J. Eur. Math. Soc. 8, 123–138 

M. Belloni · B. Kawohl · P. Juutinen

# The $p$-Laplace eigenvalue problem as $p \to \infty$ in a Finsler metric

Received July 10, 2004 and in revised form April 3, 2005

**Abstract.** We consider the $p$-Laplacian operator on a domain equipped with a Finsler metric. We recall relevant properties of its first eigenfunction for finite $p$ and investigate the limit problem as $p \to \infty$.

**Keywords.** $p$-Laplace, eigenfunction, Finsler metric

## 1. Introduction

Imagine a nonlinear elastic membrane, fixed on a boundary $\partial\Omega$ of a plane domain $\Omega$. If $u(x)$ denotes its vertical displacement, and if its deformation energy is given by $\int_\Omega |\nabla u|^p\, dx$, then a minimizer of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega |u|^p\, dx}$$

on $W_0^{1,p}(\Omega)$ satisfies the Euler–Lagrange equation

$$-\Delta_p u = \lambda_p |u|^{p-2} u \quad \text{in } \Omega, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the well known $p$-Laplace operator. This eigenvalue problem has been extensively studied in the literature. A somewhat surprising recent result is that (as $p \to \infty$) the limit equation reads

$$\min\{|\nabla u| - \Lambda_\infty u,\ -\Delta_\infty u\} = 0. \tag{1.2}$$

Here $\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i x_j}$, $\Lambda_\infty = \lim_{p\to\infty} \Lambda_p$ and $\Lambda_p = \lambda_p^{1/p}$ (see [18, 13]). Although the function $\operatorname{dist}(x, \partial\Omega)$ minimizes $\|\nabla u\|_\infty / \|u\|_\infty$, it is not always a viscosity solution of (1.2) (see [18]).

M. Belloni: Dipartimento di Matematica, Università di Parma, Parco Area delle Scienze 53/A, I-43100 Parma, Italy; e-mail: marino.belloni@unipr.it

B. Kawohl: Mathematisches Institut, Universität zu Köln, D-50923 Köln, Germany; e-mail: kawohl@math.uni-koeln.de

P. Juutinen: Department of Mathematics and Statistics, P.O. Box 35, FIN-40014 University of Jyväskylä, Finland; e-mail: peanju@maths.jyu.fi

*Mathematics Subject Classification (2000):* 35J60, 49R50, 35B40

Now suppose that the membrane is not isotropic. It is for instance woven out of elastic strings like a piece of material. Then the deformation energy can be anisotropic (see [5]). Another way to describe this effect is by stating that the Euclidean distance in $\Omega$ is somehow distorted. It is the purpose of the present paper to generalize the result on eigenfunctions for the $p$-Laplacian to the situation where $\Omega \subset \mathbb{R}^n$ is no longer equipped with the Euclidean norm, but instead with a general norm $|\cdot|$, for instance with $|x| = (\sum_{i=1}^n |x_i|^q)^{1/q}$ and $q \in (1, \infty)$. In that case a Lipschitz continuous function $u : \Omega \to \mathbb{R}$ (in a convex domain $\Omega$) has Lipschitz constant $L = \sup_{z\in\Omega} |\nabla u(z)|^*$, where $|\cdot|^*$ denotes the dual norm to $|\cdot|$, because $|u(x) - u(y)| \leq L|x - y|$ with this $L$. In order to give a meaningful definition of viscosity solutions, we assume throughout the paper that the dual norm $H : \mathbb{R}^n \to [0, \infty)$ defined by $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$.

It is well known that the $\infty$-Laplacian operator $\Delta_\infty$ is closely related to finding a minimal Lipschitz extension of a given function $\phi \in C^{0,1}(\partial\Omega)$ into $\Omega$. In [2] this result on minimal Lipschitz extensions was generalized from the Euclidean to a general norm (see also [25]). In [6] the eigenvalue problem was carried over to a general norm and studied for finite $p$, while in [5] the eigenvalue problem was investigated first for finite $p$ and the special non-Euclidean norm $|x| = (\sum_{i=1}^n |x_i|^{p'})^{1/p'}$ with $p'$ conjugate to $p$, and then for the limit $p \to \infty$.

Moreover, the $\infty$-Laplacian operator plays an important role in problems of optimal transportation. For technical reasons it is often approximated by $p$-Laplacians with large $p$ (see for instance [12], [8]).

Our paper is organized as follows. In Section 2 we recall the existence, uniqueness and regularity of weak and viscosity solutions for finite $p$. In Section 3 we derive the limit equation for $p \to \infty$. In Section 4 we provide some instructive examples.

## 2. Existence, uniqueness and regularity of solutions

If we minimize the functional

$$I_p(v) = \int_\Omega \left(|\nabla u|^*\right)^p \, dx \quad \text{on} \quad K := \{v \in W_0^{1,p}(\Omega) : \|v\|_{L^p(\Omega)} = 1\}, \tag{2.1}$$

then via standard arguments (see [6]) a minimizer $u_p$ exists for every $p > 1$ and it is a weak solution to the equation

$$-Q_p u := -\operatorname{div}((|\nabla u_p|^*)^{p-2} J(\nabla u_p)) = \lambda_p |u_p|^{p-2} u_p, \tag{2.2}$$

that is,

$$\int_\Omega (|\nabla u_p|^*)^{p-2} \langle J(\nabla u_p), \nabla v\rangle \, dx = \lambda_p \int_\Omega |u_p|^{p-2} u \cdot v \, dx \tag{2.3}$$

for any $v \in W_0^{1,p}(\Omega)$. Here $\lambda_p = I_p(u_p)$ and

$$J_i(\xi) := \frac{\partial}{\partial \xi_i}\left(\frac{(|\xi|^*)^2}{2}\right). \tag{2.4}$$

Clearly (2.4) is well defined if the dual norm $H(\eta) = |\eta|^*$ is of class $C^1(\mathbb{R}^n \setminus \{0\})$. Recall that (2.4) is well defined (and single-valued) if and only if the norm $|\cdot|$ is strictly convex, i.e. if its unit sphere $\{x : |x| = 1\}$ contains no nontrivial line segments (see [26, p. 400]). Note further that in this case $J(0) = 0$ and that for the Euclidean norm the duality map reduces to the identity $J(\nabla u) = \nabla u$. Note finally that $\Lambda_p := \lambda_p^{1/p}$ is the minimum of the Rayleigh quotient

$$R_p(v) := \frac{(\int_\Omega (|\nabla v|^*)^p \, dx)^{1/p}}{\|v\|_p} \tag{2.5}$$

on $W_0^{1,p}(\Omega) \setminus \{0\}$. Without loss of generality we may assume that $u_p$ is nonnegative. Otherwise we can replace it by its modulus.

