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On the number of positive solutions of singularly perturbed 1D nonlinear Schrödinger equations

Dedicated to Professor Antonio Ambrosetti on the ocassion of his 60th birthday

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Abstract. We study singularly perturbed 1D nonlinear Schrödinger equations (1.1). When $V(x)$ has multiple critical points, (1.1) has a wide variety of positive solutions for small ε and the number of positive solutions increases to ∞ as $\varepsilon \to 0$. We give an estimate of the number of positive solutions whose growth order depends on the number of local maxima of $V(x)$. Envelope functions or equivalently adiabatic profiles of high frequency solutions play an important role in the proof.

Keywords. Nonlinear Schrödinger equations, singular perturbations, adiabatic profiles

1. Introduction

In this paper we study the following nonlinear Schrödinger equation in $\mathbb{R}$:

$$\begin{aligned} -\varepsilon^2 u_{xx} + V(x)u &= u^p \quad \text{in } \mathbb{R}, \\ u(x) &> 0 \quad \text{in } \mathbb{R}, \\ u(x) &\in H^1(\mathbb{R}). \end{aligned} \tag{1.1}$$

Here $\varepsilon > 0$, $p \in (0, \infty)$ and $V \in C^1(\mathbb{R})$ satisfies

$$0 < \inf_{x\in\mathbb{R}} V(x) \le \sup_{x\in\mathbb{R}} V(x) < \infty.$$

The study of the existence and the profile of solutions of (1.1) was originated by Floer–Weinstein [11], Oh [16, 17] and developed by Ambrosetti–Badiale [1], Ambrosetti–Badiale–Cingolani [2], del Pino–Felmer [5], del Pino–Felmer–Tanaka [6], Gui [12], Kang–Wei [13], Rabinowitz [18], Wang [19]. In particular, they succeeded in proving the existence of solutions with finitely many peaks concentrating close to critical points of the potential $V(x)$ as $\varepsilon \to 0$. In particular, Kang–Wei [13] find positive solutions with any

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prescribed number of peaks clustering around each given local maximum point or saddle point of the potential $V(x)$. We also refer to Ambrosetti–Malchiodi–Ni [3, 4], Malchiodi–Montenegro [15] and del Pino–Kowalczyk–Wei [7] for the existence of solutions which concentrate on spheres or curves for related Neumann boundary problems. These results suggest that if $V(x)$ has multiple critical points, then the number of positive solutions of (1.1) increases as $\varepsilon \to 0$. The main purpose of this paper is to give an estimate of the number of positive solutions.

Recently a similar question for Neumann boundary value problems has been studied by Lin–Ni–Wei [14]. More precisely, they study

$$\begin{aligned} -\varepsilon^2\Delta u + u = u^p \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $\Omega \subset \mathbb{R}$ is a bounded domain with a smooth boundary $\partial\Omega$ and where $p \in (1, (N+2)/(N-2))$ if $N \geq 3$, and $p \in (1, \infty)$ if $N = 1, 2$. They show that for any integer K satisfying $1 \leq K \leq \alpha/\varepsilon^N |\log\varepsilon|^N$, where $\alpha = \alpha_{N,\Omega,p} > 0$ is a constant depending only on N, Ω, p, the problem (1.2) has a positive solution with K interior peaks. In particular they show that

$$\liminf_{\epsilon\to 0} \varepsilon^N |\log\varepsilon|^N n_\varepsilon > 0,$$

where n_ε is the number of positive solutions of (1.2).

In this paper we show that for 1D nonlinear Schrödinger equations there is a strong effect of the potential $V(x)$ on the number n_ε of positive solutions and if $V(x)$ has k local maxima, then n_ε grows at least with order $1/\varepsilon^k$.

In what follows we say an interval $[\alpha, \beta] \subset \mathbb{R}$ is a *local maximum* of $V(x)$ if

(m1) $V(x)$ is constant in $[\alpha, \beta]$,
(m2) there is a constant $\delta > 0$ such that $V(x) < V(\alpha)$ for all $x \in [\alpha-\delta, \alpha) \cup (\beta, \beta+\delta]$.

We also say a point $\alpha \in \mathbb{R}$ is a local maximum of $V(x)$ if (m2) holds with $\beta = \alpha$.

The main result of this paper is the following

Theorem 1.1. *Suppose that $V(x)$ has k local maxima and let n_ε be the number of positive solutions of* (1.1). *Then there exists a constant $c_1(V) > 0$ depending only on $V(x)$ such that*

$$\liminf_{\varepsilon\to 0} \varepsilon^k n_\varepsilon \geq c_1(V). \tag{1.3}$$

Remark 1.2. The constants $c_1(V)$ and $c_2(V)$, which will appear in (1.5) below, will be expressed explicitly in (3.2) and Remark 3.3.

A similar result holds also for Neumann boundary value problems:

$$
\begin{aligned}
-\varepsilon^2 u_{xx} + V(x)u = u^p \quad & \text{in } (0,1), \\
u > 0 \quad & \text{in } (0,1), \\
u_x = 0 \quad & \text{at } x = 0, 1.
\end{aligned} \tag{1.4}
$$

We say $[\alpha,\beta] \subset [0,1]$ is a *local maximum* of $V(x)$ in $[0,1]$ if either $[\alpha,\beta] \subset (0,1)$ and (m1)–(m2) hold, or $[\alpha,\beta] = [0,\beta] \subset [0,1)$ ($[\alpha,\beta] = [\alpha,1] \subset (0,1]$ respectively) and $[\alpha,\beta]$ satisfies (m1) and

(m3) there exists a constant $\delta > 0$ such that $V(x) < V(\beta)$ for all $x \in (\beta, \beta+\delta]$ ($V(x) < V(\alpha)$ for all $x \in [\alpha-\delta, \alpha)$).

Theorem 1.3. *Suppose that $V(x)$ has k local maxima in* $[0,1]$ *and let n_ε be the number of positive solutions of* (1.4) *without peaks on the boundary, that is, solutions which do not have local maxima on the boundary of* $[0,1]$. *Then there exists a constant $c_2(V) > 0$ depending only on $V(x)$ such that*

$$\liminf_{\varepsilon\to 0} \varepsilon^k n_\varepsilon \geq c_2(V). \tag{1.5}$$

Remark 1.4. When $V(x)$ has a local maximum at 0 or 1, that is, the boundary of I, there exists a family of positive solutions with a peak at 0 or 1, that is, a family of solutions which have a local maximum at 0 or 1. Let $n_{0,\varepsilon}$, $n_{1,\varepsilon}$, $n_{0,1,\varepsilon}$ be the numbers of positive solutions of (1.4) with a peak only at 0, a peak at 1, or peaks at 0 and 1, respectively. Then (1.5) also holds for $n_{0,\varepsilon}$, $n_{1,\varepsilon}$, $n_{0,1,\varepsilon}$.

