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Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity

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Abstract. We study the existence, nonexistence and multiplicity of positive solutions for the family of problems $-\Delta u = f_\lambda(x, u)$, $u \in H_0^1(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$ is a parameter. The results include the well-known nonlinearities of the Ambrosetti–Brezis–Cerami type in a more general form, namely $\lambda a(x)u^q + b(x)u^p$, where $0 \leq q < 1 < p \leq 2^* - 1$. The coefficient $a(x)$ is assumed to be nonnegative but $b(x)$ is allowed to change sign, even in the critical case. The notions of local superlinearity and local sublinearity introduced in [9] are essential in this more general framework. The techniques used in the proofs are lower and upper solutions and variational methods.

Keywords. Multiplicity, semilinear elliptic problem, local sub- and superlinear nonlinearities, concave-convex nonlinearities, critical exponent, upper and lower solutions, variational method

1. Introduction

This paper is concerned with the existence, nonexistence and multiplicity of solutions for the family of problems

$$\begin{cases} -\Delta u = f_\lambda(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $\lambda > 0$ is a parameter. An important feature of this family is its monotone dependence on λ, i.e. $f_\lambda(x, s) \leq f_{\lambda'}(x, s)$ if $\lambda < \lambda'$.

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There are several motivations to our study. One of them comes from the following example:

$$\begin{cases} -\Delta u = \lambda a(x) u^q + b(x) u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $0 \leq q < 1 < p$. This example was extensively studied in [1] when $a(x) \equiv 1$, $b(x) \equiv 1$; it was in particular shown there that if $p \leq 2^* - 1$ where $2^* = 2N/(N-2)$, then there exists $0 < \Lambda < \infty$ such that (1.2) has at least two solutions for $\lambda < \Lambda$, at least one solution for $\lambda = \Lambda$, and no solution for $\lambda > \Lambda$. In this paper we extend this result of [1] to the case of variable coefficients $a(x)$ and $b(x)$, with $a(x) \geq 0$ but $b(x)$ possibly indefinite. This is partly carried out along the lines of our previous work [9] where the notions of local superlinearity and local sublinearity were introduced. The main difference here with respect to [9], as far as example (1.2) is concerned, is the assumption $a(x) \geq 0$ in Ω. This allows in particular the use of the strong maximum principle. We emphasize that $b(x)$ in (1.2) is allowed to change sign even in the critical case where $p = 2^* - 1$. As observed in [5, p. 454], critical problems become more delicate in the presence of variable coefficients. In this respect, our basic assumption on $b(x)$ above in the critical case requires that $b(x)$ remains equal or sufficiently close to $\|b\|_\infty$ on a small ball (cf. condition (b) in Theorem 4.2).

Our results relative to (1.1) apply as well to several situations rather different from example (1.2). We can handle for instance a problem like

$$\begin{cases} -\Delta u = \lambda c(x)(u+1)^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < p \leq 2^* - 1$ and $c(x) \geq 0$. This problem was studied in [5] and [11] when $c(x) \equiv 1$.

Our present approach to obtain multiple solutions to (1.1) is different from that in [9]. We follow here the classical method of obtaining a first solution via upper-lower solutions and a second one via the mountain pass theorem. The H^1 versus C^1 minimization result of [6] plays an important role in this approach. In the critical case we use some of the techniques developed in [5] and [1] to handle the (PS) condition.

Our results relative to (1.1) are stated in detail in Section 2 and their proofs given in Section 3. Their application to problems (1.2) and (1.3) is dealt with in Section 4.

2. Statement of results

In this section we state our results relative to (1.1), first for a nonlinearity of arbitrary growth, then in the subcritical case, and finally in the critical case.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$. Our general assumption on the family $f_\lambda(x, s)$ is:

(H) For each $\lambda > 0$, $f_\lambda : \Omega \times [0, \infty) \to \mathbb{R}$ is a Carathéodory function with the property that for any $s_0 > 0$, there exists a constant A such that

$$|f_\lambda(x, s)| \leq A$$

for a.e. $x \in \Omega$ and all $s \in [0, s_0]$. Moreover if $\lambda < \lambda'$, then $f_\lambda(x, s) \leq f_{\lambda'}(x, s)$ for a.e. $x \in \Omega$ and all $s \geq 0$.

The following assumption concerns the behavior of $f_\lambda(x, s)$ near $s = 0$; it implies $f_\lambda(x, 0) \geq 0$ and, as assumption (H), will be assumed throughout the paper:

(H_0) For each $\lambda > 0$ and each $s_0 > 0$, there exists $B > 0$ such that

$$f_\lambda(x, s) \geq -Bs$$

for a.e. $x \in \Omega$ and all $s \in [0, s_0]$.

We will always understand that $f_\lambda(x, s)$ has been extended for $s < 0$ by putting $f_\lambda(x, s) = f_\lambda(x, 0)$ for $\lambda > 0$, a.e. $x \in \Omega$ and $s < 0$.

Observe that, at this stage, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies the equation $-\Delta u = f_\lambda(x, u)$ in the $H_0^1(\Omega)$ sense, then (H) and the standard regularity theory imply $u \in W^{2,r}(\Omega)$ for any $r < \infty$ and so $u \in C^1(\overline{\Omega})$. Moreover $u \geq 0$ (in fact, take $-u^-$ as a test function in the equation and use $f_\lambda(x, 0) \geq 0$); in addition, we have $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$ if $u \not\equiv 0$ (this follows from (H_0) and the strong maximum principle). Here ν denotes the exterior normal. Observe also that the associated functional

$$I_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F_\lambda(x, u),$$

where $F_\lambda(x, s) := \int_0^s f_\lambda(x, t)\, dt$, is well defined for $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The following two assumptions will be used in our first result:

(H_e) There exist $\lambda > 0$ and a nondecreasing function g with $\inf\{g(s)/s : s > 0\} < 1/\|e\|_\infty$ such that

$$f_\lambda(x, s) \leq g(s)$$

for a.e. $x \in \Omega$ and all $s \geq 0$; here e is the solution of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$, and $\| \ \|_\infty$ denotes the $L^\infty(\Omega)$ norm.

(H_{Ω_1}) For any $\lambda > 0$ there exists a smooth subdomain Ω_1, $s_1 > 0$ and $\theta_1 > \lambda_1(\Omega_1)$ such that

$$f_\lambda(x, s) \geq \theta_1 s$$

for a.e. $x \in \Omega_1$ and all $s \in [0, s_1]$; here $\lambda_1(\Omega_1)$ denotes the principal eigenvalue of $-\Delta$ on $H_0^1(\Omega_1)$.

Here are some comments on the above two assumptions. Assumption (H_e) is a rather standard condition to guarantee the existence of an upper solution (cf. e.g. [10]). This condition is motivated by the fact that an upper solution for an equation of the type $-\Delta u = f(u)$ can be obtained if one has an upper solution for another equation of the form $-\Delta u = g(u)$ with $f(s) \le g(s)$ for all s. Assumption (H_{Ω_1}) is a local sublinearity condition at 0, which is satisfied for instance if the following stronger condition holds:

$$\lim_{\substack{s \to 0 \\ s>0}} \frac{f_\lambda(x,s)}{s} = \infty,$$

uniformly for $x \in \Omega_1$. Assumption (H_{Ω_1}) is used to construct a lower solution.

