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Upper bounds for singular perturbation problems involving gradient fields

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Abstract. We prove an upper bound for the Aviles–Giga problem, which involves the minimization of the energy $E_\varepsilon(v) = \varepsilon \int_\Omega \left|\nabla^2 v\right|^2 dx + \varepsilon^{-1} \int_\Omega (1 - |\nabla v|^2)^2\, dx$ over $v \in H^2(\Omega)$, where $\varepsilon > 0$ is a small parameter. Given $v \in W^{1,\infty}(\Omega)$ such that $\nabla v \in BV$ and $|\nabla v| = 1$ a.e., we construct a family $\{v_\varepsilon\}$ satisfying: $v_\varepsilon \to v$ in $W^{1,p}(\Omega)$ and $E_\varepsilon(v_\varepsilon) \to \frac{1}{3}\int_{J_{\nabla v}} |\nabla^+ v - \nabla^- v|^3\, d\mathcal{H}^{N-1}$ as ε goes to 0.

1. Introduction

Consider the energy functional

$$E_\varepsilon(v) = \varepsilon \int_\Omega |\nabla^2 v|^2 + \frac{1}{\varepsilon}\int_\Omega (1 - |\nabla v|^2)^2 \tag{1.1}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^N$, v is a scalar function and ε is a small parameter. Energies similar to (1.1) appear in different physical situations: smectic liquid crystals, blisters in thin films, micromagnetics (see [13] and the references therein). Clearly, one expects that any limit of the minimizers to (1.1) should satisfy the eikonal equation

$$|\nabla v| = 1 \quad \text{a.e. in } \Omega. \tag{1.2}$$

Aviles and Giga [3] made a conjecture, based on a certain ansatz for the minimizers, that the limiting energy should take the form

$$E(v) = \frac{1}{3}\int_{J_{\nabla v}} |\nabla^+ v - \nabla^- v|^3\, d\mathcal{H}^{N-1},$$

where $J_{\nabla v}$ is the jump set of ∇v and $\nabla^\pm v$ are the traces of ∇v on the two sides of the jump set (see Section 2 below for the exact definitions of the notions needed from the theory of functions of bounded variation). Most of the results on this problem treat the two-dimensional case $N = 2$ (an example due to De Lellis [6] shows that the Aviles–Giga ansatz does not hold for $N \geq 3$), so we assume $N = 2$ in the review of the known results below. Support for the Aviles–Giga conjecture was given in the work of Jin and

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Kohn [13] who gave a lower bound for E_ε under the boundary conditions $v = 0$ and $\partial v/\partial n = -1$ on $\partial\Omega$. Aviles and Giga [4] refined the method of [13]. They defined a functional $\mathcal{J}$ on $W^{1,3}(\Omega)$ that coincides with the functional E on functions u satisfying (1.2) such that ∇u has bounded variation. Recall that there exist functions u in $W^{1,3}(\Omega)$ satisfying $\mathcal{J}(u) < \infty$, for which ∇u is not in BV (see [1]). Another important contribution is due to Ambrosio, De Lellis and Mantegazza [1] and DeSimone, Kohn, Müller and Otto [7], who independently proved that for any family $\{v_\varepsilon\}$ satisfying $E_\varepsilon(v_\varepsilon) \le C$, $\{\nabla v_\varepsilon\}$ is pre-compact in $L^3(\Omega)$. It was shown by Aviles and Giga [4] that $\mathcal{J}$ is lower semicontinuous in the strong topology of $W^{1,3}(\Omega)$. However, the Γ-convergence problem for the functionals $\{E_\varepsilon\}$ is still open, since for a given u satisfying $\mathcal{J}(u) < \infty$ and (1.2), it is not known whether there exists a family $\{u_\varepsilon\}$ satisfying $u_\varepsilon \to u$ and $E_\varepsilon(u_\varepsilon) \to \mathcal{J}(u)$ as $\varepsilon \to 0^+$. The main contribution of the present article is the construction of such a family for u satisfying $\nabla u \in \mathrm{BV}$ and (1.2).

In general, proving upper bounds is considered an easier task than proving lower bounds, since explicit constructions may be used. However, in problems involving highly nonregular functions as in our case, the proof of an effective upper bound is far from being obvious. In fact, to our knowledge, for the minimization problem (1.1) an upper bound was proved only for very special cases, like the case of u which is the distance to the boundary of an ellipse (see Jin and Kohn [13]; see also Ercolani, Indik, A. C. Newell and T. Passot [8] for a related result).

