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# Concentration phenomena for Liouville's equation in dimension four

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## 1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^4$ and let $u_k$ be solutions to the equation

$$\Delta^2 u_k = V_k e^{4u_k} \quad \text{in } \Omega, \tag{1}$$

where

$$V_k \to 1 \quad \text{uniformly in } \Omega, \tag{2}$$

as $k \to \infty$. Throughout the paper we denote as $\Delta = -\sum_i (\partial/\partial x^i)^2$ the Laplacian with the geometers' sign convention. Continuing the analysis of [19], here we study the compactness properties of equation (1).

Equation (1) is the fourth order analogue of Liouville's equation. Thus, for problem (1), (2) we may expect similar results to those obtained by Brézis–Merle [3] in the two-dimensional case. Recall the following result from [3] and its improvement by Li–Shafrir [11].

**Theorem 1.1.** *Let $\Sigma$ be a bounded domain of $\mathbb{R}^2$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of solutions to the equation*

$$\Delta u_k = V_k e^{2u_k} \quad \text{in } \Sigma, \tag{3}$$

*where $V_k \to 1$ uniformly in $\Sigma$ as $k \to \infty$, and satisfying the uniform bound*

$$\int_\Sigma V_k e^{2u_k}\, dx \le \Lambda \tag{4}$$

*for some $\Lambda > 0$. Then either*

(i) *$(u_k)_{k\in\mathbb{N}}$ is locally bounded in $C^{1,\alpha}$ on $\Sigma$ for every $\alpha < 1$, or*
(ii) *there exists a subsequence $K \subset \mathbb{N}$ such that $u_k \to -\infty$ locally uniformly in $\Omega$ as $k \to \infty$, $k \in K$, or*

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(iii) *there exist a subsequence $K \subset \mathbb{N}$ and at most finitely many points $x^{(i)} \in \Omega$, $1 \le i \le I$, with corresponding numbers $\beta_i \in 4\pi\mathbb{N}$ such that $V_k e^{2u_k}\,dx \rightharpoonup \sum_{i=1}^{I} \beta_i \delta_{x^{(i)}}$ weakly in the sense of measures while $u_k \to -\infty$ locally uniformly in $\Omega \setminus \{x^{(i)}; 1 \le i \le I\}$ when $k \to \infty$, $k \in K$.*

*Moreover, near any concentration point $x^{(i)}$, after rescaling*

$$v_k(x) = u_k(x_k + r_k x) + \log r_k, \qquad W_k(x) = V_k(x_k + r_k x) \tag{5}$$

*with suitable sequences $x_k \to x^{(i)}$ and $r_k \to 0$ as $k \to \infty$, a subsequence satisfies $v_k \to v$ locally uniformly in $C^{1,\alpha}$ on $\mathbb{R}^2$, where $v$ is a solution of Liouville's equation*

$$\Delta u = e^{2u} \quad \text{on } \mathbb{R}^2 \quad \text{with} \quad \int_{\mathbb{R}^2} e^{2u}\,dx < \infty. \tag{6}$$

Geometrically speaking, the solutions $u_k$ to equation (3) correspond to conformal metrics $g_k = e^{2u_k} g_{\mathbb{R}^2}$ on $\Sigma$ with Gauss curvature $V_k$. The fact that all solutions $u$ of equation (6) by a result of Chen-Li [5] are induced by conformal metrics $e^{2u} g_{\mathbb{R}^2}$ on $\mathbb{R}^2$ that are obtained by stereographic projection of the standard sphere then gives rise to the observed quantization. Multiple blow-up at a point is possible, as shown by X. Chen [6].

