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On the necessity of gaps

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Abstract. Recent papers have studied the existence of phase transition solutions for Allen–Cahn type equations. These solutions are either single or multi-transition spatial heteroclinics or homoclinics between simpler equilibrium states. A sufficient condition for the construction of the multi-transition solutions is that there are gaps in the ordered set of single transition solutions. In this paper we explore the necessity of these gap conditions.

1. Introduction

Recent papers have established the existence of phase transition states for model equations of Allen–Cahn type [1]–[2], [6]–[7]. Mathematically these states are single or multi-transition spatially homoclinic or heteroclinic solutions of the equation

$$-\Delta u + G_u(x, y, u) = 0, \quad (x, y) \in \mathbb{R}^2. \tag{PDE}$$

The function G satisfies

$$G \in C^2(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}) \text{ and is 1-periodic in } x \text{ and } y; \tag{G$_1$}$$

$$\begin{aligned} &G(x, y, 0) = 0 = G(x, y, 1), \\ &G(x, y, z) > 0 \quad \text{for } (x, y) \in \mathbb{R}^2 \text{ and } z \in (0, 1); \end{aligned} \tag{G$_2$}$$

$$G(x, y, z) \geq 0 \text{ for all } x, y, z. \tag{G$_3$}$$

Hypothesis (G$_2$) implies (PDE) possesses constant solutions $u \equiv 0$ and $u \equiv 1$.

We will only consider solutions of (PDE) with $0 \leq u \leq 1$. Indeed, the solutions $u \equiv 0$ and $u \equiv 1$ of (PDE) behave like geodesics for the minimization arguments of [6]–[7] and these arguments yield a variety of solutions of (PDE) with $0 \leq u \leq 1$.

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The single transition states of (PDE) are solutions which are heteroclinic from 0 to 1 or from 1 to 0 in one of the variables x and y, and are 1-periodic in the other variable. Multi-transition solutions undergo multiple transitions between the constant states. Generally they shadow the single transition solutions, i.e. they are near them on large regions. Each of the sets of one transition solutions is ordered.

The existence proofs for the multi-transition solutions are carried out assuming there are gaps in the ordered sets of simpler 1-transition solutions. A goal of this paper is to show that these gap conditions for the simpler states are necessary as well as sufficient for the existence of the more complex states.

In §2, some results from [6]–[7] will be recalled briefly and a theorem showing the necessity of these gaps for the existence of multi-transition solutions will be proved.

Another result obtained in [1]–[2] and [6]–[7] is the existence of solutions of (PDE) that e.g. as functions of y are heteroclinic between a distinct pair of the basic 1-transition heteroclinics in x mentioned above. A gap condition is again required to obtain these solutions. In §3, it will be shown that this gap condition is also necessary for such doubly heteroclinic states to exist.

See also Bangert [3] for some related results.

2. The first necessity result

Consider (PDE) under the hypotheses (G_1)–(G_3). The functions $u = 0$ and $u = 1$ are solutions of (PDE) called *pure states*. Other solutions called *mixed states* which are heteroclinic from 0 to 1 in x and 1-periodic in y are obtained by minimizing the functional associated with (PDE). More precisely, let

$$L(u) = \frac{1}{2}|\nabla u|^2 + G(x, y, u)$$

and set

$$I(u) = \int_{-\infty}^{\infty} \int_0^1 L(u)\, dx\, dy.$$

For $i \in \mathbb{Z}$, let $S_i = [i, i+1] \times T^1$ where T^1 is the 1-torus. Thus I is defined on the class of functions

$$\Gamma(0,1) = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times T^1, \mathbb{R}) \mid 0 \le u \le 1;\ \|u\|_{L^2(S_i)} \to 0,\ i \to -\infty;\\ \|1-u\|_{L^2(S_i)} \to 0,\ i \to \infty\}.$$

The elements of $\Gamma(0,1)$ are 1-periodic in y and satisfy the desired asymptotic conditions in x in a weak form.

Define

$$c(0,1) = \inf_{u \in \Gamma(0,1)} I(u). \tag{2.1}$$

Then it was shown in [6]–[7] that I has minimizers in $\Gamma(0,1)$ which are solutions of (PDE).

Theorem 2.2. *Let G satisfy* (G_1)–(G_2). *Then*

1° $\mathcal{M}(0,1) \equiv \{u \in \Gamma(0,1) \mid I(u) = c(0,1)\} \neq \emptyset$.
2° *If $u \in \mathcal{M}(0,1)$, then u is a classical solution of* (PDE).
3° $\|u\|_{C^2(S_i)} \to 0$ *as* $i \to -\infty$ *and* $\|1-u\|_{C^2(S_i)} \to 0$ *as* $i \to \infty$.
4° $u(x,y) < u(x+1,y) \equiv \tau_{-1}u(x,y)$ *for all* $(x,y) \in \mathbb{R} \times T^1$ *and* $\tau_{-1}u \in \mathcal{M}(0,1)$.
5° $\mathcal{M}(0,1)$ *is an ordered set, i.e.* $v, w \in \mathcal{M}(0,1)$ *implies* $v \equiv w$, $v < w$, *or* $v > w$.
6° $u \in \mathcal{M}(0,1)$ *has a minimality property: for any bounded open* $\mathcal{D} \subset \mathbb{R} \times T^1$, *if* $U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times T^1, \mathbb{R})$ *with* $U = u$ *in* $(\mathbb{R} \times T^1) \setminus \mathcal{D}$, *then*

$$\int_{\mathcal{D}} (L(u) - L(U))\, dx\, dy \leq 0. \tag{2.3}$$

Remark 2.4. (i) 4° will be referred to as the *monotonicity property* for u.
(ii) By 5°, either $\mathcal{M}(0,1)$ foliates $\mathbb{R} \times T^1 \times [0,1]$ or there are gaps in $\mathcal{M}(0,1)$, i.e. there are $v, w \in \mathcal{M}(0,1)$ with $v < w$ and no members of $\mathcal{M}(0,1)$ lie between v and w. The existence of such a gap pair is the *gap condition* referred to in §1.
(iii) An equivalent form of the minimality property is that for all $\varphi \in W^{1,2}(\mathbb{R} \times T^1, \mathbb{R})$ with compact support,

$$\int_{\mathbb{R} \times T^1} (L(u) - L(u+\varphi))\, dx\, dy \leq 0. \tag{2.5}$$

(iv) Any function $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times T^1, \mathbb{R})$ which satisfies (2.3) for some $\mathcal{D}$ is in fact a solution of (PDE) in $\mathcal{D}$. This follows from standard elliptic regularity arguments [4]. Thus the minimality property for u for all $\mathcal{D}$ as above implies u is a solution of (PDE) on $\mathbb{R} \times T^1$.
(v) Actually a stronger form of 6° was proved in [6]–[7]. Namely viewing the domain of u to be $\mathbb{R}^2$, for any bounded open $\mathcal{O} \subset \mathbb{R}^2$, if $U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ with $U = u$ in $\mathbb{R}^2 \setminus \mathcal{O}$, then

$$\int_{\mathcal{O}} (L(u) - L(U))\, dx\, dy \leq 0.$$

As a consequence of Theorem 2.2 and the companion result for $\mathcal{M}(1,0)$, the existence of multi-transition solutions of (PDE) can be obtained.