Theorem 2.1 (Existence of one solution without growth condition)**.** *Under the assumptions (H), (H_0), (H_e) and (H_{Ω_1}), there exists $0 < \Lambda \le \infty$ such that problem* (1.1) *has at least one solution u (with $I_\lambda(u) < 0$) for $0 < \lambda < \Lambda$ and no solution for $\lambda > \Lambda$.*

We remark that in the present generality, Λ can be ∞. One trivial example is provided by a family as above such that, for each $\lambda > 0$, there exists $M_\lambda > 0$ with $f_\lambda(x, M_\lambda) < 0$ for a.e. x. In this case the constant M_λ is an upper solution.

Theorem 2.2 (Nonexistence for λ large)**.** *In addition to the hypotheses of Theorem* 2.1*, assume:*

$(H_{\tilde\Omega})$ There exist a function h with $h(\lambda) \to \infty$ as $\lambda \to \infty$, a smooth subdomain $\tilde\Omega$ and $\tilde m \in L^\infty(\tilde\Omega)$ with $\tilde m \ge 0$, $\tilde m \not\equiv 0$, such that

$$f_\lambda(x,s) \ge h(\lambda)\tilde m(x) s$$

for all $\lambda > 0$, a.e. $x \in \tilde\Omega$ and all $s \ge 0$.

Then $\Lambda < \infty$.

Assumption $(H_{\tilde\Omega})$ can be looked at as a localized version of the trivial sufficient condition of nonexistence for $-\Delta u = l(u)$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, namely that $\inf\{l(s)/s : s > 0\} > \lambda_1(\Omega)$.

Due to the absence of growth condition, we have up to now defined a solution as a function in $H_0^1(\Omega) \cap L^\infty(\Omega)$. However, if the following growth condition with respect to s in the nonlinearity $f_\lambda(x,s)$ is assumed, then one can speak of an $H_0^1(\Omega)$ solution in the usual sense:

(G) For any $[r, R] \subset \{\lambda > 0\}$, there exist d_1, d_2 and $\sigma \le 2^* - 1$ such that

$$|f_\lambda(x,s)| \le d_1 + d_2 s^\sigma$$

for all $\lambda \in [r, R]$, a.e. $x \in \Omega$ and all $s \ge 0$.

If $\sigma < 2^* - 1$ in (G), then a standard bootstrap argument gives that any $u \in H_0^1(\Omega)$ which solves $-\Delta u = f_\lambda(x,u)$ belongs to $W^{2,r}(\Omega)$ for any $r < \infty$ and consequently to $C^1(\overline\Omega)$. This conclusion also holds if σ in (G) is equal to $2^* - 1$, by using a result of [4]. Condition (G) (with $\sigma \le 2^* - 1$) also implies that the functional $I_\lambda(u)$ is well defined for $u \in H_0^1(\Omega)$.

Aiming now to prove the existence of a solution for $\lambda = \Lambda$, we will assume the following condition:

$(AR)_d$ For any $[r, R] \subset \{\lambda > 0\}$, there exist $\theta > 2$, $\rho < 2$, $d \geq 0$ and $s_0 \geq 0$ such that

$$\theta F_\lambda(x, s) \leq s f_\lambda(x, s) + d s^\rho$$

for all $\lambda \in [r, R]$, a.e. $x \in \Omega$ and all $s \geq s_0$.

This condition $(AR)_d$ is a weakening of the classical superquadraticity condition of Ambrosetti–Rabinowitz [2]. It was introduced in [9] in order to handle indefinite non-linearities.

Theorem 2.3 (Existence of one solution for $\lambda = \Lambda$)**.** *In addition to the hypotheses of Theorem* 2.2*, assume* (G)*,* $(AR)_d$ *and the continuity of* $f_\lambda(x, s)$ *with respect to* λ *(for a.e.* x *and uniformly for* s *bounded). Then problem* (1.1) *has at least one solution* u *(with* $I_\lambda(u) \leq 0$*) for* $\lambda = \Lambda$.

Remark. The uniformity with respect to $\lambda \in [r, R]$ in (G) and $(AR)_d$ is used only in Theorem 2.3 to deal with the limiting case $\lambda = \Lambda$. It is not needed in the following Theorems 2.4–2.6, where $\lambda < \Lambda$ will be fixed.

Now we discuss multiplicity for subcritical families, namely the ones satisfying (G) with $\sigma < 2^* - 1$. Our purpose is to prove the existence of at least two solutions when $\lambda < \Lambda$. For that matter we have to strengthen a little bit some of the hypotheses of Theorem 2.1. Condition (H_0) is replaced by

$(H_0)'$ For any $\lambda > 0$ and any $s_0 > 0$, there exists $B \geq 0$ such that for a.e. $x \in \Omega$,

$$s \mapsto f_\lambda(x, s) + Bs$$

is nondecreasing on $[0, s_0]$; moreover $f_\lambda(x, 0) \geq 0$ for all $\lambda > 0$ and a.e. $x \in \Omega$.

Condition $(H_0)'$ is a classical requirement when dealing with upper-lower solutions. The monotonicity of the family f_λ is also assumed to be strict in the following sense:

(M) For any $\lambda < \lambda'$ and any $u \in C_0^1(\overline{\Omega})$ with $u > 0$ in Ω,

$$f_\lambda(x, u(x)) \leq\not\equiv f_{\lambda'}(x, u(x)).$$

We will also assume:

(H_{Ω_2}) For any $\lambda > 0$, there exist a subdomain Ω_2, s_2 and $\theta_2 > 0$ such that

$$F_\lambda(x, s) \geq \theta_2 s^2$$

for a.e. $x \in \Omega_2$ and all $s \geq s_2$.

Condition (H_{Ω_2}) is implied by a local superlinearity condition at ∞ of the form

$$\lim_{s \to \infty} \frac{f_\lambda(x, s)}{s} = \infty$$

uniformly for $x \in \Omega_2$. It is used in conjunction with $(AR)_d$ to derive the geometry of the mountain pass.

Theorem 2.4 (Existence of a second solution in the subcritical case)**.** *In addition to the hypotheses of Theorem* 2.1*, assume* (G) *with* $\sigma < 2^* - 1$ *as well as* $(AR)_d$*,* $(H_0)'$*,* (M) *and* (H_{Ω_2})*. Then problem* (1.1) *has at least two solutions* u, v *for* $0 < \lambda < \Lambda$*, with* $u < v$ *in* Ω*,* $\partial u/\partial \nu > \partial v/\partial \nu$ *on* $\partial\Omega$ *and* $I_\lambda(u) < 0$*.*

Finally, we consider multiplicity for critical families. This means that $f_\lambda(x, s)$ behaves at ∞ like $b(x)s^p$ with $p = 2^* - 1$. We thus write the function f_λ as

$$f_\lambda(x, s) = h_\lambda(x, s) + b(x)s^p \tag{2.1}$$

and we distinguish two cases: (i) h_λ satisfies (G) with $\sigma < 1$, $b(x)$ may change sign, (ii) h_λ satisfies (G) with $\sigma < 2^* - 1$, $b(x) \geq 0$ in Ω.

