(K_\beta) = \log(1 + \lfloor\beta\rfloor)$. Further, since $\mathbb{P}$ is the unique measure of maximal entropy on D, we see that ν_β is the only K_β-invariant measure with support Z and maximal entropy $\log(1 + \lfloor\beta\rfloor)$, i.e. any other K_β-invariant measure with support Z has entropy strictly less than $\log(1 + \lfloor\beta\rfloor)$. We now investigate the entropy of K_β-invariant measures μ for which $\mu(Z^c) > 0$.

Lemma 5. *Let μ be a K_β-invariant measure for which $\mu(Z^c) > 0$. Then $h_\mu(K_\beta) < \log(1 + \lfloor\beta\rfloor)$.*

Proof. Since Z and Z^c are K_β-invariant, there exist $0 \le \alpha < 1$ and K_β-invariant measures μ_1, μ_2 concentrated on Z and Z^c respectively, such that $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$. Then $h_\mu(K_\beta) = \alpha h_{\mu_1}(K_\beta) + (1-\alpha)h_{\mu_2}(K_\beta)$. From Remark 2, we have $h_{\mu_1}(K_\beta) \le \log(1 + \lfloor\beta\rfloor)$. We now show that $h_{\mu_2}(K_\beta) < \log(1 + \lfloor\beta\rfloor)$. To this end, let

$$G = \{x \in [0, \lfloor\beta\rfloor/(\beta - 1)] : x \text{ has a unique } \beta\text{-expansion}\}.$$

Then $\Omega \times G \subseteq K_\beta^{-1}(\Omega \times G)$, and $\bigcup_{i=0}^{\infty} K_\beta^{-i}(\Omega \times G) = Z^c$. From the above we see that $\mu_2(\Omega \times G) = 1$, hence it is enough to study the entropy of the map K_β restricted to $\Omega \times G$. On this set K_β has the form $I_\Omega \times T_\beta$, where I_Ω is the identity map on Ω, and T_β the greedy map restricted to G. On G we consider the Borel σ-algebra $G \cap \mathcal{B}$. Notice that $\mu_2 \circ \pi_2^{-1}$ is a T_β-invariant measure with support G, hence $h_{\mu_2\circ\pi_2^{-1}}(T_\beta) \le \log\beta$.

Let $\mathcal{F}$ and $\mathcal{G}$ be any two measurable partitions of Ω and G respectively. For any $n \ge 1$,

$$\bigvee_{i=0}^{n-1} K_\beta^{-i}(\mathcal{F} \times \mathcal{G}) = \bigvee_{i=0}^{n-1} (I_\Omega \times T_\beta)^{-i}(\mathcal{F} \times \mathcal{G}) = \mathcal{F} \times \bigvee_{i=0}^{n-1} T_\beta^{-i}\mathcal{G}$$

modulo sets of μ_2-measure 0. Hence,

$$\begin{aligned} H_{\mu_2}\Big(\Omega \times \bigvee_{i=0}^{n-1} T_\beta^{-i}\mathcal{G}\Big) &\le H_{\mu_2}\Big(\bigvee_{i=0}^{n-1} K_\beta^{-i}(\mathcal{F} \times \mathcal{G})\Big) \\ &\le H_{\mu_2}(\mathcal{F} \times G) + H_{\mu_2}\Big(\Omega \times \bigvee_{i=0}^{n-1} T_\beta^{-i}\mathcal{G}\Big). \end{aligned}$$

Now, dividing by n and taking the limit as $n \to \infty$, we get

$$h_{\mu_2}(K_\beta, \mathcal{F} \times \mathcal{G}) = h_{\mu_2}(K_\beta, \Omega \times \mathcal{G}) = h_{\mu_2\circ\pi_2^{-1}}(T_\beta, \mathcal{G}) \le \log\beta.$$

Since $\mathcal{F}$ and $\mathcal{G}$ are arbitrary partitions, we have

$$h_{\mu_2}(K_\beta) \le \log\beta < \log(1 + \lfloor\beta\rfloor).$$

Therefore, $h_\mu(K_\beta) < \log(1 + \lfloor\beta\rfloor)$. □

From Remark 2 and Lemma 5 we arrive at the following theorem.

