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C. A. Stuart

# Uniqueness and stability of ground states for some nonlinear Schrödinger equations

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**Abstract.** We discuss the orbital stability of standing waves of a class of nonlinear Schrödinger equations in one space dimension. The crucial feature for our treatment is the presence of a non-constant linear potential that is even and decreasing away from the origin in space. This enables us to establish the orbital stability of all ground states over the whole range of frequencies for which such solutions exist.

## 1. Introduction

Standing waves are simple time harmonic solutions of the nonlinear Schrödinger equation (NLS) that decay at infinity in space. Ground states are defined as standing waves that minimize the action with respect to other standing waves of the same frequency. This paper is concerned with a class of nonlinear Schrödinger equations for which we can give a complete description of all ground states including their stability.

To be more precise, consider a function $\Phi : \mathbb{R}^2 \to \mathbb{C}$ that satisfies the nonlinear Schrödinger equation

$$i\partial_t \Phi + \partial_{xx}^2 \Phi + V(x)\Phi + g(|\Phi|^2)\Phi = 0 \quad \text{for } (t,x) \in \mathbb{R}^2 \tag{NLS}$$

where $V : \mathbb{R} \to \mathbb{R}$ is the potential and the function $g : [0,\infty) \to \mathbb{R}$ defines the nonlinearity. We are interested in solutions such that $\Phi(t,\cdot) \in H = H^1(\mathbb{R},\mathbb{C})$ for all $t \in \mathbb{R}$. To formulate the hypotheses on the smoothness of the nonlinearity we set

$$f(s) = g(s^2)s \quad \text{for } s \in \mathbb{R} \tag{1.1}$$

and assume throughout that

$$\begin{aligned} &\text{(Hi)} \quad V \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}), \\ &\text{(Hii)} \quad f \in C^1(\mathbb{R}) \text{ with } f(0) = f'(0) = 0. \end{aligned}$$

Noting that $\Phi$ satisfies (NLS) if and only if $\Psi(t,x) = e^{i\omega t}\Phi(t,x)$ satisfies

$$i\partial_t \Psi + \partial_{xx}^2 \Psi + [V(x) + \omega]\Psi + g(|\Psi|^2)\Psi = 0 \quad \text{for } (t,x) \in \mathbb{R}^2$$

C. A. Stuart: Station 8, IACS-FSB, EPFL, CH-1015 Lausanne, Switzerland;
e-mail: charles.stuart@epfl.ch

the hypothesis (Hi) enables us to assume without loss of generality that

$$\inf_{x\in\mathbb{R}} V(x) = 0. \tag{1.2}$$

Let $\Lambda$ denote the infimum of the spectrum of the self-adjoint operator $-\partial_{xx}^2 - V : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Then

$$\Lambda = \inf\left\{ \frac{\int_{-\infty}^{\infty} (|z'|^2 - V|z|^2)\, dx}{\int_{-\infty}^{\infty} |z|^2\, dx} : z \in H \setminus \{0\} \right\} \le 0$$

and $\Lambda = 0$ if and only if $V \equiv 0$.

Solutions of (NLS) of the form

$$\Phi(t,x) = e^{-i\lambda t} z(x) \quad \text{where } z \not\equiv 0$$

for some $\lambda \in \mathbb{R}$ and $z \in H$ are called *standing waves* and, for such $z$, the orbit $\Theta(z) \subset H$ of the associated standing wave is defined by

$$\Theta(z) = \{e^{it} z : t \in \mathbb{R}\} = \Theta(e^{i\theta} z) \quad \text{for all } \theta \in \mathbb{R}.$$

For standing waves, (NLS) is equivalent to

$$z \in H \setminus \{0\} \quad \text{and} \quad \lambda z + z'' + Vz + g(|z|^2)z = 0 \quad \text{in } H^{-1} \tag{1.3}$$

and in Section 2 we begin by formulating hypotheses ensuring the existence of ground states. Note that if $z$ satisfies (1.3) then so does $\overline{z}$. We show that, if $e^{-i\lambda t} z(x)$ is a ground state, then there exists a real-valued, strictly positive solution $u$ of (1.3) such that $\Theta(z) = \Theta(u)$, and consequently $\Theta(z) = \Theta(\overline{z})$. Therefore, in Section 3, we focus on the problem

$$\lambda u + u'' + Vu + g(u^2)u = 0 \quad \text{where } u \in H^1(\mathbb{R}) \text{ with } u > 0 \text{ on } \mathbb{R} \tag{1.4}$$

and review some joint work with Hélène Jeanjean [9], in which we were able to show that all solutions of (1.4) form a smooth curve $C = \{(\lambda, U(\lambda)) : \lambda < \Lambda\}$ in $\mathbb{R} \times H^2(\mathbb{R})$ with $\lim_{\lambda\to-\infty} \|U(\lambda)\|_{H^2} = \infty$ and $\lim_{\lambda\to\Lambda} \|U(\lambda)\|_{H^2} = 0$. In view of what is proved in Section 2 this result gives a complete description of all ground states for (NLS).

In Section 4 we consider the stability of these ground states starting from the general criteria established by Grillakis, Shatah and Strauss [8]. A crucial condition is the monotonicity of $\|U(\lambda)\|_{L^2}$ with respect to $\lambda$. Under the hypotheses used in Section 3 to obtain the curve $C$, this monotonicity need not hold and some of the ground states can be unstable. In collaboration with J. B. McLeod and W. C. Troy [12] we have found additional conditions on $g$ that ensure that $\frac{d}{d\lambda}\|U(\lambda)\|_{L^2} < 0$ for all $\lambda < \Lambda$ and consequently that all ground states are stable.

The results from [9] and [12] that we have recalled here are proved in greater generality. We have chosen the special form $V(x)\Phi + g(|\Phi|^2)\Phi$ in (NLS) in order to state the hypotheses briefly, but our conclusions are available in a broader context. On the other hand, in higher dimensions, where $x \in \mathbb{R}^N$ with $N \ge 2$, and even for $V(x)\Phi + g(|\Phi|^2)\Phi$ when $V$ is not constant, there does not seem to be a proof of the stability of all ground

states. In [13] and [8], perturbation methods are used to establish stability for $\lambda$ near $\Lambda$ and $\lambda$ near $-\infty$, respectively. For $V$ constant, there is a complete discussion of the stability of all ground states, but the definition of orbit, and hence of stability, has to be modified to accommodate the invariance under translation. These comments also apply to the variational approach to the stability of standing waves initiated by Cazenave and P.-L. Lions.