Our main result, Theorem 1.1, establishes an upper bound for a more general energy functional than (1.1), and the latter case is then deduced in Corollary 1.1.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary of class C^2. Let $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}$ be a C^2 function such that $F(a, b, c) \ge 0$ for all a, b and c. Let $f \in \mathrm{BV}(\Omega, \mathbb{R}^q) \cap L^\infty(\Omega, \mathbb{R}^q)$ and $v \in W^{1,\infty}(\Omega, \mathbb{R})$ be such that $\nabla v \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ and $F(\nabla v(x), v(x), f(x)) = 0$ a.e. in Ω.*

(i) *For every $p \ge 1$ there exists a family $\{v_\varepsilon\} \subset C^2(\mathbb{R}^N)$ of functions satisfying*

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = v(x) \quad \text{in } W^{1,p}(\Omega)$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\{\varepsilon\int_\Omega |\nabla^2 v_\varepsilon(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x))\,dx\right\}\\
&\quad = 2\int_{J_{\nabla v}} |\nabla v^+(x) - \nabla v^-(x)|\\
&\qquad \times \inf_{\tau\in[0,1]}\left\{\int_0^\tau \sqrt{F(s\nabla v^-(x) + (1-s)\nabla v^+(x), v(x), f^+(x))}\,ds\right.\\
&\qquad \left. + \int_\tau^1 \sqrt{F(s\nabla v^-(x) + (1-s)\nabla v^+(x), v(x), f^-(x))}\,ds\right\} d\mathcal{H}^{N-1}(x),
\end{aligned}$$

where we assume that the orientation of J_f coincides $\mathcal{H}^{N-1}$-a.e. with the orientation of $J_{\nabla v}$ on $J_f \cap J_{\nabla v}$.

(ii) *Moreover, if there exists $h \in C^2(\mathbb{R}^N)$ which satisfies the boundary conditions $h(x) = v(x)$ and $\nabla h(x) = T\nabla v(x)$ on $\partial\Omega$, then we can choose v_ε that satisfies the same boundary conditions, $v_\varepsilon(x) = v(x)$ and $\nabla v_\varepsilon(x) = T\nabla v(x)$ on $\partial\Omega$. If v satisfies the additional condition $v \geq 0$ in Ω then also $v_\varepsilon \geq 0$ in Ω.*

In Section 4 we prove assertion (i) of Theorem 1.1 under a slightly more general condition, namely, that Ω is a BVG-domain. The next corollary is a direct consequence of Theorem 1.1.

Corollary 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain. Let $v \in W^{1,\infty}(\Omega)$ satisfy $\nabla v \in \mathrm{BV}(\Omega, \mathbb{R}^N)$ and $|\nabla v| = 1$ a.e. in Ω.*

(i) *For every $p \geq 1$ there exists a family $\{v_\varepsilon\} \subset C^2(\mathbb{R}^N)$ of functions satisfying*

$$\lim_{\varepsilon\to 0^+} v_\varepsilon(x) = v(x) \quad \text{in } W^{1,p}(\Omega)$$

and

$$\lim_{\varepsilon\to 0}\left\{\varepsilon\int_\Omega |\nabla^2 v_\varepsilon(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega (1-|\nabla v_\varepsilon(x)|^2)^2\,dx\right\} = \frac{1}{3}\int_{J_{\nabla v}} |\nabla v^+(x) - \nabla v^-(x)|^3\,d\mathcal{H}^{N-1}(x).$$

(ii) *Moreover, if v satisfies the boundary conditions $v = 0$ and $\partial v/\partial n = -1$ on $\partial\Omega$, then we can choose v_ε to satisfy the same boundary conditions, $v_\varepsilon = 0$ and $\partial v_\varepsilon/\partial n = -1$ on $\partial\Omega$. If v satisfies the additional condition $v \geq 0$ in Ω then also $v_\varepsilon \geq 0$ in Ω.*

Our main tool for constructing the family $\{v_\varepsilon\}$ is convolution with a smoothing kernel. This is of course a standard technique. The new ingredient in our method, however, is the special choice of the kernel, which is adapted to the particular functional, using an optimization process. We emphasize that although the results are stated and proved in any dimension N, their interest is mainly for the two-dimensional case.

