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Recovering an algebraic curve using its projections from different points

Applications to static and dynamic computational vision

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Abstract. We study some geometric configurations related to projections of an irreducible algebraic curve embedded in $\mathbb{CP}^3$ onto embedded projective planes. These configurations are motivated by applications to static and dynamic computational vision.

More precisely, we study how an irreducible closed algebraic curve X embedded in $\mathbb{CP}^3$, of degree d and genus g, can be recovered using its projections from points onto embedded projective planes. The embeddings are unknown. The only input is the defining equation of each projected curve. We show how both the embeddings and the curve in $\mathbb{CP}^3$ can be recovered modulo some action of the group of projective transformations of $\mathbb{CP}^3$.

In particular in the case of two projections, we show how in a generic situation, a characteristic matrix of the pair of embeddings can be recovered. In the process we address dimensional issues and as a result find the minimal number of irreducible algebraic curves required to compute this characteristic matrix up to a finite-fold ambiguity, as a function of their degrees and genus. Then we use this matrix to recover the class of the couple of maps and as a consequence to recover the curve. In a generic situation, two projections define a curve with two irreducible components. One component has degree $d(d-1)$ and the other has degree d, being the original curve.

Then we consider another problem. N projections, with known projection operators and $N \gg 1$, are considered as an input and we want to recover the curve. The recovery can be done by linear computations in the dual space and in the Grassmannian of lines in $\mathbb{CP}^3$. Those computations are respectively based on the dual variety and on the variety of intersecting lines. In both cases a simple lower bound for the number of necessary projections is given as a function of the degree and the genus. A closely related question is also considered. Each point of a finite closed subset of an irreducible algebraic curve is projected onto a plane from a point. For each point the projection center is different. The projection operators are known. We show when and how the recovery of the algebraic curve is possible, in terms of the degree of the curve, and of the degree of the curve of minimal degree generated by the projection centers.

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Finally, we show how these questions are motivated by applications to static and dynamic computational vision. A second part of this work is devoted to several applications to this field. The results in this paper solve a long standing problem in computer vision that could not be solved without algebraic-geometric methods.

Keywords. Plane and space curves, projections, machine vision, structure from motion

1. Introduction

1.1. Problem definition

Consider an irreducible closed algebraic curve $X \subset \mathbb{CP}^3$ (in what follows we simply write $\mathbb{P}^n$ for $\mathbb{CP}^n$). This curve is projected onto several projective planes embedded in $\mathbb{P}^3$ from several projection centers, say $\{\mathbf{O}_i\}$, $i = 1, \dots, n$. Each projection mapping, denoted by $\pi_i : \mathbb{P}^3 \setminus \{\mathbf{O}_i\} \to \mathbb{P}^2$, is represented by a 3×4 matrix $\mathbf{M}_i$ defined modulo multiplication by a non-zero scalar. So each point $\mathbf{P}$ different from $\mathbf{O}_i$ is mapped by π_i to $\mathbf{M}_i\mathbf{P}$. Each projection operator π_i, via its matrix, can be regarded as a point in $\mathbb{P}^{11}$. Let $Y_i = \pi_i(X)$ be the different projections of the curve X. Below we always deal with generic configurations, even when not mentioned explicitly.

When we consider the problem of recovering the projection maps from the projected curves, we will show that the recovery is possible only modulo some action of the group of projective transformations of $\mathbb{P}^3$ on the set of projection maps. To define this action we refer to a projection map as a point in $\mathbb{P}^{11}$. Assume that we have n projections, and consider the projective variety

$$\mathbb{V} = \overbrace{\mathbb{P}^{11} \times \cdots \times \mathbb{P}^{11}}^{n \text{ times}}.$$

Let Pr_3 be the group of projective transformations of $\mathbb{P}^3$. We define an action of Pr_3 on $\mathbb{V}$ as follows: $\theta_n : \mathrm{Pr}_3 \to \mathrm{Mor}(\mathbb{V}, \mathbb{V})$, $\mathbf{A} \mapsto ((\mathbf{Q}_1, \dots, \mathbf{Q}_n) \mapsto (\mathbf{M}_1\mathbf{A}^{-1}, \dots, \mathbf{M}_n\mathbf{A}^{-1}))$, where each matrix $\mathbf{M}_i$ is built from the coordinates of $\mathbf{Q}_i = [Q_{i,1}, \dots, Q_{i,12}]^T$ as follows:

$$\mathbf{M}_i = \begin{bmatrix} Q_{i,1} & Q_{i,2} & Q_{i,3} & Q_{i,4} \\ Q_{i,5} & Q_{i,6} & Q_{i,7} & Q_{i,8} \\ Q_{i,9} & Q_{i,10} & Q_{i,11} & Q_{i,12} \end{bmatrix}.$$

The geometric meaning of this action is that if we change the projective basis in $\mathbb{P}^3$ by the transformation $\mathbf{A}$, we need to change the projection maps accordingly for the projected curves to be invariant.

We first investigate the case of two projections. Given the projected curves Y_1 and Y_2 as the only data, our first problem is to compute the characteristic matrix (to be defined below) of the two projection maps, π_1 and π_2, up to a finite-fold ambiguity. It is shown that this is equivalent to finding a necessary and sufficient condition on X for the action of θ_2 to have a finite number of orbits. Then we show that for each orbit we can recover the curve X modulo Pr_3. More precisely each orbit induces a curve embedded in $\mathbb{P}^3$ containing two irreducible components, one of degree $d(d-1)$ and the other of degree d. The latter is the curve we are looking for.

Then we turn to another problem. The projection maps π_i, $i = 1, \ldots, n$, are now assumed to be known, in addition to the projected curves Y_i. We want to recover the curve X. This can be performed by linear computations using either the dual variety or the variety of lines intersecting X. In both cases a simple lower bound on the minimal number of projections is deduced.

With the variety of lines intersecting X another problem is also handled. Consider a set of N points in $\mathbb{P}^3$ and let X be the curve generated by these points. Each of these points is projected on a different plane by a different projection operator π_i. Those projection operators are known. We want to recover X by linear computations. Let Z be the curve, of minimal degree, generated by the projection centers. We give a formula for the number of constraints obtained on X as a function of the degree of Z.

1.2. Applications to computational vision

All the problems above are directly related to applications in computer vision.

In order to define what are the contributions of our research to computer vision, we shall first recall a rough and common classification of vision algorithms. Three kinds of processes are generally identified in computer vision: early or low-level vision, middle-level vision and high-level vision [12].

Low-level vision deals with feature extraction and matching. It is strongly related to image processing. Given geometric features, that is, primitives extracted from images, and correspondences between these primitives over several images, a set of more sophisticated computations are possible. They are generally defined in a mathematical language and constitute what is commonly called middle-level vision. One can mention examples like camera motion recovery, three-dimensional structure recovery, trajectory recovery of moving points etc. Finally, one aspires to develop a complete artificial vision system that is able to accomplish high-level tasks, like object recognition, classification etc.

Our work belongs to middle-level vision. It gives a mathematical modelization of what can be computed from primitives like algebraic curves. We also discuss implementation issues.

However, we assume that the extraction of the curves from the images as well as the correspondences between them has been done in a pre-process. The question of detection and fitting of algebraic curves has its own complexity and is not treated here. We refer to [27, 28, 4] for more details.

The contributions of our work can be summarized as follows:

1. Given two images of the same algebraic curve, we show how to compute the camera matrices defined in Sections 5.2 and 5.3. We give a necessary and sufficient condition on the degree and the genus of the curve, for the camera matrices to be defined up to a finite-fold ambiguity.
2. Given the camera matrices, we show how to reconstruct the space curve from two views. In that case, we show that the two viewing cones define, in a generic situation, a curve with two irreducible components. One of these components is the right solution

of the reconstruction problem and can be extracted if the degree of the curve is at least three.

3. Given the camera matrices of a sequence of images of the same algebraic curve, we show that the reconstruction can be done linearly either in the dual space or using the variety of lines which intersect the curve in space. We give simple lower bounds on the number of necessary images as functions of the degree and genus of the curve.
4. Given the camera matrices of a sequence of images of a moving point, we show how to recover linearly the trajectory of the point. The trajectory is regarded as a piecewise algebraic curve. We also address a theoretical analysis, and get a necessary and sufficient condition on the trajectory of the camera center, enabling the complete trajectory triangulation of the moving point.

In the past two decades, geometric computer vision, namely the theory of multiple-view, has been considerably developed. A large body of research has been devoted to the case of a scene consisting of point and line features. A summary of the past decade of work in this area can be found in [8, 16].

The theory is somewhat sparse and fragmented when it comes to curve features. Our work presents a coherent theory of multiple-view geometry for algebraic curves. Moreover it paves the way for a more comprehensive use of extremely rich and powerful tools of algebraic geometry in computer vision.

Typically the first three contributions of our work (mentioned above) can be used for the reconstruction of hand-made objects. The fourth contribution has a wide field of applications. It allows the triangulation of the trajectory of a moving point while the camera centers are also moving along arbitrary trajectories. While computer vision has reached significant achievements [8, 16] in the context of static scenes, the results on dynamic scenes are very sparse and our work solves a long standing problem, namely general trajectory triangulation.

1.3. Paper organization

The paper is organized as follows. Sections 2, 3 and 4 contain the mathematical results and present the theoretical contributions of our work in the theory of multiple-view geometry of algebraic curves. Section 5 concerns applications in vision, with many examples demonstrating the practical potential of our approach.

Since our computations will occur in $\mathbb{P}^3$, we fix $[X, Y, Z, T]^T$ as homogeneous coordinates, and $T = 0$ as the plane at infinity.

2. Projection operators

Let π be a projection operator from $\mathbb{P}^3$ to an embedded projective plane $i(\mathbb{P}^2)$ from a point $\mathbf{O}$. This projection can be represented by a 3×4 matrix $\mathbf{M}$. It has several simple, but very useful properties. The kernel of $\mathbf{M}$ is exactly the projection center. The transpose of $\mathbf{M}$ maps a line in $i(\mathbb{P}^2)$ to the plane it defines together with the projection center, given as a point of the dual space $\mathbb{P}^{3*}$. This can be deduced by a duality argument and a simple computation.

There exists a matrix $\widehat{\mathbf{M}}$, a polynomial function of $\mathbf{M}$, which maps a point in $i(\mathbb{P}^2)$ to the Plücker coordinates of the line it generates together with the projection center [8]. If the matrix $\mathbf{M}$ is decomposed as follows:

$$\mathbf{M} = \begin{bmatrix} \Gamma^T \\ \Lambda^T \\ \Theta^T \end{bmatrix},$$

then for $\mathbf{p} = [x, y, z]^T$, the line $\mathbf{L_p} = \widehat{\mathbf{M}}\mathbf{p}$ is given by the extensor $\mathbf{L_p} = x\Lambda \wedge \Theta + y\Theta \wedge \Gamma + z\Gamma \wedge \Lambda$, where $\wedge$ denotes the meet operator in the Grassmann–Cayley algebra (see [2]). By duality, the matrix $\widetilde{\mathbf{M}} = \widehat{\mathbf{M}}^T$ maps lines in $\mathbb{P}^3$ to lines in $i(\mathbb{P}^2)$.

