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A symmetry problem in the calculus of variations

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Abstract. We consider the integral functional

$$J(u) = \int_\Omega [f(|Du|) - u]\, dx, \qquad u \in W_0^{1,1}(\Omega),$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a nonempty bounded connected open subset of $\mathbb{R}^n$ with smooth boundary, and $\mathbb{R} \ni s \mapsto f(|s|)$ is a convex, differentiable function. We prove that if J admits a minimizer in $W_0^{1,1}(\Omega)$ depending only on the distance from the boundary of Ω, then Ω must be a ball.

Keywords. Minimizers of integral functionals, distance function, Euler equation

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smooth domain, and let J_2 be the integral functional defined in $H_0^1(\Omega)$ by

$$J_2(u) = \int_\Omega \left(\frac{1}{2}|Du|^2 - u\right) dx, \qquad u \in H_0^1(\Omega).$$

It is well known that J_2 has a unique minimum point in $H_0^1(\Omega)$, which is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In recent papers [11, 10], the following question arises in connection with the estimate of the minimum of J_2: If the minimizer u_0 of J_2 depends only on the distance from the boundary of Ω, what can be said about the geometry of Ω?

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In order to answer to this question, let us recall a celebrated result by J. Serrin (see [19]) which states that if the overdetermined Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \left|\dfrac{\partial u}{\partial \nu}\right| = c & \text{on } \partial\Omega, \end{cases} \tag{2}$$

admits a solution, then Ω must be a ball. (Here $\partial u/\partial\nu$ denotes the derivative with respect to the outer normal to the boundary of Ω.)

Now, assume that the minimizer u_0 of J_2 depends only on the distance d_Ω from the boundary of Ω, that is, $u_0(x) = \phi(d_\Omega(x))$, $x \in \Omega$. Since the outer normal to $\partial\Omega$ is given by $\nu(y) = -Dd_\Omega(y)$ for every $y \in \partial\Omega$, we have

$$\left|\frac{\partial u_0}{\partial \nu}(y)\right| = |\phi'(0) Dd_\Omega(y) \cdot \nu(y)| = |\phi'(0)|, \qquad \forall y \in \partial\Omega,$$

hence u_0 is a solution to the overdetermined problem (2) with $c = |\phi'(0)|$. From Serrin's result, we conclude that Ω is a ball.

We remark that the above argument works also for the functional

$$J_p(u) = \int_\Omega \left(\frac{1}{p}|Du|^p - u\right) dx, \qquad u \in W_0^{1,p}(\Omega),\ 1 < p < \infty,$$

using the analog of Serrin's result for the p-Laplace operator (see [3, 13, 12]).

Our aim is to extend this kind of symmetry results to the general functional

$$J(u) = \int_\Omega [f(|Du|) - u]\, dx, \qquad u \in W_0^{1,1}(\Omega), \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth domain and $\mathbb{R} \ni s \mapsto f(|s|)$ is a convex, differentiable function. Let us define the set of so-called *web functions* (or *radial functions*)

$$\mathcal{W}(\Omega) = \{u \in W_0^{1,1}(\Omega);\ u \text{ depends only on the distance from } \partial\Omega\}. \tag{4}$$

We prove that if the minimum problem

$$\min\{J(u);\ u \in W_0^{1,1}(\Omega)\} \tag{5}$$

admits a solution belonging to $\mathcal{W}(\Omega)$, then Ω is a ball (see Theorem 1). The converse is also true (see Remark 1).

Our approach is not based on the analysis of an associated overdetermined problem. Indeed, the Euler equation associated to the functional J is, at least formally, the nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(f'(|Du|)\dfrac{Du}{|Du|}\right) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

For such general equations there are no results concerning the overdetermined problem with

$$\left|\frac{\partial u}{\partial \nu}\right| = c \quad \text{on } \partial\Omega, \tag{7}$$

similar to Serrin's result for (2).

On the other hand, the requirement that the integral functional J has a minimizer in $\mathcal{W}(\Omega)$ is much stronger than the boundary condition (7). We shall fully exploit this assumption in order to obtain the following results.

(a) An explicit representation formula for the solutions to the minimum problem

$$\min\{J(u);\ u \in \mathcal{W}(\Omega)\}$$

(see Theorem 2 below).

(b) The validity in the sense of distributions of the Euler–Lagrange equation (6) associated to the minimum problem (5), for minima belonging to $\mathcal{W}(\Omega)$ (see Theorem 3 below).

(c) Existence, uniqueness and explicit representation of the solution v of the equation

$$-\operatorname{div}(v(x)Dd_\Omega(x)) = 1 \quad \text{in } \Omega,$$

which is related to the Euler–Lagrange equation (6) when $u \in \mathcal{W}(\Omega)$.

We remark that the results in (c) were obtained in [5, 6] in the framework of mathematical models for sandpile growth.

Combining (a), (b) and (c) we shall prove that the mean curvature of $\partial\Omega$ is constant. This implies that Ω must be a ball, thanks to a fundamental result in differential geometry by A. D. Aleksandrov [1].

The paper is organized as follows. In Section 2 we state the main result of the paper (see Theorem 1 below). The existence and characterization of minimizers of J in the space $\mathcal{W}(\Omega)$ is established in Section 3, whereas the validity of the Euler–Lagrange equation for minimizers of J in $W_0^{1,1}(\Omega)$, belonging to $\mathcal{W}(\Omega)$, is proven in Section 4. Finally, in Section 5 we recall the result proven in [6] and complete the proof of Theorem 1.

2. Notation and statement of the result

In what follows, Ω will denote a smooth domain in $\mathbb{R}^n$, that is, a nonempty bounded open connected subset of $\mathbb{R}^n$ with C^2 boundary. We denote by $\partial\Omega$ the boundary of Ω, and by $d_\Omega : \overline{\Omega} \to \mathbb{R}$ the distance function to $\partial\Omega$. The symbol r_Ω will denote the *inradius* of Ω, that is, the supremum of the radii of the balls contained in Ω. It is easily seen that $r_\Omega = \max\{d_\Omega(x);\ x \in \Omega\}$.