Theorem 5. *The measure ν_β is the unique K_β-invariant measure of maximal entropy.*

An interesting consequence of the above theorems is that if $\beta, \beta' > 1$ are non-integers, then

$$\lfloor\beta\rfloor = \lfloor\beta'\rfloor \quad \text{if and only if} \quad (K_\beta, \nu_\beta) \text{ is isomorphic to } (K_{\beta'}, \nu_{\beta'}).$$

As before, let $\pi_2 : \Omega \times [0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$ be the natural projection $\pi_2(\omega, x) = x$. We are interested in identifying the projection of the measure ν_β on the second coordinate, that is, the measure $\nu_\beta \circ \pi_2^{-1}$ defined on $[0, \lfloor\beta\rfloor/(\beta-1)]$. To do that, we consider the purely discrete measures $\{\delta_i\}_{i\geq 1}$ defined on $\mathbb{R}$ as follows:

$$\delta_i(\{0\}) = \frac{1}{\lfloor\beta\rfloor+1}, \quad \ldots, \quad \delta_i(\{\lfloor\beta\rfloor\beta^{-i}\}) = \frac{1}{\lfloor\beta\rfloor+1}.$$

Let δ be the corresponding infinite Bernoulli convolution,

$$\delta = \lim_{n\to\infty} \delta_1 * \ldots * \delta_n.$$

Theorem 6. $\nu_\beta \circ \pi_2^{-1} = \delta$.

Proof. Let $h : D \to [0, \lfloor\beta\rfloor/(\beta-1)]$ be given by $h(y) = \sum_{i=1}^\infty y_i/\beta^i$, where $y = (y_1, y_2, \ldots)$. Then $\pi_2 = h \circ \varphi$ and $\delta = \mathbb{P} \circ h^{-1}$. Since $\mathbb{P} = \nu_\beta \circ \varphi^{-1}$, it follows that $\nu_\beta \circ \pi_2^{-1} = \delta$. □

If $\beta \in (1,2)$ then δ is an Erdős measure on $[0, 1/(\beta-1)]$, and lots of things are already known. For example, if β is a Pisot number, then δ is singular with respect to Lebesgue measure λ ([E1], [E2], [S]). Further, for almost all $\beta \in (1,2)$ the measure δ is equivalent to λ ([So], [MS]). There are many generalizations of these results to the case of an arbitrary digit set (see [PSS] for more references and results).

4. Finite greedy expansion of 1 with positive coefficients, and the Markov property of the random β-expansion

We now assume that the greedy expansion of 1 in base β satisfies $1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n$ with $b_i \geq 1$ for $i = 1, \ldots, n$ and $n \geq 2$ (notice that $\lfloor\beta\rfloor = b_1$). We show that in this case the dynamics of K_β can be identified with a subshift of finite type with an irreducible adjacency matrix. As a result the unique measure of maximal entropy ν_β obtained in the previous section is Markov.

The analysis of the case $\beta^2 = b_1\beta + 1$ needs some adjustments. For this reason, we assume here that $\beta^2 \neq b_1\beta + 1$, and refer the reader to the end of this section (Remarks 6(2)) for the appropriate modifications needed for the case $\beta^2 = b_1\beta + 1$.

We begin by a proposition that is an immediate consequence of Proposition 1 and Lemma 2, and which plays a crucial role in finding the Markov partition describing the dynamics of the map K_β, as defined in Section 1.

Proposition 2. *Suppose* 1 *has a finite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n$. *If* $b_j \geq 1$ *for* $1 \leq j \leq n$, *then*

(i) $T_\beta^i 1 = S_\beta^i 1 \in E_{b_{i+1}}$, $0 \leq i \leq n-2$.

(ii) $T_\beta^{n-1} 1 = S_\beta^{n-1} 1 = b_n/\beta \in S_{b_n}$, $T_\beta^n 1 = 0$, *and* $S_\beta^n 1 = 1$.

(iii) $T_\beta^i\left(\frac{b_1}{\beta-1} - 1\right) = S_\beta^i\left(\frac{b_1}{\beta-1} - 1\right) \in E_{b_1 - b_{i+1}}$, $0 \leq i \leq n-2$.