Consider now two projection operators π_1 and π_2. Let $\mathbf{O}_1$ and $\mathbf{O}_2$ be the projection centers and $i_1(\mathbb{P}^2)$ and $i_2(\mathbb{P}^2)$ the projection planes. Let $\mathbf{e}_j$ be the point of intersection of $i_j(\mathbb{P}^2)$ with the line $\overline{\mathbf{O}_1\mathbf{O}_2}$. Let $\sigma(\mathbf{e}_2)$ be the pencil of lines in $i_2(\mathbb{P}^2)$ through $\mathbf{e}_2$. It is easy to define a map from $i_1(\mathbb{P}^2)\setminus\{\mathbf{e}_1\}$ to $\sigma(\mathbf{e}_2)$ as follows. Each point $\mathbf{p}$ is sent to the line given by $\pi_2(\pi_1^{-1}(\mathbf{p}))$. This map is linear and its matrix is $\mathbf{F} = \widetilde{\mathbf{M}}_2\widehat{\mathbf{M}}_1$. Following the standard terminology used in computational vision, we will call the matrix $\mathbf{F}$ the *fundamental matrix* of the pair of projections π_1 and π_2, and the points $\mathbf{e}_1$ and $\mathbf{e}_2$ will be respectively called the *first* and the *second epipole*. The lines in the first (second) projection plane passing through the first (second) epipole are called the *epipolar lines*. Clearly $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{e}_2^T\mathbf{F} = \mathbf{0}^T$.

Proposition 1. *The knowledge of the fundamental matrix* $\mathbf{F}$ *of a couple of projection operators allows the recovery of the matrices* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *of the projections modulo the action* θ_2. *More precisely, the couple* $(\mathbf{M}_1, \mathbf{M}_2)$ *is equivalent to* $([\mathbf{I}; \mathbf{0}], [\mathbf{H}; \mathbf{e}_2])$, *where* $\mathbf{H} = -([\mathbf{e}_2]/\|\mathbf{e}_2\|^2)\mathbf{F}$ *and the matrix* $[\mathbf{e}_2]$ *is defined to be* $\tau(\mathbf{e}_2)$, *where* τ *maps any vector* $\mathbf{x}$ *of* $\mathbb{C}^3$ *to the matrix that represents the cross-product by* $\mathbf{x}$. *We have*

$$\tau(\mathbf{x}) = [\mathbf{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix},$$

and $\mathbf{x} = [x_1, x_2, x_3]$.

Proof. We are looking for a matrix $\mathbf{A} \in \mathrm{Pr}_3$ such that

$$\mathbf{M}_1 = [\mathbf{I}; \mathbf{0}]\mathbf{A}^{-1}, \qquad \mathbf{M}_2 = [\mathbf{H}; \mathbf{e}_2]\mathbf{A}^{-1}.$$

Let us write $\mathbf{B} = \mathbf{A}^{-1}$ as follows:

$$\mathbf{B} = \begin{bmatrix} \boldsymbol{\Omega} & \vdots & \mathbf{u} \\ \cdots & \cdots & \cdots \\ \mathbf{v}^T & \vdots & 0 \end{bmatrix}$$

Let us write $\mathbf{M}_i = [\overline{\mathbf{M}}_i, \mathbf{m}_i]$. Then it follows immediately from the definition of $\mathbf{F}$ that $\mathbf{F} = [\mathbf{e}_2]\overline{\mathbf{M}}_2\overline{\mathbf{M}}_1^{-1}$. Using the following algebraic identity for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^3$: $(\mathbf{x}^T\mathbf{y})\mathbf{I} = \mathbf{x}\mathbf{y}^T - [\mathbf{x}][\mathbf{y}]$, it is easy to prove that it is sufficient to take

$$\boldsymbol{\Omega} = \overline{\mathbf{M}}_1, \quad \mathbf{u} = \mathbf{m}_1, \quad \mathbf{v} = \frac{1}{\|\mathbf{e}_2\|^2}\overline{\mathbf{M}}_2\mathbf{e}_2, \quad \lambda = \frac{1}{\|\mathbf{e}_2\|^2}(\mathbf{e}_2^T\mathbf{m}_2 - \mathbf{e}_2^T\mathbf{H}\mathbf{m}_1). \qquad \square$$

This shows that in order to characterize a couple of projections modulo the action θ_2, we only need to compute the fundamental matrix. In what follows we show how to recover $\mathbf{F}$ from two projections of an algebraic curve.

3. Two projections with unknown projection operators

In this section we deal with the first problem. A smooth and irreducible curve X embedded in $\mathbb{P}^3$ is projected onto two generic planes from two generic points. The projection operators π_1 and π_2 are unknown. First we want to recover the fundamental matrix of the couple (π_1, π_2) from the projected curves $Y_1 = \pi_1(X)$ and $Y_2 = \pi_2(X)$.

3.1. Single projection

We mention a few well known facts about generic projections. Let X be a smooth irreducible algebraic curve embedded in $\mathbb{P}^3$, and Y its projection on a generic plane from a center $\mathbf{O}$.

1. The curve Y will always contain singularities. Furthermore, for a generic position of the projection center, the only singularities of Y will be nodes.
2. The *class* of a planar curve is defined to be the degree of its dual curve. Let m be the class of Y. Then m is constant for a generic position of the projection center.
3. If d and g are the degree and genus of X, they are respectively, for a generic position of $\mathbf{O}$, the degree and the genus of Y, and the Plücker formula yields

$$m = d(d-1) - 2(\sharp\text{nodes}),$$
$$g = \frac{(d-1)(d-2)}{2} - (\sharp\text{nodes}),$$

where $\sharp$nodes denotes the number of nodes of Y. Hence the genus, degree and class are related by

$$m = 2d + 2g - 2.$$

3.2. Fundamental matrix construction

We are now ready to investigate the recovery of the fundamental matrix of a couple of projections (π_1, π_2) when we only know the projections of a smooth irreducible curve.

As before, let X be a smooth irreducible curve embedded in $\mathbb{P}^3$, which cannot be embedded in a plane. The degree of X is $d \geq 3$. Let $\mathbf{M}_i$, $i = 1, 2$, be the projection matrices. Let $\mathbf{F}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ be as defined before. We will need to consider the following two mappings: $\mathbf{p} \overset{\gamma}{\mapsto} \mathbf{e}_1 \vee \mathbf{p}$ and $\mathbf{p} \overset{\xi}{\mapsto} \mathbf{F}\mathbf{p}$, where $\vee$ is the join operator [2], which is equivalent to the cross-product in that case. Both maps are defined on the first projection plane.

Let Y_1 and Y_2 be the projected curves. Assume that they are defined by the polynomials f_1 and f_2. Let $Y_1^\star$ and $Y_2^\star$ denote the dual curves, whose polynomials are respectively ϕ_1 and ϕ_2.

Theorem 1. *For a generic position of the projection centers with respect to the curve X, there exists a non-zero scalar λ such that for all points $\mathbf{p}$ in the projection plane, the following equality holds:*

$$\phi_2(\xi(\mathbf{p})) = \lambda\phi_1(\gamma(\mathbf{p})). \tag{1}$$

For reasons that will be clear later, we shall call this equation the *generalized Kruppa equation*.

Proof. Let ϵ_i be the set of epipolar lines tangent to the curve in image i. We start with the following lemma:

Lemma 1. *The two sets ϵ_1 and ϵ_2 are projectively equivalent. Moreover for each corresponding pair of epipolar lines $(\mathbf{l}, \mathbf{l}') \in \epsilon_1 \times \epsilon_2$, the multiplicities of $\mathbf{l}$ and $\mathbf{l}'$ as points of $Y_1^\star$ and $Y_2^\star$ are the same.*

Proof. Consider the following three pencils:

- $\sigma(\mathbf{L}) \approx \mathbb{P}^1$, the pencil of *epipolar planes*, that is, planes containing the baseline joining the two projection centers,
- $\sigma(\mathbf{e}_1) \approx \mathbb{P}^1$, the pencil of epipolar lines in the first projection plane,
- $\sigma(\mathbf{e}_2) \approx \mathbb{P}^1$, the pencil of epipolar lines in the second projection plane.

Thus we have $\epsilon_i \subset \sigma(\mathbf{e}_i)$. Moreover if E is the set of planes in $\sigma(\mathbf{L})$ tangent to the curve in space, then there exists a one-to-one mapping between E and each ϵ_i which leaves the multiplicities unchanged. This completes the proof. □

This lemma implies that both sides of equation (1) define the same algebraic set, that is, the union of epipolar lines tangent to Y_1. Since ϕ_1 and ϕ_2, in the generic case, have the same degree (as stated in 3.1), each side can be factorized as follows:

$$\phi_1(\gamma(x, y, z)) = \prod_i (\alpha_{1i}x + \alpha_{2i}y + \alpha_{3i}z)^{a_i},$$

$$\phi_2(\xi(x, y, z)) = \prod_i \lambda_i(\alpha_{1i}x + \alpha_{2i}y + \alpha_{3i}z)^{b_i},$$

where $\sum_i a_i = \sum_j b_j = m$. By the previous lemma we also have $a_i = b_i$ for all i. □

By eliminating the scalar λ from the generalized Kruppa equation (1) we obtain a set of bi-homogeneous equations in $\mathbf{F}$ and $\mathbf{e}_1$. Hence they define a variety in $\mathbb{P}^2 \times \mathbb{P}^8$. We now turn to the dimensional analysis of this variety. We wish to exhibit conditions under which this variety is discrete.

3.3. Dimensional analysis

Let $\{E_i(\mathbf{F}, \mathbf{e}_1)\}_i$ be the set of bi-homogeneous equations on $\mathbf{F}$ and $\mathbf{e}_1$, extracted from the generalized Kruppa equation (1). Our first concern is to determine whether all solutions of equation (1) are admissible, that is, whether they satisfy the usual constraint $\mathbf{F}\mathbf{e}_1 = 0$. Indeed, we prove the following statement:

Proposition 2. *As long as there are at least two distinct lines through* $\mathbf{e}_1$ *tangent to* Y_1*, equation* (1) *implies that* $\operatorname{rank} \mathbf{F} = 2$ *and* $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$.

Proof. The variety defined by $\phi_1(\gamma(\mathbf{p}))$ is then a union of at least two distinct lines through $\mathbf{e}_1$. If equation (1) holds, $\phi_2(\xi(\mathbf{p}))$ must define the same variety.

There are two cases to exclude: If $\operatorname{rank} \mathbf{F} = 3$, then the curve defined by $\phi_2(\xi(\mathbf{p}))$ is projectively equivalent to the curve defined by ϕ_2, which is $Y_2^\star$. In particular, it is irreducible.