## 2. Existence of ground states

Amongst all standing waves, those called ground states are most likely to be stable. They are defined as follows. For $\lambda \in \mathbb{R}$, set

$$A_\lambda = \{z \in H \setminus \{0\} : \lambda z + z'' + Vz + g(|z|^2)z = 0 \text{ in } H^{-1}\}$$

and let

$$S(z) = \int_{-\infty}^{\infty} (|z'|^2 - \lambda|z|^2 - V|z|^2 - G(|z|^2))\, dx \quad \text{where} \quad G(s) = \int_0^s g(\tau)\, d\tau$$

denote the action of the standing wave $\Phi^z(t,x) = e^{-i\lambda t} z(x)$ associated with $z \in A_\lambda$. Then both $z \in A_\lambda$ and the associated standing wave $\Phi^z$ are referred to as *ground states* of (NLS) if $\lambda < \Lambda$ and $S(z) \le S(w)$ for all $w \in A_\lambda$. Let

$$G_\lambda = \{z \in A_\lambda : S(z) \le S(w) \text{ for all } w \in A_\lambda\}$$

denote the set of all ground states with frequency $|\lambda|$. It turns out that the minimality of the action of a ground state also pertains to a much larger set that is sometimes referred to as the *Nehari manifold*. For $\lambda < \Lambda$, let

$$J(z) = \int_{-\infty}^{\infty} (|z'|^2 - \lambda|z|^2 - V|z|^2 - g(|z|^2)|z|^2)\, dx$$
$$N_\lambda = \{z \in H \setminus \{0\} : J(z) = 0\} \text{ and then } M_\lambda = \{z \in N_\lambda : S(z) \le S(w) \text{ for all } w \in N_\lambda\}.$$

Clearly, $A_\lambda$ is a subset of the Nehari manifold $N_\lambda$. The conditions (Hi) and (Hii) ensure that $S$ and $J$ are in $C^1(H, \mathbb{R})$ with

$$\begin{aligned} S'(z)w &= 2\,\mathrm{Re} \int_{-\infty}^{\infty} (z'\overline{w'} - \{\lambda + V + g(|z|^2)\} z\overline{w})\, dx \\ J'(z)w &= 2\,\mathrm{Re} \int_{-\infty}^{\infty} (z'\overline{w'} - \{\lambda + V + g(|z|^2) + g'(|z|^2)\} z\overline{w})\, dx \\ &= S'(z)w - 2\,\mathrm{Re} \int_{-\infty}^{\infty} g'(|z|^2) z\overline{w}\, dx \qquad \text{for all } z, w \in H. \end{aligned}$$

Thus $S'(z)z = 2J(z)$ and $J'(z)z = 2J(z) - 2\int_{-\infty}^{\infty} g'(|z|^2)|z|^2\, dx$. Note also that

$$S'(z) = 0 \quad \text{if and only if} \quad z \in A_\lambda \cup \{0\}.$$

**Lemma 2.1.** *Suppose that* (Hi) *and* (Hii) *are satisfied and consider* $\lambda < \Lambda$.

(i) *Then* $A_\lambda \subset H^2(\mathbb{R}, \mathbb{C}) \cap C^2(\mathbb{R}, \mathbb{C})$ *and*

$$z'' + \{\lambda + V + g(|z|^2)\}z = 0 \quad \text{on } \mathbb{R} \text{ for } z \in A_\lambda.$$

(ii) *If* $z \in A_\lambda$, *then* $z$ *has only simple zeros.*

(iii) *If, in addition,*

$$\text{(Hiii)} \quad f'(s) > \frac{f(s)}{s} \ [\text{equivalently } g'(s) > 0] \quad \text{for all } s > 0,$$

*then* $M_\lambda \subset G_\lambda$ *and* $u = |z| \in M_\lambda$ *with* $u > 0$ *whenever* $z \in M_\lambda$. *Furthermore there exists* $\theta \in [0, 2\pi)$ *such that* $z(x) = e^{i\theta}u(x)$ *for all* $x \in \mathbb{R}$.

(iv) *If* (Hi) *to* (Hiii) *hold and* $M_\lambda \neq \emptyset$, *then* $M_\lambda = G_\lambda$.

*Proof.* (i) $H \subset L^\infty \cap C$ and so $z'' = -\{\lambda + V + g(|z|^2)\}z \in L^2 \cap C$.

(ii) If $z \in A_\lambda \cup \{0\}$ and $z(y) = z'(y) = 0$ for some $y$, then $z \equiv 0$ by the uniqueness of the solution to the initial value problem for the equation (1.3).

(iii) For any $z \in N_\lambda$, we have

$$J'(z)z = 2J(z) - 2\int_{-\infty}^{\infty} g'(|z|^2)|z|^2\,dx = -2\int_{-\infty}^{\infty} g'(|z|^2)|z|^2\,dx < 0.$$

Thus, if $z \in M_\lambda$, there is a Lagrange multiplier $\xi$ such that $S'(z) = \xi J'(z)$ and hence $S'(z)z = \xi J'(z)z$. But $S'(z)z = 2J(z) = 0$ and $J'(z)z < 0$ so we must have $\xi = 0$, showing that $S'(z) = 0$. Thus $M_\lambda \subset G_\lambda$.

If $z \in H$, then $u \in H$ with $S(u) = S(z)$ and $J(u) = J(z)$. Therefore, if $z \in M_\lambda$ we see that $u \in M_\lambda \subset G_\lambda$. But then $u \in C^2(\mathbb{R})$ and has only simple zeros by parts (i) and (ii). Since $u = |z| \geq 0$, it follows that in fact $u > 0$ on $\mathbb{R}$. We have

$$\lambda z + z'' + Vz + g(|z|^2)z = 0,$$
$$\lambda u + u'' + Vu + g(u^2)u = 0,$$

since $z$ and $u \in G_\lambda \subset A_\lambda$. Hence $z''u - zu'' = 0$ on $\mathbb{R}$ and so there is a constant $C$ such that $z'u - zu' = C$ on $\mathbb{R}$. But $z, u \in H^2$ by part (i) and therefore $z, z', u$ and $u'$ all tend to zero as $x \to \infty$ so $C = 0$. This means that $v = z/u$ is also constant on $\mathbb{R}$ and since $|v| = 1$, there exists $\theta \in [0, 2\pi)$ such that $v = e^{i\theta}$.

(iv) Suppose that $\widehat{z} \in M_\lambda$. Then $\widehat{z} \in G_\lambda$ and so $S(z) = S(\widehat{z})$ for any $z \in G_\lambda$. Since $G_\lambda \subset N_\lambda$ this means that $z \in M_\lambda$ and so $G_\lambda \subset M_\lambda$. □

We now give conditions ensuring that $M_\lambda \neq \emptyset$.

**Remark.** Under the hypotheses of Lemma 2.1, it can happen that $A_\lambda$ and hence $G_\lambda$ is empty for all $\lambda$. For example, if in addition $V \in C^1(\mathbb{R})$ with $V' > 0$ and $z \in A_\lambda$, then

$$\frac{d}{dx}\{\lambda|z|^2 + |z'|^2 + V|z|^2 + G(|z|^2)\} = 2\,\mathrm{Re}[\{\lambda z + z'' + Vz + g(|z|^2)z\}\overline{z'}] + V'|z|^2 = V'|z|^2$$

and so $\lambda|z|^2 + |z'|^2 + V|z|^2 + G(|z|^2)$ is nondecreasing on $\mathbb{R}$. But $z$ and $z'$ tend to zero at infinity because $z \in H^2$ and so $\lambda|z|^2 + |z'|^2 + V|z|^2 + G(|z|^2) \equiv 0$. Thus $V'|z|^2 \equiv 0$ and $z \equiv 0$, a contradiction. Thus $A_\lambda = \emptyset$ if $V' > 0$ on $\mathbb{R}$.

