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Arbitrary number of positive solutions for an elliptic problem with critical nonlinearity

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Abstract. We show that the critical nonlinear elliptic Neumann problem

$$\Delta u - \mu u + u^{7/3} = 0 \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded and smooth domain in $\mathbb{R}^5$, has arbitrarily many solutions, provided that $\mu > 0$ is small enough. More precisely, for any positive integer K, there exists $\mu_K > 0$ such that for $0 < \mu < \mu_K$, the above problem has a nontrivial solution which blows up at K interior points in Ω, as $\mu \to 0$. The location of the blow-up points is related to the domain geometry. The solutions are obtained as critical points of some finite-dimensional reduced energy functional. No assumption on the symmetry, geometry nor topology of the domain is needed.

Keywords. Semilinear elliptic Neumann problems, critical Sobolev exponent, blow-up

1. Introduction

Lin and Ni [28] considered the following nonlinear elliptic equation:

$$\Delta u - \mu u + u^q = 0 \quad \text{on } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $\mu > 0$ and $1 < q \leq (N+2)/(N-2)$ are parameters. Such problems arise in mathematical models of chemotaxis [29] and biological pattern formation [17], [32].

The situation is known to depend highly on the parameter μ. Ni and Takagi showed that for μ large enough and $1 < q < (N+2)/(N-2)$, i.e. in the subcritical case, a nontrivial least energy solution exists, which concentrates at a boundary point maximizing the mean curvature of the frontier [34], [35] as μ goes to infinity. Higher energy solutions also exist, which concentrate at one or several points, located on the boundary [7], [37],

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[13], [20], [24], [26], [49], [50], in the interior of the domain [8], [12], [14], [18], [19], [22], [48], or some of them on the boundary and others in the interior [23].

Many works have also been devoted to the critical case, i.e. $q = (N+2)/(N-2)$. As in the subcritical case, nonconstant solutions exist for μ large enough [1], [43], and the least energy solution blows up, as μ goes to infinity, at a unique point which maximizes the mean curvature of the boundary [3], [33]. Higher energy solutions have also been exhibited, blowing up at one [2], [44], [39], [21] or several (separated) boundary points [15], [30], [45], [46]. The question of interior blow-up is still open. However, in contrast with the subcritical situation, at least one blow-up point has to lie on the boundary [16], [40]. Some a priori estimates for those solutions are given in [21], [27].

In the case of small μ, Lin, Ni and Takagi [29] proved in the subcritical case that problem (1.1) admits only the trivial solution (i.e., $u \equiv \mu^{1/(p-1)}$). Based on this, Lin and Ni [28] asked:

Lin–Ni's Conjecture. For μ small and $q = (N+2)/(N-2)$, problem (1.1) admits only the constant solution.

The above conjecture was studied by Adimurthi–Yadava [4], [5] and Budd–Knapp–Peletier [10] in the case $\Omega = B_R(0)$ and u radial. Namely, they considered the following problem:

$$\begin{cases} \Delta u - \mu u + u^{(N+2)/(N-2)} = 0 \quad \text{in } B_R(0), \quad u > 0 \quad \text{in } B_R(0), \\ u \text{ is radial}, \quad \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_R(0). \end{cases} \tag{1.2}$$

The following results were proved:

Theorem A ([4], [5], [6], [10]). *For μ sufficiently small*

(1) *if $N = 3$ or $N \geq 7$, problem* (1.2) *admits only the constant solution;*
(2) *if $N = 4, 5$ or 6, problem* (1.2) *admits a nonconstant solution.*

Theorem A reveals that Lin–Ni's conjecture depends very sensitively on the dimension N. A natural question is: what about general domains? (For Dirichlet boundary conditions, Brezis and Nirenberg proved that a qualitative difference occurs between $N = 3$ and $N \geq 4$ [9].) The proofs of Theorem A use radial symmetry to reduce the problem to an ODE boundary value problem. Consequently, they do not carry over to general domains. In the general three-dimensional domain case, M. Zhu [52] proved:

Theorem B ([52], [51]). *The conjecture is true if $N = 3$ ($q = 5$) and Ω is convex.*

Zhu's proof relies strongly on a priori estimates. Recently, Wei and Xu [51] gave a direct proof of Theorem B, using only integration by parts.

The purpose of this paper is to establish a result similar to (2) of Theorem A in *general* five-dimensional domains, with important additional information about *multiplicity* and shape of solutions. Namely, we consider the problem

$$\Delta u - \mu u + u^{7/3} = 0 \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^5$ and $\mu > 0$ is small. Our main result can be stated as follows:

Main Theorem. *For any integer $K \in \mathbb{N}^*$, there exists μ_K such that for $0 < \mu < \mu_K$, problem* (1.3) *has a solution u_μ which blows up at exactly K interior points in Ω. As a consequence, for μ small, problem* (1.3) *has an arbitrary number of nonconstant distinct positive solutions.*

In order to make this statement more precise, some notations have to be introduced. Let $G(x, Q)$ be the Green's function defined as

$$\Delta_x G(x, Q) + \delta_Q - \frac{1}{|\Omega|} = 0 \quad \text{in } \Omega, \qquad \frac{\partial G}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \qquad \int_\Omega G(x, Q)\, dx = 0. \tag{1.4}$$

We decompose

$$G(x, Q) = K(|x - Q|) - H(x, Q),$$

where

$$K(r) = \frac{1}{c_5 r^3}, \qquad c_5 = 3|S^4|, \tag{1.5}$$

is the fundamental solution of the Laplacian operator in $\mathbb{R}^5$ ($|S^4|$ denotes the area of the unit sphere).

For $\delta > 0$ sufficiently small, we define a configuration space as

$$\mathcal{M}_\delta := \{\mathbf{Q} = (Q_1, \ldots, Q_K) \in \Omega^K \mid \min_i d(Q_j, \partial\Omega) > \delta,\ \min_{i \neq j} |Q_i - Q_j| > \delta\}. \tag{1.6}$$

Let $\mathbf{Q} = (Q_1, \ldots, Q_K) \in \mathcal{M}_\delta$. We set

$$F(\mathbf{Q}) = \sum_{j=1}^{K} H(Q_j, Q_j) - \sum_{i \neq j} G(Q_i, Q_j) - K F_0 \sum_{j=1}^{K} \int_\Omega \frac{1}{|x - Q_j|^3}\, dx \tag{1.7}$$

where $F_0 > 0$ is a constant which depends on Ω only.

For normalization reasons, we consider throughout the paper the following equation:

$$\Delta u - \mu u + 15 u^{7/3} = 0, \quad u > 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{1.8}$$

instead of the original one. The solutions are identical, up to the multiplicative constant $15^{-3/4}$. We recall that, according to [11], the functions

$$U_{\varepsilon,Q}(x) = \frac{\varepsilon^{3/2}}{(\varepsilon^2 + |x - Q|^2)^{3/2}}, \qquad \varepsilon > 0,\ Q \in \mathbb{R}^5, \tag{1.9}$$

are the only solutions to the problem

$$-\Delta u = 15 u^{7/3}, \quad u > 0 \quad \text{in } \mathbb{R}^5. \tag{1.10}$$

Our main result can be stated precisely as follows:

Theorem 1.1. *Let Ω be any smooth and bounded domain in $\mathbb{R}^5$, and $K \in \mathbb{N}^*$. There exists $\mu_K > 0$ such that for $0 < \mu < \mu_K$, problem* (1.8) *has a nontrivial solution u_μ with the following properties:*

(1) *u_μ has K local maximum points Q_i^μ, $i = 1, \ldots, K$, such that*

$$F(Q_1^\mu, \ldots, Q_K^\mu) \to \max_{\mathbf{Q}\in\mathcal{M}_\delta} F(\mathbf{Q}) \quad \text{as } \mu \to 0,$$

(2) *$u_\mu(x) = \sum_{j=1}^K U_{\mu^2\Lambda_j, Q_j^\mu}(x) + O(\mu^2)$, where $\Lambda_j \to \Lambda_0$, and $\Lambda_0 > 0$ is some generic constant. As a consequence, $u_\mu(Q_j^\mu) \sim \mu^{-3}$ and $u_\mu(x) \to 0$ for any $x \in \Omega \setminus \bigcup_{i=1}^K B_\delta(Q_i^\mu)$, where $\delta > 0$ is any small number, and u_μ blows up at K points $Q_1, \ldots, Q_K$ in Ω such that $\mathbf{Q} = (Q_1, \ldots, Q_K)$ maximizes F in $\mathcal{M}_\delta$.*

Remarks. 1. The existence of a global maximum for the function $F(\mathbf{Q})$ in $\mathcal{M}_\delta$ follows from the properties of the Green's function—see the proof of Lemma 6.1.

2. We believe that Theorem 1.1 should also be true in dimensions $N = 4$ and $N = 6$. When $N = 4$, our computations show that the blow-up rate should be e^{c_1/μ^2} for some $c_1 > 0$ (instead of μ^{-3} here). When $N = 6$, the blow-up rate should be μ^{-2}. In both cases, the blow-up rate also depends on the location of the blow-up points. We shall come back to this question in a future work.

3. There have been many works on the multiplicity of solutions for elliptic equations with critical nonlinearity—see [31], [30], [44], [45], [46] and references therein. As far as the authors know, all the multiplicity results are proved with some additional assumptions either on the symmetry, geometry, or topology of the domain. In Theorem 1.1, no condition is required.

As we commented earlier, PDE methods have to be used to prove Theorem 1.1. Note that the least energy solution has to be constant if μ is small (see [52] and [29]). Therefore, the solutions in Theorem 1.1 must have higher energy. To capture such solutions, we use the so-called "localized energy method", a combination of the Lyapunov–Schmidt reduction method and variational techniques. Namely, we first use the Lyapunov–Schmidt method to reduce the problem to a finite-dimensional one, with some *reduced energy*. Then the solutions in Theorem 1.1 turn out to be generated by critical points of the reduced energy functional. This idea has been used in [22] to study the interior spike solutions of problem (1.1) when μ is large and q is subcritical. This kind of argument has been applied in many other papers (see [12], [36], [19], [22], [24], [41], [42] and references therein). However, a new functional setting has to be introduced, and an appropriate variational argument to be developed to make the approach followed in our earlier works [41], [42] successful.

We set

$$\varepsilon = \mu^2, \quad \Omega_\varepsilon := \Omega/\varepsilon = \{z \mid \varepsilon z \in \Omega\}. \tag{1.11}$$

Through the transformation $u(x) \mapsto \varepsilon^{-3/2}u(x/\varepsilon)$, (1.8) becomes the rescaled problem we shall work with:

$$\Delta u - \varepsilon^{5/2}u + 15u^{7/3} = 0, \quad u > 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{1.12}$$

We set

$$S_\varepsilon[u] := -\Delta u + \varepsilon^{5/2} u - 15u_+^{7/3}, \quad u_+ = \max(u, 0), \tag{1.13}$$

and we introduce the following functional defined in $H^1(\Omega_\varepsilon)$:

$$J_\varepsilon[u] = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{\varepsilon^{5/2}}{2}\int_{\Omega_\varepsilon} u^2 - \frac{9}{2}\int_{\Omega_\varepsilon} u_+^{10/3} \tag{1.14}$$

whose nontrivial critical points are solutions to (1.12) ($J'_\varepsilon[u] = S_\varepsilon[u]$).