Moreover as shown in [6] any nonnegative weak solution of (2.3) is necessarily bounded and positive in $\Omega$. If $p > n$, then $u_p$ is Hölder continuous because of the Sobolev embedding theorem and the equivalence of the usual Sobolev norm and

$$\|u\|_{1,p} := \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p} + \left(\int_\Omega (|\nabla u(x)|^*)^p \, dx\right)^{1/p}. \tag{2.6}$$

But even for general $p \geq 2$, one can show its $C^{1,\alpha}$ regularity as in [6]. For the reader's convenience let us briefly repeat the arguments. The function $u_p$ minimizes $I_p$ in (2.1) and the theory for quasiminima in [14] implies that minimizers of $I_p$ are bounded ([14, Thm. 7.5]), Hölder continuous ([14, Thm. 7.6]), and satisfy a strong maximum principle ([14, Thm. 7.12]). Therefore $u_p$ is positive. Once positivity is known, the uniqueness follows from a simple convexity argument (see [4] or [6]). Moreover $u_p \in C^{1,\alpha}(\Omega)$ according to [23], [24] or [11]. Let us summarize these statements.

**Theorem 2.1.** *Suppose that $H(\eta) = |\eta|^*$ is of class $C^1(\mathbb{R}^n \setminus \{0\})$ or that the norm $|\cdot|$ is strictly convex. Then for every $p \in [2, \infty)$, the nonnegative minimizer $u_p$ of* (2.1) *is unique, positive and of class $C^{1,\alpha}$. It solves* (2.2) *in the weak sense of* (2.3).

The next item will be viscosity solutions. As in [18] and [5] we plan to show that every weak solution is a viscosity solution. For every $z \in \mathbb{R}$, $q \in \mathbb{R}^n$ and for every real symmetric $n \times n$ matrix $X$ we consider the function

$$\begin{aligned}\tilde{F}_p(z, \xi, X) = &-(p-2)(|\xi|^*)^{p-4}\langle XJ(\xi), J(\xi)\rangle \\ &-(|\xi|^*)^{p-2} X \otimes DJ(\xi) - \lambda_p |z|^{p-2} z,\end{aligned}$$

where $X \otimes DJ(\xi)$ is shorthand for $\sum_{i,j=1}^n X_{ij} \frac{\partial J_i}{\partial \xi_j}(\xi)$. Now $(|\xi|^*)^2/2$ is convex and homogeneous of degree 2 and its first derivative $J(\xi)$ is homogeneous of degree 1. Therefore its second derivative $DJ(\xi)$ exists almost everywhere and is essentially bounded. If we assume that $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$, then $DJ$ is well defined and continuous outside the origin, so that $\tilde{F}_p$ is well defined and continuous for $\xi \neq 0$. To define $F_p$ at $\xi = 0$ we use the homogeneity of the norm $|\cdot|^*$ and see that for any $t > 0$ and $\xi \neq 0$,

$$J(t\xi) = tJ(\xi) \quad \text{implies} \quad DJ(\xi) = DJ\,(t\xi)\,.$$

So if we assume that the dual norm is of class $C^2$ outside the origin, then one easily sees that for $p > 2$ the function

$$\tilde{F}_p = -(|\xi|^*)^{p-2}\left[(p-2)\left\langle XJ\left(\frac{\xi}{|\xi|^*}\right), J\left(\frac{\xi}{|\xi|^*}\right)\right\rangle + X \otimes DJ(\xi)\right] \\ -\lambda_p |z|^{p-2} z \tag{2.7}$$

has a continuous extension to $\xi = 0$. So now we can define

$$F_p(z, \xi, X) := \begin{cases} \tilde{F}_p(z, \xi, X) & \text{if } \xi \neq 0, \\ -\lambda_p |z|^{p-2} z & \text{if } \xi = 0, \end{cases} \tag{2.8}$$

and the upper and lower semicontinuous envelopes $F_p^*$ and $F_{p*}$ of $F_p$ coincide with $F_p$ for $p > 2$. Notice that the case $p = 2$ is more delicate, because $\tilde{F}_2(z, \xi, X) = X \otimes DJ(\xi) - \lambda_2 z$ is not continuous at $\xi = 0$. This problem was overcome in [22] for $p \in (1, 2)$ by multiplying $F_p$ with $|\nabla u|$ and by studying the modified differential equation, but since we are interested in the limit $p \to \infty$ we do not investigate the range $p \in (1, 2]$ any further.

**Definition 2.2.** *Let $F_p$ be as in* (2.8)*. We call $u \in C(\Omega)$ a* viscosity subsolution *(resp.* supersolution*) of $F_p = 0$ if*

$$F_p(\phi(x), D\phi(x), D^2\phi(x)) \leq 0 \quad (\textit{resp. } F_p(\phi(x), D\phi(x), D^2\phi(x)) \geq 0) \tag{2.9}$$

*for every $\phi \in C^2(\Omega)$ with $u - \phi$ attaining a local maximum (resp. minimum) zero at $x$. We call $u$ a* viscosity solution *of $F_p = 0$ if it is both a viscosity subsolution and a viscosity supersolution.*

**Lemma 2.3.** *Suppose that $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. Then for $p > 2$ every (weak) solution of* (2.3) *is a viscosity solution of $F_p = 0$ with $F_p$ given by* (2.8)*.*

*Proof.* We omit the subscript $p$ on $u_p$ and check first if $u$ is a viscosity subsolution. Without loss of generality fix $x_0 \in \Omega$ and choose $\phi \in C^2(\Omega)$ such that $u(x_0) = \phi(x_0)$ and $u(x) < \phi(x)$ for $x \neq x_0$. We want to show that

$$-(p-2)(|\nabla\phi(x_0)|^*)^{p-4}\langle D^2\phi(x_0) J(\nabla\phi(x_0)), J(\nabla\phi(x_0))\rangle \\ - (|\nabla\phi(x_0)|^*)^{p-2} D^2\phi(x_0) \otimes DJ(\nabla\phi(x_0)) - \lambda_p |\phi(x_0)|^{p-2}\phi(x_0) \leq 0 \tag{2.10}$$

and argue by contradiction. Otherwise there exists a small ball $B_r(x_0)$ in which (2.10) is violated. Set $M = \sup\{\phi(x) - u(x) : x \in \partial B_r(x_0)\}$ and $\Phi = \phi - M/2$. Then $\Phi > u$ on $\partial B_r(x_0)$, $\Phi(x_0) < u(x_0)$ and

$$-(p-2)(|\nabla\Phi|^*)^{p-4}\langle D^2\Phi J(\nabla\Phi), J(\nabla\Phi)\rangle \\ - (|\nabla\Phi|^*)^{p-2} D^2\Phi \otimes DJ(\nabla\Phi) > \lambda_p\, |\phi|^{p-2}\, \phi \quad \text{in } B_r(x_0). \tag{2.11}$$