The following examples show that the existence of local maxima of $V(x)$ is necessary for estimates like (1.3) and (1.5) to hold with $k \geq 2$.

- If $V \in C^1(\mathbb{R})$ is a strictly monotone function in $\mathbb{R}$, for example $V_x(x) \neq 0$ for all $x \in \mathbb{R}$, then (1.1) has no positive solutions.
- If $V \in C^1(\mathbb{R})$ satisfies $V(-x) = V(x)$ and $xV_x(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$, then (1.1) has a unique positive solution.
- In the setting of the Neumann boundary problem (1.4), if $V(x) \equiv 1$, then (1.4) has exactly $2[\sqrt{p-1}/\pi\varepsilon]$ positive solutions, where $[n]$ denotes the greatest integer which is less than n. As a consequence, we have

$$\lim_{\varepsilon\to 0} \varepsilon n_\varepsilon = 2\sqrt{p-1}/\pi.$$

In Section 4, we will give more precise information on estimates of the number of positive solutions.

Theorems 1.1 and 1.3 are consequences of our recent studies [8, 9, 10] on high frequency solutions of 1D semilinear problems. In [10] we deal with 1D nonlinear Schrödinger equations and we consider a family (u_ε) of solutions which is uniformly bounded but becomes highly oscillatory as $\varepsilon \to 0$. We show that it is possible to describe its behavior by means of an envelope function, which is the asymptotic amplitude of solutions (u_ε), or equivalently by an adiabatic profile. Conversely, we also prove that for a given envelope function or adiabatic profile, there exists a family of solutions having such behavior. When $V(x)$ has k local maxima, there exists an adiabatic profile whose support is a union of k intervals near local maxima. A family (u_ε) corresponding to such an adiabatic profile has k clusters of peaks and each cluster C_i $(i = 1, \dots, k)$ has n_ε^i peaks, where n_ε^i satisfies $\lim_{\varepsilon\to 0} \varepsilon n_\varepsilon^i = \alpha_i$ and $\alpha_i > 0$ is determined by an adiabatic profile. This is the key to the proof of our theorems.

2. Adiabatic profiles and solutions with clusters of peaks

To introduce adiabatic profiles, first we consider the following x-independent problem:

$$\begin{aligned} -v'' + Vv &= |v|^{p-1}v, \qquad s \in \mathbb{R}, \\ v(0) &= y_0, \\ v'(0) &= y_1, \end{aligned} \tag{2.1}$$

where $V \in (0, \infty)$ and $y_0, y_1 \in \mathbb{R}$. This equation appears as a limit equation when we take the limit as $\varepsilon \to 0$ in (1.1) after a suitable scaling. We denote the solution of (2.1) by $v = v(V, y_0, y_1; s)$ and remark that

- $v(V, y_0, y_1; s)$ is periodic and has constant sign if $\frac{1}{2}y_1^2 - \frac{V}{2}y_0^2 + \frac{1}{p+1}y_0^{p+1} < 0$,
- $v(V, y_0, y_1; s)$ is periodic and sign-changing if $\frac{1}{2}y_1^2 - \frac{V}{2}y_0^2 + \frac{1}{p+1}y_0^{p+1} > 0$,
- $v(V, y_0, y_1; s)$ is homoclinic to 0, or identically 0, if $\frac{1}{2}y_1^2 - \frac{V}{2}y_0^2 + \frac{1}{p+1}y_0^{p+1} = 0$.

We denote by $T(V, y_0, y_1)$ the period of $v(V, y_0, y_1; s)$ if $\frac{1}{2}y_1^2 - \frac{V}{2}y_0^2 + \frac{1}{p+1}y_0^{p+1} \neq 0$ and set $T(V, y_0, y_1) = \infty$ if $\frac{1}{2}y_1^2 - \frac{V}{2}y_0^2 + \frac{1}{p+1}y_0^{p+1} = 0$. Now we define

$$A(V, y_0, y_1) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^{T(V,y_0,y_1)} |v'(V, y_0, y_1; s)|^2\, ds & \text{if } \dfrac{1}{2}y_1^2 - \dfrac{V}{2}y_0^2 + \dfrac{1}{p+1}y_0^{p+1} > 0, \\ \displaystyle\int_0^{T(V,y_0,y_1)} |v'(V, y_0, y_1; s)|^2\, ds & \text{if } \dfrac{1}{2}y_1^2 - \dfrac{V}{2}y_0^2 + \dfrac{1}{p+1}y_0^{p+1} < 0, \\ A_0(V) & \text{if } \dfrac{1}{2}y_1^2 - \dfrac{V}{2}y_0^2 + \dfrac{1}{p+1}y_0^{p+1} = 0, \end{cases}$$

where

$$A_0(V) = \int_{-\infty}^{\infty} \left| v'\left(V, \left(\frac{p+1}{2}V\right)^{1/(p-1)}, 0; s\right)\right|^2 ds.$$

We remark that $A(V, y_0, y_1)$ is the area (or half the area) enclosed by the orbit $(v(s), v'(s))$ in the phase plane and is a function of class C^1. We also remark that $A_0(V)$ is the area enclosed by a homoclinic orbit and is an increasing function of V.

Remark 2.1. It is easily seen that $T(V, y_0, y_1)$ and $A(V, y_0, y_1)$ are functions of V and $E = \frac{1}{2}y_1^2 - \frac{V}{2}y_0^2 + \frac{1}{p+1}y_0^{p+1}$. So we may write them as $T(V, E)$ and $A(V, E)$. We remark that for fixed $V > 0$, $E \mapsto A(V, E)$ is a strictly increasing function and thus E can be regarded as a function of V and the area A. Therefore $T(V, E)$ can also be regarded as a function of V and A. We denote it by

$$T = \tilde{T}(V, A).$$

We remark that $A \mapsto \tilde{T}(V, A)$ is strictly increasing when $A < A_0(V)$ and strictly decreasing when $A > A_0(V)$. We also remark that $A < A_0(V)$ ($A > A_0(V)$ respectively) if and only if $E < 0$ ($E > 0$ respectively) and moreover the corresponding solution $v(s)$ of (2.1) has constant sign (is sign-changing, respectively).

In what follows we mainly work in the setting of Theorem 1.3. Theorem 1.1 requires only minor modifications.

We set $I = [0, 1]$ and for a given potential $V : I \to (0, \infty)$ we define the trivial action function by

$$a_0(x) = A_0(V(x)),$$

which is the area enclosed by a homoclinic orbit of (2.1) with $V = V(x)$ and is a C^1-function of x.

