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On polynomials and surfaces of variously positive links

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Abstract. It is known that the minimal degree of the Jones polynomial of a positive knot is equal to its genus, and the minimal coefficient is 1, with a similar relation for links. We extend this result to almost positive links and partly identify the next three coefficients for special types of positive links. We also give counterexamples to the Jones polynomial-ribbon genus conjectures for a quasipositive knot. Then we show that the Alexander polynomial completely detects the minimal genus and fiber property of canonical Seifert surfaces associated to almost positive (and almost alternating) link diagrams.

Keywords. Positive link, quasipositive link, almost positive link, almost alternating link, Alexander polynomial, Jones polynomial, fiber surface, ribbon genus

1. Introduction

A link is called *quasipositive* if it is the closure of a braid which is the product of conjugates of the Artin generators σ_i [Ru2]. (We call such conjugates and their inverses positive resp. negative *bands*.) It is called *strongly quasipositive* if these conjugates are positive *embedded bands* in the band representation of [Ru2]. It is called *positive* if it has a diagram with all crossings positive (in the skein sense), and *braid positive* (or a *positive braid link*) if it has a braid representation which is a positive word in the Artin generators. It is called *fibered* if its complement in S^3 is a surface bundle over the circle.

We have

$$\begin{aligned} \{\text{quasipositive links}\} \supset \{\text{strongly quasipositive links}\} \supset \{\text{positive links}\} \\ \supset \{\text{fibered positive links}\} \supset \{\text{braid positive links}\}. \end{aligned} \quad (1)$$

The only non-obvious inclusions are the second and fourth one. The fourth inclusion is a well-known fact (it follows e.g. from [Ga1]), and the second inclusion follows, as observed by Rudolph [Ru1] and Nakamura [N], by applying the algorithm of Yamada

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[Y] or Vogel [Vo] to a positive diagram. Links in some of the above classes have been studied, besides their intrinsic knot-theoretical interest, with different motivations and in a variety of contexts, including singularity theory [A, BoW, Mi], algebraic curves [Ru2, Ru3], dynamical systems [BW] and (in some vague and yet-to-be understood way) in 4-dimensional QFTs [Kr].

A different class related to positive links is the almost positive links, those with almost positive diagrams, which are, however, not positive. (A diagram is *almost positive* if it has exactly one negative crossing.)

Let g be the genus of a knot, g_s the slice genus, and g_r the ribbon genus. (The definitions are recalled below.) For links similarly write χ , χ_r and χ_s for the (Seifert), ribbon and slice Euler characteristic resp. As any Seifert surface is a ribbon surface, and any ribbon surface is (deformable into) a slice surface, one has the inequalities $g \geq g_r \geq g_s$ and $\chi \leq \chi_r \leq \chi_s$.

For knots we also have $u \geq g_r$, with u being the *unknotting number* [Li]. By the work of Kronheimer–Mrowka [KM1, KM2] and Rudolph [Ru2], it is now known that the slice genus is estimated below by the slice Bennequin inequality (a version of [Be, Theorem 3] with g replaced by g_s), implying that for a strongly quasipositive knot $g = g_s$, so that $u \geq g = g_r = g_s$. For positive braid knots $u \leq g$ was known by [BoW]. Thus $u = g$ in this case.

Let V be the Jones polynomial [J]. Fiedler [Fi] proved that $\min \deg V = g$ for a positive braid knot, and that $\min \text{cf } V = 1$. For positive braid links L of $n = n(L)$ components, $\min \text{cf } V = (-1)^{n-1}$ and $2 \min \deg V = 1 - \chi$. This follows more generally for positive links L by virtue of the fact that positive diagrams are semiadequate (see [LT]). Fiedler further conjectured (his Conjecture 1) that for arbitrary knots and links L which have a band representation on s strands with b bands,

$$\min \deg V \leq \frac{b - s + 1}{2}.$$

He made a second conjecture (Conjecture 2), whose truth would imply that equality in the above inequality is achieved only for quasipositive links L .

In the paper of Kawamura [K], the theorems of Fiedler and Kronheimer–Mrowka–Rudolph have been found to imply that for a positive braid knot, $\min \deg V = u$, with a similar (ly obvious) relation for links. Then Kawamura quoted a special case of Fiedler’s first conjecture, asking whether it is true (at least) for quasipositive links, and observing that the slice Bennequin inequality would then imply the relation $\min \deg V \leq u$ for a quasipositive knot. (That $\min \deg V = u$ does not extend to quasipositive knots is easy to observe.)

In this paper, we will investigate several properties of polynomials of the above link classes. We will start in §3 by giving counterexamples to both Fiedler conjectures of several special types, in particular the case of the first conjecture addressed by Kawamura. Then, in §4, we will partly identify up to three of the coefficients of the Jones polynomial of a positive link following the minimal one, including a handy criterion (Theorem 1) to single out positive braid links, even among fibered positive links. We will also extend Fiedler’s result to almost positive links in Theorem 5. Some consequences are derived for

the skein polynomial [H], in particular a description of up to two more coefficients of its values on positive links (Corollary 6). For almost positive links, we obtain a proof of the inequality conjectured by Morton [Mo2] (for which in the general case counterexamples are now known [St3]).

In some of the proofs we will use the even valence graph version of the Alexander polynomial studied in [MS] with K. Murasugi. Applying this method, we can also show that the Alexander polynomial completely determines the minimal genus (Corollary 5) and fiber property (Theorem 7) of canonical Seifert surfaces associated to almost positive (and almost alternating) link diagrams. Thus we extend work of Hirasawa [Hi] and, with a significantly shorter proof, Goda–Hirasawa–Yamamoto [GHY]. At the end we will give a few examples showing that many of the possible extensions of these theorems are not true, and mention some problems.

2. Preliminaries

Link polynomials. The *skein polynomial* P is a Laurent polynomial in two variables l and m of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1} P(\text{crossing}) + l P(\text{crossing}) = -m P(\text{cup}) P(\text{cap}). \tag{2}$$

For a diagram D of a link L , we will use all of the notations $P(D) = P_D = P_D(l, m) = P(L)$ etc. for its skein polynomial, with the self-suggestive meaning of indices and arguments.

The *Jones polynomial* V and (one-variable) *Alexander polynomial* Δ are obtained from P by the substitutions

$$V(t) = P(-it, i(t^{-1/2} - t^{1/2})), \tag{3}$$

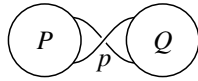
$$\Delta(t) \doteq P(i, i(t^{1/2} - t^{-1/2})), \tag{4}$$

hence these polynomials also satisfy corresponding skein relations. The sign “ \doteq ” means that the Alexander polynomial is defined only up to units in $\mathbb{Z}[t, t^{-1}]$; we will choose the normalization depending on the context.

In the following we denote the coefficient of t^m in $V(t)$ by $[V(t)]_m$. In the case of a 2-variable polynomial, we index the bracket by the whole monomial, and not just the power of the variables. The *minimal* or *maximal degree* $\min \deg V$ or $\max \deg V$ is the minimal resp. maximal exponent of t with non-zero coefficient in V . An explicit (one-variable) polynomial may be denoted by the convention of [LM] by its coefficient list, with bracketing its absolute term to indicate its minimal degree, e.g. $(-3 [1] 2) = -3/t + 1 + 2t$. The *minimal* or *leading coefficient* $\min \text{cf } V$ of V is $[V]_{\min \deg V}$.

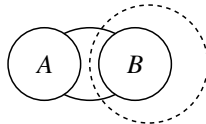
For an account on these link polynomials we refer to the papers [LM, J]. (Note: our convention for P differs from [LM] by interchange of l and l^{-1} , that is, our $P(l, m)$ is Lickorish and Millett’s $P(l^{-1}, m)$.)

Link diagrams. A crossing p in a link diagram D is called *reducible* (or *nugatory*) if D can be represented in the form



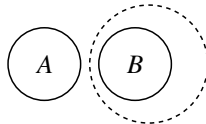
D is called *reducible* if it has a reducible crossing, otherwise it is called *reduced*.

A link diagram D is *composite* if there is a closed curve γ intersecting (transversely) the curve of D in two points, such that both the interior and exterior of γ contain crossings of D , that is, D has the form



Otherwise D is *prime*. A link is prime if in any composite diagram replacing one of A and B by a trivial (0-crossing) arc gives an unknot diagram.

The diagram is *split* if there is a closed curve not intersecting it, but which contains parts of the diagram in both its interior and exterior:



Otherwise D is *connected* or *non-split*. A link is split if it has a split diagram, and otherwise non-split.

We call a diagram D *k-almost positive* if D has exactly k negative crossings. A link L is *k-almost positive* if it has a k -almost positive diagram, but no l -almost positive one for any $l < k$. We call a diagram or link *positive* if it is 0-almost positive (see [Cr1, O, Yo, Zu]), and *almost positive* if it is 1-almost positive [St2]. Similarly one defines *k-almost negative*, and in particular *almost negative* and *negative* links and diagrams to be the mirror images of their k -almost positive (or almost positive or positive) counterparts, and (*k*-)almost alternating diagrams and links [Ad1, Ad2]. The *valency* of a Seifert circle s is the number of crossings attached to s . We call such crossings also *adjacent* to s .

Link surfaces. A *Seifert* resp. *slice* surface of $L \subset S^3 = \partial B^4 \subset B^4$ is a smoothly embedded compact orientable surface $S \subset S^3$ resp. $S \subset B^4$ with $\partial S = L$. A *ribbon surface* is a smoothly immersed compact orientable surface $S \subset S^3$ with $\partial S = L$, embedded except at a finite number of double transverse (ribbon) singularities, whose preimages are two arcs, one lying entirely in the interior $\text{int } S$ of S , and the other one too, except for its two endpoints, which lie on ∂S . A *canonical (Seifert) surface* is a Seifert surface obtained by Seifert's algorithm (see [Ro]). We may allow (for links) all these surfaces to be disconnected, but they should have no closed ($\partial = \emptyset$) components.

