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# On covering and quasi-unsplit families of curves

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**Abstract.** Given a covering family $V$ of effective 1-cycles on a complex projective variety $X$, we find conditions allowing one to construct a geometric quotient $q : X \to Y$, with $q$ regular on the whole of $X$, such that every fiber of $q$ is an equivalence class for the equivalence relation naturally defined by $V$. Among other results, we show that on a normal and $\mathbb{Q}$-factorial projective variety $X$ with canonical singularities and $\dim X \leq 4$, every covering and quasi-unsplit family $V$ of rational curves generates a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$ of classes of effective 1-cycles and that the associated Mori contraction yields a geometric quotient for $V$.

**Keywords.** Covering families of curves, extremal curves, quotient

## 1. Introduction

Let $X$ be a normal complex projective variety. Let $V$ be an irreducible and closed subset of $\mathrm{Chow}(X)$ such that any element of $V$ is a 1-cycle, and such that for any point $x \in X$, there exists an element of $V$ passing through $x$. We call $V$ a *covering family of* 1-*cycles* on $X$.

The framework is Campana's notion of $V$-equivalence, and the construction of the related rational map defined on $X$: two points $x, x'$ are said to be *$V$-equivalent* if there exist $v_1, \dots, v_m \in V$ such that some connected component of $C_{v_1} \cup \dots \cup C_{v_m}$ contains $x$ and $x'$, where $C_v \subset X$ is the curve corresponding to $v \in V$. By Campana's results (see Section 2), there exists a dominant almost holomorphic map $q : X \dashrightarrow Y$, $Y$ a normal projective variety, whose general fibers are $V$-equivalence classes. Let $f_V := \dim X - \dim Y$.

A morphism $q' : X \to Y'$ onto a normal projective variety $Y'$ will be called a *geometric quotient* for $V$ if *every* fiber of $q'$ is a $V$-equivalence class. If such a quotient exists, then it is clearly unique up to isomorphism. On the other hand, even if $X$ is smooth, a

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geometric quotient for $V$ does not necessarily exist (see Example 2). We refer to [KS02] for a general introduction to this question and related ones.

The main problem is to find sufficient conditions on $X$ and $V$ for the geometric quotient to exist.

Let $\mathcal{N}_1(X)_{\mathbb{R}}$ (respectively $\mathcal{N}_1(X)_{\mathbb{Q}}$) be the vector space of 1-cycles in $X$ with real (respectively rational) coefficients, modulo numerical equivalence. In $\mathcal{N}_1(X)_{\mathbb{R}}$, let $\overline{\mathrm{NE}}(X)$ be the closure of the convex cone generated by classes of effective 1-cycles in $X$.

We say that $V$ is *quasi-unsplit* if there exists a half-line $R_V \subseteq \overline{\mathrm{NE}}(X)$ such that the numerical class of every irreducible component of every cycle in $V$ belongs to $R_V$ [CO04, Definition 2.13]. Equivalently, $V$ is quasi-unsplit if all irreducible components of the cycles parametrized by $V$ are numerically proportional.

The first result is the following theorem.

**Theorem 1.** *Let $X$ be a normal and $\mathbb{Q}$-factorial complex projective variety of dimension $n$ and let $V$ be a covering and quasi-unsplit family of 1-cycles on $X$. If $f_V \geq n-2$, then there exists a geometric quotient for $V$.*

Next, we specialize to rational 1-cycles on $X$, that is, cycles whose irreducible components are rational curves, and look at related questions.

A *geometric extremal ray* of the Mori cone $\overline{\mathrm{NE}}(X)$ is a half-line $R \subseteq \overline{\mathrm{NE}}(X)$ such that if $\gamma_1 + \gamma_2 \in R$ for some $\gamma_1, \gamma_2 \in \overline{\mathrm{NE}}(X)$, then $\gamma_1, \gamma_2 \in R$.

Let $V$ be a covering and quasi-unsplit family of rational 1-cycles on $X$. Is $R_V$ a geometric extremal ray of $\overline{\mathrm{NE}}(X)$?

Note that this question is natural, since any family of rational 1-cycles such that the general member generates a geometric extremal ray of $\overline{\mathrm{NE}}(X)$ is quasi-unsplit. If $V$ is not assumed to be covering, the preceding statement is not true by looking at a smooth blow-down of a smooth projective variety to a nonprojective one: contracted curves do not define a geometric extremal ray.

**Theorem 2.** *Let $X$ be a normal and $\mathbb{Q}$-factorial complex projective variety of dimension $n$ with canonical singularities. Let $V$ be a covering and quasi-unsplit family of rational 1-cycles on $X$, and let $f_V$ be the dimension of a general $V$-equivalence class. If $f_V \geq n-3$, then $R_V$ is extremal in the sense of Mori theory and the associated contraction yields a geometric quotient for $V$.*

We then immediately get the following corollary.

**Corollary 1.** *Let $X$ be a normal and $\mathbb{Q}$-factorial complex projective variety of dimension $\leq 4$ with canonical singularities and let $V$ be a covering and quasi-unsplit family of rational 1-cycles on $X$. Then $R_V$ is extremal in the sense of Mori theory and the associated contraction yields a geometric quotient for $V$.*

We finally consider the toric case, where we can prove both extremality and existence of a geometric quotient for a quasi-unsplit family of 1-cycles in any dimension.