If $A \subset \mathbb{R}^n$, we denote by $|A|$ and $\mathcal{H}^{n-1}(\partial A)$, respectively, the Lebesgue measure of A and the $(n-1)$-dimensional Hausdorff measure of ∂A.

Let J be the functional defined in (3). Our assumptions on f are the following:

(F1) $f : [0, b) \to \mathbb{R}$, $b \in (0, \infty]$, is a convex, differentiable, nondecreasing function. (If $b < \infty$, for notational convenience we set $f(s) = \infty$ for every $s \geq b$.)

(F2) If $0 < b < \infty$,

$$\lim_{s \to b^-} f(s) = \infty\,;$$

if $b = \infty$,

$$\lim_{s\to\infty} \frac{f(s)}{s} = \lim_{s\to\infty} f'(s) > \frac{1}{n} \left(\frac{|\Omega|}{v_n}\right)^{1/n},$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$.

(F3) $f'_+(0) := \lim_{s\to 0^+} \dfrac{f(s) - f(0)}{s} = 0.$

We remark that, in the case $b = \infty$, assumption (F2) is certainly satisfied if f is a superlinear function, that is, if

$$\lim_{s\to\infty} \frac{f(s)}{s} = \infty.$$

Assumption (F3) is equivalent to the differentiability of the map $s \mapsto f(|s|)$ at $s = 0$.

Our main result is the following.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, let J be the functional defined in* (3)*, and assume that f satisfies assumptions* (F1)–(F3)*. If J admits a minimizer in $W_0^{1,1}(\Omega)$ that depends only on the distance from the boundary of Ω, then Ω must be a ball.*

The proof of this theorem is postponed to Section 5.

Remark 1. Of course, the converse of Theorem 1 also holds. Namely, if Ω is a ball and f satisfies assumptions (F1)–(F3), then the functional J admits a unique minimizer in $W_0^{1,1}(\Omega)$, which is radially symmetric (see for example [8]).

Remark 2. In Theorem 1, the assumption that f be differentiable cannot be dropped. Namely, let $f(s) = \max\{0, s - \rho\}$, $s \geq 0$, where $\rho \geq 0$ is a fixed parameter. Then f is a convex nondecreasing function in $[0, \infty)$, and, if $\rho > 0$, it is not differentiable at $s = \rho$. Let $\Omega \subset \mathbb{R}^n$ be a smooth domain. Assume in addition that Ω is a convex set, and that $|\Omega| \leq v_n$, so that (F2) clearly holds. Under these assumptions, in [20] it was proven that the function $u_0(x) = \rho\, d_\Omega(x)$ is a minimizer of J in $W_0^{1,1}(\Omega)$. (In [6] the same result is proven also in the case of nonconvex domains.) This assertion also holds in the case $\rho = 0$. We remark that, in this case, f does not satisfy assumption (F3). This example shows that, for such f, the functional J admits minimizers depending only on the distance from $\partial\Omega$ even if Ω is not a ball.

3. Existence of minimizers in $\mathcal{W}(\Omega)$

The aim of this section is to prove, under the assumptions of Theorem 1, that the functional J has a minimizer in the space $\mathcal{W}(\Omega)$ of web functions. Moreover, we give an explicit representation of the minimizers and we prove that they satisfy a suitable Euler–Lagrange inclusion.

Throughout this section, Ω will be a smooth domain of $\mathbb{R}^n$, although Theorem 2 below still holds under a weaker regularity assumption on the boundary of Ω (see Remark 4 below).

We recall that d_Ω is a Lipschitz continuous function, with gradient satisfying $|Dd_\Omega(x)| = 1$ for a.e. $x \in \Omega$. It is clear from the definition that $0 \leq d_\Omega(x) \leq r_\Omega$ for every $x \in \overline{\Omega}$, where r_Ω denotes the inradius of Ω. For every $i = 1, \dots, n-1$, denote by $\kappa_i(y)$ the i-th principal curvature of $\partial\Omega$ at the point $y \in \partial\Omega$, corresponding to a principal direction $e_i(y)$ orthogonal to $Dd_\Omega(y)$, with the sign convention $\kappa_i(y) \geq 0$ if the normal section of Ω along the direction e_i is convex. Let Σ denote the singular set of d_Ω, that is, the set of points $x \in \Omega$ for which d_Ω is not differentiable. The set $\overline{\Sigma}$ is also known as *ridge* or *cut locus*. From Rademacher's theorem, Σ has vanishing n-dimensional Lebesgue measure. Introducing the projection $\Pi(x)$ of $x \in \Omega$ on $\partial\Omega$, Σ is also the set of points x for which $\Pi(x)$ is not a singleton. We extend κ_i, $i = 1, \dots, n-1$, to $\overline{\Omega} \setminus \overline{\Sigma}$ by setting $\kappa_i(x) = \kappa_i(\Pi(x))$ for every $x \in \Omega \setminus \overline{\Sigma}$. Define the *normal distance* to the cut locus of Ω by

$$\tau(x) = \begin{cases} \min\{t \geq 0;\ x + t\, Dd_\Omega(x) \in \overline{\Sigma}\} & \text{if } x \in \overline{\Omega} \setminus \overline{\Sigma}, \\ 0, & \text{if } x \in \overline{\Sigma}. \end{cases} \tag{8}$$

It is known that if $\partial\Omega$ is of class $C^{2,1}$, then τ is Lipschitz continuous on $\partial\Omega$ (see [18, 17]); for less regular domains the Lipschitz continuity may fail, but continuity is preserved if $\partial\Omega$ is of class C^2 (see [5, 6]).

The function $f\colon [0,\infty) \to [0,\infty]$ appearing in the definition of J will be a lower semicontinuous, nondecreasing convex function. We remark that, for the results established in this section, f need not be differentiable, and can take the value ∞. With some abuse of notation, we will denote by $f^*\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ the conjugate function of the map $s \mapsto f(|s|)$, $s \in \mathbb{R}$. As is customary, the symbol $\partial f(s)$ will denote the subgradient of f at s, in the sense of convex analysis.