The (*Seifert*) *genus* g , *slice genus* g_s , *canonical genus* \tilde{g} and *ribbon genus* g_r are defined to be the minimal genera of Seifert, slice, canonical resp. ribbon surfaces of L . Similarly one can define the (Seifert), slice, canonical resp. ribbon *Euler characteristic* χ , χ_s , $\tilde{\chi}$, and χ_r to be the maximal Euler characteristic of such surfaces of L .

In [Be, Theorem 3], Bennequin shows that for a braid β on $s(\beta)$ strands, with writhe (exponent sum) $w(\beta)$ and with closure $\hat{\beta} = K$, we have an estimate for the Euler characteristic $\chi(K)$ of K :

$$1 - \chi(K) \geq w(\beta) - s(\beta) + 1.$$

This is easily observed to extend by means of the algorithm of Yamada [Y] or Vogel [Vo] to an inequality for arbitrary link diagrams D of K :

$$1 - \chi(K) \geq w(D) - s(D) + 1 =: b(D), \tag{5}$$

with $w(D)$ being the writhe of D , and $s(D)$ the number of its Seifert circles. We call the r.h.s. of (5) the *Bennequin number* of D . (It clearly depends a lot on the diagram for a given link.)

Rudolph [Ru3] later improved this inequality, by replacing $\chi(K)$ by $\chi_s(K)$:

$$1 - \chi_s(K) \geq b(D). \tag{6}$$

Recently, he obtained a further improvement, this time by increasing the r.h.s. [Ru2]:

$$1 - \chi_s(K) \geq w(D) - s(D) + 1 + 2s_-(D) =: rb(D), \tag{7}$$

with $s_-(D)$ being the number of (≥ 2 -valent) Seifert circles of D , to which only negative non-nugatory crossings are adjacent. We call the new quantity on the right the *Rudolph–Bennequin number* of D . Again $rb(D)$ heavily depends on the diagram, even more than $b(D)$. (For example, unlike $b(D)$, $rb(D)$ is no longer invariant under flypes and mutations.) Thus again one is interested in choosing for a given link K the diagram D so that $rb(D)$ is as large as possible.

3. Counterexamples to the Jones polynomial-ribbon genus conjectures

3.1. Preparations

While the improvement (7), as compared to (6), may not seem significant at first sight, it has the advantage of eliminating the minimal l -degree in the skein polynomial $\min \deg_l P$ as an obstruction to increasing the estimate by proper choice of the diagram D , since by [Mo1] we always have $b(D) \leq \min \deg_l P(K)$.

A practical example where this turned out helpful was given in [St4], and is recalled below, as it will be used. (The notation for knots we apply is the one of Rolfsen’s tables [Ro, appendix] for ≤ 10 crossings, and of the knot table program KnotScape [HT] for 11 to 16 crossings. By $!K$ we will denote the obverse, or mirror image, of K .)

Example 1. The knot 13_{6374} has $\min \deg_l P = 0$ and Alexander polynomial $\Delta = 1$. It has many diagrams D with $b(D) = 0$, but it cannot have any such diagram with $b(D) > 0$, because of Morton’s inequality. However, it does have diagrams D with $rb(D) > 0$, thus showing it not to be slice.

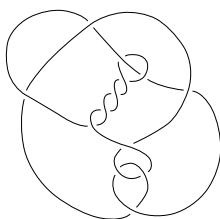
13₆₃₇₄

Fig. 1. A non-slice knot with unit Alexander polynomial.

In order to construct our counterexamples, we need a few more simple lemmas.

Lemma 1. *If K is quasipositive, then $\min \deg_l P(K) \geq 1 - \chi_s(K)$.*

Proof. If D is a diagram of a quasipositive braid representation of K , then $1 - \chi_s(K) = b(D)$, and $b(D) \leq \min \deg_l P(K)$ by Morton's inequality. \square

In the following $K_1 \# K_2$ denotes the connected sum of K_1 and K_2 , and $\#^n K$ denotes the connected sum of n copies of K .

Lemma 2. *If $K_{1,2}$ have diagrams $D_{1,2}$ which are not negative, then $K_1 \# K_2$ has a diagram D with $rb(D) = rb(D_1) + rb(D_2)$.*

Proof. We apply the connected sum of $D_{1,2}$ so that the Seifert circles of $D_{1,2}$ affected by the operation have at least one positive crossing adjacent to them. \square

Lemma 3. *If K is strongly quasipositive, then $\chi(K) = \chi_s(K)$.*

Proof. For the Seifert surface S associated to a strongly quasipositive braid representation diagram D of K , we have

$$1 - \chi(K) \leq 1 - \chi(S) = b(D) \leq rb(D) \leq 1 - \chi_s(K) \leq 1 - \chi(K),$$

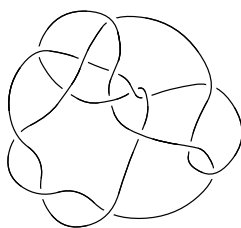
implying equality everywhere. \square

3.2. Degree inequality conjecture

Fiedler's first conjecture was whether

$$\min \deg V_L \leq \frac{b - s + 1}{2}$$

if L has a b -band representation on s strands, and Kawamura's (weaker) question was whether it is true at least if this band representation is positive.



!15₁₆₂₅₀₈

Fig. 2

Example 2. Consider the knot !15₁₆₂₅₀₈ (see Figure 2). Using the method described in [St5, appendix], it was found that it is ribbon (and hence slice), and one calculates $\min \deg V = 1$. It turns out to have the quasipositive 5-braid representation

$$(\sigma_1^{-1} \sigma_2^{-1} \sigma_3 \sigma_4 \sigma_3^{-1} \sigma_2 \sigma_1) (\sigma_2^{-1} \sigma_1 \sigma_2) (\sigma_2 \sigma_3^{-1} \sigma_4 \sigma_3 \sigma_2^{-1}) \sigma_3.$$

(The knot can be identified from this representation by the tool knotfind included in [HT]. Note that this representation also directly shows sliceness.) Thus it is a slice example answering negatively Kawamura’s question, and hence also a counterexample to Fiedler’s first conjecture.

Another special type of example is

Example 3. Consider the knot K in Figure 3, which is the closure of the 4-braid

$$\sigma_1^2 (\sigma_1 \sigma_2 \sigma_1^{-1}) \sigma_2 \sigma_1 \sigma_3 (\sigma_1 \sigma_2 \sigma_1^{-1}) \sigma_2 (\sigma_2 \sigma_3 \sigma_2^{-1}) (\sigma_1 \sigma_2 \sigma_1^{-1}) (\sigma_2 \sigma_3 \sigma_2^{-1}). \tag{8}$$

This braid is quasipositive, in fact, strongly quasipositive. The diagram of K in Figure 3 was obtained from that representation. One easily sees that $g = g_s = 4$. But $\min \deg V = 5$. Thus $g_s < \min \deg V$. In fact, this knot has unknotting number 4. (Switch the encircled crossings in the diagram of Figure 3.) Thus even the weaker inequality, in which Kawamura was interested, $\min \deg V \leq u$, is not always true.

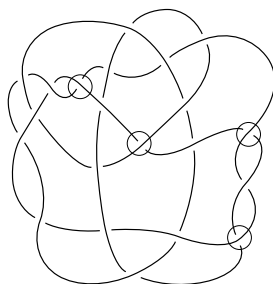


Fig. 3. The knot K in Example 3.

Remark 1. $!15_{162508}$ is surely not strongly quasipositive, as $g > 0 = g_s$. Thus the above example K is the most special in the hierarchy (1).

The only case of some interest, remaining not covered by the above examples, is that of a slice knot with $u < \min \deg V$. Very likely such examples exist, too, although I have not found any.

Remark 2. If one is interested in a *general* knot K with $\min \deg V > u$, then there is a much simpler and well-known example, $!10_{132}$. It has $u = 1$, but $\min \deg V = 2$. However, $!10_{132}$ is not quasipositive. As it is not ribbon, or slice (its determinant 5 is not a square), it has 4-genus 1, and a quasipositive representation of n strands would have $n + 1$ bands. Then the untwisted 2-cable link $(!10_{132})_2$ would have a representation on $2n$ strands of writhe $2n + 2$. Thus by [Mo1], $\min \deg_l P((!10_{132})_2) \geq 3$, but from the calculation of [MoS] we know $\min \deg_l P((!10_{132})_2) = 1$.

Remark 3. In a preprint [Ta], T. Tanaka has independently claimed counterexamples to Fiedler's first conjecture. On the opposite end, M. Ishikawa [I] proved Fiedler's inequality for some links obtained by A'Campo's method [A].

3.3. Extremal property conjecture

Fiedler also conjectured (his Conjecture 2) that if a link L has a b -band s -strand band representation with

$$\min \deg V_L = \frac{b - s + 1}{2}, \quad (9)$$

then it is quasi-positive. (Fiedler's formulation is slightly different, but easily implies the one given here.) We will now construct a counterexample also to this conjecture, albeit some more effort is necessary, and we must use the example found previously in a related context in [St4]. Our counterexample likely has crossing number 58.

Proposition 1. *The knot $K' = 13_{6374} \# \#^3(!15_{162508})$ is not quasipositive, yet it has a band representation with equality in (9).*

Proof. We first discuss the prime factors separately.