**Theorem 3.** *Let $X$ be a toric and $\mathbb{Q}$-factorial complex projective variety, and let $V$ be a covering and quasi-unsplit family of 1-cycles in $X$. Then $R_V$ is an extremal ray of $\overline{\mathrm{NE}}(X)$ in the sense of Mori theory and the associated contraction yields an equivariant and equidimensional morphism $q' \colon X \to Y'$ onto a toric and $\mathbb{Q}$-factorial projective variety $Y'$ which is a geometric quotient for $V$.*

The following is an immediate application of Theorems 2 and 3.

**Corollary 2.** *Let $X \subset \mathbb{P}^N$ be a normal and $\mathbb{Q}$-factorial variety such that through any point of $X$ there is a line contained in $X$. Assume that either $X$ is toric, or has canonical singularities and* $\dim X \leq 4$. *Let $V$ be an irreducible family of lines covering $X$. Then there exists a geometric quotient for $V$.*

Note that if $X \subset \mathbb{P}^N$ has sufficiently small degree then $X$ is covered by lines (see [KNS05] for a precise statement).

Finally, we point out that our results are related to the construction in [BCE$^+$02] of the reduction morphism for a nef line bundle $L \in \operatorname{Pic} X$ (see also [Tsu00]). This is an almost holomorphic rational map $f \colon X \dashrightarrow T$, dominant with connected fibers, such that:

- $L \cdot C = 0$ for any curve $C$ contained in a proper fiber $F$ of $f$ with $\dim F = \dim X - \dim Y$;
- $L \cdot C > 0$ for every irreducible curve $C$ passing through a general point of $X$.

The map $f$ is unique up to birational equivalence of $T$, and the dimension of $T$ is called the *nef dimension* of $L$.

It is still quite unclear in which circumstances $f$ can be chosen holomorphic (see [BCE$^+$02, §2.4]); it is always so if the nef dimension of $L$ is at most one. Theorem 1 gives a partial answer for nef line bundles with nef dimension two.

**Corollary 3.** *Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety with* $\dim X \geq 3$, *and* $L \in \operatorname{Pic} X$ *be a nef line bundle with nef dimension* 2. *Assume that* $\{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma \cdot L = 0\}$ *is a half-line. Then there exists a nef reduction morphism $q \colon X \to Y$ onto a normal projective surface $Y$ such that for any curve $C \subset X$ we have $q(C) = \{pt\}$ if and only if* $C \cdot L = 0$.

## 2. Set-up on families of 1-cycles

Let $X$ be a normal, irreducible, $n$-dimensional complex projective variety. For any curve $C \subset X$, we denote by $[C] \in \mathcal{N}_1(X)_{\mathbb{R}}$ its numerical class.

If $R \subset \overline{\mathrm{NE}}(X)$ is a half-line and $D$ is a divisor in $X$, we write $D \cdot R > 0$, $D \cdot R = 0$, or $D \cdot R < 0$ if respectively $D \cdot \gamma > 0$, $D \cdot \gamma = 0$, or $D \cdot \gamma < 0$ for a nonzero element $\gamma \in R$.

Let $V$ be a covering family of 1-cycles on $X$. We have a diagram given by the incidence variety $\mathcal{C}$ associated to $V$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ V & & \end{array} \tag{1}$$

where $\pi$ and $F$ are proper and surjective. Set $C_v := F(\pi^{-1}(v))$ for any $v \in V$.

The relation of $V$-equivalence on $X$ induced by such a family was introduced and studied in [Cam81]; we refer the reader to [Cam04], [Deb01, §5.4], [KMM92] or [Kol96, §IV.4] for more details. By [Deb01, Theorem 5.9] there exists a closed and irreducible subset of $\mathrm{Chow}(X)$ whose normalization $Y$ has the following properties:

(a) if $Z \subset Y \times X$ is the restriction of the universal family,

$$\begin{array}{ccc} Z & \xrightarrow{\ e\ } & X \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle q} & \\ Y & & \end{array} \tag{2}$$

then $e$ is a birational morphism and $q = p \circ e^{-1}$ is almost holomorphic (which means that the exceptional locus of $e$ does not dominate $Y$);

(b) a general fiber of $q$ is a $V$-equivalence class,

(c) a general fiber of $q$, hence of $p$, is irreducible.

As a consequence of the existence of the map $q$, a general $V$-equivalence class is a closed subset of $X$. We denote by $f_V$ its dimension, so that $\dim Y = n - f_V$. Moreover, it is well known that any $V$-equivalence class is a countable union of closed subsets of $X$.

**Example 1** (see [Kac97, Example 11.1], and references therein)**.** Fix a point $x_0$ in $\mathbb{P}^3$ and let

$$P_0 := \{\Pi \in (\mathbb{P}^3)^* \mid x_0 \in \Pi\} \simeq \mathbb{P}^2$$

be the variety of 2-planes in $\mathbb{P}^3$ containing $x_0$. Consider the variety $X \subset \mathbb{P}^3 \times P_0$ defined as

$$X := \{(x, \Pi) \in \mathbb{P}^3 \times P_0 \mid x \in \Pi\}.$$

Then $X$ is a smooth Fano 4-fold, with Picard number 2 and pseudo-index 2. The two elementary extremal contractions are given by the projections on the two factors.