Similarly, the solutions $u_k$ to (1) induce conformal metrics $g_k = e^{2u_k} g_{\mathbb{R}^4}$ on $\Omega$ having $Q$-curvature proportional to $V_k$. In contrast to the two-dimensional case, however, there is a much greater abundance of solutions to the corresponding limit equation

$$\Delta^2 u = e^{4u} \quad \text{on } \mathbb{R}^4. \tag{7}$$

In fact, by a result of Chang–Chen [4] for any $\alpha \in ]0, 16\pi^2]$ there exists a solution $u_\alpha$ of (7) of total volume $\int_{\mathbb{R}^4} e^{4u_\alpha}\,dx = \alpha$ which for $\alpha < 16\pi^2$ fundamentally differs from the solution $u(x) = \log(\sqrt{96}/(\sqrt{96} + |x|^2))$ corresponding to the metric obtained by pull-back of the spherical metric on $S^4$ under stereographic projection. Only the latter solution (and any solution obtained from $u$ by rescaling as in (5)) achieves the maximal value $\int_{\mathbb{R}^4} e^{4u}\,dx = 16\pi^2$. If we then consider a suitable sequence $u_k = u_{\alpha_k}$ with $\alpha_k \to 0$ as $k \to \infty$, normalized as in (5) so that $u_k \le u_k(0) = k$, we can even achieve that $(u_k)_{k\in\mathbb{N}}$ blows up at $x^{(1)} = 0$ in the sense that $u_k(0) \to \infty$ while $u_k(x) \to -\infty$ for any $x \ne 0$ as $k \to \infty$.

As shown in Example 3.1, solutions to equation (1) with a similar concentration behavior exist even in the radially symmetric case.

There is a further complication in the four-dimensional case, illustrated by the following simple example. Consider the sequence $(v_k)$ on $\mathbb{R}^4$, defined by letting $v_k(x) = w_k(|x^1|)$, where for $k \in \mathbb{N}$ we let $w_k$ solve the initial value problem for the ordinary differential equation $w_k'''' = e^{4w_k}$ on $0 < s < \infty$ with initial data $w_k(0) = w_k'(0) = w_k'''(0) = 0$, $w_k''(0) = -k$. Given $\Lambda > 0$, we can then find a sequence of radii $R_k > 0$ such that $\int_{B_{R_k}(0)} e^{4v_k}\,dx = \Lambda$. Observe that $R_k \to \infty$ as $k \to \infty$. Scaling as in (5), we then obtain a sequence of solutions $u_k(x) = v_k(R_k x) + \log R_k$ to (7) on $\Omega = B_1(0)$ such that $u_k(x) \to \infty$ for all $x \in S_0 = \{x \in \Omega; x^1 = 0\}$ and $u_k \to -\infty$ away from $S_0$ as $k \to \infty$. Scaling back as in (5), from $(u_k)$ we recover the normalized functions $v_k$ which fail to converge to a solution of the limit problem (7) and develop an interior layer on the hypersurface $\{x \in \mathbb{R}^4;\, x^1 = 0\}$ instead.

These comments illustrate that conclusions (i), (ii) and (iii) of Theorem 1.1 do not exhaustively describe all the possible concentration phenomena for (1). In fact, the following concentration-compactness result seems best possible.

**Theorem 1.2.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^4$ and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of solutions to* (1), (2) *above. Assume that there exists $\Lambda > 0$ such that*

$$\int_\Omega V_k e^{4u_k}\, dx \le \Lambda \tag{8}$$

*for all $k$. Then either*

(i) *a subsequence $(u_k)$ is relatively compact in $C^{3,\alpha}_{\mathrm{loc}}(\Omega)$, or*
(ii) *there exist a subsequence $(u_k)$ and a closed nowhere dense set $S_0$ of vanishing measure and at most finitely many points $x^{(i)} \in \Omega$, $1 \le i \le I \le C\Lambda$, such that, letting*

$$S = S_0 \cup \{x^{(i)};\ 1 \le i \le I\},$$

*we have $u_k \to -\infty$ locally uniformly away from $S$ as $k \to \infty$.*

*Moreover, there is a sequence of numbers $\beta_k \to \infty$ such that*

$$u_k/\beta_k \to \varphi \quad \text{in } C^{3,\alpha}_{\mathrm{loc}}(\Omega \setminus S),$$

*where $\varphi \in C^4(\Omega \setminus \{x^{(i)};\ 1 \le i \le I\})$ is such that*

$$\Delta^2\varphi = 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0,$$

*and*

$$S_0 = \{x \in \Omega \setminus \{x^{(i)};\ 1 \le i \le I\};\ \varphi(x) = 0\}.$$