Definition 2.2. *We say a function $a : I \to (0, \infty)$ is an* adiabatic profile *(or* action profile*) if it is continuous and whenever $a(x) \neq a_0(x)$, we have $a'(x) = 0$. We also define its* support *by*

$$\mathrm{supp}(a) = \{x \in I;\ a(x) \neq a_0(x)\}.$$

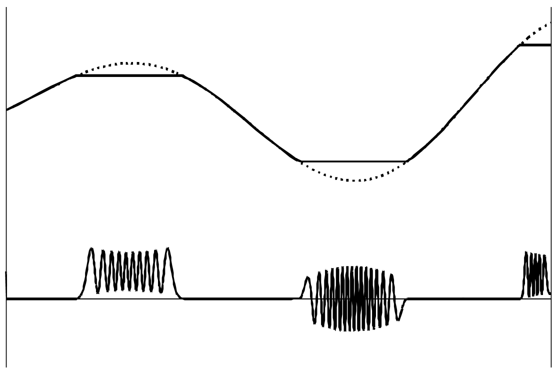

Fig. 2.1. An adiabatic profile and the corresponding solution.

For a given family (u_ε) of solutions of

$$\begin{aligned} -\varepsilon^2 u_{xx} + V(x)u &= |u|^{p-1}u \qquad \text{in } I, \\ u_x &= 0 \qquad\qquad\quad \text{at } \partial I, \end{aligned} \tag{2.2}$$

we define an approximate adiabatic profile (or action) $a_\varepsilon(x)$ by

$$a_\varepsilon(x) = A(V(x), u_\varepsilon(x), \varepsilon u_\varepsilon'(x)).$$

The following is one of the main results of [10].

Theorem 2.3. *Let* (u_ε) *be an* L^∞*-bounded family of solutions of* (1.4)*. Then after extracting a subsequence* $\varepsilon_n \to 0$*, the corresponding approximate adiabatic profile* $a_\varepsilon(x)$ *converges to an adiabatic profile* $a(x)$*. Moreover* $u_{\varepsilon_n}(x)$ *has peaks only in a neighborhood of* $\mathrm{supp}(a) \cup \{x \in I;\ V'(x) = 0\} \cup \partial I$*, that is, for any* $\delta > 0$ *there is an* $n_0(\delta) > 0$ *such that* $u_{\varepsilon_n}(x)$ *has peaks only in a* δ*-neighborhood of* $\mathrm{supp}(a) \cup \{x \in I;\ V'(x) = 0\} \cup \partial I$ *if* $n \geq n_0(\delta)$*.*

Remark 2.4. Let $(\alpha, \beta) \subset I$ be an isolated connected component of $\mathrm{supp}(a)$. Then:

(1) If $a(x) < a_0(x)$ in (α, β), then $u_{\varepsilon_n}(x)$ has constant sign in (α, β) for large n.
(2) If $a(x) > a_0(x)$ in (α, β), then $u_{\varepsilon_n}(x)$ is sign-changing in (α, β) for large n.
(3) Let $n_{\varepsilon_n}(\alpha, \beta)$ be the number of peaks (i.e., positive local maxima or negative local minima) of $u_{\varepsilon_n}(x)$ in (α, β). Then

$$\varepsilon_n n_{\varepsilon_n}(\alpha, \beta) \to \begin{cases} \displaystyle\int_\alpha^\beta \frac{1}{\tilde{T}(V(x), a(x))}\, dx & \text{if } a(x) < a_0(x) \text{ in } (\alpha, \beta), \\ \displaystyle\int_\alpha^\beta \frac{2}{\tilde{T}(V(x), a(x))}\, dx & \text{if } a(x) > a_0(x) \text{ in } (\alpha, \beta), \end{cases}$$

where $\tilde{T}(V, A)$ is defined in Remark 2.1.

Conversely, for a given adiabatic profile $a(x)$ we can construct the corresponding family of solutions of (2.2).

Theorem 2.5. *For a given adiabatic profile* $a(x)$*, there exists a family* (u_ε) *of solutions of* (2.2) *such that the corresponding approximate adiabatic profile* $a_\varepsilon(x)$ *converges to* $a(x)$ *as* $\varepsilon \to 0$*. Moreover for any* $\delta > 0$*, there exists an* $\varepsilon_0(\delta) > 0$ *such that for* $0 < \varepsilon < \varepsilon_0(\delta)$*,* $u_\varepsilon(x)$ *has peaks only in a* δ*-neighborhood of* $\mathrm{supp}(a)$*.*

Remark 2.6. The statements in Remark 2.4 hold for $u_\varepsilon(x)$ obained in Theorem 2.5 without taking a subsequence.

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3 (dealing with Theorem 1.1 requires slight modifications). Since we deal with only positive solutions of (2.2), recalling Remark 2.4, we may consider adiabatic profiles $a(x)$ satisfying $a(x) \leq a_0(x)$ for all $x \in I$.

For the proof of Theorem 1.3, assume $k \geq 2$ and that there exists an adiabatic profile $a(x)$ such that:

(a1) $a(x) \leq a_0(x)$ for all $x \in I$ and $a(x) \not\equiv a_0(x)$.
(a2) $\mathrm{supp}(a)$ consists of exactly k disjoint intervals $I_1, \ldots, I_k$. For each $j \in \{1, \ldots, k\}$, I_j is of the form (α, β) $(0 < \alpha < \beta < 1)$, $[0, \beta)$ $(\beta \in (0, 1))$, or $(\alpha, 1]$ $(\alpha \in (0, 1))$.
(a3) There exists a $\delta > 0$ such that

(1) For $I_j = (\alpha, \beta)$ $(0 < \alpha < \beta < 1)$,

$$V_x(x) > 0 \quad \text{in } (\alpha - 2\delta, \alpha + 2\delta),$$
$$V_x(x) < 0 \quad \text{in } (\beta - 2\delta, \beta + 2\delta).$$

(2) For $I_j = [0, \beta)$ $(\beta \in (0, 1))$,

$$V_x(x) < 0 \quad \text{in } (\beta - 2\delta, \beta + 2\delta).$$

(3) For $I_j = (\alpha, 1]$ $(\alpha \in (0, 1))$,

$$V_x(x) > 0 \quad \text{in } (\alpha - 2\delta, \alpha + 2\delta).$$

Remark 3.1. Recalling $A_0(V)$ is a strictly increasing function of V, we have

(1) For $I_j = (\alpha, \beta)$ $(0 < \alpha < \beta < 1)$, $V(x) > V(\alpha) = V(\beta)$ for all $x \in I_j$.
(2) For $I_j = [0, \beta)$ $(0 < \beta < 1)$, $V(x) > V(\beta)$ for all $x \in I_j$.
(3) For $I_j = (\alpha, 1]$ $(0 < \alpha < 1)$, $V(x) > V(\alpha)$ for all $x \in I_j$.

