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Jean Mawhin

The periodic Ambrosetti–Prodi problem for nonlinear perturbations of the p-Laplacian

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Abstract. We prove an Ambrosetti–Prodi type result for the periodic solutions of the equation $(|u'|^{p-2}u'))' + f(u)u' + g(x, u) = t$, when f is arbitrary and $g(x, u) \to +\infty$ or $g(x, u) \to -\infty$ when $|u| \to \infty$. The proof uses upper and lower solutions and the Leray–Schauder degree.

Keywords. Ambrosetti–Prodi problem, periodic solutions, upper and lower solutions, topological degree

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be open, bounded and smooth, and let us denote by $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ the eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$, and by $\phi > 0$ the principal eigenfunction. Consider the semilinear Dirichlet problem

$$\Delta u + f(u) = v(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $v \in C^{0,\alpha}(\overline{\Omega})$ and $f \in C^2(\mathbb{R})$. The following seminal result was proved by Ambrosetti–Prodi in 1972 [2].

Theorem 1. *Assume that f satisfies the following conditions:*

$$f''(s) > 0 \quad \text{for all } s \in \mathbb{R} \tag{2}$$

and

$$0 < \lim_{s\to-\infty} \frac{f(s)}{s} < \lambda_1 < \lim_{s\to+\infty} \frac{f(s)}{s} < \lambda_2. \tag{3}$$

Then there exists a closed connected manifold $A_1 \subset C^{0,\alpha}(\overline{\Omega})$ of codimension 1 such that $C^{0,\alpha}(\overline{\Omega}) \setminus A_1 = A_0 \cup A_2$ and (1) has exactly zero, one or two solutions according as v is in A_0, A_1 or A_2.

J. Mawhin: Département de mathématique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium; e-mail: mawhin@math.ucl.ac.be

The proof of Theorem 1 is based upon an extension of Caccioppoli's mapping theorem to some singular case. Conditions (3) mean that *the nonlinearity f crosses the first eigenvalue λ_1 of $-\Delta$ when s goes from $-\infty$ to $+\infty$.*

It is convenient to write (1) in an equivalent way. Let

$$Lu := \Delta u + \lambda_1 u, \quad g(u) := f(u) - \lambda_1 u,$$
$$v(x) = t\phi(x) + h(x) \quad \text{with} \quad \int_\Omega h(x)\phi(x)\,dx = 0,$$

so that problem (1) is equivalent to

$$Lu + g(u) = t\phi(x) + h(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{4}$$

condition (2) is equivalent to

$$g''(s) > 0 \quad \text{for all } s \in \mathbb{R}, \tag{5}$$

and condition (3) is equivalent to

$$-\lambda_1 < \lim_{s\to-\infty} \frac{g(s)}{s} < 0 < \lim_{s\to+\infty} \frac{g(s)}{s} < \lambda_2 - \lambda_1. \tag{6}$$

A cartesian representation of A_1 was given by Berger–Podolak in 1975 [4].

Theorem 2. *If conditions* (5) *and* (6) *hold, then there exists t_1 such that* (4) *has exactly zero, one or two solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.*

The proof of Theorem 2 is based upon a global Lyapunov–Schmidt reduction. The same year, using upper and lower solutions, Kazdan–Warner [9] weakened the assumptions (and the conclusions) of Berger–Podolak.

Theorem 3. *If*

$$-\infty \le \limsup_{s\to-\infty} \frac{g(s)}{s} < 0 < \liminf_{s\to+\infty} \frac{g(s)}{s} \le +\infty, \tag{7}$$

then there exists t_1 such that (4) *has zero or at least one solution according as $t < t_1$ or $t > t_1$.*

The multiplicity conclusion of Ambrosetti–Prodi (without exactness) was obtained independently by Dancer in 1978 [6] and Amann–Hess in 1979 [1] under the Kazdan–Warner condition (7), when g satisfies a suitable growth condition at $+\infty$. We state the more general result of Dancer.

Theorem 4. *If condition* (7) *holds and*

$$\lim_{s\to+\infty} \frac{g(s)}{s^\sigma} = 0, \quad \sigma = \frac{N+1}{N-1}, \tag{8}$$

then there exists t_1 such that (4) *has zero, at least one or at least two solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.*

The proof of Theorem 4 is a combination of the method of upper and lower solutions and of degree theory.

Condition (7) implies that

$$\lim_{|u|\to\infty} g(u) = +\infty. \tag{9}$$

Can we replace (7) by (9) in the Ambrosetti–Prodi problem?