*Finally, near any point $x_0 \in S$ where $\sup_{B_r(x_0)} u_k \to \infty$ for every $r > 0$ as $k \to \infty$, in particular, near any concentration point $x^{(i)}$, there exist points $x_k \to x_0$, numbers $L_k \to \infty$, and suitable radii $r_k \to 0$ such that after normalizing we have*

$$v_k(x) = u_k(x_k + r_k x) + \log r_k \le 0 \le \log 2 + v_k(0) \quad \text{for } |x| \le L_k. \tag{9}$$

*As $k \to \infty$ then either a subsequence $v_k \to v$ in $C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^4)$, where $v$ solves the limit equation* (7), *or $v_k \to -\infty$ almost everywhere and there is a sequence of numbers $\gamma_k \to \infty$ such that a subsequence satisfies*

$$v_k/\gamma_k \to \psi \quad \text{in } C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^4),$$

*where $\psi \le 0$ is a non-constant quadratic polynomial.*

We regard Theorem 1.2 as a first step towards a more complete description of the possible concentration behavior of sequences of solutions to problem (1), (2).

Considering (1) as a system of second order equations for $u_k$ and $\Delta u_k$, respectively, it is possible to obtain some partial results in this regard from the observation that (1), (2) provide uniform integral bounds for $\Delta u_k$ up to a remainder given by a harmonic function. The latter component may be controlled if one imposes, for instance, the Navier boundary conditions $u_k = \Delta u_k = 0$ on $\partial\Omega$. In fact, in this case, assuming that each $V_k$ is a constant $\lambda_k > 0$ that tends to 0 as $k \to \infty$, J. Wei [21] has shown (in the notation of Theorem 1.2) that $S_0 = \emptyset$ and that at any concentration point $x^{(i)}$ suitably rescaled functions satisfy $v_k \to v$ in $C^{3,\alpha}_{\mathrm{loc}}(\mathbb{R}^4)$, where $v$ is the profile induced by stereographic projection.

As shown by Robert [18], the same result holds if for some open subset $\emptyset \neq \omega \subset \Omega$ we have the a priori bounds

$$\|(\Delta u_k)^-\|_{L^1(\Omega)} \leq C, \quad \|(\Delta u_k)^+\|_{L^1(\omega)} \leq C,$$

for all $k \in \mathbb{N}$, where $s^\pm = \pm\max\{0, \pm s\}$. Also in the radially symmetric case there is a complete description of the possible concentration patterns; see [18].

In the geometric context similar results hold for the related problem of describing the possible concentration behavior of solutions to the equation of prescribed $Q$-curvature on a closed 4-manifold $M$. Here the bi-Laplacian in equation (1) is replaced by the Paneitz–Branson operator and $V_k$ may again be interpreted as being proportional to the $Q$-curvature of the metric $g_k = e^{2u_k} g_M$. In the case when $M = S^4$, Malchiodi–Struwe [14] have shown that any such sequence $(g_k)$ of metrics when $V_k \to 1$ uniformly either is relatively compact or blows up at a single concentration point where a round spherical metric forms after rescaling. Further compactness results and references can be found in the papers of Druet–Robert [8] and Malchiodi [13].

Related results on compactness issues for fourth order equations can be found in Hebey–Robert–Wen [10], C. S. Lin [12] and Robert [17]; concentration-compactness issues for problems with exponential nonlinearities in two dimensions have been treated in Adimurthi–Druet [1], Adimurthi–Struwe [2] and Druet [7].

In the following the letter $C$ denotes a generic constant independent of $k$ which may change from line to line and even within the same line.

## 2. Proof of Theorem 1.2

Recall the following result, obtained independently by C. S. Lin [12, Lemma 2.3] and J. Wei [21, Lemma 2.3], which generalizes Theorem 1 from [3] to higher dimensions.