The paper is organized as follows. In Section 2 we recall some known properties and results on BV-spaces which are used throughout this paper. In Section 3 we prove an upper bound for a vector-valued problem. This bound is not optimal (in the vectorial case) but it provides the necessary tool for the proof of our main result, for the second order problem, which is the subject of Section 4. In the Appendix we give the proof of two technical results which are used in Section 4.

Remark 1.1. A similar result to ours for the Aviles–Giga functional has been independently obtained by Sergio Conti and Camillo de Lellis [5].

Remark 1.2. The results of this paper were announced in [14].

2. Preliminaries

In this section we present some known results on BV-spaces. We rely mainly on the book [2] by Ambrosio, Fusco and Pallara. Other sources are the books by Hudjaev and Volpert [15], Giusti [12] and Evans and Gariepy [10]. We begin by introducing some notation. For every $\boldsymbol{\nu} \in S^{N-1}$ (the unit sphere in $\mathbb{R}^N$) and $R > 0$ we define

$$B_R^+(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : |y-x| < R,\ (y-x)\cdot\boldsymbol{\nu} > 0\}, \tag{2.1}$$

$$B_R^-(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : |y-x| < R,\ (y-x)\cdot\boldsymbol{\nu} < 0\}, \tag{2.2}$$

$$H_+^N(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : (y-x)\cdot\boldsymbol{\nu} > 0\}, \tag{2.3}$$

$$H_-^N(x,\boldsymbol{\nu}) = \{y \in \mathbb{R}^N : (y-x)\cdot\boldsymbol{\nu} < 0\}, \tag{2.4}$$

$$H_{\boldsymbol{\nu}} = \{y \in \mathbb{R}^N\ :\ y\cdot\boldsymbol{\nu} = 0\}. \tag{2.5}$$

Next we recall the definition of the space of functions with bounded variation (BV). We denote by $\mathcal{L}^N$ the Lebesgue measure in $\mathbb{R}^N$.

Definition 2.1. *Let Ω be a domain in $\mathbb{R}^N$ and let $f = (f_1,\dots,f_m)$ be a function in $L^1(\Omega,\mathbb{R}^m)$. We say that $f \in \mathrm{BV}(\Omega,\mathbb{R}^m)$ if*

$$\int_\Omega |Df| := \sup\left\{ \int_\Omega \sum_{k=1}^m f_k \operatorname{div}\delta_k \, d\mathcal{L}^N : \delta_k \in C_c^1(\Omega,\mathbb{R}^N), \forall k,\ \sum_{k=1}^m |\delta_k(x)|^2 \le 1,\ \forall x \in \Omega \right\}$$

is finite. In this case we define the BV-*norm of f by $\|f\|_{\mathrm{BV}} := \int_\Omega |f|\, d\mathcal{L}^N + \int_\Omega |Df|$.*

We recall below some basic notions of BV-functions (see [2]).

Definition 2.2. *Let Ω be a domain in $\mathbb{R}^N$. Consider a function $f \in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ and a point $x \in \Omega$.*

(i) *We say that x is a* point of approximate continuity *of f if there exists $z \in \mathbb{R}^m$ such that*

$$\lim_{\rho\to 0^+} \frac{\int_{B_\rho(x)} |f(y)-z|\,dy}{\mathcal{L}^N(B_\rho(x))} = 0.$$

In this case z is called an approximate limit *of f at x and we write $z = \tilde{f}(x)$. The set of approximate continuity of f in Ω is denoted by G_f.*

(ii) *We say that x is an* approximate jump point *of f if there exist $a, b \in \mathbb{R}^m$ and $\boldsymbol{\nu} \in S^{N-1}$ such that $a \ne b$ and*

$$\lim_{\rho\to 0^+} \frac{\int_{B_\rho^+(x,\boldsymbol{\nu})} |f(y)-a|\,dy}{\mathcal{L}^N(B_\rho(x))} = 0, \qquad \lim_{\rho\to 0^+} \frac{\int_{B_\rho^-(x,\boldsymbol{\nu})} |f(y)-b|\,dy}{\mathcal{L}^N(B_\rho(x))} = 0. \tag{2.6}$$

The triple $(a, b, \boldsymbol{\nu})$, uniquely determined by (2.6) up to a permutation of (a, b) and a change of sign of $\boldsymbol{\nu}$, is denoted by $(f^+(x), f^-(x), \boldsymbol{\nu}_f(x))$. We shall call $\boldsymbol{\nu}_f(x)$ the approximate jump vector *and we shall sometimes write simply $\boldsymbol{\nu}(x)$ if the reference to the function f is clear. The set of approximate jump points is denoted by J_f. A choice of $\boldsymbol{\nu}(x)$ for every $x \in J_f$ (which is unique up to sign) determines an orientation of J_f. At a point of approximate continuity x, we shall use the convention $f^+(x) = f^-(x) = \tilde{f}(x)$.*

We recall the following results on BV-functions that we shall use. They are all taken from [2]. In all of them Ω is a domain in $\mathbb{R}^N$ and f is a function in $\mathrm{BV}(\Omega,\mathbb{R}^m)$.