Under the assumption of Theorem 1.3, we can easily find an adiabatic profile $a(x)$ such that each I_j is a neighborhood of a local maximum of $V(x)$.

To prove Theorem 1.3, it suffices to show the following

Proposition 3.2. *Let $a(x)$ be an adiabatic profile given above and define*

$$m_j \equiv \int_{I_j} \frac{1}{\tilde{T}(V(x), a(x))}\, dx \qquad (j = 1, \ldots, k). \tag{3.1}$$

Then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $(K_j)_{j=1}^k \subset \mathbb{N}$ satisfying

$$1 \le K_j \le \frac{1}{\varepsilon} m_j \qquad (j = 1, \ldots, k),$$

there exists a positive solution $u_\varepsilon(x)$ of (1.4) *such that $u_\varepsilon(x)$ has exactly K_j interior peaks in $N_\delta(I_j)$ for $j = 1, \ldots, k$ and no peaks elsewhere.*

From the above proposition, considering combinations of $(K_1, \ldots, K_k)$, we can observe that (1.4) has at least $[m_1/\varepsilon] \times \cdots \times [m_k/\varepsilon]$ positive solutions. Thus

$$\liminf_{\varepsilon \to 0} \varepsilon^k n_\varepsilon \ge m_1 \cdots m_k > 0.$$

We remark that under the assumption of Theorem 1.3 there exists an adiabatic profile $a^{(k)}(x)$ satisfying (a1) and

(a2′) $\mathrm{supp}(a^{(k)})$ consists of exactly k disjoint intervals $I_1, \ldots, I_k$. For each $j \in \{1, \ldots, k\}$, I_j has the form (α, β) $(0 \le \alpha < \beta \le 1)$, $[0, \beta)$ $(\beta \in (0, 1])$, $(\alpha, 1]$ $(\alpha \in [0, 1))$ or $[0, 1]$.

(a4) There are no adiabatic profiles $a(x)$ whose support has exactly k disjoint intervals and

$$a(x) \le a^{(k)}(x) \qquad \text{for all } x \in I,$$
$$a(x_0) < a^{(k)}(x_0) \qquad \text{for some } x_0 \in I.$$

We call an adiabatic profile satisfying (a2′) and (a4) a *k-minimal adiabatic profile.*

We can easily see that there exists a sequence $(a_\ell)_{\ell=1}^{\infty}$ of adiabatic profiles such that

- $a_\ell(x)$ satisfies (a1)–(a3).
- $a_\ell(x) \ge a^{(k)}(x)$ for all $x \in I$,
- $a_\ell(x) \to a^{(k)}(x)$ as $\ell \to \infty$.

Writing $\operatorname{supp}(a^{(k)}) = I_1^{(k)} \cup \cdots \cup I_k^{(k)}$ and noting that $\operatorname{supp}(a_\ell) \subset \operatorname{supp}(a^{(k)})$, set

$$m_{j,\ell} = \int_{I_j^{(k)}} \frac{1}{\tilde{T}(V(x), a_\ell(x))}\, dx \qquad \text{for } \ell \in \mathbb{N}.$$

Then, repeating the previous argument, we can see that for each $\ell \in \mathbb{N}$,

$$\liminf_{\varepsilon \to 0} \varepsilon^k n_\varepsilon \ge m_{1,\ell} \cdots m_{k,\ell}.$$

Since $m_{j,\ell} \to \int_{I_j^{(k)}} \frac{1}{\tilde{T}(V(x), a^{(k)}(x))}\, dx$, we have

$$\liminf_{\varepsilon \to 0} \varepsilon^k n_\varepsilon \ge \prod_{j=1}^{k} \int_{I_j^{(k)}} \frac{1}{\tilde{T}(V(x), a^{(k)}(x))}\, dx.$$

Thus $c_2(V)$ in Theorem 1.3 is given explicitly by

$$c_2(V) = \prod_{j=1}^{k} \int_{I_j^{(k)}} \frac{1}{\tilde{T}(V(x), a^{(k)}(x))}\, dx. \tag{3.2}$$

Remark 3.3. The constant $c_1(V)$ in Theorem 1.1 is also represented by (3.2). Here $a^{(k)}(x)$ is a k-minimal adiabatic profile whose support is a bounded subset of $\mathbb{R}$.

Proof of Proposition 3.2. We argue indirectly and suppose that there exist sequences $\varepsilon_n \to 0$ and $(K_{jn})_{j=1}^{k} \subset \mathbb{N}$ such that

$$1 \le K_{jn} \le m_j/\varepsilon_n \qquad (j = 1, \ldots, k)$$

and (1.4) with $\varepsilon = \varepsilon_n$ has no solutions with the following property:

- $u(x)$ has exactly K_{jn} interior peaks in $N_\delta(I_j)$ for each $j = 1, \ldots, k$ and no peaks elsewhere.

Taking a subsequence if necessary, we may assume that

$$K_{jn}/\varepsilon_n \to \ell_j \in [0, m_j] \qquad \text{as } n \to \infty.$$

For such $(\ell_1, \ldots, \ell_k)$ we have the following

Proposition 3.4. *Let $I_1 \cup \cdots \cup I_k$ be a support of an adiabatic profile $a(x)$ and let*

$$\ell_j \in \left[0, \int_{I_j} \frac{1}{\tilde{T}(V(x), a(x))}\, dx\right] \quad (j = 1, \dots, k).$$

Then for any $\delta > 0$ and for any sequence $(\bar{K}_{j\varepsilon})_{j=1}^{k} \subset \mathbb{N}$ satisfying

$$\varepsilon \bar{K}_{j\varepsilon} \to \ell_j \quad \textit{as } \varepsilon \to 0,$$

there exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, (1.4) has a solution $u_\varepsilon(x)$ which has exactly $\bar{K}_{j\varepsilon}$ interior peaks in $N_\delta(I_j)$ for each j but no peaks elsewhere. Moreover the approximate adiabatic profile $a_\varepsilon(x)$ corresponding to $u_\varepsilon(x)$ converges to an adiabatic profile $\bar{a}(x)$ after extracting a subsequence and $\bar{a}(x)$ satisfies

$$a(x) \le \bar{a}(x) \le a_0(x) \quad \textit{for all } x \in I,$$
$$\int_{I_j} \frac{1}{\tilde{T}(V(x), \bar{a}(x))}\, dx = \ell_j \quad \textit{for } j = 1, \dots, k.$$

If we set $\ell_j = \int_{I_j} (1/\tilde{T}(V(x), a(x)))\, dx$ for $j = 1, \dots, k$, Theorem 2.5 follows from Proposition 3.4. We will give a sketch of the proof of Proposition 3.4 in Section 5.