The morphism $X \to P_0$ is a fibration in $\mathbb{P}^2$: the fiber over a point is the plane corresponding to that point.

Consider the morphism $X \to \mathbb{P}^3$. If $x \neq x_0$, the fiber over $x$ is the $\mathbb{P}^1$ of planes containing $x$ and $x_0$. But the fiber $F_0$ over $x_0$ is naturally identified with $P_0$, hence it is isomorphic to $\mathbb{P}^2$. Let $V \to \mathbb{P}^3$ be the blow-up of $x_0$ and $\mathcal{C} \to X$ be the blow-up of $F_0$:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ F=e\ } & X \\ {\scriptstyle \pi=p}\downarrow & \swarrow {\scriptstyle q} & \downarrow{\scriptstyle q'} \\ V & \xrightarrow{\ \psi\ } & \mathbb{P}^3 \end{array}$$

**Definition 1.** *We say that a subset of $X$ is $V$*-connected *if it is contained in a $V$-equivalence class.*

**Lemma 1.** *Let $X$ be a normal projective variety and $V$ be a covering family of* 1*-cycles on $X$. Consider the diagram* (2) *above. Then $e(p^{-1}(y))$ is $V$-connected for any $y \in Y$.*

*Proof.* Let $\mathcal{R} \subset X \times X$ be the graph of the equivalence relation defined by $V$; it is a countable union of closed subvarieties since $V$ is proper. The fiber product $Z \times_Y Z$ is irreducible and thus $(e \times e)(Z \times_Y Z) \subset \mathcal{R}$ thanks to properties (a) and (b) above. Therefore, for any $x \in e(p^{-1}(y))$, the cycle $e(p^{-1}(y))$ is contained in the $V$-equivalence class of $x$. □

Finally, we will need the following.

**Lemma 2.** *Let $X$ be a normal projective variety and $V$ be a covering and quasi-unsplit family of 1-cycles on $X$. Then there exists a covering and quasi-unsplit family $V'$ of 1-cycles on $X$ such that:*

- *the general cycle of $V'$ is reduced and irreducible;*
- *for any $v' \in V'$ there exists $v \in V$ such that $C_{v'} \subseteq C_v$; in particular $R_V = R_{V'}$.*

*Proof.* Let $\mathcal{C}$ be the incidence variety associated to $V$ as in (1). It is well known that every irreducible component of $\mathcal{C}$ dominates $V$; let $\mathcal{C}''$ be an irreducible component of $\mathcal{C}$ which also dominates $X$. Let $\mathcal{C}'$ be the normalization of $\mathcal{C}''$ and $\mathcal{C}' \to V'$ be the Stein factorization of the composite map $\mathcal{C}' \to \mathcal{C}'' \to V$. Since $\mathcal{C}' \to V'$ has connected fibers and $\mathcal{C}'$ is normal, the general fiber of $\mathcal{C}' \to V'$ is irreducible. Moreover, the image in $X$ of every fiber of $\mathcal{C}' \to V'$ is contained in a cycle of $V$.

Since $V'$ is normal, there is a holomorphic map $V' \to \mathrm{Chow}(X)$. Then after replacing $V'$ by its image in $\mathrm{Chow}(X)$ and $\mathcal{C}'$ by its image in $\mathrm{Chow}(X) \times X$, we get the desired family. □

## 3. Properties of the base locus

Let $V$ be a covering family of 1-cycles on $X$, and recall the diagram (2) associated to $V$. Let $E \subset Z$ be the exceptional locus of $e$, and $B := e(E) \subset X$. Observe that since $X$ is normal, $\dim B \leq n - 2$.

**Proposition 1.** *Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ be a covering and quasi-unsplit family of 1-cycles on $X$. Consider the associated diagram as in (2). Then:*

(i) *$e(p^{-1}(y))$ is a $V$-equivalence class of dimension $f_V$ for every $y \in Y \setminus p(E)$;*
(ii) *$B$ is the union of all $V$-equivalence classes of dimension greater than $f_V$.*

*Proof.* Set $X^0 := X \setminus B$ and $Y^0 := Y \setminus p(E) = q(X^0)$. Choose a very ample line bundle $H$ on $Y$, and let $U \subset |H|$ be the open subset of prime divisors $D$ such that $D \cap Y^0 \neq \emptyset$. For any $D$ in $U$, we define $\widehat{D} := \overline{q^{-1}(D \cap Y^0)}$, which is a prime Weil divisor in $X$. Since $X$ is $\mathbb{Q}$-factorial, some multiple of $\widehat{D}$ defines a line bundle $\widehat{H}$ on $X$. Observe that as $D$ varies in $U$, the divisors $\widehat{D}$ are numerically equivalent in $X$.

Observe also that a general cycle $C_v$ of $V$ is contained in a fiber of $q$ disjoint from $\widehat{D}$, so $\widehat{D} \cdot C_v = 0$. Since $V$ is quasi-unsplit, this gives $\widehat{D} \cdot R_V = 0$, meaning that for *every* irreducible component $C$ of *every* cycle of $V$ we have $\widehat{D} \cdot C = 0$.