**Proposition 2.2.** *Suppose that the assumptions* (Hi) *to* (Hiii) *are satisfied and that*

(Hiv) *$V$ is even and nonincreasing on* $[0, \infty)$ *with* $\lim_{x\to\infty} V(x) = 0$.

(Hv) $\lim_{s\to\infty} g(s) = \infty$.

*Then $M_\lambda \neq \emptyset$ for every $\lambda < \Lambda$.*

*Proof.* (a) *A norm on $H$ and its properties.* Fix $\lambda < \Lambda$ and consider $d(z) = \int_{-\infty}^{\infty}(|z'|^2 - \lambda|z|^2 - V|z|^2)\,dx$. For any $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}
d(z) &= \varepsilon \int_{-\infty}^{\infty} |z'|^2\,dx + (1-\varepsilon)\int_{-\infty}^{\infty}(|z'|^2 - V|z|^2)\,dx - \int_{-\infty}^{\infty}(\lambda + \varepsilon V)|z|^2\,dx \\
&\geq \varepsilon \int_{-\infty}^{\infty} |z'|^2\,dx + (1-\varepsilon)\Lambda\int_{-\infty}^{\infty}|z|^2\,dx - \int_{-\infty}^{\infty}(\lambda + \varepsilon V)|z|^2\,dx \\
&= \varepsilon \int_{-\infty}^{\infty} |z'|^2\,dx + \int_{-\infty}^{\infty}\{(1-\varepsilon)(\Lambda - \lambda) - \varepsilon(V + \lambda)\}|z|^2\,dx \\
&\geq \mu(\varepsilon)\int_{-\infty}^{\infty}(|z'|^2 + |z|^2)\,dx,
\end{aligned}$$

where $\mu(\varepsilon) = \min\{\varepsilon, (1-\varepsilon)(\Lambda - \lambda) - \varepsilon(V(0) + \lambda)\} > 0$ for $\varepsilon$ small enough. Hence

$$\|z\| = \left\{\int_{-\infty}^{\infty}(|z'|^2 - \lambda|z|^2 - V|z|^2)\,dx\right\}^{1/2}$$

defines a norm on $H$ which is equivalent to the usual norm

$$\|z\|_1 = \left\{\int_{-\infty}^{\infty}(|z'|^2 + |z|^2)\,dx\right\}^{1/2}.$$

For $z \in H$ we have $u = |z| \in H$ and we use $z^*$ to denote the Schwarz symmetrization (see [11]) of $|z|$. Then $z^* = u^* \in H$ with

$$\begin{aligned}
&\int_{-\infty}^{\infty}|z|^2\,dx = \int_{-\infty}^{\infty}u^2\,dx = \int_{-\infty}^{\infty}(u^*)^2\,dx = \int_{-\infty}^{\infty}(z^*)^2\,dx, \\
&\int_{-\infty}^{\infty}|z'|^2\,dx \geq \int_{-\infty}^{\infty}(u')^2\,dx \geq \int_{-\infty}^{\infty}([u^*]')^2\,dx = \int_{-\infty}^{\infty}([z^*]')^2\,dx, \\
&\int_{-\infty}^{\infty}V|z|^2\,dx = \int_{-\infty}^{\infty}Vu^2\,dx \leq \int_{-\infty}^{\infty}V^*(u^*)^2\,dx = \int_{-\infty}^{\infty}V(z^*)^2\,dx.
\end{aligned}$$

It follows from these inequalities that

$$\|z^*\| \leq \|z\| \quad \text{for all } z \in H.$$

(b) *Minimizing $S$ on $N_\lambda$.* For $z \in H$,

$$S(z) = \|z\|^2 - \int_{-\infty}^{\infty} G(|z|^2)\,dx \quad \text{and} \quad J(z) = \|z\|^2 - \int_{-\infty}^{\infty} g(|z|^2)|z|^2\,dx.$$

For $z \in N_\lambda$,

$$S(z) = \int_{-\infty}^{\infty} g(|z|^2)|z|^2 - G(|z|^2)\,dx = \int_{-\infty}^{\infty} h(|z|^2)\,dx \quad \text{where} \quad h(s) = g(s)s - G(s).$$

Noting that $h(0) = 0$ and $h'(s) = g'(s)s > 0$ for $s > 0$, we see that $S(z) > 0$ for all $z \in N_\lambda$, so setting $m_\lambda = \inf\{S(z) : z \in N_\lambda\}$ we have $m_\lambda \geq 0$. For $z \in H \setminus \{0\}$ and $t > 0$, let

$$k(t) = \frac{J(tz)}{t^2} = \|z\|^2 - \int_{-\infty}^{\infty} g(t^2|z|^2)|z|^2\,dx.$$

Using (Hiii) and (Hv), we find that $k$ is strictly decreasing on $(0, \infty)$ with

$$\lim_{t\to 0} k(t) = \|z\|^2 > 0 \quad \text{and} \quad \lim_{t\to\infty} k(t) = -\infty.$$

Hence there exists a unique $t(z) \in (0, \infty)$ such that $k(t(z)) = 0$ and $t(z)z \in N_\lambda$. Furthermore,

$$\frac{d}{dt}S(tz) = 2t\left\{\|z\|^2 - \int_{-\infty}^{\infty} g(t^2|z|^2)|z|^2\,dx\right\} = 2tk(t)$$

where $k(t) > 0$ for $t \in (0, t(z))$ and $k(t) < 0$ for $t > t(z)$. Thus we have

$$S(tz) < S(t(z)z) \quad \text{for } t \in (0, t(z)) \quad \text{and} \quad S(tz) < S(t(z)z) \quad \text{for } t > t(z).$$

For $z \in N_\lambda$,

$$S(t(z^*)z^*) = \int_{-\infty}^{\infty} h(t(z^*)^2|z^*|^2)\,dx$$

and

$$0 = J(z) = \|z\|^2 - \int_{-\infty}^{\infty} g(|z|^2)|z|^2\,dx \geq \|z^*\|^2 - \int_{-\infty}^{\infty} g(|z^*|^2)|z^*|^2\,dx$$

so that $t(z^*) \leq 1$ and

$$S(t(z^*)z^*) = \int_{-\infty}^{\infty} h(t(z^*)^2|z^*|^2)\,dx \leq \int_{-\infty}^{\infty} h(|z^*|^2)\,dx = \int_{-\infty}^{\infty} h(|z|^2)\,dx = S(z).$$

Hence if $\{z_n\} \subset N_\lambda$ is a sequence such that $S(z_n) \to m_\lambda$, by setting $w_n = t(z_n^*)z_n^*$, we obtain a sequence $\{w_n\} \subset N_\lambda$ such that $w_n = w_n^*$ and $S(w_n) \to m_\lambda$.

