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Sharp estimates for the Ambrosetti–Hess problem and consequences

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Abstract. Motivated by [3], we define the "Ambrosetti–Hess problem" to be the problem of bifurcation from infinity and of the local behavior of continua of solutions of nonlinear elliptic eigenvalue problems. Although the works in this direction underline the asymptotic properties of the nonlinearity, here we point out that this local behavior is determined by the global shape of the nonlinearity.

1. Introduction

Consider the boundary value problem

$$\begin{cases} -u''(t) = \lambda u(t) + g(u(t)), \quad t \in (0,\pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ satisfies

(H1) g is a continuous function, with $\lim_{|s|\to+\infty} g(s)/s = 0$.

For this problem, bifurcation techniques can be used, and the Global Bifurcation Theorem [7] ensures that, under condition (H1), every eigenvalue of the linearized problem with odd multiplicity, σ_k, is a bifurcation point from infinity, i.e. there exists a sequence (λ_n, u_n) in $\mathbb{R} \times C^1[0,\pi]$ of solutions of (1) such that $\lambda_n \to \sigma_k$ and $\|u_n\|_{L^2} \to +\infty$. Furthermore, this sequence has a subsequence (λ_{n_j}, u_{n_j}) such that

$$\frac{u_{n_j}}{\|u_{n_j}\|_{L^2}} \to \phi_k \quad (C^1\text{-convergence})$$

where ϕ_k is an eigenfunction associated to σ_k with $\|\phi_k\|_{L^2} = 1$.

In the particular case of bifurcation from infinity at the principal eigenvalue σ_1, both ϕ_1 and $-\phi_1$ have sequences as above. Since ϕ_1 lies in the interior of the C^1-cone of

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positive solutions, we refer to such bifurcations as "bifurcation from $(\sigma_1, +\infty)$" and "bifurcation from $(\sigma_1, -\infty)$" respectively. One can also deduce from the above convergence that, near the bifurcation point, the solutions have constant sign.

We are interested in the local behavior of bifurcations from infinity at the first eigenvalue, σ_1, particularly if the parameters λ_n of the sequence bifurcating from infinity lie either to the left or to the right of the first eigenvalue σ_1 for large values of n.

This behavior has been widely studied by several authors. In [3] the authors consider a problem similar to (1) and they prove that if

$$\liminf_{s\to+\infty} g(s) > 0 \quad (\text{resp. } \limsup_{s\to+\infty} g(s) < 0)$$

then the component of positive solutions bifurcating from $(\sigma_1, +\infty)$ goes to the left (resp. right) of σ_1. So, the study of local behavior is done provided that the nonlinearity g keeps away from zero at infinity.

Later, in [2] and [1], g is allowed to approach zero at infinity. Specifically, in those works the authors prove that the behavior of the bifurcation from infinity is decided by the sign of $\lim_{s\to+\infty} g(s)s$. If this limit is positive (resp. negative) the bifurcation starts to the left (resp. to the right).

One more step is done in [4], where the authors extend previous results by comparing $g(s)$ to $s^{-\alpha}$. Explicitly, they prove that the sign of $\lim_{s\to+\infty} g(s)s^\alpha$ determines the behavior of the bifurcation from infinity only if $\alpha \le 2$. They also prove that the result is not true for $\alpha > 2$.

In the case of the Neumann boundary condition we proved in [6] that the sign of $g(s)$ for "s close to ∞" decides the local behavior of the bifurcation from infinity, in contrast to the Dirichlet case. Example 2.3 in [6] emphasizes that, using the same nonlinearity g, the behavior of the bifurcation can depend on the boundary condition.