If we multiply (2.11) by $(u-\Phi)^+$ and integrate by parts, we obtain

$$\int_{\{u>\Phi\}} (|\nabla\Phi|^*)^{p-2}\langle J(\nabla\Phi), \nabla(u-\Phi)\rangle\, dx > \lambda_p \int_{\{u>\Phi\}} |\phi|^{p-2}\phi(u-\Phi)\, dx. \quad (2.12)$$

Now we exploit the fact that $u$ is a weak solution of (2.3) and pick $v=(u-\Phi)^+$, extended by zero outside $B_r(x_0)$, as a test function in (2.3). Then

$$\int_{\{u>\Phi\}} (|\nabla u|^*)^{p-2}\langle J(\nabla u), \nabla(u-\Phi)\rangle\, dx = \lambda_p \int_{\{u>\Phi\}} |u|^{p-2}u(u-\Phi)\, dx. \quad (2.13)$$

Subtracting (2.12) from (2.13) we obtain

$$\int_{\{u>\Phi\}} \langle[(|\nabla u|^*)^{p-2}J(\nabla u) - (|\nabla\Phi|^*)^{p-2}J(\nabla\Phi)], \nabla(u-\Phi)\rangle\, dx$$
$$< \lambda_p \int_{\{u>\Phi\}} (|u|^{p-2}u - |\phi|^{p-2}\phi)(u-\Phi)\, dx. \quad (2.14)$$

But the right hand side of (2.14) is nonpositive, while the left hand side is nonnegative because the functional $\int(|\nabla v|^*)^p\, dx$ is convex in $v$. So $u(x_0) \le \Phi(x_0)$, a contradiction to $\Phi(x_0) < u(x_0)$. This proves that $u$ is a viscosity subsolution. The proof that $u$ is also a viscosity supersolution is left to the reader.

Note that, as a byproduct of this proof, there are no admissible test functions $\phi$ that touch $u_p$ at a critical point from below. This shows that $u_p$ is not of class $C^2$.

## 3. The limit eigenvalue equation for $p \to \infty$

In this section we study the sequence $(\Lambda_p, u_p)$ of eigenvalues and normalized eigenfunctions as $p \to \infty$. In particular we will derive the equation which is satisfied by the cluster points $u_\infty$ of $u_p$. Consider a bounded domain $\Omega \subset \mathbb{R}^n$. The distance function to the boundary $\delta(x) := \inf_{y\in\partial\Omega} |x-y|$ is Lipschitz continuous, satisfies $|\nabla\delta(x)|^* = 1$ almost everywhere in $\Omega$ and it is equal to zero on the boundary of $\Omega$. For every $\varphi \in W_0^{1,\infty}(\Omega)$ and $y \in \partial\Omega$ we then have

$$|\varphi(x)| = |\varphi(x) - \varphi(y)| \le \|\,|\nabla\varphi|^*\|_\infty \delta(x),$$

which implies

$$\frac{1}{\|\delta\|_\infty} \le \frac{\|\,|\nabla\varphi|^*\|_\infty}{\|\varphi\|_\infty}. \quad (3.1)$$

Now define

$$\Lambda_\infty := \frac{\|\,|\nabla\delta|^*\|_\infty}{\|\delta\|_\infty} \left(= \frac{1}{\|\delta\|_\infty}\right). \quad (3.2)$$

Then $\Lambda_\infty$ is a geometric quantity related to $\Omega$. It is the inverse of the radius of the largest (in general non-Euclidean) ball inside $\Omega$. We can now prove the following lemma, which explains the analytic meaning of $\Lambda_\infty$.

**Lemma 3.1.** *The following limit holds:*

$$(\lim_{p\to\infty} \lambda_p^{1/p} =) \lim_{p\to\infty} \Lambda_p = \Lambda_\infty.$$

*Here $\Lambda_p = R_p(u_p)$ and the Rayleigh quotient $R_p$ is given by* (2.5).

*Proof.* From the definition of the Rayleigh quotient and $\delta(x)$ we get

$$\Lambda_p \le \frac{|\Omega|^{1/p}}{\|\delta\|_p},$$

which implies

$$\limsup_{p\to\infty} \Lambda_p \le \Lambda_\infty.$$

In order to obtain the opposite inequality, we observe that $\|\nabla u_p\|_p \le C < \infty$ uniformly in $p$, because $\delta(x)$ can be used as a test function in any of the Rayleigh quotients. But then (see also [7] and [18]) Hölder's inequality allows us to conclude that $\|\nabla u_p\|_m \le C < \infty$ for $p > m > n$. We can thus select a subsequence (still denoted by $\{u_p\}$) converging strongly in $C^\alpha$ and weakly in $W^{1,m}$ to a cluster point $u_\infty$ of the original sequence. Without loss of generality we may assume that each $u_p$ has $L^\infty$ norm 1. Then by the convergence in $C^\alpha$, $\lim u_p = u_\infty$ has $L^\infty$ norm 1 and positive $L^m$ norm. From the lower semicontinuity of the Rayleigh quotient we now get

$$\frac{(\int_\Omega (|\nabla u_\infty|^*)^m\, dx)^{1/m}}{\|u_\infty\|_m} \le \liminf_{p\to\infty} \frac{(\int_\Omega (|\nabla u_p|^*)^m\, dx)^{1/m}}{\|u_p\|_m}.$$

Multiplying and dividing the last inequality by $\|u_p\|_p$, by Hölder's inequality for $p > m$ we get

$$\frac{(\int_\Omega (|\nabla u_\infty|^*)^m\, dx)^{1/m}}{\|u_\infty\|_m} \le \liminf_{p\to\infty}\left( \Lambda_p \frac{\|u_p\|_p}{\|u_p\|_m} |\Omega|^{(p-m)/pm}\right).$$

By taking first the limit over $p$ and next over $m$ and using (3.1) we conclude that $\Lambda_\infty \le \liminf_{p\to\infty} \Lambda_p$, which completes the proof of the lemma.

Before we derive the limit equation which a nontrivial cluster point $u_\infty$ of the sequence $u_p$ must satisfy, let us show that $u_\infty$ is positive in $\Omega$. The functions $u_p$ are viscosity supersolutions of $H_p(\nabla u, D^2 u) = 0$, where

$$H_p(\xi, X) := -\langle XJ(\xi), J(\xi)\rangle - \frac{(|\xi|^*)^2}{p-2} X \otimes DJ(\xi)$$

is elliptic and continuous for $p > 2$ by assumption. Therefore by a well known stability theorem [9] supersolutions converge to a supersolution of the limiting problem, i.e. to a supersolution $u_\infty$ of the equation

$$H_\infty(\xi, X) = -\langle XJ(\xi), J(\xi)\rangle = 0$$

in the viscosity sense. As we saw above, $u_\infty \not\equiv 0$. Now the positivity of $u_\infty$ follows from a comparison result of Barles and Busca (see [3, Lemma 3.2]).