(iv) $T_\beta^{n-1}\left(\dfrac{b_1}{\beta-1}-1\right) = S_\beta^{n-1}\left(\dfrac{b_1}{\beta-1}-1\right) = \dfrac{b_1}{\beta(\beta-1)} + \dfrac{b_1-b_n}{\beta} \in S_{b_1-b_n+1}$,
$T_\beta^{n}\left(\dfrac{b_1}{\beta-1}-1\right) = \dfrac{b_1}{\beta-1}-1, \quad S_\beta^{n}\left(\dfrac{b_1}{\beta-1}-1\right) = \dfrac{b_1}{\beta-1}$.

Moreover, by Proposition 1 and Lemma 2, one has

$$T_\beta^i 1 = S_\beta^i 1 > \frac{b_1}{\beta-1}-1,$$
$$T_\beta^i\left(\frac{b_1}{\beta-1}-1\right) = S_\beta^i\left(\frac{b_1}{\beta-1}-1\right) < 1 \quad \text{for all } i=1,\dots,n-1.$$

To find the Markov chain behind the map K_β, one starts by refining the partition

$$\mathcal{E} = \{E_0, S_1, E_1, \dots, S_{b_1}, E_{b_1}\}$$

of $[0, b_1/(\beta-1)]$, using the orbits of 1 and $b_1/(\beta-1)-1$ under the transformation T_β. We place the endpoints of $\mathcal{E}$ together with $T_\beta^i 1$, $T_\beta^i(b_1/(\beta-1)-1)$, $i=0,\dots,n-2$, in increasing order. We use these points to form a new partition $\mathcal{C}$ which is a refinement of $\mathcal{E}$, consisting of intervals. We write $\mathcal{C}$ as

$$\mathcal{C} = \{C_0, C_1, \dots, C_L\}.$$

We choose $\mathcal{C}$ to satisfy the following. For $0 \le i \le n-2$,

— $T_\beta^i 1 \in C_j$ if and only if $T_\beta^i 1$ is a left endpoint of C_j,
— $T_\beta^i(b_1/(\beta-1)-1) \in C_j$ if and only if $T_\beta^i(b_1/(\beta-1)-1)$ is a right endpoint of C_j.

Notice that this choice is possible, since the points $T_\beta^i 1$, $T_\beta^i(b_1/(\beta-1)-1)$ for $0 \le i \le n-2$ are all different.

Recall that the map $\ell : [0, \lfloor\beta\rfloor/(\beta-1)] \to [0, \lfloor\beta\rfloor/(\beta-1)]$ defined by $\ell(x) = \lfloor\beta\rfloor/(\beta-1)-x$ satisfies $T_\beta \circ \ell = \ell \circ S_\beta$. Thus, if $x \in E_i$ for some i, then $T_\beta x = S_\beta x$ and $T_\beta \ell(x) = \ell T_\beta(x)$. From the dynamics of K_β on this refinement, one reads the following properties of $\mathcal{C}$.

p1. $C_0 = [0, b_1/(\beta-1)-1]$ and $C_L = [1, b_1/(\beta-1)]$.
p2. For $i = 0, 1, \dots, b_1$, E_i can be written as a finite disjoint union $\bigcup_{j\in M_i} C_j$ with $M_0, M_1, \dots, M_{b_1}$ disjoint subsets of $\{0, 1, \dots, L\}$. Further, the number of elements in M_i equals the number of elements in M_{b_1-i}.
p3. To each S_i there corresponds exactly one $j \in \{0, 1, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k$ such that $S_i = C_j$. This is possible since the T_β-orbits of 1 and $b_1/(\beta-1)-1$ never hit the interior of $\bigcup_{i=1}^{b_1} S_i$.
p4. If $C_j \subset E_i$, then $T_\beta(C_j) = S_\beta(C_j)$ is a finite disjoint union of elements of $\mathcal{C}$, say $T_\beta(C_j) = C_{i_1} \cup \dots \cup C_{i_l}$. Since $\ell(C_j) = C_{L-j} \subset E_{b_1-i}$, it follows that $T_\beta(C_{L-j}) = C_{L-i_1} \cup \dots \cup C_{L-i_l}$.
p5. If $C_j = S_i$, then $T_\beta(C_j) = C_0$ and $S_\beta(C_j) = C_L$.