1. Consider 13_{6374} . By switching one of the crossings in the clasp in the lower right part of the diagram in Figure 1, one obtains 4_1 . Thus by two crossing changes $\#^2 13_{6374}$ turns into the slice knot $4_1 \# 4_1$. Hence $1 - \chi_s(\#^2 13_{6374}) \leq 4$. On the other hand, as 13_{6374} has a diagram D with $rb(D) = 2$, Lemma 2 shows that $1 - \chi_s(\#^2 13_{6374}) = 4$. By the above,

$$\min \deg_l P(\#^2 13_{6374}) = 0 = \frac{1 - \chi_s(\#^2 13_{6374})}{2} - 2.$$

Also $\min \deg V = -1$ by calculation. As $\max \deg_m P(13_{6374}) = 4$ and it has crossing number < 15 , by [St1] 13_{6374} has a diagram of canonical genus 2, and thus by applying Yamada’s algorithm [Y] to it, we obtain an (embedded) band representation with

$$\frac{b - s + 1}{2} = 2 = \min \deg V + 3.$$

2. For $!15_{162508}$, we have a quasipositive band representation as 5-braid with 4 bands, and it is slice. Thus

$$1 - \chi_s = 0 = \frac{\min \deg_l P}{2} \quad \text{and} \quad \frac{b - s + 1}{2} = 0 = \min \deg V - 1.$$

In summary we have the following situation for proper diagrams D and b -band s -braid representations:

	13_{6374}	$!15_{162508}$
$\frac{\min \deg_l P}{2} - \frac{rb(D)}{2}$	-1	0
$\min \deg V - \frac{b - s + 1}{2}$	-3	1

Since both quantities are additive under connected sum for proper diagrams and band representations (by Lemma 1 resp. in the obvious way), we obtain for K' a band representation with

$$\frac{b - s + 1}{2} = \min \deg V(K'),$$

but also a diagram D of $K' \# K'$ with

$$\min \deg_l P(K' \# K') < rb(D) \leq 1 - \chi_s(K' \# K')$$

(in fact $rb(D) = 1 - \chi_s(K' \# K')$), so that $K' \# K'$ is not quasipositive by Lemma 1. Then K' cannot be quasipositive either. \square

One can also obtain a counterexample to an “embedded band” version of Fiedler’s conjecture, namely that a knot K with an *embedded* band representation achieving equality in (9) is *strongly* quasipositive.

Proposition 2. *The knot $K' = \#^3 K \# 6_1$, with K being the knot in Example 3, is not strongly quasipositive, yet it has an embedded band representation satisfying equality in (9).*

Proof. As 6_1 has canonical genus 1, it has an embedded band representation with

$$\frac{b - s + 1}{2} = 1 = \min \deg V + 3.$$

Now consider K . It has a strongly quasipositive band representation with $b = 11$ bands on $s = 4$ strands, so that

$$1 - \chi_s = \frac{b - s + 1}{2} = 4.$$

However, $\min \deg V = 5$. Thus K' has an embedded band representation satisfying (9). As genus is additive under connected sum, we have $g(K') = 13$. However, as g_s is subadditive under connected sum, and 6_1 is slice, we have $g_s(K') \leq 12$, so that $g > g_s$, and so K' is not strongly quasipositive by Lemma 3. \square

There is an exponentiated version of Fiedler's conjecture, namely asking about (non-strong) quasipositivity if one assumes (9) for an embedded band representation. We conclude this section by showing how to construct counterexamples also for this sharpest case. The problem reduces to replacing $!15_{162508}$ by a strongly quasipositive knot with $\min \deg V > g$. Then the same argument as in the proof of Proposition 1 goes through with embedded band representations.

Example 4. Consider the (apparently) 17-crossing knot of Figure 4. It has a band representation with 7 bands on 4 strands,

$$((\sigma_2 \sigma_3 \sigma_2^{-1})(\sigma_1 \sigma_2 \sigma_1^{-1}))^3 \sigma_1.$$

(The diagram in Figure 4 was obtained again using KnotScape.) Thus its genus is $g = 2$. Also $\min \deg_l P = 4$, but $\min \deg V = 3$.

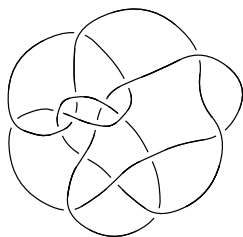


Fig. 4

Remark 4. It is clear from Example 4 that in fact we could have used it also as counterexample to Fiedler's first conjecture. However, unlike for the knot K in Example 3, I cannot show $u = g (= 2)$ here. On the other hand, K cannot be used in Example 4, because it has $\min \deg_l P = 10$. (K was found as a counterexample to Morton's conjecture, as reported in [St3].) This way, any of the previous knots has its independent significance.

4. The coefficients of the Jones polynomial

Convention. It is convenient to assume in what follows that all diagrams we consider are non-split. In particular, since non-split positive diagrams represent non-split links, we assume all positive links to be non-split.

Definition 1. A separating Seifert circle is a Seifert circle with non-empty interior and exterior. (That is, both interior and exterior contain crossings, or equivalently, other Seifert circles.) A diagram with no separating Seifert circles is called special.

Any diagram decomposes as Murasugi sum along its separating Seifert circles into special diagrams (see [Cr1, §1]). For any diagram, any two of the properties: positive, alternating and special imply the third. We call these diagrams special alternating to conform to the classical terminology of Murasugi [Mu3].

4.1. Positive braids

It is known (e.g. from [Fi]) that the minimal coefficient of the Jones polynomial of a positive braid link is ± 1 . We will show here a statement on the next three coefficients.

Theorem 1. Let L be a non-split braid positive link of $n(L)$ components. Then

$$(-1)^{n(L)-1} t^{(\chi(L)-1)/2} V_L(t) = 1 + pt^2 + kt^3 + (\text{higher order terms}),$$

with $p = p(L)$ being the number of prime factors of L and

$$-p \leq k \leq \frac{3}{2}(1 - \chi(L) - p), \tag{10}$$

where $\chi(L)$ is the Euler characteristic of L .

Note that it is a rather unusual situation to be able to read the number of prime factors off a polynomial. This is, for example, not possible for alternating links as shown by the well-known pair 8_9 and $4_1\#4_1$ —the first knot is prime and the other composite, yet they have the same Jones polynomial. Two more interesting examples of this type are as follows:

Example 5. With some effort one also finds such pairs of positive (or special) alternating knots: 12_{420} (Figure 5) and $!3_1\#9_{13}$ or 14_{4132} and $!5_2\#!9_9$.

Example 6. Even more complicated, but still existent, are such examples of fibered positive knots. The simplest group I found is a triple consisting of 14_{39977} , $!3_1\#14_{33805}$ and $!3_1\#14_{37899}$ (see Figure 5).

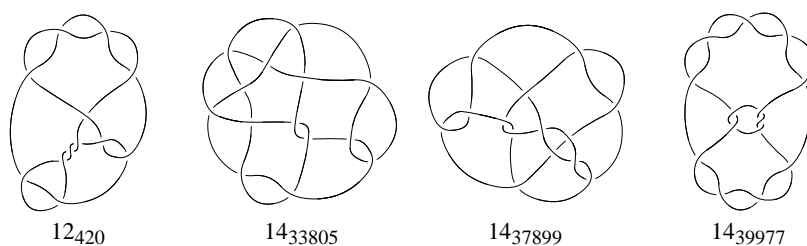


Fig. 5

Poof of Theorem 1. If β is a positive braid diagram of L , then by the result of [Cr2] the number of prime factors of the link $\hat{\beta}$ is equal to the number of prime factors of the diagram $\hat{\beta}$. By [BoW] one can always choose β so that it contains a σ_i^2 . Apply the skein relation at one of the crossings. Then

$$V_+ = t^2V_- + (t^{3/2} - t^{1/2})V_0,$$

with L_- and L_0 both braid positive. Let $p_* = p(L_*)$. By induction on the crossing number of the braid we have

$$\begin{aligned} t^2V_- &= (-1)^{n(L)-1}t^{(1-\chi(L))/2} \cdot ([0] \quad 1 \quad 0 \quad p_- \quad \dots) \\ (t^{3/2} - t^{1/2})V_0 &= (-1)^{n(L)-1}t^{(1-\chi(L))/2} \cdot ([1] \quad -1 \quad p_0 \quad k_0 - p_0 \quad \dots) \\ V_+ &= (-1)^{n(L)-1}t^{(1-\chi(L))/2} \cdot ([1] \quad 0 \quad p_0 \quad k_0 - p_0 + p_- \quad \dots) \end{aligned} \tag{11}$$

As $p_+ = p_0$ and $0 \leq p_- - p_+ = p_- - p_0 \leq 2$, the claim follows by induction, once it is checked directly for connected sums of trefoils and Hopf links, except for the right inequality in (10), which follows only with the constant $3/2$ replaced by 2 . (Note that k and p are both additive under connected sum.) To prove $k \leq \frac{3}{2}(1 - \chi - p)$, we need to show that after a smoothing with $p_- = p_+ + 2$ we can choose another one with $p_- \leq p_+ + 1$.

Write

$$\beta = \prod_{k=1}^l \sigma_i^{m_k} w_k$$

with all w_k containing no σ_i but some of $\sigma_{i\pm 1}$. Then one of the k_i , say k_1 , is equal to 2 , $k_2 \geq 2$ and $l = 2$. Then after smoothing out one of the crossings in the clasp, we have $k_1 = 1$, and then applying the skein relation at the other clasp, we have $p_- - p_0 \leq 1$, as desired. \square

From the proof it is clear that the second inequality in (10) is not sharp, and with some work it may be improvable. Candidates for the highest ratio $k/(1 - \chi - p)$ are braids of the form $(\sigma_1^2 \sigma_2^2 \cdots \sigma_l^2)^2$ for which this ratio converges upward to 1 as $l \rightarrow \infty$.

In contrast, the first inequality is clearly sharp, namely for connected sums of $(2, \cdot)$ -torus links.

Question 1. Are connected sums of $(2, \cdot)$ -torus links the only links with $p + k = 0$?

4.2. Fibered positive links

We shall now prove a result on almost positive diagrams which shows a weaker version of Theorem 1 for fibered positive links. We need a definition.

Definition 2. The Seifert graph \bar{S}_D of a diagram D is a graph obtained by putting a vertex for each Seifert circle of D and connecting two vertices by an edge if a crossing joins the two corresponding Seifert circles. (If two Seifert circles are connected by several crossings, \bar{S}_D has multiple edges.) The reduced Seifert graph S_D of D is obtained by removing edges of \bar{S}_D such that (a) S_D has no multiple edge and (b) two vertices are connected by an edge in \bar{S}_D iff they are so in S_D .