Taking $\bar{K}_{j\varepsilon} = K_{jn}$ in Proposition 3.4, we get a contradiction to our assumption that (1.4) has no solutions with exactly K_{jn} interior peaks in $N_\delta(I_j)$, and this completes the proof of Proposition 3.2. □

Remark 3.5. In Propositions 3.2 and 3.4, if $0 \in \mathrm{supp}(a)$ ($1 \in \mathrm{supp}(a)$, $\{0, 1\} \subset \mathrm{supp}(a)$ respectively), then we can construct a positive solution with exactly K_j interior peaks in $N_\delta(I_j)$ and a peak at the boundary point 0 (a peak at 1, peaks at both 0 and 1, respectively).

Remark 3.6. The uniqueness of solutions $u_\varepsilon(x)$ obtained in Propositions 3.2 and 3.4 is an important problem. If the solutions are unique for all $(K_j)_{j=1}^{k}$ and ε small, we believe that the following is true:

$$\limsup_{\varepsilon \to 0} \varepsilon^{k_0+1} n_\varepsilon = 0.$$

Here k_0 is the number of local maxima of $V(x)$ in $[0, 1]$.

4. Number of positive solutions with prescribed number of peaks

Let k be the number of local maxima of $V(x)$ in $[0, 1]$ and let $a^{(k)}(x)$ be the corresponding k-minimal adiabatic profile. From the proof it is clear that Theorem 1.3 estimates just the number of solutions corresponding to adiabatic profiles which are less than $a^{(k)}(x)$. Such solutions have at most $(1/\varepsilon) \int_I (1/\tilde{T}(V(x), a^{(k)}(x)))\, dx$ peaks in I. We remark that there are solutions with more peaks. In this section we study the number of such solutions.

We use the following notation: for an adiabatic profile $a(x)$, we set

$$p(a) = \int_I \frac{1}{\tilde{T}(V(x), a(x))}\, dx.$$

For $0 < \nu_1 < \nu_2$ we also denote by $\tilde{n}_\varepsilon(\nu_1, \nu_2)$ the number of positive solutions of (1.4) which have no peaks at the boundary ∂I and the number of interior peaks is between $(1/\varepsilon)\nu_1$ and $(1/\varepsilon)\nu_2$.

With this notation, Theorem 1.3 shows

$$\liminf_{\varepsilon \to 0} \varepsilon^k \tilde{n}_\varepsilon(0, p(a^{(k)})) > 0.$$

Our next result is the following

Theorem 4.1. *Assume that there exists an adiabatic profile $a(x)$ whose support consists of exactly ℓ intervals. Then for any $\delta > 0$,*

$$\liminf_{\varepsilon \to 0} \varepsilon^\ell \tilde{n}_\varepsilon(p(a) - \delta, p(a)) > 0.$$

Proof. Let $\delta > 0$ be a given number and let $\text{supp}(a) = J_1 \cup \cdots \cup J_\ell$ ($J_i \cap J_j = \emptyset$ for $i \neq j$). As in the proof of Theorem 1.3, we take another adiabatic profile $\bar{a}(x)$ such that

- $\bar{a}(x)$ satisfies (a1)–(a3).
- $\text{supp}(\bar{a})$—denoted by $J_1' \cup \cdots \cup J_\ell'$—is slightly smaller than $\text{supp}(a)$ and

$$p(a) - \delta/2 < p(\bar{a}) < p(a). \tag{4.1}$$

We can adapt the previous argument to $\bar{a}(x)$. We remark that we may restrict the number of peaks in J_i' between

$$\frac{1}{\varepsilon}\left(\int_{J_i} \frac{1}{\tilde{T}(V(x), \bar{a}(x))}\, dx - \frac{\delta}{2\ell}\right) \quad \text{and} \quad \frac{1}{\varepsilon}\int_{J_i} \frac{1}{\tilde{T}(V(x), \bar{a}(x))}\, dx,$$

thus we have

$$\liminf_{\varepsilon \to 0} \varepsilon^\ell \tilde{n}_\varepsilon(p(\bar{a}) - \delta/2, p(\bar{a})) \geq \left(\frac{\delta}{2\ell}\right)^\ell > 0.$$

By (4.1), we have $\tilde{n}_\varepsilon(p(a) - \delta, p(a)) \geq \tilde{n}_\varepsilon(p(\bar{a}) - \delta/2, p(\bar{a}))$ and we have the conclusion of Theorem 4.1. □

In the following example we say for $\ell > 2$ that $a(x)$ is an *ℓ-minimal adiabatic profile* if $a(x)$ satisfies (a1), (a2′) and (a4) with $k = \ell$.

Example 4.2. We consider the following situation: $V \in C^1([0, 1])$ has exactly five critical points $0 < s_1 < t_1 < s_2 < t_2 < s_3 < 1$ and s_1, s_2, s_3 are local maxima and t_1, t_2 are local minima of $V(x)$. We also assume that

$$V(0) = V(1) < V(t_2) < V(t_1).$$

We remark that in this situation we can find unique points $\tau_{11} \in (0, s_1)$, $\tau_{12} \in (s_2, t_2)$ and $\tau_{21} \in (0, \tau_{11})$, $\tau_{22} \in (s_3, 1)$ such that

$$V(\tau_{11}) = V(t_1) = V(\tau_{12}), \qquad V(\tau_{21}) = V(t_2) = V(\tau_{22}).$$

We can easily see that adiabatic profiles $a^{(2)}(x)$, $a^{(3)}(x)$ satisfying

$$\mathrm{supp}(a^{(2)}) = (\tau_{21}, t_2) \cup (t_2, \tau_{22}), \qquad \mathrm{supp}(a^{(3)}) = (\tau_{11}, t_1) \cup (t_1, \tau_{12}) \cup (t_2, \tau_{22})$$

are 2-minimal and 3-minimal respectively. We also have $a^{(2)}(x) \le a^{(3)}(x)$ for all $x \in I$, $a^{(2)}(x) \not\equiv a^{(3)}(x)$ and $p(a^{(2)}) > p(a^{(3)})$. It is also clear that for any $0 < \nu_3 \le p(a^{(3)}) < \nu_2 \le p(a^{(2)})$ there exist adiabatic profiles $a_3(x)$ and $a_2(x)$ such that

$$p(a_3) = \nu_3 \quad \text{and} \quad p(a_2) = \nu_2$$

and whose supports consist of exactly three and two intervals respectively. Thus by Theorem 4.1 we have

$$\liminf_{\varepsilon \to 0} \varepsilon^3 \tilde{n}_\varepsilon(\nu_3 - \delta, \nu_3) > 0, \qquad \liminf_{\varepsilon \to 0} \varepsilon^2 \tilde{n}_\varepsilon(\nu_2 - \delta, \nu_2) > 0$$

for all $0 < \nu_3 \le p(a^{(3)}) < \nu_2 \le p(a^{(2)})$ and $\delta > 0$.