Theorem 2.1 (Theorems 3.69 and 3.78 from [2]).

(i) $\mathcal{H}^{N-1}$*-almost every point in* $\Omega \setminus J_f$ *is a point of approximate continuity of* f.

(ii) *The set* J_f *is a countably* $\mathcal{H}^{N-1}$*-rectifiable Borel set, oriented by* $\boldsymbol{\nu}(x)$. *In other words,* J_f *is* σ*-finite with respect to* $\mathcal{H}^{N-1}$ *and there exist countably many* $(N-1)$*-dimensional* C^1 *hypersurfaces* $\{S_k\}_{k=1}^{\infty}$ *such that* $\mathcal{H}^{N-1}(J_f - \bigcup_{k=1}^{\infty} S_k) = 0$ *and for any* k, *for* $\mathcal{H}^{N-1}$*-almost every* $x \in J_f \cap S_k$, *the approximate jump vector* $\boldsymbol{\nu}(x)$ *is normal to* S_k *at the point* x.

(iii) $[(f^+ - f^-) \otimes \boldsymbol{\nu}_f](x) \in L^1(J_f, d\mathcal{H}^{N-1})$.

Theorem 2.2 (Theorems 3.92 and 3.78 from [2]). *The distributional gradient* Df *can be decomposed as a sum of three Borel regular finite matrix measures on* Ω,

$$Df = D^a f + D^c f + D^j f$$

with

$$D^a f = (\nabla f)\,\mathcal{L}^N \quad \text{and} \quad D^j f = (f^+ - f^-) \otimes \boldsymbol{\nu}_f \mathcal{H}^{N-1} \llcorner J_f,$$

where $D^a f$, $D^c f$ *and* $D^j f$ *are the absolutely continuous part, the Cantor and the jump part of* Df, *respectively, and* $\nabla f \in L^1(\Omega, \mathbb{R}^{m\times N})$ *is the approximate differential of* f. *The three parts are singular to each other. We have the following properties:*

(i) $(D^a f)(A) = 0$ *for every Borel set* $A \subset \Omega$ *such that* $\mathcal{L}^N(A) = 0$.

(ii) *The support of* $D^c f$ *is concentrated on a set of* $\mathcal{L}^N$*-measure zero, but* $(D^c f)(B) = 0$ *for any Borel set* $B \subset \Omega$ *which is* σ*-finite with respect to* $\mathcal{H}^{N-1}$.

(iii) *The support of* $D^j f$ *is concentrated on a countably* $\mathcal{H}^{N-1}$*-rectifiable set. Moreover,* $(D^a f)(f^{-1}(H)) = 0$ *and* $(D^c f)(\tilde{f}^{-1}(H)) = 0$ *for all* $H \subset \mathbb{R}^m$ *satisfying* $\mathcal{H}^1(H) = 0$.

Theorem 2.3 (Volpert chain rule, Theorems 3.96 and 3.99 from [2]). *Let* $\Phi : \mathbb{R}^m \to \mathbb{R}^q$ *be a Lipschitz function satisfying* $\Phi \in C^1$ *if* $m > 1$ *and* $\Phi(0) = 0$ *if* $|\Omega| = \infty$. *Then* $v(x) = (\Phi \circ f)(x)$ *belongs to* $\mathrm{BV}(\Omega, \mathbb{R}^q)$ *and we have*

$$D^a v = \nabla\Phi(f)\,\nabla f\,\mathcal{L}^N, \qquad D^c v = \nabla\Phi(\tilde{f})\,D^c f,$$
$$D^j v = [\Phi(f^+) - \Phi(f^-)] \otimes \boldsymbol{\nu}_f\,\mathcal{H}^{N-1} \llcorner J_f.$$

We also recall that the trace operator T is continuous between $\mathrm{BV}(\Omega)$ endowed with the strong topology (or more generally, the topology induced by strict convergence) and $L^1(\partial\Omega, \mathcal{H}^{N-1} \llcorner \partial\Omega)$, provided that Ω has a bounded Lipschitz boundary (see [2, Theorems 3.87 and 3.88]).