In this work we fix our attention on the Dirichlet case, and we find that the key to decide how the bifurcation behaves is an integral condition on g. In a first approach, under suitable conditions (see (H2) below) we calculate explicitly

$$\lim_{n\to\infty} (\sigma_1 - \lambda_n)\|u_n\|^3_{L^2} = \frac{2}{\phi_1'(0)} \int_0^{+\infty} g(s)s\, ds \tag{2}$$

for any sequence (λ_n, u_n) of solutions of (1) bifurcating from $(\sigma_1, +\infty)$. Later, by using comparison arguments, we weaken hypothesis (H2), obtaining a natural extension of all the previously known asymptotic conditions on the nonlinearity (see hypothesis (H3) and Theorem 2 in Section 2).

Finally, in Section 3, by using the same approach as in [4], we give an application to strongly resonant problems.

In what follows, for the sake of simplicity we write $\|\cdot\|$ for $\|\cdot\|_{L^2}$.

2. The main result

Let (λ_n, u_n) be a sequence of solutions of (1) bifurcating from $(\sigma_1, +\infty)$. Multiplying the equation in (1) by ϕ_1, integrating by parts, and using the symmetry of positive solutions we obtain the following identity:

$$(\sigma_1 - \lambda_n) \int_0^{\pi} u_n(t)\phi_1(t)\,dt = \int_0^{\pi} g(u_n(t))\phi_1(t)\,dt = 2\int_0^{\pi/2} g(u_n)\phi_1. \tag{3}$$

Since $u_n/\|u_n\| \to \phi_1$, one has $\int_0^{\pi} (u_n/\|u_n\|)\phi_1 \to \int_0^{\pi} \phi_1^2 = 1$ and so

$$\lim_{n\to\infty} (\sigma_1 - \lambda_n)\|u_n\|^3 = 2 \lim_{n\to\infty} \|u_n\|^2 \int_0^{\pi/2} g(u_n)\phi_1. \tag{4}$$

So, to calculate the limit (2) we are led to calculate

$$\lim_{n\to\infty} \|u_n\|^2 \int_0^{\pi/2} g(u_n)\phi_1.$$

For a first version, we consider g approaching zero at infinity very quickly. This condition turns out to be so restrictive that no previous asymptotic result can be applied (see (H2) below).

In the next lemma we prepare the arguments to show that this limit concentrates on a neighborhood of the boundary, where the change of variables $s = u_n(t)$ is almost linear.

Lemma 1. *Assume that the function g satisfies hypothesis* (H1) *and*

(H2) $\lim_{|s|\to+\infty} g(s)s^2 = 0$ *and* $g(s)s \in L^1([0,\infty))$.

Consider a sequence u_n in $C^1[0,\pi]$ such that $\|u_n\| \to \infty$ and $u_n/\|u_n\| \to \phi_1$ (C^1-convergence). Fix $\varepsilon > 0$. Then there exists $t_0 \in (0,\pi/2)$ such that:

(i) *for n sufficiently large,*

$$\left| \frac{\|u_n\|}{u_n'(t)} - \frac{1}{\phi_1'(0)} \right| < \varepsilon, \qquad \forall t \in (0,t_0).$$

(ii) $\lim_{n\to\infty} \|u_n\|\phi_1(t)/u_n(t) = 1$ *uniformly in $t \in (0,t_0)$. In particular, for n large,*

$$\left| \frac{\|u_n\|\phi_1(t)}{u_n(t)} - 1 \right| < \varepsilon, \qquad \forall t \in (0,t_0).$$

(iii) $\lim_{n\to\infty} \|u_n\|^2 g(u_n(t)) = 0$ *uniformly in $t \in [t_0,\pi/2]$. In particular, for n large,*

$$\left| \|u_n\|^2 \int_{t_0}^{\pi/2} g(u_n)\phi_1 \right| < \varepsilon.$$

Proof. (i) Use the uniform convergence $u_n'/\|u_n\| \to \phi_1'$ in $[0,\pi]$, and the continuity of ϕ_1' at zero, with $\phi_1'(0) \neq 0$. We remark that it is here that t_0 is determined. The following items work for any t_0 small.