In the following, a major role will be played by the function $\alpha\colon [0, r_\Omega] \to \mathbb{R}$ defined by

$$\alpha(t) = \begin{cases} \dfrac{|\Omega_t|}{\mathcal{H}^{n-1}(\partial\Omega_t)} & \text{if } t \in [0, r_\Omega), \\ 0 & \text{if } t = r_\Omega, \end{cases} \tag{9}$$

where $\Omega_t = \{x \in \Omega;\ d_\Omega(x) > t\}$. (We recall that r_Ω denotes the inradius of Ω.)

Theorem 2. *Let Ω be a smooth domain in $\mathbb{R}^n$. Let J be the functional defined in* (3), *where $f\colon [0,\infty) \to [0,\infty]$ is a lower semicontinuous, nondecreasing convex function, satisfying*

$$f(s) \geq Ms - a, \qquad \forall s \geq 0, \tag{10}$$

for some positive constants M and a, with

$$M > \frac{1}{n}\left(\frac{|\Omega|}{v_n}\right)^{1/n}. \tag{11}$$

Then, for every measurable selection

$$\gamma(t) \in \partial f^*(\alpha(t)), \qquad t \in [0, r_\Omega], \tag{12}$$

the function

$$u_0(x) = \int_0^{d_\Omega(x)} \gamma(t)\, dt, \qquad x \in \Omega,$$

belongs to $W^{1,\infty}(\Omega)$ and is a minimizer of J in the set $\mathcal{W}(\Omega)$ defined in (4)*. Conversely, if $u_0 \in \mathcal{W}(\Omega)$ is a minimizer of J in $\mathcal{W}(\Omega)$, then u_0 belongs to $W^{1,\infty}(\Omega)$ and satisfies the Euler–Lagrange inclusion*

$$|Du_0(x)| \in \partial f^*(\alpha(d_\Omega(x))), \qquad a.e.\ x \in \Omega. \tag{13}$$

Remark 3. Under the additional assumption of convexity of Ω, Theorem 2 was proved in [9], without convexity assumptions on f.

The remaining part of this section will be devoted to the proof of Theorem 2. We start by proving a simple estimate on the function α defined in (9).

Lemma 1. *For every $t \in [0, r_\Omega)$ we have*

$$0 < \alpha(t) \le \frac{1}{n}\left(\frac{|\Omega_t|}{v_n}\right)^{1/n} \le \frac{1}{n}\left(\frac{|\Omega|}{v_n}\right)^{1/n}. \tag{14}$$

(Recall that v_n is the volume of the unit ball in $\mathbb{R}^n$.) As a consequence, $\lim_{t\to r_\Omega} \alpha(t) = \alpha(r_\Omega) = 0$.

Proof. The fact that α is a positive function on $[0, r_\Omega)$ follows from its very definition. Concerning the upper bound, we recall that the isoperimetric inequality

$$\mathcal{H}^{n-1}(\partial A) \ge n v_n^{1/n} |A|^{(n-1)/n} \tag{15}$$

holds for every bounded measurable set $A \subset \mathbb{R}^n$ (see [4, §14.3 and §14.6]). Applying (15) to the bounded measurable set Ω_t we get

$$0 < \alpha(t) = |\Omega_t|^{1/n} \frac{|\Omega_t|^{(n-1)/n}}{\mathcal{H}^{n-1}(\partial\Omega_t)} \le \frac{1}{n v_n^{1/n}} |\Omega_t|^{1/n} \le \frac{1}{n}\left(\frac{|\Omega|}{v_n}\right)^{1/n},$$

and the proof is complete.

Let us define the set

$$\mathcal{K} = \left\{ \phi \in AC_{\mathrm{loc}}[0, r_\Omega) \;\middle|\; \begin{array}{l} \phi(0) = 0 \\ t \mapsto \mathcal{H}^{n-1}(\partial\Omega_t)\phi(t) \in L^1(0, r_\Omega) \\ t \mapsto \mathcal{H}^{n-1}(\partial\Omega_t)\phi'(t) \in L^1(0, r_\Omega) \end{array} \right\}, \tag{16}$$

where $AC_{\mathrm{loc}}[0, r_\Omega)$ denotes the set of absolutely continuous functions in $[0, r]$ for every $r \in (0, r_\Omega)$.

Lemma 2. *A function u belongs to $\mathcal{W}(\Omega)$ if and only if $u = \phi \circ d_\Omega$ for some $\phi \in \mathcal{K}$.*

Proof. Let $u \in \mathcal{W}(\Omega)$. By definition of $\mathcal{W}(\Omega)$, there exists a measurable function $\phi\colon [0, r_\Omega] \to \mathbb{R}$ such that $u(x) = \phi(d_\Omega(x))$ for every $x \in \Omega$.

The tricky part of the proof that $\phi \in \mathcal{K}$ consists in showing that ϕ belongs to $AC_{\mathrm{loc}}[0, r_\Omega)$. We shall use a local coordinate system in Ω whose properties were proved in [16, p. 236]. More precisely, since Ω has C^2 boundary, we can choose a finite family $\mathcal{U}_1, \dots, \mathcal{U}_N$ of bounded open sets in $\mathbb{R}^n$ so that $\partial\Omega \subseteq \bigcup_{i=1}^N \mathcal{U}_i$, and such that, for each $i = 1, \dots, N$, in a suitable coordinate system in $\mathbb{R}^n$ we have

$$\Omega \cap \mathcal{U}_i = \{(x', t);\ x' \in V_i,\ t > \Phi_i(x')\} \cap \mathcal{U}_i, \tag{17}$$

where $V_i = \{x' \in \mathbb{R}^{n-1};\ (x', t) \in \mathcal{U}_i \text{ for some } t \in \mathbb{R}\}$ is an open set, and Φ_i is a C^2 function on $\mathbb{R}^{n-1}$. Define the maps $G_i\colon V_i \times \mathbb{R} \to \mathbb{R}^n$, $i = 1, \dots, N$, by

$$G_i(x', t) = y + t\, Dd_\Omega^s(y), \qquad \text{where } y = (x', \Phi_i(x')) \in \partial\Omega, \tag{18}$$

and d_Ω^s denotes the signed distance to $\partial\Omega$ defined by

$$d_\Omega^s(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

We collect here the main properties of the maps G_i, $i = 1, \dots, N$ (see [16, Lemmas 14 and 15]):

(a) G_i is Lipschitz continuous on bounded subsets of $V_i \times \mathbb{R}$.
(b) The Jacobian JG_i is a locally bounded measurable function, and

$$JG_i(x', t) = \sqrt{1 + |D\Phi_i(x')|^2}\, \prod_{j=1}^{n-1} (1 - \kappa_j t),$$

where $\kappa_1, \dots, \kappa_{n-1}$ are the principal curvatures of $\partial\Omega$ at $G_i(x', 0)$.
(c) G_i is one-to-one on the set

$$U_i = \{(x', t);\ x' \in V_i,\ t \in (0, \tau(G_i(x', 0)))\} \subset \mathbb{R}^n.$$

(d) $\Omega \setminus \Sigma = \bigcup_{i=1}^N G_i(U_i)$.