We first deal with case (i).

Theorem 2.5 (Existence of a second solution in the critical case with $\sigma < 1$)**.** *In addition to the hypotheses of Theorem* 2.1*, assume that* $f_\lambda(x, s)$ *satisfies* $(H_0)'$ *and* (M)*. Suppose also that* $f_\lambda(x, s)$ *can be written as in* (2.1) *with* $p = 2^* - 1$*,* $h_\lambda(x, s)$ *satisfying* (G) *with* $\sigma < 1$*, and* $h_\lambda(x, s)$ *nondecreasing with respect to* s *for any* $\lambda > 0$ *and a.e.* x*. Suppose also that* $b(x)$ *in* (2.1) *is* $\not\equiv 0$*, belongs to* $L^\infty(\Omega)$ *and satisfies*

(b) *for some* $x_0 \in \Omega$*, some ball* $B_1 \subset \Omega$ *around* x_0*, some constant* M *and some* γ *with* $\gamma > 2^*$ *when* $N \geq 5$*,* $\gamma \geq 2^*$ *when* $N = 4$*,* $\gamma > 3/5$ *when* $N = 3$*, one has*

$$0 \leq \|b\|_\infty - b(x) \leq M|x - x_0|^\gamma$$

for a.e. $x \in B_1$*. (Recall that* $\| \ \|_\infty$ *denotes the* $L^\infty(\Omega)$ *norm.)*

Then the conclusion of Theorem 2.4 *holds.*

Assumption (b) implies $\|b^-\|_\infty \leq \|b^+\|_\infty$, with in addition some limitation on the way $b(x)$ approaches $\|b\|_\infty$. It trivially holds if $b(x) = \|b\|_\infty$ a.e. on a small ball.

We now deal with the critical case (ii).

Theorem 2.6 (Existence of a second solution in the critical case with $\sigma < 2^* - 1$)**.** *In addition to the hypotheses of Theorem* 2.1*, assume that* $f_\lambda(x, s)$ *satisfies* $(H_0)'$ *and* (M)*. Suppose also that* $f_\lambda(x, s)$ *can be written as in* (2.1) *with* $p = 2^* - 1$*,* $h_\lambda(x, s)$ *satisfying* (G) *with* $\sigma < 2^* - 1$*,* $h_\lambda(x, s)$ *nondecreasing with respect to* s *for any* $\lambda > 0$ *and a.e.* x*, and* $h_\lambda(x, s)$ *satisfying* $(AR)_d$*. Suppose that* b *in* (2.1) *is* $\not\equiv 0$*,* ≥ 0 *in* Ω*, belongs to* $L^\infty(\Omega)$ *and satisfies condition* (b) *above. Then the conclusion of Theorem* 2.4 *holds.*

In Theorem 2.6, $h_\lambda(x, s)$ is allowed any subcritical growth, at the expense of assuming $(AR)_d$ for $h_\lambda(x, s)$ and $b(x) \geq 0$.

3. Proofs

This section is devoted to the proofs of all theorems stated above. It will be convenient from now on to denote (1.1) as $(1.1)_\lambda$.

Proof of Theorem 2.1. We start by proving the existence of an upper solution of $(1.1)_\lambda$ for the value of λ provided by (H_e). The construction is inspired from [3] (see also [1], [10]). One takes the solution e of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$. With λ and g given by (H_e), there exists $M > 0$ such that

$$1/\|e\|_\infty \geq g(M\|e\|_\infty)/(M\|e\|_\infty)$$

and so one has

$$-\Delta(Me) = M \geq g(M\|e\|_\infty) \geq g(Me) \geq f_\lambda(x, Me).$$

This shows that Me is a classical upper solution of $(1.1)_\lambda$.

We now construct a lower solution for $(1.1)_\lambda$ by using the subdomain Ω_1 provided by (H_{Ω_1}). Denote by φ_1 the positive principal eigenfunction of $-\Delta$ on $H_0^1(\Omega_1)$. Extend φ_1 by 0 on $\Omega \setminus \Omega_1$; the extended function, still denoted by φ_1, belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. One then argues as in [9, pp. 464–465] to show that for $\varepsilon > 0$ sufficiently small, $\varepsilon\varphi_1$ is a weak lower solution of $(1.1)_\lambda$ which satisfies $\varepsilon\varphi_1 \leq Me$ in Ω.

It follows that Theorem 2.4 of [13] can be applied; it yields the existence of a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of $(1.1)_\lambda$ for the value of λ provided by (H_e). So at this stage we have proved that

$$\Lambda := \sup\{\lambda > 0 : (1.1)_\lambda \text{ has a solution}\} > 0.$$

It remains to show that for each $0 < \lambda < \Lambda$, $(1.1)_\lambda$ has a solution u with $I_\lambda(u) < 0$. Let $0 < \lambda < \Lambda$ and take $\overline{\lambda}$ such that $\lambda < \overline{\lambda} < \Lambda$ and $(1.1)_{\overline{\lambda}}$ has a solution $\overline{u}$; this is clearly possible by the definition of Λ. One has, by the monotonicity of the family f_λ,

$$-\Delta\overline{u} = f_{\overline{\lambda}}(x, \overline{u}) \geq f_\lambda(x, \overline{u}),$$

which shows that $\overline{u}$ is an upper solution for $(1.1)_\lambda$. A previous argument involving the subdomain Ω_1 from (H_{Ω_1}) shows that for $\varepsilon > 0$ sufficiently small, $\varepsilon\varphi_1$ is a weak lower solution of $(1.1)_\lambda$ which satisfies $\varepsilon\varphi_1 \leq \overline{u}$ in Ω. Theorem 2.4 from [13] then yields the existence of a solution $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of $(1.1)_\lambda$ which satisfies

$$I_\lambda(u_0) = \min\{I_\lambda(u) : u \in H_0^1(\Omega) \text{ and } \varepsilon\varphi_1 \leq u \leq \overline{u}\}. \tag{3.1}$$