**Theorem 2.1.** *Let $v$ be a solution to the equation*

$$\Delta^2 v = f \quad \text{in } B_R(x_0) \subset \mathbb{R}^4 \tag{10}$$

*with*

$$v = \Delta v = 0 \quad \text{on } \partial B_R(x_0), \tag{11}$$

*where $f \in L^1(B_R(x_0))$ satisfies*

$$\|f\|_{L^1} = \alpha < 8\pi^2.$$

*Then for any $p < 8\pi^2/\alpha$ we have $e^{4p|v|} \in L^1(B_R(x_0))$ with*

$$\int_{B_R(x_0)} e^{4p|v|}\,dx \le C(p)R^4.$$

The following characterization of biharmonic functions, due to Pizzetti [16], can be found in [15]. Denote by $\fint_{B_R(y)} h\,dx$ the average of $h$ over $B_R(y)$, etc.

**Lemma 2.2.** *For any $n \in \mathbb{N}$, any solution $h$ of*

$$\Delta^2 h = 0 \quad \text{in } B_R(y) \subset \mathbb{R}^n \tag{12}$$

*satisfies*

$$h(y) - \fint_{B_R(y)} h(z)\,dz = \frac{R^2}{2(n+2)}\Delta h(y). \tag{13}$$

*Proof.* For convenience, we indicate a short proof. We may assume that $B_R(y) = B_R(0) = B_R$. For $0 < r < R$ let $G_r$ be the fundamental solution of the operator $\Delta^2$ on $B_r$ satisfying $G_r = \Delta G_r = 0$ on $\partial B_r$. Note that $G_r(x) = r^{4-n}G_1(x/r)$. (If $n = 4$, we have $G_r(x) = c_0(\log \frac{r}{|x|} - \frac{r^2-|x|^2}{4r^2})$.) Applying the mean value formula to the harmonic function $\Delta h$, for some constants $c_1, c_2$ we have

$$\begin{aligned}
0 &= \int_{B_r} G_r \Delta^2 h\,dx = h(0) + \int_{\partial B_r} \left(\frac{\partial}{\partial n} G_r \Delta h + \frac{\partial}{\partial n}\Delta G_r h\right) do \\
&= h(0) - \fint_{\partial B_r} (c_1 r^2 \Delta h + c_2 h)\,do = h(0) - c_1 r^2 \Delta h(0) - c_2 \fint_{\partial B_r} h\,do;
\end{aligned}$$

that is, for some constants $c_3, c_4$,

$$nr^{n-1}h(0) = c_3 r^{n+1}\Delta h(0) + c_4 \int_{\partial B_r} h\,do.$$

Integrating over $0 < r < R$ and dividing by $R^n$, we obtain the identity

$$h(0) = c_5 R^2 \Delta h(0) + c_6 \fint_{B_R} h\,dx$$

with uniform constants $c_5, c_6$ for all biharmonic functions $h$ on $B_R$. Inserting a harmonic function $h$, we obtain the value $c_6 = 1$, whereas the choice $h(x) = |x|^2$ yields $c_5 = 1/2(n+2)$. ◻

Lemma 2.2 gives rise to a Liouville property for biharmonic functions on $\mathbb{R}^n$. To see this first recall the following result for harmonic functions.

**Theorem 2.3.** *Suppose that the function $H$ is harmonic on $\mathbb{R}^n$ with $H(x) \leq C(1+|x|^l)$ for some $l \in \mathbb{N}$. Then $d^{l+1}H \equiv 0$; that is, $H$ is a polynomial of degree at most $l$.*

*Proof.* From the mean value property of the harmonic function $d^{l+1}H$, where $d^k$ now denotes any partial derivative of order $k$, for any $x$ and $R > 0$ we have

$$|d^{l+1}H(x)| \leq CR^{-(l+1)} \fint_{B_R(x)} |H(y)|\, dy; \tag{14}$$

see for instance Evans [9, Theorem 2.2.7, p. 29]. But if we assume $H(x) \leq C(1+|x|^l)$, the right hand side by the mean value poperty of $H$ up to an error of order $R^{-1}$ and up to a multiplicative constant equals $R^{-(l+1)}H(x)$, and the latter tends to 0 as $R \to \infty$ for any fixed $x$. □

Together with Lemma 2.2 we now obtain the following result.