Theorem 2.6 ([6]–[7]). *Suppose G satisfies* (G_1)–(G_3) *and* $\mathcal{M}(0,1)$ *and* $\mathcal{M}(1,0)$ *contain gaps. Then for each* $k \in \mathbb{N}$, $k \geq 2$, *there exist infinitely many solutions of* (PDE) *in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times T^1, \mathbb{R})$ *which undergo k transitions and satisfy* $\|u\|_{C^2(S_i)} \to 0$, $i \to -\infty$.

Remark 2.7. (i) The same conclusion obtains for solutions which satisfy $\|1-u\|_{C^2(S_i)} \to 0$ as $i \to -\infty$.
(ii) For U as in Theorem 2.6, if k is even, $\|u\|_{C^2(S_i)} \to 0$ as $i \to \infty$ and the solutions are homoclinic to 0 in x, while if k is odd, $\|1-u\|_{C^2(S_i)} \to 0$ as $i \to \infty$ and the solutions are heteroclinic from 0 to 1 in x.

(iii) The simplest case of Theorem 2.6 is for $k = 2$ where the solutions are near 0 for $|x|$ large and are near 1 for an intermediate x interval (and all y). The infinitude of solutions is distinguished by the length of this intermediate x interval.

Theorem 2.6 is proved by minimizing I over a set of functions in $0 \le u \le 1$ satisfying several integral constraints. The precise nature of these constraints is not of importance here except the fact that the constraints involve a compact subset of $\mathbb{R} \times T^1$. Therefore the arguments of [6]–[7] show that a solution of (PDE) given by Theorem 2.6 does not satisfy (2.3) or (2.5), i.e. is not minimal in the sense of 6° of Theorem 2.2. However it does have a partial minimality property.

Definition. *We say u satisfies the* asymptotic minimality property *if there is an $x_0 > 0$ such that if $\mathcal{D}$ is contained in $\{x \ge x_0\} \times T^1$ or in $\{x \le -x_0\} \times T^1$, then* (2.3) *or equivalently* (2.5) *holds.*

Now the main result of this section can be stated:

Theorem 2.8. *Let G satisfy* (G_1)–(G_3). *Suppose $u \in C^2(\mathbb{R} \times T^1, \mathbb{R})$ is a solution of* (PDE) *and the asymptotic minimality condition holds. Then*

1° (a) $\|u\|_{C^2(S_i)} \to 0$ *or* (b) $\|1 - u\|_{C^2(S_i)} \to 0$ *as* $i \to -\infty$.
2° (a) $\|u\|_{C^2(S_i)} \to 0$ *or* (b) $\|1 - u\|_{C^2(S_i)} \to 0$ *as* $i \to \infty$.
3° *If* 1°(a) *and* 2°(a) *(resp.* 1°(b) *and* 2°(b)*) hold and $u \not\equiv 0$ (resp. $u \not\equiv 1$), then $\mathcal{M}(0, 1)$ and $\mathcal{M}(1, 0)$ have gaps.*
4° *If* 1°(a) *and* 2°(b) *hold and $u \notin \mathcal{M}(0, 1)$, then $\mathcal{M}(0, 1)$ has gaps.*
5° *If* 1°(b) *and* 2°(a) *hold and $u \notin \mathcal{M}(1, 0)$, then $\mathcal{M}(1, 0)$ has gaps.*
6° *If u is minimal and $u \not\equiv 0, \not\equiv 1$, then either* 1°(a) *and* 2°(b) *hold and $u \in \mathcal{M}(0, 1)$, or* 1°(b) *and* 2°(a) *hold and $u \in \mathcal{M}(1, 0)$.*

The proof of Theorem 2.8 involves several steps. The first is to get the appropriate asymptotics for u.

Proposition 2.9. *Let u be a solution of* (PDE) *satisfying the asymptotic minimality property for $\{x \ge x_0\} \times T^1$. Then 2° of Theorem* 2.8 *holds.*

Proof. For $i \in \mathbb{N}$, set $\psi_i(x, y) = u(x + i, y)$. Let $\alpha \in (0, 1)$. Since $0 \le u \le 1$ and G is a C^2 function of its arguments, by standard estimates from elliptic regularity theory [4], the functions ψ_i are bounded in $C^{2,\alpha}(S_0)$. Therefore as $i \to \infty$, $\psi_i \to \psi$ in $C^2(S_0)$ along a subsequence, where ψ is a solution of (PDE). Moreover

$$\int_{S_0} L(\psi_i)\, dx\, dy \to \int_{S_0} L(\psi)\, dx\, dy$$

as $i \to \infty$ along the subsequence. If $0 \not\equiv \psi \not\equiv 1$, there is a $\beta > 0$ such that

$$\int_{S_0} L(\psi)\, dx\, dy \ge \beta.$$

Hence there is an $i_0 \in \mathbb{N}$, $i_0 \geq x_0$, such that for $i \geq i_0$ and i in the subsequence, then

$$\int_{S_0} L(\psi_i)\,dx\,dy \geq \beta/2.$$

Since

$$\int_{S_0} L(\psi_i)\,dx\,dy \geq 0$$

for all $i \in \mathbb{N}$, if $\ell_j \geq \ell_i \geq i_0$ and are in the subsequence,

$$\int_{\ell_i}^{\ell_j+1} L(u)\,dx\,dy \geq \frac{\ell_j - \ell_i}{2}\beta. \tag{2.10}$$

On the other hand, define

$$u^*(x) = \begin{cases} u(x), & x \leq \ell_i, \\ (\ell_i + 1 - x)u(x), & \ell_i \leq x \leq \ell_i + 1, \\ 0, & \ell_i + 1 \leq x \leq \ell_j, \\ (x - \ell_j)u(x), & \ell_j \leq x \leq \ell_j + 1, \\ u(x), & \ell_j + 1 \leq x. \end{cases} \tag{2.11}$$

Then by (2.3) and (2.10),

$$\begin{aligned} \frac{\ell_j - \ell_i}{2}\beta &\leq \sum_{k=\ell_i}^{\ell_j} \int_{S_k} L(u)\,dx\,dy \leq \sum_{k=\ell_i}^{\ell_j} \int_{S_k} L(u^*)\,dx\,dy \\ &= \int_{S_{\ell_i}} L(u^*)\,dx\,dy + \int_{S_{\ell_j}} L(u^*)\,dx\,dy. \end{aligned} \tag{2.12}$$

Setting $\theta(x, y) = x\psi(x, y)$, we get, as $\ell_j \to \infty$,

$$\int_{S_{\ell_j}} L(u^*)\,dx\,dy \to \int_{S_0} L(\theta)\,dx\,dy.$$

Consequently, the right hand side of (2.12) is bounded as $\ell_j \to \infty$ while the left hand side tends to ∞, a contradiction. Thus $\psi \equiv 0$ or $\psi \equiv 1$. Since any subsequence of ψ_i has a further subsequence converging in $C^2(S_0)$ to 0 or 1, Proposition 2.9 follows. □

Remark 2.13. 1° of Theorem 2.8 is proved by the same argument.