Let now $N := h^0(H)$, and let $s_1, \dots, s_N$ be general global sections generating $H$. For each $i = 1, \dots, N$, let $D_i \in |H|$ be the divisor of zeros of $s_i$ and $\widehat{D}_i$ in $X$ as defined above.

Let us show that $\widehat{D}_1 \cap \cdots \cap \widehat{D}_N = B$. If $x \notin B$, then $q$ is defined at $x$ and there is some $i_0 \in \{1, \dots, N\}$ such that $q(x) \notin D_{i_0}$, so $x \notin \widehat{D}_{i_0}$. Conversely, let $x \in B$ and fix $i \in \{1, \dots, N\}$. Then $e^{-1}(x)$ has positive dimension; let $C \subset Z$ be an irreducible curve such that $e(C) = x$. Recall that $Z \subset Y \times X$, so $p(C)$ is a curve in $Y$. Then $D_i \cap p(C) \neq \emptyset$ and $p^{-1}(D_i) \cap C \neq \emptyset$. Now observe that $p^{-1}(D_i)$ does not contain any component of $E$, hence $e(p^{-1}(D_i))$ is a divisor in $X$ which coincides with $\widehat{D}_i$ over $X \setminus B$. Then $\widehat{D}_i = e(p^{-1}(D_i))$ and $x \in \widehat{D}_i$.

**Claim.** *Let $C$ be an irreducible curve in $X$ such that $\widehat{D} \cdot C = 0$ for some $D \in U$. Then either $C \subseteq B$, or $C \cap B = \emptyset$ and $q(C) = \{pt\}$.*

In fact, assume that $C$ is not contained in $B$. Since $B = \widehat{D}_1 \cap \cdots \cap \widehat{D}_N$, there exists $i \in \{1, \dots, N\}$ such that $C$ is not contained in $\widehat{D}_i$. Then $C \cap \widehat{D}_i = \emptyset$, because $C \cdot \widehat{D}_i = 0$. Thus $C \cap B = \emptyset$.

Moreover if $q(C)$ is a curve, there exists $D_0 \in U$ such that $D_0$ intersects $q(C)$ in a finite number of points. Then $\widehat{D}_0$ intersects $C$ without containing it, a contradiction, again because $C \cdot \widehat{D}_0 = 0$.

This shows that $B$ is closed with respect to $V$-equivalence. In fact, let $C$ be an irreducible component of a cycle of $V$ such that $C \cap B \neq \emptyset$. We have $C \cdot \widehat{D} = 0$, so the Claim above implies that $C \subseteq B$.

Consider now a $V$-equivalence class $F \subseteq X$. Since $B$ is closed with respect to $V$-equivalence, either $F \cap B = \emptyset$, or $F \subseteq B$.

Assume that $F \cap B = \emptyset$, and choose an irreducible component $C$ of a cycle of $V$ such that $C \subseteq F$. We have $\widehat{D} \cdot C = 0$ for any $D \in U$, hence $q(C)$ is a point by the Claim above. By definition of $V$-equivalence, any two points of $F$ can be joined by a chain of components of cycles of $V$, so we have $q(F) = y_0 \in Y$, and $F \subseteq e(p^{-1}(y_0))$. On the other hand, $e(p^{-1}(y))$ is $V$-connected by Lemma 1, so $F = e(p^{-1}(y_0))$. Finally, since $F \cap B = \emptyset$, we must have $y_0 \in Y^0$, so $F$ is a proper fiber of $q$ of dimension $f_V$.

For any $x \in X$, let $Y_x := p(e^{-1}(x))$ be the family of cycles parametrized by $Y$ and passing through $x$, and $\mathrm{Locus}(Y_x) := e(p^{-1}(Y_x))$. Observe that for any $y \in Y_x$, the subset $e(p^{-1}(y))$ contains $x$ and is $V$-connected by Lemma 1. Hence $\mathrm{Locus}(Y_x)$ is $V$-connected for any $x \in X$.

Since $Z \subset Y \times X$, we have $\dim Y_x = \dim e^{-1}(x)$. Thus $\dim Y_x > 0$ if and only if $x \in B$, by Zariski's main theorem. If so, $\mathrm{Locus}(Y_x)$ has dimension at least $f_V + 1$.

Now let $F$ be a $V$-equivalence class contained in $B$, and $x \in F$. Then $\mathrm{Locus}(Y_x)$ has dimension at least $f_V + 1$ and is contained in $F$, hence $\dim F \geq f_V + 1$. □

Let us remark that in general, if $V$ is not quasi-unsplit, $B$ is not closed with respect to $V$-equivalence.

**Example 2.** In $\mathbb{P}^2$ fix two points $x, y$ and the line $L = \overline{xy}$. Consider $\mathbb{P}^2 \times \mathbb{P}^2$ with the projections $\pi_1, \pi_2$ on the two factors, and fix three curves $R_x, R_y, L'$ such that:

- $R_x$ is a line in $\mathbb{P}^2 \times x$ and $R_y$ is a line in $\mathbb{P}^2 \times y$;
- $\pi_1(R_x) \cap \pi_1(R_y)$ is a point $z \in \mathbb{P}^2$;
- $L' := z \times L$ is the unique line dominating $L$ via $\pi_2$ and intersecting both $R_x$ and $R_y$.