(c) *Boundedness of a minimizing sequence.* Let us show that $\{w_n\}$ is bounded. Suppose that $\|w_n\| \to \infty$. Let $c = \sqrt{m_\lambda + 1}$ and then set

$$v_n = \frac{c}{\|w_n\|}w_n.$$

Since $w_n \in N_\lambda$, we have $S(v_n) \le S(w_n)$, and since $v_n = v_n^*$, for all $y > 0$,

$$\int_{-\infty}^{\infty} v_n^2\,dx \ge \int_{-y}^{y} v_n^2\,dx \ge 2y v_n^2(y).$$

But there exists a constant $K > 0$ such that

$$\int_{-\infty}^{\infty} z^2\,dx \le \|z\|_1^2 \le K\|z\|^2 \quad \text{for all } z \in H$$

and hence

$$v_n^2(y) \le \frac{Kc^2}{2y} \quad \text{for all } y > 0 \text{ and all } n \in \mathbb{N}.$$

Since $\{v_n\}$ is bounded in $H$, by passing to a subsequence, we may assume that there exists $v \in H$ such that $v_n \rightharpoonup v$ weakly in $H$. If $v \not\equiv 0$, there exist $\delta > 0$ and an interval $[a, b]$ with $a < b$ such that $v \ge \delta$ on $[a, b]$. Then there exists $n_0 \in \mathbb{N}$ such that $v_n \ge \delta/2$ on $[a, b]$ for all $n \ge n_0$ because $\{v_n\}$ converges uniformly to $v$ on $[a, b]$. But then, for $n \ge n_0$,

$$\int_a^b \frac{G(w_n^2(x))}{\|w_n\|^2}\,dx = \int_a^b \frac{G(\frac{\|w_n\|^2}{c^2} v_n(x)^2)}{\|w_n\|^2}\,dx \ge \int_a^b \frac{G(\frac{\|w_n\|^2\delta^2}{4c^2})}{\|w_n\|^2}\,dx$$

and

$$\lim_{n\to\infty} \int_a^b \frac{G(w_n^2(x))}{\|w_n\|^2}\,dx = \infty \quad \text{since} \quad \lim_{s\to\infty} \frac{G(s)}{s} = \infty \quad \text{by (Hv).}$$

On the other hand,

$$\int_a^b \frac{G(w_n^2(x))}{\|w_n\|^2}\,dx \le \int_{-\infty}^{\infty} \frac{G(w_n^2(x))}{\|w_n\|^2}\,dx = \frac{\|w_n\|^2 - S(w_n)}{\|w_n\|^2} \to 1,$$

since $S(w_n) \to m_\lambda$ and $\|w_n\| \to \infty$. Thus we must have $v \equiv 0$. Then, for all $y > 0$, we find that $v_n \to 0$ uniformly on $[-y, y]$ and

$$\begin{aligned}
\limsup_{n\to\infty} \int_{-\infty}^{\infty} G(v_n^2)\,dx &= \limsup_{n\to\infty} \int_{|x|\ge y} G(v_n^2)\,dx \le \limsup_{n\to\infty} \int_{|x|\ge y} g(v_n^2) v_n^2\,dx \\
&\le \limsup_{n\to\infty} g\left(\frac{Kc^2}{2y}\right) \int_{|x|\ge y} v_n^2\,dx \le g\left(\frac{Kc^2}{2y}\right) \int_{-\infty}^{\infty} v_n^2\,dx \\
&\le g\left(\frac{Kc^2}{2y}\right) Kc^2.
\end{aligned}$$

Letting $y \to \infty$, we find that

$$\int_{-\infty}^{\infty} G(v_n^2)\,dx \to 0 \quad \text{and so} \quad S(v_n) \to c^2 = m_\lambda + 1.$$

But we have seen that $S(v_n) \le S(w_n)$ and $S(w_n) \to m_\lambda$ and we again have a contradiction. This proves that $\{w_n\}$ is bounded in $H$.

(d) *Existence of a minimizer.* By passing to a subsequence we may now assume that there exists $w \in H$ such that $w_n \rightharpoonup w$ weakly in $H$. Let $B > 0$ be such that $\|w_n\| \le B$ for all $n$. Then, as in part (c),

$$w_n^2(y) \le \frac{KB^2}{2y} \quad \text{and} \quad w^2(y) \le \frac{KB^2}{2y} \quad \text{for all } y > 0 \text{ and all } n \in \mathbb{N}.$$

Hence

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} G(w_n^2)\,dx - \int_{-\infty}^{\infty} G(w^2)\,dx \right| \\
&\qquad \le \int_{-y}^{y} |G(w_n^2) - G(w^2)|\,dx + \int_{|x|\ge y} (g(w_n^2)w_n^2 + g(w^2)w^2)\,dx \\
&\qquad \le \int_{-y}^{y} |G(w_n^2) - G(w^2)|\,dx + g\left(\frac{KB^2}{2y}\right) \int_{-\infty}^{\infty} [w_n^2 + w^2]\,dx \\
&\qquad \le \int_{-y}^{y} |G(w_n^2) - G(w^2)|\,dx + g\left(\frac{KB^2}{2y}\right) 2KB^2
\end{aligned}$$

and so

$$\limsup_{n\to\infty} \left| \int_{-\infty}^{\infty} G(w_n^2)\,dx - \int_{-\infty}^{\infty} G(w^2)\,dx \right| \le g\left(\frac{KB^2}{2y}\right) 2KB^2$$

since $w_n \to w$ uniformly on $[-y, y]$. Letting $y \to \infty$, we see that

$$\int_{-\infty}^{\infty} G(w_n^2)\,dx \to \int_{-\infty}^{\infty} G(w^2)\,dx \quad \text{as } n \to \infty,$$

from which it follows that

$$S(w) \le \liminf_{n\to\infty} S(w_n) = m_\lambda.$$

A similar argument shows that

$$\int_{-\infty}^{\infty} g(w_n^2)w_n^2\,dx \to \int_{-\infty}^{\infty} g(w^2)w^2\,dx \quad \text{and} \quad J(w) \le \liminf_{n\to\infty} J(w_n) = 0.$$

Furthermore, $w \ne 0$ since otherwise

$$\|w_n\|^2 = \int_{-\infty}^{\infty} g(w_n^2)w_n^2\,dx \to \int_{-\infty}^{\infty} g(w^2)w^2\,dx = 0,$$

whereas

$$\int_{-\infty}^{\infty} w_n^2\,dx \le K\|w_n\|^2 = K\int_{-\infty}^{\infty} g(w_n^2)w_n^2\,dx \le Kg(\max w_n^2)\int_{-\infty}^{\infty} w_n^2\,dx$$

and so

$$1 \le Kg(\max w_n^2) \le Kg(\|w_n\|_1^2) \le Kg\left(\frac{\|w_n\|^2}{\mu(\varepsilon)}\right),$$

in contradiction with $\|w_n\| \to 0$. From our observation that $J(w) \le 0$ it follows that $t(w) \le 1$ and then

$$\begin{aligned} m_\lambda &\le S(t(w)w) = \int_{-\infty}^{\infty} h(t(w)^2 w^2)\,dx \le \int_{-\infty}^{\infty} h(w^2)\,dx \\ &= \int_{-\infty}^{\infty} (g(w^2)w^2 - G(w^2))\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} (g(w_n^2)w_n^2 - G(w_n^2))\,dx \\ &= \lim_{n\to\infty} S(w_n) = m_\lambda. \end{aligned}$$

Thus $t(w)w \in M_\lambda$ and the proof is complete. □

**Remark.** In fact, the proof yields some extra information. If $t(w) < 1$, we have $\int_{-\infty}^{\infty} h(t(w)^2 w^2)\,dx < \int_{-\infty}^{\infty} h(w^2)\,dx$ and then $m_\lambda < m_\lambda$, a contradiction. Hence $t(w) = 1$. This means that

$$\|w\|^2 = \int_{-\infty}^{\infty} g(w^2)w^2\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} g(w_n^2)w_n^2\,dx = \lim_{n\to\infty} \|w_n\|^2,$$

showing that the minimizing sequence $\{w_n\}$ converges strongly in $H$ to the minimizer $z$ which belongs to $N_\lambda$.