Remark 4.3. Formally $a^{(1)}(x) \equiv 0$ can be regarded as a 1-minimal adiabatic profile and $\tilde{T}(V(x), 0) = \pi/\sqrt{(p-1)V(x)}$. Setting $p(a^{(1)}) = (2/\pi) \int_I \sqrt{(p-1)V(x)}\, dx$, we also have

$$\liminf_{\varepsilon \to 0} \varepsilon \tilde{n}_\varepsilon(\nu_1 - \delta, \nu_1) > 0$$

for all $p(a^{(2)}) < \nu_1 \le p(a^{(1)})$ and $\delta > 0$.

5. Existence: proof of Proposition 3.4

This section is devoted to the proof of Proposition 3.4. For simplicity, we consider $I = (0, 1)$ and we assume that the adiabatic profile $a(x)$ has support $\mathrm{supp}(a) = (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) \subset (0, 1)$. Fix $\delta > 0$ small such that $0 < \alpha_1 - \delta$, $\beta_1 + \delta < \alpha_2 - \delta$, $\beta_2 + \delta < 1$ and $V'(x) > 0$ in $(\alpha_i - \delta, \alpha_i + \delta)$, $V'(x) < 0$ in $(\beta_i - \delta, \beta_i + \delta)$ for $i = 1, 2$.

We want to show that if $\varepsilon \bar{K}_{j\varepsilon} \to \ell_j$ as $\varepsilon \to 0$ for $j = 1, 2$ with

$$0 \le l_j \le \int_{\alpha_j}^{\beta_j} \frac{1}{\tilde{T}(V(x), a(x))}\, dx,$$

then, for sufficiently small $\varepsilon > 0$, we can find a family (u_ε) of solutions of (2.2) which has exactly $\bar{K}_{j\varepsilon}$ peaks in $(\alpha_j - \delta, \beta_j + \delta)$ for $j = 1, 2$.

We consider an adiabatic profile $\bar{a}(x)$ such that $a(x) \le \bar{a}(x) \le a_0(x)$ for all $x \in I$ and

$$\int_{\alpha_j}^{\beta_j} \frac{1}{\tilde{T}(V(x), \bar{a}(x))}\, dx = \ell_j \quad \text{for } j = 1, 2. \tag{5.1}$$

This result will be proved by maximizing a finite-dimensional functional of Nehari type. In the proof we will have to know precisely the behavior of the oscillatory solutions of (2.2). Assume we have functions $u_n : [a_n, b_n] \to \mathbb{R}$ satisfying (2.2) for $\varepsilon = \varepsilon_n$ and $I = [a_n, b_n]$, where $\varepsilon_n \to 0$ as $n \to \infty$. Suppose that u_n is positive, a_n and b_n are local minima of u_n, $\lim_{n\to\infty} a_n = \bar{a}$ and $\lim_{n\to\infty} b_n = \bar{b}$ with $\bar{a} < \bar{b}$.

Let $a_n < y_n^0 < y_n^1 < \cdots < y_n^{s_n-1} < y_n^{s_n} < b_n$ be the local maximum points of u_n in $[a_n, b_n]$ and assume that $s_n \to \infty$ as $n \to \infty$. Considering a subsequence, if necessary, we define

$$\alpha = \lim_{n\to\infty} y_n^0 \quad \text{and} \quad \beta = \lim_{n\to\infty} y_n^{s_n}.$$

The following proposition, corresponding to Proposition 4.1 of [10], is crucial to prove our result.

Proposition 5.1. *Assume $V'(x)$ is positive in $[\bar{a}, \bar{b}]$. Then for any interval $[x_1, x_2] \subset (\alpha, \beta)$ there exists n_0 such that for every $n \geq n_0$ the solution u_n has at least one maximum point and one minimum point in $[x_1, x_2]$. Moreover:*

(i) *If $\alpha > \bar{a}$, then the approximate adiabatic profile $a_{\varepsilon_n}(y_n^0)$ tends to $a_0(\alpha)$.*
(ii) *If $y_n^{i_n} \to \bar{x} \in (\alpha, \bar{b}]$, then $\limsup_{n\to\infty} |a_{\varepsilon_n}(y_n^{i_n})| < a_0(\bar{x})$.*
(iii) *$\bar{b} = \beta$.*

An analogous statement holds if $V'(x)$ is negative in $[\bar{a}, \bar{b}]$.

We start by defining an auxiliary adiabatic profile $\tilde{a}$ such that $\operatorname{supp}(\tilde{a}) = (\tilde{\alpha}_1, \tilde{\beta}_1) \cup (\tilde{\alpha}_2, \tilde{\beta}_2)$ and

$$\overline{\operatorname{supp}(a)} \subset \operatorname{supp}(\tilde{a}) \subset (\alpha_1 - \delta/2, \beta_1 + \delta/2) \cup (\alpha_2 - \delta/2, \beta_2 + \delta/2).$$

For $x, y \in I$ we define

$$d(x, y) = \frac{1}{\varepsilon} \int_x^y \frac{2}{\tilde{T}(V(x), \tilde{a}(x))}\, dx.$$

Set $N_\varepsilon^1 = 2\bar{K}_{1\varepsilon}$, $N_\varepsilon^2 = 2\bar{K}_{2\varepsilon}$, $N_\varepsilon = N_\varepsilon^1 + N_\varepsilon^2 - 1$ and $x_0 = 0$, $x_{N_\varepsilon+1} = 1$, and define the domain $\Delta_\varepsilon \subset \mathbb{R}^{N_\varepsilon}$ as

$$\begin{aligned} \Delta_\varepsilon = \{(x_1, \ldots, x_{N_\varepsilon});\, & x_0 \leq x_1 \leq \cdots \leq x_{N_\varepsilon+1}, \\ & d(x_i, x_{i+1}) \geq 1 \text{ for } i = 0, \ldots, N_\varepsilon, \\ & x_{N_\varepsilon^1-1} \leq \beta_1 + \delta/2,\ x_{N_\varepsilon^1+1} \geq \alpha_2 - \delta/2\}. \end{aligned} \tag{5.2}$$