From properties (a), (b) and (c) we deduce that, for every $i = 1, \dots, N$ and every $\epsilon > 0$, G_i is a bi-Lipschitz map on the set

$$U_i^\epsilon = \{(x', t);\ x' \in V_i,\ t \in (0, \max\{0, \tau(G_i(x', 0)) - \epsilon\})\}.$$

Since the restriction of u to the set $A_i^\epsilon = G_i(U_i^\epsilon)$ belongs to $W^{1,1}(A_i^\epsilon)$, from Theorem 2.2.2 in [21] we see that the restriction of the composite map $v = u \circ G_i$ to U_i^ϵ belongs to $W^{1,1}(U_i^\epsilon)$.

Finally, let us prove that, for every fixed $\epsilon \in (0, r_\Omega)$, the map ϕ belongs to $AC[0, r_\Omega - \epsilon]$. From property (d) and the fact that the n-dimensional Lebesgue measure of Σ is zero, there exists an index $i \in \{1, \dots, N\}$ such that

$$\max\{\tau(G_i(x', 0));\ x' \in V_i\} = r_\Omega.$$

From Theorem 2.1.4 in [21] we can assume that the function $v = u \circ G_i$ is absolutely continuous on the line segment

$$\Lambda(x') = \{(x', t);\ t \in [0, \max\{0, \tau(G_i(x', 0)) - \epsilon/2\}]\}$$

for almost every $x' \in V_i$. Since τ is a continuous map, there exists $x' \in V_i$ such that $T = \tau(G_i(x', 0)) > r_\Omega - \epsilon$ and the restriction of v to $\Lambda(x')$ is absolutely continuous. By the very definitions of the functions G_i and v we conclude that the map $t \mapsto u(y + tDd_\Omega(y)) = \phi(t)$, with $y = (x', \Phi_i(x'))$, is absolutely continuous in $[0, T]$; in particular, $\phi \in AC[0, r_\Omega - \epsilon]$. Furthermore, the boundary condition on u implies that $\phi(0) = 0$.

Let us conclude the proof of the lemma. From the change of variables formula (see [14, §3.4.3]) it follows that

$$\int_\Omega |u(x)|\, dx = \int_0^{r_\Omega} \mathcal{H}^{n-1}(\partial\Omega_t)|\phi(t)|\, dt,$$
$$\int_\Omega |Du(x)|\, dx = \int_0^{r_\Omega} \mathcal{H}^{n-1}(\partial\Omega_t)|\phi'(t)|\, dt,$$

hence $\phi \in \mathcal{K}$. Conversely, from the above formulas it is easily seen that if $\phi \in \mathcal{K}$, then $u = \phi \circ d_\Omega$ belongs to $W_0^{1,1}(\Omega)$.

Remark 4. Following [16, Section 3], it can be easily proved that the conclusion of Lemma 2 holds (with minor modifications in the proof) under a weaker assumption on the regularity of the boundary of Ω. More precisely, it is enough that the bounded open set $\Omega \subset \mathbb{R}^n$ is *of positive reach* (see [15]), that is, there exists $r > 0$ with the following property: for every $y \in \partial\Omega$ there exists a closed ball $B \subset \mathbb{R}^n \setminus \Omega$, of radius r, such that $B \cap \overline{\Omega} = \{y\}$.

As a consequence of Lemma 2, a function $u_0 = \phi_0 \circ d_\Omega$ is a minimizer of J in $\mathcal{W}(\Omega)$ if and only if ϕ_0 is a minimizer of the functional

$$F(\phi) = \int_0^{r_\Omega} \mathcal{H}^{n-1}(\partial\Omega_t)[f(|\phi'(t)|) - \phi(t)]\, dt \tag{19}$$

in $\mathcal{K}$. In order to simplify the subsequent analysis, it is convenient to rewrite the term in ϕ using the following integration-by-parts formula.

Lemma 3. *If $\phi \in \mathcal{K}$, then the map $t \mapsto |\Omega_t|\phi'(t)$ belongs to $L^1(0, r_\Omega)$ and*

$$\int_0^{r_\Omega} |\Omega_t|\phi'(t)\,dt = \int_0^{r_\Omega} \mathcal{H}^{n-1}(\partial\Omega_t)\phi(t)\,dt. \tag{20}$$

Proof. From the isoperimetric inequality (15) we have

$$|\Omega_t| = |\Omega_t|^{1/n}|\Omega_t|^{(n-1)/n} \le |\Omega|^{1/n}\frac{1}{n v_n^{1/n}}\mathcal{H}^{n-1}(\partial\Omega_t),$$

hence

$$|\Omega_t|\,|\phi'(t)| \le \frac{1}{n}\left(\frac{|\Omega|}{v_n}\right)^{1/n}\mathcal{H}^{n-1}(\partial\Omega_t)|\phi'(t)|, \qquad \text{a.e. } t \in [0, r_\Omega].$$

Since $t \mapsto \mathcal{H}^{n-1}(\partial\Omega_t)\phi'(t)$ belongs to $L^1(0, r_\Omega)$, so does $t \mapsto |\Omega_t|\phi'(t)$. If we recall that $t \mapsto |\Omega_t|$ is absolutely continuous in $[0, r_\Omega]$ and $\frac{d}{dt}\,|\Omega_t| = -\mathcal{H}^{n-1}(\partial\Omega_t)$ for a.e. t, formula (20) now follows from a standard integration by parts.