Definition 3. For a link diagram D , let $\chi(D) = s(D) - c(D)$, where $s(D)$ is the number of Seifert circles, and $c(D)$ the number of crossings of D . $\chi(D)$ is the canonical Euler characteristic of D .

Theorem 2. Let D be an almost positive diagram of a link L with $n(L)$ components, with negative crossing p . If there is another crossing in D joining the same two Seifert circles as p , then $\min \deg V_L \geq (1 - \chi(D))/2$. Otherwise, $\min \deg V_L = (1 - \chi(D))/2 - 1$ and $\min \text{cf } V_L = (-1)^{n(L)-1}$.

Recall that the Kauffman bracket $[D]$ (see [Ka]) of a link diagram D is a Laurent polynomial in a variable A , obtained by summing the terms

$$A^{\#A(S) - \#B(S)} (-A^2 - A^{-2})^{|S|-1}, \tag{12}$$

over all states S , where a state is a choice of splittings of type A or B for any single crossing (see Figure 6), $\#A(S)$ and $\#B(S)$ denote the number of type A (resp. type B) splittings, and $|S|$ the number of (disjoint) circles obtained after all splittings in a state.

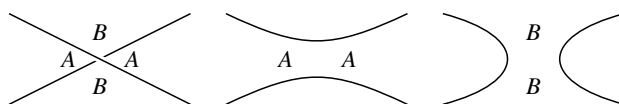


Fig. 6. The A - and B -corners of a crossing, and both of its splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.

The Jones polynomial of a link L is related to the Kauffman bracket of some diagram D of it by

$$V_L(t) = (-t^{-3/4})^{-w(D)} [D]_{A=t^{-1/4}}, \tag{13}$$

$w(D)$ being the writhe of D .

Proof of Theorem 2. The maximal possible degree of A in

$$[D] = \sum_{S \text{ state}} A^{\#A(S) - \#B(S)} (-A^2 - A^{-2})^{|S|-1} \tag{14}$$

is that of the A -state (the state with all crossings A -split), because under any splitting switch $A \rightarrow B$, the power of A in the first factor in (12) goes down by 2, and the maximal

power of A in the second factor in (12) increases at most by 2. If D is almost positive with negative crossing p , then the maximal possible power of A in (14) is $A^{c(D)+2(s(D)-2)}$, as the A -state S_A has $s(D) - 1$ loops. They are the Seifert circles not adjacent to p , and a loop consisting of the two Seifert circles, call them a and b , adjacent to p .

Now we must consider which states contribute to $A^{c(D)+2(s(D)-2)}$ in (14). These are exactly the states with the property that whenever the state is obtained from the A -state by successively switching $A \rightarrow B$ splittings, $|\cdot|$ increases under any such switch.

Let $\langle S : k \rangle \in \{A, B\}$ be the split of k in S , and let $s_k(S)$ be the state obtained by switching splitting $A \rightarrow B$ at crossing k in S , assuming $\langle S : k \rangle = A$. Then if $|s_k(S_A)| < |S_A|$, any state S with $\langle S : k \rangle = B$ is irrelevant for the highest term in (14). Clearly, this happens whenever k is a crossing connecting one or two Seifert circles not adjacent to p . Thus the only terms contributing to $A^{c(D)+2(s(D)-2)}$ in (14) are those for which $\langle S : k \rangle = B$ implies that k has the same two adjacent Seifert circles a and b as p has.

Let $p_1, \dots, p_k = p$ be these crossings. Since any splitting switch $A \rightarrow B$ in $s_p(S_A)$ reduces $|\cdot|$, the only state S with $\langle S : p \rangle = B$ relevant for the highest term in (14) is $s_p(S_A)$, whose contribution to the coefficient of this highest term is $(-1)^{|s_p(S_A)|-1} = (-1)^{s(D)-1}$.

It is also easy to see that if $\langle S : p \rangle = A$, any of the 2^{k-1} remaining states S to consider contribute to $A^{c(D)+2(s(D)-2)}$, the coefficient being $(-1)^{s(D)+\#B(S)}$, as $|S| = s(D) - 1 + \#B(S)$. The sum over all such S of these coefficients is $(-1)^{s(D)}$ times the alternating sum of binomial coefficients. Thus this sum vanishes for $k - 1 > 0$, and cancels for $k - 1 = 0$ the coefficient $(-1)^{s(D)-1}$ of $s_p(S_A)$. The rest follows from (13) with $w(D) = c(D) - 2$, and the remark that $1 - \chi(D)$ and $n(L) - 1$ have the same parity. \square

Corollary 1. *Let L be a fibered positive link of $n(L)$ components. Then $[V_L(t)]_{(3-\chi(L))/2} = 0$, that is,*

$$(-1)^{n(L)-1} t^{(\chi(L)-1)/2} V_L(t) = 1 + kt^2 + (\text{higher order terms}),$$

with k being some integer.

Proof. This is proved just as Theorem 1 by induction on the crossing number of a positive diagram D . Apply the skein relation at any (non-nugatory) crossing p of D . Since the reduced Seifert graph of D is a tree, there is another crossing between the same two Seifert circles. Let D_0 be D with p smoothed out, and L_0 be the link D_0 represents. Then L_0 is still fibered, because D_0 is positive and connected, and its reduced Seifert graph is still a tree. Similarly let D_- be D with p switched, and let D_- represent a link L_- . Then we can apply the above theorem to L_- . So $\min \deg V_- = (1 - \chi)/2 - 1$, and the coefficients of $t^{(1-\chi)/2+1}$ in $t^2 V_-$ and $(t^{3/2} - t^{1/2}) V_0$ cancel as in (11). \square

4.3. Positive and almost positive links

Corollary 1 is a special case of the following result, describing the second coefficient of the Jones polynomial for an arbitrary positive link.

Theorem 3. *Let L be a positive link with positive diagram D . Then*

$$(-1)^{n(L)-1} [V_L]_{(3-\chi(L))/2} = s(D) - 1 - \#\{(a, b) \text{ Seifert circles : there is a crossing joining } a \text{ and } b\}.$$

In other words, if S_D is the reduced Seifert graph, then $(-1)^{n(L)} [V_L]_{(3-\chi(L))/2} = b_1(S_D)$, b_1 being the first Betti number.

Corollary 2. *For a positive diagram D , $b_1(S_D)$ is an invariant of the link represented by D . □*

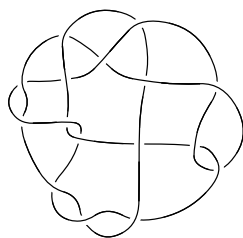
Note that for the non-reduced Seifert graph \bar{S}_D of D , $b_1(\bar{S}_D) = 1 - \chi(D) = 1 - \chi(L)$ is also a link invariant.

Corollary 3. *For a positive link L of $n(L)$ components, $(-1)^{n(L)} [V_L]_{(3-\chi(L))/2} \geq 0$, and this coefficient is 0 iff L is fibered. □*

Of course, this fiberedness condition is not very useful when a positive diagram of L is given, since to decide then about fiberedness is trivial. However, applied in the opposite direction, it can prove that L is not positive. This happens sometimes in a quite non-trivial way, as shown by the following example.

Example 7. The knot $16_{1059787}$ in Figure 7 satisfies all conditions on positivity known about its ∇ , V , P and F polynomials. It seems useful to list all properties that hold, even if they involve invariants we did not consider here. See the given references for an accurate account. (However, keep in mind that the conventions there differ from the ones we use; for F we conjugate in the a variable.)

- $2 \min \deg V = \min \deg_l P = \max \deg_m P = \max \deg \nabla = \min \deg_a F = 4$ [Cr1, Fi, Yo, Zu],
- $[P]_{m^4}(\sqrt{-l})$ and $\nabla(z)$ are positive (that is, all coefficients are non-negative) [Cr1],
- $\tilde{P}_i(l) := \sqrt{-1}^i [P]_{m^i}(\sqrt{-l})$ take only positive values at $l \in (0, 1)$ for $i = 0, 2, 4$ [CM].
- $[V]_{l^2} = 1$ [Fi, Zu].
- $[F]_{a^4}(l) = [P]_{m^4}(l)$ [Yo].



16₁₀₅₉₇₈₇

Fig. 7

- $[F]_{z^k a^l} \geq 0$ for all (k, l) with $l - k$ maximal among the (k, l) for which $[F]_{z^k a^l} \neq 0$ (that is, “critical line” polynomials are positive) [Th2].
- $16_{1059787}$ does have diagrams (with canonical Seifert surface) of genus 2, so that $\tilde{g} = g = \max \deg \Delta = 2$ [Cr1]. Here Δ is normalized so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$.
- The signature $\sigma = 4$ [CG, St2], so that by Murasugi’s inequality [Mu2], $g_s = g = 2$ [Ru1, St6].

However, now $[V]_{f^3} = 0$, so that if $16_{1059787}$ is positive, it must be fibered. But $[\Delta(t)]_{f^2} = 2$ contradicts this property.

Proof of Theorem 3. This is proved just as Theorem 2 using the bracket. The term $s(D) - 1$ comes from the A -state, while for every pair of Seifert circles joined by (at least) one crossing, a -1 comes from an alternating sum of binomial coefficients coming from states in which a B -splitting is applied at some (non-empty) set of crossings linking a and b . □

Corollary 4. *Let L be an almost positive link with an almost positive diagram D such that there is no positive crossing q joining the same two Seifert circles as the negative crossing p . Then*

$$\min \deg V_L = \frac{1 - \chi(D)}{2} \quad \text{and} \quad \min \text{cf } V_L = (-1)^{n(L)-1}.$$

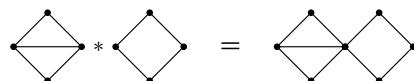
Proof. Apply the skein relation at the negative crossing p and use Theorem 3 for D_+ and D_0 (they have the same reduced Seifert graph). □

The following theorem is the key step needed to extend Fiedler’s result to almost positive links.