Corollary 2.14. *Let u be as in Theorem* 2.8. *Then $I(u) < \infty$.*

Proof. Since $u \in C^2(\mathbb{R} \times T^1, \mathbb{R})$, it suffices to show

$$\sum_{i \geq x_0} \int_{S_i} L(u)\,dx\,dy < \infty. \tag{2.15}$$

Then similarly

$$\sum_{i<-x_0} \int_{S_i} L(u)\,dx\,dy < \infty.$$

To verify (2.15), let u^* be as in (2.11) with ℓ_i, ℓ_j replaced by i, j. By (2.12) for $i \geq i_0$,

$$\sum_{k=i}^{j} \int_{S_k} L(u)\,dx\,dy \leq \int_{S_i} L(u^*)\,dx\,dy + \int_{S_j} L(u^*)\,dx\,dy. \tag{2.16}$$

Letting $j \to \infty$, by Proposition 2.9 and Remark 2.13,

$$\int_{S_j} L(u)\,dx\,dy \to 0, \quad j \to \infty.$$

Hence

$$\sum_{i}^{\infty} \int_{S_k} L(u)\,dx\,dy \leq \int_{S_i} L(u^*)\,dx\,dy < \infty.$$

□

The next result extends the asymptotic minimality property to allow $\mathcal{O}$ to be unbounded.

Proposition 2.17. *Suppose that u satisfies the asymptotic minimality property with associated x_0. Then*

$$\int_{[x_0,\infty)\times T^1} (L(u) - L(u+\varphi))\,dx\,dy \leq 0$$

for all $\varphi \in W_0^{1,2}([x_0,\infty)\times T^1, \mathbb{R})$.

Proof. If

$$\int_{[x_0,\infty)\times T^1} L(u+\varphi)\,dx\,dy = \infty, \tag{2.18}$$

the result is trivially true via Corollary 2.14. So suppose the integral in (2.18) is finite. If Proposition 2.17 is false, there is a φ as above and $\gamma > 0$ such that

$$\int_{[x_0,\infty)\times T^1} (L(u) - L(u+\varphi))\,dx\,dy \geq \gamma. \tag{2.19}$$

Since

$$\int_{[x_0,\infty)\times T^1} L(u)\,dx\,dy, \quad \int_{[x_0,\infty)\times T^1} L(u+\varphi)\,dx\,dy < \infty,$$

there is an $n_0 \in \mathbb{N}$, $n_0 \geq x_0$, such that

$$\int_{[n_0+1,\infty)\times T^1} L(u)\,dx\,dy, \quad \int_{[n_0+1,\infty)\times T^1} L(u+\varphi)\,dx\,dy \leq \frac{\gamma}{4}. \tag{2.20}$$

Hence

$$\int_{[x_0,n_0+1]\times T^1} (L(u) - L(u+\varphi))\,dx\,dy \geq \frac{\gamma}{2}. \tag{2.21}$$

By Proposition 2.9, $\|u\|_{C^2(S_{n_0})}$ or $\|1-u\|_{C^2(S_{n_0})}$ is small for n_0 large. Likewise $\varphi \in W_0^{1,2}([x_0,\infty)\times T^1,\mathbb{R})$ implies $\|\varphi\|_{W^{1,2}(S_{n_0})}$ is small. Set

$$\psi(x) = \begin{cases} \varphi(x), & x_0 \le x \le n_0, \\ (n_0+1-x)\varphi(x), & n_0 \le x \le n_0+1, \\ 0, & n_0+1 \le x. \end{cases}$$

Then the above observations show

$$\left| \int_{S_{n_0}} (L(u+\varphi) - L(u+\psi))\, dx\, dy \right| \le \frac{\gamma}{4} \tag{2.22}$$

for n_0 sufficiently large. Consequently

$$\begin{aligned} &\int_{[x_0,n_0+1]\times T^1} (L(u) - L(u+\varphi))\, dx\, dy \\ &= \int_{[x_0,n_0+1]\times T^1} [(L(u) - L(u+\psi)) + (L(u+\psi) - L(u+\varphi))]\, dx\, dy \le \frac{\gamma}{4} \end{aligned} \tag{2.23}$$

via (2.5) and (2.22). But (2.23) is contrary to (2.21) and Proposition 2.17 is verified. □

One final preliminary is needed to prove Theorem 2.8.

Proposition 2.24. *Let u be as in* 1°(a) *and* 2°(a) *of Theorem* 2.8. *Suppose $v \in \mathcal{M}(0,1)$ and $v \ge u$ for $x \ge -x_0$ where u is asymptotically minimal in $\{x \le -x_0\}$. Then $v > u$ on $\mathbb{R}\times T^1$ and there is a $\delta > 0$ such that $v \ge u + \delta$ for $x \ge -x_0$.*

Proof. We have $\max(u,v) \in \Gamma(0,1)$. Therefore by (2.1),

$$I(\max(u,v)) \ge c(0,1) = I(v). \tag{2.25}$$

Let $A = \{(x,y) \in \mathbb{R}\times T^1 \mid u(x,y) > v(x,y)\}$. If $A \ne \emptyset$, then $A \subset \{x \le -x_0\} \times T^1$, and by (2.25),

$$I(\max(u,v)) - I(v) = \int_A (L(u) - L(v))\, dx\, dy \ge 0. \tag{2.26}$$

By Proposition 2.17 with $\varphi = v - u$,

$$\int_A (L(v) - L(u))\, dx\, dy \ge 0. \tag{2.27}$$

Combining (2.26)–(2.27) gives

$$\int_A L(u)\, dx\, dy = \int_A L(v)\, dx\, dy$$

and

$$\int_{\mathbb{R}\times T^1} L(\max(u,v))\, dx\, dy = c(0,1). \tag{2.28}$$

Therefore $\max(u, v) \in \mathcal{M}(0,1)$. By 5° of Theorem 2.2, $\mathcal{M}(0,1)$ is an ordered set. Thus either (a) $\max(u, v) > v$ or (b) $\max(u, v) \equiv v$. But (a) implies $u = \max(u, v) > v$ contrary to $v \geq u$ for $x \geq -x_0$. If (b) occurs, $v = \max(u, v) \geq u$ so $A = \emptyset$. It follows that $v \geq u$ on $\mathbb{R} \times T^1$.

Suppose there is a point $(x^*, y^*) \in \mathbb{R} \times T^1$ such that $v(x^*, y^*) = u(x^*, y^*)$. Set $w = v - u$. Since each of v and u are solutions of (PDE), w satisfies the linear elliptic partial differential equation

$$-\Delta w + H(x, y)w = 0 \tag{2.29}$$

where

$$H(x, y) = \begin{cases} \dfrac{G_u(x, y, v(x, y)) - G_u(x, y, u(x, y))}{v(x, y) - u(x, y)} & \text{if } v(x, y) > u(x, y), \\ G_{uu}(x, y, v(x, y)) & \text{if } v(x, y) = u(x, y). \end{cases}$$

Thus H is continuous, $w \geq 0$ and $w(x^*, y^*) = 0$. Consequently, the maximum principle implies $w \equiv 0$, i.e. $u \equiv v$. But as $x \to \infty$, $u(x, y) \to 0$ while $v(x, y) \to 1$. Thus $v > u$ on $\mathbb{R} \times T^1$.