If $\operatorname{rank} \mathbf{F} < 2$, or $\operatorname{rank} \mathbf{F} = 2$ and $\mathbf{F}\mathbf{e}_1 \neq \mathbf{0}$, then there is some $\mathbf{a}$, not a multiple of $\mathbf{e}_1$, such that $\mathbf{F}\mathbf{a} = \mathbf{0}$. Then the variety defined by $\phi_2(\xi(\mathbf{p}))$ is a union of lines through $\mathbf{a}$. In neither case can this variety contain two distinct lines through $\mathbf{e}_1$, so we must have $\operatorname{rank} \mathbf{F} = 2$ and $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$. ◻

As a result, in a generic situation every solution of $\{E_i(\mathbf{F}, \mathbf{e}_1)\}_i$ is admissible. Let V be the subvariety of $\mathbb{P}^2 \times \mathbb{P}^8 \times \mathbb{P}^2$ defined by the equations $\{E_i(\mathbf{F}, \mathbf{e}_1)\}_i$ together with $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{e}_2^T \mathbf{F} = \mathbf{0}^T$. We next compute a lower bound on the dimension of V, after which we will be ready for the calculation itself.

Proposition 3. *If* V *is non-empty, the dimension of* V *is at least* $7 - m$.

Proof. Choose any line $\mathbf{l}$ in $\mathbb{P}^2$ and restrict $\mathbf{e}_1$ to the affine piece $\mathbb{P}^2 \setminus \mathbf{l}$. Let (x, y) be homogeneous coordinates on $\mathbf{l}$. If $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$, the two sides of equation (1) are both unchanged upon replacing $\mathbf{p}$ by $\mathbf{p} + \alpha\mathbf{e}_1$. So equation (1) will hold for all $\mathbf{p}$ if it holds for all $\mathbf{p} \in \mathbf{l}$. Therefore equation (1) is equivalent to the equality of two homogeneous polynomials of degree m in x and y, which in turn is equivalent to the equality of $m + 1$ coefficients. After eliminating λ, we have m algebraic conditions on $(\mathbf{e}_1, \mathbf{F}, \mathbf{e}_2)$ in addition to $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$, $\mathbf{e}_2^T \mathbf{F} = \mathbf{0}^T$.

The space of all epipolar geometries, that is, solutions to $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$, $\mathbf{e}_2^T \mathbf{F} = \mathbf{0}^T$, is irreducible of dimension 7. Therefore, V is at least $(7 - m)$-dimensional. ◻

For the calculation of the dimension of V we introduce some additional notations. Given a triplet $(\mathbf{e}_1, \mathbf{F}, \mathbf{e}_2) \in \mathbb{P}^2 \times \mathbb{P}^8 \times \mathbb{P}^2$, let $\{\mathbf{q}_{1\alpha}(\mathbf{e}_1)\}$ (respectively $\{\mathbf{q}_{2\alpha}(\mathbf{e}_2)\}$) be the tangency points of the epipolar lines through $\mathbf{e}_1$ (respectively $\mathbf{e}_2$) to the first (respectively second) projected curve. Let $\mathbf{Q}_\alpha(\mathbf{e}_1, \mathbf{e}_2)$ be the 3D points projected onto $\{\mathbf{q}_{1\alpha}(\mathbf{e}_1)\}$ and $\{\mathbf{q}_{2\alpha}(\mathbf{e}_2)\}$. Let $\mathbf{L}$ be the baseline joining the two projection centers. We next provide a sufficient condition for V to be discrete.

Proposition 4. *For a generic position of the projection centers, the variety* V *will be discrete if, for any point* $(\mathbf{e}_1, \mathbf{F}, \mathbf{e}_2) \in V$*, the union of* $\mathbf{L}$ *and the points* $\mathbf{Q}_\alpha(\mathbf{e}_1, \mathbf{e}_2)$ *is not contained in any quadric surface.*

Proof. For generic projections, there will be m distinct points $\{\mathbf{q}_{1\alpha}(\mathbf{e}_1)\}$ and $\{\mathbf{q}_{2\alpha}(\mathbf{e}_2)\}$, and we can regard $\mathbf{q}_{1\alpha}$, $\mathbf{q}_{2\alpha}$ locally as smooth functions of $\mathbf{e}_1$, $\mathbf{e}_2$.

We let W be the affine variety in $\mathbb{C}^3 \times \mathbb{C}^9 \times \mathbb{C}^3$ defined by the same equations as V. Let $\Theta = (\mathbf{e}_1, \mathbf{F}, \mathbf{e}_2)$ be a point of W corresponding to a non-isolated point of V. Then there is a tangent vector $\vartheta = (\mathbf{v}, \Phi, \mathbf{v}')$ to W at Θ with Φ not a multiple of $\mathbf{F}$.

If χ is a function on W, then $\nabla_{\Theta,\vartheta}(\chi)$ will denote the derivative of χ in the direction defined by ϑ at Θ. For

$$\chi_\alpha(\mathbf{e}_1, \mathbf{F}, \mathbf{e}_2) = \mathbf{q}_{2\alpha}(\mathbf{e}_2)^T \mathbf{F}\mathbf{q}_{1\alpha}(\mathbf{e}_1),$$

the generalized Kruppa equation implies that χ_α vanishes identically on W, so its derivative must also vanish. This yields

$$\nabla_{\Theta,\vartheta}(\chi_\alpha) = (\nabla_{\Theta,\vartheta}(\mathbf{q}_{2\alpha}))^T \mathbf{F}\mathbf{q}_{1\alpha} + \mathbf{q}_{2\alpha}^T \Phi \mathbf{q}_{1\alpha} + \mathbf{q}_{2\alpha}^T \mathbf{F}(\nabla_{\Theta,\vartheta}(\mathbf{q}_{1\alpha})) = 0. \tag{2}$$

We shall prove that $\nabla_{\Theta,\vartheta}(\mathbf{q}_{1\alpha})$ is in the linear span of $\mathbf{q}_{1\alpha}$ and $\mathbf{e}_1$. Consider $\kappa(t) = f(\mathbf{q}_{1\alpha}(\mathbf{e}_1 + t\mathbf{v}))$, where f is the polynomial defining the image curve Y_1. Since $\mathbf{q}_{1\alpha}(\mathbf{e}_1 + t\mathbf{v}) \in Y_1$, we have $\kappa \equiv 0$, so the derivative $\kappa'(0)$ is 0. On the other hand, $\kappa'(0) = \nabla_{\Theta,\vartheta}(f(\mathbf{q}_{1\alpha})) = \mathrm{grad}_{\mathbf{q}_{1\alpha}}(f)^T \nabla_{\Theta,\vartheta}(\mathbf{q}_{1\alpha})$.

Thus we have $\mathrm{grad}_{\mathbf{q}_{1\alpha}}(f)^T \nabla_{\Theta,\vartheta}(\mathbf{q}_{1\alpha}) = 0$. But also $\mathrm{grad}_{\mathbf{q}_{1\alpha}}(f)^T \mathbf{q}_{1\alpha} = 0$ and $\mathrm{grad}_{\mathbf{q}_{1\alpha}}(f)^T \mathbf{e}_1 = 0$. Since $\mathrm{grad}_{\mathbf{q}_{1\alpha}}(f) \neq \mathbf{0}$ ($\mathbf{q}_{1\alpha}$ is not a singular point of the curve), this shows that $\nabla_{\Theta,\vartheta}(\mathbf{q}_{1\alpha})$, $\mathbf{q}_{1\alpha}$, and $\mathbf{e}_1$ are linearly dependent. As $\mathbf{q}_{1\alpha}$ and $\mathbf{e}_1$ are linearly independent, $\nabla_{\Theta,\vartheta}(\mathbf{q}_{1\alpha})$ must be in their linear span.

We have $\mathbf{q}_{2\alpha}^T \mathbf{F}\mathbf{e}_1 = \mathbf{q}_{2\alpha}^T \mathbf{F}\mathbf{q}_{1\alpha} = 0$, so $\mathbf{q}_{2\alpha}^T \mathbf{F}\nabla_{\Theta,\vartheta}(\mathbf{q}_{1\alpha}) = 0$, i.e. the third term of (2) vanishes. In a similar way, the first term of equation (2) vanishes, leaving

$$\mathbf{q}_{2\alpha}^T \Phi \mathbf{q}_{1\alpha} = 0. \tag{3}$$

The derivative of $\chi(\mathbf{e}_1, \mathbf{F}, \mathbf{e}_2) = \mathbf{F}\mathbf{e}_1$ must also vanish, which yields

$$\mathbf{e}_2^T \Phi \mathbf{e}_1 = 0. \tag{4}$$

From (3), we deduce that for every $\mathbf{Q}_\alpha$, we have

$$\mathbf{Q}_\alpha^T \mathbf{M}_2^T \Phi \mathbf{M}_1 \mathbf{Q}_\alpha = 0.$$

From (4), we deduce that every point $\mathbf{P}$ lying on the baseline must satisfy

$$\mathbf{P}^T \mathbf{M}_2^T \Phi \mathbf{M}_1 \mathbf{P} = 0.$$

The fact that Φ is not a multiple of F implies that $\mathbf{M}_2^T \Phi \mathbf{M}_1 \neq 0$, so together the last two equations mean that the union $\mathbf{L} \cup \{\mathbf{Q}_\alpha\}$ lies on a quadric surface. Thus if there is no such quadric surface, every point in V must be isolated. □

Observe that this result is consistent with the previous proposition, since there always exists a quadric surface containing a given line and six given points. However, in general there is no quadric containing a given line and seven given points. Therefore we can deduce the following theorem.

Proposition 5. *For a generic position of the projection centers, there is no quadric containing the line L and the tangency points $\mathbf{Q}_\alpha(\mathbf{e}_1, \mathbf{e}_2)_{\alpha=1,\ldots,m}$.*

Proof. First, observe that when the genus g of X is equal to or greater than 2, then the result is obvious. Indeed, the intersection of a quadric and X has at most $2d$ points (by Bézout's theorem) and there are $m > 2d$ points of tangency, which are distinct for a generic line L.

To handle the general case, let us introduce some notations. Consider the product of m copies of $\mathbb{P}^3$: $H = \mathbb{P}^3 \times \cdots \times \mathbb{P}^3$. Then an m-tuple $(Q_1, \ldots, Q_m) \in H$ is such that the points Q_i lie on a quadric if the matrix $\mathbf{W}$ has rank at most 9, where

$$\mathbf{W} = \begin{bmatrix} x_1^2 & \ldots & x_m^2 \\ x_1 y_1 & \ldots & x_m y_m \\ x_1 z_1 & \ldots & x_m z_m \\ x_1 t_1 & \ldots & x_m t_m \\ y_1^2 & \ldots & y_m^2 \\ y_1 z_1 & \ldots & y_m z_m \\ y_1 t_1 & \ldots & y_m t_m \\ z_1^2 & \ldots & z_m^2 \\ z_1 t_1 & \ldots & z_m t_m \\ t_1^2 & \ldots & t_m^2 \end{bmatrix}$$

and $[x_i, y_i, z_i, t_i]$ are the homogeneous coordinates of Q_i. This defines a closed subvariety of H, which we shall denote by S.

Consider now the following set:

$$\Sigma = \{(L, \mathbf{Q}_1(\mathbf{e}_1, \mathbf{e}_2), \ldots, \mathbf{Q}_m(\mathbf{e}_1, \mathbf{e}_2)) \in \mathbb{G}(1, 3) \times H\},$$

where $\mathbb{G}(1, 3)$ is the Grassmannian of lines in $\mathbb{P}^3$. The set Σ of course depends on both L (that is, on the projection center) and X. Let us show that Σ is an algebraic variety.