In 1986, a positive answer was given in [7] for a second ordinary differential equation with periodic boundary conditions. We describe the result in the special case

$$u'' + cu' + g(u) = t + h(x), \quad u(0) - u(T) = u'(0) - u'(T) = 0, \tag{10}$$

where $c \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, $h : [0, T] \to \mathbb{R}$ are continuous and $\int_0^T h(x)\, dx = 0$. Notice that 0 is the principal eigenvalue of $-d^2/dx^2 - cd/dx$ with the T-periodic boundary conditions.

Theorem 5. *If condition* (9) *holds, then there exists t_1 such that* (10) *has zero, at least one or at least two solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.*

The nonlinearities

$$g(u) = |u|^{1/2}, \quad g(u) = \log(1 + |u|)$$

satisfy condition (9) but are such that

$$\lim_{u\to-\infty} \frac{g(u)}{u} = \lim_{u\to+\infty} \frac{g(u)}{u} = 0.$$

There is no crossing of the zero eigenvalue!

A similar conclusion holds for the Neumann problem

$$\Delta u + g(u) = t + h(x) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{11}$$

with $g : \mathbb{R} \to \mathbb{R}$ and $h : \overline{\Omega} \to \mathbb{R}$ Hölder continuous, and $\int_\Omega h(x)\, dx = 0$, as shown in 1987 in [11], with the following result.

Theorem 6. *Assume that condition* (9) *holds and*

$$\lim_{u\to+\infty} \frac{g(u)}{u^\sigma} = 0, \quad \sigma = \frac{N}{N-2} \quad \textit{when } N \geq 3. \tag{12}$$

Then there exists t_1 such that (11) *has zero, at least one or at least two solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.*

A natural question was to know if condition (9) could also replace condition (6) in the Dirichlet problem. In the case of dimension $N = 1$,

$$u'' + u + g(u) = t(2/\pi)^{1/2} \sin x + h(x), \quad u(0) = u(\pi) = 0, \tag{13}$$

with $g : \mathbb{R} \to \mathbb{R}$ and $h : [0, \pi] \to \mathbb{R}$, continuous, and $\int_0^\pi h(x) \sin x\, dx = 0$, the following result was proved in 1987 in [5].

Theorem 7. *If condition* (9) *holds, then there exists* $t_1 \leq t_2$ *such that* (13) *has zero, at least one or at least two solutions according as* $t < t_1$, $t \in [t_1, t_2]$ *or* $t > t_2$. *If* $u \mapsto Mu + g(u)$ *is nondecreasing in a neighborhood of* 0 *for some* M, *then* $t_1 = t_2$.

The problems of knowing if $t_1 = t_2$ without an extra condition upon g (even if $N = 1$) and of extending Theorem 7 to higher dimensions are still open. A partial answer to the second question for the Dirichlet problem can be found in a 1987 paper of Kannan–Ortega [8], for sufficiently smooth g and h.

Theorem 8. *If*

$$|g(u)| \leq \gamma |u|^{\sigma} + \beta, \quad \sigma < \frac{N+1}{N-1} \quad \text{when } N > 2, \tag{14}$$

$$\lim_{s \to -\infty} [\lambda_1 s + g(s)] = +\infty, \quad \lim_{s \to +\infty} g(s) = +\infty, \tag{15}$$

then there exists t_1 *such that* (4) *has zero, at least one or at least two solutions according as* $t < t_1$, $t = t_1$ *or* $t > t_1$.

The stability of T-periodic solutions obtained in [7] was considered by Ortega in 1989 [14, 15].

Theorem 9. *Assume that* $c > 0$, $g \in C^1(\mathbb{R})$ *is strictly convex and satisfies condition* (9), *and*

$$0 < g'(+\infty) \leq \left(\frac{\pi}{T}\right)^2 + \frac{c^2}{4}. \tag{16}$$

Then, for each $t > t_1$, *one solution of* (10) *is asymptotically stable and the other unstable.*

The proof is based upon the use of Poincaré's operator and Brouwer degree.

The delicate case of almost periodic solutions of (10) was studied by Ortega–Tarallo in 2003 [16].

Theorem 10. *Assume that* $h \in C(\mathbb{R}, \mathbb{R})$ *is almost periodic,* $g \in C^1(\mathbb{R})$ *is strictly convex and satisfies*

$$-\infty \leq g'(-\infty) < 0 < g'(+\infty) \leq \frac{c^2}{4}. \tag{17}$$

Then there exists t_1 *such that* (10) *has zero, at most one or exactly two almost periodic solutions according as* $t < t_1$, $t = t_1$ *or* $t > t_1$.