Lastly, to prove the second assertion of Proposition 2.24, the different asymptotic behaviors of v and u imply there is a $\beta > 0$ such that $v(x, y) > u(x, y) + 1/2$ for $x \geq \beta$. Since $v > u$ for $-x_0 \leq x \leq \beta$, there is a $\delta > 0$ such that $v \geq u + \delta$ for $x \geq -x_0$. □

Now we are ready for the

Proof of Theorem 2.8. 1° and 2° of the theorem follow from Proposition 2.9 and Remark 2.13. To prove 3°, set

$$\Lambda = \{v \in \mathcal{M}(0,1) \mid v \geq u \text{ for } x \geq -x_0\}.$$

The difference in the asymptotic behavior of u and v as $x \to \infty$ shows $\Lambda \neq \emptyset$. Define

$$V(x, y) = \inf_{v \in \Lambda} v(x, y). \tag{2.30}$$

We claim that $V \in \Lambda$. Certainly $V(x, y) \geq u(x, y)$ for $x \geq -x_0$. Since any $v \in \mathcal{M}(0,1)$ is a solution of (PDE) lying between 0 and 1, standard elliptic estimates [4] give $C^{2,\alpha}_{\text{loc}}(\mathbb{R} \times T, \mathbb{R})$ estimates for v which are independent of v. Choose a sequence $(v_j) \subset \Lambda$ such that $v_j(-x_0, y_0) \to V(-x_0, y_0)$ as $j \to \infty$ for some $y_0 \in T^1$. Passing to a subsequence if necessary, it can be assumed that v_j converges in $C^2_{\text{loc}}(\mathbb{R} \times T^1, \mathbb{R})$ to $W \in C^2(\mathbb{R} \times T^1, \mathbb{R})$. The C^2_{loc} convergence and 4° and 6° of Theorem 2.2 imply W is a solution of (PDE) which is minimal and monotone. Since $W(-x_0, y_0) = V(-x_0, y_0) \geq u(-x_0, y_0) > 0$, the monotonicity of W and 2° of Theorem 2.8 imply $\|1 - W\|_{C^2(S_i)} \to 0$ as $i \to \infty$. By 1° of Theorem 2.8, either (a) $\|W\|_{C^2(S_0)} \to 0$ as $i \to -\infty$ or (b) $\|1 - W\|_{C^2(S_i)} \to 0$ as $i \to -\infty$. If (b) holds, the monotonicity of W implies $W \equiv 1$. But $W(-x_0, y_0) < 1$ so (a) must hold. Therefore $W \in \Gamma(0,1)$.

For $\ell \in \mathbb{N}$,

$$\int_{[-\ell,\ell]\times T^1} L(W)\,dx\,dy = \lim_{j\to\infty}\int_{[-\ell,\ell]\times T^1} L(v_j)\,dx\,dy$$
$$\leq \lim_{j\to\infty}\int_{\mathbb{R}\times T^1} L(v_j)\,dx\,dy = c(0,1). \tag{2.31}$$

Letting $\ell \to \infty$ shows $I(W) \leq c(0,1)$. But $W \in \Gamma$ so $I(W) = c(0,1)$ and $W \in \mathcal{M}(0,1)$. The C^2_{loc} convergence of (v_j) also implies $W \in \Lambda$. Thus $V \leq W$. If $V(x^*, y^*) < W(x^*, y^*)$ for some $(x^*, y^*) \in \mathbb{R} \times T^1$, there is a $v \in \Lambda$ such that

$$V(x^*, y^*) \leq v(x^*, y^*) < W(x^*, y^*).$$

Since $\mathcal{M}(0,1)$ is ordered,

$$V(-x_0, y_0) \leq v(-x_0, y_0) < W(-x_0, y_0) = V(-x_0, y_0),$$

a contradiction. Consequently, $V \equiv W \in \Lambda$. By Proposition 2.24, there is a $\delta > 0$ such that $V \geq u + \delta$ for $x \geq -x_0$. Suppose that $\mathcal{M}(0,1)$ does not possess a gap. Consider the ordered connected set

$$\mathcal{C} = \{w \in \mathcal{M}(0,1) \mid V(x-1, y) \leq w(x,y) \leq V(x,y)\}.$$

There is an $R > 0$ such that $V(x-1, y) \geq u(x,y) + \delta$ for $x \geq R$. Therefore any $w \in \mathcal{C}$ near V satisfies $V \geq w \geq u + \frac{1}{2}\delta$ for $x \geq -x_0$. But this contradicts the definition of V. Thus $\mathcal{M}(0,1)$ must contain gaps.

Replacing v in Proposition 2.24 by $w \in \mathcal{M}(1,0)$ where $w \geq u$ for $x \leq x_0$ yields a variant of that result with $w \geq u + \delta$ for $x \leq x_0$. This fact and the argument just given shows $\mathcal{M}(1,0)$ also must contain gaps and 3° is proved.

Parts 4°–5° of Theorem 2.8 are proved in the same way so only 4° will be treated here. Thus suppose $u \notin \mathcal{M}(0,1)$ is a solution of (PDE) which satisfies 1°(a) and 2°(b) and is asymptotically minimal in $\{x \leq -x_0\}$ and $\{x \geq x_0\}$. Set

$$\Lambda^* = \{v \in \mathcal{M}(0,1) \mid v \geq u \text{ for } (x,y) \in [-x_0, x_0] \times T^1\}.$$

Define

$$V^*(x,y) = \inf_{v\in\Lambda^*} v(x,y).$$

As in the proof above of 3°, $V^* \in \mathcal{M}(0,1)$. By the argument associated with (2.29), $V^* > u$ for $(x,y) \in (-x_0, x_0)\times T^1$. If $\mathcal{M}(0,1)$ has no gaps, $V^* > u$ for $[-x_0, x_0]\times T^1$ is not possible as in the proof of 3°. Thus $V^*(x^*, y^*) = u(x^*, y^*)$ for some $x^* \in \{-x_0, x_0\}$ and $y^* \in T^1$. Let

$$A^+ = \{(x,y) \in (x_0, \infty) \times T^1 \mid u(x,y) > V^*(x,y)\},$$
$$A^- = \{(x,y) \in (-\infty, -x_0) \times T^1 \mid u(x,y) > V^*(x,y)\}.$$

The argument of Proposition 2.24 shows $A^+ = \emptyset = A^-$. Therefore $V^* \geq u$ with equality at (x^*, y^*). But the maximum principle argument of (2.29) again shows this is not possible. Hence $\mathcal{M}(0,1)$ must contain gaps.

Lastly to prove 6° of Theorem 2.8, observe that by 1°–2°, there are four possibilities for the asymptotic behavior of u. If 1°(a) and 2°(a) occur, minimality implies $u \equiv 0$. Likewise if 1°(b) and 2°(b) hold, then $u \equiv 1$. These cases are excluded by the hypotheses of 6°. Thus suppose that 1°(a) and 2°(b) occur. Then $u \in \Gamma(0,1)$.