The paper is organized as follows: In Section 3, we construct suitable approximate K-bubble solutions W, and list their properties. In Section 4, we solve the linearized problem at W in a finite-codimensional space. Then, in Section 4, we are able to solve the nonlinear problem in that space. In Section 5, we study the remaining finite-dimensional problem and solve it in Section 6, finding critical points of the reduced energy functional. The proof of two technical lemmas may be found in Appendices A and B.

Throughout the paper, the letters C, C_i will denote various positive constants independent of ε small. δ will always denote a small constant.

2. Approximate bubble solutions

This section is devoted to the construction of suitable approximate K-bubble solutions, in the neighbourhood of which solutions of Theorem 1.1 will be found.

Let ε be as defined at (1.11). We consider $Q \in \Omega$, $\Lambda > 0$ a constant, and $U_{\Lambda,Q/\varepsilon}$ as defined in (1.9). In view of (1.10) and (1.9), $U_{\Lambda,Q/\varepsilon}$ provides us with a first approximate solution to (1.8) as ε goes to zero (equivalently, μ goes to zero). However, because of the additional linear term μu in (1.8), such an approximation has to be improved. To this end, we consider the equation

$$\Delta\Psi + U_\Lambda = 0, \quad \Psi_\Lambda(x) \to 0 \quad \text{as } |x| \to +\infty \tag{2.1}$$

where U_Λ denotes $U_{\Lambda,0}$. It is known that there exists a unique radially symmetric solution Ψ_Λ, which satisfies

$$\Psi_\Lambda(x) = \frac{B}{|x|}\left(1 + O\left(\frac{1}{|x|^2}\right)\right) \quad \text{for } |x| > 1 \tag{2.2}$$

where $B = \Lambda^{3/2}/2 > 0$. For $a \in \mathbb{R}^5$, we set

$$\Psi_{\Lambda,a}(x) = \Psi_\Lambda(x - a).$$

(Note that $\partial_\Lambda \Psi_{\Lambda,a} = O(|x-a|^{-1})$ and $\partial_{a_i}\Psi_{\Lambda,a} = O(|x-a|^{-2})$ as $|x-a|$ goes to infinity.)

An additional correction is necessary, in order to obtain approximate solutions which satisfy the required boundary conditions. With this aim in view, we define

$$\hat{U}_{\Lambda,Q/\varepsilon}(z) = -\Psi_{\Lambda,Q/\varepsilon}(z) - c_5\varepsilon^{1/2}\Lambda^{3/2}H(\varepsilon z, Q) + R_{\varepsilon,\Lambda,Q}(z)\chi(\varepsilon z) \tag{2.3}$$

where $R_{\varepsilon,\Lambda,Q}$ is defined by $\Delta R_{\varepsilon,\Lambda,Q} - \varepsilon^2 R_{\varepsilon,\Lambda,Q} = 0$ in Ω_ε and

$$\frac{\partial R_{\varepsilon,\Lambda,Q}}{\partial\nu} = \frac{\partial}{\partial\nu}[U_{\Lambda,Q/\varepsilon} - \varepsilon^{5/2}\Psi_{\Lambda,Q/\varepsilon} - c_5\varepsilon^3\Lambda^{3/2}H(\varepsilon z, Q)] \quad \text{on } \partial\Omega_\varepsilon. \tag{2.4}$$

Lastly, $\chi(x)$ is a smooth cut-off function in Ω such that $\chi(x) = 1$ for $d(x, \partial\Omega) < \delta/4$ and $\chi(x) = 0$ for $d(x, \partial\Omega) > \delta/2$.

We notice that (2.2), an expansion of $U_{\Lambda,Q/\varepsilon}$ and the definition of H imply that the normal derivative of $R_{\varepsilon,Q}$ is of order $\varepsilon^{9/2}$ on the boundary of Ω_ε, from which we deduce

$$|R_{\varepsilon,\Lambda,Q}| + |\varepsilon^{-1}\nabla_z R_{\varepsilon,\Lambda,Q}| + |\varepsilon^{-2}\nabla_z^2 R_{\varepsilon,\Lambda,Q}| \le C\varepsilon^{7/2}. \tag{2.5}$$

Such an estimate also holds for the derivatives of $R_{\varepsilon,\Lambda,Q}$ with respect to Λ and Q. It will ensure that $R_{\varepsilon,\Lambda,Q}$ play no role in further computations, being negligible.

We are now able to define the appropriate approximate K-bubble solutions we are looking for. Let $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_K)$ and $\mathbf{Q} = (Q_1, \ldots, Q_K)$ be such that

$$1/C_0 \le |\mathbf{\Lambda}| \le C_0, \qquad \mathbf{Q} \in \mathcal{M}_\delta. \tag{2.6}$$

In view of the rescaling, we write

$$\bar{Q}_i = \frac{1}{\varepsilon} Q_i, \qquad \bar{\mathbf{Q}} = (\bar{Q}_1, \ldots, \bar{Q}_K) \tag{2.7}$$

and we define our approximate solutions as

$$W_{\varepsilon,\Lambda,\bar{\mathbf{Q}}} := \sum_{j=1}^{K} (U_j + \varepsilon^{5/2}\hat{U}_j) + \eta\varepsilon^{5/2} \tag{2.8}$$

with

$$\eta = \frac{c_5}{|\Omega|} \sum_{j=1}^{K} \Lambda_j^{3/2}. \tag{2.9}$$

To simplify our notations, we wrote U_j and $\hat{U}_j$ instead of $U_{\Lambda_j,Q_j/\varepsilon}$ and $\hat{U}_{\Lambda_j,Q_j/\varepsilon}$. For the same reason, we shall also omit the dependence of W on ε, $\mathbf{\Lambda}$, $\bar{\mathbf{Q}}$. The last term $\eta\varepsilon^{5/2}$ in (2.8) has been added to cancel, in the Laplacian of W, the Laplacian of H introduced through the $\hat{U}_j$'s. By construction, the normal derivative of W vanishes on the boundary of Ω_ε, and W satisfies

$$-\Delta W + \varepsilon^{5/2} W = 15 \sum_{j=1}^{K} U_j^{7/3} + \varepsilon^5 \sum_{j=1}^{K} \hat{U}_j - \varepsilon^{5/2}\Delta(R_{\varepsilon,Q}\chi(\varepsilon\cdot)) \quad \text{in } \Omega_\varepsilon. \tag{2.10}$$

According to (2.5), the last term occurring in that equation is dominated by ε^8.

We note that W depends smoothly on Λ, $\bar{\mathbf{Q}}$. Setting, for $z \in \Omega_\varepsilon$,

$$\langle z - \bar{\mathbf{Q}} \rangle = \min_{j=1}^{K} (1 + |z - \bar{Q}_j|^2)^{1/2}$$

we derive from the definition of W the inequalities

$$|W(z)| \leq C(\varepsilon^{5/2} + \langle z - \bar{\mathbf{Q}}\rangle^{-3}), \tag{2.11}$$

$$|D_{\mathbf{\Lambda}} W(z)| \leq C(\varepsilon^{5/2} + \langle z - \bar{\mathbf{Q}}\rangle^{-3}) \tag{2.12}$$

and

$$|D_{\bar{\mathbf{Q}}} W(z)| \leq C(\varepsilon^{3} + \langle z - \bar{\mathbf{Q}}\rangle^{-4}) \tag{2.13}$$

where $D_{\mathbf{\Lambda}}$ and $D_{\bar{\mathbf{Q}}}$ denote the first partial derivatives with respect to $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_K)$ and $\bar{\mathbf{Q}} = (\bar{Q}_1, \ldots, \bar{Q}_K)$ respectively.

By our choice of W, we have the following error and energy estimates, proved in Appendix A.

Lemma 2.1. *We have*

$$|S_\varepsilon[W](z)| \leq C(\varepsilon^{5/2}\langle z - \bar{\mathbf{Q}}\rangle^{-4} + \varepsilon^5\langle z - \bar{\mathbf{Q}}\rangle^{-1/2}). \tag{2.14}$$

The same estimate holds for $D_{\mathbf{\Lambda}} S_\varepsilon[W](z)$ *and* $D_{\bar{\mathbf{Q}}} S_\varepsilon[W](z)$*, and*

$$\begin{aligned} J_\varepsilon[W] = A_0 + \varepsilon^{5/2}\beta(\mathbf{\Lambda}) + \varepsilon^3 E_0\Bigg[\sum_{j=1}^{K} \Lambda_j^3 H(Q_j, Q_j) - \sum_{i\neq j} \Lambda_i^{3/2}\Lambda_j^{3/2} G(Q_i, Q_j) \\ - F_0\Big(\sum_{j=1}^{K}\Lambda_j^{3/2}\Big)\sum_{j=1}^{K}\Lambda_j^{3/2}\int_\Omega \frac{dx}{|x - Q_j|^3}\Bigg] + o(\varepsilon^3). \end{aligned} \tag{2.15}$$

Moreover

$$D_{\mathbf{\Lambda}}(J_\varepsilon[W]) = \varepsilon^{5/2} D_{\mathbf{\Lambda}}\beta(\mathbf{\Lambda}) + O(\varepsilon^3) \tag{2.16}$$

where $\beta(\mathbf{\Lambda})$ *is defined by*

$$\beta(\mathbf{\Lambda}) = -B_0\Big(\sum_{j=1}^{K}\Lambda_j^{3/2}\Big)^2 + D_0\sum_{j=1}^{K}\Lambda_j^2. \tag{2.17}$$

A_0, B_0, D_0, E_0, F_0 *are all generic strictly positive constants.*

3. Finite-dimensional reduction: a linear problem

According to our general strategy, we first consider the linearized problem at W, and we solve it in a finite-codimensional subspace, i.e. the orthogonal space to the finite-dimensional subspace generated by the derivatives of W with respect to the parameters Λ_j and $\bar{Q}_{j,i}$. Namely, we equip $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u, v)_\varepsilon = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + \varepsilon^{5/2} uv).$$

Orthogonality to the functions

$$Y_{j,0} = \frac{\partial W}{\partial \Lambda_j}, \quad j = 1, \ldots, K, \quad Y_{j,i} = \frac{\partial W}{\partial \bar{Q}_{j,i}}, \quad 1 \le i \le 5, \ j = 1, \ldots, K, \tag{3.1}$$

in that space is equivalent to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle \cdot, \cdot \rangle$, to the functions $Z_{j,i}$, $1 \le j \le K$, $0 \le i \le 5$, defined as

$$\begin{cases} Z_{j,0} = -\Delta \dfrac{\partial W}{\partial \Lambda_j} + \varepsilon^{5/2} \dfrac{\partial W}{\partial \Lambda_j}, \\ Z_{j,i} = -\Delta \dfrac{\partial W}{\partial \bar{Q}_{j,i}} + \varepsilon^{5/2} \dfrac{\partial W}{\partial \bar{Q}_{j,i}}, \quad 1 \le i \le 5, \ j = 1, \ldots, K. \end{cases} \tag{3.2}$$

Note that differentiating (2.10) with respect to Λ_j and $\bar{Q}_{j,i}$ and straightforward computations provide us with the estimate

$$|Z_{j,i}(z)| \le C(\varepsilon^{11/2} + \langle z - \bar{\mathbf{Q}} \rangle^{-7}). \tag{3.3}$$

Now, we consider the following problem: given h, find a function ϕ which satisfies

$$\begin{cases} -\Delta\phi + \varepsilon^{5/2}\phi - 35 W_+^{4/3}\phi = h + \sum_{j,i} c_{j,i} Z_{j,i} & \text{in } \Omega_\varepsilon, \\ \partial\phi/\partial\nu = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_{j,i}, \phi \rangle = 0, & 0 \le i \le 5, \ 1 \le j \le K, \end{cases} \tag{3.4}$$

for some numbers $c_{j,i}$.