Define the partition $\mathcal{P}$ of $\Omega \times [0, b_1/(\beta-1)]$ by

$$\mathcal{P} = \left\{\Omega \times C_j : j \in \bigcup_{k=0}^{b_1} M_k\right\} \cup \{\{\omega_1 = i\} \times S_j : i = 0, 1,\ j = 1, \ldots, b_1\}.$$

From **p4** and **p5** we conclude that $\mathcal{P}$ is a Markov partition underlying the map K_β.

To define the underlying subshift of finite type associated with K_β, we consider the $(L+1) \times (L+1)$ matrix $A = (a_{i,j})$ with entries in $\{0, 1\}$ defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } i \in \bigcup_{k=0}^{b_1} M_k \text{ and } \lambda(C_j \cap T_\beta(C_i)) = \lambda(C_j), \\ 0 & \text{if } i \in \bigcup_{k=0}^{b_1} M_k \text{ and } C_i \cap T_\beta^{-1} C_j = \emptyset, \\ 1 & \text{if } i \in \{0, \ldots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = 0, L, \\ 0 & \text{if } i \in \{0, \ldots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L. \end{cases} \tag{1}$$

Remark 3. Because of our assumption $\beta^2 \neq b_1\beta + 1$, we have $\lambda(C_j \cap T_\beta(C_i)) = \lambda(C_j)$ if and only if $C_j \subseteq T_\beta(C_i)$. However, for the analysis of the case $\beta^2 = b_1\beta + 1$, we need the definition of the matrix A as given in equation (1).

Let Y denote the topological Markov chain (or the subshift of finite type) determined by the matrix A, that is, $Y = \{y = (y_i) \in \{0, 1, \ldots, L\}^{\mathbb{N}} : a_{y_i y_{i+1}} = 1\}$. We let σ_Y be the left shift on Y. For ease of notation, we denote by $s_1, \ldots, s_{b_1}$ the states $j \in \{0, \ldots, L\} \setminus \bigcup_{k=0}^{b_1} M_k$ corresponding to the switch regions $S_1, \ldots, S_{b_1}$ respectively.

To each $y \in Y$, we associate a sequence $(e_i) \in \{0, 1, \ldots, b_1\}^{\mathbb{N}}$ and a point $x \in [0, b_1/(\beta-1)]$ as follows. Let

$$e_j = \begin{cases} i & \text{if } y_j \in M_i, \\ i & \text{if } y_j = s_i \text{ and } y_{j+1} = 0, \\ i-1 & \text{if } y_j = s_i \text{ and } y_{j+1} = L. \end{cases} \tag{2}$$

Now set

$$x = \sum_{j=1}^{\infty} \frac{e_j}{\beta^j}. \tag{3}$$

Our aim is to define a map $\psi : Y \to \Omega \times [0, b_1/(\beta-1)]$ that intertwines the actions of K_β and σ_Y. Given $y \in Y$, equations (2) and (3) describe what the second coordinate of ψ should be. In order to be able to associate an $\omega \in \Omega$, one needs that $y_i \in \{s_1, \ldots s_{b_1}\}$ infinitely often. For this reason it is not possible to define ψ on all of Y, but only on an invariant subset. To be more precise, let

$$Y' = \left\{y = (y_1, y_2, \ldots) \in Y : y_i \in \{s_1, \ldots, s_{b_1}\} \text{ for infinitely many } i\text{'s}\right\}.$$

Define $\psi : Y' \to \Omega \times [0, b_1/(\beta-1)]$ as follows. Let $y = (y_1, y_2, \ldots) \in Y'$, and define x as in (3). To define a point $\omega \in \Omega$ corresponding to y, we first locate the indices $n_i = n_i(y)$ where the realization y of the Markov chain is in state s_r for some $r \in \{1, \ldots, b_1\}$.

That is, let $n_1 < n_2 < \cdots$ be the indices such that $y_{n_i} = s_r$ for some $r = 1, \ldots, b_1$. Define

$$\omega_j = \begin{cases} 1 & \text{if } y_{n_j+1} = 0, \\ 0 & \text{if } y_{n_j+1} = L. \end{cases}$$

Now set $\psi(y) = (\omega, x)$.