Since by (H_{Ω_1}),

$$I_\lambda(\varepsilon\varphi_1) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla\varphi_1|^2 - \int_\Omega F_\lambda(x, \varepsilon\varphi_1) < 0 \tag{3.2}$$

for ε sufficiently small (so that $\varepsilon\varphi_1 \leq s_1$), one deduces $I_\lambda(u_0) < 0$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. We have to prove that for λ sufficiently large, $(1.1)_\lambda$ has no solution. The subdomain $\tilde{\Omega}$ provided by $(H_{\tilde{\Omega}})$ will be used here. Suppose that $(1.1)_\lambda$ admits a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Denoting by $\tilde{\varphi}$ the positive eigenfunction associated to the principal eigenvalue $\lambda_1(\tilde{m}, \tilde{\Omega})$ of $-\Delta$ on $H_0^1(\tilde{\Omega})$ for the weight $\tilde{m}$ and extending $\tilde{\varphi}$ by 0 on $\Omega \setminus \tilde{\Omega}$, one argues as in [9, p. 466] to get

$$\int_\Omega \nabla u \nabla \tilde{\varphi} = \int_{\partial\tilde{\Omega}} u \frac{\partial \tilde{\varphi}}{\partial \nu} + \int_{\tilde{\Omega}} u(-\Delta\tilde{\varphi}) \leq \lambda_1(\tilde{m}, \tilde{\Omega}) \int_{\tilde{\Omega}} \tilde{m} u \tilde{\varphi}. \tag{3.3}$$

On the other hand, by $(H_{\tilde{\Omega}})$,

$$\int_\Omega \nabla u \nabla \tilde{\varphi} = \int_\Omega f_\lambda(x, u)\tilde{\varphi} \geq h(\lambda) \int_{\tilde{\Omega}} \tilde{m} u \tilde{\varphi}. \tag{3.4}$$

Since $\int_{\tilde{\Omega}} \tilde{m} u \tilde{\varphi}$ is > 0, one deduces from (3.3) and (3.4) that $h(\lambda) \leq \lambda_1(\tilde{m}, \tilde{\Omega})$. The conclusion follows since $h(\lambda) \to \infty$ as $\lambda \to \infty$. □

Proof of Theorem 2.3. We have to prove that $(1.1)_\lambda$ has at least one solution u with $I_\lambda(u) \leq 0$ for $\lambda = \Lambda$. The continuity of f_λ with respect to λ as well as the fact that (G) and $(AR)_d$ hold uniformly for $\lambda \in [r, R]$ will be used here. Let $\lambda_k \to \Lambda$ with $0 < \lambda_k < \Lambda$ and λ_k increasing, and let u_k be a solution of $(1.1)_{\lambda_k}$ with $I(u_k) < 0$.

We first show that the sequence (u_k) remains bounded in $H_0^1(\Omega)$. Indeed, using $I_{\lambda_k}(u_k) < 0$ and $(AR)_d$, one obtains

$$\frac{\theta}{2}\|u_k\|^2 - \int_\Omega u_k f_{\lambda_k}(x, u_k) \leq d \int_\Omega u_k^\rho + c_1$$

for some constant c_1, where $\|v\|$ denotes $(\int_\Omega |\nabla v|^2)^{1/2}$. But $\int_\Omega u_k f_{\lambda_k}(x, u_k) = \|u_k\|^2$ by $(1.1)_{\lambda_k}$, and consequently

$$\left(\frac{\theta}{2} - 1\right)\|u_k\|^2 \leq c_2\|u_k\|^\rho + c_1$$

for some constant c_2. This implies the desired bound since $\theta > 2$ and $\rho < 2$.

Bootstrapping that bound using (G), one sees in particular that for a subsequence, $u_k \to u$ in $H_0^1(\Omega) \cap C(\overline{\Omega})$. The bootstrapping here is the standard one when $\sigma < 2^* - 1$, and is based on [4] (see also [7]) when $\sigma = 2^* - 1$.

Clearly u solves $-\Delta u = f_\Lambda(x, u)$ in Ω, $u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$, and one has $I_\Lambda(u) \leq 0$. It remains to see that $u \not\equiv 0$. Assume by contradiction $u \equiv 0$. We will use (H_{Ω_1}) for $\lambda = \lambda_1$, the first element of the increasing sequence λ_k. Let as before Ω_1 be the corresponding subdomain and φ_1 the positive eigenfunction associated to the principal eigenvalue $\lambda_1(\Omega_1)$ of $-\Delta$ on $H_0^1(\Omega_1)$. We have

$$\int_\Omega \nabla u_k \nabla \varphi_1 = \int_{\Omega_1} f_{\lambda_k}(x, u_k)\varphi_1 \geq \int_{\Omega_1} f_{\lambda_1}(x, u_k)\varphi_1 \geq \theta_1 \int_{\Omega_1} u_k \varphi_1 \tag{3.5}$$

for k sufficiently large (so that $0 \leq u_k(x) \leq s_1$ for $x \in \Omega_1$, which is possible since $u_k \to 0$ uniformly on $\overline{\Omega}$). On the other hand,

$$\int_{\Omega_1} \nabla u_k \nabla \varphi_1 = \int_{\partial\Omega_1} u_k \frac{\partial \varphi_1}{\partial \nu} + \int_{\Omega_1} u_k(-\Delta \varphi_1) \leq \lambda_1(\Omega_1) \int_{\Omega_1} u_k \varphi_1, \tag{3.6}$$

and a contradiction follows from (3.5), (3.6) since $\theta_1 > \lambda_1(\Omega_1)$ and $\int_{\Omega_1} u_k \varphi_1 > 0$. This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. We have to prove the existence of a second solution of $(1.1)_\lambda$ for each $0 < \lambda < \Lambda$. Fix such a λ. Proceeding exactly as at the end of the proof of Theorem 2.1 above, introducing $\overline{\lambda}$, $\overline{u}$ and considering the solution u_0 of $(1.1)_\lambda$ constructed there, we start by showing that

$$\underline{u} < u_0 < \overline{u} \quad \text{in } \Omega, \tag{3.7}$$

$$\partial \underline{u}/\partial \nu > \partial u_0/\partial \nu > \partial \overline{u}/\partial \nu \quad \text{on } \partial\Omega, \tag{3.8}$$

where $\underline{u}$ denotes $\varepsilon\varphi_1$, with φ_1 a positive principal eigenfunction of $-\Delta$ on $H_0^1(\Omega_1)$ (extended by 0 outside Ω_1).

The inequalities of (3.7), (3.8) involving $\underline{u}$ and u_0 are obtained in the following way. Since $\underline{u}$ is the extension by 0 on $\Omega \setminus \Omega_1$ of a $C_0^1(\overline{\Omega}_1)$ function and since u_0 is a solution, these inequalities clearly hold on $\Omega \setminus \Omega_1$ and on $\partial\Omega \setminus \partial\Omega_1$ respectively. On the other hand $\underline{u} \not\equiv u_0$ in Ω_1; moreover, using $(H_0)'$, one gets for a suitable B,

$$\begin{cases} -\Delta(u_0 - \underline{u}) \geq f_\lambda(x, u_0) - f_\lambda(x, \underline{u}) \geq -B(u_0 - \underline{u}) & \text{on } \Omega_1, \\ u_0 - \underline{u} \geq 0 & \text{on } \Omega_1. \end{cases}$$

Consequently, by the strong maximum principle, $u_0 - \underline{u} > 0$ in Ω_1 and $\partial(u_0 - \underline{u})/\partial\nu < 0$ on $\partial\Omega_1$. The proof of the inequalities in (3.7), (3.8) involving u_0 and $\overline{u}$ is simpler since both functions belong to $C_0^1(\overline{\Omega})$; the fact that $u_0 \not\equiv \overline{u}$ in Ω here follows from (M).