A point $Q \in \mathbb{P}^3$ is a tangency point of a plane containing L with X if and only if the following two conditions are satisfied: (i) $Q \in X$ and (ii) the tangent to X at Q intersects L (in projective space).

The Plücker coordinates of the tangent T_Q to X at Q are homogeneous polynomial functions of the coordinates of Q (given by the Gauss map). T_Q intersects L if and only if the join $T_Q \vee L$ vanishes [2], which yields a bi-homogeneous equation on the coordinates of Q and of those of L. We shall denote this equation by $\phi(Q, L) = 0$.

For a polynomial $F \in R[X, Y, Z, T]$, where R is some ring, we write F_i for the polynomial in $R[X_i, Y_i, Z_i, T_i]$, obtained from F by substituting the variables X, Y, Z, T by X_i, Y_i, Z_i, T_i. Then the set Σ is formed by the common zeros of the polynomials $F_{11}, \ldots, F_{r1}, \ldots, F_{1m}, \ldots, F_{rm}, \phi_1, \ldots, \phi_m$, where $F_1, \ldots, F_r$ are the polynomials defining X. Thus Σ can be viewed as a closed subvariety of $\mathbb{G}(1, 3) \times H$.

Let π_1 and π_2 be the canonical projections: $\pi_1 : \Sigma \to \mathbb{G}(1, 3)$ and $\pi_2 : \Sigma \to H$. Therefore for a line L, if there exists a quadric containing L and the tangency points $\mathbf{Q}_1(\mathbf{e}_1, \mathbf{e}_2), \ldots, \mathbf{Q}_m(\mathbf{e}_1, \mathbf{e}_2)$ then $\pi_2(\pi_1^{-1}(L))$ is included in the closed subvariety S defined above.

Thus a line L for which there exists a quadric containing L and the tangency points must lie in $\pi_1(\pi_2^{-1}(S))$. This is a subset of a proper closed subvariety of $\mathbb{G}(1, 3)$. This completes the proof. □

We conclude this section with the following corollary.

Corollary 1. *For a generic position of the projection centers, the generalized Kruppa equation defines the epipolar geometry up to a finite-fold ambiguity if and only if* $m \geq 7$.

Since different curves in generic position give rise to independent equations, this result means that the sum of the classes of the projected curves must be at least 7 for V to be a finite set.

3.4. Recovering the curve

Let the projection matrices be $\mathbf{M}_1$ and $\mathbf{M}_2$. Hence the two cones defined by the projected curves and the projection centers are given by $\Delta_1(\mathbf{P}) = f_1(\mathbf{M}_1\mathbf{P})$ and $\Delta_2(\mathbf{P}) = f_2(\mathbf{M}_2\mathbf{P})$. The reconstruction is defined as the curve whose equations are $\Delta_1 = 0$ and $\Delta_2 = 0$. This curve has two irreducible components as the following theorem states.

Theorem 2. *For a generic position of the projection centers, namely when no epipolar plane is tangent twice to the curve* X*, the curve defined by* $\{\Delta_1 = 0,\ \Delta_2 = 0\}$ *has two irreducible components. One has degree* d *and is the actual solution of the reconstruction. The other one has degree* $d(d-1)$.

Proof. For a line $\mathbf{L} \subset \mathbb{P}^3$, we write $\sigma(\mathbf{L})$ for the pencil of planes containing $\mathbf{L}$. For a point $\mathbf{p} \in \mathbb{P}^2$, we write $\sigma(\mathbf{p})$ for the pencil of lines through $\mathbf{p}$. There is a natural isomorphism between $\sigma(\mathbf{e}_i)$, the epipolar lines in image i, and $\sigma(\mathbf{L})$, the planes containing both projection centers. Consider the following covers of $\mathbb{P}^1$:

1. $X \xrightarrow{\eta} \sigma(\mathbf{L}) \cong \mathbb{P}^1$, taking a point $x \in X$ to the epipolar plane that it defines together with the projection centers.
2. $Y_1 \xrightarrow{\eta_1} \sigma(\mathbf{e}_1) \cong \sigma(\mathbf{L}) \cong \mathbb{P}^1$, taking a point $y \in Y_1$ to its epipolar line in the first projection plane.
3. $Y_2 \xrightarrow{\eta_2} \sigma(\mathbf{e}_2) \cong \sigma(\mathbf{L}) \cong \mathbb{P}^1$, taking a point $y \in Y_2$ to its epipolar line in the second projection plane.

If ρ_i is the projection $X \to Y_i$, then $\eta = \eta_i \rho_i$. Let $\mathcal{B}$ the union of the sets of branch points of η_1 and η_2. It is clear that the branch points of η lie in $\mathcal{B}$. Let $S = \mathbb{P}^1 \setminus \mathcal{B}$, pick $t \in S$, and write $X_S = \eta^{-1}(S)$, $X_t = \eta^{-1}(t)$. Let μ_{X_S} be the monodromy $\pi_1(S, t) \to \mathrm{Perm}(X_t)$, where $\mathrm{Perm}(Z)$ is the group of permutations of a finite set Z (see [22]). It is well known that the path-connected components of X are in one-to-one correspondence with the orbits of the action of $\mathrm{im}(\mu_{X_S})$ on X_t. Since X is assumed to be irreducible, it has only one component and $\mathrm{im}(\mu_{X_S})$ acts transitively on X_t. Then if $\mathrm{im}(\mu_{X_S})$ is generated by transpositions, this will imply that $\mathrm{im}(\mu_{X_S}) = \mathrm{Perm}(X_t)$. In order to show that $\mathrm{im}(\mu_{X_S})$ is actually generated by transpositions, consider a loop in $\mathbb{P}^1$ based at t, say l_t. If l_t does not go round any branch point, then l_t is homotopic to the constant path in S and then $\mu_{X_S}([l_t]) = 1$. Now in $\mathcal{B}$, there are three types of branch points:

1. branch points that come from nodes of Y_1: these are not branch points of η,
2. branch points that come from nodes of Y_2: these are not branch points of η,
3. branch points that come from epipolar lines tangent either to Y_1 or to Y_2: these are genuine branch points of η.

If the loop l_t goes round a point of the first two types, then it is still true that $\mu_{X_S}([l_t]) = 1$. Now suppose that l_t goes round a genuine branch point of η, say b (and goes round no other points in $\mathcal{B}$). By genericity, b is a simple two-fold branch point, hence $\mu_{X_S}([l_t])$ is a transposition. This shows that $\mathrm{im}(\mu_{X_S})$ is actually generated by transpositions and so $\mathrm{im}(\mu_{X_S}) = \mathrm{Perm}(X_t)$.

Now consider the curve $\tilde{X}$ defined by $\{\Delta_1 = 0,\ \Delta_2 = 0\}$. By Bézout's theorem $\tilde{X}$ has degree d^2. Let $\tilde{x} \in \tilde{X}$. It is projected onto a point y_i in Y_i such that $\eta_1(y_1) = \eta_2(y_2)$. Hence $\tilde{X} \cong Y_1 \times_{\mathbb{P}^1} Y_2$; restricting to the inverse image of the set S, we have $\tilde{X}_S \cong X_S \times_S X_S$. We can therefore identify $\tilde{X}_t$ with $X_t \times X_t$. The monodromy $\mu_{\tilde{X}_S}$ can then be given by $\mu_{\tilde{X}_S}(x, y) = (\mu_{X_S}(x), \mu_{X_S}(y))$. Since $\mathrm{im}(\mu_{X_S}) = \mathrm{Perm}(X_t)$, the action of $\mathrm{im}(\mu_{\tilde{X}_S})$ on $X_t \times X_t$ has two orbits, namely $\{(x, x)\} \cong X_t$ and $\{(x, y) \mid x \neq y\}$. Hence $\tilde{X}$ has two irreducible components. One has degree d and is X, the other has degree $d^2 - d = d(d - 1)$. □

This result provides a way to find the right solution for the recovery in a generic configuration, except in the case of conics, where the two components of the reconstruction are both admissible.

4. The $N \gg 1$ projections problem with known projection operators

Now we turn to the second problem. $N \gg 1$ projection maps $\{\pi_i\}_{i=1,\ldots,N}$ given by N matrices $\{\mathbf{M}_i\}_{i=1,\ldots,N}$ are known. Therefore N projections of an irreducible smooth algebraic curve are also provided. The problem is to recover the original curve by linear computations as much as possible.

4.1. Curve presentation in the dual space

Let $X^\star$ be the dual variety of X. Since X is supposed not to be a line, the dual variety $X^\star$ must be a hypersurface of the dual space [15]. Our first concern is to determine the degree of $X^\star$.

Proposition 6. *The degree of $X^\star$ is m, that is, the common degree of the dual projected curves.*

Proof. Since $X^\star$ is a hypersurface of $\mathbb{P}^{3\star}$, its degree is the number of points where a generic line in $\mathbb{P}^{3\star}$ meets $X^\star$. By duality it is the number of planes in a generic pencil that are tangent to X. Hence it is the degree of the dual projected curve. Another way to express the same fact is the observation that the dual projected curve is the intersection of $X^\star$ with a generic plane in $\mathbb{P}^{3\star}$. Note that this provides a new proof that the degree of the dual projected curve is constant for a generic position of the projection center. □

For the recovery of $X^\star$ from multiple projections, we will need to consider the mapping from a line $\mathbf{l}$ of the projection plane to the plane that it defines together with the projection center. Let $\mu : \mathbf{l} \mapsto \mathbf{M}^T \mathbf{l}$ denote this mapping. Let Υ be a generator of the ideal of $X^\star$. There exists a link between Υ, μ and ϕ, the polynomial of the dual projected curve: $\Upsilon(\mu(\mathbf{l})) = 0$ whenever $\phi(\mathbf{l}) = 0$. Since these two polynomials have the same degree (because μ is linear) and ϕ is irreducible, there exists a scalar λ such that

$$\Upsilon(\mu(\mathbf{l})) = \lambda\phi(\mathbf{l})$$

for all lines $\mathbf{l} \in \mathbb{P}^{2\star}$. Eliminating λ, we get $\binom{m+2}{m} - 1$ linear equations on Υ. Since the number of coefficients in Υ is $\binom{m+3}{m}$, we can state the following result:

Proposition 7. *The recovery in the dual space can be done linearly using at least* $k \geq \frac{m^2+6m+11}{3(m+3)}$ *projections.*

4.2. Curve presentation in the Grassmannian of lines in $\mathbb{P}^3$

Let $\mathbb{G}(1, 3)$ be the Grassmannian of lines in $\mathbb{P}^3$. Consider the set of lines in $\mathbb{P}^3$ intersecting the curve X of degree d. This defines a subvariety of $\mathbb{G}(1, 3)$ which is the intersection of $\mathbb{G}(1, 3)$ with a hypersurface of degree d in $\mathbb{P}^5$, given by a homogeneous polynomial Γ, defined modulo the dth graded piece $I(\mathbb{G}(1, 3))_d$ of the ideal of $\mathbb{G}(1, 3)$ and modulo scalars. However, picking one representative of this equivalence class is sufficient to recover the curve X entirely without any ambiguity. In our context, we shall call any representative of this class the *Chow polynomial* of the curve. We need to compute the class of Γ in the homogeneous coordinate ring of $\mathbb{G}(1, 3)$, or more precisely in its dth graded piece, $S(\mathbb{G}(1, 3))_d$, whose dimension is $N_d = \binom{d+5}{d} - \binom{d-2+5}{d-2}$.