The proof uses separation conditions, Opial's method of ordered upper and lower solutions and a special case of a result on nonordered upper and lower solutions given in [13].

Let $p > 1$,

$$\phi : \mathbb{R} \to \mathbb{R}, \quad s \mapsto |s|^{p-1}s,$$

$f : \mathbb{R} \to \mathbb{R}$ be continuous, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be T-periodic in x for some $T > 0$ and continuous, and let $t \in \mathbb{R}$. In this paper, we are interested in studying the 'p-Laplacified'

Ambrosetti–Prodi problem for the T-periodic solutions of the equation

$$(\phi(u'))' + f(u)u' + g(x, u) = t, \tag{18}$$

in terms of the value of the forcing term t. A *T-periodic solution* of (18) is a periodic function $u \in C^1(\mathbb{R})$ of period T such that $\phi \circ u' \in C^1(\mathbb{R})$ and which satisfies (18). Using an approach similar to that of [7], but with substantial technical differences due to the presence of the p-Laplacian, we prove here the following result.

Theorem 11. *If*

$$\lim_{|s|\to\infty} g(x, s) = +\infty \quad \text{uniformly in } x \in \mathbb{R}, \tag{19}$$

then there exists t_1 such that (18) *has zero, at least one or at least two T-periodic solutions according as $t < t_1$, $t = t_1$ or $t > t_1$.*

This theorem is a consequence of Lemmas 4, 6 and 7. Let us mention that, very recently, Arcoya and Ruiz [3] have extended the conditions of Amann–Hess for the Ambrosetti–Prodi problem to perturbations of the p-Laplacian in $\Omega \subset \mathbb{R}^N$ with Dirichlet conditions, when $p \geq 2$. It is interesting to notice that, in the case where $1 < p < 2$, their conclusion is similar to the one in [5].

We use the following notations. For $k \geq 0$ integer, let

$$C_T^k = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is of class } C^k \text{ and } T\text{-periodic}\}.$$

If $v \in C_T^0$, and $p \geq 1$, we set

$$\bar{v} := \frac{1}{T}\int_0^T v(x)\,dx, \quad \tilde{v} = v - \bar{v},$$

$$\|v\|_\infty = \max_{\mathbb{R}} |v|, \quad \|v\|_p = \left(\frac{1}{T}\int_0^T |v(x)|^p\,dx\right)^{1/p}.$$

If $\Omega \subset X$ is an open bounded set of a normed space X and if $\mathcal{S} : \overline{\Omega} \subset X \to X$ is compact and such that $0 \notin (I - \mathcal{S})(\partial\Omega)$, the *Leray–Schauder degree* of $I - S$ with respect to Ω and 0 is denoted by $d_{\mathrm{LS}}[I - \mathcal{S}, \Omega, 0]$.

2. Periodic upper and lower solutions and degree

We need the following results on the method of upper and lower solutions.

Definition 1. *A* T-periodic lower solution α *(resp.* T-periodic upper solution β*) of* (18) *is a C^1 T-periodic function such that $\phi \circ \alpha' \in C^1(\mathbb{R})$ (resp. $\phi \circ \beta' \in C^1(\mathbb{R})$) and*

$$(\phi(\alpha'(x)))' + f(\alpha(x))\alpha'(x) + g(x, \alpha(x)) \geq t \tag{20}$$

(resp.

$$(\phi(\beta'(x)))' + f(\beta(x))\beta'(x) + g(x, \beta(x)) \geq t) \tag{21}$$

for all $x \in \mathbb{R}$. A lower (resp. upper) solution is strict *if the strict inequality holds in* (20) *(resp.* (21)*).*

If the T-periodic lower solution α and the T-periodic upper solution β of (18) are such that $\alpha(x) \le \beta(x)$ for all $x \in \mathbb{R}$, let us define the bounded continuous map $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$r(x,u) = \begin{cases} \alpha(x) & \text{if } u < \alpha(x), \\ u & \text{if } \alpha(x) \le u \le \beta(x), \\ \beta(x) & \text{if } u > \beta(x), \end{cases}$$

and consider the modified equation

$$(\phi(u'))' - [\phi(u) - \phi[r(x,u)]] + f[r(x,u)]u' + g[x, r(x,u)] = t. \tag{22}$$

The following result is classical. We give its simple proof for completeness.