It follows from (3.7) and (3.8) that $\{u \in H_0^1(\Omega) : \underline{u} \leq u \leq \overline{u}\}$ contains a $C_0^1(\overline{\Omega})$ neighborhood of u_0 and consequently, by (3.1), u_0 is a local minimizer of I_λ on $C_0^1(\overline{\Omega})$. Theorem 1 of [6] then shows that u_0 is also a local minimizer of I_λ on $H_0^1(\Omega)$ (assumption (G), with $\sigma \leq 2^* - 1$, is used here).

The second solution will be constructed in the form $u_0 + w$ where u_0 is the first solution above and w satisfies

$$\begin{cases} -\Delta w = g_\lambda(x, w) & \text{in } \Omega, \\ w \not\equiv 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $g_\lambda(x, s) := f_\lambda(x, u_0(x) + s^+) - f_\lambda(x, u_0(x))$. This is a device already considered in [1] for (1.2) with $a(x) \equiv b(x) \equiv 1$. Clearly any solution w of (3.9) is ≥ 0 (in fact, multiply by $-w^-$ and conclude), and so, by the strong maximum principle and $(H_0)'$, w satisfies $w > 0$ in Ω and $\partial w/\partial\nu < 0$ on $\partial\Omega$. Consequently, $u_0 + w$ will be a second solution of $(1.1)_\lambda$ which fulfills the requirements of Theorem 2.4. Writing $G_\lambda(x,s) := \int_0^s g_\lambda(x,t)\,dt$ and

$$J_\lambda(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_\Omega G_\lambda(x,w), \tag{3.10}$$

we are thus led to look for a nonzero critical point of J_λ on $H_0^1(\Omega)$.

One easily verifies, using

$$G_\lambda(x,s) = F_\lambda(x, u_0(x) + s^+) - F_\lambda(x, u_0(x)) - f_\lambda(x, u_0(x))s^+$$

and the fact that u_0 solves $(1.1)_\lambda$, that for $w \in H_0^1(\Omega)$,

$$J_\lambda(w) = I_\lambda(u_0 + w^+) - I_\lambda(u_0) + \frac{1}{2}\|w^-\|^2. \tag{3.11}$$

It follows from (3.11) that 0 is a local minimizer of J_λ on $H_0^1(\Omega)$, i.e., for some $r > 0$,

$$J_\lambda(0) \le J_\lambda(w) \tag{3.12}$$

for all $w \in B(0,r)$, the ball of center 0 and radius r in $H_0^1(\Omega)$.

Assumption (G) with $\sigma < 2^* - 1$ and $(AR)_d$ imply that I_λ satisfies the (PS) condition on $H_0^1(\Omega)$, as shown in [9, p. 460]. On the other hand, one easily verifies that if w_k is a (PS) sequence for J_λ at level c, then $\|w_k^-\| \to 0$ and $u_0 + w_k^+$ is a (PS) sequence for I_λ at level $c + I_\lambda(u_0)$. It follows that J_λ satisfies the (PS) condition on $H_0^1(\Omega)$.

Now comes an alternative connected with (3.12). Either there exists $w \in B(0,r)$ with $w \neq 0$ and $J_\lambda(w) = 0$, or the strict inequality holds in (3.12) for all $w \in B(0,r)$ with $w \neq 0$. In the first case this w is a nonzero local minimizer for J_λ and so a critical point of J_λ, and the proof is finished. In the second case, Theorem 5.10 from [8] applies to guarantee that for each $r > 0$ sufficiently small,

$$J_\lambda(0) < \inf\{J_\lambda(w) : w \in H_0^1(\Omega) \text{ and } \|w\| = r\}, \tag{3.13}$$

i.e. there is a “mountain range” around 0. We aim at applying the mountain pass theorem. For that purpose we look for some $u_2 \in H_0^1(\Omega)$ such that $J_\lambda(tu_2) \to -\infty$ as $t \to \infty$. Assumption (H_{Ω_2}) will be used here. In fact, as shown in [9, p. 462], (H_{Ω_2}) and $(AR)_d$ imply that for some s_3 and some $c > 0$,

$$F_\lambda(x,s) \ge cs^\theta$$

for a.e. $x \in \Omega_2$ and all $s \ge s_3$, where $\theta > 2$ comes from $(AR)_d$. This inequality clearly implies the same type of inequality for G_λ:

$$G_\lambda(x,s) \ge c's^\theta$$

for some s_3' and $c' > 0$, and a.e. $x \in \Omega_2$ and all $s \ge s_3'$. One then takes a smooth function u_2 with support in Ω_2 and $u_2 \ge 0$, $\not\equiv 0$. Calculating as in [9, p. 462], one finds that $J_\lambda(tu_2) \to -\infty$ as $t \to \infty$. The usual mountain pass theorem can thus be applied. This concludes the proof of Theorem 2.4. □

Proof of Theorem 2.5. Fix λ with $0 < \lambda < \Lambda$. Proceeding exactly as at the beginning of the proof of Theorem 2.4, one has a first solution u_0 which is a local minimizer of I_λ on $H_0^1(\Omega)$, and one is reduced to proving the existence of a solution w of (3.9), where $g_\lambda(x,s)$ now reads

$$g_\lambda(x,s) := h_\lambda(x, u_0(x)+s^+) - h_\lambda(x, u_0(x)) + b(x)[(u_0(x)+s^+)^p - u_0(x)^p].$$

The associated functional J_λ has again the form given in (3.10), with now

$$\begin{aligned} G_\lambda(x,s) := {} & H_\lambda(x, u_0(x)+s^+) - H_\lambda(x, u_0(x)) - h_\lambda(x, u_0(x))s^+ \\ & + b(x)\left[\frac{(u_0(x)+s^+)^{p+1} - u_0(x)^{p+1}}{p+1} - u_0(x)^p s^+\right], \end{aligned}$$

where $H_\lambda(x,s) := \int_0^1 h_\lambda(x,t)\,dt$. As before 0 is a local minimizer of J_λ on $H_0^1(\Omega)$, and we are reduced to proving the existence of a nonzero critical point for J_λ.

Assume by contradiction that 0 is the only critical point of J_λ. Then, for some ball $B(0,r)$ in $H_0^1(\Omega)$,

$$J_\lambda(0) < J_\lambda(w) \tag{3.14}$$

for all $w \in B(0,r)$. The following lemma will be proved below.

Lemma 3.1. *Assume* 0 *is the only critical point of J_λ. Then J_λ satisfies the* $(\mathrm{PS})_c$ *condition for all levels c with*

$$c < c_0 := S^{N/2}/(N\|b\|_\infty^{(N-2)/2}), \tag{3.15}$$

where S is the best Sobolev constant.