Let f be the polynomial defining the projected curve, Y. Consider the mapping that associates to a point in the projection plane the line it generates together with the projection center: $\nu : \mathbf{p} \mapsto \widehat{\mathbf{M}}\mathbf{p}$. The polynomial $\Gamma(\nu(\mathbf{p}))$ vanishes whenever $f(\mathbf{p})$ does. Since they have the same degree and f is irreducible, there exists a scalar λ such that for every point $\mathbf{p} \in \mathbb{P}^2$, we have

$$\Gamma(\nu(\mathbf{p})) = \lambda f(\mathbf{p}).$$

This yields $\binom{d+2}{d} - 1$ linear equations on Γ.

Hence a similar statement to that in Proposition 7 can be made:

Proposition 8. *The recovery in* $\mathbb{G}(1, 3)$ *can be done linearly using at least* $k \geq \frac{1}{6}\frac{d^3+5d^2+8d+4}{d}$ *projections.*

4.3. Family of projection operators and finite closed subsets of points

Consider now a finite collection of points $\mathbf{P}_i$ in $\mathbb{P}^3$. Each point is projected by a different projection map. The N projection maps are known and so are the projected points.

Let X be the smooth irreducible curve generated by the points $\mathbf{P}_i$, and Y be the smooth irreducible curve, of minimal degree, generated by the projection centers.

Each projected point $\mathbf{p}_i$ yields one linear equation on the variety of intersecting lines of X, namely $\Gamma(\pi_i^{-1}(\mathbf{p}_i)) = 0$, where Γ is the Chow polynomial of X as before.

Let $\mathbf{d}$ and $\mathbf{d}'$ be respectively the degrees of X and Y. We compute the number of constraints obtained on Γ from the projected points as a function of $\mathbf{d}$ and $\mathbf{d}'$. In other words, we want to compute the maximal number of constraints that one can extract on a smooth irreducible curve X embedded in $\mathbb{P}^3$ from a finite number of lines in the join of X with a known curve Y.

Proposition 9. *The maximal number of constraints is*

$$N_d - (h^0(\mathcal{O}_{\mathbb{P}^5}(d-d')) - h^0(\mathcal{O}_{\mathbb{P}^5}(d-d'-2)) + 1),$$ [1]

where $N_d = \dim(S(\mathbb{G}(1,3))_d)$ *is the dimension of the* d*-th graded piece of the homogeneous coordinate ring of* $\mathbb{G}(1,3)$.

Proof. Each projected point generates a line together with the projection center. Let $\mathbf{L}_1, \ldots, \mathbf{L}_n$ be these n lines joining X and Y. Let Γ_X and Γ_Y be the Chow polynomials of X and Y respectively. We shall denote by $Z(\Gamma_X)$ and $Z(\Gamma_Y)$ the sets where they vanish. Let $V = Z(\Gamma_X) \cap Z(\Gamma_Y) \cap \mathbb{G}(1,3)$. For $n \gg 1$, we have

$$\{\Gamma \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d)) : \Gamma(L_i) = 0,\ i = 1, \ldots, n\} = \{\Gamma \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d)) : \Gamma_V \equiv 0\} = I_{V,\mathbb{P}^5}(d).$$

So, we want to compute $\dim(I_{V,\mathbb{P}^5}(d))$, or equivalently, $h^0(V, \mathcal{O}_V(d)) = h^0(\mathcal{O}_{\mathbb{P}^5}(d)) - \dim(I_{V,\mathbb{P}^5}(d))$. Since V is a complete intersection of degree $(d, d', 2)$ in $\mathbb{P}^5$, the dimension of $I_{V,\mathbb{P}^5}(d)$ should be equal to

$$h^0(\mathcal{O}_{\mathbb{P}^5}(d-2)) + h^0(\mathcal{O}_{\mathbb{P}^5}(d-d')) - h^0(\mathcal{O}_{\mathbb{P}^5}(d-d'-2)) + 1.$$

As a consequence,

$$h^0(V, \mathcal{O}_V(d)) = N_d - (h^0(\mathcal{O}_{\mathbb{P}^5}(d-d')) - h^0(\mathcal{O}_{\mathbb{P}^5}(d-d'-2)) + 1). \qquad \square$$

5. Applications to static and dynamic computational vision

The results obtained above were motivated by some applications to computational vision. We now proceed to show how these results can be applied to this field. We start by a quick survey on linear computational vision. More details can be found in [7, 16, 8]. Some of the terminology was introduced before in Section 2.

[1] As usual, $h^0(\mathcal{O}_{\mathbb{P}^5}(k))$ denotes the dimension of the cohomology group $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(k))$ (see [17]).

5.1. *Foundations of linear computational vision*

Projective algebraic geometry provides a natural framework to geometric computer vision. However, one has to keep in mind that the geometric entities to be considered are in fact embedded in the physical three-dimensional Euclidean space. Euclidean space is provided with three structures defined by three groups of transformations: the orthogonal group Euc_3 (which defines the Euclidean structure and which is included in the affine group), Aff_3 (defining the affine structure and itself included in the projective group), and Pr_3 (defining the projective structure). We fix $[X, Y, Z, T]^T$ as homogeneous coordinates, and $T = 0$ as the plane at infinity.

5.2. *A single camera system*

Computational vision starts with images captured by cameras. The camera components are the following:

- a plane $\mathcal{R}$, called the *retinal plane* or *image plane*;
- a point $\mathbf{O}$, called the *optical center* or *camera center*, which does not lie on the plane $\mathcal{R}$.

The plane $\mathcal{R}$ is regarded as a two-dimensional projective space embedded into $\mathbb{P}^3$. Hence it is also denoted by $i(\mathbb{P}^2)$. The camera is a projection machine: $\pi : \mathbb{P}^3 \setminus \{\mathbf{O}\} \to i(\mathbb{P}^2), \mathbf{P} \mapsto \overline{\mathbf{OP}} \cap i(\mathbb{P}^2)$. The projection π is determined (up to a scalar) by a 3×4 matrix $\mathbf{M}$ (the image of $\mathbf{P}$ being $\lambda\mathbf{MP}$).

The physical properties of a camera imply that $\mathbf{M}$ can be decomposed as follows:

$$\mathbf{M} = \begin{bmatrix} f & s & u_0 \\ 0 & \alpha f & v_0 \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{R}; \mathbf{t}],$$

where (f, α, s, u_0, v_0) are the so-called internal parameters of the camera, whereas the rotation $\mathbf{R}$ and the translation $\mathbf{t}$ are the external parameters.

It is easy to see that:

- The camera center $\mathbf{O}$ is given by $\mathbf{MO} = \mathbf{0}$.
- The matrix $\mathbf{M}^T$ maps a line in $i(\mathbb{P}^2)$ to the unique plane containing both the line and $\mathbf{O}$.
- There exists a matrix $\widehat{\mathbf{M}} \in \mathcal{M}_{6\times 3}(\mathbb{R})$, which is a polynomial function of $\mathbf{M}$, that maps a point $\mathbf{p} \in i(\mathbb{P}^2)$ to the line $\overline{\mathbf{Op}}$ (optical ray), represented by its Plücker coordinates in $\mathbb{P}^5$. If the camera matrix is decomposed as follows:

$$\mathbf{M} = \begin{bmatrix} \Gamma^T \\ \Lambda^T \\ \Theta^T \end{bmatrix},$$

  then for $\mathbf{p} = [x, y, z]^T$, the optical ray $\mathbf{L_p} = \widehat{\mathbf{M}}\mathbf{p}$ is given by the extensor $\mathbf{L_p} = x\Lambda \wedge \Theta + y\Theta \wedge \Gamma + z\Gamma \wedge \Lambda$, where $\wedge$ denotes the meet operator in the Grassmann–Cayley algebra (see [2]).
- The matrix $\widetilde{\mathbf{M}} = \widehat{\mathbf{M}}^T$ maps lines in $\mathbb{P}^3$ to lines in $i(\mathbb{P}^2)$.

Moreover we will need to consider the projection of the *absolute conic* onto the image plane. The absolute conic is simply defined by the following equations:

$$\begin{cases} X^2 + Y^2 + Z^2 = 0, \\ T = 0. \end{cases}$$

By definition, the absolute conic is left invariant under Euclidean transformations. Therefore its projection onto the image plane, defined by the matrix ω, is a function of the internal parameters only. By Cholesky decomposition $\omega = \mathbf{LU}$, where $\mathbf{L}$ (respectively $\mathbf{U}$) is a lower (respectively an upper) triangular matrix. Hence it is easy to see that $\mathbf{U} = \overline{\mathbf{M}}^{-1}$, where $\overline{\mathbf{M}}$ is the 3×3 matrix of the internal parameters of $\mathbf{M}$.

5.3. *A system of two cameras*

Given two cameras, $(\mathbf{O}_j, i_j(\mathbb{P}^2))_{j=1,2}$ are their components where $i_1(\mathbb{P}^2)$ and $i_2(\mathbb{P}^2)$ are two generic projective planes embedded into $\mathbb{P}^3$, and $\mathbf{O}_1$ and $\mathbf{O}_2$ are two generic points in $\mathbb{P}^3$ not lying on the above planes. As in 5.2, let $\pi_j : \mathbb{P}^3 \backslash \{\mathbf{O}_j\} \to i_j(\mathbb{P}^2), \mathbf{P} \mapsto \overline{\mathbf{O}_j\mathbf{P}} \cap i_j(\mathbb{P}^2)$, be the respective projections. The camera matrices are $\mathbf{M}_i$, $i = 1, 2$.

5.3.1. Homography between two images of the same plane. Consider the case where the two cameras are looking at the same plane in space, denoted by Δ. Let

$$\mathbf{M}_i = \begin{bmatrix} \Gamma_i^T \\ \Lambda_i^T \\ \Theta_i^T \end{bmatrix}$$

be the camera matrices, decomposed as above. Let $\mathbf{P}$ be a point lying on Δ. We shall denote the projections of $\mathbf{P}$ by $\mathbf{p}_i = [x_i, y_i, z_i]^T \cong \mathbf{M}_i\mathbf{P}$, where $\cong$ means equality modulo multiplication by a non-zero scalar.