Theorem 4. *Let p be a crossing in a reduced special alternating diagram D such that there is no crossing q joining the same two Seifert circles as p does. Let D_p be D with p smoothed out. Then $\Delta_{D_p}(0) < \Delta_D(0)$, where Δ is the Alexander polynomial normalized so that $\min \deg \Delta = 0$ and $\min \text{cf } \Delta = \Delta(0) > 0$.*

The proof will use the machinery of even valence graphs [MS]. We recall the basic notions from that paper.

Definition 4. *The join (or block sum) $*$ of two graphs is defined by*



This operation depends on the choice of a vertex in each graph. We call this vertex the join vertex.

A cut vertex is a vertex which disconnects the graph when removed together with all its incident edges. (A join vertex is always a cut vertex.) Analogously a 2-cut of G is a pair of edges of G whose deletion disconnects G .

Definition 5. A cell C is the boundary of a connected component of the complement of a graph G in the plane. It consists of a set of edges. If p is among these edges, then we say that C contains p or p bounds C . By $G \setminus C$ we mean the graph obtained from G by deleting all edges in C .

A cycle C in a graph G is a set $\{p_1, \dots, p_n\}$ of edges such that the pairs (p_1, p_n) and (p_i, p_{i+1}) for $1 \leq i < n$ share a common vertex, and all these vertices are different. The plane complement of a cycle in a planar graph has two components. The bounded one is called the interior $\text{int}(C)$ of C , and the unbounded one the exterior $\text{ext}(C)$. (A cell is a cycle with one of interior or exterior being empty, that is, containing no edges.)

Before we make the next definition, first note that the Seifert graph \bar{S}_D of any diagram D is always planar(ly embeddable). Namely \bar{S}_D is the join of the Seifert graphs corresponding to the special diagrams in the Murasugi sum decomposition of D along its separating Seifert circles, the join vertex corresponding to the separating Seifert circle. The join of planar graphs is planar, and if D is a special diagram, then \bar{S}_D has a natural planar embedding (shrink the Seifert circles into vertices and turn crossings into edges).

Definition 6. Assume for a special diagram D that \bar{S}_D is planarly embedded in the natural way. Its dual is called the even valence graph G_D of D (as the name says, all its vertices have even valence). Alternatively, G_D is the checkerboard graph with vertices corresponding to the non-Seifert circle regions of D .

A canonical orientation is an orientation of the edges of G_D so that all edges bounding a cell are oriented the same way, clockwise or counterclockwise, as seen from inside the cell. (The canonical orientation is unique up to reversal of orientation of all edges in a connected component of the graph.)

Proof of Theorem 4. Consider the planar even valence graph G_D associated to D . Then $G_{D_p} = (G_D)_p$, where G_p is G with edge p contracted. Both G and G_p are connected by assumption. We shall assume from now on that a canonical orientation is chosen in $G = G_D$, and hence also on G_p .

By the matrix-tree theorem (see Theorem 2 of [MS]), $\Delta_D(0) = \min \text{cf } \Delta$ is the number of index-0 spanning rooted trees of G , i.e. trees in which each edge, oriented as in G , points towards the root of the tree. We will also call such trees *arborescences*. Importantly, the number of arborescences does not depend on the choice of root vertex. We will exploit this property several times in the following.

Let v_0 be the source and v_1 the target of p in G . In G_p , v_0 and v_1 are identified to a vertex we call v . By the proof of Proposition 1, part (3), of [MS], we have

$$\begin{aligned} \# \{ \text{index-0 sp. rooted trees with root } v \text{ in } G_p \} \\ = \# \{ \text{index-0 sp. rooted trees with root } v_1 \text{ in } G \text{ containing } p \}. \end{aligned}$$

Thus the statement of the theorem is equivalent to saying that G has an index-0 spanning rooted tree with root v_1 not containing p . The assumption of the theorem in terms of even

valence graphs means that each edge of G bounds a cell not containing p , or equivalently, p is in no 2-cut of G . In particular, both v_0 and v_1 have valence at least 4 in G .

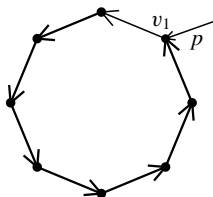
It is easy to see that any planar even valence graph G can be built up from the empty one by adding directed cycles. Moreover, if G is connected, then we can achieve that all intermediate graphs are connected (more exactly, all their connected components except one are trivial, i.e. isolated vertices of valence 0). Also, one can start the building-up with any particular cycle in G .

Let E be a cell (cycle with empty interior) in G containing p . We claim that then $\hat{G} = G \setminus E$ is still connected. This requires a little argument. We will show that if a disconnected graph \hat{G} is connected by adding a cell E , then each edge in E forms a 2-cut with another edge in E . To see this, first reduce the problem to \hat{G} having two components \hat{G}_1 and \hat{G}_2 . If G has further components $\hat{G}_3, \dots, \hat{G}_n$, one can connect them to \hat{G}_2 by adding cells, and a 2-cut of edges in E would still remain one if we undo this connecting. It is also easy to see that one can assume there are no valence-2 vertices of E in G (that is, each vertex of E is attached to one of \hat{G}_1 or \hat{G}_2 in G). Then we show that there are at most two edges of E connecting \hat{G}_1 and \hat{G}_2 . Since E is oriented, one can easily distinguish between interior or exterior of E depending on the (left or right) side in orientation direction. If ≥ 4 edges connect \hat{G}_1 and \hat{G}_2 , one must attach vertices of \hat{G}_1 and \hat{G}_2 to E from different sides, and E will not be a cell in G .

Let E' be some other cycle passing through v_1 such that $p \notin E'$. (Such a cycle exists because $\text{val}_G(v_1) > 2$.)

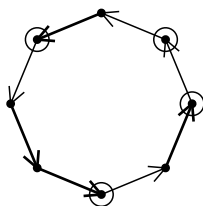
Then build up G by adding cycles E_n such that we start with $E_1 = E'$ and finish with $E_z = E$, and all intermediate graphs G_n are connected. We construct successively in each G_n an index-0 spanning rooted tree T_n with root v_1 such that in the final stage in $G_z = G$ the tree $T_z = T$ does not contain p .

In $G_1 = E'$, fix the root to be v_1 and let T_1 consist of all edges in E' except the one outgoing from v_1 .



Now, given an index-0 spanning rooted tree T_n of G_n , we construct an index-0 spanning rooted tree T_{n+1} of $G_{n+1} = G_n \cup E_{n+1}$ as follows. Let w_1, \dots, w_k be the vertices of the cycle E_{n+1} in cyclic order, so that w_i and w_{i+1} are connected by a (directed) edge p_i . Then there is a non-empty set $S \subset \{1, \dots, k\}$ such that $w_s \in G_n$ for all $s \in S$, and w_s is a trivial connected component (isolated vertex) in G_n otherwise. Then add the following vertices to T_n to obtain T_{n+1} : for each $i, j \in S$ such that $(i, j) \cap S = \emptyset$ add $\{p_m : m \in (i + 1, j - 1)\}$. Here (i, j) is the interval of numbers between (but not including) i and j , in the cyclic order of \mathbb{Z}_k .

Here is an example of a cycle E_{n+1} , with the vertices in G_n encircled, and the edges in $T_{n+1} \setminus T_n$ thickened.



Then T_{n+1} is an index-0 spanning rooted tree with root v_1 in G_{n+1} .

It remains to see why $p \notin T_z = T$. For this note that $E = E_z \ni p$ is added last, and $\text{val}_G(v_0), \text{val}_G(v_1) \geq 4$, so that $v_0, v_1 \in G_{z-1}$. \square

Corollary 5. *If D is an almost positive diagram with negative crossing p such that there is no (positive) crossing q joining the same two Seifert circles as p , then $\min \deg \Delta_D(t) = 1 - \chi(D)$, where Δ is normalized so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$. In particular, the canonical Seifert surface associated to D is of minimal genus.*

Proof. Apply the skein relation for Δ at the negative crossing to obtain the result for special diagrams. Then use the multiplicativity of $[\Delta(D)]_{(1-\chi(D))/2}$ under Murasugi sum of diagrams [Mu2] to obtain the general case. \square

This corollary improves the result of Hirasawa [Hi, Theorem 2.1] stating that this Seifert surface is incompressible.

Now we have all the preparations together to obtain the extension of Fiedler’s result.

Theorem 5. *If L is an almost positive link, then*

$$\min \deg V_L = \frac{1}{2}(1 - \chi(L)) \quad \text{and} \quad \min \text{cf } V_L = (-1)^{n(L)-1}.$$

Proof. Let D be an almost positive diagram of D with negative crossing p and canonical Seifert surface S . One can easily reduce the proof to the situation that D is connected. We then distinguish two cases.

(a) There is a (positive) crossing q joining the same two Seifert circles as p . By Theorem 2 we must show that

$$\frac{1 - \chi(L)}{2} = \frac{1 - \chi(D)}{2} - 1.$$

Clearly, $(1-\chi(L))/2 \leq (1-\chi(D))/2$, and by Bennequin’s inequality $(1-\chi(L))/2 \geq (1-\chi(D))/2 - 1$. Thus assume that $(1-\chi(L))/2 = (1-\chi(D))/2$, i.e. S is a minimal genus surface. By [Ga2], this is true for the Murasugi summand of S , which is the canonical Seifert surface associated to an almost positive (or almost alternating) special diagram. However, by assumption this surface is clearly not of minimal genus, a contradiction.

(b) There is no such crossing q . Then we must show by Corollary 4 that $(1-\chi(L))/2 = (1-\chi(D))/2$, i.e. S is a minimal genus surface. This follows again from [Ga1], using Corollary 5. \square

4.4. Skein polynomial and Morton's inequalities

The results on the Jones polynomial and their proofs allow also some applications to the skein polynomial [H]. First, we can identify two more coefficients of the polynomial of some positive links.