**Theorem 3.2.** *If $H(\eta) := |\eta|^*$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$ then every cluster point $u_\infty$ of the sequence $\{u_p\}$ is a viscosity solution of the equation*

$$F_\infty(u, \nabla u, D^2 u) = \min\{|\nabla u|^* - \Lambda_\infty u, -Q_\infty u\} = 0$$

*with $Q_\infty u = \langle D^2 u J(\nabla u), J(\nabla u)\rangle$ representing the $\infty$-Laplacian in the Finsler metric.*

*Proof.* We show first the result for viscosity supersolutions. We consider a subsequence $\{u_p\}$ converging uniformly in $\Omega$ to a function $u_\infty$. Fix a point $\xi \in \Omega$ and a function $\varphi \in C^2$ such that $u_\infty(\xi) = \varphi(\xi)$ and $u_\infty(x) > \varphi(x)$ for $x \neq \xi$. Also fix $B_{2R}(\xi) \subseteq \Omega$. If $0 < r < R$ we have

$$\inf\{u_\infty(x) - \varphi(x) : x \in B_R(\xi) \setminus B_r(\xi)\} > 0.$$

The sequence $\{u_p\}$ converges uniformly, so for sufficiently large $p$ we have

$$\inf\{u_p(x) - \varphi(x) : x \in B_R(\xi) \setminus B_r(\xi)\} > u_p(\xi) - \varphi(\xi).$$

For those $p$ we have

$$\inf\{u_p(x) - \varphi(x) : x \in B_R(\xi)\} = u_p(x_p) - \varphi(x_p)$$

with $x_p \in B_r(\xi)$, and obviously $x_p \to \xi$ as $p \to \infty$. The function $u_p$ is a viscosity solution of (2.2), therefore

$$\begin{aligned} -(p-2)(|\nabla\varphi(x_p)|^*)^{p-4}\langle D^2\varphi(x_p) J(\nabla\varphi(x_p)), J(\nabla\varphi(x_p))\rangle \\ - (|\nabla\varphi(x_p)|^*)^{p-2} D^2\varphi(x_p) \otimes DJ(\nabla\varphi(x_p)) \geq \Lambda_p^p |\varphi(x_p)|^{p-2}\varphi(x_p). \end{aligned} \tag{3.3}$$

Now $u_\infty(\xi) > 0$, but then also $\varphi(x_p) > 0$ for sufficiently large $p$ and by (3.3), $\nabla\varphi(x_p) \neq 0$ for large $p$. Dividing both members of (3.3) by $(p-2)(|\nabla\varphi(x_p)|^*)^{p-4}$ we obtain

$$\begin{aligned} -\langle D^2\varphi(x_p) J(\nabla\varphi(x_p)), J(\nabla\varphi(x_p))\rangle - \frac{(|\nabla\varphi(x_p)|^*)^2}{p-2} D^2\varphi(x_p) \otimes DJ(\nabla\varphi(x_p)) \\ \geq \frac{\Lambda_p^4 |\varphi(x_p)|^3}{p-2}\left(\frac{|\varphi(x_p)|\Lambda_p}{|\nabla\varphi(x_p)|^*}\right)^{p-4}. \end{aligned} \tag{3.4}$$

Letting $p \to \infty$ in (3.4), we obtain the necessary condition

$$\frac{\Lambda_\infty \varphi(\xi)}{|\nabla\varphi(\xi)|^*} \leq 1, \tag{3.5}$$

and taking into account (3.5) and letting $p \to \infty$ in (3.4) we obtain

$$-Q_\infty\varphi(\xi) = -\langle D^2\varphi(\xi) J(\nabla\varphi(\xi)), J(\nabla\varphi(\xi))\rangle \geq 0. \tag{3.6}$$

Inequalities (3.5) and (3.6) must hold together, and therefore the cluster points $u_\infty$ of the sequence $u_p$ must satisfy, in the viscosity sense, the equation

$$\min\{|\nabla u(\xi)|^* - \Lambda_\infty u(\xi), -Q_\infty u(\xi)\} \geq 0. \tag{3.7}$$

This shows that $u_\infty$ is a viscosity supersolution of

$$F_\infty(u, \nabla u, D^2u) = \min\{|\nabla u|^* - \Lambda_\infty u,\ -Q_\infty u\} = 0.$$

Let us run the proof for subsolutions. Fix a point $\xi \in \Omega$ and a function $\varphi \in C^2$ such that $u_\infty(\xi) = \varphi(\xi)$ and $u_\infty(x) < \varphi(x)$ for $x \neq \xi$. We have to show that

$$\min\{|\nabla u(\xi)|^* - \Lambda_\infty u(\xi),\ -Q_\infty u(\xi)\} \leq 0.$$

Clearly if $|\nabla u(\xi)|^* - \Lambda_\infty u(\xi) \leq 0$, then there is nothing to prove. Therefore we assume $|\nabla u(\xi)|^* - \Lambda_\infty u(\xi) > 0$, i.e.

$$\frac{\varphi(\xi)\Lambda_\infty}{|\nabla\varphi(\xi)|^*} < 1 - \varepsilon. \tag{3.8}$$

By continuity, this inequality remains true (for every sufficiently large $p$) if $\Lambda_\infty$ is replaced by $\Lambda_p$ and $\xi$ by $x_p$, and $x_p$ is now the maximum point of $u_p(x) - \varphi(x)$. As in the supersolution case, repeating step by step the proof but reversing the inequality between the left and right member, we get

$$\begin{aligned} -\langle D^2\varphi(x_p)J(\nabla\varphi(x_p)), J(\nabla\varphi(x_p))\rangle - \frac{(|\nabla\varphi(x_p)|^*)^2}{p-2} D^2\varphi(x_p) \otimes DJ(\nabla\varphi(x_p)) \\ \leq \frac{\Lambda_p^4\varphi(x_p)^3}{p-2}\left(\frac{|\varphi(x_p)|\Lambda_p}{|\nabla\varphi(x_p)|^*}\right)^{p-4}. \end{aligned} \tag{3.9}$$

Letting $p \to \infty$ and taking into account (3.8) we get

$$-Q_\infty\varphi(\xi) \leq 0,$$

which ends the proof.