3. An upper bound construction

In this section we establish two basic estimates needed in the proof of an upper bound for a first order problem. The construction is not optimal for a general vector-valued problem, but will give the sharp estimate for the Aviles–Giga problem involving gradient fields (see Corollary 1.1). In the first estimate (see Theorem 3.1), we compute the limit of the energies of functions defined by convolution with a fixed smoothing kernel. In the second (see Theorem 3.2 and Corollary 3.1), we compute the infimum of this expression over all smoothing kernels.

Throughout this section we assume that Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Next we define a special class of mollifiers that we shall use in the upper bound construction. Note that in contrast with standard mollifiers, our mollifiers depend on two variables.

Definition 3.1. *The class $\mathcal{V}$ consists of all the functions $\eta \in C^2(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ which satisfy the following:*

$$\int_{\mathbb{R}^N} \eta(z, x)\, dz = 1, \qquad \forall x \in \Omega,$$

and there exist $R > 0$ and a bounded open set $\Omega' \supset\supset \Omega$ such that

$$\operatorname{supp} \eta \subset B_R(0) \times \Omega'. \tag{3.1}$$

We shall denote by $\nabla_1 \eta$ and $\nabla_2 \eta$ the gradient of η with respect to the variables z and x respectively. For $\varphi \in BV(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ and $\eta \in \mathcal{V}$ let $R > 0$ be given as in Definition 3.1. By [2, Proposition 3.21] we may extend φ to a function $\bar{\varphi} \in BV(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\bar{\varphi} = \varphi$ a.e. in Ω, $\operatorname{supp} \bar{\varphi}$ is compact and $\|D\bar{\varphi}\|(\partial\Omega) = 0$ (from the proof of Proposition 3.21 in [2] it follows that if φ is bounded then its extension is also bounded). For every $\varepsilon > 0$ and every $x \in \mathbb{R}^N$ define a function $\psi_\varepsilon \in C^1(\mathbb{R}^N)$ by

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y - x}{\varepsilon}, x\right) \bar{\varphi}(y)\, dy = \int_{\mathbb{R}^N} \eta(z, x) \bar{\varphi}(x + \varepsilon z)\, dz.$$

Next we prove:

Lemma 3.1. $\int_\Omega |\psi_\varepsilon(x) - \varphi(x)|\, dx = O(\varepsilon)$ *as* $\varepsilon \to 0$.

Proof. By the definition of ψ_ε we have

$$\begin{aligned} \int_\Omega |\psi_\varepsilon(x) - \varphi(x)|\, dx &\le \int_\Omega \int_{\mathbb{R}^N} |\eta(z, x)| \cdot |\bar{\varphi}(x + \varepsilon z) - \bar{\varphi}(x)|\, dz\, dx \\ &\le \int_{B_R(0)} \{ \sup_{(z,x) \in B_R(0) \times \Omega} |\eta(z, x)| \} \left(\int_\Omega |\bar{\varphi}(x + \varepsilon z) - \bar{\varphi}(x)|\, dx \right) dz. \end{aligned} \tag{3.2}$$

Since

$$\int_\Omega |\bar{\varphi}(x + \varepsilon z) - \bar{\varphi}(x)|\, dx \le \varepsilon |z| \cdot \|D\bar{\varphi}\|(\mathbb{R}^N)$$

(see for example [2, Exercise 3.3]), we conclude from (3.2) that

$$\int_\Omega |\psi_\varepsilon(x) - \varphi(x)|\, dx \le \varepsilon C \|D\bar{\varphi}\|(\mathbb{R}^N) \int_{B_R(0)} |z|\, dz = O(\varepsilon). \qquad \square$$

The next lemma provides an estimate for the gradient of ψ_ε.

Lemma 3.2. *If $U \subset \mathbb{R}^N$ is an open bounded set and $U' \subset \mathbb{R}^N$ is an open set satisfying $U \subset\subset U'$, then for any $0 < \varepsilon < (1/R)\,\mathrm{dist}(U, \mathbb{R}^N \setminus U')$ we have*

$$\int_U |\nabla \psi_\varepsilon(x)|\,dx \le C\|\bar{\varphi}\|_{\mathrm{BV}(U')},$$

where C depends on N and η only.