(ii) This also follows from the uniform convergence $u_n'/\|u_n\| \to \phi_1'$ in $[0,\pi]$. In fact, by the Cauchy mean value theorem, for all $n \geq 1$ and $t \in (0,t_0)$, there exists $c_{n,t} \in (0,t)$ such that

$$\frac{\|u_n\|\phi_1(t)}{u_n(t)} = \frac{\|u_n\|\phi_1'(c_{n,t})}{u_n'(c_{n,t})},$$

which converges to 1 uniformly in $(0,t_0)$.

(iii) Since ϕ_1 is strictly positive in $[t_0,\pi/2]$, there are $\exists\delta, M > 0$ such that for n large enough,

$$0 < \delta \leq \phi_1(t) \leq M, \qquad 0 < \delta \leq \frac{u_n(t)}{\|u_n\|} \leq M, \qquad \forall t \in [t_0,\pi/2].$$

Consequently, $\delta\|u_n\| \leq u_n(t)$ and $u_n(t) \to \infty$ uniformly in $t \in [t_0,\pi/2]$. In particular, by hypothesis (H2),

$$0 \leq |\|u_n\|^2 g(u_n)| \leq \left|\frac{1}{\delta^2}\, g(u_n)u_n^2\right| \to 0 \quad \text{uniformly in } t \in [t_0,\pi/2].$$

The proof of the lemma is finished.

It is important to remark here that in the lemma we have only used the fact that $\|u_n\| \to \infty$ and $u_n/\|u_n\| \to \phi_1$ (C^1-convergence). The lemma and the next theorem hold even if u_n are not solutions of (1).

Theorem 1. *Under hypotheses* (H1)–(H2), *take a sequence u_n in $C^1[0,\pi]$ such that $\|u_n\| \to \infty$ and $u_n/\|u_n\| \to \phi_1$ (C^1-convergence). Then*

$$\lim_{n\to\infty} \|u_n\|^2 \int_0^{\pi/2} g(u_n)\phi_1 = \frac{1}{\phi_1'(0)} \int_0^{+\infty} g(s)s\,ds.$$

Proof. Write $I = \int_0^{+\infty} g(s)s\,ds$. Then

$$\begin{aligned}
&\left|\|u_n\|^2 \int_0^{\pi/2} g(u_n)\phi_1 - \frac{1}{\phi_1'(0)} I\right| \\
&\quad \leq \left|\|u_n\|^2 \int_0^{t_0} g(u_n)\phi_1 - \frac{1}{\phi_1'(0)} I\right| + \left|\|u_n\|^2 \int_{t_0}^{\pi/2} g(u_n)\phi_1\right| \\
&\quad \leq \left|\|u_n\| \int_0^{t_0} g(u_n)u_n - \frac{1}{\phi_1'(0)} I\right| + \left|\|u_n\| \int_0^{t_0} g(u_n)u_n\left(\frac{\|u_n\|\phi_1}{u_n} - 1\right)\right| \\
&\qquad + \left|\|u_n\|^2 \int_{t_0}^{\pi/2} g(u_n)\phi_1\right| \\
&\quad = I_1 + I_2 + I_3,
\end{aligned}$$

where, using the change of variables $s = u_n(t)$, and (i) of the previous lemma,

$$\begin{aligned} I_1 &= \left| \|u_n\| \int_0^{t_0} g(u_n)u_n - \frac{1}{\phi_1'(0)} I \right| \\ &= \left| \int_0^{u_n(t_0)} g(s)s \frac{\|u_n\|}{u_n'(u_n^{-1}(s))}\, ds - \frac{1}{\phi_1'(0)} \int_0^{\infty} g(s)s\, ds \right| \\ &\le \int_0^{u_n(t_0)} |g(s)s| \left| \frac{\|u_n\|}{u_n'(u_n^{-1}(s))} - \frac{1}{\phi_1'(0)} \right| ds + \left| \frac{1}{\phi_1'(0)} \int_{u_n(t_0)}^{\infty} g(s)s\, ds \right| \\ &\le \varepsilon \|g(s)s\|_{L^1([0,\infty))} + \frac{1}{\phi_1'(0)} \int_{u_n(t_0)}^{\infty} |g(s)s|\, ds \\ &\le \varepsilon (\|g(s)s\|_{L^1([0,\infty))} + 1) \quad \text{(for } n \text{ large).} \end{aligned}$$