For $X = (x_1, \ldots, x_{N_\varepsilon}) \in \Delta_\varepsilon$ we let $u_i : [x_i, x_{i+1}] \to \mathbb{R}$ be a solution of

$$\begin{aligned} &\varepsilon^2 u_i'' - f(x, u_i) = 0, \quad u_i'(x_i) = 0 = u_i'(x_{i+1}), \\ &(-1)^i u_i' > 0,\, u_i > 0 \quad \text{in } [x_i, x_{i+1}], \end{aligned} \tag{5.3}$$

for $i = 0, \ldots, N_\varepsilon$. Since $d(x_i, x_{i+1}) \geq 1$, by Theorem 5.1 of [10], the function u_i is well defined, and u_i as a function of (x_i, x_{i+1}) is of class C^1. We define the functional $g_\varepsilon : \Delta_\varepsilon \to \mathbb{R}$ as

$$g_\varepsilon(X) = \sum_{i=0}^{N_\varepsilon} \int_{x_i}^{x_{i+1}} E_\varepsilon(x, u_i(x))\, dx,$$

where

$$E_\varepsilon(x,u) = \varepsilon^2 \frac{u'^2(x)}{2} + V(x)\frac{u^2}{2} - \frac{u^{p+1}}{p+1}.$$

The functional g_ε is of class C^1 and it is easy to check that

$$\frac{\partial g_\varepsilon}{\partial x_i}(X) = F(x_i, u_{i-1}(x_i)) - F(x_i, u_i(x_i)), \quad 1 \le i \le N_\varepsilon,$$

with $F(x,u) = V(x)\frac{u^2}{2} - \frac{u^{p+1}}{p+1}$. Thus, if $\nabla g_\varepsilon(X) = 0$ then the function u_ε, defined as

$$u_\varepsilon(x) = u_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N_\varepsilon, \tag{5.4}$$

is a solution of (2.2). In view of these considerations, Proposition 3.4 will be proved if we show that the maximum of g_ε is achieved in $\mathrm{Int}(\Delta_\varepsilon)$.

Proof of Proposition 3.4. We proceed by contradiction. Suppose there exist sequences $\varepsilon_n \to 0$ and $X_n = (x_1^n, \ldots, x_{N^{\varepsilon_n}}^n) \in \partial\Delta_{\varepsilon_n}$ such that $g_{\varepsilon_n}(X_n) \ge g_{\varepsilon_n}(X)$ for all $X \in \Delta_{\varepsilon_n}$. For simplicity we write $x_i = x_i^n$, $\Delta_n = \Delta_{\varepsilon_n}$, $g_n = g_{\varepsilon_n}$, $N_n^j = N_{\varepsilon_n}^j$ for $j = 1, 2$, and $N_n = N_{\varepsilon_n}$. We let $\hat{a}$ be an adiabatic profile with $\mathrm{supp}(\hat{a}) = (\hat{\alpha}_1, \hat{\beta}_1) \cup (\hat{\alpha}_2, \hat{\beta}_2)$ and such that $\tilde{\alpha}_j < \hat{\alpha}_j < \alpha_j$, $\beta_j < \hat{\beta}_j < \tilde{\beta}_j$ for $j = 1, 2$.

For $j = 1, 2$ we define $B_n^j = \{i;\ [x_i, x_{i+1}] \cap (\hat{\alpha}_j, \hat{\beta}_j) \neq \emptyset\}$.

Step 1. *For some $\kappa > 0$ there exist $j_n^1 \in B_n^1$ and $j_n^2 \in B_n^2$ such that up to a subsequence,*

$$\lim_{n\to\infty} d(x_{j_n^k}, x_{j_n^k+1}) > 1 + \kappa \quad \textit{for } k = 1, 2. \tag{5.5}$$

Suppose that (5.5) does not hold for $k = 1$. Then for some sequence $\gamma_n \to 0$ we have

$$|B_n^1| = \frac{1+\gamma_n}{\varepsilon_n}\int_{\hat{\alpha}_1}^{\hat{\beta}_1} \frac{2}{\tilde{T}(V(x), \tilde{a}(x))}\, dx,$$

which contradicts $B_n^1 \subset \{0, \ldots, N_n^1\}$.

We write $B_n^1 = \{i_1^1, \ldots, i_l^1\}$, $B_n^2 = \{i_1^2, \ldots, i_l^2\}$.

Step 2. *The function u_{ε_n} defined in* (5.4) *is a solution of* (2.2) *in* $(x_{i_1^1}, x_{i_l^1+1}) \cup (x_{i_1^2}, x_{i_l^2+1})$.

Suppose that u_{ε_n} is not a solution in $(x_{i_1^1}, x_{i_l^1})$. Then there is a sequence of integers k_n so that $i_1^1 < k_n \le j_n^1$ (or $j_n^1 + 1 \le k_n < i_l^1 + 1$) such that $\partial g_n(X_n)/\partial x_{k_n} \neq 0$ and u_{ε_n} is a solution of (2.2) in $(x_{k_n}, x_{j_n^1+1})$ (or in $(x_{j_n^1}, x_{k_n})$). Note that by Theorem 2.3, $d(x_{k_n}, x_{k_n+1}) > 1 + \bar{\kappa}$ for some $\bar{\kappa} > 0$. We have to analyze two cases:

(a) If for a subsequence $d(x_{k_n-1}, x_{k_n}) > 1$ for all n, then we can choose a point $Y_n = (y_1, \ldots, y_{N_\varepsilon})$ with $y_i = x_i$ if $i \neq k_n$ and y_{k_n} close to x_{k_n} such that $Y_n \in \Delta_n$ and $\frac{\partial g_n(X_n)}{\partial x_{k_n}}(y_{k_n} - x_{k_n}) > 0$, contradicting the maximality of X_n.

(b) If for a subsequence $d(x_{k_n-1}, x_{k_n}) = 1$ for all n, then after a simple computation we can prove that for n large,

$$\frac{\partial g_n(X_n)}{\partial x_{k_n}} = F(x_{k_n}, u_{k_n-1}(x_{k_n})) - F(x_{k_n}, u_{k_n}(x_{k_n})) > 0,$$

thus, constructing a Y_n as above we contradict the maximality of X_n.

Therefore, u_{ε_n} is a solution of (2.2) in $(x_{i_1^1}, x_{i_l^1})$. Similarly, we can conclude that u_{ε_n} is a solution in $(x_{i_1^2}, x_{i_l^2})$.