Using this lemma and Theorem 5.10 in [8] (which only requires the $(\mathrm{PS})_c$ condition to hold at the level of the strict local minimum, here the level $J_\lambda(0) = 0 < c_0$), one deduces from (3.14) that (3.13) holds for all $r > 0$ sufficiently small. We aim again at applying the mountain pass theorem. For this purpose we will show the existence of $u_1 \in H_0^1(\Omega)$ such that $J_\lambda(u_1) < 0$ and the infmax value of J_λ over the family of all continuous paths from 0 to u_1 is $< c_0$. Once this is done, the usual mountain pass theorem yields the existence of a nonzero critical point for J_λ, a contradiction which will complete the proof of Theorem 2.5.

To construct a u_1 as above, we consider as in [1] functions of the form $t\psi_\mu$ with $t > 0$ and

$$\psi_\mu(x) := d\zeta(x)\left(\frac{\mu}{\mu^2 + |x-x_0|^2}\right)^{(N-2)/2}$$

where $\mu > 0$, x_0 comes from assumption (b), ζ is a fixed smooth nonnegative function with $\zeta \equiv 1$ near x_0 and support in a small ball B_2 around x_0 (with B_2 chosen such that $\overline{B}_2 \subset B_1$ and $b(x) \geq$ some $\varepsilon > 0$ a.e. on B_2), and the normalizing constant $d > 0$ is taken so that ψ_1 satisfies $-\Delta\psi_1 = \psi_1^{(N+2)/(N-2)}$ near x_0. Since h_λ satisfies (G) with $\sigma < 1$ (in fact $\sigma < p$ suffices in this part of the argument), one finds that for each $\mu > 0$, $J_\lambda(t\psi_\mu) \to -\infty$ as $t \to \infty$, and consequently there exists $t = t_\mu > 0$ such that $J_\lambda(t_\mu\psi_\mu) < 0$. The following lemma implies that for μ sufficiently small, the infmax value of J_λ over the family of all continuous paths from 0 to $u_1 = t_\mu\psi_\mu$ is indeed $< c_0$.

Lemma 3.2. *One has*

$$\sup_{t>0} J_\lambda(t\psi_\mu) < c_0$$

for $\mu > 0$ *sufficiently small.*

The above two lemmas, to be proved below, complete the proof of Theorem 2.5. □

Proof of Lemma 3.1. Let w_n be a $(PS)_c$ sequence with $c < c_0$, i.e.

$$\frac{1}{2}\|w_n\|^2 - \int_\Omega G_\lambda(x, w_n) \to c, \tag{3.16}$$

$$\int_\Omega \nabla w_n \cdot \nabla\varphi - \int_\Omega g_\lambda(x, w_n)\varphi \le \varepsilon_n\|\varphi\|, \quad \forall\varphi \in H_0^1(\Omega), \tag{3.17}$$

where $\varepsilon_n \to 0$. We first observe that w_n remains bounded in $H_0^1(\Omega)$. This follows by multiplying (3.17) with $\varphi = u_0 + w_n$ by $1/(p+1)$ and subtracting from (3.16); the terms of power $p+1$ cancel and the remaining dominating term is $\|w_n\|^2$, which easily yields the boundedness of w_n. Note that the assumption that h_λ satisfies (G) with $\sigma < 1$ is used in this argument. So, for a subsequence, $w_n \rightharpoonup w_0$ in $H_0^1(\Omega)$ and $w_n \to w_0$ in $L^r(\Omega)$ for any $r < 2^*$. From (3.17) it follows that w_0 solves

$$\begin{cases} -\Delta w = g_\lambda(x, w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

and consequently, by the assumption of the lemma, $w_0 = 0$. We now go back to (3.17) with $\varphi = u_0 + w_n$, multiply again by $1/(p+1)$ and subtract from (3.16) to get

$$\lim \|w_n\|^2 = cN. \tag{3.18}$$

There are two possibilities: either $c = 0$ or $c \neq 0$. If $c = 0$ then w_n converges in $H_0^1(\Omega)$ by (3.18) and we are done. We will now see that $c \neq 0$ leads to a contradiction. For that purpose we deduce from (3.17) with $\varphi = w_n$ that

$$\lim \|w_n\|^2 = \lim \int_\Omega g_\lambda(x, w_n)w_n = \lim \int_\Omega b(x)(w_n^+)^{p+1}. \tag{3.19}$$

By definition of S,

$$\|w_n\|^2 \ge S\left(\int_\Omega |w_n|^{2^*}\right)^{2/2^*} \ge \frac{S}{\|b\|_\infty^{2/2^*}}\left(\int_\Omega b(x)(w_n^+)^{2^*}\right)^{2/2^*}, \tag{3.20}$$

where the latter integral is > 0 for n sufficiently large (by (3.18), (3.19) and $c > 0$). It follows from (3.18)–(3.20) that

$$cN \ge \frac{S}{\|b\|_\infty^{2/2^*}}(cN)^{2/2^*},$$

i.e., $c \ge c_0$, as $c > 0$. This contradicts (3.15) and completes the proof of Lemma 3.1. □

Proof of Lemma 3.2 when $N \geq 4$. We start as in [1, p. 537] observing that for some positive constant α,

$$g_\lambda(x,s) \geq b(x)[(s^+)^p + \alpha(u_0(x))^{p-1}s^+]$$

a.e. on B_2. Note that the assumption that h_λ is nondecreasing is used here; note also that B_2 was introduced just before the statement of Lemma 3.2. Consequently,

$$J_\lambda(t\psi_\mu) \leq \frac{t^2}{2}\|\psi_\mu\|^2 - \frac{t^{p+1}}{p+1}\int_\Omega b(x)\psi_\mu^{p+1} - \frac{t^2}{2}\alpha'\|\psi_\mu\|_2^2$$

for some other positive constant α'. Computing the maximum of the right-hand side for $t > 0$ yields

$$\sup_{t>0} J_\lambda(t\psi_\mu) \leq \frac{1}{N}[(\|\psi_\mu\|^2 - \alpha'\|\psi_\mu\|_2^2)^+]^{N/2}/\left[\int_\Omega b(x)\psi_\mu^{2^*}\right]^{(N-2)/2}. \tag{3.21}$$

We will use the following estimates from [5] (see also [12, 14]) for $\mu \to 0$:

$$\begin{aligned} &\|\psi_\mu\|^2 = S^{N/2} + O(\mu^{N-2}) && \text{when } N \geq 3,\\ &\|\psi_\mu\|_{2^*}^{2^*} = S^{N/2} + O(\mu^N) && \text{when } N \geq 3,\\ &\|\psi_\mu\|_2^2 = \begin{cases} k_1\mu^2 + O(\mu^{N-2}) & \text{when } N \geq 5,\\ k_2\mu^2|\log\mu^2| + O(\mu^2) & \text{when } N = 4, \end{cases} \end{aligned} \tag{3.22}$$

where k_1, k_2 are positive constants. To estimate the denominator in the right-hand side of (3.21), we call $b_0 := \|b\|_\infty$, introduce a ball $B_{\mu^\delta} = B(x_0, \mu^\delta)$ with $0 < \delta < 1$ to be determined later and write

$$\int_\Omega b(x)\psi_\mu^{2^*} = \int_{B_{\mu^\delta}}(b(x)-b_0)\psi_\mu^{2^*} + \int_{\Omega\setminus B_{\mu^\delta}}(b(x)-b_0)\psi_\mu^{2^*} + b_0\|\psi_\mu\|_{2^*}^{2^*}.$$