We do not know how to prove uniqueness of solutions to the Dirichlet problem for $F_\infty(u, \nabla u, D^2u) = 0$, but as in [18], we are able to obtain a comparison result. In the setting of viscosity solutions given in [10], the function $F_\infty$ is degenerate elliptic but not proper. Therefore the standard theory cannot be applied directly. The strict positivity of $u_p$ for $1 < p \leq \infty$ allows us to consider in place of $F_\infty(u, \nabla u, D^2u) = 0$ a new equation satisfied by $w_\infty = \log u_\infty$ (see [5], [18]). Let us write

$$G_\infty(\nabla w, D^2w) = 0, \tag{3.10}$$

where

$$G_\infty(\nabla w, D^2w) := \min\{|\nabla w|^* - \Lambda_\infty,\ -Q_\infty w - (|\nabla w|^*)^4\}$$

and $Q_\infty$ is defined as before. We claim that if $u$ is a viscosity supersolution (resp. subsolution) of $F_\infty(u, \nabla u, D^2u) = 0$, then $w = \log u$ is a viscosity supersolution (resp. subsolution) $G_\infty(\nabla w, D^2w) = 0$. Take $\xi \in \Omega$ and $\varphi \in C^2$ such that $\varphi(\xi) = w(\xi)$ and $\varphi(x) < w(x)$ for $x \neq \xi$. The function $\theta(x) = e^{\varphi(x)}$ is a good test function for $u$ at $\xi$. Then we have

$$\min\{|\nabla\theta(\xi)|^* - \Lambda_\infty\theta(\xi),\ -Q_\infty\theta(\xi)\} \geq 0.$$

We write the last inequality in terms of $\varphi(x)$ as

$$\min\{e^{\varphi}(|\nabla\varphi|^* - \Lambda_\infty)(\xi),\ -e^{3\varphi}(Q_\infty\varphi + \langle\nabla\varphi, J(\nabla\varphi)\rangle^2)(\xi)\} \geq 0,$$

and the claim follows from the observation that $\langle y, J(y)\rangle = (|y|^*)^2$. The proof for subsolutions is symmetric.

Now we can study $G_\infty(\nabla w, D^2 w) = 0$, which (in contrast to $F_\infty = 0$) is now proper.

**Theorem 3.3.** *Let $\Omega$ be a bounded domain, and suppose that $u$ is a uniformly continuous viscosity subsolution and $v$ a uniformly continuous viscosity supersolution of* (3.10) *in $\Omega$. Then*

$$\sup_{x\in\overline{\Omega}}(u(x) - v(x)) = \sup_{x\in\partial\Omega}(u(x) - v(x)). \tag{3.11}$$

*Proof.* There is no loss of generality if we assume $u, v \geq 0$. Otherwise we add constants to $u$ and $v$. We proceed by contradiction. Suppose that (3.11) is false; then

$$\sup_{x\in\overline{\Omega}}(u(x) - v(x)) > \sup_{x\in\partial\overline{\Omega}}(u(x) - v(x)). \tag{3.12}$$

To obtain a contradiction, we construct a new supersolution $w$ having the following properties:

(i) $\|v - w\|_\infty$ is small enough to preserve the inequality (3.12);
(ii) $w$ is a *strict* supersolution of (3.10).

With those properties in mind, we introduce the function (see [18])

$$f(z) = \frac{1}{\alpha}\log(1 + A(e^{\alpha z} - 1)),$$

where $\alpha, A > 1$. In [18] this function was shown to satisfy (a) through (d) below:

(a) $f'(z) > 1$ for every $z > 0$;
(b) $f_A$ is invertible and $(f_A)^{-1} = f_{A^{-1}}$ for every $z > 0$;
(c) $1 - [f'(z)]^{-1} + [f'(z)]^{-2} f''(z) < 0$ for every $z > 0$;
(d) $0 < f(z) - z < (A - 1)/\alpha$ for every $z > 0$.

We define $w = f(v)$. For $A$ sufficiently close to 1, property (i) holds easily. We check (ii). Let $\xi \in \Omega$ and $\varphi \in C^2$ be such that $\varphi(\xi) = w(\xi)$ and $\varphi(x) \leq w(x)$ for $x \neq \xi$. Set $\theta = f^{-1}(\varphi)$. The function $f^{-1}$ is increasing, and so $\theta$ is a good test function for $v$ at $\xi$. But $v$ is a supersolution of (3.10), therefore

$$\min\{|\nabla\theta(\xi)|^* - \Lambda_\infty,\ -Q_\infty\theta(\xi) - (|\nabla\theta(\xi)|^*)^4\} \geq 0. \tag{3.13}$$

It follows from (3.13) that

$$|\nabla\theta(\xi)|^* - \Lambda_\infty \geq 0, \tag{3.14}$$

$$-Q_\infty\theta(\xi) - (|\nabla\theta(\xi)|^*)^4 \geq 0. \tag{3.15}$$

But if we write explicitly

$$\theta_{x_j} = [f'(\theta)]^{-1}\varphi_{x_j}, \quad \theta_{x_i x_j} = [f'(\theta)]^{-1}\varphi_{x_i x_j} - [f'(\theta)]^{-3} f''(\theta)\varphi_{x_i}\varphi_{x_j},$$

from (3.14) we get

$$|\nabla\varphi(\xi)|^* \geq f'(\theta(\xi))\Lambda_\infty \tag{3.16}$$

or

$$|\nabla\varphi(\xi)|^* - \Lambda_\infty \geq [f'(\theta(\xi)) - 1]\Lambda_\infty > 0. \tag{3.17}$$

With some calculus we obtain

$$D^2\varphi = f'(\theta)D^2\theta + f''(\theta)\nabla\theta \otimes \nabla\theta$$

so that (because $J$ is homogeneous of degree one)

$$-Q_\infty\varphi = \langle D^2\varphi J(\nabla\varphi), J(\nabla\varphi)\rangle = -f'(\theta)^3 Q_\infty\theta - f''(\theta)f'(\theta)^2(|\nabla\theta|^*)^4.$$

Together with (3.15) this implies

$$-Q_\infty\varphi(\xi) - (|\nabla\varphi(\xi)|^*)^4 \geq (f'^3 - f''f'^2 - f'^4)(\theta(\xi))(|\nabla\theta(\xi)|^*)^4$$

whose right hand side is positive because of (d). Thus we have shown

$$-Q_\infty\varphi(\xi) - (|\nabla\varphi(\xi)|^*)^2 \geq f'^4\left(\frac{1}{f'} - \frac{f''}{f'^2} - 1\right)(v(\xi))\Lambda_\infty^4. \tag{3.18}$$

From (a), (3.17) and (3.18) we conclude

$$\min\{|\nabla\varphi(\xi)|^* - \Lambda_\infty, -Q_\infty\varphi(\xi) - (|\nabla\varphi(\xi)|^*)^4\} \geq \rho(\xi) > 0, \tag{3.19}$$

where we have defined

$$\rho(x) := \min\left\{[f'(v(x)) - 1]\Lambda_\infty, \left(\frac{1}{f'} - \frac{f''}{f'^2} - 1\right)(v(x))\Lambda_\infty^4\right\}.$$

Inequality (3.19) and properties (a) and (c) tell us that $w$ is a strict supersolution.