The following two lemmas reflect the fact that the dynamics of K_β is essentially the same as that of the Markov chain Y. These lemmas are generalizations of Lemmas 1 and 2 in [DK2], and the proofs are slight modifications of the arguments there.

Lemma 6. *Let $y \in Y'$ be such that $\psi(y) = (\omega, x)$. Then*

(i) $y_1 = k$ *for some* $k \in \bigcup_{i=0}^{b_1} M_i \Rightarrow x \in C_k$.
(ii) $y_1 = s_i, y_2 = 0 \Rightarrow x \in S_i$ *and* $\omega_1 = 1$ *for* $i = 1, \ldots, b_1$.
(iii) $y_1 = s_i, y_2 = L \Rightarrow x \in S_i$ *and* $\omega_1 = 0$ *for* $i = 1, \ldots, b_1$.

Lemma 7. *For $y \in Y'$, we have*

$$\psi \circ \sigma_Y(y) = K_\beta \circ \psi(y).$$

Remark 4. From Lemmas 6 and 7 we have the following. If $y \in Y'$ with $\psi(y) = (\omega, x)$, then for any $i \geq 1$ and any $k \in \{0, 1, \ldots, L\}$,

$$y_i = k \ \Rightarrow\ \pi_2(K_\beta^{i-1}(\omega, x)) \in C_k.$$

Having defined the map ψ with the above properties, we now consider the measure Q of maximal entropy on Y. This measure is unique since the adjacency matrix $A = (a_{i,j})$, as defined in (1), is irreducible [W, Theorem 8.10]. In order to describe Q explicitly, we first study the matrix A. From the dynamics of K_β as well as properties **p1**–**p5** one easily sees that A has the following properties:

(i) $a_{i,j} = a_{L-i,L-j}$ for all $i, j = 0, 1, \ldots, L$,
(ii) $\sum_{i=0}^{L} a_{i,j} = b_1 + 1$ for all $j = 0, 1, \ldots, L$.

By induction one can easily show that if $A^k = (a_{i,j}^{(k)})$, then A^k satisfies

(iii) $a_{i,j}^{(k)} = a_{L-i,L-j}^{(k)}$ for all $i, j = 0, 1, \ldots, L$,
(iv) $\sum_{i=0}^{L} a_{i,j}^{(k)} = (b_1 + 1)^k$ for all $j = 0, 1, \ldots, L$.

Since A is an irreducible, non-negative integral matrix, we calculate the topological entropy $h(Y)$ of Y by the formula

$$h(Y) = \lim_{n\to\infty} \frac{1}{n} \log |B_n(Y)|,$$

where $B_n(Y)$ denotes the collection of blocks of length n in the shift space Y. According to property (iv) above $|B_n(Y)| = \sum_{i,j} a_{i,j}^{(n)} = (L+1)(b_1+1)^n$. Hence $h(Y) = \log(b_1+1)$. It follows that the Perron eigenvalue λ_A equals $b_1 + 1$ (i.e. the largest positive eigenvalue of the matrix A). To determine the measure of maximal entropy we need to find a positive left eigenvector u and a positive right eigenvector v. According to property (ii) above a left eigenvector is given by $u = (1, 1, \dots, 1)$. For the positive right eigenvector v, we choose v to satisfy $\sum_{i=0}^{L} v_i = 1$. Using the technique developed by Parry, we show that the measure Q of maximal entropy is the Markov measure generated by the transition matrix $P = (p_{i,j})$, where $p_{i,j} = a_{i,j} \frac{v_j}{(b_1+1)v_i}$, and stationary distribution $p = v$. We equip the space Y with the σ-algebra $\mathcal{H}$ generated by the cylinders. We have the following theorem.