Existence and uniqueness of ϕ will follow from an inversion procedure in suitable function spaces. Just as del Pino, Felmer and Musso in [36], we use weighted Hölder spaces, defining here (among other possible choices) the two norms:

$$\|\phi\|_* = \|\langle z - \bar{\mathbf{Q}} \rangle^{3/2} \phi(z)\|_\infty, \quad \|f\|_{**} = \varepsilon^{-4}|\bar{f}| + \|\langle z - \bar{\mathbf{Q}} \rangle^{7/2} f(z)\|_\infty, \tag{3.5}$$

where $\|f\|_\infty = \max_{z \in \Omega_\varepsilon} |f(z)|$ and $\bar{f} = |\Omega_\varepsilon|^{-1} \int_{\Omega_\varepsilon} f(z)\, dz$ denotes the average of f in Ω_ε.

Before stating an existence result for ϕ, we need the following lemma, whose proof is given in Appendix B:

Lemma 3.1. *Let u and f satisfy*

$$-\Delta u = f \quad \textit{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial \nu} = 0 \quad \textit{on } \partial\Omega_\varepsilon, \quad \bar{u} = \bar{f} = 0.$$

Then

$$|u(x)| \le C \int_{\Omega_\varepsilon} \frac{|f(y)|}{|x - y|^3}\, dy. \tag{3.6}$$

As a corollary, we have:

Corollary 3.1. *Suppose u and f satisfy*

$$-\Delta u + \varepsilon^{5/2} u = f \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

Then

$$\|u\|_* \le C\|f\|_{**}. \tag{3.7}$$

Proof. Integrating the equation yields $\bar{u} = \varepsilon^{-5/2}\bar{f}$. We may write

$$\Delta(u - \bar{u}) = \varepsilon^{5/2}(u - \bar{u}) - (f - \bar{f}).$$

Lemma 3.1 gives

$$|u(y) - \bar{u}| \le C\varepsilon^{5/2} \int_{\Omega_\varepsilon} \frac{|u(x) - \bar{u}|}{|x - y|^3}\, dx + C \int_{\Omega_\varepsilon} \frac{|f(x) - \bar{f}|}{|x - y|^3}\, dx.$$

Since

$$\langle y - \bar{\mathbf{Q}}\rangle^{3/2} \int_{\mathbb{R}^5} \frac{1}{|x - y|^3} \langle x - \bar{\mathbf{Q}}\rangle^{-7/2}\, dx < \infty$$

we obtain

$$\begin{aligned}\|\langle y - \bar{\mathbf{Q}}\rangle^{3/2}|u - \bar{u}|\|_\infty &\le C\varepsilon^{5/2}\|\langle y - \bar{\mathbf{Q}}\rangle^{7/2}|u - \bar{u}|\|_\infty + C\|\langle y - \bar{\mathbf{Q}}\rangle^{7/2}(f - \bar{f})\|_\infty \\ &\le C\varepsilon^{1/2}\|\langle y - \bar{\mathbf{Q}}\rangle^{3/2}|u - \bar{u}|\|_\infty + C\|\langle y - \bar{\mathbf{Q}}\rangle^{7/2}(f - \bar{f})\|_\infty,\end{aligned}$$

which gives

$$\|\langle y - \bar{\mathbf{Q}}\rangle^{3/2}|u - \bar{u}|\|_\infty \le C\|\langle y - \bar{\mathbf{Q}}\rangle^{7/2}|f - \bar{f}|\|_\infty,$$

whence

$$\begin{aligned}\|\langle y - \bar{\mathbf{Q}}\rangle^{3/2} u\|_\infty &\le C\|\langle y - \bar{\mathbf{Q}}\rangle^{3/2}\|_\infty\, |\bar{u}| + C\varepsilon^{-7/2}|\bar{f}| + \|\langle y - \bar{\mathbf{Q}}\rangle^{7/2} f\|_\infty \\ &\le C\|f\|_{**}.\end{aligned}$$

□

We now state the main result of this section:

Proposition 3.1. *There exists $\varepsilon_0 > 0$ and a constant $C > 0$, independent of ε, $\mathbf{\Lambda}$ and $\bar{\mathbf{Q}}$ satisfying* (2.6)*, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem* (3.4) *has a unique solution $\phi \equiv L_\varepsilon(h)$. Furthermore*

$$\|L_\varepsilon(h)\|_* \le C\|h\|_{**}, \qquad |c_{j,i}| \le C\|h\|_{**}. \tag{3.8}$$

Moreover, the map $L_\varepsilon(h)$ is C^1 with respect to $\mathbf{\Lambda}$, $\bar{\mathbf{Q}}$ and the L^∞_*-norm, and

$$\|D_{(\mathbf{\Lambda}, \bar{\mathbf{Q}})} L_\varepsilon(h)\|_* \le C\|h\|_{**}. \tag{3.9}$$

The argument follows closely the ideas in [36], [41] and [42]. We repeat it since we use different norms. The proof relies on the following result:

Lemma 3.2. *Assume that ϕ_ε solves* (3.4) *for $h = h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$.*

Proof. Arguing by contradiction, we may assume that $\|\phi_\varepsilon\|_* = 1$. Multiplying the first equation in (3.4) by $Y_{k,l}$ and integrating in Ω_ε, we find

$$\sum_{j,i} c_{j,i}\langle Z_{j,i}, Y_{k,l}\rangle = \langle -\Delta Y_{k,l} + \varepsilon^{5/2} Y_{k,l} - 35 W_+^{4/3} Y_{k,l}, \phi_\varepsilon\rangle - \langle h_\varepsilon, Y_{k,l}\rangle.$$

On the one hand we check, in view of the definition of $Z_{j,i}$, $Y_{k,l}$,

$$\begin{cases} \langle Z_{j,0}, Y_{j,0}\rangle = \|Y_{j,0}\|_\varepsilon^2 = \gamma_0 + o(1), & 1 \le j \le K, \\ \langle Z_{j,i}, Y_{j,i}\rangle = \|Y_{j,i}\|_\varepsilon^2 = \gamma_1 + o(1), & 1 \le i \le 5, \end{cases} \tag{3.10}$$

where γ_0, γ_1 are strictly positive constants, and

$$\langle Z_{j,i}, Y_{k,l}\rangle = o(1), \qquad j \ne k,\ i \ne l. \tag{3.11}$$

On the other hand, in view of the definition of $Y_{k,l}$ and W, straightforward computations yield

$$\langle -\Delta Y_{k,l} + \varepsilon^{5/2} Y_{k,l} - 35 W_+^{4/3} Y_{k,l}, \phi_\varepsilon\rangle = o(\|\phi_\varepsilon\|_*)$$

and

$$\langle h_\varepsilon, Y_{k,l}\rangle = O(\|h_\varepsilon\|_{**}).$$

Consequently, inverting the quasidiagonal linear system solved by the $c_{j,i}$'s, we find

$$c_{j,i} = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_*). \tag{3.12}$$

In particular, $c_{j,i} = o(1)$ as ε goes to zero.

Since $\|\phi_\varepsilon\|_* = 1$, elliptic theory shows that along some subsequence, the functions $\phi_{\varepsilon,j}(y) = \phi_\varepsilon(y - \bar{Q}_j)$ converge uniformly in any compact subset of $\mathbb{R}^5$ to a nontrivial solution of

$$-\Delta\phi_j = 35 U_{\Lambda_j,0}^{4/3}\phi_j.$$

Moreover, $|\phi_j(y)| \le C(1+|y|)^{-3/2}$. A bootstrap argument (see e.g. Proposition 2.2 of [47]) implies $|\phi_j(y)| \le C(1+|y|)^{-3}$. As a consequence, ϕ_j can be written as

$$\phi_j = \alpha_0 \frac{\partial U_{\Lambda_j,0}}{\partial \Lambda_j} + \sum_{i=1}^{5} \alpha_i \frac{\partial U_{\Lambda_j,0}}{\partial y_i}$$

(see [38]). On the other hand, the equalities $\langle Z_{j,i}, \phi_\varepsilon\rangle = 0$ yield

$$\int_{\mathbb{R}^5} -\Delta \frac{\partial U_{\Lambda_j,0}}{\partial \Lambda_j}\phi_j = \int_{\mathbb{R}^5} U_{\Lambda_j,0}^{4/3} \frac{\partial U_{\Lambda_j,0}}{\partial \Lambda_j}\phi_j = 0,$$
$$\int_{\mathbb{R}^5} -\Delta \frac{\partial U_{\Lambda_j,0}}{\partial y_i}\phi_j = \int_{\mathbb{R}^5} U_{\Lambda_j,0}^{4/3} \frac{\partial U_{\Lambda_j,0}}{\partial y_i}\phi_j = 0, \qquad 1 \le i \le 5.$$

As we also have

$$\int_{\mathbb{R}^5} \left|\nabla \frac{\partial U_{\Lambda_j,0}}{\partial \Lambda_j}\right|^2 = \gamma_0 > 0, \qquad \int_{\mathbb{R}^5} \left|\nabla \frac{\partial U_{\Lambda_j,0}}{\partial y_i}\right|^2 = \gamma_1 > 0, \qquad 1 \le i \le 5,$$

and
$$\int_{\mathbb{R}^5} \nabla \frac{\partial U_{\Lambda_j,0}}{\partial \Lambda_j} . \nabla \frac{\partial U_{\Lambda_j,0}}{\partial y_i} = \int_{\mathbb{R}^5} \nabla \frac{\partial U_{\Lambda_j,0}}{\partial y_{i'}} \cdot \nabla \frac{\partial U_{\Lambda_j,0}}{\partial y_i} = 0, \quad i \neq i',$$
the α_i's solve a homogeneous quasidiagonal linear system, yielding $\alpha_i = 0$, $0 \leq i \leq N$, and $\phi_j = 0$. So $\phi_\varepsilon(z - \bar{Q}_j) \to 0$ in $C^1_{\mathrm{loc}}(\Omega_\varepsilon)$.