**Corollary 2.3.** *Under the assumptions* (Hi) *to* (Hv), *for each* $\lambda < \Lambda$, $M_\lambda = G_\lambda \ne \emptyset$, *and for any ground state* $z \in G_\lambda$ *there is a ground state* $u \in \Theta(z)$ *such that* $u > 0$.

The hypotheses (Hi) to (Hv) do not imply that there is a unique orbit of ground states with frequency $|\lambda|$. In fact, for $V \equiv 0$ and $z \in G_\lambda$ the translate $z_y = z(\cdot + y)$ clearly belongs to $G_\lambda$ for any $y \in \mathbb{R}$. But, if $z_y \in \Theta(z)$, there exists $t \in \mathbb{R}$ such that $z_y = e^{-i\lambda t} z$ and so $|z|$ is periodic with period $|y|$. Since $z \in H^1$, it follows that $z_y \in \Theta(z)$ if and only if $y = 0$.

Since the homogeneous case $V \equiv 0$ is well understood [4], we eliminate this situation and then, as we see in the next section, we do indeed have uniqueness of the orbits of ground states for a given frequency.

## 3. Uniqueness and properties of ground states

In this section we recall some results obtained in collaboration with Hélène Jeanjean [9] concerning the problem (1.4). In particular we showed that, for each $\lambda < \Lambda$, there is a unique solution $u_\lambda$. The case where $V$ is constant has to be excluded for this to hold. For convenience we restate the hypotheses that have been used so far as follows.

(V) $V$ is even and $V \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $V' \le 0$ on $(0, \infty)$ but $V' \not\equiv 0$.
(F) $f \in C^1(\mathbb{R})$ with $f(0) = f'(0) = 0$, $f'(s) > f(s)/s$ for all $s > 0$ and $\lim_{s\to\infty} f(s)/s = \infty$.

Compared to the conditions (Hi) to (Hv), the assumptions (V) and (F) simply require some extra regularity of $V$ and exclude the case where $V$ is constant. The function $f(s)/s$ is strictly increasing on $[0, \infty)$ so (F) implies that $f(s) > 0$ for all $s > 0$.

It follows from (V) and (1.2) that $-V(0) < \Lambda < 0$ and $\Lambda$ is a simple eigenvalue of $L = -\partial_{xx}^2 - V$ with an eigenfunction that is Schwarz symmetric.

**Theorem 3.1** (Theorem 1 of [9]). *Let* (V) *and* (F) *be satisfied and let* $(\lambda, u)$ *be a solution of* (1.4). *Then* $\lambda < \Lambda$ *and* $u$ *is even with* $u' < 0$ *on* $(0, \infty)$.

**Remark.** Both monotonicity and evenness of $V$ are required to get evenness of $u$. Akhmediev [2] was the first to show that evenness of $V$ does not imply that $u$ is even (or odd). There are now many other examples of this ([1], [3], [6], [7]).

Since all solutions of (1.4) are even it is enough to deal with the problem on $[0, \infty)$. Let

$$W = \{u \in H^2((0, \infty)) : u'(0) = 0\}$$

and $F : \mathbb{R} \times W \to L^2((0, \infty))$ where

$$F(\lambda, u) = u'' + Vu + f(u) + \lambda u.$$

**Theorem 3.2** (Theorems 2, 3 and 5 of [9]). *Let* (V) *and* (F) *be satisfied.*

(i) (Existence) *There exists* $w \in C^1((-\infty, \Lambda), W)$ *such that*

$$\begin{aligned}
&F(\lambda, w(\lambda)) = 0 \quad \textit{for all } \lambda < \Lambda,\\
&w(\lambda)(x) > 0 \quad \textit{and} \quad \frac{d}{dx} w(\lambda)(x) < 0 \quad \textit{for all } x > 0,\\
&\lim_{\lambda \to \Lambda} \|w(\lambda)\|_{H^2} = 0, \qquad \lim_{\lambda \to -\infty} \|w(\lambda)\|_{H^2} = \infty,\\
&\frac{d}{d\lambda} \|w(\lambda)\|_{L^\infty} = \frac{d}{d\lambda} w(\lambda)(0) < 0 \quad \textit{for all } \lambda < \Lambda.
\end{aligned}$$

(ii) (Uniqueness) *If* $(\lambda, u)$ *is a solution of* (1.4) *then* $\lambda < \Lambda$ *and* $u = w(\lambda)$ *on* $[0, \infty)$.

For $\lambda < \Lambda$ we define a function $U : \mathbb{R} \to H^2(\mathbb{R})$ by setting

$$U(\lambda)(x) = \begin{cases} w(\lambda)(x) & \text{for } x \geq 0,\\ w(\lambda)(-x) & \text{for } x < 0. \end{cases}$$

**Corollary 3.3.** *Let* (V) *and* (F) *be satisfied.*

(i) *For* $\lambda < \Lambda$, $G_\lambda = \Theta(U(\lambda))$. *In particular, if* $z$ *is a ground state then* $|z|$ *is even and strictly decreasing on* $[0, \infty)$ *and* $z \in \Theta(U(\lambda))$ *for some* $\lambda < \Lambda$.

(ii) $U \in C^1(\mathbb{R}, H^2(\mathbb{R}))$ *with*

$$\begin{aligned}
&\lim_{\lambda \to \Lambda} \|U(\lambda)\|_{H^2(\mathbb{R})} = 0, \qquad \lim_{\lambda \to -\infty} \|U(\lambda)\|_{H^2(\mathbb{R})} = \infty,\\
&\frac{d}{d\lambda} \|U(\lambda)\|_{L^\infty} = \frac{d}{d\lambda} U(\lambda)(0) < 0 \quad \textit{for all } \lambda < \Lambda.
\end{aligned}$$

A slight modification of arguments in [9] yields the following additional information which is crucial for the stability analysis in the next section.

**Lemma 3.4.** *Under the hypotheses* (V) *and* (F), *consider the self-adjoint operator* $S_\lambda : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *defined by* $S_\lambda = -\partial_{xx}^2 - V - f'(U(\lambda)) - \lambda$. *Let* $\sigma(S_\lambda)$ *and* $\sigma_e(S_\lambda)$ *denote its spectrum and essential spectrum. For all* $\lambda < \Lambda$,

(i) *all eigenvalues of* $S_\lambda$ *are simple,*
(ii) $0 \notin \sigma(S_\lambda)$ *and* $\inf \sigma_e(S_\lambda) = |\lambda| > 0$,
(iii) $S_\lambda$ *has exactly one negative eigenvalue* $a_\lambda$.