Now a comparison argument as in earlier results shows $u \in \mathcal{M}(0,1)$. Choose any $v \in \mathcal{M}(0,1)$ and $k \in \mathbb{N}$. Define

$$U_k(x,y) = \begin{cases} v(x,y), & |x| \leq k, \\ u(x,y), & |x| \geq k+1, \end{cases}$$

and interpolate linearly in x for the intermediate region. By the minimality of u,

$$I(U_k) - I(u) = \int_{-k-1}^{k+1} \int_0^1 (L(U_k) - L(u))\,dx\,dy \geq 0. \tag{2.32}$$

But

$$\begin{aligned} &\int_{-k-1}^{k+1} \int_0^1 (L(U_k) - L(u))\,dx\,dy \\ &\quad = \int_{-k-1}^{-k} \int_0^1 (L(U_k) - L(u))\,dx\,dy + \int_k^{k+1} \int_0^1 (L(U_k) - L(u))\,dx\,dy \\ &\qquad + \int_{-k}^{k} \int_0^1 (L(v) - L(u))\,dx\,dy \\ &\quad = \int_{-k-1}^{-k} \int_0^1 (L(U_k) - L(u))\,dx\,dy + \int_k^{k+1} \int_0^1 (L(U_k) - L(u))\,dx\,dy \\ &\qquad - \int_{-\infty}^{-k} \int_0^1 (L(v) - L(u)) - \int_k^{\infty} \int_0^1 (L(v) - L(u))\,dx\,dy + I(v) - I(u). \end{aligned} \tag{2.33}$$

Letting $k \to \infty$ and combining (2.32)–(2.33) shows

$$c(0,1) = I(v) \geq I(u). \tag{2.34}$$

Therefore $I(u) = c(0,1)$ and $u \in \mathcal{M}(0,1)$.

The remaining case of 6° is treated in the same way. □

3. A second necessity result

In addition to the multi-transition solutions of (PDE) that are periodic in y, it was further shown in [6]–[7] that there are solutions, U, that are heteroclinic from 0 to 1 in x and from v to w in y where $v < w$ belong to $\mathcal{M}(0,1)$. This existence result assumes that $\mathcal{M}(0,1)$ contains gaps and v, w is an associated gap pair. Our main result in this section is that this sufficient condition is also necessary. Some notation is needed before a theorem can be formulated. For $i \in \mathbb{Z}$, let $T_i = \mathbb{R} \times [i, i+1]$. For $v, w \in \mathcal{M}(0,1)$ with $v < w$, define

$$\Gamma(v,w) = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}) \mid v \le u \le w,\ \|u - v\|_{L^2(T_i)} \to 0 \text{ as } i \to -\infty,\\ \|u - w\|_{L^2(T_i)} \to 0 \text{ as } i \to \infty\}.$$

Thus $\Gamma(v,w)$ contains candidates for the doubly heteroclinic solutions of (PDE) mentioned above. Such solutions cannot be obtained directly by minimizing $\int_{\mathbb{R}^2} L(u)\,dx\,dy$ over $\Gamma(v,w)$ since this functional will be infinite on all members of $\Gamma(v,w)$. Consequently, the functional has to be renormalized in some fashion to subtract this infinity from it.

For $p \le q \in \mathbb{Z}$ and $u \in \Gamma(v,w)$, set

$$J_{p,q}(u) = \sum_{i=p}^{q} \left(\int_{T_i} L(u)\,dx\,dy - c(0,1) \right)$$

and define

$$J(u) = \varliminf_{\substack{p \to -\infty \\ q \to \infty}} J_{p,q}(u).$$

It was shown in [6]–[7] that if $J(u) < \infty$, then

$$J(u) = \lim_{\substack{p \to -\infty \\ q \to \infty}} J_{p,q}(u)$$

and

$$\|u - v\|_{W^{1,2}(T_i)} \to 0 \quad \text{as } i \to -\infty, \qquad \|u - w\|_{W^{1,2}(T_i)} \to 0 \quad \text{as } i \to \infty.$$

Moreover

Theorem 3.1 ([6]–[7]). *Suppose that G satisfies* (G_1)–(G_3) *and v, w is a gap pair in $\mathcal{M}(0,1)$. Set*

$$c(v,w) = \inf_{u \in \Gamma(v,w)} J(u). \tag{3.2}$$

Then

1° $\mathcal{M}(v,w) \equiv \{u \in \Gamma(v,w) \mid J(u) = c(v,w)\} \ne \emptyset$.

If $U \in \mathcal{M}(v,w)$, then:

2° *U is a classical solution of* (PDE).
3° $\|U - v\|_{C^2(T_i)} \to 0$ *as* $i \to -\infty$, *and* $\|U - w\|_{C^2(T_i)} \to 0$ *as* $i \to \infty$.
4° $v(x,y) < U(x,y) < U(x+1,y) < w(x,y)$,
$v(x,y) < U(x,y) < U(x,y+1) < w(x,y)$.
5° *$\mathcal{M}(v,w)$ is an ordered set.*
6° *U is minimal in the sense of* (v) *of Remark* 2.4.

Now the main theorem of this section can be stated:

Theorem 3.3. *Let G satisfy* (G_1)–(G_2). *For $v < w \in \mathcal{M}(0,1)$, suppose there is a $U \in \Gamma(v,w)$ such that*

$$J(U) = \inf_{u \in \Gamma(v,w)} J(u). \tag{3.4}$$

Then v, w is a gap pair.

Remark 3.5. Once we know $v < w$ is a gap pair, Theorem 3.1 applies so U is a minimal solution of (PDE) which is monotone in y in the sense of 4°.

Proof of Theorem 3.3. Suppose that v, w is not a gap pair in $\mathcal{M}(0,1)$. Then there is a $\varphi \in \mathcal{M}(0,1)$ with $v < \varphi < w$. We claim this implies

(A) $\min(U,\varphi) \in \Gamma(v,\varphi)$ and $\max(U,\varphi) \in \Gamma(\varphi,w)$,
(B) $J(\min(U,\varphi)) = c(v,\varphi)$, $J(\max(U,\varphi)) = c(\varphi,w)$,
(C) $\min(U,\varphi)$ and $\max(U,\varphi)$ are solutions of (PDE) in $\mathbb{R}^2$.

Assume (A)–(C) for now. Then since

$$\|U - w\|_{L^2(T_i)} \to 0 \quad \text{as } i \to \infty,$$

for $x \in [0,1]$ and $y \gg 1$, $\min(U,\varphi)(x,y) = \varphi(x,y)$. But φ and $\min(U,\varphi)$ are solutions of (PDE) in $\mathbb{R}^2$ with $\varphi \ge \min(U,\varphi)$ with equality for $x \in [0,1]$ and $y \gg 1$. Therefore by the maximum principle argument centered at (2.29), $\varphi \equiv \min(U,\varphi)$ so $U \ge \varphi$. On the other hand,

$$\|U - v\|_{L^2(T_i)} \to 0 \quad \text{as } i \to -\infty \tag{3.6}$$

so for $x \in [0,1]$ and $y \ll -1$ we have $\min(U,\varphi)(x,y) = U(x,y) < \varphi(x,y) \equiv \min(U,\varphi)(x,y)$, a contradiction. Thus such a φ cannot exist and v, w is a gap pair.