Lemma 1. *Each possible T-periodic solution u of* (22) *is such that*

$$\alpha(x) \le u(x) \le \beta(x) \quad (x \in \mathbb{R}).$$

Proof. We prove the first inequality, the other case being similar. If the conclusion does not hold, $u - \alpha$ reaches a negative minimum, say at ξ, so that

$$u(\xi) < \alpha(\xi), \quad u'(\xi) = \alpha'(\xi). \tag{23}$$

Hence, $r(\xi, u(\xi)) = \alpha(\xi)$, and, by (1),

$$\begin{aligned}(\phi(u'(\xi)))' - [\phi(u(\xi)) - \phi(\alpha(\xi))] + f(\alpha(\xi))\alpha'(\xi) + g(\xi, \alpha(\xi)) \\ = t < (\phi(\alpha'(\xi)))' + f(\alpha(\xi))\alpha'(\xi) + g(\xi, \alpha(\xi)),\end{aligned}$$

so that

$$(\phi(u'(\xi)))' - (\phi(\alpha'(\xi)))' < \phi(u(\xi)) - \phi(\alpha(\xi)) < 0.$$

By continuity, there exists $\varepsilon > 0$ such that

$$(\phi(u'(x)))' - (\phi(\alpha'(x)))' < 0 \quad \text{whenever} \quad x \in [\xi - \varepsilon, \xi + \varepsilon],$$

and $\phi \circ u' - \phi \circ \alpha$ is decreasing on $[\xi - \varepsilon, \xi + \varepsilon]$, and vanishes at ξ. This easily implies that $(u - \alpha)' < 0$ on $]\xi, \xi + \varepsilon]$ and $(u - \alpha)' > 0$ on $[\xi - \varepsilon, \xi[$, a contradiction with $u - \alpha$ reaching a minimum. □

Remark 1. If α and β are respectively T-periodic lower and upper solutions of (18) such that $\alpha(x) < \beta(x)$ for all $x \in \mathbb{R}$, a similar proof shows that each possible T-periodic solution of (22) is such that

$$\alpha(x) < \beta(x) \quad (x \in \mathbb{R}). \tag{24}$$

The following result will be useful in proving the existence of a T-periodic solution of the modified equation.

Lemma 2. *Given $t^* \in \mathbb{R}$, there exist $R, R' > 0$ such that, for each $\lambda \in [0,1]$, each t with $|t| \leq t^*$ and each possible T-periodic solution of*

$$(\phi(u'))' - [\phi(u) - \lambda\phi[r(x,u)]] + \lambda f[r(x,u)]u' + \lambda g[x, r(x,u)] = \lambda t \tag{25}$$

one has

$$\|u\|_\infty < R, \quad \|u'\|_\infty < R'. \tag{26}$$

Proof. Let $\lambda \in [0,1]$ and u be a possible T-periodic solution of (25). If we multiply both members of (25) by u, integrate over $[0,T]$ and use integration by parts and the T-periodicity, we get

$$\begin{aligned} &- \|u'\|_p^p - \|u\|_p^p - \frac{\lambda}{T}\int_0^T u(x)\phi(r(x,u(x)))\,dx \\ &\qquad + \frac{\lambda}{T}\int_0^T u(x) f(r(x,u(x))u'(x)\,dx + \frac{\lambda}{T}\int_0^T u(x) g(x, r(x,u(x))\,dx \\ &\quad = \frac{\lambda}{T}\int_0^T u(x) t\,dx. \end{aligned}$$

Hence, for some constants $M,\ M'$ we have, using the Hölder inequality,

$$\|u'\|_p^p + \|u\|_p^p \leq M\|u\|_p\|u'\|_p + M'\|u\|_p + |t^*|\,\|u\|_p.$$

This easily implies the existence of $S = S(t^*)$ and $S' = S'(t^*)$ such that

$$\|u\|_p + \|u'\|_p < S, \quad \|u\|_\infty < S'. \tag{27}$$

Now, there exists ξ such that $u'(\xi) = 0$, so that, integrating (25) between ξ and x we obtain

$$\begin{aligned} \phi(u'(x)) + \int_\xi^x [\phi(u(s)) - \phi(r(s,u(s)) + \lambda f(r(s,u(s))u'(s) + \lambda g(s, r(s,u(s))]\,ds \\ = \int_\xi^x t\,ds. \end{aligned}$$

Hence, using (27) we get $|u'(x)|^{p-1} < S''$ for all $x \in [0,T]$ and some $S'' = S''(t^*)$. □

Lemma 3. *For each $h \in C_T$ there exists a unique T-periodic solution u of*

$$(\phi(u'))' - \phi(u) = h(x). \tag{28}$$

Furthermore, the mapping

$$\mathcal{H} : C_T \to C_T^1, \quad h \mapsto u, \tag{29}$$

is completely continuous.