*Proof.* (i) Let $u$ and $v$ be eigenfunctions of $S_\lambda$ associated with an eigenvalue $\mu$. Then $[uv' - u'v]' = uv'' - u''v = 0$ and so there is a constant $C$ such that $uv' - u'v = C$ on $\mathbb{R}$. But $\lim_{|x|\to\infty}[uv' - u'v] = 0$ since $u, v \in H^2(\mathbb{R})$. Thus $C = 0$ and $u$ and $v$ are linearly dependent.

(ii) We have $\Lambda < 0$ and

$$\inf \sigma_e(S_\lambda) = \lim_{|x|\to\infty} \{-V - f'(U(\lambda)) - \lambda\} = -\lambda > 0.$$

It is sufficient to show that $\ker S_\lambda = \{0\}$. Suppose that $v \in \ker S_\lambda$. Let $u = U(\lambda)$ and $w = \frac{d}{dx}U(\lambda)$. Then

$$\begin{aligned} u'' + \left\{\lambda + V + \frac{f(u)}{u}\right\}u &= 0, \\ v'' + \{\lambda + V + f'(u)\}v &= 0, \\ w'' + \{\lambda + V + f'(u)\}w &= -V'u, \end{aligned}$$

where $u$ is even and positive on $\mathbb{R}$ and $w$ is negative in $(0, \infty)$. Since $f'(s) > f'(s)/s$ for all $s > 0$, $v$ must have at least one zero. We also have

$$\int_y^\infty \{\lambda + V + f'(u)\}v^2\,dx = -\int_y^\infty v''v\,dx = v'(x)v(x) + \int_y^\infty (v')^2\,dx$$

where $\lim_{x\to\infty}\{\lambda+V+f'(u)\} = \lambda < 0$ and so there exists $Z > 0$ such that $v'(y)v(y) < 0$ for all $y \geq Z$. Thus $v$ has a largest zero which we denote by $x_0$. Using the evenness of $u$ and $V$, we may suppose that $x_0 \geq 0$ and that $v(x) > 0$ for $x > x_0$. Then $v'(x_0) \geq 0$ and it suffices to show that $v'(x_0) = 0$. For this we note that for all $y > x_0$,

$$\begin{aligned} w(x_0)v'(x_0) &= -w'(y)v(y) + w(y)v'(y) + \int_{x_0}^y (w''v - wv'')\,dx \\ &= -w'(y)v(y) + w(y)v'(y) - \int_{x_0}^y V'uw\,dx \leq 0 \end{aligned}$$

where $\lim_{y\to\infty}\{-w'(y)v(y) + w(y)v'(y)\} = 0$. Hence

$$w(x_0)v'(x_0) = -\int_{x_0}^\infty V'uw\,dx \leq 0.$$

If $x_0 = 0$, then $w(x_0) = u'(0) = 0$ and so

$$\int_0^\infty V'uw\,dx = 0$$

where $V'uw \geq 0$ and $uw < 0$ on $(0, \infty)$, implying that $V' \equiv 0$ on $(0, \infty)$, which is excluded by (V). Hence $x_0 > 0$ and $v'(x_0) = 0$ since then $w(x_0) < 0$. Thus $v \equiv 0$.

(iii) We have

$$\begin{aligned}\inf \sigma(S_\lambda) &= \inf\left\{\frac{\int_{-\infty}^\infty ((u')^2 - \{V + f'(U(\lambda)) + \lambda\}u^2)\,dx}{\int_{-\infty}^\infty u^2\,dx} : u \in H^1 \setminus \{0\}\right\}\\ &\leq \frac{\int_{-\infty}^\infty ((U(\lambda)')^2 - \{V + f'(U(\lambda)) + \lambda\}U(\lambda)^2)\,dx}{\int_{-\infty}^\infty U(\lambda)^2\,dx}\\ &< \frac{\int_{-\infty}^\infty ((U(\lambda)')^2 - \{V + \frac{f(U(\lambda))}{U(\lambda)} + \lambda\}U(\lambda)^2)\,dx}{\int_{-\infty}^\infty U(\lambda)^2\,dx} = 0 < \inf \sigma_e(S_\lambda).\end{aligned}$$

Hence $S_\lambda$ has at least one negative eigenvalue. Let $n(\lambda)$ denote the number of negative eigenvalues of $S_\lambda$. Then $1 \leq n(\lambda) < \infty$ since $\inf \sigma_e(S_\lambda) > 0$. But

$$\|f'(U(\lambda)) - f'(U(\mu))\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } \mu \to \lambda < \Lambda$$

and so $n : (-\infty, \Lambda) \to \mathbb{N}$ is continuous at $\lambda$, and consequently equal to a constant $n$ on $(-\infty, \Lambda)$. However $-\partial_{xx}^2 - V - \Lambda$ has no negative eigenvalues and

$$\|f'(U(\lambda))\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } \lambda \to \Lambda-,$$

so $n = 1$.

## 4. Orbital stability of ground states

A standing wave $\Phi^z$ of (NLS) is said to be *orbitally stable* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\Phi_0 \in H$ with $\|\Phi_0 - z\|_H < \delta$,

(a) there is a global solution $\Phi \in C(\mathbb{R}, H) \cap C^1(\mathbb{R}, H^{-1})$ of (NLS) with $\Phi(0, \cdot) = \Phi_0$,
(b) for all $t \geq 0$ there exists $\theta(t) \in [0, 2\pi)$ such that

$$\|\Phi(t, \cdot) - e^{i\theta(t)}z(\cdot)\|_H < \varepsilon, \quad \text{that is,} \quad \operatorname{dist}(\Phi(t, \cdot), \Theta(z)) < \varepsilon.$$

Therefore, before discussing the stability of standing waves we need some basic properties of the initial-value problem for (NLS). This has been thoroughly investigated and the following result gives all the information we require.