Step 3. *Up to a subsequence, the appropriate adiabatic profiles $a^1_{\varepsilon_n}$, $a^2_{\varepsilon_n}$, defined in $(x_{i_1^1}, x_{i_l^1})$ and $(x_{i_1^2}, x_{i_l^2})$ respectively, converge to $\underline{a}(x)$ with support in $(\underline{\alpha}_1, \underline{\beta}_1) \cup (\underline{\alpha}_2, \underline{\beta}_2)$ with $\hat{\alpha}_k < \underline{\alpha}_k$, $\underline{\beta}_k < \hat{\beta}_k$ for $k = 1, 2$.*

From Remark 2.4, we obtain

$$\int_{\underline{\alpha}_k}^{\underline{\beta}_k} \frac{1}{\tilde{T}(V(x), \underline{a}(x))}\, dx = \lim_{n\to\infty} \varepsilon_n |B^k_n| \le \int_{\alpha_k}^{\beta_k} \frac{1}{\tilde{T}(V(x), \tilde{a}(x))}\, dx,$$

for $k = 1, 2$. Using this inequality and proceeding as above, we can easily see that $\hat{\alpha}_k < \underline{\alpha}_k$, $\underline{\beta}_k < \hat{\alpha}_k$.

Step 4. *$x_{i_1^1}, x_{i_l^1+1}, x_{i_1^2}, x_{i_l^2+1}$ are all local minima of u_{ε_n}.*

Suppose that $x_{i_l^1+1}$ is a maximum. Then $i_l^1 + 1 < N_n^1$ and since $\tilde{\beta}_1 > \hat{\beta}_1$ we can easily prove that $d(x_{i_l^1}, x_{i_l^1+1}) > 1$. We analyze three possible cases:

(a) For a subsequence, $d(x_{i_l^1+1}, x_{i_l^1+2}) > 1$.

(b) For a subsequence, $d(x_{i_l^1+1}, x_{i_l^1+2}) = 1$ and $x_{i_l+2} \to \bar{x} < \tilde{\beta}_1$.

(c) For a subsequence, $d(x_{i_l^1+1}, x_{i_l^1+2}) = 1$ and $x_{i_l+2} \to \bar{x} \ge \tilde{\beta}_1$.

For (a) we can use the same argument as in Step 2(a) to prove that u_{ε_n} defined as in (5.4) is a solution of (2.2) in $(x_{i_1^1}, x_{i_l^1+2})$, but this cannot happen by Proposition 5.1. To prove that (b) does not hold, we can proceed as in Step 2(b).

Suppose that (c) holds. In this case $x_{i_l^1+1} \to \tilde{\beta}_1$, and for n large, $x_{i_l^1+1} - x_{i_l^1} > c$ for some positive and fixed c. We define $Y_n = (y_1, \ldots, y_{N_\varepsilon}) \in \Delta_n$ by setting $y_i = x_i$ if $i \ne i_l^1+1$ and $y_{i_l^1+1} = x_{i_l^1+1} - \zeta$ for $\zeta > 0$ small. If we rescale we obtain $g(X_n) - g(Y_n) = \varepsilon_n I_n$ with $I_n \to I$ given by

$$I = 2\int_0^\infty (|z'|^2/2 + F(\tilde{\beta}_1, z))\, dx - 2\int_0^\infty (|w'|^2/2 + F(\tilde{\beta}_1 - \zeta, w))\, dx,$$

where z and w satisfy

$$\begin{aligned} z'' - V(\tilde{\beta}_1)z + z^p = 0, &\qquad z'(0) = 0, \qquad z(\infty) = 0, \\ w'' - V(\tilde{\beta}_1 - \zeta)w + w^p = 0, &\qquad w'(0) = 0, \qquad w(\infty) = 0. \end{aligned}$$

It is easy to check that

$$I = C(V(\tilde{\beta}_1)^{\frac{p+3}{2(p-1)}} - V(\tilde{\beta}_1 - \zeta)^{\frac{p+3}{2(p-1)}}) < 0. \tag{5.6}$$

This contradicts the maximality of X_n.

All the remaining cases can be handled in the same way.

Step 5. $i_1^1 = 0, i_l^1 + 1 = i_1^2 = N_n^1, i_l^2 + 1 = N_n + 1.$

We start by proving that $i_l^1 + 1 = N_n^1$. If this is not the case then $i_l^1 + 2 \le N_n^1$, and since $x_{i_l^1}$ is a minimum we have $x_{i_l^1+1} - x_{i_l^1} > c$ for some fixed $c > 0$. If $d(x_{i_l^1+1}, x_{i_l^1+2}) > 1$, then arguing as in Step 2(a), we find that u_{ε_n} as defined by (5.4) is a solution of (2.2) in $(x_{i_1^1}, x_{i_l^1+2})$ with an isolated peak. This cannot happen by Proposition 5.1.

If $d(x_{i_l^1+1}, x_{i_l^1+2}) = 1$ and $x_{i_l^1+2} \to \bar{x} < \tilde{\beta}_1$ we can proceed as in Step 2(b) to reach a contradiction. When $d(x_{i_l^1+1}, x_{i_l^1+2}) = 1$ and $x_{i_l^1+2} \to \bar{x} \ge \tilde{\beta}_1$ then we define $Y_n = (y_1, \dots, y_{N_\varepsilon}) \in \Delta_n$ as $y_i = x_i$ if $i \ne i_l^1 + 1, i_l^1 + 2$ and $y_{i_l^1+1} = x_{i_l^1+1} - \zeta$, $y_{i_l^1+2} = x_{i_l^1+2} - \zeta$ for $\zeta > 0$ small. Then as before we have $g(X_n) - g(Y_n) = \varepsilon_n I_n$ with $I_n \to I$ given by

$$I = 2\int_0^\infty (|z'|^2/2 + F(\tilde{\beta}_1, z))\,dx - 2\int_0^\infty (|w'|^2/2 + F(\tilde{\beta}_1 - \zeta, w))\,dx,$$

with z, w as in Step 4(c). Thus, by (5.6) we conclude that $g(Y_n) > g(X_n)$, which contradicts the maximality of X_n.

All the other cases can be argued similarly.

Step 6. $x_{N_n^1-1} < \beta_1 + \delta/2$, $x_{N_n^1+1} > \alpha_2 + \delta/2$ *and* $d(x_{N_n^1\pm 1}, x_{N_n^1}) > 1$.

By the arguments given above we have shown that u_{ε_n} is a solution in $[x_0, x_{N_n^1}) \cup (x_{N_n^1}, x_{N_n+1}]$ satisfying $u'_{\varepsilon_n}(x_{N_n^1+1}) = 0$. Therefore, by Proposition 5.1, Theorem 2.3 and Remark 2.4 we have $x_{N_n^1-1} \to \bar{\beta}_1 < \tilde{\beta}_1$ and $x_{N_n^1+1} \to \bar{\alpha}_2 > \tilde{\alpha}_2$. Therefore, $d(x_{N_n^1-1}, x_{N_n^1}) > 1$ and $d(x_{N_n^1}, x_{N_n^1+1}) > 1$. This concludes the proof.

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