Proof. Let $\boldsymbol{\sigma}(x) \in C_0^1(U, \mathbb{R}^N)$ be a vector field satisfying $|\boldsymbol{\sigma}| \le 1$ in U. Then, for all $0 < \varepsilon < (1/R)\,\mathrm{dist}(U, \mathbb{R}^N \setminus U')$ we have

$$\begin{aligned}
\int_U \nabla\psi_\varepsilon(x)\cdot\boldsymbol{\sigma}(x)\,dx &= -\int_U \psi_\varepsilon(x)\,\mathrm{div}\,\boldsymbol{\sigma}(x)\,dx \\
&= -\int_U\int_{B_R(0)} \eta(z,x)\bar{\varphi}(x+\varepsilon z)\,\mathrm{div}\,\boldsymbol{\sigma}(x)\,dz\,dx \\
&= -\int_{B_R(0)}\int_{U'} \eta(z,x-\varepsilon z)\bar{\varphi}(x)\,\mathrm{div}_x\,\boldsymbol{\sigma}(x-\varepsilon z)\,dx\,dz \\
&= -\int_{B_R(0)}\int_{U'} \bar{\varphi}(x)\,\mathrm{div}_x(\eta(z,x-\varepsilon z)\boldsymbol{\sigma}(x-\varepsilon z))\,dx\,dz \\
&\quad + \int_{B_R(0)}\int_{U'} \bar{\varphi}(x)\nabla_2\eta(z,x-\varepsilon z)\cdot\boldsymbol{\sigma}(x-\varepsilon z)\,dx\,dz \\
&\le C\|D\bar{\varphi}(x)\|(U') + C\int_{U'}|\bar{\varphi}(x)|\,dx.
\end{aligned} \tag{3.3}$$

The result follows by taking the supremum in (3.3) over all $\boldsymbol{\sigma}$ as above. □

The next proposition gives an estimate for $\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx$.

Proposition 3.1. *For every $\varphi \in \mathrm{BV}(\Omega)\cap L^\infty(\Omega)$ we have*

$$\lim_{\varepsilon\to 0}\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx = \int_{J_\varphi}\Big(\int_{\mathbb{R}} p^2(t,x)\,dt\Big)\cdot(\varphi^+(x)-\varphi^-(x))^2\,d\mathcal{H}^{N-1}(x), \tag{3.4}$$

where

$$p(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} \eta(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^{N-1}(y). \tag{3.5}$$

Proof. We have

$$\begin{aligned}
\nabla\psi_\varepsilon(x) &= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla_x\left\{\eta\left(\frac{y-x}{\varepsilon},x\right)\right\}\bar{\varphi}(y)\,dy \\
&= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\left\{-\frac{1}{\varepsilon}\nabla_1\eta\left(\frac{y-x}{\varepsilon},x\right)+\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\right\}\bar{\varphi}(y)\,dy \\
&= \int_{\mathbb{R}^N}\left\{-\frac{1}{\varepsilon}\nabla_z\eta(z,x)+\nabla_x\eta(z,x)\right\}\bar{\varphi}(x+\varepsilon z)\,dz.
\end{aligned} \tag{3.6}$$

In particular, $\nabla\psi_\varepsilon \in C^1(\mathbb{R}^N)$. By (3.6) we get

$$\begin{aligned}\nabla\psi_\varepsilon(x) &= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\left\{-\frac{1}{\varepsilon}\nabla_1\eta\left(\frac{y-x}{\varepsilon},x\right)+\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\right\}\bar\varphi(y)\,dy\\ &= -\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla_y\eta\left(\frac{y-x}{\varepsilon},x\right)\bar\varphi(y)\,dy+\int_{\mathbb{R}^N}\nabla_x\eta(z,x)\bar\varphi(x+\varepsilon z)\,dz\\ &= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\left(\frac{y-x}{\varepsilon},x\right)d[D\bar\varphi](y)+\int_{\mathbb{R}^N}\nabla_x\eta(z,x)\bar\varphi(x+\varepsilon z)\,dz.\end{aligned}$$

Then

$$\begin{aligned}\int_\Omega\varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx &= \varepsilon\int_\Omega\nabla\psi_\varepsilon(x)\cdot\left(\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\left(\frac{y-x}{\varepsilon},x\right)d[D\bar\varphi](y)\right)dx\\ &\quad+\int_\Omega\left(\int_{\mathbb{R}^N}\bar\varphi(x+\varepsilon z)\nabla_x\eta(z,x)\,dz\right)\cdot\varepsilon\nabla\psi_\varepsilon(x)\,dx.\end{aligned}\tag{3.7}$$