Now, we remark that Corollary 3.1 provides us with the inequality
$$\|\phi_\varepsilon\|_* \leq C \|W_+^{4/3} \phi_\varepsilon\|_{**} + C\|h_\varepsilon\|_{**} + C \sum_{j,i} |c_{j,i}| \|Z_{j,i}\|_{**}. \tag{3.13}$$

Let us estimate the right hand side. We deduce from (2.11) that
$$|\langle z - \bar{\mathbf{Q}}\rangle^{7/2} W_+^{4/3} \phi_\varepsilon| \leq C\varepsilon^{10/3} \langle z - \bar{\mathbf{Q}}\rangle^2 \|\phi_\varepsilon\|_* + C \langle z - \bar{\mathbf{Q}}\rangle^{-1/2} |\phi_\varepsilon|.$$

Since $\|\phi_\varepsilon\|_* = 1$, the first term on the right hand side is dominated by $\varepsilon^{4/3}$. The last term goes uniformly to zero in any ball $B_R(\bar{Q}_j)$, and is also dominated by $\langle z - \bar{\mathbf{Q}}\rangle^{-2} \|\phi_\varepsilon\|_* = \langle z - \bar{\mathbf{Q}}\rangle^{-2}$, which, through the choice of R, can be made as small as desired in $\Omega_\varepsilon \setminus \bigcup_j B_R(\bar{Q}_j)$. Consequently,
$$|\langle z - \bar{\mathbf{Q}}\rangle^{7/2} W_+^{4/3} \phi_\varepsilon| = o(1)$$
as ε goes to zero, uniformly in Ω_ε. (2.11) also yields
$$\begin{aligned} \varepsilon^{-4} \overline{W_+^{4/3} \phi_\varepsilon} &\leq C\varepsilon \int_{\Omega_\varepsilon} (\varepsilon^{10/3} + \langle z - \bar{\mathbf{Q}}\rangle^{-4}) |\phi_\varepsilon| \\ &\leq \varepsilon \int_{\Omega_\varepsilon} (\varepsilon^{10/3} \langle z - \bar{\mathbf{Q}}\rangle^{-3/2}) + \langle z - \bar{\mathbf{Q}}\rangle^{-11/2}) \|\phi_\varepsilon\|_* \leq \varepsilon^{5/6}. \end{aligned}$$

Finally, we obtain
$$\|W_+^{4/3} \phi_\varepsilon\|_{**} = o(1).$$
At the same time, (3.3) yields
$$\langle z - \bar{\mathbf{Q}}\rangle^{7/2} |Z_{j,i}| \leq C(\varepsilon^{11/2} \langle z - \bar{\mathbf{Q}}\rangle^{7/2} + \langle z - \bar{\mathbf{Q}}\rangle^{-7/2}) = O(1)$$
and
$$\varepsilon^{-4} \overline{Z_{j,i}} \leq \varepsilon \int_{\Omega_\varepsilon} (\varepsilon^{11/2} \langle z - \bar{\mathbf{Q}}\rangle^{-1} + \langle z - \bar{\mathbf{Q}}\rangle^{-7}) = O(\varepsilon).$$
Then, coming back to (3.13), we find $\|\phi_\varepsilon\|_* = o(1)$ contrary to the assumption that $\|\phi_\varepsilon\|_* = 1$. □

Proof of Proposition 3.1. We set
$$H = \{\phi \in H^1(\Omega_\varepsilon) \mid \langle Z_{j,i}, \phi\rangle = 0,\ 0 \leq i \leq 5,\ 1 \leq j \leq K\},$$
equipped with the scalar product $(\cdot, \cdot)_\varepsilon$. Problem (3.4) is equivalent to finding $\phi \in H$ such that
$$(\phi, \theta)_\varepsilon = \langle 35 W_+^{4/3} \phi + h,\ \theta\rangle \quad \forall \theta \in H,$$

that is,

$$\phi = T_\varepsilon(\phi) + \tilde{h}, \tag{3.14}$$

$\tilde{h}$ depending linearly on h, and T_ε being a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $\mathrm{Id} - T_\varepsilon$ is reduced to 0. We notice that any $\phi_\varepsilon \in \mathrm{Ker}(\mathrm{Id} - T_\varepsilon)$ solves (3.4) with $h = 0$. Thus, we deduce from Lemma 3.2 that $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero. As $\mathrm{Ker}(\mathrm{Id} - T_\varepsilon)$ is a vector space, it is $\{0\}$. The inequalities (3.8) follow from Lemma 3.2 and (3.12). This completes the proof of the first part of Proposition 3.1.

The smoothness of L_ε with respect to $\mathbf{\Lambda}$ and $\bar{\mathbf{Q}}$ is a consequence of the smoothness of T_ε and $\tilde{h}$, which occur in the implicit definition (3.14) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequality (3.9) is obtained by differentiating (3.4), writing the derivatives of ϕ with respect $\mathbf{\Lambda}$ and $\bar{\mathbf{Q}}$ as linear combinations of the Z_i's and an orthogonal part, and estimating each term using the first part of the proposition—see [36], [25] for detailed computations. ◻

4. Finite-dimensional reduction: a nonlinear problem

In this section, we turn our attention to the nonlinear problem, which we solve in the finite-codimensional subspace orthogonal to the $Z_{j,i}$'s. Let $S_\varepsilon[u]$ be as defined at (1.13). Then (1.12) is equivalent to

$$S_\varepsilon[u] = 0 \quad \text{in } \partial\Omega_\varepsilon, \qquad u_+ \not\equiv 0, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{4.1}$$

Indeed, if u satisfies (4.1) the Maximum Principle ensures that $u > 0$ in Ω_ε and (1.12) is satisfied. Observe that

$$S_\varepsilon[W + \phi] = -\Delta(W + \phi) + \varepsilon^{5/2}(W + \phi) - 15(W + \phi)_+^{7/3}$$

may be written as

$$S_\varepsilon[W + \phi] = -\Delta\phi + \varepsilon^{5/2}\phi - 35W_+^{4/3}\phi + R^\varepsilon - 15N_\varepsilon(\phi) \tag{4.2}$$

with

$$N_\varepsilon(\phi) = (W + \phi)_+^{7/3} - W^{7/3} - \frac{7}{3}W_+^{4/3}\phi \tag{4.3}$$

and

$$R^\varepsilon = S_\varepsilon[W] = -\Delta W + \varepsilon^{5/2}W - 15W^{7/3}. \tag{4.4}$$

From Lemma 2.1 we derive estimates of R^ε:

$$\|R^\varepsilon\|_{**} + \|D_{(\mathbf{\Lambda},\bar{\mathbf{Q}})}R^\varepsilon\|_{**} \le \varepsilon^{3/2}. \tag{4.5}$$

We now consider the following nonlinear problem: find ϕ such that, for some numbers $c_{j,i}$,

$$\begin{cases} -\Delta(W+\phi)+\varepsilon^{5/2}(W+\phi)-15(W+\phi)_+^{7/3}=\sum_{j,i}c_{j,i}Z_{j,i} \quad \text{in } \Omega_\varepsilon, \\ \partial\phi/\partial\nu=0 \quad \text{on } \partial\Omega_\varepsilon, \\ \langle Z_{j,i},\phi\rangle=0, \quad 1\le j\le K,\ 0\le i\le 5. \end{cases} \tag{4.6}$$

The first equation in (4.6) reads

$$-\Delta\phi+\varepsilon^{5/2}\phi-35W^{4/3}\phi=15N_\varepsilon(\phi)+R^\varepsilon+\sum_{j,i}c_{j,i}Z_{j,i} \tag{4.7}$$

for some numbers $c_{j,i}$. The functional N_ε may be estimated as follows:

Lemma 4.1. *There exist $\varepsilon_1>0$, independent of $\mathbf{\Lambda},\bar{\mathbf{Q}}$, and C, independent of ε, $\mathbf{\Lambda},\bar{\mathbf{Q}}$, such that for $\varepsilon\le\varepsilon_1$ and $\|\phi\|_*\le\varepsilon$,*

$$\|N_\varepsilon(\phi)\|_{**}\le C\varepsilon^{5/6}\|\phi\|_*, \tag{4.8}$$

and for $\|\phi_i\|_*\le 1$,

$$\|N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)\|_{**}\le C\varepsilon^{5/6}\|\phi_1-\phi_2\|_*. \tag{4.9}$$

Proof. We deduce from (4.3) that

$$|N_\varepsilon(\phi)|\le C(W_+^{1/3}|\phi|^2+|\phi|^{7/3}). \tag{4.10}$$

In view of (2.11), we compute

$$\begin{aligned}\varepsilon^{-4}\overline{W_+^{1/3}|\phi|^2+|\phi|^{7/3}} &\le C\varepsilon\int_{\Omega_\varepsilon}((\varepsilon^{5/6}+\langle z-\bar{\mathbf{Q}}\rangle^{-1})|\phi|^2+|\phi|^{7/3}) \\ &\le C\varepsilon\int_{\Omega_\varepsilon}((\varepsilon^{5/6}\langle z-\bar{\mathbf{Q}}\rangle^{-3}+\langle z-\bar{\mathbf{Q}}\rangle^{-4})\|\phi\|_*^2+\langle z-\bar{\mathbf{Q}}\rangle^{-7/2}\|\phi\|_*^{7/3}) \\ &\le C(\varepsilon^{-1/6}\|\phi\|_*^2+\varepsilon^{-1/2}\|\phi\|_*^{7/3})\le C\varepsilon^{5/6}\|\phi\|_*.\end{aligned}$$

On the other hand,

$$\|\langle z-\bar{\mathbf{Q}}\rangle^{7/2}(W_+^{1/3}|\phi|^2+|\phi|^{7/3})\|_\infty\le C\|\phi\|_*^2$$

and (4.8) follows. Concerning (4.9), we write

$$N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)=\partial_\eta N_\varepsilon(\eta)(\phi_1-\phi_2)$$

for some $\eta=x\phi_1+(1-x)\phi_2$, $x\in[0,1]$. From

$$\partial_\eta N_\varepsilon(\eta)=\frac{7}{3}((W+\eta)_+^{4/3}-W_+^{4/3})$$

we deduce

$$|\partial_\eta N_\varepsilon(\eta)|\le C(W_+^{1/3}|\eta|+|\eta|^{4/3}) \tag{4.11}$$

and the proof of (4.9) is similar to the previous one. □

We now state the following result:

Proposition 4.1. *There exists C, independent of ε and $\mathbf{\Lambda}$, $\bar{\mathbf{Q}}$ satisfying* (2.6)*, such that for small ε problem* (4.6) *has a unique solution $\phi = \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \varepsilon)$ with*

$$\|\phi\|_* \le C\varepsilon^{3/2}. \tag{4.12}$$

Moreover, $(\mathbf{\Lambda}, \bar{\mathbf{Q}}) \mapsto \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \varepsilon)$ is C^1 with respect to the $*$-norm, and

$$\|D_{(\mathbf{\Lambda},\bar{\mathbf{Q}})}\phi\|_* \le C\varepsilon^{3/2}. \tag{4.13}$$