Proof. The existence of at least one T-periodic solution for (28) follows from Corollary 4.1 in [10] and the fact that -1 is not an eigenvalue of the p-Laplacian with periodic boundary conditions, or from Remark 2.1 in [12]. For the uniqueness, if u and v are T-periodic and such that

$$(\phi(u'))' - \phi(u) = h(x), \quad (\phi(v'))' - \phi(v) = h(x),$$

then

$$(\phi(u') - \phi(v'))' - [\phi(u) - \phi(v)] = 0. \tag{30}$$

Now, it is easily checked that, for all $r, s \in \mathbb{R}$, one has

$$[\phi(r) - \phi(s)](r - s) \geq (|r|^{p-1} - |s|^{p-1})(|r| - |s|)$$

and hence, integrating (30), we obtain

$$0 \geq \int_0^T (|u'|^{p-1} - |v'|^{p-1})(|u'| - |v'|) + \int_0^T (|u|^{p-1} - |v|^{p-1})(|u| - |v|) \geq 0.$$

Consequently, for all $x \in \mathbb{R}$,

$$|u'(x)| = |v'(x)|, \quad |u(x)| = |v(x)|. \tag{31}$$

Hence (30) can be written as

$$(|u'|^{p-2}(u' - v'))' - |u|^{p-2}(u - v) = 0,$$

which gives, by integration after multiplication by $u - v$,

$$\int_0^T [|u'|^{p-2}(u' - v')^2 + |u|^{p-2}(u - v)^2] = 0,$$

and hence, together with (31), implies that $u = v$. Now it follows from an argument analogous to the one used in the proof of Lemma 2 that

$$-\|u'\|_p^p - \|u\|_p^p = \frac{1}{T}\int_0^T u(x)h(x)\,dx,$$

so that by the Hölder inequality,

$$\|u'\|_p^p + \|u\|_p^p \leq \|h\|_\infty [\|u'\|_p^p + \|u\|_p^p]^{1/p},$$

which gives

$$\|u\|_p \leq \|h\|_\infty^{1/p-1}, \quad \|u'\|_p \leq \|h\|_\infty^{1/p-1}, \tag{32}$$

and hence, for some constant C depending only upon T,

$$\|u\|_\infty \leq C\|h\|_\infty^{1/p-1}. \tag{33}$$

Now, there exists ξ such that $u'(\xi) = 0$, so that integrating (28) between ξ and x we obtain

$$\phi(u'(x)) + \int_\xi^x \phi(u(s))\,ds = \int_\xi^x h(s)\,ds,$$

and hence, for all $x \in [0, T]$,

$$\|u'(x)\|^{p-1} \leq T(C^{p-1} + 1)\|h\|_\infty,$$

which gives, for some constant C' only depending upon T,

$$\|u'\|_\infty \leq C'\|h\|_\infty^{1/p-1}. \tag{34}$$

Let (h_n) be a sequence in C_T such that

$$\|h_n\|_\infty \leq R \tag{35}$$

for all $n \geq 1$ and some $R > 0$. Let $u_n := \mathcal{H}(h_n)$. From relations (33), (34) and Ascoli–Arzelà's theorem, we can assume, passing to a subsequence if necessary, that $u_n \to u \in C_T$ uniformly on $\mathbb{R}$. Now, if $\xi_n \in [0, T]$ is such that $u_n'(\xi_n) = 0$, we have, for all $x \in [0, T]$,

$$\phi(u_n'(x)) = -\int_{\xi_n}^{x} \phi(u_n(s))\,ds + \int_{\xi_n}^{x} h_n(s)\,ds, \tag{36}$$

and, from relations (33), (35) and Ascoli–Arzelà's theorem, we can assume, passing to a subsequence if necessary, that the right-hand member of (36) converges to some $z \in C_T$ uniformly on $[0, T]$. Consequently, (u_n') converges uniformly on $[0, T]$ to $\phi^{-1}(z)$, and so $\mathcal{H}$ is completely continuous. □

Define

$$\mathcal{G}_t : C_T^1 \to C_T, \quad u \mapsto -\phi(u) - f(u)u' - g(\cdot, u) + t, \tag{37}$$

and, for $\alpha, \beta \in C_T$ such that $\alpha(x) < \beta(x)$ for all $x \in \mathbb{R}$, and $R' > 0$, define the open bounded set $\Omega \subset C_T^1$ by

$$\Omega := \{u \in C_T^1 : \alpha(x) < u(x) < \beta(x),\ -R' < u'(x) < R'\ (x \in \mathbb{R})\}. \tag{38}$$