Now the contradiction follows easily by standard techniques for viscosity solutions (see [10]). Let us sketch the conclusion. We consider a maximum point $(x_t, y_t)$ of the function

$$u(x) - w(y) - \frac{t}{2}|x - y|^2$$

in $\overline{\Omega} \times \overline{\Omega}$. Up to a subsequence, we have

$$x_t \to \xi \quad \text{and} \quad y_t \to \xi,$$

where $\xi \in \overline{\Omega}$ is a maximum point of $u - w$ in $\overline{\Omega}$. But inequality (3.12) holds, so $\xi$ lies in the interior. We apply the max principle for semicontinuous functions (see Chapter 3 in [10] for this result and for the definition of the semijets $\overline{J}^{2,+}(u(x_t))$ and $\overline{J}^{2,-}(w(x_t)))$, which ensures the existence of real symmetric matrices $X_t,\ Y_t$ such that

$$(t(x_t - y_t); X_t) \in \overline{J}^{2,+}(u(x_t)), \qquad (t(x_t - y_t); Y_t) \in \overline{J}^{2,-}(w(x_t)),$$
$$(X_t \nu, \nu) - (Y_t \mu, \mu) \le 3t|\nu - \mu|^2.$$

Now $u$ is a subsolution of $G_\infty = 0$, so

$$G_\infty(t(x_t - y_t); X_t) \le 0. \tag{3.20}$$

Since $w$ is a strict supersolution of $G_\infty = 0$, from (3.19) we get

$$G_\infty(t(x_t - y_t); Y_t) \ge \rho(x_t) > 0. \tag{3.21}$$

Now (3.20) and (3.21) give after some calculation $\rho(x_t) \le 0$, which is obviously a contradiction. This completes the proof.

**Remark 3.4.** Theorem 3.3 also holds when one of the functions takes the value $-\infty$ on the whole boundary.

It is well known that for any $1 < p < \infty$, the eigenvalue $\lambda_p$ can be characterized by the property that $\lambda = \lambda_p$ is the only real number for which the equation

$$-\operatorname{div}((|\nabla u_p|^*)^{p-2} J(\nabla u_p)) = \lambda |u_p|^{p-2} u_p$$

has a continuous positive solution with zero boundary value. We will show next that $\Lambda_\infty$ has an analogous characterization.

**Theorem 3.5.** *Let $\Omega$ be any bounded domain and suppose that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. If $u$ is a continuous positive viscosity solution in $\Omega$ of*

$$\min\{|\nabla u|^* - \Lambda u, -Q_\infty u\} = 0$$

*with zero boundary value, then $\Lambda = \Lambda_\infty$.*

To prove this, we need the following gradient estimate. For the standard Euclidean norm this was derived in [21]. Using a perturbation argument due to Crandall, we show that the general case follows from the results in [2].

**Theorem 3.6.** *Suppose that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. Let $u$ be a nonnegative viscosity supersolution of $-Q_\infty u = 0$ in $\Omega$, and let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. Then*

$$|\nabla u(x)|^* \le \frac{u(x)}{\delta(x)} \qquad \text{for a.e. } x \in \Omega. \tag{3.22}$$

*Proof.* It suffices to verify that $u$ enjoys the following *comparison with cones from below* property in $\Omega$ (see [2]):

> Whenever $V \subset\subset \Omega$ is an open set and $C(x) = a|x - z| + b$ with $a, b \in \mathbb{R}$, $z \notin V$ is a cone function such that $u \ge C$ on $\partial V$, then $u \ge C$ in $V$.

Indeed, for functions that enjoy comparison with cones from below, (3.22) is Remark 2.17 in [2].

To show that viscosity supersolutions of $-Q_\infty u = 0$ enjoy comparison with cones from below, we argue as in the proof of Theorem 4.13 in [2]. Suppose $u$ does not enjoy comparison with cones from below in $\Omega$. Then there is an open set $V \subset\subset \Omega$ and a cone function $C(x) = a|x - z| + b$ with $a, b \in \mathbb{R}$ and $z \notin V$ such that $u = C$ on $\partial V$ and $u < C$ in $V$. If for each $\varepsilon > 0$ we can find a perturbation $P \in C^2(\overline{V})$ such that $|P| \leq \varepsilon$ in $V$ and

$$-Q_\infty(C + P) \leq -\delta < 0 \quad \text{in } V, \tag{3.23}$$

we will be done. Indeed, for $\varepsilon > 0$ small enough, the function $u - (C + P)$ has an interior local minimum point $x_0 \in V$. Since $u$ is a viscosity supersolution and $C + P \in C^2(V)$, this implies

$$-Q_\infty(C + P)(x_0) \geq 0,$$

contradicting (3.23).

Since we are assuming that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$, suitable perturbations can be explicitly constructed using this norm. Suppose, without loss of generality, that $z = 0$ and put $P = \gamma|x|^2$ and $\gamma > 0$. Then $C(x) + P(x) = g(|x|)$ where $g(s) = as + \gamma s^2 + b$. A direct computation shows that

$$-Q_\infty g(|x|) = -g'(|x|)^3 \langle D^2|x| J(\nabla|x|), J(\nabla|x|)\rangle - g''(|x|)g'(|x|)^2 \langle \nabla|x|, J(\nabla|x|)\rangle^2.$$

Since $\langle \nabla|x|, x\rangle = |x|$ by homogeneity and $J(\nabla|x|) = x/|x|$ for $x \neq 0$, this reduces to

$$-Q_\infty g(|x|) = -(g')^3 |x|^{-2} \langle D^2|x| x, x\rangle - g''(g')^2.$$

Next observe that by the linearity of $h(t) = |tx|$ we have $0 = h''(1) = \langle D^2|x| x, x\rangle$, so that

$$-Q_\infty(C + P)(x) = -g''(|x|)g'(|x|)^2 = -2\gamma(2\gamma|x| + a)^2.$$

This is strictly negative in $V$ if either $a \geq 0$, or $a < 0$ and $\gamma > 0$ is sufficiently small. If $\gamma$ is sufficiently small we also attain $|P| \leq \epsilon$ in $V$.

For the proof of Theorem 3.5, we will also need the following auxiliary comparison result.