Theorem 7. *The dynamical systems* $(\Omega \times [0, b_1/(\beta - 1)], \mathcal{A} \times \mathcal{B}, Q \circ \psi^{-1}, K_\beta)$ *and* $(Y, \mathcal{H}, Q, \sigma_Y)$ *are measurably isomorphic.*

Proof. We show that the map $\psi : Y' \to Z$ is the required isomorphism. From Lemma 7 we find that ψ intertwines the actions of K_β and σ_Y. Furthermore, it is easily checked that $\psi : Y' \to Z$ is a bimeasurable bijection. The inverse $\psi^{-1} : Z \to Y'$ is given by $\psi^{-1}(\omega, x) = y$, where $y_i = k$ if $\pi_2(K_\beta^{i-1}(\omega, x)) \in C_k$. □

Remark 5. The proof of the above theorem shows that $Q \circ \psi^{-1}$ is a K_β-invariant measure on $\Omega \times [0, b_1/(\beta - 1)]$ with support Z, and of maximal entropy $\log(1 + \lfloor \beta \rfloor)$. By Theorem 5 it follows that $Q \circ \psi^{-1} = \nu_\beta$. In Theorem 6, the projection of this measure on the second coordinate was identified as an infinite convolution of Bernoulli measures.

Let $\pi_1 : \Omega \times [0, \lfloor \beta \rfloor/(\beta - 1)] \to \Omega$ be the canonical projection onto the first coordinate. Consider the measure $Q' = \nu_\beta \circ \pi_1^{-1}$ on Ω. Then $Q' = Q \circ \alpha^{-1}$, where $\alpha = \pi_1 \circ \psi : Y' \to \Omega$.

Theorem 8. *The measure* Q' *is the uniform Bernoulli measure on* $\{0, 1\}^{\mathbb{N}}$.

Proof. Define the stopping times $(T_i)_{i \geq 1}$ on Y' recursively as follows:

$$T_1 = \min\{m \geq 2 : y_{m-1} \in \{s_1, \dots, s_{b_1}\}\},$$
$$T_i = \min\{m > T_{i-1} : y_{m-1} \in \{s_1, \dots, s_{b_1}\}\}, \quad i \geq 2.$$

An application of the Strong Markov Property shows that the stopped process $y_{T_1}, y_{T_2}, \dots$ is also a Markov chain with state space $\{0, L\}$ and transition probabilities given by $q_{ij} = 1/2$ for $i, j \in \{0, L\}$. Therefore, if $j_1, \dots, j_l \in \{0, L\}$, then

$$Q(\{y_{T_1} = j_1, \dots, y_{T_l} = j_l\}) = 1/2^l.$$

Define $\chi : \{0, L\} \to \{0, 1\}$ by $\chi(0) = 1$, $\chi(L) = 0$. It follows that

$$Q'(\{\omega_1 = \chi(j_1), \dots, \omega_l = \chi(j_l)\}) = Q(\{y_{T_1} = j_1, \dots, y_{T_l} = j_l\}) = 1/2^l. \quad \square$$

Remarks 6. (1) If 1 has a finite greedy expansion $1 = b_1/\beta + \cdots + b_n/\beta^n$ with some of the coefficients b_i equal to zero, then one is able to find examples of such β's where the map K_β has an underlying Markov partition similar to the one described above, i.e. determined by the random orbits of 1 and $b_1/(\beta - 1) - 1$. On the other hand, one is also able to find examples where K_β has no such Markov partition. For example, for $n \geq 2$, let $\beta_n \in (1, 2)$ be the unique solution to the equation

$$\beta^n = \beta^{n-1} + 1.$$

Then 1 has a greedy expansion $1 = 1/(\beta) + 1/(\beta^n)$. For $n = 2, 3, 4, 5$, it is not hard to see that K_β has a natural underlying Markov partition (one might need to divide the switch regions as well). However, for n sufficiently large this is not the case. For in [EK] it was shown that for *each* β sufficiently close to 1, there exists a sequence (ϵ_i) of zeros and ones satisfying $\sum_{i=1}^{\infty} \epsilon_i/\beta^i = 1$ and containing all possible finite variations of the digits 0 and 1. Now, it is easy to check that $\beta_n \downarrow 1$ as $n \to \infty$. Hence, if β_n is sufficiently close to 1, then by Theorem 2 there is an $\omega \in \Omega$ such that $\epsilon_i = d_i(\omega, 1)$ for each i. Since each block of zeros and ones appears in $(d_i(\omega, 1))_{i\geq 1}$ this implies that

$$\overline{\{\pi_2(K_{\beta_n}^m(\omega, 1)) : m \geq 0\}} = \left[0, \frac{1}{\beta_n - 1}\right].$$

Hence, there is no underlying Markov partition (determined by the random orbits of 1 and $1/(\beta_n - 1) - 1$) for the map K_β.