Let $\sigma\colon W \to \mathbb{P}^2 \times \mathbb{P}^2$ be the blow-up of $R_x$ and $R_y$. In $W$, the strict transform of $L'$ is a smooth rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$. Let $X$ be the variety obtained by "flipping" this curve. Then $X$ is a smooth toric Fano 4-fold with $\rho_X = 4$ (this is $Z_2$ in Batyrev's list, see [Bat99, Proposition 3.3.5]).

$$\begin{array}{ccc} X & \dashrightarrow & W \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle \pi_2\circ\sigma} \\ & & \mathbb{P}^2 \end{array}$$

The strict transform of a general line in a fiber of $\pi_2$ gives a covering family $V$ of rational curves on $X$. The birational map $X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ is an isomorphism over $\mathbb{P}^2 \times (\mathbb{P}^2 \setminus L)$; if $U \subset X$ is the corresponding open subset, then $U$ is closed with respect to $V$-equivalence and every fiber of $q\colon U \to \mathbb{P}^2 \setminus L$ is a $V$-equivalence class isomorphic to $\mathbb{P}^2$. Thus $f_V = 2$.

Let $T_x$ and $T_y$ be the images in $X$ of the exceptional divisors of $\sigma$ in $W$. These two divisors are $V$-connected, and they cannot be contained in $B$ because $\dim B \leq 2$. Moreover, $P := T_x \cap T_y$ is the $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ obtained under the flip. The map $q\colon X \dashrightarrow \mathbb{P}^2$ cannot be defined over $P$, so $P \cap B \neq \emptyset$. Therefore $B$ cannot be closed with respect to $V$-equivalence.

Observe that the numerical class of $V$ lies in the interior of $\overline{\mathrm{NE}}(X)$, hence the unique morphism, onto a projective variety, which contracts curves in $V$, is $X \to \{\mathrm{pt}\}$.

*Proof of Theorem 1.* If $B$ is not empty, Proposition 1 gives $\dim B \geq f_V + 1 \geq n - 1$, which is impossible because $X$ is normal. Hence $B$ is empty and $q\colon X \to Y$ is an equidimensional morphism, whose fibers are $V$-equivalence classes. □

*Proof of Corollary 3.* Let $V$ be a covering family of 1-cycles having intersection zero with $L$. Such a family exists because $L$ has nef dimension two and the dimension of $X$ is at least three.

Since $R := \{\gamma \in \overline{\mathrm{NE}}(X) \mid \gamma \cdot L = 0\}$ is an extremal ray of $\overline{\mathrm{NE}}(X)$ and the general 1-cycle of $V$ has numerical class in $R$, the family $V$ is quasi-unsplit and $R_V = R$.

Consider the nef reduction $f\colon X \dashrightarrow T$ of $L$ and let $F$ be a general fiber. For any irreducible curve $C \subseteq F$ we have $C \cdot L = 0$, so $[C] \in R$. Using the Claim in the proof of Proposition 1, we see that $C \cap B = \emptyset$ and $q(C)$ is a point. This is true for all curves in $F$, so $F \cap B = \emptyset$ and $q(F)$ must be point. Then $F$ is contained in a $V$-equivalence class, hence $f_V \geq \dim F = n - \dim T \geq n - 2$. Now the statement follows from Theorem 1.

□

## 4. Extremality for covering families of rational 1-cycles

We now consider extremality properties of $R_V$ for a covering and quasi-unsplit family of rational 1-cycles.

The following well known remark will be of constant use (see [Kol96, Proposition IV.3.13.3], or [ACO04, Corollary 4.2]).

**Remark 1.** Assume that $V$ is a covering family of rational 1-cycles, and let $Z \subset X$ be $V$-connected. Then every curve contained in $Z$ is numerically equivalent in $X$ to a linear combination with rational coefficients of irreducible components of cycles in $V$. In particular, if $V$ is quasi-unsplit and $C$ is a curve contained in a $V$-connected subset of $X$, then $[C] \in R_V$.

A key observation is the following.

**Proposition 2.** *Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ a covering and quasi-unsplit family of rational 1-cycles on $X$. If every connected component of $B$ is $V$-connected, then there exists a nef divisor $\widehat{D}$ on $X$ such that for any curve $C$ in $X$, $\widehat{D} \cdot C = 0$ if and only if $[C] \in R_V$.*

*Proof.* We use the same notation as in the proof of Proposition 1. So $H$ is a very ample line bundle on $Y$, $U \subset |H|$ is the open subset of divisors $D$ that are irreducible and such that $D \not\subseteq p(E)$, and for any $D$ in $U$, we set $\widehat{D} := \overline{q^{-1}(D \setminus p(E))}$. Recall that $\widehat{D} \cdot R_V = 0$, and that $B = \widehat{D}_1 \cap \cdots \cap \widehat{D}_N$ for some $D_1, \ldots, D_N \in U$.

Let us show that $\widehat{D}$ is nef. By contradiction, suppose that there exists an irreducible curve $C$ with $C \cdot \widehat{D} < 0$. Then $C$ must be contained in $\widehat{D}_1, \ldots, \widehat{D}_N$, hence $C \subseteq B$. But $B$ is $V$-connected, so by Remark 1, $C$ should be numerically proportional to $V$, which is impossible because $\widehat{D} \cdot R_V = 0$.