Proposition 1. *If* (18) *has T-periodic lower and upper solutions α, β such that $\alpha(x) \leq \beta(x)$ for all $x \in \mathbb{R}$, then it has a T-periodic solution u such that $\alpha(x) \leq u(x) \leq \beta(x)$ for all $x \in \mathbb{R}$. Furthermore, if α and β are strict and if $\alpha(x) < \beta(x)$ for all $x \in \mathbb{R}$, then*

$$d_{\mathrm{LS}}[I - \mathcal{H}\mathcal{G}_t, \Omega, 0] = 1. \tag{39}$$

Proof. By Lemma 1, the existence conclusion follows from the existence of a T-periodic solution to (22). Let

$$\widehat{\Omega} := \{u \in C_T^1 : \|u\|_\infty < R,\ \|u'\|_\infty < R'\}$$

where R and R' are given by Lemma 2, and let

$$\widehat{\mathcal{G}} : C_T^1 \times [0, 1] \to C_T, \quad (u, \lambda) \mapsto -\lambda\phi(r(\cdot, u)) - \lambda f(r(\cdot, u))u' - \lambda g(\cdot, u) + \lambda t.$$

It is clear from Lemma 3 that the T-periodic solutions of (22) are the fixed points of $\mathcal{H}\widehat{\mathcal{G}}(\cdot, 1)$ in C_T^1. The homotopy invariance of the Leray–Schauder degree gives

$$d_{\mathrm{LS}}[I - \mathcal{H}\widehat{\mathcal{G}}(\cdot, 1), \widehat{\Omega}, 0] = d_{\mathrm{LS}}[I - \mathcal{H}\widehat{\mathcal{G}}(\cdot, 0), \widehat{\Omega}, 0] = d_{\mathrm{LS}}[I, \widehat{\Omega}, 0] = 1,$$

and the excision property of the Leray–Schauder degree gives

$$d_{\mathrm{LS}}[I - \mathcal{H}\widehat{\mathcal{G}}(\cdot, 1), \widehat{\Omega}, 0] = d_{\mathrm{LS}}[I - \mathcal{H}\widehat{\mathcal{G}}(\cdot, 1), \Omega, 0] = d_{\mathrm{LS}}[I - \mathcal{H}\mathcal{G}, \Omega, 0]. \quad □$$

3. Existence of the first solution

Assume now that

$$g(x,u) \to +\infty \quad \text{as } |u| \to \infty, \text{ uniformly in } x \in \mathbb{R}. \tag{40}$$

Let

$$\sigma := \min_{u\in\mathbb{R},\, x\in\mathbb{R}} g(x,u). \tag{41}$$

Lemma 4. *If condition* (40) *holds, then there exists* $t_1 \geq \sigma$ *such that* (18) *has no* T*-periodic solution if* $t < t_1$ *and at least one* T*-periodic solution if* $t > t_1$.

Proof. We first notice that, for $t \geq t^* := \max_{\mathbb{R}} g(x,0)$, 0 is an upper solution for (18) (a strict upper solution if $t > t^*$). Given $t \geq t^*$, it follows from condition (40) that there exists $R_t > 0$ such that

$$g(x,u) > t \quad \text{whenever} \quad |u| \geq R_t,\ x \in \mathbb{R},$$

so that $-R_t$ (or any smaller number) is a strict lower solution for (18). Hence, from Proposition 1, for each $t \geq t^*$, this equation has at least one T-periodic solution such that $-R_t < u(x) < 0$ for all $x \in \mathbb{R}$. Let us now show that if (18) has a T-periodic solution $\widehat{u}$ for some $\widehat{t} < t^*$, then it has a T-periodic solution for all $t \in [\widehat{t}, t^*]$. Indeed, for such a t, we have

$$(\phi(\widehat{u}'(x)))' + f(\widehat{u}(x))\widehat{u}'(x) + g(x,\widehat{u}(x)) = \widehat{t} \leq t,$$

which shows that $\widehat{u}$ is an upper solution for (18). Furthermore, by the reasoning above, there exists $R_t > -\min_{\mathbb{R}} \widehat{u}$ such that $\min_{x\in\mathbb{R}} g(x,-R_t) > t$ so that $-R_t < \min_{\mathbb{R}} \widehat{u}$ is a lower solution for (18). Again, this implies the existence of a T-periodic solution for (18). Consequently, the set of $t \in \mathbb{R}$ such that (18) has a T-periodic solution is an interval unbounded from above. Let

$$t_1 = \inf\{t \in \mathbb{R} : (18) \text{ has a } T\text{-periodic solution}\}. \tag{42}$$