Notice that β_5 is the smallest Pisot number. One might conjecture that for $\beta \in (1, \beta_5)$, one cannot construct a Markov partition similar to the one described in this section.

(2) We now consider the case $\beta^2 = b_1\beta + 1$. Notice that $\mathcal{C} = \mathcal{E}$, since 1 and $b_1/(\beta - 1) - 1$ are already endpoints of intervals in $\mathcal{E}$. For ease of notation, we denote the alphabet of Y by $\{e_0, s_1, e_1, \ldots, s_{b_1}, e_{b_1}\}$. For any $1 \leq i \leq b_1$,

$$T_\beta(S_i) = \overline{E}_0 = [0, 1/\beta], \qquad S_\beta(S_i) = \overline{E}_{b_1} = \left[1, \frac{b_1}{\beta - 1}\right].$$

As a result, Lemmas 6 and 7 do not hold for elements in Y' corresponding to endpoints of elements of $\mathcal{E}$. To be precise, for $1 \leq i \leq b_1$ we define the sequences $x^{(i)}, y^{(i)}, q^{(i)}$ and $r^{(i)}$ as follows.

— Let $x^{(i)} = (s_i, e_{b_1}, s_1, e_{b_1}, s_1, \ldots)$. Then $\psi(x^{(i)}) = (\omega^{(0)}, i/\beta)$, where $\omega^{(0)} = (0, 0, 0, \ldots)$. We have $x_{2m+1}^{(i)} = s_1$ for $m \geq 1$, while for $j \geq 2$,

$$\pi_2\left(K_\beta^{(j)}\left(\omega^{(0)}, \frac{i}{\beta}\right)\right) = \frac{b_1}{\beta - 1}.$$

— Let $y^{(i)} = (e_i, s_1, e_{b_1}, s_1, e_{b_1}, \ldots)$. Then

$$\psi(y^{(i)}) = \left(\omega^{(0)}, \frac{b_1}{\beta(\beta - 1)} + \frac{i - 1}{\beta}\right).$$

We have $y_{2m}^{(i)} = s_1$ for $m \geq 1$, while for $j \geq 1$,

$$\pi_2\left(K_\beta^{(j)}\left(\omega^{(0)}, \frac{b_1}{\beta(\beta - 1)} + \frac{i - 1}{\beta}\right)\right) = \frac{b_1}{\beta - 1}.$$

— Let $q^{(i)} = (e_{i-1}, s_{b_1}, e_0, s_{b_1}, e_0, \ldots)$. Then $\psi(q^{(i)}) = (\omega^{(1)}, i/\beta)$, where $\omega^{(1)} = (1, 1, 1, \ldots)$. We have $q_{2m}^{(i)} = s_{b_1}$ for $m \geq 1$, while for $j \geq 1$,

$$\pi_2\left(K_\beta^{(j)}\left(\omega^{(1)}, \frac{i}{\beta}\right)\right) = 0.$$

— Let $r^{(i)} = (s_i, e_0, s_{b_1}, e_0, s_{b_1}, \ldots)$. Then $\psi(r^{(i)}) = (\omega^{(1)}, \frac{b_1}{\beta(\beta-1)} + \frac{i-1}{\beta})$. We have for $m \geq 1$, $r^{(i)}_{2m+1} = s_{b_1}$, while for $j \geq 2$,

$$\pi_2(K_\beta^{(j)}(\omega^{(1)}, \frac{b_1}{\beta(\beta-1)} + \frac{i-1}{\beta})) = 0.$$

Except for these points, the analysis used in this section remains valid. So, the only modification needed is the removal of a set of measure zero from the domain of Y', namely all points whose orbit under σ_Y eventually equals $x^{(i)}$, $y^{(i)}$, $q^{(i)}$ or $r^{(i)}$ for some $i = 1, \ldots, b_1$.