The optical ray generated by $\mathbf{p}_1$ is given by $\mathbf{L}_{\mathbf{p}_1} = x_1\Lambda_1 \wedge \Theta_1 + y_1\Theta_1 \wedge \Gamma_1 + z_1\Gamma_1 \wedge \Lambda_1$. Hence $\mathbf{P} = \mathbf{L}_{\mathbf{p}_1} \wedge \Delta = x_1\Lambda_1 \wedge \Theta_1 \wedge \Delta + y_1\Theta_1 \wedge \Gamma_1 \wedge \Delta + z_1\Gamma_1 \wedge \Lambda_1 \wedge \Delta$. So $\mathbf{p}_2 \cong \mathbf{M}_2\mathbf{P}$ is given by the expression $\mathbf{p}_2 \cong \mathbf{H}_\Delta\mathbf{p}_1$, where

$$\mathbf{H}_\Delta = \begin{bmatrix} \Gamma_2^T(\Lambda_1 \wedge \Theta_1 \wedge \Delta) & \Gamma_2^T(\Theta_1 \wedge \Gamma_1 \wedge \Delta) & \Gamma_2^T(\Gamma_1 \wedge \Lambda_1 \wedge \Delta) \\ \Lambda_2^T(\Lambda_1 \wedge \Theta_1 \wedge \Delta) & \Lambda_2^T(\Theta_1 \wedge \Gamma_1 \wedge \Delta) & \Lambda_2^T(\Gamma_1 \wedge \Lambda_1 \wedge \Delta) \\ \Theta_2^T(\Lambda_1 \wedge \Theta_1 \wedge \Delta) & \Theta_2^T(\Theta_1 \wedge \Gamma_1 \wedge \Delta) & \Theta_2^T(\Gamma_1 \wedge \Lambda_1 \wedge \Delta) \end{bmatrix}.$$

This yields the expression of the collineation $\mathbf{H}_\Delta$ between two images of the same plane.

Definition 1. *The previous collineation is called the* homography *between the two images, through the plane* Δ.

5.3.2. Epipolar geometry

Definition 2. *Let* $(\mathbf{O}_j, i_j(\mathbb{P}^2), \mathbf{M}_j)_{j=1,2}$ *be as defined before. Given a pair* $(\mathbf{p}_1, \mathbf{p}_2) \in i_1(\mathbb{P}^1) \times i_2(\mathbb{P}^2)$*, we say that it is a pair of* corresponding *or* matching points *if there exists* $\mathbf{P} \in \mathbb{P}^3$ *such that* $\mathbf{p}_j = \pi_j(\mathbf{P})$ *for* $j = 1, 2$.

Consider a point $\mathbf{p} \in i_1(\mathbb{P}^2)$. Then $\mathbf{p}$ can be the image of any point lying on the fiber $\pi_1^{-1}(\mathbf{p})$. The matching point in the second image must lie on $\pi_2(\pi_1^{-1}(\mathbf{p}))$, which is, for a generic point $\mathbf{p}$, a line on the second image. Since π_1 and π_2 are both linear, there exists a matrix $\mathbf{F} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ such that $\xi(\mathbf{p}) = \pi_2(\pi_1^{-1}(\mathbf{p})) = \mathbf{F}\mathbf{p}$ for all but one point in the first image.

Definition 3. *The matrix* $\mathbf{F}$ *is called the* fundamental matrix*, whereas the line* $\mathbf{l}_\mathbf{p} = \mathbf{F}\mathbf{p}$ *is called the* epipolar line *of* $\mathbf{p}$.

Let $\mathbf{e}_1 = \overline{\mathbf{O}_1\mathbf{O}_2} \cap i_1(\mathbb{P}^2)$ and $\mathbf{e}_2 = \overline{\mathbf{O}_1\mathbf{O}_2} \cap i_2(\mathbb{P}^2)$. Those two points are respectively called the *first* and the *second epipole*. It is easy to see that $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$, since $\pi_1^{-1}(\mathbf{e}_1) = \overline{\mathbf{O}_1\mathbf{O}_2}$ and $\pi_2(\overline{\mathbf{O}_1\mathbf{O}_2}) = \mathbf{e}_2$. Observe that by symmetry $\mathbf{F}^T$ is the fundamental matrix of the reverse couple of images. Hence $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$. Since the only point in the first image that is mapped to zero by $\mathbf{F}$ is the first epipole, $\mathbf{F}$ has rank 2.

Now we want to deduce an expression of $\mathbf{F}$ as a function of the camera matrices. By the previous analysis, it is clear that $\mathbf{F} = \widetilde{\mathbf{M}}_2\widehat{\mathbf{M}}_1$. Moreover we have the following properties:

Proposition 10. *For any plane* Δ*, not passing through the camera centers, the following equalities hold:*

(i)
$$\mathbf{F} \cong [\mathbf{e}_2]_\times \mathbf{H}_\Delta,$$
where $[\mathbf{e}_2]_\times$ *is the matrix associated with the cross-product as follows: for any vector* $\mathbf{p}$, $\mathbf{e}_2 \times \mathbf{p} = [\mathbf{e}_2]_\times\mathbf{p}$. *Hence*
$$[\mathbf{e}_2]_\times = \begin{bmatrix} 0 & -\mathbf{e}_{23} & \mathbf{e}_{22} \\ \mathbf{e}_{23} & 0 & -\mathbf{e}_{21} \\ -\mathbf{e}_{22} & \mathbf{e}_{21} & 0 \end{bmatrix}.$$
In particular, $\mathbf{F} = [\mathbf{e}_2]_\times\mathbf{H}_\infty$*, where* $\mathbf{H}_\infty$ *is the homography between the two images through the plane at infinity.*

(ii)
$$\mathbf{H}_\Delta^T\mathbf{F} + \mathbf{F}^T\mathbf{H}_\Delta = \mathbf{0}. \tag{5}$$

Proof. The first equality is clear from its geometric meaning. Given a point $\mathbf{p}$ in the first image, $\mathbf{F}\mathbf{p}$ is its epipolar line in the second image. The optical ray $\mathbf{L}_\mathbf{p}$ passing trough $\mathbf{p}$ meets the plane Δ in a point $\mathbf{Q}$ whose projection in the second image is $\mathbf{H}_\Delta\mathbf{p}$. Hence the epipolar line must be $\mathbf{e}_2 \vee \mathbf{H}_\Delta\mathbf{p}$. This gives the required equality. The second equality is easily deduced from the first one by a short calculation. □

Proposition 11. *For a generic plane* Δ,
$$\mathbf{H}_\Delta\mathbf{e}_1 \cong \mathbf{e}_2.$$

Proof. The image of $\mathbf{e}_1$ by the homography must be the projection on the second image of the point defined as the intersection of the optical ray generated by $\mathbf{e}_1$ and the plane Δ. Hence $\mathbf{H}_\Delta\mathbf{e}_1 = \mathbf{M}_2(\mathbf{L}_{\mathbf{e}_1} \wedge \Delta)$. But $\mathbf{L}_{\mathbf{e}_1} = \overline{\mathbf{O}_1\mathbf{O}_2}$. Thus the result must be $\mathbf{M}_2\mathbf{O}_1$ (except when the plane is passing through $\mathbf{O}_2$), that is, the second epipole $\mathbf{e}_2$. □

5.3.3. Canonical stratification of the reconstruction. Three-dimensional reconstruction can be achieved from a system of two cameras, once the camera matrices are known. However, a typical situation is that the camera matrices are unknown. Then we face a double problem: recovering the camera matrices and the actual object. There exists an inherent ambiguity. Consider a pair of camera matrices $(\mathbf{M}_1, \mathbf{M}_2)$. If you change the world coordinate system by a transformation $\mathbf{V} \in \mathrm{Pr}_3$, the camera matrices are mapped to $(\mathbf{M}_1\mathbf{V}^{-1}, \mathbf{M}_2\mathbf{V}^{-1})$. Therefore we define the following equivalence relation:

Definition 4. *Given a group of transformations G, two pairs of camera matrices, say $(\mathbf{M}_1, \mathbf{M}_2)$ and $(\mathbf{N}_1, \mathbf{N}_2)$, are said to be* equivalent modulo G *if there exists $\mathbf{V} \in G$ such that $\mathbf{M}_1 = \mathbf{V}\mathbf{N}_1$ and $\mathbf{M}_2 = \mathbf{V}\mathbf{N}_2$.*

Note that this definition is similar to that of θ_2 below. Any reconstruction algorithm will always yield a reconstruction modulo some group of transformations. More precisely there exist three levels of reconstruction according to the information that can be extracted from the two images and from a priori knowledge of the world.

Projective stratum. When the only available information is the fundamental matrix, the reconstruction is done modulo Pr_3. Indeed, from $\mathbf{F}$, the so-called intrinsic homography $\mathbf{S} = -(\mathbf{e}_2/\|\mathbf{e}_2\|)\mathbf{F}$ is computed and the camera matrices are equivalent to $([\mathbf{I}; \mathbf{0}], [\mathbf{S}; \mathbf{e}_2])$, as shown in Proposition 1.

Affine stratum. When, in addition to the epipolar geometry, the homography between the two images through the plane at infinity, denoted by $\mathbf{H}_\infty$, can be computed, the reconstruction can be done modulo the group of affine transformations. Then the two camera matrices are equivalent to $([\mathbf{I}; \mathbf{0}], [\mathbf{H}_\infty; \mathbf{e}_2])$ (see [8]).

Euclidean stratum. The Euclidean stratum is obtained by the data of the projection of the absolute conic Ω onto the image planes, which allows the recovery of the internal parameters of the cameras. Once these parameters of the cameras are known, the relative motion between the cameras expressed by a rotation $\mathbf{R}$ and a translation $\mathbf{t}$ can be extracted from the fundamental matrix. However, only the direction of $\mathbf{t}$, not the norm, can be recovered. Then the camera matrices are equivalent, modulo the group of similarity transformations, to $(\overline{\mathbf{M}}_1[\mathbf{I}; \mathbf{0}], \overline{\mathbf{M}}_2[\mathbf{R}; \mathbf{t}])$, where $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$ are the matrices of internal parameters (see [8]).

Note that the projection of the absolute conic on the image can be computed using some a priori knowledge of the world. Moreover there exists a famous equation linking ω_1 and ω_2, the two matrices defining the projection of the absolute conic onto the images, when the epipolar geometry is given. This is the so-called *Kruppa equation*, defined in the following proposition.

Proposition 12. *The projections of the absolute conic onto two images are related as follows. There exists a non-zero scalar λ such that*

$$[\mathbf{e}_1]_\times^T \omega_1^* [\mathbf{e}_1]_\times = \lambda \mathbf{F}^T \omega_2^* \mathbf{F},$$

where $[\mathbf{e}_1]_\times$ is the matrix representing the cross-product by $\mathbf{e}_1$ and ω_i^* is the adjoint matrix of ω_i.

Let ϵ_i be the tangents to $\pi_i(\Omega)$ through $\mathbf{e}_i$. The Kruppa equation simply states that ϵ_1 and ϵ_2 are projectively isomorphic.