We now show that (18) has no T-periodic solution for $t < \sigma$. Indeed, if u were a T-periodic solution of (18) for some $t < \sigma$, and if $u(\xi) = \min_{\mathbb{R}} u$, then $u'(\xi) = 0$, and

$$(\phi(u'(\xi)))' = t - g(\xi, u(\xi)) < \sigma - g(\xi, u(\xi)) \leq 0.$$

By continuity, there exists $\varepsilon > 0$ such that

$$(\phi(u'(x)))' < 0 \quad \text{for } x \in [\xi - \varepsilon, \xi + \varepsilon],$$

so that $\phi \circ u'$ is decreasing on $[\xi - \varepsilon, \xi + \varepsilon]$, and the same is true for u'. This contradicts the fact that u reaches its minimum at ξ. Consequently, $t_1 \geq \sigma$. □

4. A priori estimates

We now prove an a priori estimate for the possible T-periodic solutions of (18) when t is bounded from above.

Lemma 5. *For each $t_2 > t_1$, there exist $M(t_2) > 0$ and $N(t_2) > 0$ such that, for each $t \in [t_1, t_2]$ and each possible T-periodic solution u of* (18)*, one has*

$$\|u\|_\infty < M(t_2), \tag{43}$$

$$\|u'\|_\infty < N(t_2). \tag{44}$$

Proof. Let $t \in [t_1, t_2]$ and let u be a T-periodic solution of (18). Integrating both members of the equation over $[0, T]$ gives

$$\frac{1}{T}\int_0^T g(x, u(x))\,dx = t. \tag{45}$$

We deduce from (18) that

$$\widetilde{u}(\phi(u'))' + \widetilde{u}f(u)u' + \widetilde{u}g(x, u) = \widetilde{u}t,$$

which, integrated over $[0, T]$, gives, by the T-periodicity of u,

$$-\|u'\|_p^p + \frac{1}{T}\int_0^T g(x, u(x))\widetilde{u}(x)\,dx = 0,$$

and hence, using (45), with σ defined in (41),

$$\begin{aligned} \|u'\|_p^p &= \frac{1}{T}\int_0^T [g(x, u(x)) - \sigma]\widetilde{u}(x)\,dx \\ &\le \frac{1}{T}\int_0^T [g(x, u(x)) - \sigma]|\widetilde{u}(x)|\,dx \\ &\le \|\widetilde{u}\|_\infty(t - \sigma) \le \|\widetilde{u}\|_\infty(t_2 - \sigma). \end{aligned} \tag{46}$$

Now, if ξ is such that $\widetilde{u}(\xi) = 0$, we have for each $x \in \mathbb{R}$, using the Hölder inequality,

$$|\widetilde{u}(x)| = \left|\int_\xi^x u'(s)\,ds\right| \le T\|u'\|_p,$$

so that (46) implies that

$$\|u'\|_p \le [T(t_2 - \sigma)]^{1/(p-1)}. \tag{47}$$

Now, there exists $R_2 > 0$ such that $g(x, u) > t_2$ whenever $|u| \ge R_2$ and $x \in \mathbb{R}$. Consequently, if $|u(x)| \ge R_2$ for all $x \in \mathbb{R}$, we have, by (45),

$$t = \frac{1}{T}\int_0^T g(x, u(x)\,dx > t_2,$$

which is impossible. Hence $|u(\xi)| < R_2$ for some $\xi \in \mathbb{R}$, which implies

$$\begin{aligned}|u(x)| &\leq |u(\xi)| + \left|\int_\xi^x u'(s)\,ds\right| < R_2 + T\|u'\|_p \\ &\leq R_2 + [T(t_2 - \sigma)]^{1/(p-1)} =: M(t_2).\end{aligned} \tag{48}$$

Now, there exists $\xi \in \mathbb{R}$ such that $u'(\xi) = 0$. If we set

$$F(u) = \int_0^u f(s)\,ds, \tag{49}$$

we can write (18) in the form

$$[\phi(u') + F(u)]' = t - g(x, u),$$

so that, integrating from ξ to x, we get

$$\phi(u'(x)) + F(u(x)) = F(u(\xi)) + \int_\xi^x [t - g(s, u(s))]\,ds,$$

which gives, by (48), for each $x \in \mathbb{R}$,

$$\begin{aligned}|\phi(u'(x))| &\leq 2 \max_{|u| \leq M(t_2)} |F(u)| + \int_0^T |g(s, u(s)) - t|\,ds \\ &\leq 2 \max_{|u| \leq M(t_2)} |F(u)| + \int_0^T [|g(s, u(s)) - \sigma| + |\sigma - t|]\,ds \\ &\leq 2 \max_{|u| \leq M(t_2)} |F(u)| + T[(t - \sigma) + |\sigma - t|] \\ &\leq 2[\max_{|u| \leq M(t_2)} |F(u)| + T(t_2 - \sigma)] := S(t_2),\end{aligned}$$

and this immediately yields (44) for any $N(t_2) > [S(t_2)]^{1/(p-1)}$. □

This result allows us to prove the existence of at least one solution for $t = t_1$.