In view of Lemma 3, the functional F can be rewritten as

$$F(\phi) = \int_0^{r_\Omega} g(t, \phi'(t))\,dt, \qquad \phi \in \mathcal{K}, \tag{21}$$

where $g\colon [0, r_\Omega] \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined by

$$g(t, \xi) = \mathcal{H}^{n-1}(\partial\Omega_t)\,f(|\xi|) - |\Omega_t|\,\xi\,. \tag{22}$$

The advantage of rewriting F in this way lies in the fact that the lagrangean g does not depend on ϕ. We remark that $g(t, \cdot)$ is a convex function for every $t \in [0, r_\Omega]$. From assumption (10) and estimate (14) we have

$$\begin{aligned} g(t, \xi) &\ge \mathcal{H}^{n-1}(\partial\Omega_t)[M - \alpha(t)]\,|\xi| - \mathcal{H}^{n-1}(\partial\Omega_t)a \\ &\ge \mathcal{H}^{n-1}(\partial\Omega_t)\left[M - \frac{1}{n}\left(\frac{|\Omega|}{v_n}\right)^{1/n}\right]|\xi| - \mathcal{H}^{n-1}(\partial\Omega_t)a, \end{aligned}$$

hence from the assumption (11),

$$\lim_{|\xi|\to\infty} g(t, \xi) = \infty$$

for every $t \in [0, r_\Omega)$. This implies that, for every such t, the convex function $g(t, \cdot)$ has a nonempty compact set of minimizers. Recalling that ξ is a minimizer of $g(t, \cdot)$ if and only if $0 \in \partial g(t, \xi)$, that is, if and only if $\xi \in \partial g^*(t, 0)$, we conclude that the convex set $\partial g^*(t, 0)$ is nonempty and compact for every $t \in [0, r_\Omega)$. Moreover, a simple computation shows that

$$\partial g^*(t, 0) = \partial f^*(\alpha(t)), \qquad \forall t \in [0, r_\Omega).$$

The following lemma will be used in order to prove the Lipschitz regularity of the minimizers of J in $\mathcal{W}(\Omega)$.

Lemma 4. *Let* $g\colon [0, r_\Omega] \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *be the function defined in* (22), *where* f *satisfies the assumptions of Theorem 2. Then there exists a positive constant C with the following property. If* $\xi\colon [0, r_\Omega] \to \mathbb{R}$ *is a measurable selection of the multifunction* $t \mapsto \partial g^*(t, 0)$, *then* $|\xi(t)| \le C$ *for a.e.* $t \in [0, r_\Omega]$.

Proof. Denote by

$$M_n = \frac{1}{n}\left(\frac{|\Omega|}{v_n}\right)^{1/n}$$

the constant appearing in (11). From (10) we deduce that the open interval $(-M, M)$ is contained in the essential domain of f^* (that is, f^* is finite in that interval). Since $M > M_n$ by assumption (11), and $0 < \alpha(t) \le M_n$ by (14), from the monotonicity of the subgradient we know that $|\xi| \le (f^*)'_+(M_n) < \infty$ for every $\xi \in \partial f^*(\alpha(t))$. The conclusion now follows by choosing $C = (f^*)'_+(M_n)$.

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. Let $g\colon [0, r_\Omega] \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the function defined in (22), and let $\mathcal{K}$ be the set defined in (16). From the discussion above, it is enough to prove that the functional

$$F(\phi) = \int_0^{r_\Omega} g(t, \phi'(t))\, dt, \qquad \phi \in \mathcal{K},$$

admits minimizers, and $\phi_0 \in \mathcal{K}$ is a minimizer of F if and only if

$$\phi'(t) \in \partial g^*(t, 0), \qquad \text{a.e. } t \in [0, r_\Omega]. \tag{23}$$

We have already shown that the multifunction

$$t \mapsto \partial g^*(t, 0) = \partial f^*(\alpha(t)), \qquad t \in [0, r_\Omega], \tag{24}$$

has nonempty, compact convex values for every $t \in [0, r_\Omega)$. Moreover, from Lemma 4, if $\xi(t)$ is a measurable selection of that multifunction, then $\xi \in L^\infty(0, r_\Omega)$. Hence, the function

$$\phi_0(t) := \int_0^t \xi(s)\, ds$$

belongs to $\mathcal{K} \cap W^{1,\infty}(0, r_\Omega)$. Let us show that ϕ_0 is a minimizer of F in $\mathcal{K}$. Since $\phi_0'(t) = \xi(t) \in \partial g^*(t, 0)$, we deduce that $0 \in \partial g(t, \phi_0'(t))$ for a.e. $t \in [0, r_\Omega]$, so that

$$F(\phi) - F(\phi_0) = \int_0^{r_\Omega} [g(t, \phi'(t)) - g(t, \phi_0'(t))]\, dt \ge 0$$

for every $\phi \in \mathcal{K}$.

Conversely, let $\phi_0 \in \mathcal{K}$ be a minimizer of F in $\mathcal{K}$. Let $\xi(t)$ be a measurable selection of the multifunction (24), and define $\phi(t) = \int_0^t \xi(s)\, ds$, $t \in [0, r_\Omega]$. From the first part of the proof, ϕ is a minimizer of F, so that $F(\phi) = F(\phi_0)$. Moreover, $\phi'(t)$ is a minimum point of $g(t, \cdot)$ for a.e. $t \in [0, r_\Omega]$, so that

$$g(t, \phi'(t)) - g(t, \phi_0'(t)) \le 0, \qquad \text{a.e. } t \in [0, r_\Omega].$$

Since $F(\phi) = F(\phi_0)$, we must have

$$g(t, \phi'(t)) = g(t, \phi_0'(t)), \qquad \text{a.e. } t \in [0, r_\Omega].$$

Hence $\phi_0'(t)$ must be a minimum point of $g(t, \cdot)$ for a.e. t, that is, (23) holds.