Let us finally show that $C \cdot \widehat{D} = 0$ if and only if $[C] \in R_V$: actually, if $C \cdot \widehat{D} = 0$, the Claim in the proof of Proposition 1 shows that either $C \subset B$ or $C$ is contained in a fiber of $q$, both are $V$-connected, hence $[C] \in R_V$ by Remark 1. □

Unfortunately, $B$ is not $V$-connected in general as shown by the following example.

**Example 3.** Let us go back to Example 1. We have $\mathcal{N}_{F_0/X} = \Omega^1_{\mathbb{P}^2}(1)$ and $(-K_X)_{|F_0} = \mathcal{O}_{F_0}(2)$. Observe that $V$ is a family of extremal irreducible rational curves of anticanonical degree 2.

If we consider $X \times \mathbb{P}^1$ with the same family of curves, we have $\dim Y = 4$, $f_V = 1$ and $B = F_0 \times \mathbb{P}^1$, which is not $V$-connected.

We finally get the following result: if $B$ has the smallest possible dimension, then it is $V$-connected.

**Lemma 3.** *Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ be a covering and quasi-unsplit family of 1-cycles on $X$. If $\dim B = f_V + 1$, then every connected component of $B$ is a $V$-equivalence class.*

*Proof.* By Proposition 1, we know that $B$ is the union of all $V$-equivalence classes whose dimension is $f_V + 1$. Since each of these equivalence classes must contain an irreducible component of $B$, they are in a finite number, and each is contained in a connected component of $B$.

So if $B_0$ is a connected component of $B$, we have $B_0 = F_1 \cup \cdots \cup F_r$, where each $F_i$ is a $V$-equivalence class. We want to show that $r = 1$.

Assume by contradiction that $r > 1$. Observe that the $F_i$'s are disjoint and $B_0$ is connected, hence at least one $F_i$ is not a closed subset of $X$; assume it is $F_1$.

Then $F_1$ is a countable union of closed subsets. Considering the decomposition of $B_0$ as a union of irreducible components, we find an irreducible component $T$ of $B_0$ such that

$$T = \bigcup_{m \in \mathbb{N}} K_m$$

where each $K_m$ is a nonempty proper closed subset of $T$. Since $T$ is an irreducible complex projective variety, this is impossible. □

We reformulate in a single result what we proved so far.

**Proposition 3.** *Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ a covering and quasi-unsplit family of rational 1-cycles on $X$. Then:*

(i) *either $B = \emptyset$ or $\dim B \geq f_V + 1$,*
(ii) *if $B = \emptyset$ or if $\dim B = f_V + 1$ then there exists a nef divisor $\widehat{D}$ on $X$ such that for any curve $C$ in $X$, $\widehat{D} \cdot C = 0$ if and only if $[C] \in R_V$.*

## 5. Existence of a geometric quotient

Let $V$ be a covering and quasi-unsplit family of 1-cycles on $X$. Observe that the geometric quotient $q' \colon X \to Y'$ for $V$, provided it exists, has the following property: *for any irreducible curve $C$ in $X$, $q'(C)$ is a point if and only if $[C] \in R_V$.*

Conversely, we show that a morphism with the property above is quite close to being a geometric quotient.

**Proposition 4.** *Let $X$ be a normal and $\mathbb{Q}$-factorial projective variety, and $V$ a covering and quasi-unsplit family of 1-cycles on $X$. Assume that there exists a morphism with connected fibers $q' \colon X \to Y'$ onto a complete and normal algebraic variety $Y'$, such that for any irreducible curve $C$ in $X$, $q'(C)$ is a point if and only if $[C] \in R_V$. Then there exists a birational morphism $\psi \colon Y \to Y'$ that fits into the commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow{\ e\ } & X \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle q} & \downarrow{\scriptstyle q'} \\ Y & \xrightarrow{\ \psi\ } & Y' \end{array} \qquad (3)$$

*Moreover, if $B' := q'(B)$, we have $(q')^{-1}(B') = B$, and*

$$B' = \{y \in Y' \mid \dim(q')^{-1}(y) > f_V\} = \{y \in Y' \mid \dim \psi^{-1}(y) > 0\}.$$

*In particular, every fiber of $q'$ over $Y' \setminus B'$ is a $V$-equivalence class.*

Observe that in Example 1, $\psi$ is not an isomorphism.

*Proof.* Let $C \subset X$ be an irreducible curve contained in a fiber $F$ of $q'$. Then $[C] \in R_V$, so the Claim in the proof of Proposition 1 gives that either $C \subseteq B$, or $C \cap B = \emptyset$ and $q(C)$ is a point. Since $F$ is connected, we see that either $F \cap B = \emptyset$, or $F \subseteq B$. This means that $(q')^{-1}(q'(B)) = B$.

The existence of $\psi$ as in (3) follows easily from the normality of $Y$ and the fact that $q'$ contracts all curves in $V$, hence all $V$-equivalence classes. Observe that $\psi$ is surjective with connected fibers.

Let us show that $p$ contracts to a point any fiber of $q' \circ e$ over $Y' \setminus B'$.