Proof. Following [36], we consider the map A_ε from $\mathcal{F} = \{\phi \in H^1(\Omega_\varepsilon) \mid \|\phi\|_* \le C'\varepsilon^{3/2}\}$ to $H^1(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi) = L_\varepsilon(15N_\varepsilon(\phi) + R^\varepsilon).$$

Here C' is a large number, to be determined later, and L_ε is given by Proposition 3.1. We remark that finding a solution ϕ to problem (4.6) is equivalent to finding a fixed point of A_ε. On the one hand, we have for $\phi \in \mathcal{F}$, using (4.5), Proposition 3.1 and Lemma 4.1,

$$\begin{aligned}\|A_\varepsilon(\phi)\|_* &\le \|L_\varepsilon(N_\varepsilon(\phi))\|_* + \|L_\varepsilon(R^\varepsilon)\|_* \le C_1(\|N_\varepsilon(\phi)\|_{**} + \varepsilon^{3/2})\\ &\le C_2C'\varepsilon^{3/2+5/6} + C_1\varepsilon^{3/2} \le C'\varepsilon^{3/2}\end{aligned}$$

for $C' = 2C_1$ and ε small enough, implying that A_ε sends $\mathcal{F}$ into itself. On the other hand, A_ε is a contraction. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we write

$$\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \le C\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \le C\varepsilon^{5/6}\|\phi_1 - \phi_2\|_* \le \frac{1}{2}\|\phi_1 - \phi_2\|_*$$

for ε small enough. The Contraction Mapping Theorem implies that A_ε has a unique fixed point in $\mathcal{F}$, that is, problem (4.6) has a unique solution ϕ such that $\|\phi\|_* \le C'\varepsilon^{3/2}$.

In order to prove that $(\mathbf{\Lambda}, \bar{\mathbf{Q}}) \mapsto \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}})$ is C^1, we remark that if we set for $\eta \in \mathcal{F}$,

$$B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \eta) \equiv \eta - L_\varepsilon(15N_\varepsilon(\eta) + R^\varepsilon)$$

then ϕ is defined as

$$B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \phi) = 0. \tag{4.14}$$

We have

$$\partial_\eta B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \eta)[\theta] = \theta - 15L_\varepsilon(\theta\partial_\eta N_\varepsilon)(\eta)).$$

Using Proposition 3.1 and (4.11) we write

$$\begin{aligned}\|L_\varepsilon(\theta(\partial_\eta N_\varepsilon)(\eta))\|_* &\le C\|\theta(\partial_\eta N_\varepsilon)(\eta)\|_{**} \le C\|\langle z - \bar{\mathbf{Q}}\rangle^{-3/2}(\partial_\eta N_\varepsilon)(\eta)\|_{**}\|\theta\|_*\\ &\le C\|\langle z - \bar{\mathbf{Q}}\rangle^{-3/2}(|W_+^{1/3}|\eta| + |\eta|^{4/3})\|_{**}\|\theta\|_*.\end{aligned}$$

In view of (3.5), (2.11) and $\eta \in \mathcal{F}$, we obtain

$$\|L_\varepsilon(\theta(\partial_\eta N_\varepsilon)(\eta))\|_* \le C\varepsilon^{3/2}\|\theta\|_*.$$

Consequently, $\partial_\eta B(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \phi)$ is invertible with uniformly bounded inverse. Then the fact that $(\mathbf{\Lambda}, \bar{\mathbf{Q}}) \mapsto \phi(\mathbf{\Lambda}, \bar{\mathbf{Q}})$ is C^1 follows from the fact that $(\mathbf{\Lambda}, \bar{\mathbf{Q}}, \eta) \mapsto L_\varepsilon(N_\varepsilon(\eta))$ is C^1 and the implicit function theorem.

Finally, let us show how estimate (4.13) may be obtained. Differentiating (4.14) with respect to $\mathbf{\Lambda}$, we find

$$\partial_{\mathbf{\Lambda}}\phi = (\partial_\eta B(\Lambda, \xi, \phi))^{-1}((\partial_{\mathbf{\Lambda}} L_\varepsilon)(N_\varepsilon(\phi)) + L_\varepsilon((\partial_{\mathbf{\Lambda}} N_\varepsilon)(\phi)) + L_\varepsilon(\partial_{\mathbf{\Lambda}} R^\varepsilon)),$$

whence, according to Proposition 3.1,

$$\|\partial_{\mathbf{\Lambda}}\phi\|_* \le C(\|N_\varepsilon(\phi)\|_{**} + \|(\partial_{\mathbf{\Lambda}} N_\varepsilon)(\phi)\|_{**} + \|\partial_{\mathbf{\Lambda}} R^\varepsilon\|_{**}).$$

From Lemma 4.1 and (4.12) we know that

$$\|N_\varepsilon(\phi)\|_{**} \le C\varepsilon^{3/2}.$$

Concerning the next term, we notice that according to the definition of N_ε,

$$|(\partial_{\mathbf{\Lambda}} N_\varepsilon)(\phi)| = \frac{7}{3}\left|(W+\phi)_+^{4/3} - W_+^{4/3} - \frac{4}{3}W_+^{1/3}\phi\right||\partial_{\mathbf{\Lambda}} W|,$$

whence again, using (2.11), (2.12) and (4.12),

$$\|(\partial_{\mathbf{\Lambda}} N_\varepsilon)(\phi)\|_{**} \le C\varepsilon^{3/2}.$$

Finally, using (4.5), we obtain

$$\|\partial_{\mathbf{\Lambda}}\phi\|_* \le C\varepsilon^{3/2}.$$

The derivative of ϕ with respect to $\bar{\mathbf{Q}}$ may be estimated in the same way. This concludes the proof of Proposition 4.1. □

5. Finite-dimensional reduction: reduced energy

Let us define a reduced energy functional as

$$I_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) \equiv J_\varepsilon[W_{\mathbf{\Lambda},\bar{\mathbf{Q}}} + \phi_{\varepsilon,\mathbf{\Lambda},\bar{\mathbf{Q}}}]. \tag{5.1}$$

Then we state:

Proposition 5.1. *The function $u = W + \phi$ is a solution to problem* (1.12) *if and only if* $(\mathbf{\Lambda}, \bar{\mathbf{Q}})$ *is a critical point of I_ε.*

Proof. We notice that $u = W + \phi$ being a solution to (1.12) is equivalent to being a critical point of J_ε. It is also equivalent to the cancellation of the $c_{j,i}$'s in (4.6) or, in view of (3.10) and (3.11),

$$J_\varepsilon'[W+\phi][Y_{j,i}] = 0, \quad 1 \le j \le K,\ 0 \le i \le 5. \tag{5.2}$$

On the other hand, we deduce from (5.1) that $I_\varepsilon'(\mathbf{\Lambda},\mathbf{Q})=0$ is equivalent to the cancellation of $J_\varepsilon'(W+\phi)$ applied to the derivatives of $W+\phi$ with respect to $\mathbf{\Lambda}$ and $\bar{\mathbf{Q}}$. According to the definition (3.1) of the $Y_{j,i}$'s and Proposition 4.1 we have

$$\frac{\partial(W+\phi)}{\partial\Lambda_j}=Y_{j,0}+y_{j,0},\quad 1\le j\le K,\qquad \frac{\partial(W+\phi)}{\partial\bar{Q}_{j,i}}=Y_{j,i}+y_{j,i},\quad 1\le i\le 5,$$

with $\|y_{j,i}\|_*=o(1)$, $1\le j\le K$, $0\le i\le 5$. Writing

$$y_{j,i}=y'_{j,i}+\sum_{k,l}a_{ji,kl}Y_{k,l},\qquad \langle y'_{j,i},Z_{k,l}\rangle=(y'_{j,i},Y_{j,i})_\varepsilon=0,\qquad 0\le i\le 5,\ 1\le j\le K,$$

and

$$J_\varepsilon'[W+\phi][Y_{j,i}]=\alpha_{j,i}$$

it turns out that $I_\varepsilon'(\mathbf{\Lambda},\bar{\mathbf{Q}})=0$ is equivalent, since $J_\varepsilon'[W+\phi][\theta]=0$ for $\langle\theta,Z_{j,i}\rangle=(\theta,Y_{j,i})_\varepsilon=0$, $1\le j\le K$, $0\le i\le 5$, to

$$(\mathrm{Id}+[a_{ji,kl}])[\alpha_{ji}]=0.$$

As $a_{ji,kl}=O(\|y_{k,l}\|_*)=o(1)$, we see that $I_\varepsilon'(\mathbf{\Lambda},\mathbf{Q})=0$ means exactly that (5.2) is satisfied. ◻

In view of Proposition 5.1, to prove the theorem, we have to find critical points of I_ε. We establish an expansion of I_ε.

Proposition 5.2. *For ε sufficiently small, we have*

$$I_\varepsilon(\mathbf{\Lambda},\mathbf{Q})=J_\varepsilon[W]+\varepsilon^3\sigma_\varepsilon(\mathbf{\Lambda},\mathbf{Q}) \tag{5.3}$$

where $\sigma_\varepsilon=o(1)$ and $D_{\mathbf{\Lambda}}\sigma_\varepsilon=O(1)$ as ε goes to zero, uniformly with respect to $\mathbf{\Lambda},\mathbf{Q}$ satisfying (2.6)*.*

Proof. We first prove that

$$I_\varepsilon(\mathbf{\Lambda},\mathbf{Q})-J_\varepsilon[W]=o(\varepsilon^3). \tag{5.4}$$

Actually, in view of (5.1), a Taylor expansion and the fact that $J_\varepsilon'[W+\phi][\phi]=0$ yield

$$\begin{aligned}I_\varepsilon(\mathbf{\Lambda},\mathbf{Q})-J_\varepsilon[W]&=J_\varepsilon[W+\phi]-J_\varepsilon[W]=-\int_0^1 J_\varepsilon''(W+t\phi)[\phi,\phi]t\,dt\\&=-\int_0^1\left(\int_{\Omega_\varepsilon}(|\nabla\phi|^2+\varepsilon^{5/2}\phi^2-35(W+t\phi)_+^{4/3}\phi^2)\right)t\,dt,\end{aligned}$$

whence

$$\begin{aligned}&I_\varepsilon(\mathbf{\Lambda},\mathbf{Q})-J_\varepsilon[W]\\&=-\int_0^1\left(15\int_{\Omega_\varepsilon}\left(N_\varepsilon(\phi)\phi+\frac{7}{3}[W_+^{4/3}-(W+t\phi)_+^{4/3}]\phi^2\right)\right)t\,dt-\int_{\Omega_\varepsilon}R^\varepsilon\phi.\end{aligned} \tag{5.5}$$

From (4.3), (2.11) and Proposition 4.1, we deduce that the first term on the right hand side satisfies

$$\left|\int_{\Omega_\varepsilon} N_\varepsilon(\phi)\phi\right| \le C\int_{\Omega_\varepsilon}(W_+^{1/3}|\phi|^3 + |\phi|^{10/3}) \le C\varepsilon^4.$$

Similarly, for the second term on the right hand side we obtain

$$\left|\int_{\Omega_\varepsilon}(W_+^{4/3} - (W+t\phi)_+^{4/3})\phi^2\right| \le C\int_{\Omega_\varepsilon}(W_+^{1/3}|\phi|^3 + |\phi|^{10/3}) \le C\varepsilon^4.$$

Concerning the last integral, we remark that according to (2.14),

$$|R^\varepsilon| \le C\varepsilon^{5/2}\langle z - \bar{\mathbf{Q}}\rangle^{-4} + C\varepsilon^5\langle z - \bar{\mathbf{Q}}\rangle^{-1/2}$$

uniformly in Ω_ε. Therefore

$$\left|\int_{\Omega_\varepsilon} R^\varepsilon\phi\right| \le C\|\phi\|_*\int_{\Omega_\varepsilon}\varepsilon^{5/2}\langle z - \bar{\mathbf{Q}}\rangle^{-11/2} + \varepsilon^5\|\phi\|_*\int_{\Omega_\varepsilon}\langle z - \bar{\mathbf{Q}}\rangle^{-2} \le C\varepsilon^{7/2}.$$

This concludes the proof of (5.4).