Remark 5. From the proof of Theorem 2 it is clear that a function $u_0(x) = \phi_0(d_\Omega(x))$, $\phi_0 \in \mathcal{K}$, is a minimizer of J in $\mathcal{W}(\Omega)$ if and only if ϕ_0 satisfies the Euler–Lagrange inclusion

$$\phi_0'(t) \in \partial f^*(\alpha(t)), \qquad \text{a.e. } t \in [0, r_\Omega], \tag{25}$$

where α is the function defined in (9). Since α is strictly positive in $[0, r_\Omega)$, and $\partial f^*(p) \subset [0, \infty)$ for every $p > 0$, the differential inclusion (25) implies that $\phi_0' \geq 0$ almost everywhere.

The last inequality can also be deduced directly from the fact that ϕ_0 is a minimizer of the functional F defined in (19). Namely, consider the function

$$\phi_1(t) = \int_0^t |\phi_0'(s)|\, ds, \qquad t \in [0, r_\Omega].$$

Since $\phi_0 \in \mathcal{K}$, it is easy to check that also $\phi_1 \in \mathcal{K}$, and $0 \leq \phi_1'(t) = |\phi_0'(t)|$ for a.e. $t \in [0, r_\Omega]$. Then $\phi_1(t) \geq \phi_0(t)$ for every $t \in [0, r_\Omega]$. Assume by contradiction that the set $E = \{t \in [0, r_\Omega];\ \phi_0'(t) < 0\}$ has positive Lebesgue measure. In this case, $\phi_1 > \phi_0$ on some interval. Hence $F(\phi_1) < F(\phi_0)$, in contradiction with the fact that ϕ_0 is a minimizer of F.

Remark 6. Assume that f satisfies (F1)–(F3), and extend f over all $\mathbb{R}$ by setting $f(s) = f(|s|)$ when $s < 0$. We remark that (F3) is equivalent to the differentiability of this extension at $s = 0$ (where we have $f'(0) = 0$). Hence f is differentiable everywhere in $(-b, b)$. Under this differentiability assumption, the inclusion (25) can be written in the equivalent form

$$\alpha(t) = f'(\phi_0'(t)), \qquad \text{a.e. } t \in [0, r_\Omega].$$

4. Validity of the Euler–Lagrange equation

The aim of this section is to establish a result concerning the validity of the Euler–Lagrange equation associated to the minimum problem (5), for minimizers belonging to the space $\mathcal{W}(\Omega)$.

Theorem 3. *Let Ω be a smooth domain in $\mathbb{R}^n$, let f satisfy* (F1)–(F3), *and let $u_0 \in \mathcal{W}(\Omega)$ be a minimizer of J in $W_0^{1,1}(\Omega)$. Then u_0 satisfies the Euler–Lagrange equation*

$$\int_\Omega [f'(|Du_0(x)|)\langle Dd_\Omega(x), D\varphi(x)\rangle - \varphi(x)]\, dx = 0, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{26}$$

Proof. As a first step, let us prove that

$$\lim_{\epsilon\to 0}\int_\Omega \frac{f(|Du_0+\epsilon D\varphi|)-f(|Du_0|)}{\epsilon}\,dx=\int_\Omega f'(|Du_0|)\langle Dd_\Omega, D\varphi\rangle\,dx \tag{27}$$

for every $\varphi\in C_0^\infty(\Omega)$. Since u_0 is also a minimizer of J in $\mathcal{W}(\Omega)$, from Theorem 2 we know that $u_0\in W^{1,\infty}(\Omega)$, hence there exists a positive constant C_0 such that

$$|Du_0(x)|\le C_0<b,\qquad \text{a.e. } x\in\Omega. \tag{28}$$

Now, let φ be a fixed function in $C_0^\infty(\Omega)$. From (28) there exists $\epsilon_0>0$ such that

$$|Du_0(x)+\epsilon D\varphi(x)|\le C<b,\qquad \text{a.e. } x\in\Omega, \tag{29}$$

for every $|\epsilon|\le\epsilon_0$. From the mean value theorem, for every such ϵ there exists a function θ_ϵ such that

$$\frac{f(|Du_0+\epsilon D\varphi|)-f(|Du_0|)}{\epsilon}=f'(|Du_0|+\theta_\epsilon)\langle Dd_\Omega, D\varphi\rangle$$

and $|\theta_\epsilon(x)|\le\epsilon|D\varphi(x)|$ for every $x\in\Omega$. This last estimate, together with (29), implies that

$$\left|\frac{f(|Du_0+\epsilon D\varphi|)-f(|Du_0|)}{\epsilon}\right|\le f'(C)\|D\varphi\|_\infty,\qquad \forall|\epsilon|\le\epsilon_0.$$

On the other hand, since f is a convex differentiable function, its derivative is continuous, hence

$$\lim_{\epsilon\to 0}\frac{f(|Du_0+\epsilon D\varphi|)-f(|Du_0|)}{\epsilon}=f'(|Du_0|)\langle Dd_\Omega, D\varphi\rangle.$$

The equality (27) now follows from the Lebesgue dominated convergence theorem.

Let us prove (26). For every $\varphi\in C_0^\infty(\Omega)$ and every $\epsilon>0$, since u_0 is a minimizer of J we have

$$0\le\frac{J(u_0+\epsilon\varphi)-J(u_0)}{\epsilon}=\int_\Omega\frac{f(|Du_0+\epsilon D\varphi|)-f(|Du_0|)-\epsilon\varphi}{\epsilon}\,dx.$$

Passing to the limit as $\epsilon\to 0^+$, from (27) we deduce that

$$\int_\Omega[f'(|Du_0|)\langle Dd_\Omega, D\varphi\rangle-\varphi]\,dx\ge 0.$$

Since this inequality also holds if we replace φ with $-\varphi$, (26) follows.

Remark 7. Without the assumption that u_0 depends only on the distance from $\partial\Omega$, the validity of the Euler–Lagrange equation can be established provided that f satisfies suitable growth conditions. See [7] for details.

5. Proof of Theorem 1

We will use the following result.