Let $F$ be a fiber of $q'$ over $Y' \setminus B'$; then we have $F \subset X \setminus B$. Choose an irreducible curve $C \subseteq e^{-1}(F)$. Then $e(C) \subseteq F$ and $e(C) \cap B = \emptyset$, so $q(e(C)) = p(C)$ is a point. Since $e^{-1}(F)$ is connected, we have shown that $p$ contracts $e^{-1}(F)$ to a point. Since $Y$ and $Y'$ are normal, this implies that $\psi$ is an isomorphism over $Y' \setminus B'$.

Finally, let $y \in B'$ and let $F' = (q')^{-1}(y)$. Then $F' \subseteq B$, so $e$ has positive-dimensional fibers on $F'$, and $\dim e^{-1}(F') > \dim F' \geq f_V$. Since $e^{-1}(F') = p^{-1}(\psi^{-1}(y))$ and $p$ has all fibers of dimension $f_V$, we must have $\dim \psi^{-1}(y) > 0$. □

*Proof of Theorem 2.* If $B$ is empty, then the statement is clear. Assume that $B$ is not empty. Then Proposition 3 and Lemma 3 show that $\dim B = f_V + 1 = n - 2$, every connected component of $B$ is a $V$-equivalence class, and there exists a nef divisor $\widehat{D}$ on $X$ such that for any curve $C$ in $X$, $\widehat{D} \cdot C = 0$ if and only if $[C] \in R_V$.

We have to show that $-K_X \cdot R_V > 0$. Let $V'$ be the covering family of rational 1-cycles on $X$ given by Lemma 2, and consider a resolution of singularities $f\colon X' \to X$. The family $V'$ determines a covering family $V''$ of rational 1-cycles in $X'$. If $C_0 \subset X$ is a general element of the family $V'$, then $C' := \overline{f^{-1}(C_0 \setminus \mathrm{Sing}(X))}$ is a general element of $V''$, and $C_0 = f_*(C')$.

Since $C_0$ is reduced and irreducible, so is $C'$. Moreover $V''$ is covering, so $C'$ is a free curve in $X'$, and it has positive anticanonical degree.

Let $m \in \mathbb{Z}_{>0}$ be such that $mK_X$ is Cartier. Since $X$ has canonical singularities, we have

$$mK_{X'} = f^*(mK_X) + \sum_i a_i E_i,$$

where $E_i$ are exceptional divisors of $f$ and $a_i \in \mathbb{Z}_{\geq 0}$. Then

$$-mK_X \cdot C_0 = -f^*(mK_X) \cdot C' = -mK_{X'} \cdot C' + \sum_i a_i E_i \cdot C' > 0.$$

This gives $-K_X \cdot R_{V'} > 0$ and thus $-K_X \cdot R_V > 0$.

Since $X$ has canonical singularities, the cone theorem and the contraction theorem hold for $X$ (see [Deb01, Theorems 7.38 and 7.39]). Moreover, the extremal ray $R_V$ lies in the $K_X$-negative part of the Mori cone, hence it can be contracted.

Let $q'\colon X \to Y'$ be the extremal contraction; then $Y'$ is a normal, projective variety, and it is $\mathbb{Q}$-factorial by [Deb01, Proposition 7.44].

Applying Proposition 4, we see that all fibers of $q'$ over $Y' \setminus q'(B)$ are $V$-equivalence classes. Since connected components of $B$ are $V$-equivalence classes, they are exactly the fibers of $q'$ over $q'(B)$, and we have the statement. □

## 6. The toric case: proof of Theorem 3

*Step 1: the case with Picard number one.*

If $\rho_X = 1$, the statement is just that $X$ is $V$-connected. This is well known, and can be seen as follows. Consider a divisor $\widehat{D}$ on $X$ constructed as in the proof of Proposition 1. This is an effective divisor which cannot be ample because $\widehat{D} \cdot R_V = 0$. Since $\rho_X = 1$, the only possibility is that $Y$ is a point and $\widehat{D} = 0$.

Recall the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ V & & \end{array}$$

Recall also that if $D \subset X$ is a prime invariant Weil divisor, there is a natural inclusion $i_D \colon \mathcal{N}_1(D)_{\mathbb{R}} \hookrightarrow \mathcal{N}_1(X)_{\mathbb{R}}$.

*Step 2: let $D \subset X$ be a prime invariant Weil divisor such that $D \cdot R_V = 0$. Then there exists a covering and quasi-unsplit family $V_D$ of 1-cycles in $D$ such that $i_D(R_{V_D}) = R_V$.*

Choose an irreducible component $W$ of $F^{-1}(D)$ which dominates $D$. Set $V'_D := \pi(W)$, and let $\mathcal{C}'_D$ be an irreducible component of $\pi^{-1}(V'_D)$ containing $W$. Consider the normalization $\mathcal{C}_D$ of $\mathcal{C}'_D$, and let $\pi_D \colon \mathcal{C}_D \to V_D$ be the Stein factorization of the composite map $\mathcal{C}_D \to \mathcal{C}'_D \to V'_D$. Finally, let $F_D \colon \mathcal{C}_D \to X$ be the induced map.