An estimate for the derivative with respect to $\mathbf{\Lambda}$ is established exactly in the same way, differentiating the right hand side in (5.5) and estimating each term separately, using (4.3), (4.5) and Lemma 2.1 (see Proposition 3.4 in [25] for detailed computations). □

6. Proof of Theorem 1.1

In view of Proposition 5.1, proving Theorem 1.1 turns out to be equivalent to proving the existence of a critical point of $I_\varepsilon(\mathbf{\Lambda}, \mathbf{Q})$. According to Proposition 5.2 and Lemma 2.1, setting

$$K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) := \frac{I_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) - A_0}{\varepsilon^{5/2}}$$

we have the expansion

$$\begin{aligned} K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) = \beta(\mathbf{\Lambda}) + \varepsilon^{1/2}E_0\Bigg[\sum_{j=1}^{K}\Lambda_j^3 H(Q_j, Q_j) - \sum_{i\ne j}\Lambda_i^{3/2}\Lambda_j^{3/2}G(Q_i, Q_j) \\ - F_0\Big(\sum_{j=1}^{K}\Lambda_j^{3/2}\Big)\sum_{j=1}^{K}\Lambda_j^{3/2}\int_\Omega \frac{dx}{|x - Q_j|^3}\Bigg] + o(\varepsilon^{1/2}) \end{aligned} \tag{6.1}$$

and

$$D_{\mathbf{\Lambda}}K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) = D_{\mathbf{\Lambda}}\beta(\mathbf{\Lambda}) + O(\varepsilon^{1/2}) \tag{6.2}$$

with

$$\beta(\mathbf{\Lambda}) = -B_0\Big(\sum_{j=1}^{K}\Lambda_j^{3/2}\Big)^2 + D_0\sum_{j=1}^{K}\Lambda_j^2.$$

We notice that $\beta(\mathbf{\Lambda}) \to -\infty$ as $|\mathbf{\Lambda}| \to \infty$. Except for $K = 1$, the maximum points of β in $\overline{\mathbb{R}_+^K}$ lie on the boundary of this set. However, computing the first derivatives

$$\partial_{\Lambda_i} \beta(\mathbf{\Lambda}) = -3B_0 \Big(\sum_{j=1}^{K} \Lambda_j^{3/2} \Big) \Lambda_i^{1/2} + 2D_0 \Lambda_i \tag{6.3}$$

we see that, in any case, β has a (unique) critical point $\hat{\mathbf{\Lambda}}_0$ in the interior of $\mathbb{R}_+^K$ such that

$$\hat{\mathbf{\Lambda}}_0 = (\Lambda_0, \dots, \Lambda_0), \qquad \Lambda_0 = \frac{2D_0}{3B_0 K}, \qquad \beta(\hat{\mathbf{\Lambda}}_0) = \frac{4D_0^3}{27B_0^2 K}. \tag{6.4}$$

We compute

$$\partial^2_{\Lambda_i \Lambda_j} \beta(\hat{\mathbf{\Lambda}}_0) = D_0(-3/K + \delta_{ij}).$$

Thus, the eigenvalues of β'' are $\lambda^+ = D_0$, with multiplicity $K - 1$, and $\lambda^- = -2D_0$, with multiplicity one. Consequently, $\hat{\mathbf{\Lambda}}_0$ is a maximum point in the $(1, \dots, 1)$ direction, corresponding to λ^-, and a minimum point in the orthogonal hyperplane (when $K \geq 2$).

We also remark that for $\mathbf{\Lambda} = \hat{\mathbf{\Lambda}}_0$, the term in square brackets in the expansion (6.1) of K_ε can be written as $\Lambda_0^3 F(\mathbf{Q})$ with

$$F(\mathbf{Q}) = \sum_{j=1}^{K} H(Q_j, Q_j) - \sum_{i \neq j} G(Q_i, Q_j) - F_0 K \sum_{j=1}^{K} \int_\Omega \frac{dx}{|x - Q_j|^3}. \tag{6.5}$$

Note also that F achieves its maximum $\hat{F}$ in the interior of $\mathcal{M}_\delta$. More precisely, we shall prove:

Lemma 6.1. *There exists a constant $C > 0$ such that*

$$\sup_{\mathbf{Q} \in \partial \mathcal{M}_\delta} F(\mathbf{Q}) \leq -C/\delta^3 \quad \textit{as } \delta \to 0. \tag{6.6}$$

Considering these facts, our aim is to prove that for ε small enough, K_ε has a critical point $(\hat{\mathbf{\Lambda}}, \hat{\mathbf{Q}})$, with $\hat{\mathbf{\Lambda}}$ close to $\hat{\mathbf{\Lambda}}_0$ and $\hat{\mathbf{Q}}$ close to a maximum point of F. In order to use a linking argument, we set

$$\Sigma = \{(\mathbf{\Lambda}, \mathbf{Q}) \mid \mathbf{Q} \in \mathcal{M}_\delta,\ 1/C_0 < \Lambda_i < C_0,\ 1 \leq i \leq K\}$$

where C_0 is a large constant. We also define a closed subset of Σ,

$$\mathcal{B} = \{(\mathbf{\Lambda}, \mathbf{Q}) \mid \mathbf{Q} \in \mathcal{U},\ |\mathbf{\Lambda} - \hat{\mathbf{\Lambda}}_0| \leq \alpha\},$$

where $\mathcal{U}$ is a closed contractible neighbourhood of a maximum point of F, and $\alpha > 0$ is a fixed small number. Lastly, we define $\mathcal{B}_0$, a closed subset of $\mathcal{B}$, as

$$\mathcal{B}_0 = \{(\mathbf{\Lambda}, \mathbf{Q}) \mid \mathbf{Q} \in \mathcal{U},\ |\mathbf{\Lambda} - \hat{\mathbf{\Lambda}}_0| = \alpha,\ (\mathbf{\Lambda} - \hat{\mathbf{\Lambda}}_0) \cdot \hat{\mathbf{\Lambda}}_0 = 0\}.$$

In view of the behaviour of β at $\hat{\mathbf{\Lambda}}_0$, α is chosen small enough so that for any $(\mathbf{\Lambda}, \mathbf{Q}) \in \mathcal{B}_0$, $\beta(\mathbf{\Lambda}) > \beta(\hat{\mathbf{\Lambda}}_0)$. Finally, we set

$$\Gamma = \{\varphi \in C^0(\mathcal{B}, \Sigma) \mid \varphi|_{\mathcal{B}_0} = \mathrm{Id}\}, \qquad c = \max_{\varphi \in \Gamma} \min_{(\mathbf{\Lambda}, \mathbf{Q}) \in \mathcal{B}} K_\varepsilon(\varphi(\mathbf{\Lambda}, \mathbf{Q})).$$

We show that c is a critical value of K_ε. To this end, standard deformation arguments ensure that it is sufficient to prove:

(H1) $\min_{(\mathbf{\Lambda}, \mathbf{Q}) \in \mathcal{B}_0} K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) > c$.

(H2) *For all* $(\mathbf{\Lambda}, \mathbf{Q}) \in \partial\Sigma$ *such that* $K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) = c$*, there exists* $\tau_{(\mathbf{\Lambda}, \mathbf{Q})}$*, a tangent vector to* $\partial\Sigma$ *at* $(\mathbf{\Lambda}, \mathbf{Q})$*, such that*

$$\partial_{\tau_{(\mathbf{\Lambda}, \mathbf{Q})}} K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) \neq 0.$$

Before proving (H1) and (H2), we need to estimate c. We remark that for any φ in Γ, there exists some $(\mathbf{\Lambda}', \mathbf{Q}') = \varphi(\mathbf{\Lambda}, \mathbf{Q})$, $(\mathbf{\Lambda}, \mathbf{Q}) \in \mathcal{B}$, such that $\mathbf{\Lambda}'$ is proportional to $(1, \dots, 1)$. (This follows from the fact that $\varphi \in C^0(\mathcal{B}, \Sigma)$ and $\varphi|_{\mathcal{B}_0} = \mathrm{Id}$.) Then, according to (6.1) and (6.5),

$$K_\varepsilon(\mathbf{\Lambda}', \mathbf{Q}') = \beta(\mathbf{\Lambda}') + \varepsilon^{1/2} E_0 \Lambda'^3 F(\mathbf{Q}') + o(\varepsilon^{1/2}).$$

Maximizing the right hand side with respect to $\mathbf{\Lambda}'$ proportional to $(1, \dots, 1)$ and $\mathbf{Q}'$ in $\mathcal{M}_\delta$, we see that for any φ in Γ, there exists some $(\mathbf{\Lambda}', \mathbf{Q}')$ such that

$$K_\varepsilon(\mathbf{\Lambda}', \mathbf{Q}') \leq \beta(\hat{\mathbf{\Lambda}}_0) + \varepsilon^{1/2} E_0 \Lambda_0^3 \hat{F} + o(\varepsilon^{1/2}),$$

whence also

$$c \leq \beta(\hat{\mathbf{\Lambda}}_0) + \varepsilon^{1/2} E_0 \Lambda_0^3 \hat{F} + o(\varepsilon^{1/2}). \tag{6.7}$$

On the other hand, we consider a special φ such that, if we set $(\mathbf{\Lambda}', \mathbf{Q}') = \varphi(\mathbf{\Lambda}, \mathbf{Q})$ for $(\mathbf{\Lambda}, \mathbf{Q}) \in \mathcal{B}$, then $\mathbf{\Lambda}'$ is the orthogonal projection of $\mathbf{\Lambda}$ over the disk $D = \{\mathbf{\Lambda} \mid |\mathbf{\Lambda} - \hat{\mathbf{\Lambda}}_0| \leq \alpha,\ (\mathbf{\Lambda} - \hat{\mathbf{\Lambda}}_0) \cdot \hat{\mathbf{\Lambda}}_0 = 0\}$. Moreover, we choose φ in such a way that, for $|\mathbf{\Lambda} - \hat{\mathbf{\Lambda}}_0| \leq \alpha/2$, $\mathbf{Q}'$ is a maximum point of F (this is possible, since we assumed that $\mathcal{U}$ is a closed contractible neighbourhood of a maximum point of F). In view of (6.1) and the behaviour of β, for such φ and ε small enough we have

$$\min_{(\mathbf{\Lambda}, \mathbf{Q}) \in \mathcal{B}} K_\varepsilon(\varphi(\mathbf{\Lambda}, \mathbf{Q})) = \beta(\hat{\mathbf{\Lambda}}_0) + \varepsilon^{1/2} E_0 \Lambda_0'^3 \hat{F} + o(\varepsilon^{1/2}),$$

whence the reverse inequality to (6.7), and the final estimate

$$c = \beta(\hat{\mathbf{\Lambda}}_0) + \varepsilon^{1/2} E_0 \Lambda_0'^3 \hat{F} + o(\varepsilon^{1/2}). \tag{6.8}$$

Let us now show that (H1) and (H2) are satisfied. In view of (6.8), the inequality in (H1) follows directly from the expansion (6.1), the definition of $\mathcal{B}_0$ and the properties of β, provided that ε is small enough.