Theorem 4 (see [6]). *Let Ω be a smooth domain. Then the function $v\colon \overline{\Omega} \to \mathbb{R}$ defined by $v(x) = 0$ for every $x \in \overline{\Sigma}$ and*

$$v(x) = \int_0^{\tau(x)} \prod_{i=1}^{n-1} \frac{1 - (d_\Omega(x) + s)\kappa_i(x)}{1 - d_\Omega(x)\kappa_i(x)}\, ds, \qquad \forall x \in \overline{\Omega} \setminus \overline{\Sigma}, \tag{30}$$

is continuous in $\overline{\Omega}$ and it is the unique solution to

$$\int_\Omega [v(x)\langle Dd_\Omega(x), D\varphi(x)\rangle - \varphi(x)]\, dx = 0, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{31}$$

We recall that, for every $x \in \Omega \setminus \overline{\Sigma}$, $\kappa_i(x)$, $i = 1, \dots, n-1$, denotes the i-th principal curvature of $\partial\Omega$ at $\Pi(x)$, and τ is the normal distance to the cut locus defined in (8).

Remark 8. In the case $n = 2$, Theorem 4 was proven in [5].

Let us denote by $H_1(y)$ the mean curvature of $\partial\Omega$ at a point y, that is,

$$H_1(y) = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i(y), \qquad y \in \partial\Omega\,. \tag{32}$$

Under the assumptions of Theorem 1, we shall show that H_1 is constant on $\partial\Omega$, that is, every connected component S of $\partial\Omega$ is a compact embedded hypersurface without boundary with constant mean curvature H_1. From a celebrated result of Aleksandrov [1], it follows that S is a hypersphere of radius $R = 1/H_1$. Since Ω is connected, we conclude that Ω must be a ball of radius R.

Proof of Theorem 1. Let $u_0 = \phi_0 \circ d_\Omega$ be a minimizer of J, depending only on the distance from $\partial\Omega$. Under the assumptions of Theorem 1, from Theorem 2 we find that ϕ_0 is Lipschitz continuous on $[0, r_\Omega]$. Furthermore, the Euler–Lagrange equation

$$\alpha(t) = f'(|\phi_0'(t)|), \qquad \text{a.e. } t \in [0, r_\Omega], \tag{33}$$

holds, where $\alpha\colon [0, r_\Omega] \to \mathbb{R}$ is the function defined in (9) (see Remark 6).

From Theorem 3 we know that u_0 satisfies the Euler–Lagrange equation (26). On the other hand, from Theorem 4 we deduce that $v(x) = f'(|Du_0(x)|)$. From (33), we then have

$$v(x) = \alpha(d_\Omega(x)), \qquad \text{a.e. } x \in \Omega.$$

More precisely, from the continuity of v it follows that v is constant on the level sets $\partial\Omega_t$ of the distance d_Ω, for every $t \in [0, r_\Omega]$.

Let $x \in \overline{\Sigma}$, that is, $v(x) = 0$, and let $t = d_\Omega(x)$. From the discussion above we have $v(y) = 0$ for every $y \in \partial\Omega_t$, that is, $\partial\Omega_t \subseteq \overline{\Sigma}$.

We claim that $t = r_\Omega$. Assume by contradiction that there exists a point $z \in \Omega$ with $d_\Omega(z) > t$, and let $y \in \Pi(z)$. The function d_Ω is differentiable at any point of the line segment (y, z) (see [2]), which is in contradiction with the fact that (y, z) must intersect $\partial\Omega_t$, that is, (y, z) must contain at least one singular point of d_Ω.

We have thus proven that

$$x \in \overline{\Sigma} \ \Leftrightarrow\ d_\Omega(x) = r_\Omega.$$

As a consequence, d_Ω is regular on the set $\{x \in \Omega;\ 0 < d_\Omega(x) < r_\Omega\}$, and then the distance to the cut locus is $\tau(y) = r_\Omega$ for every $y \in \partial\Omega$. From the explicit representation (30) of v, for every $t \in [0, r_\Omega)$ we have

$$v(y) = \alpha(t) = \int_t^{r_\Omega} \prod_{i=1}^{n-1} \frac{1 - s\kappa_i(y)}{1 - t\kappa_i(y)}\, ds, \qquad \forall y \in \partial\Omega_t.$$

From this formula we deduce that the function α is of class C^∞ on $[0, r_\Omega)$. By a direct computation we get

$$\alpha'(t) = -1 + \sum_{j=1}^{n-1} \int_t^{r_\Omega} \left(\prod_{i=1}^{n-1} \frac{1 - s\kappa_i(y)}{1 - t\kappa_i(y)} \right) \cdot \frac{\kappa_j(y)}{1 - t\kappa_j(y)}\, ds \tag{34}$$

for every $t \in [0, r_\Omega)$ and $y \in \partial\Omega_t$. Evaluating this derivative at $t = 0$ we obtain

$$\alpha'(0) = -1 + \alpha(0) \sum_{j=1}^{n-1} \kappa_j(y) = -1 + (n-1)\alpha(0) H_1(y), \qquad \forall y \in \partial\Omega,$$

so that H_1 is constant on $\partial\Omega$. From a result of Aleksandrov [1] we conclude that the connected set Ω is a ball.

Remark 9. The term $\kappa_j(y)/(1 - t\kappa_j(y))$, appearing in the integral in (34), is the j-th principal curvature of the set $\partial\Omega_t$ at y (see [16]).

Remark 10. The fact that d_Ω is regular on $\{x \in \Omega;\ 0 < d_\Omega(x) < r_\Omega\}$ alone is not enough to conclude that Ω is a ball. For example, the set $\Omega = B_R(0) \setminus \overline{B}_r(0)$, $0 < r < R$, is a connected set with C^∞ boundary, with inradius $r_\Omega = (R - r)/2$, and the singular set Σ coincides with $\partial B_{(r+R)/2}(0) = \{x \in \Omega;\ d_\Omega(x) = r_\Omega\}$.

Among convex sets, if we relax the assumption on the regularity of the boundary, an example can be constructed in the following way. Let $\Sigma \subset \mathbb{R}^n$ be a nonempty compact convex set without interior points (in the language of convex geometry, its dimension must be at most $n - 1$). Let $r > 0$, and define $\Omega = \bigcup_{x \in \Sigma} B_r(x)$. Then Ω is an open convex set with inradius $r_\Omega = r$, and the singular set of d_Ω coincides with $\{x \in \Omega;\ d_\Omega(x) = r_\Omega\} = \Sigma$.