It remains to prove (A)–(C). □

Proof of (A). It will be shown that $\min(U,\varphi) \in \Gamma(v,\varphi)$. That $\max(U,\varphi) \in \Gamma(\varphi,w)$ follows similarly. Certainly $v \le \min(U,\varphi) \le \varphi$ and $\min(U,\varphi) \in W^{1,2}_{\text{loc}}(\mathbb{R}^2,\mathbb{R})$ so all that need be proved is $\min(V,\varphi)$ has the desired asymptotic behavior:

$$\|v - \min(U,\varphi)\|_{L^2(T_0)} \to 0, \quad i \to -\infty, \tag{3.7}$$
$$\|\varphi - \min(U,\varphi)\|_{L^2(T_i)} \to 0, \quad i \to \infty. \tag{3.8}$$

To verify (3.7), note that

$$\|v - \min(U,\varphi)\|_{L^\infty(\mathbb{R}^2,\mathbb{R})} \le 1.$$

Therefore

$$\int_{T_i} |v - \min(U,\varphi)|^2 \, dx\, dy \le \int_{T_i} (\min(U,\varphi) - v) \, dx\, dy. \tag{3.9}$$

Now for any $p \in \mathbb{N}$,

$$\int_{T_i} (\min(U,\varphi) - v)\,dx\,dy = \int_{[-\infty,-p]\times[i,i+1]} + \int_{[-p,p]\times[i,i+1]} + \int_{[p,\infty]\times[i,i+1]} \equiv I_1 + I_2 + I_3 \tag{3.10}$$

with $I_1, I_2, I_3 \geq 0$. If we write I_2 as

$$I_2 = \int_{[-p,p]\times[0,1]} (\min(U(x,y+i),\varphi(x,y)) - v(x,y))\,dx\,dy, \tag{3.11}$$

then (3.6) and the Dominated Convergence Theorem imply $I_2 \to 0$ as $i \to -\infty$. To analyze I_1 and I_3, let $j \in \mathbb{N}$ and note that $v(x+j,y) \to 1$ as $j \to \infty$ uniformly for $y \in T^1$. Therefore for j sufficiently large, $v(j,0) > \varphi(1,0)$. Since $v(\cdot + j,\cdot) \in \mathcal{M}(0,1)$ and $\mathcal{M}(0,1)$ is an ordered set,

$$v(x,y) < \varphi(x,y) < v(x+j,y).$$

Thus

$$\begin{aligned} I_3 &\leq \int_{[p,\infty)\times[0,1]} (v(x+j,y) - v(x,y))\,dx\,dy \\ &= \lim_{q\to\infty} \int_{[p,q]\times[0,1]} (v(x+j,y) - v(x,y))\,dx\,dy \\ &= \lim_{q\to\infty} \left[\int_{[q-j,q]\times[0,1]} v(x+j,y)\,dx\,dy - \int_{[p,p+j]\times[0,1]} v(x,y)\,dx\,dy \right] \\ &= j - \int_{[p,p+j]\times[0,1]} v(x,y)\,dx\,dy. \end{aligned} \tag{3.12}$$

Let $\epsilon > 0$. Since $v(x,y) \to 1$ as $x \to \infty$ uniformly in $y \in [0,1]$, p can be chosen so large that the right hand side above is $\leq \epsilon$. Similarly $I_1 \leq \epsilon$. Thus

$$\overline{\lim_{i\to\infty}} \int_{T_i} |v - \min(U,\varphi)|^2\,dx\,dy \leq 2\epsilon.$$

Since ϵ is arbitrary, (3.7) and likewise (3.8) follows. □

Proof of (B). First observe that

$$\begin{aligned} &J_{p,q}(\min(U,\varphi)) + J_{p,q}(\max(U,\varphi)) \\ &\qquad = \sum_{i=p}^{q} \left[\int_{T_i} (L(\min(U,\varphi)) + L(\max(U,\varphi)))\,dx\,dy - 2c(0,1) \right]. \end{aligned} \tag{3.13}$$

Since

$$\int_{T_i} (L(\min(U,\varphi)) + L(\max(U,\varphi)))\,dx\,dy$$
$$= \int_{T_i} (L(U) + L(\varphi))\,dx\,dy = \int_{T_i} L(U)\,dx\,dy + c(0,1), \tag{3.14}$$

substituting (3.14) in (3.13) and letting $p \to -\infty$, $q \to \infty$ yields

$$J(\min(U,\varphi)) + J(\max(U,\varphi)) = J(U). \tag{3.15}$$

By (A), $\min(U,\varphi) \in \Gamma(v,\varphi)$ and $\max(U,\varphi) \in \Gamma(\varphi,w)$. Therefore (3.15) implies

$$c(v,\varphi) + c(\varphi,w) \le J(U) = c(v,w). \tag{3.16}$$

If there is equality in (3.16), then (3.15) shows $J(\min(U,\varphi)) = c(v,\varphi)$ and $J(\max(U,\varphi)) = c(\varphi,w)$, i.e. (B) holds. To verify equality in (3.16), arguing indirectly, suppose

$$\epsilon \equiv c(v,w) - c(v,\varphi) - c(\varphi,w) > 0.$$

Choose $f \in \Gamma(v,\varphi)$ and $g \in \Gamma(\varphi,w)$ such that

$$J(f) < c(v,\varphi) + \epsilon/3, \qquad J(g) < c(\varphi,w) + \epsilon/3. \tag{3.17}$$

Since $u \in \Gamma(\psi,\chi)$ implies $u(\cdot,\cdot + j) \in \Gamma(\psi,\chi)$ for any $j \in \mathbb{Z}$, it can be assumed that

$$J_{-\infty,0}(f) < c(v,\varphi) + \epsilon/3, \qquad J_{1,\infty}(g) < c(\varphi,w) + \epsilon/3 \tag{3.18}$$

and

$$\begin{cases} \|f-\varphi\|_{W^{1,2}(T_i)} \le \sigma, & i \ge 0,\\ \|g-\varphi\|_{W^{1,2}(T_i)} \le \sigma, & i \le 1, \end{cases} \tag{3.19}$$

where σ is free for the moment. Define

$$h(x,y) = \begin{cases} f(x,y), & y \le 0,\\ yg(x,y) + (1-y)f(x,y), & 0 \le y \le 1,\\ g(x,y), & y \ge 1, \end{cases} \tag{3.20}$$

so $h \in \Gamma(v,w)$. Then for σ sufficiently small,

$$\left| \int_{T_0} L(h)\,dx\,dy - c(0,1) \right| < \frac{\epsilon}{3} \tag{3.21}$$

via (3.19). Hence by (3.18)–(3.21),

$$J(h) < c(v,\varphi) + c(\varphi,w) + \epsilon = c(v,w). \tag{3.22}$$

But $h \in \Gamma(v,w)$ implies

$$J(h) \ge c(v,w). \tag{3.23}$$

Thus (3.22)–(3.23) show $\epsilon > 0$ is impossible and (B) has been established. □

Proof of (C). It suffices to prove the more general result:

Proposition 3.24. *Let $f < g$ with $f, g \in \mathcal{M}(0,1)$. Suppose there is a $u \in \mathcal{M}(f,g)$ such that*

$$J(u) = c(f,g) = \inf_{\Gamma(v,w)} J.$$

Then u is a solution of (PDE) *in $\mathbb{R}^2$.*

Proof. It suffices to show that there is an $r > 0$ such that for any $(x^*, y^*) \in \mathbb{R}^2$,

$$J(u) \le J(u + t\zeta) \tag{3.25}$$

for all $\zeta \in C^1(\mathbb{R}^2, \mathbb{R})$ with support in $B_r(x^*, y^*)$ (the open ball of radius r about (x^*, y^*)) and for all $t \in [0, t_0(\zeta)]$ where $t_0(\varphi) > 0$. Indeed, if this is the case, suppose $B_r(x^*, y^*) \subset \mathbb{R} \times [p, q+1]$. Then by (3.25),

$$J_{p,q}(u) \le J_{p,q}(u + t\zeta), \tag{3.26}$$

which in turn implies

$$\int_{B_r(x^*,y^*)} L(u)\,dx\,dy \le \int_{B_r(x^*,y^*)} L(u + t\zeta)\,dx\,dy. \tag{3.27}$$

Now (3.27) and standard elliptic regularity arguments imply $u \in C^2(B_r(x^*, y^*))$ and satisfies (PDE) in $B_r(x^*, y^*)$.