We are left with the proof of (H2). We note that $K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) = c$ implies, through (6.1), that

$$\beta(\mathbf{\Lambda}) = c + O(\varepsilon^{1/2}). \tag{6.9}$$

As already stated, $\beta(\mathbf{\Lambda}) \to -\infty$ as soon as some Λ_i goes to infinity. Therefore, (6.9) implies that $\Lambda_i \le C_1$, $1 \le i \le K$, for some constant C_1. On the other hand, suppose that Λ_i goes to zero for some indices, say $1 \le i \le m$. If $m = K$, then $\beta(\mathbf{\Lambda})$ goes to zero, a contradiction with (6.9). If $m < K$, there exists some index $j \ge m+1$ such that $\partial_{\Lambda_j}\beta(\mathbf{\Lambda}) \ne 0$. Indeed, if not, in view of (6.3) we would obtain

$$\Lambda_j = \frac{2D_0}{3B_0(K-m)} + o(1), \qquad m+1 \le j \le K,$$

whence

$$\beta(\mathbf{\Lambda}) = \frac{4D_0^3}{27B_0^2(K-m)} + o(1),$$

and again, comparing with (6.4), a contradiction with (6.9). Consequently, there exists an index $j \ge m+1$ such that $\partial_{\Lambda_j}\beta(\mathbf{\Lambda}) \ne 0$, implying through (6.2) that $\partial_{\Lambda_j}K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) \ne 0$ for ε small enough. Then we see that if we choose $C_0 > C_1$ large enough in the definition of Σ, (H2) is satisfied when $(\mathbf{\Lambda}, \mathbf{Q}) \in \partial\Sigma$ with $K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) = c$ is such that $\Lambda_i = C_0$ (impossible) or $\Lambda_i = 1/C_0$ (taking $\tau_{(\mathbf{\Lambda},\mathbf{Q})} = \partial_{\Lambda_j}$ for some appropriate index j).

It only remains to consider the case $1/C_0 < \Lambda_j < C_0$, $1 \le j \le K$, and $\mathbf{Q} \in \partial\mathcal{M}_\delta$. If there exists some index j such that $\partial_{\Lambda_j}K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) \ne 0$, then (H2) holds. If not, it follows from (6.2) and (6.3) that

$$\mathbf{\Lambda} = \hat{\mathbf{\Lambda}}_0 + O(\varepsilon^{1/2}) \quad \text{and} \quad \beta(\mathbf{\Lambda}) = \beta(\hat{\mathbf{\Lambda}}_0) + O(\varepsilon).$$

Thus, (6.1) yields

$$K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) = \beta(\hat{\mathbf{\Lambda}}_0) + \varepsilon^{1/2}E_0\Lambda_0^3F(\mathbf{Q}) + o(\varepsilon^{1/2}).$$

Then the assumption $K_\varepsilon(\mathbf{\Lambda}, \mathbf{Q}) = c$, together with (6.8), implies that $F(\mathbf{Q}) = \hat{F} + o(1)$, a contradiction with Lemma 6.1, provided δ is chosen small enough. This concludes the proof of (H2).

Proof of Lemma 6.1. We first note the existence of a positive constant C independent of $Q \in \Omega$ such that

$$\int_\Omega \frac{1}{|x-Q|^3}\,dx \le C. \tag{6.10}$$

So the integral term in $F(\mathbf{Q})$ is uniformly bounded in δ.

Let $Q \in \Omega$ be close to $\partial\Omega$, and Q_0 be the nearest point of $\partial\Omega$ to Q. It is easily checked that

$$H(x, Q) = -\frac{1}{c_5|x-Q^*|^3} + O\left(\frac{1}{(d(Q, \partial\Omega))^2}\right) \quad \text{as } d(Q, \partial\Omega) \to 0$$

uniformly in Ω, where Q^* is the reflection of Q across the boundary, that is, the symmetric point to Q with respect to Q_0 (see Appendix B). In particular,

$$H(Q, Q) = -\frac{1}{8c_5(d(Q, \partial\Omega))^3} + O\left(\frac{1}{(d(Q, \partial\Omega))^2}\right).$$

On the other hand, we have

$$G(Q_i, Q_j) = \frac{1}{c_5|Q_i - Q_j|^3} - H(Q_i, Q_j).$$

Then, in view of (6.5), we see that

$$\max_{\mathbf{Q}\in\mathcal{M}_\delta} F(\mathbf{Q}) \le -C/\delta^3 \quad \text{as } \delta \to 0$$

where C is some strictly positive constant. □

Proof of Theorem 1.1 completed. We proved that for ε small enough, I_ε has a critical point $(\mathbf{\Lambda}^\varepsilon, \mathbf{Q}^\varepsilon)$.

Let $u_\varepsilon = W_{\mathbf{\Lambda}^\varepsilon,\bar{\mathbf{Q}}^\varepsilon,\varepsilon} + \phi_{\mathbf{\Lambda}^\varepsilon,\bar{\mathbf{Q}}^\varepsilon,\varepsilon}$. Then u_ε is a nontrivial solution to problem (1.12). The strong maximum principle shows that $u_\varepsilon > 0$ in Ω_ε. Let $u_\mu = \varepsilon^{-3/2}u_\varepsilon(x/\varepsilon)$. By our construction, u_μ has all the properties of Theorem 1.1. □

7. Appendix A: Proof of Lemma 2.1

From the definition (2.8) of W, (2.10) and (2.5), we know that

$$\begin{aligned} S_\varepsilon[W] &= -\Delta W + \varepsilon^{3/2}W - 15W_+^{7/3} \\ &= 15\sum_{j=1}^{K} U_j^{7/3} + \varepsilon^5 \sum_{j=1}^{K} \hat{U}_j - 15\Big(\sum_{j=1}^{K}(U_j + \varepsilon^{5/2}\hat{U}_j) + \eta\varepsilon^{5/2}\Big)^{7/3} + O(\varepsilon^8) \\ &= \varepsilon^5 \sum_{j=1}^{K} \hat{U}_j + O\Big[\sum_{i\neq j} U_j^{4/3}(U_i + \varepsilon^{5/2}) + \varepsilon^{10/3}\sum_{j=1}^{K} U_j + \varepsilon^{35/6}\Big]. \end{aligned}$$

According to the definition of $U_j = U_{\Lambda_j, Q_j/\varepsilon}$ and the fact that in $\mathcal{M}_\delta$ the points Q_j remain far apart, we have

$$U_j = O(\langle z - \bar{\mathbf{Q}}\rangle^{-3}), \quad U_j^{4/3}U_i = O(\varepsilon^3\langle z - \bar{\mathbf{Q}}\rangle^{-4}) \quad \text{for } i \neq j. \tag{7.1}$$

From (2.3), (2.2) and (2.5), we also have

$$\hat{U}_j = O(\langle z - \bar{\mathbf{Q}}\rangle^{-1/2}). \tag{7.2}$$

Combining these facts yields estimate (2.14). Estimates for $D_{\mathbf{\Lambda}}S_\varepsilon[W]$ and $D_{\bar{\mathbf{Q}}}S_\varepsilon[W]$ are obtained exactly in the same way.

We now turn to the proof of the energy estimate (2.15). From (2.10) and (2.11) we deduce that

$$\int_{\Omega_\varepsilon} |\nabla W|^2 + \varepsilon^{5/2}\int_{\Omega_\varepsilon} W^2 = 15\sum_{j=1}^{K}\int_{\Omega_\varepsilon} U_j^{7/3}W + \varepsilon^5\sum_{j=1}^{K}\int_{\Omega_\varepsilon}\hat{U}_j W + o(\varepsilon^3). \tag{7.3}$$

The definition (2.8) of W and (7.2) yield $|W - \eta\varepsilon^{5/2}| = O(\langle z - \bar{\mathbf{Q}}\rangle^{-3})$, whence, in view of (7.2) and (2.2),

$$\begin{aligned}\varepsilon^5 \sum_{j=1}^{K} \int_{\Omega_\varepsilon} \hat{U}_j W &= \eta\varepsilon^{5/2}\varepsilon^5 \int_{\Omega_\varepsilon} \sum_{j=1}^{K} (-\Psi_j - c_5\varepsilon^{1/2}\Lambda_j^{3/2} H(\varepsilon z, Q_j)) + o(\varepsilon^3)\\ &= -c_5\eta\varepsilon^3 \sum_{j=1}^{K} \Lambda_j^{3/2} \int_\Omega H(x, Q_j)\, dx + o(\varepsilon^3)\\ &= -\eta\varepsilon^3 \sum_{j=1}^{K} \Lambda_j^{3/2} \int_\Omega \frac{1}{|x - Q_j|^3}\, dx + o(\varepsilon^3).\end{aligned}$$

Concerning the first terms on the right hand side of (7.3), we remark that in view of the definitions of U_i, $\hat{U}_i$ and (2.2), for $i \neq j$ we have on $B_j = B(\bar{Q}_j, \delta/2\varepsilon)$,

$$(U_i + \varepsilon^{5/2}\hat{U}_i)(z) = \frac{\varepsilon^3\Lambda_i^{3/2}}{|Q_j - Q_i|^3} - c_5\varepsilon^3\Lambda_i^{3/2} H(Q_j, Q_i) + O(\varepsilon^4|z - \bar{Q}_j| + \varepsilon^{7/2}).$$