Corollary 6. *If L is a fibered positive link of $n(L)$ components, then*

$$[P_L]_{l^{1-\chi(L)}m^{-1-\chi(L)}} = (-1)^{n(L)}(1 - \chi(L)),$$

and if L is prime and braid positive, then

$$[P_L]_{l^{1-\chi(L)}m^{-3-\chi(L)}} = (-1)^{n(L)-1} \chi(L) \frac{\chi(L) + 1}{2}.$$

Proof. Murasugi and Przytycki showed in [MP] that $[P_D]_{m^{1-\chi(D)}}$ is multiplicative under Murasugi sum. (That

$$\max \deg_m P_D \leq 1 - \chi(D) \tag{15}$$

was shown by Morton [Mo1].) Since any positive diagram of a fibered positive link decomposes as Murasugi sum of connected sums of $(2, \cdot)$ -torus links, we have, for any fibered positive link L ,

$$[P_L]_{m^{1-\chi(L)}} = l^{1-\chi(L)} \cdot (-1)^{n(L)-1}.$$

Now apply Corollary 1 and the conversion (3). □

Remark 5. A formula for the first of the coefficients in the corollary can be written for an arbitrary positive link using Theorem 3 instead of corollary 1.

The proof of Theorem 5 can also be applied for P .

Theorem 6. *If L is an almost positive link, then*

$$\max \deg_m P(L) = 1 - \chi(L). \tag{16}$$

Proof. Consider the two cases in the proof of Theorem 5.

If q shares its Seifert circles with another (positive) crossing in D , then one of the special Murasugi sum components D' of D can be reduced to a diagram D'' with $\chi(D'') > \chi(D)$. Thus by (15),

$$\max \deg_m P(D') = \max \deg_m P(D'') \leq 1 - \chi(D'') < 1 - \chi(D'),$$

so that $[P_{D'}]_{m^{1-\chi(D')}} = 0$. Then by [MP] the same holds for D . Since we know that

$$1 - \chi(L) = -1 - \chi(D) \tag{17}$$

from the proof of Theorem 5, the inequality

$$\max \deg_m P(L) \leq 1 - \chi(L) \tag{18}$$

follows. Now use the fact that, as a consequence of [LM, Proposition 21], for an arbitrary link L , $\min \deg_l P \leq \max \deg_m P$. From Morton's inequalities [Mo1] we then have, for an almost positive diagram D of L ,

$$-1 - \chi(D) = w(D) - s(D) + 1 \leq \min \deg_l P(D) \leq \max \deg_m P(L). \quad (19)$$

Now (19) and (17) show that the inequality (18) is an equality.

If q does not share its Seifert circles with another crossing in D , then combining [Mo1] and the argument in the proof of Theorem 5, we have

$$1 - \chi(D) = 2 \max \deg \Delta(D) \leq \max \deg_m P(D) \leq 1 - \chi(D) = 1 - \chi(L),$$

so we have the equality in (16). □

Further we have

Corollary 7. *If L is an almost positive link, then $\min \deg_l P(L) \leq 1 - \chi(L)$.*

Proof. Use again the above mentioned consequence of [LM, Proposition 21]. □

This is another special case of Morton's conjectured inequality [Mo2] (disproved now in [St3] for arbitrary links). There is, though, much experimental evidence that we have in fact equality in Morton's inequality.

Question 2. Is it true that for any almost positive link L , $\min \deg_l P(L) = 1 - \chi(L)$?

Note that in one case in the proof of (16), we did obtain this equality, namely when the almost positive diagram D is not of minimal genus. The latter property is understood to mean that the canonical Seifert surface does not realize the (Seifert) genus of L , i.e. $\chi(D) > \tilde{\chi}(L)$. Question 2 is thus related to the question: Does any almost positive link L have an almost positive diagram D which is not of minimal genus?

As we later found, the answer to this question is negative, and a counterexample is the knot $!12_{1930}$ (which nevertheless satisfies $\min \deg_l P = 1 - \chi$). It is displayed in Figure 8 of [St6] (and occurs also later in this paper as L_4 in the proof of Corollary 8). Besides its obvious two almost positive diagrams (considered also in the proof below), there are no other (reduced) ones. The proof of this fact will be presented elsewhere, as it requires, apart from some computation, several tools (developed in [St6, St2, St1]) that go beyond the scope of the present paper.

The opposite situation to the last question is not less interesting, in particular because positive diagrams are always genus-minimizing.

Question 3. Does any almost positive link L have an almost positive diagram D of minimal genus?

A positive answer to this question will show that Morton's inequality for $\tilde{\chi}(L)$,

$$\max \deg_m P(L) \leq 1 - \tilde{\chi}(L) \quad (20)$$

(which is a direct consequence of (15)) is sharp. It would not be a surprise, as knots with strict inequality are hard to find. So far two methods apply: unity root values of V [St1] and Gabai’s foliation algorithm [Ga3] to show that in fact $\max \deg_m P(L) < 1 - \chi(L)$ [St3]. The latter option seems unlikely to work for almost positive links, and the former requires considerable extension of the calculations. Out of the ≈ 4500 non-alternating prime knots K of ≤ 16 crossings with $\max \deg_m P(K) \leq 4$, in [St1] we obtained 28 such knots with $4 = \max \deg_m P(K) < 2\tilde{g}(K)$ using values of the Jones and Q polynomial at roots of unity (and one further undecided case). An easy check shows that none of these 28 knots is almost positive.

5. Almost positive diagrams with canonical fiber Seifert surfaces

The even valence graphs can be used to give a description of almost positive diagrams whose canonical Seifert surfaces are fiber surfaces. The restriction to canonical surfaces is suggestive, since in general establishing the fiber property of a link or a surface may be difficult, even though both algebraic and geometric methods are known. Our result is closely related to the result for almost alternating diagrams due to Goda–Hirasawa–Yamamoto [GHY]. Our main motivation here was in fact to use the present (and quite different) tools to extend and simplify the proof of their criterion. We succeed almost completely, with the exception that we cannot recover combinatorially the fact (see their Proposition 5.1) that instead of general Murasugi sum decomposability of the fiber into Hopf bands in part (i) of the theorem below we have in fact stronger plumbing decomposability. On the other hand, we show in part (iii) that the fiberedness condition for the Alexander polynomial is exact. Due to the copious ways to calculate the Alexander polynomial, this makes the fiberedness property even easier to detect than by the classification result (iv) for such diagrams. (Our version of this result is also more explicit than in the form given in [GHY].) In the next section we will give examples showing (together with the examples in [GHY]) that one cannot extend the result much further.

In the following Δ will be normalized so that $\Delta(t) = \Delta(t^{-1})$ and $\min \text{cf } \Delta > 0$.

Theorem 7. *Let D be a connected almost positive link diagram with canonical Seifert surface S . Then the following conditions are equivalent:*

- (i) *S decomposes under iterated Murasugi sum (not necessarily plumbing) completely into Hopf bands (of one full twist).*
- (ii) *S is a fiber surface.*
- (iii) *$2 \max \deg \Delta(D) = 1 - \chi(D)$ and $\min \text{cf } \Delta(D) = 1$.*
- (iv) *Decompose D along its separating Seifert circles (Seifert circles with non-empty interior and exterior) as Murasugi sum of special diagrams, and those special diagrams into prime factors. Then all these prime factors are special alternating diagrams of $(2, n)$ -torus links (with parallel orientations), except for one, which after reductions of the type*

$$\begin{array}{ccc}
 \text{Diagram 1} & \longrightarrow & \text{Diagram 2} \\
 \text{(crossing with arrows)} & & \text{(crossing with arrows)}
 \end{array} \tag{21}$$

becomes an almost positive special diagram of the following forms:

- (a) a special diagram whose (even valence) checkerboard graph G can be obtained as follows: take a chain of circles of positive edges



and attach to it from the outside a cell (cycle with empty exterior) with one negative edge which joins interior points of the two outermost loops in (22) (the negative edge corresponds to the crossing to be switched); see e.g. Figure 8;

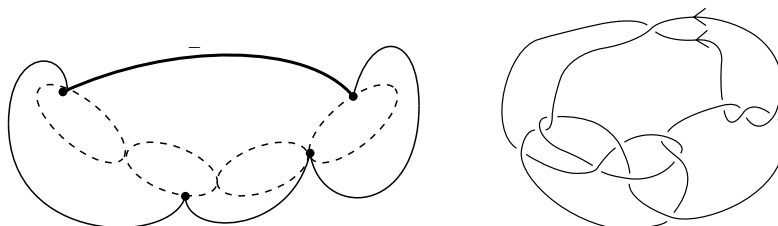


Fig. 8. A checkerboard graph and its diagram illustrating case (iv) of Theorem 7.

or (non-exclusively)

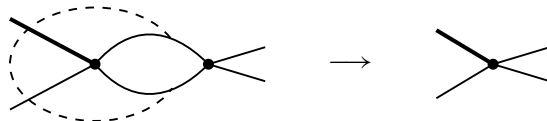
- (b) a diagram of a $(2, \dots, 2)$ -pretzel link (at least two 2's), oriented to be special, with one crossing changed.

Proof. In the following pictures, we assume graphs to be canonically oriented, but do not draw edge orientation if it is not necessary. The edge of the (only) crossing p of negative checkerboard sign will be distinguished by being drawn as a thickened or dashed line.