**Theorem 2.4.** *Suppose that the function $h$ is biharmonic on $\mathbb{R}^n$ with $h(x) \leq C(1+|x|)$ for some $C \in \mathbb{R}$. Then $\Delta h \equiv$ const $\geq 0$ and $h$ is a polynomial of degree $\leq 2$.*

*Proof.* From Lemma 2.2 and the assumption $h(y) \leq C(1+|y|)$ we obtain the equation

$$\begin{aligned} \Delta h(x) &= 2(n+2) \lim_{R\to\infty} R^{-2} \fint_{B_R(x)} |h(y)|\, dy \\ &= 2(n+2) \lim_{R\to\infty} R^{-2} \fint_{B_R(0)} |h(y)|\, dy = \Delta h(0) =: 2na \end{aligned} \tag{15}$$

for every $x \in \mathbb{R}^n$, where $a \geq 0$. The function $H(x) = h(x) + a|x|^2$ is then harmonic with $H(x) \leq C(1+|x|^2)$ and the claim follows from Theorem 2.3. □

*Proof of Theorem 1.2.* Choose a subsequence $k \to \infty$ and a maximal number of points $x^{(i)} \in \Omega$, $1 \leq i \leq I$, such that for each $i$ and any $R > 0$,

$$\liminf_{k\to\infty} \int_{B_R(x^{(i)})} V_k e^{4u_k}\, dx \geq 8\pi^2.$$

By (8) we then have $I \leq C\Lambda$. Moreover, given $x_0 \in \Omega \setminus \{x^{(i)};\ 1 \leq i \leq I\})$, we can choose a radius $R > 0$ such that

$$\limsup_{k\to\infty} \int_{B_R(x_0)} V_k e^{4u_k}\, dx < 8\pi^2. \tag{16}$$

For such $x_0$ and $R > 0$ decompose

$$u_k = v_k + h_k \quad \text{on } B_R(x_0),$$

where $v_k$ satisfies

$$\Delta^2 v_k = V_k e^{4u_k} \quad \text{in } B_R(x_0), \qquad v_k = \Delta v_k = 0 \quad \text{on } \partial B_R(x_0),$$

and with $\Delta^2 h_k = 0$ in $B_R(x_0)$.

By (8) and Theorem 2.1 we then have

$$\|h_k^+\|_{L^1(B_R(x_0))} \le \|u_k^+\|_{L^1(B_R(x_0))} + \|v_k\|_{L^1(B_R(x_0))} \le C, \tag{17}$$

uniformly in $k$.

We now distinguish the following cases.

**Case 1:** Suppose that $\|h_k\|_{L^1(B_{R/2}(x_0))} \le C$, uniformly in $k$. Then Lemma 2.2 shows that for all $x \in B_{R/8}(x_0)$ we can bound

$$|\Delta h_k(x)| = \left| \fint_{B_{R/8}(x)} \Delta h_k(y)\,dy \right| \le CR^{-2} \fint_{B_{R/2}(x_0)} |h(z)|\,dz \le C,$$

uniformly in $k$ and $x$, and $(h_k)$ is locally bounded in $C^4$ on $B_{R/8}(x_0)$. But then from Lemma 2.2 and (17) we also obtain

$$\fint_{B_R(x_0)} |h(x)|\,dx \le C - \fint_{B_R(x_0)} h(x)\,dx = C + \frac{1}{12} R^2 \Delta h_k(x_0) - h_k(x_0) \le C.$$

By repeating the first step of the argument on any ball contained in $B_R(x_0)$ we then infer that $(h_k)$ is locally bounded in $C^4$ on $B_R(x_0)$.

But then by Theorem 2.1 and (16) we see that

$$\Delta^2 v_k = V_k e^{4u_k} = (V_k e^{4h_k}) e^{4v_k}$$

is locally bounded in $L^p$ on $B_R(x_0)$ for some uniform number $p > 1$. Since Theorem 2.1 also yields uniform $L^1$-bounds for $v_k$, we may conclude that $(v_k)$ is locally bounded in $C^{3,\alpha}$ on $B_R(x_0)$ for any $\alpha < 1$, and hence so is $(u_k)$.