As $U_i + \varepsilon^{5/2}\hat{U}_i = O(\langle z - \bar{\mathbf{Q}}\rangle^{-3} + \varepsilon^{5/2})$ and, outside B_j, $U_j^{7/3} = O(\varepsilon^7)$, we obtain, for $i \neq j$,

$$15 \int_{\Omega_\varepsilon} U_j^{7/3}(U_i + \varepsilon^{5/2}\hat{U}_i) = c_5^2\varepsilon^3\Lambda_i^{3/2}\Lambda_j^{3/2} G(Q_i, Q_j) + o(\varepsilon^3),$$

noticing that

$$15 \int_{\mathbb{R}^5} U_j^{7/3} = c_5\Lambda_j^{3/2}. \tag{7.4}$$

In the same way we find, for $i = j$,

$$\begin{aligned}&15 \int_{\Omega_\varepsilon} U_j^{7/3}(U_j + \varepsilon^{5/2}\hat{U}_j)\\ &\qquad = 15 \int_{\mathbb{R}^5} U_{1,0}^{10/3} - 15\varepsilon^{5/2} \int_{\Omega_\varepsilon} U_j^{7/3}\Psi_j - c_5^2\varepsilon^3\Lambda_j^3 H(Q_j, Q_j) + o(\varepsilon^3).\end{aligned}$$

Thus we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla W|^2 + \varepsilon^{5/2} \int_{\Omega_\varepsilon} W^2 &= 15K \int_{\mathbb{R}^5} U_{1,0}^{10/3} - 15\varepsilon^{5/2} \sum_{j=1}^{K} \int_{\mathbb{R}^5} U_j^{7/3}\Psi_j\\ &\quad - c_5^2\varepsilon^3\Big[\sum_{j=1}^{K} \Lambda_j^3 H(Q_j, Q_j) - \sum_{i\neq j} \Lambda_i^{3/2}\Lambda_j^{3/2} G(Q_i, Q_j)\Big]\\ &\quad + \eta\varepsilon^{5/2} c_5 \sum_{j=1}^{K} \Lambda_j^{3/2} - \eta\varepsilon^3 \sum_{j=1}^{K} \Lambda_j^{3/2} \int_\Omega \frac{1}{|x - Q_j|^3}\, dx + o(\varepsilon^3).\end{aligned} \tag{7.5}$$

It only remains to estimate

$$\begin{aligned}
\int_{\Omega_\varepsilon} W_+^{10/3} &= \int_{\Omega_\varepsilon} \Big(\sum_{j=1}^{K} (U_j + \varepsilon^{5/2} \hat{U}_j) + \eta \varepsilon^{5/2} \Big)_+^{10/3} \\
&= \int_{\Omega_\varepsilon} \Big(\sum_{j=1}^{K} U_j \Big)^{10/3} + \frac{10}{3} \varepsilon^{5/2} \int_{\Omega_\varepsilon} \Big(\sum_{j=1}^{K} U_j \Big)^{7/3} \Big(\sum_{j=1}^{K} \hat{U}_j \Big) \\
&\quad + \frac{10}{3} \eta \varepsilon^{5/2} \int_{\Omega_\varepsilon} \Big(\sum_{j=1}^{K} U_j \Big)^{7/3} + O\Big(\int_{\Omega_\varepsilon} \Big(\varepsilon^5 \sum_{j=1}^{K} U_j^{4/3} + \varepsilon^{25/3} \Big) \Big) \\
&= \sum_{j=1}^{K} \int_{\Omega_\varepsilon} U_j^{10/3} + \frac{10}{3} \sum_{i \neq j} \int_{\Omega_\varepsilon} U_j^{7/3} (U_i + \hat{U}_i) + \frac{10}{3} \varepsilon^{5/2} \sum_{j=1}^{K} \int_{\Omega_\varepsilon} U_j^{7/3} \hat{U}_j \\
&\quad + \frac{10}{3} \eta \varepsilon^{5/2} \sum_{j=1}^{K} \int_{\Omega_\varepsilon} U_j^{7/3} + o(\varepsilon^3)
\end{aligned}$$

since, as a consequence of the definition of the U_j's and the fact that the Q_j's remain far apart in $\mathcal{M}_\delta$ (see for instance (7.1)),

$$\varepsilon^5 \int_{\Omega_\varepsilon} U_j^{4/3} = O(\varepsilon^4)$$

and, for $i \neq j$,

$$\int_{\Omega_\varepsilon} U_j^{4/3} U_i = O(\varepsilon^2), \qquad \int_{\Omega_\varepsilon} U_j^{4/3} U_i^2 = O(\varepsilon^4).$$

Therefore, the same computations as above yield

$$\begin{aligned}
\int_{\Omega_\varepsilon} W_+^{10/3} &= K \int_{\mathbb{R}^5} U_{1,0}^{10/3} - \frac{10}{3} \varepsilon^{5/2} \sum_{j=1}^{K} \int_{\mathbb{R}^5} U_j^{7/3} \Psi_j + \frac{2}{9} \eta \varepsilon^{5/2} c_5 \sum_{j=1}^{K} \Lambda_j^{3/2} \\
&\quad - \frac{2}{9} c_5^2 \varepsilon^3 \Big[\sum_{j=1}^{K} \Lambda_j^3 H(Q_j, Q_j) - \sum_{i \neq j} \Lambda_i^{3/2} \Lambda_j^{3/2} G(Q_i, Q_j) \Big] + o(\varepsilon^3).
\end{aligned}$$

Combining this expansion with (7.5), we obtain

$$\begin{aligned}
J_\varepsilon[W] &= \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla W|^2 + \frac{\varepsilon^{5/2}}{2} \int_{\Omega_\varepsilon} W^2 - \frac{9}{2} \int_{\Omega_\varepsilon} W^{10/3} \\
&= 3K \int_{\mathbb{R}^5} U_{1,0}^{10/3} + \frac{15}{2} \varepsilon^{5/2} \sum_{j=1}^{K} \int_{\mathbb{R}^5} U_j^{7/3} \Psi_j - \frac{1}{2} \eta \varepsilon^{5/2} c_5 \sum_{j=1}^{K} \Lambda_j^{3/2} \\
&\quad + \frac{1}{2} c_5^2 \varepsilon^3 \Big[\sum_{j=1}^{K} \Lambda_j^3 H(Q_j, Q_j) - \sum_{i \neq j} \Lambda_i^{3/2} \Lambda_j^{3/2} G(Q_i, Q_j) \Big] \\
&\quad - \frac{c_5}{2|\Omega|} \varepsilon^3 \Big(\sum_{j=1}^{K} \Lambda_j^{3/2} \Big) \sum_{j=1}^{K} \Lambda_j^{3/2} \int_{\Omega} \frac{1}{|x - Q_j|^3}\, dx + o(\varepsilon^3).
\end{aligned}$$

Lastly, we notice that in view of (2.1),

$$15\int_{\mathbb{R}^5} U_j^{7/3}\Psi_j = \int_{\mathbb{R}^5} U_j^2 = \Big(\int_{\mathbb{R}^5} U_{1,0}^2\Big)\Lambda_j^2 = \frac{c_5\pi}{16}\Lambda_j^2,$$

whence, according to the definition (2.9) of η,

$$15\sum_{j=1}^K\int_{\mathbb{R}^5} U_j^{7/3}\Psi_j - \eta c_5\sum_{j=1}^K \Lambda_j^{3/2} = \frac{c_5\pi}{16}\sum_{j=1}^K\Lambda_j^2 - \frac{c_5^2}{|\Omega|}\Big(\sum_{j=1}^K\Lambda_j^{3/2}\Big)^2.$$

Finally, we obtain

$$\begin{aligned}J_\varepsilon[W] = A_0 - \varepsilon^{5/2}D_0\Big(\sum_{j=1}^K\Lambda_j^{3/2}\Big)^2 + \varepsilon^{5/2}B_0\sum_{j=1}^K\Lambda_j^2 + \varepsilon^3 E_0\Big[\sum_{j=1}^K\Lambda_j^3 H(Q_j,Q_j)\\ - \sum_{i\neq j}\Lambda_i^{3/2}\Lambda_j^{3/2}G(Q_i,Q_j) - F_0\Big(\sum_{j=1}^K\Lambda_j^{3/2}\Big)\sum_{j=1}^K\Lambda_j^{3/2}\int_\Omega\frac{1}{|x-Q_j|^3}\,dx\Big] + o(\varepsilon^3)\end{aligned}$$

where $A_0, B_0, D_0, E_0, F_0 > 0$ are all generic constants which can be traced back from the computations, namely:

$$A_0 = \frac{3\pi c_5}{256},\quad B_0 = \frac{\pi c_5}{32},\quad D_0 = \frac{c_5^2}{2|\Omega|},\quad E_0 = \frac{c_5^2}{2},\quad F_0 = \frac{1}{c_5|\Omega|}.$$

To prove estimate (2.16), we observe that

$$D_{\Lambda_j}J_\varepsilon[W] = \int_\Omega S_\varepsilon[W]\partial_{\Lambda_j}W = \int_\Omega S_\varepsilon[W]\partial_{\Lambda_j}(U_j + \varepsilon^{5/2}\hat{U}_j + \eta\varepsilon^{5/2}) + O(\varepsilon^3).$$

Then the rest of the proof is similar to the previous one. (Note that here we just need an error in $O(\varepsilon^3)$.)

8. Appendix B: Proof of Lemma 3.1

To prove (3.6), we show that there exists a constant C, independent of x and y, such that

$$|G(x,y)| \le \frac{C}{|x-y|^3}.$$

We recall the decomposition of G:

$$G(x,y) = K(|x-y|) - H(x,y)$$

where $K(|x-y|)$ is the singular part of G and $H(x,y)$ is the regular part. Since $|K(|x-y|)| = \frac{1}{c_5|x-y|^3}$, it remains to show that

$$|H(x,y)| \le \frac{C}{|x-y|^3}. \tag{8.1}$$

Note that if for some fixed $d_0 > 0$, $d(x, \partial\Omega) > d_0$ or $d(y, \partial\Omega) > d_0$, then $|H(x, y)| \le C$ and (8.1) holds. So we just need to estimate $H(x, y)$ for $d(x, \partial\Omega)$ and $d(y, \partial\Omega)$ small. For $y \in \Omega$ such that $d = d(y, \partial\Omega)$ is sufficiently small, there exists a unique point $\bar{y} \in \partial\Omega$ such that $d = |y - \bar{y}|$. Let y^* be the reflection point of y through the boundary, i.e. $y^* - y = 2(\bar{y} - y)$, and consider the auxiliary function

$$H^*(x, y) = K(|x - y^*|).$$

Then H^* satisfies $\Delta H^* = 0$ in Ω and, on $\partial\Omega$,

$$\frac{\partial}{\partial\nu}(H^*(x, y)) = -\frac{\partial}{\partial\nu}(K(|x - y|)) + O\left(\frac{1}{d^2}\right).$$

Since both $\overline{K(|x - y|)}$ and $\overline{K(|x - y^*|)}$ are uniformly bounded, we derive that

$$H(x, y) = -H^*(x, y) + O\left(\frac{1}{d^2}\right),$$

which proves (8.1) for $x, y \in \Omega$. This implies, for $x \in \Omega$,

$$|u(x)| \le C \int_\Omega \frac{|f(y)|}{|x - y|^3}\, dy. \tag{8.2}$$

If $x \in \partial\Omega$, we consider a sequence of points $x_i \in \Omega$ with $x_i \to x \in \partial\Omega$ and take the limit in (8.2). Lebesgue's Dominated Convergence Theorem applies and (3.6) is proved. □

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