5.4. *Applications of the previous results to computational vision*

As mentioned in the introduction, the mathematical material presented in this paper was motivated by applications to static and dynamic configurations in computer vision. Applications of the previous results (Sections 3 and 4) are related to different contexts:

1. The recovery of the epipolar geometry from two images of the same smooth irreducible curve. Theorem 1 generalizes the Kruppa equation to algebraic curves. Section 3.3 provides a necessary and sufficient condition on the degree of the curve for the epipolar geometry to be defined up to finite-fold ambiguity. Note that the case of conic sections was first introduced in [18, 19].
2. The 3D reconstruction of a curve from two images is possible in a generic situation as shown in Proposition 2. The case of conics was also treated in [19, 23, 24]. Note that [11] presents an algorithm for curve reconstruction using a blow-up of the projected curve. This nice result, however, does not provide any information about the relative position of the curve in $\mathbb{P}^3$ with respect to other elements of the scene. On the other hand, our approach based on two images allows reconstructing the curve in the context of the whole scene. Furthermore the problem of curve reconstruction was also considered in [3] from the point of view of global optimization and bundle adjustment. Our approach, on the contrary, is based on looking at algebraic curves for which the representation is more compact.
3. The 3D reconstruction of a curve from $N \gg 1$ projections is linear using the dual space or the Grassmannian of lines $\mathbb{G}(1, 3)$ (Sections 4.1 and 4.2). The formalism of the dual space in the case of conics or quadrics was also used in [13, 20].
4. The trajectory recovery of a moving point viewed by a moving camera whose matrix is known over time is a linear problem when using the variety of intersecting lines of the curve generated by the motion of the point. Moreover this gives rise to the problem of counting the number of constraints that can be obtained. This is done in Proposition 9. Note that our algorithm for trajectory recovery or triangulation is a complete generalization of [1].

5.5. *Experiments and discussion*

Now we are in a position to perform some experiments relating to different applications mentioned above. The algorithms induced by our theoretical analysis involve either solving systems of polynomial equations or estimating high-dimensional parameters that appear linearly in equations built also from noisy data.

Solving a system of polynomial equations is a hard task when the system has many variables or the equation has high degree. There are roughly three methods to handle this problem: (i) computing a Gröbner basis of the ideal defined by the equation,

(ii) proceeding in the dual space of the coordinate ring of an affine piece of the variety via computation of resultants (see [6] for a detailed presentation), (iii) building a homotopy that defines the deformation from a system of polynomial equations whose solutions are known to the system you need to solve [25]. In our work the computations were done using Gröbner basis methods.

Note that numerical optimization tools like Newton–Raphson or Levenberg–Marquet optimization are not considered here because (i) zero-dimensional polynomial systems which are not overdetermined have more than one root and these optimization methods are designed to extract a single solution, (ii) the convergence to a solution with these tools is well behaved only when one starts in a small enough neighborhood of the solution.

The use of symbolic tools (either Gröbner bases or resultants) for computer vision applications is not without challenges. First, symbolic computations require large amounts of computational and memory resources. There is the issue of computational efficiency, scalability to large problems and the question of effectiveness in the presence of measurement errors. The full answer to these questions is far beyond the scope of this work. The field of symbolic computations for solving polynomial systems is a very active field of research where major progress has been made in the past decade [6, 14, 26]. For example, throughout this paper, the experiments were performed with one of the latest symbolic tools "FastGB" developed by Jean-Charles Faugère for efficient and robust Gröbner basis computation. With those latest tools, one can achieve a high degree of scalability and efficiency in the computations.

A second challenging problem is the sensitivity to noise (approximate polynomial equations). It is related to perturbation theory. It is necessary to note that since the computations are symbolic, they do not add any perturbation to the solution. Therefore, as opposed to numerical methods, there is no additional error due to possible truncation during the computations. However, there is very little research on measurement error sensitivity and their propagation throughout symbolic computations. Such research would be of great interest to the computer vision community and more generally for applications of algebraic geometry, but this topic is largely open. However, the development of interval arithmetic constitutes a first step toward both a theoretical and practical approach to this issue.

The second question, mentioned above, in connection to experiments, is related to the estimation of high-dimensional parameters, which appear linearly in equation built from noisy data. This is a typical case of heteroscedastic estimation [21] and will be discussed below.

Recovering epipolar geometry from a rational cubic and two conics. We proceed to a synthetic experiment, where the epipolar geometry is computed from a rational cubic and two conics. The curves are randomly chosen, as are the cameras.

The cubic is defined by the following system:

$$\begin{aligned}&226566665956452626ZX - 191485499323608 6169ZT - 791130248041963297YZ\\&- 1198609868087508022Z^2 + 893468169675527814XT + 2859405018489194 22T^2\\&- 1796326150569700 90YT + 277960038226472656Y^2 = 0,\end{aligned}$$

$$
\begin{aligned}
&5559200764529453 12XY + 6564944204577765614ZX - 1755155973545148735YZ\\
&- 1749154450800074954Z^2 + 984240461094724954XT - 61309565864179510YT\\
&- 1802588912007356295ZT + 2913197457767954 74T^2 = 0,
\end{aligned}
$$

$$
\begin{aligned}
&1111840152905890624X^2 - 2905335341664005486ZX - 793850352563738017YZ\\
&+ 1286890161434843658Z^2 + 1713207647519936006XT - 2487988473063282 02YT\\
&- 2942349361064284313ZT + 3988143869515851 34T^2 = 0.
\end{aligned}
$$

The first and the second conic are respectively defined by:

$$25X + 9Y + 40Z + 61T = 0,$$
$$40X^2 - 78XY + 62ZX + 11XT + 88Y^2 + YZ + 30YT + 81Z^2 - 5ZT - 28T^2 = 0,$$

and

$$4X - 11Y + 10Z + 57T = 0,$$
$$-82X^2 - 48XY - 11ZX + 38XT - 7Y^2 + 58YZ - 94YT - 68Z^2 + 14ZT - 35T^2 = 0.$$

The camera matrices are

$$\mathbf{M}_1 = \begin{bmatrix} -87 & 79 & 43 & -66 \\ -53 & -61 & -23 & -37 \\ 31 & -34 & -42 & 88 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} -76 & -65 & 25 & 28 \\ -61 & -60 & 9 & 29 \\ -66 & -32 & 78 & 39 \end{bmatrix}.$$

Then we form the extended Kruppa equations for each curve. From the computational point of view, it is crucial to enforce the constraint that each λ is different from zero. Mathematically this means that the computation is done in the localization with respect to each λ.

As expected, we get a zero-dimensional variety of degree one. Thus there is a single solution to the epipolar geometry given by the following fundamental matrix:

$$\mathbf{F} = \begin{bmatrix} -\frac{511443}{13426} & -\frac{2669337}{13426} & -\frac{998290}{6713} \\ \frac{84845}{2329} & \frac{23737631}{114121} & \frac{14061396}{114121} \\ \frac{1691905}{228242} & \frac{3426650}{114121} & \frac{8707255}{228242} \end{bmatrix}.$$

Recovering epipolar geometry from points and conics. We proceed to the recovery of the epipolar geometry from conics and point correspondences extracted from real images. The extraction has been done manually and the conics were fitted by classical least square optimization.

The recovery of the epipolar geometry has been done using four conics and one point. First the fundamental matrix is computed using three conics and one point, which leads to a finite number of solutions. Then the additional conic is used to select the right solution.

The images used for the experiments together with results and comments are presented in Figure 1.

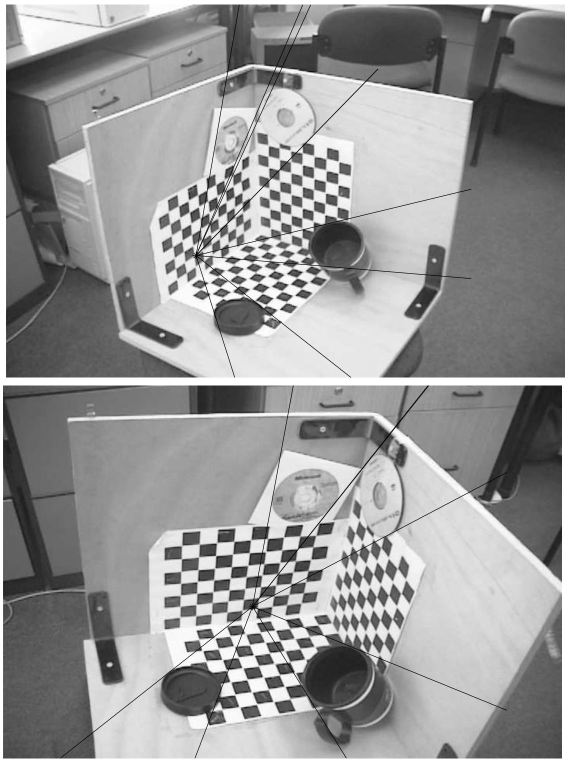

Fig. 1. The two images that were used. The epipoles and the corresponding epipolar lines tangent to the conics are drawn on the images.

Reconstruction of a spatial quartic in $\mathbb{P}^3$. Consider the curve X, drawn in Figure 2, defined by the following equations:

$$F_1(x, y, z, t) = x^2 + y^2 - t^2,$$
$$F_2(x, y, z, t) = xt - (z - 10t)^2.$$

The curve X is smooth and irreducible, and has degree 4 and genus 1. We define two camera matrices:

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & -1 & 0 & -10 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -10 \end{bmatrix}.$$

Then the curve is reconstructed from the two projections. As expected there are two irreducible components. One has degree 4 and is the original curve, while the other has degree 12.

Reconstruction using the Grassmannian. For the next experiment, we consider six images of an electric wire—one of the views is shown in Figure 3 and the image curve

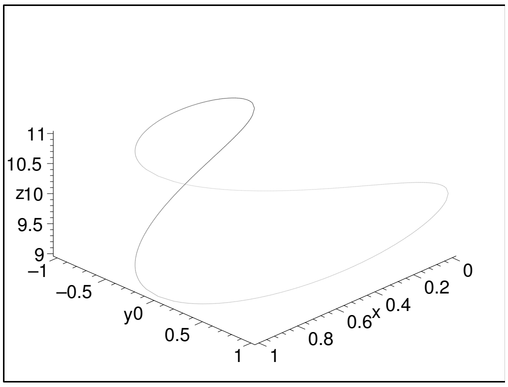

Fig. 2. A spatial quartic.

after segmentation and thinning is shown in Figure 4. Hence for each of the images, we extract a set of points lying on the thread. No fitting is performed in the image space. For each image, the camera matrix is calculated using the calibration pattern. Then we compute the Chow polynomial Γ of the curve in space. The curve X has degree 3. Once Γ is computed, a reprojection is easily performed, as shown in Figure 5.

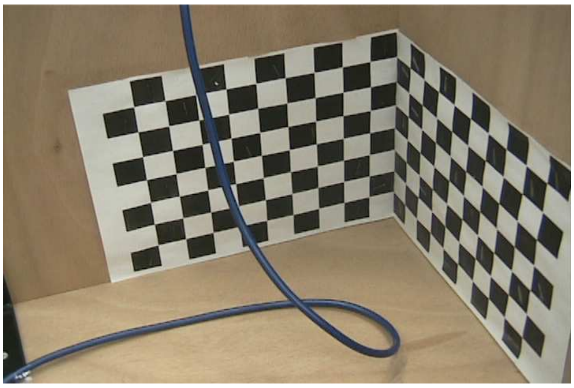

Fig. 3. One of the six views of an electric thread that were used to perform the reconstruction.

The computation of the Chow polynomial involves an estimation problem. Moreover as mentioned above, the Chow polynomial is not uniquely defined. In order to get a unique solution, we have to add some constraints to the estimation problem which do not distort the geometric meaning of the Chow polynomial. This is done by requiring the Chow polynomial to vanish over W_d additional arbitrary points of $\mathbb{P}^5$ which do not lie on $\mathbb{G}(1, 3)$. The number of additional points necessary to get a unique solution is $W_d = \binom{d+5}{d} - N_d$, where d is the degree of the Chow polynomial.