**Theorem 4.1** (see [4, Section 3.5]). *Suppose that* (Hi) *and* (Hii) *hold and that there exist* $C > 0$ *and* $\alpha \in [0, 4)$ *such that*

$$|g(s^2)| \le C(1 + s^\alpha) \quad \textit{for all } s \ge 0. \tag{4.1}$$

*Then, for any initial condition* $\Phi_0 \in H$, *there exists a unique function* $\Phi \in C(\mathbb{R}, H) \cap C^1(\mathbb{R}, H^{-1})$ *such that* $\Phi$ *satisfies* (NLS) *and* $\Phi(0, \cdot) = \Phi_0$. *Furthermore, the following quantities are independent of* $t \in \mathbb{R}$*:*

$$\int_{-\infty}^{\infty} (|\partial_x \Phi(t, \cdot)|^2 - V|\Phi(t, \cdot)|^2 - G(|\Phi(t, \cdot)|^2))\, dx \quad \textit{and} \quad \int_{-\infty}^{\infty} |\Phi(t, \cdot)|^2\, dx.$$

We base our discussion of stability on the work of Grillakis, Shatah and Strauss [8]. They deal with real infinite-dimensional Hamiltonian systems in the form

$$\frac{d}{dt} Y(t) = J E'(Y(t)) \tag{4.2}$$

where $X$ is a real Hilbert space, $Y : \mathbb{R} \to X$, $J : X^* \to X$ is skew-symmetric and $E : X \to \mathbb{R}$ is the Hamiltonian. To express (NLS) using this formalism we set

$$\begin{aligned}
X &= H^1(\mathbb{R}) \times H^1(\mathbb{R}), \qquad R : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R}) \text{ is the Riesz isomorphism,}\\
J &= \begin{pmatrix} 0 & R^{-1} \\ -R^{-1} & 0 \end{pmatrix} : X^* \to X,\\
Y(t) &= (\operatorname{Re} z(t, \cdot), \operatorname{Im} z(t, \cdot)), \qquad Q(Y) = \frac{1}{2} \int_{-\infty}^{\infty} |Y|^2\, dx,\\
E(Y) &= \frac{1}{2} \int_{-\infty}^{\infty} (|\partial_x Y|^2 - V|Y|^2 - G(|Y|^2))\, dx \qquad \text{for } Y \in X,
\end{aligned}$$

where our hypothesis (Hi) and (Hii) of Section 1 ensure that $E \in C^2(X, \mathbb{R})$.

*Assumption 1* of [8] concerns the initial value problem for (NLS) and Theorem 4.1 shows that it is satisfied where the conserved quantities are $Q$ and $E$.

If we set

$$T(t) = \begin{pmatrix} (\cos t)I & (\sin t)I \\ -(\sin t)I & (\cos t)I \end{pmatrix},$$

then standing waves of (NLS) are solutions of (4.2) of the form $T(\lambda t)Y$ for some $\lambda \in \mathbb{R}$ and $Y \in X$ and are referred to as bound states in [8]. For such solutions, (4.2) reduces to the equation

$$\lambda Y = E'(Y), \tag{4.3}$$

which is equivalent to (1.3) for $Y = (\operatorname{Re} z, \operatorname{Im} z) \in X \setminus \{0\}$.

*Assumption 2* of [8] concerns the existence of a smooth branch $\lambda \mapsto Y_\lambda$ of nontrivial solutions of (4.3) and this is ensured by our Corollary 3.3 if we set

$$Y_\lambda = (U(\lambda), 0) \qquad \text{for } \lambda < \Lambda.$$

The stability criteria for the standing waves $T(\lambda t)Y_\lambda$ are formulated in [8] in terms of the real function $d : (-\infty, \Lambda) \to \mathbb{R}$ defined by

$$d(\lambda) = E(Y_\lambda) - \lambda Q(Y_\lambda)$$

and the bounded linear operator $H_\lambda : X \to X^*$ defined by

$$H_\lambda = E''(Y_\lambda) - \lambda Q''(Y_\lambda).$$

*Assumption 3* of [8] concerns the spectrum $\sigma(H_\lambda)$ of $H_\lambda$ where

$$\sigma(H_\lambda) = \left\{\mu \in \mathbb{R} : H_\lambda - \mu \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} : X \to X^* \text{ is not an isomorphism}\right\}.$$

It is required that, for all $\lambda$ in some open interval $J$ in $(-\infty, \Lambda)$,

(I) $\sigma(H_\lambda) \cap (-\infty, 0) = \{a_\lambda\}$ where $\dim\ker\left[H_\lambda - a_\lambda \left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right)\right] = 1$,
(II) $\ker H_\lambda = \operatorname{span}\{JY_\lambda\} = \operatorname{span}\{(0, U(\lambda))\}$,
(III) there exists $\varepsilon_\lambda > 0$ such that $\sigma(H_\lambda) \setminus \{a_\lambda, 0\} \subset [\varepsilon_\lambda, \infty)$.

Under these assumptions, Theorem 2 of [8] proves that, for $\lambda \in J$, the standing wave $T(\lambda t)Y_\lambda$ is orbitally stable if and only if $d$ is convex on some neighbourhood of $\lambda$.

Now

$$d'(\lambda) = E'(Y_\lambda)\frac{dY_\lambda}{d\lambda} - Q(Y_\lambda) - \lambda Q'(Y_\lambda)\frac{dY_\lambda}{d\lambda} = -Q(Y_\lambda)$$

since $E'(Y_\lambda) - \lambda Q'(Y_\lambda) = 0$ by (4.3), and so the stability of $T(\lambda t)Y_\lambda$ is established if we show that (I) to (III) hold and that

$$\frac{d}{d\lambda}\int_{-\infty}^{\infty} U(\lambda)^2\,dx < 0 \tag{4.4}$$

since $Q(Y_\lambda) = \int_{-\infty}^{\infty} U(\lambda)^2\,dx$.

In particular contexts such as (NLS), the relevance of (4.4) as a criterion for stability has been known at least since the work of Vakhitov–Kolokolov in 1973 ([15], [10]) and its rigorous justification for some cases of (NLS) was given by M. I. Weinstein [16], [13].

We begin our discussion by showing that the conditions (I) to (III) are satisfied by (NLS) under the hypotheses of Section 3. Allowing for some abuse of the notation, we find that, for $(\varphi, \psi) \in X$,

$$H_\lambda(\varphi, \psi) = -\begin{pmatrix} \varphi'' + \{\lambda + V + f'(U(\lambda)^2)\}\varphi \\ \psi'' + \{\lambda + V + g(U(\lambda)^2)\}\psi \end{pmatrix}^T \in X^* = H^{-1}(\mathbb{R}) \times H^{-1}(\mathbb{R}).$$

**Lemma 4.2.** *Suppose that* (V) *and* (F) *are satisfied. Then $H_\lambda$ has the properties* (I) *to* (III) *for all $\lambda < \Lambda$. Consequently, if* (4.1) *also holds, then the standing wave $\Phi^{U(\lambda)}$ is orbitally stable if and only if the function $d$ is convex on some open neighbourhood of $\lambda$.*

*Proof.* For any $\lambda < \Lambda$, $(\partial_{xx}^2 + (\lambda+\mu)R, \partial_{xx}^2 + (\lambda+\mu)R) : X \to X^*$ is an isomorphism for $\mu < |\lambda|$ and

$$\lim_{|x|\to\infty} \{V + f'(U(\lambda)^2)\} = \lim_{|x|\to\infty} \{V + g(U(\lambda)^2)\} = 0$$

so that $H_\lambda - \mu R : X \to X^*$ is a Fredholm operator of index zero for $\mu < |\lambda|$. Hence $\sigma(H_\lambda) \cap (-\infty, |\lambda|)$ consists only of isolated eigenvalues of finite multiplicity.