**Lemma 3.7.** *Suppose that the norm $|\cdot|$ is of class $C^2(\mathbb{R}^n \setminus \{0\})$. If $u$ is a continuous positive viscosity solution of*

$$\min\{|\nabla u|^* - \Lambda u, -Q_\infty u\} = 0 \tag{3.24}$$

*in a bounded domain $\Omega$ with zero boundary values, normalized so that* $\sup u = 1/\Lambda$, *then*

$$u(x) \leq \operatorname{dist}(x, \partial\Omega) \quad \textit{for every } x \in \Omega.$$

*Proof.* Fix $z \in \partial\Omega$ and for $a > 1,\ \gamma > 0$ let $v(x) = a|x - z| - \gamma|x - z|^2$. Analogously to the proof of Theorem 3.6 above, we obtain $-Q_\infty v(x) > 0$ provided that $\gamma > 0$ is sufficiently small. Moreover,

$$|\nabla v(x)|^* = (a - 2\gamma|x - z|)|\nabla|x - z|\,|^* = a - 2\gamma|x - z| > 1$$

if $\gamma$ is small enough. Thus we have

$$\min\{|\nabla v|^* - 1, -Q_\infty v\} > 0. \tag{3.25}$$

Next notice that due to the assumption $\sup u = 1/\Lambda$, (3.24) implies

$$\min\{|\nabla u|^* - 1, -Q_\infty u\} \le 0 \quad \text{in the viscosity sense.} \tag{3.26}$$

Since $v \in C^2$ and $v \ge u = 0$ on $\partial\Omega$ (if $\gamma$ is small enough), it follows that $v \ge u$ in $\Omega$. Indeed, otherwise $u - v$ would have an interior local maximum point at which $v$ would be a test function for $u$ from above, contradicting (3.25) and (3.26).

We have thus shown that $u(x) \le a|x - z| - \gamma|x - z|^2$ for every $z \in \partial\Omega$, $a > 1$ and $\gamma > 0$ sufficiently small. Hence

$$u(x) \le \inf_{z\in\partial\Omega} |x - z| = \operatorname{dist}(x, \partial\Omega),$$

as desired.

**Remark 3.8.** Lemma 3.7 implies that if $u$ is any positive viscosity solution to the eigenvalue equation $F_\infty(u, \nabla u, D^2u) = 0$ with zero boundary data, it cannot be differentiable at its maximum points. To see this, normalize $u$ so that $\sup u = 1/\Lambda$. Then if $u(x_0) = \sup_{x\in\Omega} u(x)$, it follows that $\delta(x_0) = \sup_{x\in\Omega} \delta(x)$. Since $\delta$ is not differentiable at $x_0$ and $u \le \delta$, $u(x_0) = \delta(x_0)$, it is now clear that $u$ is not differentiable at $x_0$.

*Proof of Theorem 3.5.* Notice first that if $\Lambda \le 0$, then the eigenvalue equation above reduces to the equation $-Q_\infty u = 0$, whose only solution with zero boundary values is $u \equiv 0$ (see [2] or [3]).

Normalize $u$ so that $\sup u = 1/\Lambda$. Then by Lemma 3.7 we obtain $u(x) \le \delta(x) := \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$, which together with the gradient estimate (3.22) yields $|\nabla u(x)|^* \le 1$ for a.e. $x \in \Omega$. Consequently,

$$\frac{\||\nabla u|^*\|_\infty}{\|u\|_\infty} \le \frac{1}{\|u\|_\infty} = \Lambda.$$

Because

$$\Lambda_\infty = \inf\left\{\frac{\||\nabla w|^*\|_\infty}{\|w\|_\infty} : w \in W_0^{1,\infty}(\Omega) \setminus \{0\}\right\}$$

by (3.1) and (3.2), we must have $\Lambda_\infty \le \Lambda$.

To prove the reverse inequality, we approximate $v = \log u$ by its semiconcave inf-convolutions

$$v^\epsilon(x) = \inf_{y\in\overline{\Omega}_\sigma} \left\{v(y) + \frac{1}{2\varepsilon}|x - y|^2\right\}$$

for $\varepsilon > 0$ in the set $\Omega_\sigma = \{x \in \Omega : \delta(x) > \sigma\}$. Since $|\nabla v|^* \geq \Lambda$ in the viscosity sense by the assumptions and $v^\varepsilon$ is twice differentiable a.e., it follows from the properties of the inf-convolution that $|\nabla v^\varepsilon(x)|^* \geq \Lambda$ for a.e. $x$ in a smaller set $\Omega_{\sigma,\varepsilon} = \{x \in \Omega_\sigma : \operatorname{dist}(x, \partial\Omega_\sigma) > C\varepsilon\}$. Moreover, the function $e^{v^\varepsilon}$ is a positive supersolution of $-Q_\infty w = 0$ in $\Omega_{\sigma,\varepsilon}$. Thus using the gradient estimate (3.22) we obtain

$$\Lambda \leq |\nabla v^\varepsilon(x)|^* = \frac{1}{e^{v^\varepsilon}}|\nabla(e^{v^\varepsilon(x)})|^* \leq \frac{1}{\operatorname{dist}(x, \partial\Omega_{\sigma,\varepsilon})}$$

for a.e. $x \in \Omega_{\sigma,\varepsilon}$, and so, letting $\varepsilon \to 0$ and $\sigma \to 0$ gives

$$\Lambda \leq \frac{1}{\sup_{x\in\Omega} \delta(x)} = \Lambda_\infty.$$

This completes the proof.

## 4. Example and concluding remarks

If the norm under consideration for $x \in \Omega$ is the usual $\ell_q$ norm, i.e. $|x| = (\sum_{i=1}^n |x_i|^q)^{1/q}$ with $q \in (1, \infty)$, the duality map according to (2.4) is easily calculated as

$$J_i(y) = (|y|_{q'})^{2-q'}|y_i|^{q'-2}y_i,$$

with $q' = q/(q-1)$ being the conjugate exponent. Notice that this differs from the $J$ in [2, Example 5.2]. The $p$-Laplace operator in this Finsler metric is explicitly given by (see [6])

$$Q_p u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|\nabla u|_{q'}^{p-q'}\left|\frac{\partial u}{\partial x_i}\right|^{q'-2}\frac{\partial u}{\partial x_i}\right).$$

For $p > 2$ this definition is meaningful and for $q = 2$ $(= q')$ it recovers the well known $p$-Laplace operator. The operator $Q_2$ is formally given by

$$Q_2 u = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left[\frac{|u_{x_i}|}{|\nabla u|_{q'}}\right]^{q'-2}\frac{\partial u}{\partial x_i}\right).$$

However, $Q_2 u$ does not seem to be well defined at critical points of $u$. The $\infty$-Laplace operator in the same Finsler metric is explicitly given by

$$Q_\infty u = |\nabla u|_{q'}^{4-2q'} \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i x_j}\left|\frac{\partial u}{\partial x_i}\right|^{q'-2}\frac{\partial u}{\partial x_i}\left|\frac{\partial u}{\partial x_j}\right|^{q'-2}\frac{\partial u}{\partial x_j}\right)$$

and for $q = 2$ this expression reduces to the customary

$$\Delta_\infty u = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}.$$

**Remark 4.1.** It should be remarked that the distance function minimizes the Rayleigh quotient $R_\infty$, but that $\delta(x)$ is in general not a viscosity solution of the limiting eigenvalue problem, unless $\Omega$ is a "ball" in the Finsler metric (see [18], [19], [5]).