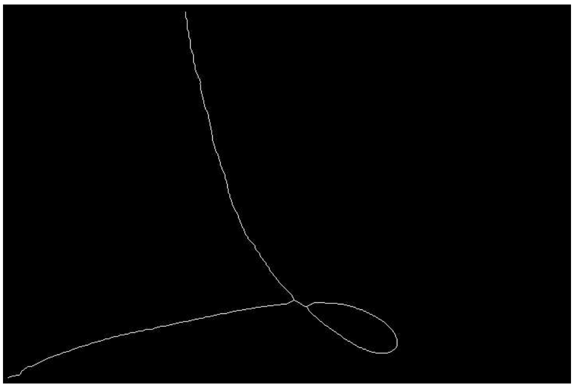

Fig. 4. An electric thread after segmentation and thinning.

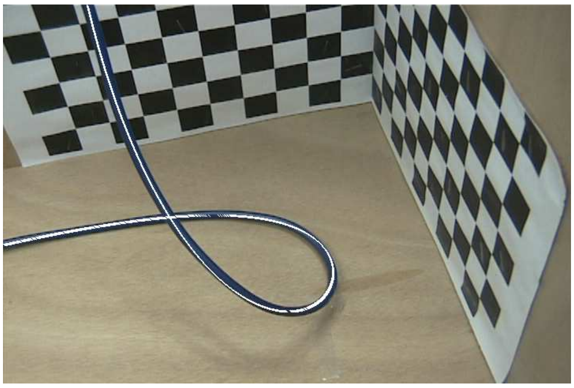

Fig. 5. Reprojection on a new image.

We shall see that the estimation of the Chow polynomial is a typical case of heteroscedastic estimation. Every 2D measurement $\mathbf{p}$ is corrupted by additive noise, which we assume to be an isotropic Gaussian noise $\mathcal{N}(0, \sigma)$. The variance is estimated to be about 2 pixels.

For each 2D point $\mathbf{p}$, we form the optical ray it generates, $\mathbf{L} = \widehat{\mathbf{M}}\mathbf{p}$. Then the estimation of the Chow polynomial is made using the optical rays $\mathbf{L}$. In order to avoid the problem of scale, the Plücker coordinates of each line are normalized in such a way that the last coordinate is equal to one. Hence the lines are represented by vectors in a five-dimensional affine space, denoted by $\mathbf{L}_a$. Hence if θ is a vector containing the coefficient of the Chow polynomial Γ, then θ is the solution of the following problem:

$$Z(\mathbf{L}_a)^T \theta = 0 \quad \text{for all optical rays,}$$

with $\|\theta\| = 1$ and $Z(\mathbf{L}_a)$ is a vector whose coordinates are monomials generated by the coordinates of $\mathbf{L}_a$. Following [5, 21], in order to obtain a reliable estimate, the solution θ is computed using a maximum likelihood estimator. This allows us to take into account the fact that each $Z(\mathbf{L}_a)$ has a different covariance matrix, or in other terms that the noise is *heteroscedastic*. More precisely, each $Z(\mathbf{L}_a)$ has the following covariance matrix:

$$\mathbf{C_L} = \mathbf{J}_\phi \mathbf{J}_n \widehat{\mathbf{M}} \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \widehat{\mathbf{M}}^T \mathbf{J}_n^T \mathbf{J}_\phi^T,$$

where $\mathbf{M}$ is the camera matrix and $\mathbf{J}_n$ and $\mathbf{J}_\phi$ are respectively the Jacobian matrices of the normalization of $\mathbf{L}$ and of the map sending $\mathbf{L}_a$ to $Z(\mathbf{L}_a)$. That is, for $\mathbf{L}(t) = [L_1, L_2, L_3, L_4, L_5, L_6]^T$, we have

$$\mathbf{J}_n = \begin{bmatrix} 1/L_6 & 0 & 0 & 0 & 0 & -L_1/L_6^2 \\ 0 & 1/L_6 & 0 & 0 & 0 & -L_2/L_6^2 \\ 0 & 0 & 1/L_6 & 0 & 0 & -L_3/L_6^2 \\ 0 & 0 & 0 & 1/L_6 & 0 & -L_4/L_6^2 \\ 0 & 0 & 0 & 0 & 1/L_6 & -L_5/L_6^2 \end{bmatrix},$$

and $\mathbf{J}_\phi$ is similarly computed. Then we use the method presented in [5] to perform the estimation. It is worth noting that the estimation is reliable because the initial guess of the algorithm was well chosen and because the number of measurements is very large. It is necessary to use a very large number of measurements for two reasons. First, the dimension of the parameter space is quite high, and secondly, the measurements are concentrated on a part of the space (over the Grassmannian $\mathbb{G}(1, 3)$).

Synthetic trajectory triangulation. Let $\mathbf{P} \in \mathbb{P}^3$ be a point moving on a cubic, as follows:

$$\mathbf{P}(t) = \begin{bmatrix} t^3 \\ 2t^3 + 3t^2 \\ t^3 + t^2 + t + 1 \\ t^3 + t^2 + t + 2 \end{bmatrix}.$$

It is viewed by a moving camera. At each instant a picture is made, we get a 2D point

$$\mathbf{p}(t) = [x(t), y(t)]^T = \left[\frac{\mathbf{m}_1^T(t)\mathbf{P}(t)}{\mathbf{m}_3^T(t)\mathbf{P}(t)}, \frac{\mathbf{m}_2^T(t)\mathbf{P}(t)}{\mathbf{m}_3^T(t)\mathbf{P}(t)} \right]^T,$$

where $\mathbf{M}^T(t) = [\mathbf{m}_1(t), \mathbf{m}_2(t), \mathbf{m}_3(t)]$ is the transpose of the camera matrix at time t.

Then we build the set of optical rays generated by the sequence. The Chow polynomial is then computed and given below:

$$\begin{aligned} \Gamma(L_1, \ldots, L_6) = \; & -72L_2^2L_3 + L_1^3 - 5L_1L_4L_5 \\ & - 18L_1L_3L_6 + 57L_2L_3L_5 + 48L_2L_4L_5 - 43L_1L_2L_4 \\ & - 10L_1L_3L_5 + 21L_1L_5L_6 - 30L_1L_4L_6 - 108L_2L_3L_6 \\ & + 41L_1L_2L_5 + 69L_1L_2L_6 - 26L_1L_2L_3 - 36L_2L_4^2 \\ & - 21L_2L_5^2 + 3L_3L_5^2 - 9L_3^2L_5 - 12L_4^2L_5 + 6L_4L_5^2 \\ & + 4L_1^2L_4 + 20L_2^3 - 13L_3^3 + 8L_4^3 - L_5^3 + 108L_2^2L_6 \\ & - 120L_2^2L_5 + 27L_3^2L_6 - 25L_1^2L_6 + 57L_2L_3^2 \\ & + 84L_2^2L_4 + 7L_1L_3^2 - L_1^2L_5 + 31L_1L_2^2 \\ & + 5L_1^2L_3 + L_1L_5^2 - 11L_1^2L_2 + 7L_1L_4^2. \end{aligned}$$

From the Chow polynomial, one can extract directly the locations of the moving point at each time instant an image was made. This is done by a two-step computation. The first step consists in giving a parametric representation of the optical ray generated by the 2D measurement. During the second step, the pencil of lines passing through a generic point on the optical ray is considered. For this generic point to be on the trajectory, the Chow polynomial must vanish over the pencil. This yields a polynomial system in one variable, whose root gives the location of the 3D moving point. We show in Figure 6 the recovered discrete locations of the point in 3D.

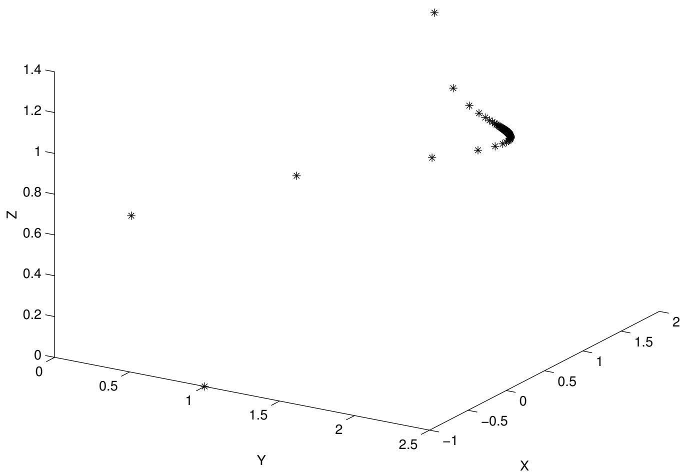

Fig. 6. The 3D locations of the point.

Trajectory triangulation from real images. A point is moving over a conic section. Four static non-synchronized cameras are looking at it. We show in Figure 7 one image of one sequence.

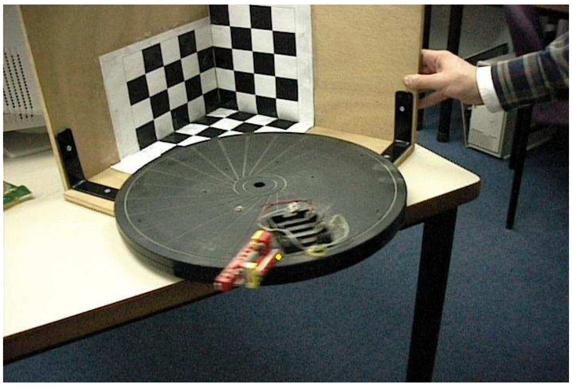

Fig. 7. A moving point over a conic section.

The camera matrices are computed using the calibration pattern. Every 2D measurement $\mathbf{p}(t)$ is corrupted by additive noise, which we assume to be an isotropic Gaussian noise $\mathcal{N}(0, \sigma)$. The variance is estimated to be about 2 pixels.

As before the estimation is done from the set of optical rays generated by the 2D moving point. The estimation is also a case of heteroscedastic estimation, which was handled with the method presented in [5]. The result is stable when starting with a good initial guess. In order to handle a more general situation we further stabilize it by incorporating some extra constraints that come from our *a priori* knowledge of the form of the solution. The final result is presented in Figure 8.

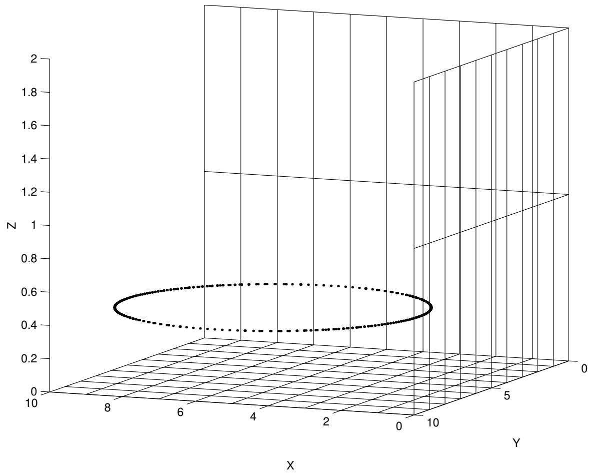

Fig. 8. The trajectory rendered in the calibration pattern.

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