To verify (3.25), let $a = \max(u + t\zeta, g)$ and $b = \min(u + t\zeta, g)$. Set

$$\Gamma(g) = \{\psi \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}) \mid f - 1 \le \psi \le g + 1 \text{ and } \|\psi - g\|_{L^2(T_i)} \to 0 \text{ as } |i| \to \infty\}.$$

Then, as was shown in [7], $J(\psi) \ge 0$. Since $a \in \Gamma(g)$ (via the argument of (A)),

$$J(b) \le J(a) + J(b) = J(u + t\zeta), \tag{3.28}$$

the latter equality following as in (3.15). Set $\Phi = \max(b, f)$ and $\Psi = \min(b, f)$. Then as above $\Phi \in \Gamma(f, g)$ and $\Psi \in \Gamma(f)$ where

$$\Gamma(f) = \{\psi \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}) \mid f - 1 \le \psi \le g + 1 \text{ and } \|\psi - f\|_{L^2(T_i)} \to 0 \text{ as } |i| \to \infty\}.$$

Again via [7], $J(\Psi) \ge 0$ so

$$J(\Phi) \le J(\Phi) + J(\Psi) = J(b). \tag{3.29}$$

Hence by (3.28)–(3.29),

$$c(f,g) = J(u) \le J(\Phi) \le J(u + t\varphi). \tag{3.30}$$

Thus Proposition 3.24, (C), and Theorem 3.3 are proved. □

Remark 3.31. Just as for Theorem 2.2, Theorem 3.1 leads to an analogue of Theorem 2.8 obtained by variationally gluing, e.g., numbers of $\mathcal{M}(v, w)$ and $\mathcal{M}(w, v)$. This construction succeeds when there are gaps in $\mathcal{M}(v, w)$ and $\mathcal{M}(w, v)$. See [7]. Our earlier arguments can be used again to show that such gap conditions are necessary.

It is natural to ask whether there is a version of Theorem 2.8 in the current setting. As was shown in [3], there are minimal solutions of (PDE) that are (a) heteroclinic from 0 to 1 in any direction θ in the x, y plane with $\tan\theta \in \mathbb{Q}$ and (b) periodic in the orthogonal direction. Thus a complete classification of solutions here based on asymptotic minimality or even minimality will be complicated. However, there is an interesting special case that is in part an application of Theorem 3.3. It will be studied next.

Suppose G satisfies (G_1)–(G_3) and $u \in C^2(\mathbb{R}^2, [0, 1])$ is a solution of (PDE) which is minimal and for all $(x, y) \in \mathbb{R}^2$,

$$u(x+1, y) \geq u(x, y) \tag{3.32}$$

$$u(x, y+1) \geq u(x, y). \tag{3.33}$$

For $k \in \mathbb{Z}$, set

$$u_k(x, y) = u(x, y+k). \tag{3.34}$$

Then by (3.33),

$$0 \leq u_k \leq u_{k+1} \leq 1. \tag{3.35}$$

As in Proposition 2.9, the functions $(u_k)_{k\in\mathbb{Z}}$ are bounded in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R} \times [0, 1], [0, 1])$. Hence there are functions $v, w \in C^{2,\alpha}(\mathbb{R} \times T^1, [0, 1])$ such that $u_k \to v$ in C^2_{loc} as $k \to -\infty$ and $u_k \to w$ in C^2_{loc} as $k \to \infty$. Moreover v and w are minimal solutions of (PDE). By (3.32), $\varphi(x+1, y) \geq \varphi(x, y)$ for $\varphi \in \{v, w\}$. Thus by 6° of Theorem 2.8, $\varphi \equiv 0$, $\varphi \equiv 1$, or $\varphi \in \mathcal{M}(0, 1)$. If for some (x_0, y_0), $v(x_0, y_0) = w(x_0, y_0) \neq 0, 1$, then $v \equiv w$ via the ordering properties of $\mathcal{M}(0, 1)$. Consequently, we have the following five possibilities for u:

(A) $v \equiv w \equiv u \in \{0, 1, \mathcal{M}(0, 1)\}$;
(B) $v \equiv 0$, $w \equiv 1$ and $0 < u < 1$;
(C) $v \equiv 0$ and $w \in \mathcal{M}(0, 1)$;
(D) $v \in \mathcal{M}(0, 1)$ and $w \equiv 1$;
(E) $v, w \in \mathcal{M}(0, 1)$ and $v < w$.

In case (B), for $k \in \mathbb{Z}$, set $U_k(x, y) = u(x+k, y)$. Then as above $U_k \to V, W$ as $k \to -\infty, \infty$ and $V \leq W$. Thus reversing the roles of x and y puts us either in case (A) when $V = W$ at some point, or case (E) when $V < W$. Cases (C) and (D) are also essentially the same with 0 and 1 interchanged. Thus, there are really only two different cases to analyze. We begin with case (E) which involves modifying arguments from [8].

Theorem 3.36. *Let G satisfy* (G_1)–(G_3). *Suppose $u \in C^2(\mathbb{R}^2, [0, 1])$ is a minimal solution of* (PDE) *with the asymptotics of case* (E). *Then v, w is a gap pair and*

$$u \in \mathcal{M}(v, w) = \{U \in \Gamma(v, w) \mid J(U) = c(v, w)\}.$$

Proof. It suffices to prove (i) $u \in \Gamma(v, w)$ and (ii) $J(u) = c(v, w)$. Then $u \in \mathcal{M}(v, w)$ and by Theorem 3.3, v, w is a gap pair.