(3) Suppose in the switch regions we decide to flip a biased coin, with $0 < \mathbb{P}(\text{Heads}) = p < 1$, in order to decide whether to use the greedy or the lazy map. The measure of maximal entropy discussed in this section does not reflect this fact. A natural invariant measure that preserves this property is obtained by considering the Markov measure Q_λ on Y with transition probabilities $p_{i,j}$, given by

$$p_{i,j} = \begin{cases} \lambda(C_i \cap T_\beta^{-1} C_j)/\lambda(C_i) & \text{if } i \in \bigcup_{k=0}^{b_1} M_k, \\ p & \text{if } i \in \{0, 1, \ldots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = 0, \\ 1 - p & \text{if } i \in \{0, 1, \ldots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = L, \\ 0 & \text{if } i \in \{0, 1, \ldots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L, \end{cases}$$

(as before, λ denotes Lebesgue measure) and initial distribution the corresponding stationary distribution (see [DK2]). Another interesting feature is that the projection of $Q_\lambda \circ \psi^{-1}$ on the second coordinate for $p = 1$ is the Parry measure μ_β, and for $p = 0$ it is the lazy measure ρ_β (see Section 1).

References

[DK1] Dajani, K., Kraaikamp, C.: From greedy to lazy expansions, and their driving dynamics. Expo. Math. **20**, 315–327 (2002) Zbl 1030.11035 MR 2003h:11089

[DK2] Dajani, K., Kraaikamp, C.: On random β-expansions. Ergodic Theory Dynam. Systems **23**, 461–479 (2003)

[E1] Erdős, P.: On a family of symmetric Bernoulli convolutions. Amer. J. Math. **61**, 974–976 (1939) Zbl 0022.35402 MR 0000311

[E2] Erdős, P.: On the smoothness properties of Bernoulli convolutions. Amer. J. Math. **62**, 180–186 (1940) Zbl 66.0511.02 MR 0000858

[EJK] Erdős, P., Joó, I., Komornik, V.: Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. Bull. Soc. Math. France **118**, 377–390 (1990) Zbl 0721.11005 MR 91j:11006

[EK] Erdős, P., Komornik, V.: Developments in non-integer bases. Acta Math. Hungar. **79**, 57–83 (1998) Zbl 0906.11008 MR 99e:11132

[G] Gel'fond, A. O.: A common property of number systems. Izv. Akad. Nauk SSSR Ser. Mat. **23**, 809–814 (1959) (in Russian) MR 0109817

[JS] Joó, I., Schnitzer, F. J.: Expansions with respect to non-integer bases. Grazer Math. Berichte **329** (1996) Zbl 0941.11003 MR 98e:11090

[KL] Komornik, V., Loreti, P.: Subexpansions, superexpansions and uniqueness properties in non-integer bases. Period. Math. Hungar. **44**, 197–218 (2002) Zbl 1017.11008 MR 2003i:11019

[MS] Mauldin, R. D., Simon, K.: The equivalence of some Bernoulli convolutions to Lebesgue measure. Proc. Amer. Math. Soc. **126**, 2733–2736 (1998) Zbl 0912.28004 MR 98i:26009

[P] Parry, W.: On the β-expansions of real numbers. Acta Math. Acad. Sci. Hungar. **11**, 401–416 (1960) Zbl 0099.28103 MR 0142719

[PSS] Peres, Y., Schlag, W., Solomyak, B.: Sixty years of Bernoulli convolutions. In: Fractal Geometry and Stochastics II, Progr. Probab. 46, Birkhäuser, 39–65 (1998) Zbl 0961.42006 MR 2001m:42020

[R1] Rényi, A.: Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493. MR 0097374

[R2] Rényi, A.: On algorithms for the generation of real numbers. Magyar Tud. Akad. Mat. Fiz. Oszt. Kzl. **7**, 265–293 (1957) MR 0097645

[S] Salem, R.: Algebraic Numbers and Fourier Analysis. Heath (1963) Zbl 0126.07802 MR 0157941

[So] Solomyak, B.: On the random series $\sum \pm\lambda^i$ (an Erdős problem). Ann. of Math. **142**, 611–625 (1995) Zbl 0837.28007 MR 97d:11125

[W] Walters, P.: An Introduction to Ergodic Theory. Springer, New York (1982) Zbl 0475.28009 MR 84e:28017