By (ii) and (i) of the previous lemma and again with $s = u_n(t)$,

$$\begin{aligned} I_2 &= \left| \|u_n\| \int_0^{t_0} g(u_n)u_n \left(\frac{\|u_n\|\phi_1}{u_n} - 1 \right) \right| \le \varepsilon \|u_n\| \int_0^{t_0} |g(u_n(t))u_n(t)|\, dt \\ &= \varepsilon \int_0^{u_n(t_0)} |g(s)s| \frac{\|u_n\|}{u_n'(u_n^{-1}(s))}\, ds \le \varepsilon \left(\frac{1}{\phi_1'(0)} + \varepsilon \right) \|g(s)s\|_{L^1([0,\infty))}. \end{aligned}$$

Finally, by (iii) of the previous lemma,

$$I_3 = \left| \|u_n\|^2 \int_{t_0}^{\pi/2} g(u_n)\phi_1 \right| \le \varepsilon.$$

So,

$$\begin{aligned} \left| \|u_n\|^2 \int_0^{\pi/2} g(u_n)\phi_1 - \frac{1}{\phi_1'(0)} I \right| &\le I_1 + I_2 + I_3 \\ &\le \varepsilon \left[\left(1 + \frac{1}{\phi_1'(0)} + \varepsilon \right) \|g(s)s\|_{L^1([0,\infty))} + 2 \right]. \end{aligned}$$

Since ε is chosen arbitrarily, the proof is complete.

Observe that using formula (4), one infers the following

Corollary 1. *Assume hypotheses* (H1)–(H2) *and let* (λ_n, u_n) *be a sequence of solutions of* (1) *bifurcating from* $(\sigma_1, +\infty)$. *Then:*

(i)

$$\lim_{n\to\infty} \|u_n\|^3(\sigma_1 - \lambda_n) = \frac{2}{\phi_1'(0)} \int_0^{+\infty} g(s)s\, ds.$$

(ii) *If* $\int_0^{+\infty} g(s)s\, ds > 0$, *then bifurcation from* $(\sigma_1, +\infty)$ *is to the left of* σ_1.

(iii) *If* $\int_0^{+\infty} g(s)s\, ds < 0$, *then bifurcation from* $(\sigma_1, +\infty)$ *is to the right of* σ_1.

We now weaken hypothesis (H2) to obtain a natural extension of the previously known results in [3]–[4].

Theorem 2. *Assume that the function g satisfies* (H1) *and*

$$\text{(H3)}\quad \begin{cases} \exists \widetilde{g} : [0,+\infty) \to \mathbb{R},\ \widetilde{g} \text{ satisfying (H1)–(H2), with} \\ \text{either (H3+)} \begin{cases} g(s) \ge \widetilde{g}(s),\ \forall s \in [0,+\infty),\ \text{and} \\ \int_0^{+\infty} \widetilde{g}(s)s\,ds > 0, \end{cases} \\ \text{or (H3–)} \begin{cases} g(s) \le \widetilde{g}(s),\ \forall s \in [0,+\infty),\ \text{and} \\ \int_0^{+\infty} \widetilde{g}(s)s\,ds < 0. \end{cases} \end{cases}$$

Then bifurcation of solutions of problem (1) *from* $(\sigma_1, +\infty)$ *is to the left provided* (H3+) *holds and to the right provided* (H3–) *holds.*

Proof. We will use comparison arguments. Take a sequence (λ_n, u_n) of solutions of problem (1) bifurcating from $(\sigma_1, +\infty)$. By using the identity (3), it is clear that for large values of n,

$$\operatorname{sign}(\sigma_1 - \lambda_n) = \operatorname{sign}\left(\int_0^{\pi/2} g(u_n(t))\phi_1(t)\,dt\right).$$