Proof of (i). It will be shown that with u_k as in (3.34),

$$\int_{T_0} (u_k - v)^2\, dx\, dy \to 0 \quad \text{as } k \to -\infty. \tag{3.37}$$

The remaining asymptotic condition follows similarly. To verify (3.37), let $p \in \mathbb{N}$. Then

$$\int_{T_0} (u_k - v)^2\, dx\, dy = \int_{(-\infty,-p]\times[0,1]} + \int_{[-p,p]\times[0,1]} + \int_{[p,\infty)\times[0,1]} \equiv I_1 + I_2 + I_3.$$

Since $u_k \to v$ in $C^2_{\mathrm{loc}}(\mathbb{R} \times [0,1], [0,1])$ as $k \to -\infty$, $I_2 \to 0$. Noting that for some $j \in \mathbb{N}$,

$$I_3 \le \int_{[p,\infty)\times[0,1]} (w - v)^2\, dx\, dy \le \int_{[p,\infty)\times[0,1]} |v(x + j, y) - v(x, y)|^2\, dx\, dy,$$

the I_1 and I_3 terms can be bounded as in (3.12) and (i) follows. □

Proof of (ii). Since $u \in \Gamma(v, w)$,

$$J(u) \ge c(v, w). \tag{3.38}$$

Thus we must show inequality in (3.38) is not possible. This involves comparison arguments for which a strengthening of (3.37) is needed. We claim

$$\|u - v\|_{W^{1,2}(T_i)}, \|u - w\|_{W^{1,2}(T_i)} \to 0 \quad \text{as } i \to -\infty. \tag{3.39}$$

To verify (3.39) for v, note that u and v are solutions of (PDE). Therefore $U = v - u$ satisfies (2.29):

$$-\Delta U + H(x, y)U = 0 \tag{3.40}$$

where $\|H\|_{L^\infty(\mathbb{R}^2)} \le \|G_{uu}\|_{L^\infty(\mathbb{R}^2\times[0,1])}$. Let η be a cut-off function with $\eta = 1$ on $\bigcup_{j=-1}^{1} T_{i+j}$, $\eta = 0$ outside of $\bigcup_{j=-2}^{2} T_{i+j}$, and $|\nabla\eta| \le 3$. Multiplying (3.40) by $\eta^2 U$ and integrating by parts yields

$$0 = \int_{\bigcup_{j=-2}^{2} T_{i+j}} (\eta^2|\nabla U|^2 + 2\eta U \nabla\eta \cdot \nabla U + H\eta^2 U^2)\, dx\, dy. \tag{3.41}$$

Hence by simple estimates,

$$\begin{aligned} \frac{1}{2}\int_{\bigcup_{j=-1}^{1} T_{i+j}} |\nabla U|^2\, dx &\le \frac{1}{2}\int_{\bigcup_{j=-2}^{2} T_{i+j}} \eta^2|\nabla U|^2\, dx\, dy \\ &\le (18 + \|H\|_{L^\infty}) \int_{\bigcup_{j=-2}^{2} T_{i+j}} U^2\, dx\, dy \end{aligned} \tag{3.42}$$

□

By (3.37), the right hand side of (3.42) tends to 0 as $i \to -\infty$. Hence (3.42) yields (3.39) for v and similarly for w.

Now suppose

$$J(u) > c(v, w). \tag{3.43}$$

Then there is a $U \in \Gamma(v, w)$ and a $\sigma > 0$ such that

$$c(v, w) \le J(U) < J(U) + \sigma < J(u). \tag{3.44}$$

Let $\epsilon > 0$. By (3.39) and the fact that $J(U) < \infty$, there is a $p = p(\epsilon) \in \mathbb{N}$ such that if $\varphi \in \{u, U\}$,

$$\begin{cases} \|\varphi - v\|_{W^{1,2}(T_i)} \le \epsilon, & i \le -p, \\ \|\varphi - w\|_{W^{1,2}(T_i)} \le \epsilon, & i \ge p. \end{cases} \tag{3.45}$$

For $i \in \mathbb{Z}$ and $y \in [i, i+1]$, set

$$g_i = (i - y)U + (y + 1 - i)u, \qquad h_i = (i - y)u + (y + 1 - i)U.$$

Then for ϵ sufficiently small and $\varphi \in \{u, U, g_i, h_i\}$, for $|i| \ge p(\epsilon)$,

$$|J_i(\varphi)| \le \sigma/6 \quad \text{where} \quad J_i(\varphi) = \int_{T_i} L(\varphi)\, dx\, dy - c(0, 1). \tag{3.46}$$

Next let $q \in \mathbb{N}$, $q > 1$. For q large enough,

$$J_{-q,q-1}(U) \le J(U) + \sigma/6. \tag{3.47}$$

Set

$$\Phi = \begin{cases} u, & |y| \ge q, \\ g_{-q}, & -q \le y \le -q+1, \\ U, & -q+1 \le y \le q-1, \\ h_q, & q-1 \le y \le q, \end{cases}$$

and consider

$$\begin{aligned} &\int_{\mathbb{R}\times[-q,q]} (L(u) - L(U))\, dx\, dy \\ &\quad = \int_{\mathbb{R}\times[-q+1,q+1]} (L(u) - L(\Phi))\, dx\, dy + J_{-q}(\Phi) - J_{-q}(U) + J_q(\Phi) - J_q(U) \end{aligned} \tag{3.48}$$

Since u is minimal, the first term on the right in (3.48) is ≤ 0 so by (3.46) the right hand side of (3.48) is $\le \frac{2}{3}\sigma$. For the left hand side of (3.48) we have

$$\begin{aligned} \int_{\mathbb{R}\times[-q,q]} (L(u) - L(U))\, dx\, dy &= \sum_{i=-q}^{q-1} \int_{T_i} (L(u) - L(U))\, dx\, dy \\ &= J_{-q,q-1}(u) - J_{-q,q-1}(U) \ge J_{-q,q-1}(u) - J(U) - \sigma/6 \end{aligned} \tag{3.49}$$

by (3.47). Now if $J(u) = \infty$, by (3.49), the left hand side of (3.48) tends to ∞ as $q \to \infty$ while if $J(u) < \infty$, by (3.44), for large w the left hand side exceeds $\frac{2}{3}\sigma$. In either event we have a contradiction. Thus (ii) and Theorem 3.36 are proved. □

We conclude this section with some remarks about cases (C) and (D) which are roughly equivalent. We do not know if (C) or (D) can occur. However, suppose u further satisfies, for each $(k_1, k_2) \in \mathbb{Z}^2$ and for all $(x, y) \in \mathbb{R}^2$, either (α) $u(x + k_1, y + k_2) \equiv u(x, y)$, (β) $u(x + k_1, y + k_2) > u(x, y)$ or (γ) $u(x + k_1, y + k_2) < u(x, y)$. This property is what Moser in [5] calls the "without self-intersection" property. Then (C) and (D) are impossible unless we are in case (A). E.g. to exclude (D), take $k_2 = 1 = -k_1$ and $p \in \mathbb{N}$. If (α) occurs, then by (3.32), for $k \in \mathbb{N}$,

$$u(x, y) = u(x - k, y + k) \geq u(x - k - p, y + k). \tag{3.50}$$

As $p \to \infty$, $u(x - k - p, y + k) \to V(x - k, y + k) = V(x, y + k)$ where V lies in the analogue of $\mathcal{M}(0, 1)$ with the roles of x and y reversed. Thus

$$u(x, y) \geq V(x, y + k) \tag{3.51}$$

and letting $k \to \infty$ yields $u(x, y) \geq 1$ so $u \equiv 1$. But then we are in case (A).

If (β) occurs for $k_2 = 1 = -k_1$, then

$$u(x, y) > u(x + k, y - k) \geq u(x + k - p, y + k). \tag{3.52}$$

Hence letting $p \to \infty$ gives $u(x, y) \geq v(x + k, y)$, and letting $k \to \infty$ shows that $u(x, y) \geq 1$ so $u \equiv 1$ as for (α). A similar argument applies for (γ).

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