For $v \in V_D$, set $G_v := F_D(\pi_D^{-1}(v))$. Then $G_v \cap D \neq \emptyset$, $G_v$ is connected, and $G_v \cdot D = 0$ because $V$ is quasi-unsplit. This implies $G_v \subseteq D$, hence $F_D(\mathcal{C}_D) \subseteq D$. Moreover, since $W$ dominates $D$, we have $F_D(\mathcal{C}_D) = D$.

Since $V_D$ is normal, there is a holomorphic map $V_D \to \mathrm{Chow}(D)$. Then after replacing $V_D$ by its image in $\mathrm{Chow}(D)$ and $\mathcal{C}_D$ by its image in $\mathrm{Chow}(D) \times X$, we get the desired family.

*Step 3: if $\rho_X > 1$, then there exists an invariant prime Weil divisor having intersection zero with $R_V$.*

In fact, let $q \colon X \dashrightarrow Y$ be the rational map associated to $V$. Since $\rho_X > 1$, $Y$ is not a point. Let $D$ be a prime divisor in $Y$ intersecting $q(X^0)$ and set $D' := \overline{q^{-1}(D)}$. Since there are curves of the family $V$ disjoint from $D'$, we have $D' \cdot R_V = 0$. Moreover, $D'$ is linearly equivalent to $\sum_i a_i D_i$, where $a_i \in \mathbb{Q}_{>0}$ and $D_i$ are invariant prime Weil divisors. Hence the statement.

*Step 4: we prove the statement.*

Let $\Sigma_X$ be the fan of $X$ in $N \cong \mathbb{Z}^n$, and let $G_X$ be the set of primitive generators of one-dimensional cones in $\Sigma_X$. It is well known that $G_X$ is in bijection with the set of invariant prime divisors of $X$; for any $x \in G_X$, we denote by $D_x$ the associated divisor. Recall that for any class $\gamma \in \mathcal{N}_1(X)_{\mathbb{Q}}$, we have

$$\sum_{x \in G_X} (\gamma \cdot D_x)x = 0 \quad \text{in } N \otimes_{\mathbb{Z}} \mathbb{Q},$$

and that the assignment $\gamma \mapsto \sum_{x \in G_X} (\gamma \cdot D_x) x$ gives a canonical identification of $\mathcal{N}_1(X)_{\mathbb{Q}}$ with the $\mathbb{Q}$-vector space of linear relations with rational coefficients among $G_X$.

Let $m_1 x_1 + \cdots + m_h x_h = 0$ be the relation corresponding to the numerical class of a general cycle of $V$, with $x_i \in G_X$ and $m_i$ nonzero rational numbers for all $i$. Since $V$ is covering and quasi-unsplit, all $m_i$'s must be positive.

The following two statements are equivalent (see [Rei83, Theorem 2.4] and [Cas03, Theorem 2.2]):

(a) there exists a $\mathbb{Q}$-factorial, projective toric variety $Y'$ and a flat, equivariant morphism $q' \colon X \to Y'$ such that for any curve $C$ in $X$, $q'(C)$ is a point if and only if $[C] \in R_V$;
(b) for any $\tau \in \Sigma_X$ such that $x_1, \ldots, x_h \notin \tau$, we have

$$\tau + \langle x_1, \ldots, \check{x}_i, \ldots, x_h \rangle \in \Sigma_X \quad \text{for all } i = 1, \ldots, h. \tag{4}$$

Let us show (b) by induction on the dimension of $X$.

If $\rho_X = 1$, we have already shown (a) and hence (b) in Step 1.

Assume that $\rho_X > 1$. Observe that if $y \in G_X$, we have $D_y \cdot R_V = 0$ if and only if $y$ is different from $x_1, \ldots, x_h$. So by Step 3, we know that $G_X \setminus \{x_1, \ldots, x_h\}$ is nonempty.

Clearly, it is enough to check (4) for any maximal $\tau$ in $\Sigma_X$ not containing any $x_i$. Since $\{x_1, \ldots, x_h\} \subsetneq G_X$, such a maximal $\tau$ will have positive dimension.

Let $y \in G_X \cap \tau$. We have $D_y \cdot R_V = 0$, so by Step 2 there exists a quasi-unsplit, covering family $V_{D_y}$ in $D_y$ such that $i_{D_y}(R_{V_{D_y}}) = R_V$.

Set $\overline{N} := N/\mathbb{Z} \cdot y$ and for any $z \in N$, write $\overline{z}$ for its image in $\overline{N}$. The fan $\Sigma_{D_y}$ of $D_y$ is given by the projections in $\overline{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ of all cones of $\Sigma_X$ containing $y$. The relation corresponding to the numerical class of a general cycle of $V_{D_y}$ is $\lambda m_1 \overline{x}_1 + \cdots + \lambda m_h \overline{x}_h = 0$ for some $\lambda \in \mathbb{Q}_{>0}$. By induction, we know that (b) holds for $V_{D_y}$ in $D_y$. In particular, the projection $\overline{\tau}$ of $\tau$ is in $\Sigma_{D_y}$, so we have

$$\overline{\tau} + \langle \overline{x}_1, \ldots, \check{\overline{x}}_i, \ldots, \overline{x}_h \rangle \in \Sigma_{D_y} \quad \text{for all } i = 1, \ldots, h.$$

This yields (4).

Finally, since $q'$ is equidimensional, all fibers must be $V$-equivalence classes and $B = \emptyset$.

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