Remark 11. Let us define the i-th order mean curvature H_i of $\partial\Omega$ to be the elementary symmetric polynomial of degree i in the principal curvatures $\kappa_1, \dots, \kappa_{n-1}$ normalized by the following identity:

$$\prod_{i=1}^{n-1}(1+\kappa_i t) = \sum_{i=0}^{n-1}\binom{n-1}{i} H_i\, t^i\,.$$

It is easily seen that $H_0 \equiv 1$ and H_1 is the mean curvature, defined in (32). Computing the first $n-1$ derivatives of α at $t = 0$ we can prove that $H_1, \dots, H_{n-1}$ are constant on $\partial\Omega$ (it is clear that, *a posteriori*, this is a consequence of the fact that Ω is a ball). For example, if $n \geq 3$, the second derivative of α is given by

$$\begin{aligned}\alpha''(t) = &-\sum_{j=1}^{n-1}\frac{\kappa_j(y)}{1-t\kappa_j(y)}\\ &+\sum_{j,h=1}^{n-1}\int_t^{r_\Omega}\left(\prod_{i=1}^{n-1}\frac{1-s\kappa_i(y)}{1-t\kappa_i(y)}\right)\cdot\frac{\kappa_j(y)\kappa_h(y)}{(1-t\kappa_j(y))(1-t\kappa_h(y))}\,ds\end{aligned}$$

for every $t \in [0, r_\Omega)$ and $y \in \partial\Omega_t$, hence

$$\alpha''(0) = -(n-1)H_1(y) + (n-1)(n-2)H_2(y)\alpha(0), \qquad \forall y \in \partial\Omega.$$

Since H_1 is constant on $\partial\Omega$, we deduce that also H_2 is constant on $\partial\Omega$.

References

[1] Aleksandrov, A. D.: Uniqueness theorems for surfaces in the large. I, II. Amer. Math. Soc. Transl. **21**, 341–388 (1962)

[2] Bardi, M., Capuzzo-Dolcetta, I.: Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations. Birkhäuser, Boston (1997) Zbl 0890.49011 MR 1484411

[3] Brock, F., Prajapat, J.: Some new symmetry results for elliptic problems on the sphere and in Euclidean space. Rend. Circ. Mat. Palermo **49**, 445–462 (2000) Zbl 1008.35002 MR 1809087

[4] Burago, Yu. D., Zalgaller, V. A.: Geometric Inequalities. Springer, Berlin (1988) Zbl 0633.53002 MR 0936419

[5] Cannarsa, P., Cardaliaguet, P.: Representation of equilibrium solutions to the table problem for growing sandpiles. J. Eur. Math. Soc. **6**, 435–464 (2004) Zbl pre02139679 MR 2094399

[6] Cannarsa, P., Cardaliaguet, P., Crasta, G., Giorgieri, E.: A boundary value problem for PDE. Mathematical models of mass transfer: representation of solutions and applications. Calc. Var. Partial Differential Equations, to appear

[7] Cellina, A.: On the validity of the Euler–Lagrange equation. J. Differential Equations **171**, 430–442 (2001) Zbl 1015.49018 MR 1818657

[8] Crasta, G.: Existence, uniqueness and qualitative properties of minima to radially symmetric non-coercive non-convex variational problems. Math. Z. **235**, 569–589 (2000) Zbl 0965.49003 MR 1800213

[9] Crasta, G.: Variational problems for a class of functionals on convex domains. J. Differential Equations **178**, 608–629 (2002) Zbl 1019.49020 MR 1879839

[10] Crasta, G.: Estimates for the energy of the solutions to elliptic Dirichlet problems on convex domains. Proc. Roy. Soc. Edinburgh Sect. A **134**, 89–107 (2004) Zbl pre02114666 MR 2039904

[11] Crasta, G., Fragalà, I., Gazzola, F.: A sharp upper bound for the torsional rigidity of rods by means of web functions. Arch. Rat. Mech. Anal. **164**, 189–211 (2002) Zbl 1021.74020 MR 1930391

[12] Crasta, G., Fragalà, I., Gazzola, F.: On the role of energy convexity in the web function approximation. NoDEA Nonlinear Differential Equations Appl. **12**, 93–109 (2005) MR 2138936

[13] Damascelli, L., Pacella, F.: Monotonicity and symmetry results for p-Laplace equations and applications. Adv. Differential Equations **5**, 1179–1200 (2000) Zbl 1002.35045 MR 1776351

[14] Evans, L. C., Gariepy, R. F.: Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton (1992) Zbl 0804.28001 MR 1158660

[15] Federer, H.: Curvature measures. Trans. Amer. Math. Soc. **93**, 418–491 (1959) Zbl 0089.38402 MR 0110078

[16] Feldman, M.: Variational evolution problems and nonlocal geometric motion. Arch. Rat. Mech. Anal. **146**, 221–274 (1999) Zbl 0955.49025 MR 1720391

[17] Itoh, J., Tanaka, M.: The Lipschitz continuity of the distance function to the cut locus. Trans. Amer. Math. Soc. **353**, 21–40 (2001) Zbl 0971.53031 MR 1695025

[18] Li, Y.Y., Nirenberg, L.: The distance function to the boundary, Finsler geometry and the singular set of viscosity solutions of some Hamilton–Jacobi equations. Comm. Pure Appl. Math. **58**, 85–146 (2005) Zbl pre02126519 MR 2094267

[19] Serrin, J.: A symmetry problem in potential theory. Arch. Rat. Mech. Anal. **43**, 304–318 (1971) Zbl 0222.31007 MR 0333220

[20] Vornicescu, M.: A variational problem on subsets of $\mathbb{R}^n$. Proc. Roy. Soc. Edinburgh Sect. A **127**, 1089–1101 (1997) Zbl 0920.49002 MR 1475648

[21] Ziemer, W. P.: Weakly Differentiable Functions. Springer, New York (1989) Zbl 0692.46022 MR 1014685