Using assumption (b) and (3.22), one has

$$\left|\int_{B_{\mu^\delta}}(b(x)-b_0)\psi_\mu^{2^*}\right| \leq \tilde{M}\mu^{\gamma\delta}[S^{N/2} + O(\mu^N)]$$

for some constant $\tilde{M}$. On the other hand, for some constant C,

$$\left|\int_{\Omega\setminus B_{\mu^\delta}}(b(x)-b_0)\psi_\mu^{2^*}\right| \leq C\int_{\Omega\setminus B_{\mu^\delta}}\psi_\mu^{2^*} = O(\mu^{N(1-\delta)}),$$

where the latter equality can be verified by using a Taylor expansion in

$$\int_{\mu^\delta}^\infty [\mu/(\mu^2+r^2)^N]r^{N-1}\,dr.$$

Let us first consider the case $N \geq 5$. Using the above estimates in (3.21), one gets, for μ sufficiently small,

$$\sup_{t>0} J_\lambda(t\psi_\mu) \leq \frac{S^{N/2}}{Nb_0^{(N-2)/2}} \frac{[1-\alpha''\mu^2 + O(\mu^{N-2})]^{N/2}}{[1+O(\mu^{\gamma\delta}) + O(\mu^{N(1-\delta)})]^{(N-2)/2}} \tag{3.23}$$

with another positive constant α''. Since $\gamma > 2^*$, one can find δ such that $\gamma\delta > 2$ and $N(1-\delta) > 2$. It follows that the quotient $[\cdots]^{N/2}/[\cdots]^{(N-2)/2}$ in (3.23) is < 1 for μ sufficiently small. This proves the lemma when $N \geq 5$.

When $N = 4$ the bracket $[\cdots]^{N/2}$ in (3.23) now reads

$$[1-\alpha''\mu^2|\log\mu^2| + O(\mu^2)]^{N/2},$$

and the same argument as above, using $\gamma \geq 2^*$, yields the conclusion. □

Proof of Lemma 3.2 when $N = 3$. We again start as in [1, p. 537] to reach here

$$J_\lambda(t\psi_\mu) \leq \frac{t^2}{2}\|\psi_\mu\|^2 - \frac{t^6}{6}\int_\Omega b(x)\psi_\mu^6 - \frac{t^5}{5}\alpha\|\psi_\mu\|_5^5 \tag{3.24}$$

for some positive constant α. The maximum of the right-hand side for $t > 0$ is achieved for $t_0 = t_0(\mu)$ satisfying

$$\|\psi_\mu\|^2 = \left(\int_\Omega b(x)\psi_\mu^6\right)t_0^4 + \alpha\|\psi_\mu\|_5^5\, t_0^3. \tag{3.25}$$

In addition to (3.22) we will use

$$\|\psi_\mu\|_5^5 = k\mu^{1/2} + O(\mu^{5/2}) \tag{3.26}$$

with k a positive constant (cf. [1]). We will also use

$$\int_\Omega b(x)\psi_\mu^6 = b_0 S^{3/2} + O(\mu^{\gamma\delta}) + O(\mu^{3(1-\delta)}), \tag{3.27}$$

which is obtained as in the proof for $N \geq 4$.

Using (3.26), (3.27) together with (3.22), one deduces from (3.25) that

$$t_0(\mu) = \frac{1}{b_0^{1/4}} - \frac{k}{4b_0S^{3/2}}\mu^{1/2} + o(\mu^{1/2})$$

provided δ is chosen so that $\gamma\delta > 1/2$ and $3(1-\delta) > 1/2$, which is possible since $\gamma > 3/5$. It then follows from (3.24) that

$$\sup_{t>0} J_\lambda(t\psi_\mu) \leq \frac{S^{3/2}}{3b_0^{1/2}} - \frac{k}{5b_0^{5/4}}\mu^{1/2} + o(\mu^{1/2}) < \frac{S^{3/2}}{3b_0^{1/2}}$$

for μ sufficiently small. This is the conclusion of Lemma 3.2 when $N = 3$. □

Proof of Theorem 2.6. The only difference with the proof of Theorem 2.5 occurs at the beginning of the proof of Lemma 3.1, at the point where one shows that any $(PS)_c$ sequence is bounded.

The argument to prove that any sequence w_n satisfying (3.16) and (3.17) is bounded here goes as follows. First observe that in our situation, $H_\lambda(x,s) \geq 0$ and so θ in the condition $(AR)_d$ for h_λ can always be chosen such that $2 < \theta < p+1$. We will estimate

$$\Phi(w_n) := J_\lambda(w_n) - \frac{1}{\theta} J'_\lambda(w_n)(u_0 + w_n).$$

By (3.16) and (3.17), one has, for some constant C,

$$\Phi(w_n) \leq C + \frac{\varepsilon_n}{\theta}\|u_0 + w_n\|. \tag{3.28}$$

On the other hand, expanding $\Phi(w_n)$, one obtains

$$\begin{aligned}\Phi(w_n) = \left(\frac{1}{2} - \frac{1}{\theta}\right)\|w_n\|^2 - \int_\Omega \left[H_\lambda(x, u_0 + w_n^+) - \frac{1}{\theta} h_\lambda(x, u_0 + w_n^+)(u_0 + w_n^+)\right] \\ - \left(\frac{1}{p+1} - \frac{1}{\theta}\right)\int_\Omega b(x)(u_0 + w_n^+)^{p+1} + A_n,\end{aligned} \tag{3.29}$$

where A_n is a first order term, i.e. satisfies $\|A_n\| \leq c_1 + c_2\|w_n\|$ for some constants c_1, c_2. Combining (3.28) and (3.29) gives

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{\theta}\right)\|w_n\|^2 = \int_\Omega \left[H_\lambda(x, u_0 + w_n^+) - \frac{1}{\theta} h_\lambda(x, u_0 + w_n^+)(u_0 + w_n^+)\right] \\ + \left(\frac{1}{p+1} - \frac{1}{\theta}\right)\int_\Omega b(x)(u_0 + w_n^+)^{p+1} + A'_n,\end{aligned}$$

for another first order term A'_n. Using $(AR)_d$, $2 < \theta < p+1$ and $b(x) \geq 0$, one easily concludes that w_n remains bounded. The proof of Theorem 2.6 is thus complete. □

4. Applications

In this section we will see how the previous theorems apply to problems (1.2) and (1.3). We start with (1.2), where $I_\lambda(u)$ now reads

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{q+1}\int_\Omega a(x)(u^+)^{q+1} - \frac{1}{p+1}\int_\Omega b(x)(u^+)^{p+1}.$$

Theorem 4.1. *Let $0 \leq q < 1 < p$ and assume that $a, b \in L^\infty(\Omega)$ with*

(i) *$a(x) \geq 0$ a.e. x in Ω,*
(ii) *$a(x) \geq \varepsilon_1 > 0$ a.e. on some ball B_1.*

Then there exists $0 < \Lambda \leq \infty$ such that problem (1.2) *has at least one solution u (with $I_\lambda(u) < 0$) for $0 < \lambda < \Lambda$ and no solution for $\lambda > \Lambda$. If in addition*

(iii) $b(x) \geq 0$ *a.e. on some ball* B_2*, with* $a(x)b(x) \not\equiv 0$ *on* B_2*,*

then $\Lambda < \infty$*. Moreover, if in addition* $p \leq 2^* - 1$*, then problem* (1.2) *has at least one solution* u *(with* $I_\lambda(u) \leq 0$*) for* $\lambda = \Lambda$.