(iv) \Rightarrow (i). The reverse of the move in (21) preserves the property of the canonical Seifert surface to be a fiber, as it corresponds to plumbing of a Hopf band. That the canonical Seifert surfaces of the diagrams in (iv) are fibers is easy to see. (The diagram D' obtained from D by removing the trivial clasp is a connected sum of Hopf links, its surface is clearly a fiber, which is unique, and $\chi(D') = \chi(D)$.) For (iv) we remark that each of the graphs described turns into

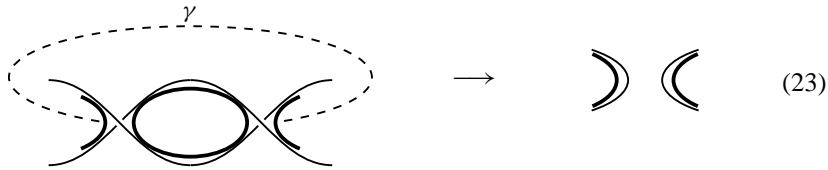


under repeating the operation



(contracting a double edge), with the dashed line having two properties: first, it is an arc passing through edges whose total sign sum is 0 (in our case the negative edge p and one

other, positive, edge), and second, it has a single vertex and either no complete edge in its interior or none in its exterior. This corresponds to the diagram move

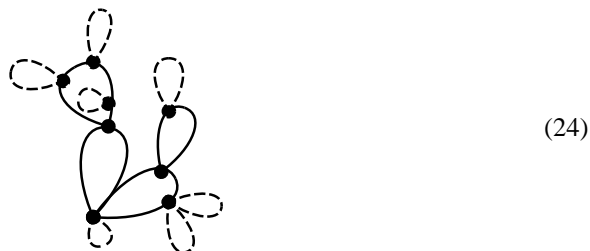


The dashed line γ can pass through the interior of Seifert circles and crossings only in such a way that the total writhe of these crossings is 0, and must have a single non-Seifert circle region, and either no complete Seifert circle in its interior or none in its exterior.

Then this is a Hopf plumbing, the Hopf band being obtained by thickening γ into a strip, and taking the union with the two half-twisted strips and one Seifert circle on the left of (23). (The first condition on γ is needed to ensure the correct twisting, while the second one is needed to have the Hopf band separated by a sphere from the rest of the surface after deplumbing.)

(i) \Rightarrow (ii) \Rightarrow (iii) are well known, so it remains to show the real result (iii) \Rightarrow (iv). As the minimal coefficient of the Alexander polynomial, when its degree is equal to $(1 - \chi)/2$, is multiplicative under Murasugi sum [Mu2], we need to consider only the almost alternating special Murasugi summand. For this we consider the canonically oriented even valence graph $G = G_D$ of D and recall the proof of Theorem 4. The condition $2 \min \deg \Delta = 1 - \chi$ implies that each edge of G bounds a cell not containing p , the edge in G of negative checkerboard sign. In particular, both v_0 and v_1 have valence ≥ 4 . Let E_1 and E_2 be the cells containing p . Then $E_1 \cap E_2 = \{p\}$, since p is in no 2-cut of G , and $G \setminus E_1$ and $G \setminus E_2$ are connected, by the argument in the proof of Theorem 4.

Now consider the condition $\min \text{cf } \Delta(D) = 1$. It means that there is only one index-0 spanning rooted tree with root v_1 not containing p . If $G \setminus E_1$ has two different index-0 spanning rooted trees with root v_1 , then by the construction in the proof of Theorem 4, we could extend them to index-0 spanning rooted trees of G with root v_1 not containing p , which would clearly still be different. Thus $G \setminus E_1$ has only one index-0 spanning rooted tree (with root v_1 or any other fixed vertex). Then, by part (5) of Theorem 3 of [MS], $G \setminus E_1$ is a join of chains (topologically, a bouquet of circles).



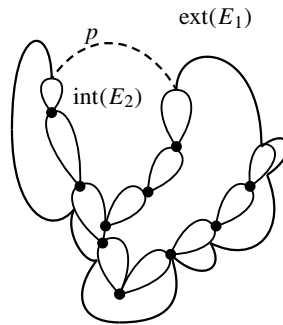
Assume without loss of generality that G is embedded so that the exterior $\text{ext}(E_1)$ of E_1 is the unbounded component. Since adding E_1 must remove all cut vertices (our diagram is prime by assumption), E_1 must touch interior points of all circles L_i with only one cut

vertex. (Here “interior” is meant to be different from the cut vertex.) We call these L_i leaf circles; in (24) they are drawn dashed.

Also, since the exterior of E_1 is the unbounded component, cut vertices coming from attaching circles inside other ones:



cannot be removed by adding E_1 , so assume there are no such inner circles. Thus we have a picture like

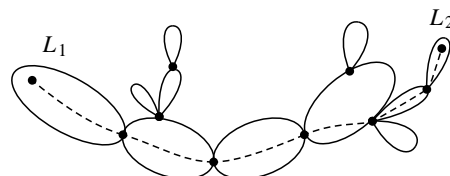


From now on, let us remove valence-2 vertices (we call this operation *unbisection*) and consider only the topological type of the tree

$$\text{---}\bullet\text{---} \quad \longrightarrow \quad \text{---} \quad (25)$$

This move on graphs corresponds to the reversed move (21). (Note that $\text{val}_G(v_{0,1}) \geq 4$ by assumption, so that both edges on the left of (25) are positively signed.)

The way between two leaf circles L_1 and L_2 is made up of those circles bounding disks whose interior is passed by a path from an interior point of the disk bounded by L_1 to an interior point of the disk bounded by L_2 . We require that this path passes only through interior points of disks bounded by loops and cut vertices, each such vertex being passed at most once.

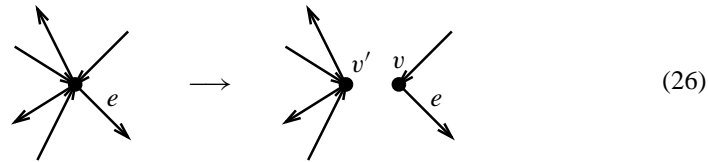


Now use the fact that $G \setminus E_2$ must also be a join of loops (or bouquet of circles). We claim that either

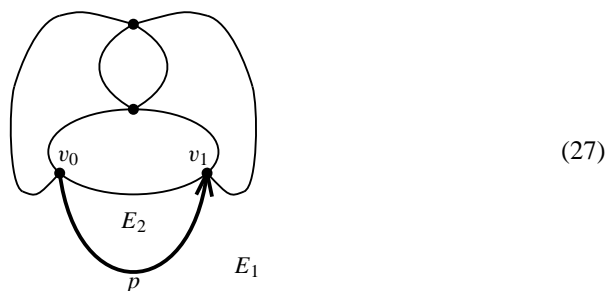
- (a) p touches interior points of two different leaf loops L_1, L_2 and all other vertices of E_1 touch only interior points or cut vertices belonging to circles *on the way* between L_1 and L_2 (as on the left of Figure 8), or

- (b) p touches two points on the same loop L (interior points or the cut vertex), and E_1 touches points only of the same loop.

Assume neither (a) nor (b) holds. We derive a contradiction showing that $G \setminus E_2$ has at least 2 arborescences. Observe that by unbisections (25) and separations



one can simplify G to obtain



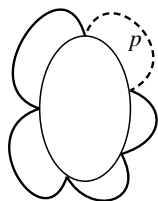
(still with E_1 being the boundary of the infinite region), in such a way that $v_{0,1}$ are not involved in any of these moves. Then removing E_2 and applying unbisections, one obtains a graph G_0 with two vertices and an edge of multiplicity 4, which has two arborescences. It suffices now to show that *bisections* (reverses of unbisections) and *deseparations* (reverses of separations (26)) do not reduce the number of arborescences. For this we specify how to map injectively arborescences of the original graph with root v_1 to arborescences of the resulting graph.

For a bisection creating a vertex $v \neq v_{0,1}$, add the outgoing edge of v to the arborescence, and do the same with the incoming one, if the original (bisected) edge was in the (original) arborescence. The same argument, but without the restriction $v \neq v_{0,1}$, finds two arborescences (with root v_1 or any other vertex) of (27), starting from those of G_0 .

For a deseperation at least one of the two vertices v, v' on the right of (26) has valence 2. Let v be such a vertex. The outgoing edge e of v is in any arborescence, since $v \neq v_{0,1}$. Remove e from the arborescence, and keep the status of the other edges, while joining v and v' .

Thus G is of type (a) or (b). In case (a) the assumption there are no cut vertices in G implies that $G \setminus E_1$ is as in (22), and p joins interior points on the two outermost circles. Note that the union of $L_{1,2}$ and all loops on the way between them forms a bouquet of type (22). Thus we arrive at case (iv, a) in Theorem 7.

In case (b), $G \setminus E_1$ must be a single loop, and we have a picture like this:



This is case (iv, b) in Theorem 7. □

What we have done allows solving an open problem in our previous work [St2] on almost positive knots and proving

Corollary 8. *There exist almost positive knots of any genus ≥ 2 .*

It was shown in [St1] that there are no almost positive knots of genus 1.

Proof. Consider the $(3, 3, \dots, 3, -1)$ -pretzel knots and links L_n . (These are 2-component links if the number n of 3's is odd; in this case we orient them so that the twists counted by the 3's are reverse.) As these diagrams of L_n come from the construction in part (iv) of Theorem 7, L_n are fibered (or see alternatively Theorem 6.7 of [Ga4]). Their diagrams reduce by one crossing to $D_n = (-2 - 1, 3, \dots, 3)$ (one 3 less), which are almost positive and of crossing number $3(1 - \chi(D_n)) = 3(1 - \chi(L_n)) = 3n$. If L_n were positive, by [Cr1, Corollary 5.1] they would have crossing number $c(L_n) \leq 2(1 - \chi)$.

To show that this is not the case, consider the crossing number inequality of [Ka, Mu1, Th1], $c(L_n) \geq \text{span } V(L_n)$. We know that $\min \deg V(L_n) = (1 - \chi)/2$. On the other hand, for $n = 1 - \chi > 2$, $\max \deg V(L_n)$ is easy to determine, as the diagram D_n is B -semiadequate, and thus only the contribution of the B -state S_B (specified by $\langle S_B : k \rangle = B$ for any crossing k) is relevant in (14). By a simple count of the loops one arrives at $\max \deg V(L_n) = \frac{7}{2}(1 - \chi) - 2$, and thus

$$c(L_n) \geq \text{span } V(L_n) = \max \deg V(L_n) - \min \deg V(L_n) = 3(1 - \chi) - 2 > 2(1 - \chi).$$

Thus L_n is almost positive for $n \geq 3$. (For $n = 1$ and $n = 2$ one obtains the Hopf link and trefoil, resp.) □

6. Some examples and problems

6.1. Showing almost positivity

The problem to show that a certain link is almost positive, but not positive, turned out to be very hard. All previously known positivity criteria are either easily provable to extend to almost positive links, or at least no examples are known where they do not. Theorem 5 is an addition to that picture.