**Case 2:** Now assume that $\beta_k := \|h_k\|_{L^1(B_{R/2}(x_0))} \to \infty$ as $k \to \infty$. Normalize

$$\varphi_k = \frac{h_k}{\|h_k\|_{L^1(B_{R/2}(x_0))}},$$

so that $\|\varphi_k\|_{L^1(B_{R/2}(x_0))} = 1$ for all $k$. By arguing as in Case 1, we then find that $(\varphi_k)$ is locally bounded in $C^4$ on $B_R(x_0)$. A subsequence as $k \to \infty$ therefore converges in $C^{3,\alpha}_{\mathrm{loc}}(B_R(x_0))$ to a limit $\varphi$ satisfying the equation $\Delta^2\varphi = 0$ in $B_R(x_0)$ and with $\|\varphi\|_{L^1(B_{R/2}(x_0))} = 1$. Clearly, the function $\varphi$ then cannot vanish identically. By (17), moreover, we have $\|\varphi^+\|_{L^1(B_R(x_0))} = 0$, and therefore $\varphi \le 0$. It then follows from Lemma 2.2 that $\Delta\varphi(x) \ne 0$ at any point $x$ where $\varphi(x) = 0$. The set $S_0 = \{x \in B_R(x_0); \varphi(x) = 0\}$ is hence of codimension $\ge 1$ and therefore also has vanishing measure; moreover, $S_0$ is closed and nowhere dense. Thus, we conclude that $\varphi < 0$ almost everywhere and hence $h_k = \beta_k\varphi_k \to -\infty$ almost everywhere and locally uniformly away from $S_0$ as $k \to \infty$. Again observing that

$$\Delta^2 v_k = V_k e^{4u_k} = (V_k e^{4h_k}) e^{4v_k}$$

is locally bounded in $L^p$ on $B_R(x_0) \setminus S_0$ for some uniform number $p > 1$, as before we conclude that $(v_k)$ is locally bounded in $C^{3,\alpha}$ for any $\alpha < 1$ on $B_R(x_0) \setminus S_0$. It follows that $u_k = v_k + h_k \to -\infty$ almost everywhere and locally uniformly away from $S_0$ as $k \to \infty$ and $u_k/\beta_k \to \varphi$.

Since Cases 1 and 2 are mutually exclusive and since the region $\Omega \setminus \{x^{(i)};\ 1 \le i \le I\}$ is connected, upon covering this region with balls $B_R(x_0)$ as above we see that either a subsequence $(u_k)$ is locally bounded in $C^{3,\alpha}$ away from $\{x^{(i)};\ 1 \le i \le I\}$ for any $\alpha < 1$, and hence $(u_k)$ is relatively compact in $C^{3,\alpha}$ on this domain for any $\alpha < 1$, or $u_k \to -\infty$ almost everywhere and locally uniformly away from $S = S_0 \cup \{x^{(i)};\ 1 \le i \le I\})$, with $(u_k/\beta_k)$ converging to a nontrivial biharmonic limit $\varphi \le 0$.

Finally, we show that whenever there is concentration only the second case can occur, that is, $u_k \to -\infty$ almost everywhere as $k \to \infty$ if $\{x^{(i)};\ 1 \le i \le I\}) \ne \emptyset$. Indeed, suppose by contradiction that there is at least one concentration point and that $u_k \to u$ in $C^{3,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^{(i)};\ 1 \le i \le I\})$ as $k \to \infty$. By Robert's result [18], or by the reasoning of Wei [21] we then have the convergence

$$V_k e^{4u_k} dx \rightharpoonup e^{4u} dx + \sum_{i=1}^{I} m_i \delta_{x^{(i)}}$$

weakly in the sense of measures, where $m_i \ge 16\pi^2$, $1 \le i \le I$. But near each $x^{(i)}$ the leading term in the Green function $G$ for the bi-Laplacian is given by

$$G(x) = \frac{1}{8\pi^2} \log\left(\frac{1}{|x - x^{(i)}|}\right).$$

By arguing as in Brézis–Merle [3, p. 1242 f.], we then conclude that

$$u(x) \ge 2 \log\left(\frac{1}{|x - x^{(i)}|}\right) - C$$

near $x^{(i)}$, and with a constant $c_0 > 0$ we find

$$e^{4u(x)} \ge c_0 |x - x^{(i)}|^{-8} \notin L^1(\Omega),$$

thus contradicting the hypothesis (8). This completes the proof of the asserted macroscopic concentration behavior of $(u_k)$.