Note that Λ can be ∞ in the first part of Theorem 4.1. This happens for instance if $b(x) \equiv -1$ (cf. the observation following Theorem 2.1).

Proof of Theorem 4.1. It suffices to verify the hypotheses of Theorems 2.1, 2.2 and 2.3. (H) and (H_0) are obvious, by (i). In (H_e) one takes $g(s) = \lambda \|a\|_\infty s^q + \|b\|_\infty s^p$ with λ sufficiently small. (H_{Ω_1}) follows from (ii). At this stage Theorem 2.1 yields the first part of Theorem 4.1. On the other hand, $(H_{\tilde{\Omega}})$ follows from (iii) by applying Lemma 3.6 from [9]. Theorem 2.2 thus yields the second part of Theorem 4.1. Finally, (G) is obvious when $p \leq 2^* - 1$, and $(AR)_d$ follows as in [9, p. 457] by taking $\theta = p + 1$, $\rho = q + 1$, $d = R(\theta/(p+1) - 1)\|a\|_\infty$ and $s_0 = 0$. (Recall that $\lambda \in [r, R]$ in $(AR)_d$.) The last part of Theorem 4.1 thus follows from Theorem 2.3. □

Theorem 4.2. *Let* $0 \leq q < 1 < p$ *and assume that* $a, b \in L^\infty(\Omega)$ *with* (i) *and* (ii) *above. Assume in addition either* $p < 2^* - 1$ *and*

(iv) $b(x) \geq \varepsilon_2 > 0$ *a.e. on some ball* B_2*,*

or $p = 2^* - 1$ *and condition* (b) *of Theorem* 2.5 *for* $b(x)$*. Then problem* (1.2) *has at least two solutions* u, v *for* $0 < \lambda < \Lambda$*, with* $u < v$ *in* Ω*,* $\partial u/\partial \nu > \partial v/\partial \nu$ *on* $\partial\Omega$ *and* $I_\lambda(u) < 0$*.*

Note that (b) is a stronger condition than (iv). Note also that $b(x)$ above is allowed to change sign in Ω.

Proof of Theorem 4.2. It suffices to verify the hypotheses of Theorems 2.4 and 2.5. As observed in the proof of Theorem 4.1, the hypotheses of Theorem 2.1 follow from (i) and (ii), and $(AR)_d$ can be verified as in [9, p. 457]. Moreover, (H_0') and (M) are obvious. Theorem 2.4 thus applies when $p < 2^* - 1$. In the critical case $p = 2^* - 1$, Theorem 2.5 clearly applies. □

We now turn to problem (1.3). The functional $I_\lambda(u)$ here reads

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{p+1}\int_\Omega c(x)(u^+ + 1)^{p+1}.$$

Theorem 4.3. *Let* $p > 1$ *and assume that* $c \in L^\infty(\Omega)$ *with*

$$c(x) \geq 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad c(x) \geq \varepsilon > 0 \quad \text{a.e. on some ball } B. \tag{4.1}$$

Then there exists $0 < \Lambda < \infty$ *such that problem* (1.3) *has at least one solution* u *(with* $I_\lambda(u) < 0$*) for* $0 < \lambda < \Lambda$ *and no solution for* $\lambda > \Lambda$*. Moreover, if* $p \leq 2^* - 1$*, then problem* (1.3) *has at least one solution* u *(with* $I_\lambda(u) \leq 0$*) for* $\lambda = \Lambda$*.*

Proof. Theorems 2.1–2.3 easily apply to yield the desired conclusions. In the verification of (H_e) one can take $g(s) = \lambda\|c\|_\infty(s+1)^p$. In the verification of $(AR)_d$ one has

$$\theta F_\lambda(x,s) - sf_\lambda(x,s) \le \lambda c(x)(s+1)^p\left[\left(\frac{\theta}{p+1}-1\right)(s+1)+1\right] \tag{4.2}$$

and so, if we choose θ with $2 < \theta < p+1$, the right-hand side of (4.2) is ≤ 0 for s sufficiently large, which yields $(AR)_d$ with $d = 0$.

Theorem 4.4. *Let $p > 1$ and assume that $a \in L^\infty(\Omega)$ with* (4.1)*. Assume in addition either $p < 2^* - 1$, or $p = 2^* - 1$ and condition (b) of Theorem* 2.5 *holds for $a(x)$. Then problem* (1.3) *has at least two solutions u, v for $0 < \lambda < \Lambda$, with $u < v$ in Ω, $\partial u/\partial\nu > \partial v/\partial\nu$ on $\partial\Omega$ and $I_\lambda(u) < 0$.*

Proof. The subcritical case $p < 2^* - 1$ follows immediately from Theorem 2.4. The critical case $p = 2^* - 1$ requires more care because the right-hand side of (1.3) is not written in the form (2.1) . However, u solves (1.3) for λ if and only if $v = \lambda^{1/(p-1)}u$ solves

$$\begin{cases} -\Delta v = c(x)(v+\mu)^p & \text{in } \Omega,\\ v > 0 & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

for $\mu = \lambda^{1/(p-1)}$. It follows in particular that (4.3) has at least one solution for $\mu < \Lambda^{1/(p-1)}$ and no solution for $\mu > \Lambda^{1/(p-1)}$. We aim at applying Theorem 2.6 to (4.3). For this purpose, we write

$$c(x)(s+\mu)^p = h_\mu(x,s) + c(x)s^p,$$

where $h_\mu(x,s) = c(x)[(s+\mu)^p - s^p]$. A simple application of the mean value theorem shows that $h_\mu(x,s)$ satisfies (G) with $\sigma = p-1$, and a calculation similar to (4.2) shows that it satisfies $(AR)_d$ with $d = 0$. The other hypotheses of Theorem 2.6 are easily verified, in the same way as they were verified earlier for (1.3). It follows that (4.3) admits a second solution for $\mu < \Lambda^{1/(p-1)}$, with negative energy. Finally, one observes that the energy of the corresponding solution of (1.3) is also negative.

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