Now, we apply Theorem 1 to the function $\widetilde{g}$ to infer that, for large values of n,

$$\operatorname{sign}\left(\int_0^{\pi/2} \widetilde{g}(u_n)\phi_1\right) = \operatorname{sign}\left(\int_0^{+\infty} \widetilde{g}(s)s\,ds\right).$$

Now, if (H3+) holds, then

$$\int_0^{\pi/2} g(u_n)\phi_1 \ge \int_0^{\pi/2} \widetilde{g}(u_n)\phi_1 > 0, \quad \text{so} \quad \sigma_1 - \lambda_n > 0 \ \text{(bifurcation to the left).}$$

On the other hand, if (H3–) holds, then

$$\int_0^{\pi/2} g(u_n)\phi_1 \le \int_0^{\pi/2} \widetilde{g}(u_n)\phi_1 < 0, \quad \text{so} \quad \sigma_1 - \lambda_n < 0 \ \text{(bifurcation to the right).}$$

Thus the proof is complete.

Remarks. 1. Observe that even if $g(s)s \notin L^1([0,+\infty))$, hypothesis (H3) gives a meaning to the expression $0 < \int_0^{+\infty} g(s)s\,ds \le \infty$ (if (H3+) holds), or $0 > \int_0^{+\infty} g(s)s\,ds \ge -\infty$ (if (H3–) holds). This fact makes Corollary 1 (items (ii) and (iii)) valid also for this case.

2. The asymptotic conditions given in all previously known results [1]–[4] allow an application of Theorem 2 for a suitable $\widetilde{g}$. In fact, in all those cases, $\int_0^{+\infty} g(s)s\,ds = +\infty$ (or $-\infty$).

3. The above results can be easily adapted for any sequence (λ_n, u_n) of solutions of (1), bifurcating from $(\sigma_1, -\infty)$. In fact, taking $v_n = -u_n$ and $h(s) = -g(-s)$, it is easy to check that (λ_n, u_n) is a solution of (1) if and only if (λ_n, v_n) is a solution of

$$\begin{cases} -v''(t) = \lambda v(t) + h(v(t)), & t \in (0, \pi), \\ v(0) = v(\pi) = 0. \end{cases}$$

So, bifurcation from $(\sigma_1, -\infty)$ for problem (1) can be considered as bifurcation from $(\sigma_1, +\infty)$ for the above problem. Consequently, the side of the bifurcation will depend on the sign of

$$\int_0^{+\infty} h(s)s\,ds = \int_{-\infty}^{0} g(s)s\,ds.$$

Essentially, under suitable conditions which are left to the reader, the result is:

(i) If $\int_{-\infty}^{0} g(s)s\,ds > 0$, then bifurcation from $(\sigma_1, -\infty)$ is to the left of σ_1.

(ii) If $\int_{-\infty}^{0} g(s)s\,ds < 0$, then bifurcation from $(\sigma_1, -\infty)$ is to the right of σ_1.

3. Application to a strongly resonant problem

Taking into account this last remark, we use the ideas in [4] for resonant problems in the first eigenvalue. The idea is to observe both bifurcations (from $+\infty$ and from $-\infty$) at σ_1 for problem (1). If both bifurcations start "to the same side", then the resonant problem

$$\begin{cases} -u''(t) = \sigma_1 u(t) + g(u(t)), & t \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{5}$$

has a solution (see [4] for details).

Theorem 3. *Assume that $g : \mathbb{R} \to \mathbb{R}$ satisfies* (H1)–(H2). *If*

$$\operatorname{sign}\left(\int_{-\infty}^{0} g(s)s\,ds\right) = \operatorname{sign}\left(\int_0^{+\infty} g(s)s\,ds\right),$$

then the resonant problem (5) *has a solution.*

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