In order to analyze the asymptotic behavior of $(u_k)$ near concentration points we adapt an argument of Schoen to our setting; see [20, proof of Theorem 2.2]. Let $x_0 \in S$ with $\sup_{B_r(x_0)} u_k \to \infty$ for every $r > 0$ as $k \to \infty$. For $r \ge 0$ denote as $K_r(x_0) = \{x;\ |x - x_0| \le r\}$ the closed $r$-ball centered at $x_0$. For $R < \mathrm{dist}(x_0, \partial\Omega)$ choose $0 \le r_k < R$ and $x_k \in K_{r_k}(x_0)$ such that

$$(R - r_k) e^{u_k(x_k)} = (R - r_k) \sup_{K_{r_k}(x_0)} e^{u_k} = \max_{0 \le r < R} \left((R - r) \sup_{K_r(x_0)} e^{u_k}\right) =: L_k. \tag{18}$$

Note that $L_k \to \infty$ as $k \to \infty$. Define $s_k = (R - r_k)/2L_k$ and similar to (5) let

$$v_k(x) = u_k(x_k + s_k x) + \log s_k,$$

satisfying

$$\sup_{K_{L_k}(0)} e^{v_k} = s_k \sup_{K_{(R-r_k)/2}(x_k)} e^{u_k} \le s_k \sup_{K_{(R+r_k)/2}(x_0)} e^{u_k} = L_k^{-1}\left(R - \frac{R+r_k}{2}\right) \sup_{K_{(R+r_k)/2}(x_0)} e^{u_k}$$
$$\le L_k^{-1}(R-r_k)e^{u_k(x_k)} = 1 = 2e^{v_k(0)}$$

in view of (18), which is equivalent to the assertion (9).

Observe that $v_k$ solves the equation

$$\Delta^2 v_k = W_k e^{4v_k}$$

in $B_{L_k}(0)$, where the sequence of balls $B_{L_k}(0)$ exhausts all of $\mathbb{R}^4$ and

$$W_k(x) = V_k(x_k + s_k x) \to 1 \quad \text{locally uniformly in } \mathbb{R}^4;$$

moreover,

$$\int_{B_{L_k}(0)} W_k e^{4v_k}\, dx \le \Lambda$$

for all $k$. By applying the previous result to the sequence of blown-up functions $v_k$, we then obtain the microscopic description of blow-up asserted in Theorem 1.2. The characterization of the limit function $\psi$ follows from Theorem 2.4. □

## 3. An example

We demonstrate the absence of quantization also in the radially symmetric case by means of the following example.

**Example 3.1.** Consider the radially symmetric function $\varphi$ with

$$\Delta^2\varphi = e^{-|x|^2/2} \quad \text{in } \mathbb{R}^4, \quad \varphi(0) = \Delta\varphi(0) = 0.$$

This function can be computed explicitly. In fact, for any $x \in \mathbb{R}^4$ we have

$$\varphi(x) = \int_0^{|x|} s^{-3}\left\{\int_0^s t^3\left[\int_0^t \sigma^{-3}\left(\int_0^\sigma \tau^3 e^{-\tau^2/2}\, d\tau\right) d\sigma\right] dt\right\} ds.$$

For $k \in \mathbb{N}$ and $x \in \mathbb{R}^4$ let

$$u_k(x) = \ln k - \frac{k^6|x|^2}{8} + k^{-8}\varphi(k^3 x).$$

Then $(u_k)$ satisfies equation (1), that is,

$$\Delta^2 u_k = V_k e^{4u_k},$$

where

$$V_k(x) = e^{-4k^{-8}\varphi(k^3x)} \to 1 \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^4) \text{ as } k \to \infty.$$

Thus, also (2) is satisfied. Finally, we compute that $V_k e^{4u_k} \rightharpoonup 0$ in the sense of measures when $k \to \infty$.

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