Lemma 6. *If condition* (40) *holds, then* (18) *has at least one T-periodic solution for $t = t_1$.*

Proof. Let (τ_k) be a sequence in $]t_1, +\infty[$ which converges to t_1, and let u_k be a T-periodic solution of (18) with $t = \tau_k$ given by Lemma 4. From Lemma 5, we know that, for all $k \geq 1$,

$$\|u_k\|_\infty < M(t_2), \quad \|u_k'\|_\infty < N(t_2), \tag{50}$$

and from Lemma 3 that, for all $k \geq 1$,

$$u_k = \mathcal{H}\mathcal{G}_{\tau_k}(u_k). \tag{51}$$

Conditions (50) and the complete continuity of $\mathcal{H}$ imply that, up to a subsequence, the right-hand member of (51) converges in C^1_T, and then (u_k) converges to some $u \in C^1_T$ such that $u = \mathcal{H}\mathcal{G}_{t_1}(u)$, i.e. to a T-periodic solution of (18). □

5. Existence of two solutions

Define

$$B(R, R') := \{u \in C^1_T : \|u\|_\infty < R,\ \|u'\|_\infty < R'\}.$$

Lemma 7. *If condition* (40) *holds, then, for each* $t > t_1$, (18) *has at least two* T*-periodic solutions.*

Proof. Let $t_2 > t_1$ and let $t \in [t_1, t_2]$. As (18) has no T-periodic solution for $t < t_1$, we have, for all $t \le t_2$, using Lemma 5,

$$d_{\mathrm{LS}}[I - \mathcal{H}\mathcal{G}_t, B(M(t_2), N(t_2)), 0] = 0. \tag{52}$$

By the reasoning in the proof of Lemma 4, there exists $R_{t_2} > 0$ such that

$$\min_{x\in\mathbb{R}} g(x, -R_{t_2}) > t_2,$$

and hence

$$\min_{x\in\mathbb{R}} g(x, -R_{t_2}) > t \qquad \text{for all } t \le t_2.$$

Thus, $-R_{t_2}$ is a strict T-periodic lower solution for (18) whenever $t \le t_2$. On the other hand, a T-periodic solution u_1 of (18) with $t = t_1$ is such that

$$(\phi(u_1'(x)))' + f(u_1(x))u_1'(x) + g(x, u_1(x)) = t_1 < t,$$

and is a strict T-periodic upper solution of (18). We can of course always increase $M(t_2)$ in such a way that

$$-M(t_2) < -R_{t_2} < u_1(x) < M(t_2)$$

for all $x \in \mathbb{R}$. Hence, if Ω_1 is the open bounded subset of $B(M(t_2), N(t_2))$ defined by

$$\{u \in C^1_T : -R_{t_2} < u(x) < u_1(x),\ -N(t_2) < u'(x) < N(t_2)\ (x \in \mathbb{R})\},$$

it follows from Proposition 1 that (18) has at least one T-periodic solution in Ω_1, and that $d_{\mathrm{LS}}[I - \mathcal{H}\mathcal{G}_t, \Omega_1, 0] = 1$. The excision property of the Leray–Schauder degree and (52) give, for $t \in]t_1, t_2]$,

$$\begin{aligned} &d_{\mathrm{LS}}[I - \mathcal{H}\mathcal{G}_t, B(M(t_2), N(t_2)) \setminus \overline{\Omega}_1, 0] \\ &\qquad = d_{\mathrm{LS}}[I - \mathcal{H}\mathcal{G}_t, B(M(t_2), N(t_2)), 0] - d_{\mathrm{LS}}[I - \mathcal{H}\mathcal{G}_t, \Omega_1, 0] = -1, \end{aligned}$$

which implies the existence of a T-periodic solution of equation (18) contained in $B(M(t_2), N(t_2)) \setminus \overline{\Omega}_1$. As $t_2 > t_1$ is arbitrary, the proof is complete. ◻

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