**Remark 4.2.** If $\Omega$ is a "ball" in $\mathbb{R}^n$ and $p = n$, then all the level sets of solutions to (2.2),

$$-Q_n u = \lambda_n |u|^{n-2} u,$$

are similar "balls"(see [6]).

**Remark 4.3.** The smoothness assumption made on the dual spheres in our paper is violated if the underlying norm is the $\ell_1$ or $\ell_\infty$ norm. However, the pde $-Q_p = 1$ and its limit as $p \to \infty$ were studied even in this case in [15]; see also [20], [7], [17] and [16] for the case of the Euclidean norm and for variants of this problem.

**Remark 4.4.** Clearly the eigenvalue $\lambda_p$ depends on $\Omega$. There is an analogue of the Faber–Krahn inequality which states that among all domains of given volume, $\lambda_p(\Omega)$ becomes minimal if $\Omega$ is a "ball" in the Finsler metric. This result is formulated in [6], but it is based on a rearrangement inequality from [1].

*Acknowledgments*. This paper was essentially written when the second author visited the first in the spring of 2004. Financial support for the second author was provided by G.N.A.M.P.A. The third author is supported by the Academy of Finland, project #80566. We thank G. Buttazzo for bringing [16] to our attention.

## References


[1] Alvino, A., Ferone, V., Trombetti, G., Lions, P. L.: Convex symmetrization and applications. Ann. Inst. H. Poincaré Anal. Non Linéaire **14**, 275–293 (1997) Zbl 0877.35040 MR 1441395

[2] Aronsson, G., Crandall, M. G., Juutinen, P.: A tour of the theory of absolutely minimizing functions. Bull. Amer. Math. Soc. **41**, 439–505 (2004) Zbl pre02108961 MR 2083637

[3] Barles, G., Busca, J.: Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term. Comm. Partial Differential Equations **26**, 2323–2337 (2001) Zbl 0997.35023 MR 1876420

[4] Belloni, M., Kawohl, B.: A direct uniqueness proof for equations involving the *p*-Laplace operator. Manuscripta Math. **109**, 229–231 (2002) Zbl pre01837448 MR 1935031

[5] Belloni, M., Kawohl, B.: The pseudo-*p*-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$. ESAIM Control Optim. Calc. Var. **10**, 28–52 (2004) Zbl pre02193988 MR 2084254

[6] Belloni, M., Ferone, V., Kawohl, B.: Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators. J. Appl. Math. Phys. (ZAMP) **54**, 771–783 (2003) Zbl pre02021111 MR 2019179

[7] Bhattacharya, T., DiBenedetto, E., Manfredi, J.: Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems. Rend. Sem. Mat. Univ. Politec. Torino **1989**, Special Issue, 15–68 MR 1155453

[8] Buttazzo, G., Oudet, E., Stepanov, E.: Optimal transportation problems with free Dirichlet regions. In: Progr. Nonlinear Differential Equations 51, Birkhäuser, 41–65 (2002) Zbl 1055.49029

[9] Crandall, M. G.: Viscosity solutions, a primer. In: Viscosity Solutions and Applications, I. Capuzzo Dolcetta and P. L. Lions (eds.), Lecture Notes in Math. 1660, Springer, 1–43 (1997) Zbl 0901.49026 MR 1462699

[10] Crandall, M. G., Ishii, H., Lions, P. L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. **27**, 1–67 (1992) Zbl 0755.35015 MR 1118699

[11] DiBenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal. **7**, 827–850 (1983) Zbl 0539.35027 MR 0709038

[12] Evans, L. C., Gangbo, W.: Differential equations methods for the Monge–Kantorovich mass transfer problem. Mem. Amer. Math. Soc. **653** (1999) Zbl 0920.49004 MR 1464149

[13] Fukagai, N., Ito, M., Narukawa, K.: Limit as $p \to \infty$ of $p$-Laplace eigenvalue problems and $L^\infty$ inequality of the Poincaré type. Differential Integral Equations **12**, 183–206 (1999) Zbl pre01858744 MR 1672746

[14] Giusti, E.: Metodi diretti nel calcolo delle variazioni. Un. Mat. Ital., Bologna (1994) Zbl 0942.49002 MR 1707291

[15] Ishibashi, T., Koike, S.: On fully nonlinear PDEs derived from variational problems of $L^p$ norms. SIAM J. Math. Anal. **33**, 545–569 (2001) Zbl 1030.35088 MR 1871409

[16] Ishii, H., Loreti, P.: Limits of solutions of $p$-Laplace equations as $p$ goes to infinity and related variational problems. SIAM J. Math Anal. **37**, 411–437 (2005)

[17] Janfalk, U.: Behaviour in the limit, as $p \to \infty$, of minimizers of functionals involving $p$-Dirichlet integrals. SIAM J. Math. Anal. **27**, 341–360 (1996) Zbl 0853.35028 MR 1377478

[18] Juutinen, P., Lindqvist, P., Manfredi, J.: The $\infty$-eigenvalue problem. Arch. Rat. Mech. Anal. **148**, 89–105 (1999) Zbl 0947.35104 MR 1716563

[19] Juutinen, P., Lindqvist, P., Manfredi, J.: The infinity Laplacian: examples and observations. In: Papers on Analysis, Rep. Univ. Jyväskylä Dept. Math. Stat. 83, Univ. Jyväskylä, Jyväskylä, 207–217 (2001) Zbl 1016.35029 MR 1886623

[20] Kawohl, B.: A family of torsional creep problems. J. Reine Angew. Math. **410**, 1–22 (1990) Zbl 0701.35015 MR 1068797

[21] Lindqvist, P., Manfredi, J.: Note on $\infty$-superharmonic functions. Rev. Mat. Univ. Complut. Madrid **10**, 471–480 (1997) Zbl 0891.35043 MR 1605682

[22] Ohnuma, M., Sato, K.: Singular degenerate parabolic equations with applications to the $p$-Laplace diffusion equation. Comm. Partial Differential Equations **22**, 381–411 (1997) Zbl 0990.35077 MR 1443043

[23] Tolksdorf, P.: On the Dirichlet problem for quasilinear equations in domains with conical boundary points. Comm. Partial Differential Equations **8**, 773–817 (1983) Zbl 0515.35024 MR 0700735

[24] Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations **51**, 126–150 (1984) Zbl 0488.35017 MR 0727034

[25] Wu, Y.: Absolute minimizers in Finsler metric. PhD Thesis, Berkeley (1995)

[26] Zeidler, E.: Nonlinear Functional Analysis and Applications III, Variational Methods and Optimization. Springer, Heidelberg (1985) Zbl 0583.47051 MR 0768749