In [St2] it was shown, in the case of knots, that any almost positive knot has only finitely many reduced almost positive diagrams. As the proof is constructive, one can, in theory, decide (for knots) about almost positivity, in the sense that for any knot one can write down a finite set of almost positive diagrams, among which one would have to check whether the knot occurs. However, this method is not generally very efficient, except for a few knots of small genus.

Cromwell's estimate $c \leq 2(1 - \chi)$ for fibered homogeneous links remains the only way known so far to circumvent these problems, at least in certain cases. Using Theorem 7, we can now construct plenty of examples of almost positive fibered link diagrams, which we can show to represent almost positive links by proving that Cromwell's inequality is violated.

However, this inequality will still not be violated in many cases, and thus one may ask whether it can be improved. Cromwell's estimate is trivially sharp for alternating (prime) links (consider the rational links $2/2 \dots 2$) and composite positive links (consider the connected sums of Hopf links). However, even for prime positive links the inequality cannot be improved much.

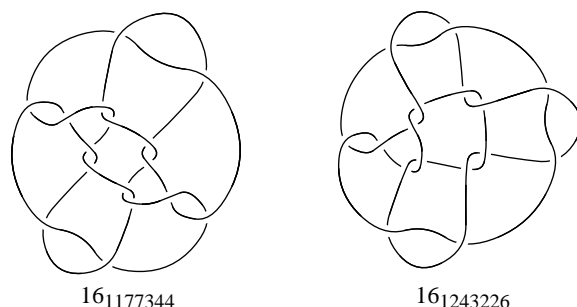


Fig. 9. The two fibered positive knots of genus 4 and crossing number 16. (The diagrams here are chosen to be positive and reveal a plumbing structure of the fiber surfaces. $16_{1243226}$ also has almost positive 16-crossing diagrams, and $16_{1177344}$ even 2-almost positive ones.)

Example 8. The $(2, 2, \dots, 2, -2, -2)$ -pretzel link, oriented so that the clasps counted by 2 are reverse, and those counted by -2 are parallel, has $c = 2(1 - \chi) - 2$. The diagram is of minimal crossing number as follows by considering linking numbers, and decomposes under Murasugi sum into connected sums of Hopf bands, thus the link is fibered. (The link is also prime by [KL].)

Example 9. Even just considering knots, there exist examples of genus 4 and crossing number 16, $16_{1177344}$ and $16_{1243226}$. (For genus 3 the maximal crossing number example is the knot 11_{550} of [St3] without a minimal positive diagram.) Apparently these examples can be generalized to higher genera (although the proof of minimality of the crossing number is not straightforward).

Thus Cromwell's estimate seems rather sharp even in our case.

A different problem in this context is the position of the class of almost positive links with respect to the hierarchy (1).

Question 4. Is any almost positive link strongly quasipositive, or at least quasipositive?

Some 2-almost positive links, like the figure-8-knot, are not quasipositive. On the other hand, all almost positive examples examined so far are strongly quasipositive.

6.2. Detecting genus and fiberedness with the Alexander polynomial

From our results in the previous two sections, we have the following

Corollary 9. *If a link L has a connected almost positive (or almost alternating) diagram (with canonical Seifert surface) of minimal genus, then*

- (a) $2 \max \deg \Delta_L = 1 - \chi(L)$, and
- (b) L is fibered if and only if $\min \text{cf } \Delta_L = \pm 1$. □

Unfortunately, we cannot decide about fiberedness if the almost positive diagram is not of minimal genus. Many almost positive knots seem to have almost positive minimal genus diagrams, but whether all have is unclear. Coming back to the inequality $g(K) \leq \tilde{g}(K)$ in Question 3, it is known that almost alternating knots may fail to realize it sharply. One of the two $\Delta = 1$ knots of 11 crossings has genus two [Ga3], and is almost alternating by the verification in [Ad1, Ad2], while the calculation in [LM, Example 11.1] gives $\max \deg_m P = 6$, so that by (20), $\tilde{g} = 3$. (A genus three canonical surface is not too hard to find.) This knot thus does not have any diagram whatsoever of minimal genus. Let us mention in contrast that among the 28 knots we found with strict Morton inequality $4 = \max \deg_m P(K) < 2\tilde{g}(K)$, none could be identified as almost alternating (although there are not enough tools to exclude it). However, there are several ≤ 2 -almost alternating knots, for example 15_{130745} (see Figure 9 of [St1]).

For almost positivity the problem to find knots with $\tilde{g} > g$ seems much harder than for almost alternating.

Question 5. What is the minimal k with a k -almost positive knot having no diagram of minimal genus?

So far it seems likely that such knots with $k = 4$ occur, but even whether $k \leq 3$ is unclear. In contrast, there is a 2-almost positive knot, $16_{1337674}$, with strict inequality (20).

Note that both statements in Corollary 9 are true for many (other) links, in particular for all knots in Rolfsen's tables [Ro, appendix]. However, the following examples show that the corollary does not extend much further.

Example 10. Consider the diagram in the middle of Figure 10. It is another diagram of the previously encountered knot 13_{6374} with unit Alexander polynomial. It is 2-almost alternating, and its canonical Seifert surface is of minimal genus (two), as can be shown by [Ga3]. Thus neither of the two criteria hold for 2-almost alternating diagrams.

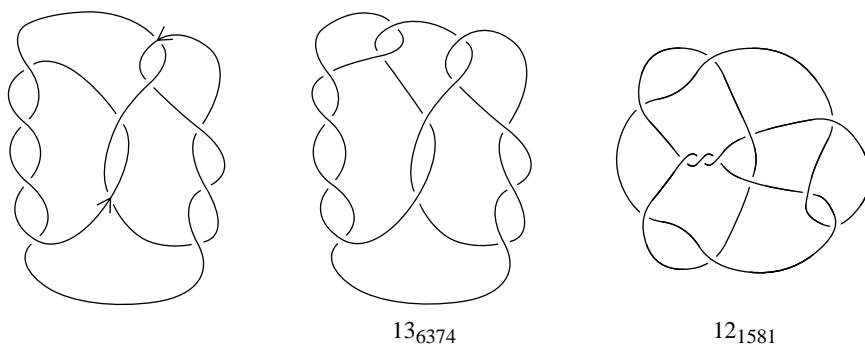


Fig. 10

Example 11. The diagram on the right of Figure 10 depicts the knot 12_{1581} with Alexander polynomial $\Delta = (2[-3]2)$. It is a (special) 2-almost positive diagram whose canonical Seifert surface is of minimal genus (again two). Thus criterion (a) in Corollary 9 is not valid for 2-almost positive diagrams.

So far I have no example of a 2-almost positive *knot* diagram for criterion (b), but one can easily obtain a link diagram.

Example 12. Consider the diagram of 13_{6374} in Example 10. It has a single separating Seifert circle, whose interior contains two crossings. By removing this interior (deplumbing a Hopf band), one arrives at the link diagram on the left of Figure 10. Its canonical Seifert surface is still of minimal genus by [Ga2], so that $1 - \chi = 3$, but one calculates that $\Delta = t^{1/2} - t^{-1/2}$.

In all the above examples we showed a surface not to be a fiber by proving that the Alexander polynomial has too small a degree. There are also examples where the degree is maximal, and thus all conditions in Corollary 9 taken together still do not suffice to determine a fiber.

Example 13. The $(-2, 4, 6)$ -pretzel link diagram, oriented to be special (all clasps reverse), has $\max \deg \Delta = 1 - \chi = 2$ and $\min \text{cf } \Delta = 1$. That its canonical surface is not a fiber follows from [Ga4, Theorem 6.7] (Case 1). Using properly signed Hopf plumbings, one can generate from it many more examples of 2-almost alternating and/or 2-almost positive diagrams, in particular (diagrams of) several genus two knots. Two such knots (for 2-almost positive diagrams) are the mirror images of 15_{120617} and 15_{159580} , displayed in Figure 11. (These two knots have in fact been found first, by a check in the tables, and the pretzel link was obtained from them by deplumbings.)

Remark 6. For genus one canonical surfaces of knots one needs 3-almost positive (and 3-almost alternating) diagrams to have such a situation, the simplest example being the $(-3, 5, 5)$ -pretzel knot [CT]. (It has the Alexander polynomial of the figure-8-knot.) For 4-almost positive (or 4-almost alternating) diagrams, even worse, one can use [Ga4, The-

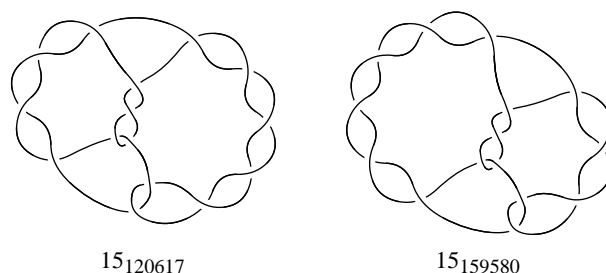


Fig. 11

orem 6.7] to construct diagrams differing by mutation, with the canonical surface of one being a fiber, and of the other not.

Acknowledgements. I wish to thank M. Khovanov for helpful remarks and discussions, in particular for suggesting Theorem 1 and, implicitly, Question 1. The idea to distinguish in the study of almost positive diagrams between having another crossing joining the same pair of Seifert circles as the negative one and having no such crossing appears previously in Hirasawa's paper [Hi], and was observed even before that in unpublished work of K. Taniyama.

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