Suppose now that $a < 0$ is an eigenvalue of $H_\lambda$. Then there exists $(\varphi, \psi) \in X \setminus \{0\}$ such that

$$-\varphi'' - \{\lambda + V + f'(U(\lambda)^2)\}\varphi = a\varphi,$$
$$-\psi'' - \{\lambda + V + g(U(\lambda)^2)\}\psi = a\psi.$$

But $U(\lambda)$ is a positive eigenfunction of $-\partial_{xx}^2 - \{\lambda + V + g(U(\lambda)^2)\}$ with eigenvalue zero and so all other eigenvalues of this operator are positive. Hence $\psi = 0$ and we must have

$$\varphi \in H^1(\mathbb{R}) \setminus \{0\} \quad \text{with} \quad -\varphi'' - \{\lambda + V + f'(U(\lambda)^2)\}\varphi = a\varphi.$$

It follows that $\varphi \in H^2(\mathbb{R})$ and so $a$ is an eigenvalue of the self-adjoint operator $-\partial_{xx}^2 - \{\lambda + V + f'(U(\lambda)^2)\} : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Property (I) of $H_\lambda$ now follows from Lemma 3.4.

We have already observed that $(0, U(\lambda)) \in \ker H_\lambda$. But if $(\varphi, \psi) \in \ker H_\lambda$ then $-\varphi'' - \{\lambda + V + f'(U(\lambda)^2)\}\varphi = 0$ and Lemma 3.4(ii) implies that $\varphi = 0$. Thus $H_\lambda$ also has property (II) since, as for $S_\lambda$, all eigenvalues of $-\partial_{xx}^2 - \{\lambda + V + g(U(\lambda)^2)\}$ are simple.

Since $\sigma(H_\lambda)$ has no accumulation points in $(-\infty, \frac{1}{2}|\lambda|]$, property (III) has also been established. □

The stability of all ground states is not ensured by the conditions (V) and (F).

**Example.** Let $V$ be any potential satisfying (V) and consider the function $f(s) = s^k$ for $s \geq 0$. Then the condition (F) is satisfied for all $k > 1$, but, as was shown in [12], $\|U(\lambda)\|_{L^2(\mathbb{R})} \to 0$ as $\lambda \to -\infty$ if $k > 5$. Since $\|U(\lambda)\|_{L^2(\mathbb{R})} \to 0$ as $\lambda \to \Lambda-$ for all $k > 1$, it follows that $d$ cannot be convex on $(-\infty, \Lambda)$ and so not all ground states $U(\lambda)$ are stable.

Recalling that $f(s)/s > 0$ for $s > 0$ when a function $f$ satisfies (F), we now make the following additional assumptions.

(H) On the interval $(0, \infty)$, $f'(s) - f(s)/s$ is nondecreasing, $sf'(s)/f(s)$ nonincreasing and $0 \leq sf'(s)/f(s) \leq 5$.

**Examples.** (i) $f(s) = s^k$ satisfies the conditions (F) and (H) if $1 < k \leq 5$.
(ii) More generally, $f(s) = s^{k+1}/(s+1)^l$ satisfies the conditions (F) and (V) if $0 \leq l < 4k^2/(4k+1)$ and $0 < k \leq 4$.

**Theorem 4.3** (see Theorem 2.1 of [12]). *Under the hypotheses* (V), (F) *and* (H) *we have* $\frac{d}{d\lambda}\|U(\lambda)\|_{L^2} < 0$ *for all* $\lambda < \Lambda$. *If, in addition, the condition* (4.1) *is satisfied, then all the ground states* $U(\lambda)$ *for* $\lambda < \Lambda$ *are orbitally stable.*

**Example.** $V$ satisfies (V) and $f(s) = s^k$, then all ground states of (NLS) are stable if $1 < k < 5$. For $k = 5$, the strict monotonicity of $\|U(\lambda)\|_{L^2}$ still holds but the condition (4.1) for global existence of the initial value problem fails.

## References


[1] Ambrosetti, A., Arcoya, D., Gámez, J. L.: Asymmetric bound states of differential equations in nonlinear optics. Rend. Sem. Mat. Padova **100**, 231–247 (1998) Zbl 0922.34020 MR 1675283

[2] Akhmediev, N. N.: A novel class of nonlinear surface waves; asymmetric modes in a symmetric layered structure. Sov. Phys. JETP **56**, 299–303 (1982)

[3] Arcoya, D., Cingolani, S., Gamez, J. L.: Asymmetric modes in symmetric nonlinear optical wave guides. SIAM J. Math. Anal. **30**, 1391–1400 (1999) Zbl 0933.34014 MR 1718307

[4] Cazenave, T.: Semilinear Schrödinger Equations. Courant Inst. Lecture Notes 10, Amer. Math. Soc., Providence (2003) Zbl 1055.35003 MR 2002047

[5] Cazenave, T., Lions, P.-L.: Orbital stability of standing waves for some nonlinear Schrödinger equations. Comm. Math. Phys. **85**, 549–561 (1982) Zbl 0513.35007 MR 0677997

[6] Cingolani, S., Gamez, J. L.: Asymmetric positive solutions for a symmetric nonlinear problem in $\mathbb{R}^N$. Calc. Var. **11**, 97–117 (2000) Zbl 0968.35043 MR 1777465

[7] Esteban, M. J.: Nonsymmetric ground states of symmetric variational problems. Comm. Pure Appl. Math. **44**, 259–274 (1991) Zbl 0826.49002 MR 1085830

[8] Grillakis, M., Shatah, J., Strauss, W.: Stability theory of solitary waves in the presence of symmetry, I. J. Funct. Anal. **74**, 160–197 (1987) Zbl 0656.35122 MR 0901236

[9] Jeanjean, H., Stuart, C. A.: Nonlinear eigenvalue problems having an unbound branch of symmetric bound states. Adv. Differential Equations **4**, 639–670 (1999) Zbl 0958.34017 MR 1696348

[10] Kivshar, Y. S., Sukhorukov, A. A.: Stability of spatial optical solitons. In: Spatial Optical Solitons, T. Torruellas and S. Trillo (eds.), Springer, New York, 211–245 (2001)

[11] Lieb, E. H., Loss, M.: Analysis. Grad. Stud. in Math. 14, Amer. Math. Soc., Providence (1997) Zbl 0873.26002 MR 1415616

[12] McLeod, J. B., Stuart, C. A., Troy, W. C.: Stability of standing waves for some nonlinear Schrödinger equations. Differential Integral Equations **16**, 1025–1038 (2003) Zbl 1037.34022 MR 1989539

[13] Rose, H. A., Weinstein, M. I.: On the bound states of the nonlinear Schrödinger equation with a linear potential. Phys. D, **30**, 207–218 (1988) Zbl 0694.35202 MR 0939275

[14] Stuart, C. A.: A global branch of solutions to a semilinear equation on an unbounded interval. Proc. Roy. Soc. Edinburgh **101A**, 273–282 (1985) Zbl 0582.34010 MR 0824226

[15] Vakhitov, M. G., Kolokolov, A. A.: Stationary solutions of the wave equation in the medium with nonlinearity saturation. Radiophys. Quantum Electron. **16**, 783 (1973)

[16] Weinstein, M. I.: Lyapunov stability of ground states for nonlinear dispersive evolution equations. Comm. Pure Appl. Math. **39**, 51–67 (1986) Zbl 0594.35005 MR 0820338