Using Lemma 3.2 we infer that

$$\left|\int_\Omega\left(\int_{\mathbb{R}^N}\bar\varphi(x+\varepsilon z)\nabla_x\eta(z,x)\,dz\right)\cdot\varepsilon\nabla\psi_\varepsilon(x)\,dx\right|\le\varepsilon C\|\bar\varphi\|_{BV(\mathbb{R}^N)}=o_\varepsilon(1).$$

Therefore, by (3.7) we have

$$\begin{aligned}\int_\Omega\varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx &= o_\varepsilon(1)+\varepsilon\int_\Omega\nabla\psi_\varepsilon(x)\cdot\left(\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\left(\frac{y-x}{\varepsilon},x\right)d[D\bar\varphi](y)\right)dx\\ &= o_\varepsilon(1)+\varepsilon\int_{\mathbb{R}^N}\frac{1}{\varepsilon^N}\left(\int_\Omega\eta\left(\frac{y-x}{\varepsilon},x\right)\nabla\psi_\varepsilon(x)\,dx\right)d[D\bar\varphi](y).\end{aligned}$$

Put $w_\varepsilon(x,y)=\varepsilon^{1-N}\eta(\frac{y-x}{\varepsilon},x)\nabla\psi_\varepsilon(x)$. Then we may write

$$\int_\Omega\varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx=o_\varepsilon(1)+D_\varepsilon+H_\varepsilon+I_\varepsilon,\tag{3.8}$$

where

$$D_\varepsilon:=\int_{\mathbb{R}^N\setminus\Omega}\left(\int_{\Omega\cap B_{R\varepsilon}(y)}w_\varepsilon(x,y)\,dx\right)d[D\bar\varphi](y),\tag{3.9}$$

$$H_\varepsilon:=\int_\Omega\left(\int_{\Omega\cap B_{R\varepsilon}(y)}w_\varepsilon(x,y)\,dx-\int_{B_{R\varepsilon}(y)}w_\varepsilon(x,y)\,dx\right)d[D\bar\varphi](y),\tag{3.10}$$

and

$$I_\varepsilon:=\int_\Omega\left(\int_{B_{R\varepsilon}(y)}w_\varepsilon(x,y)\,dx\right)d[D\bar\varphi](y).$$

For every $d>0$ put

$$(\partial\Omega)_d:=\{x\in\mathbb{R}^N:\operatorname{dist}(x,\partial\Omega)<d\}.\tag{3.11}$$

Recall that $R > 0$ is chosen so that $\eta(z, x) = 0$ for all $|z| > R$ and $x \in \mathbb{R}^N$. Therefore, by (3.9) and (3.10) we deduce that

$$D_\varepsilon = \int_{(\partial\Omega)_{R\varepsilon}\setminus\Omega} \left(\int_{\Omega\cap B_{R\varepsilon}(y)} w_\varepsilon(x, y)\, dx \right) d[D\bar{\varphi}](y), \tag{3.12}$$

$$H_\varepsilon = -\int_{(\partial\Omega)_{R\varepsilon}\cap\Omega} \left(\int_{B_{R\varepsilon}(y)\setminus\Omega} w_\varepsilon(x, y)\, dx \right) d[D\bar{\varphi}](y). \tag{3.13}$$

Since $\bar{\varphi} \in L^\infty$ we see from (3.6) that $|\varepsilon\nabla\psi_\varepsilon(\cdot)| \le C$, where $C > 0$ does not depend on ε. Therefore, by (3.12) and (3.13) we infer that

$$|D_\varepsilon| + |H_\varepsilon| \le C\|D\bar{\varphi}\|_{(\partial\Omega)_{R\varepsilon}} = o_\varepsilon(1) \quad \text{(since } \|D\bar{\varphi}\|(\partial\Omega) = 0). \tag{3.14}$$

From (3.8) and (3.14) we obtain

$$\begin{aligned}
\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\, dx &= o_\varepsilon(1) + I_\varepsilon \\
&= o_\varepsilon(1) + \int_\Omega \left(\int_{B_R(0)} \eta(z, y - \varepsilon z)\, \varepsilon\nabla\psi_\varepsilon(y - \varepsilon z)\, dz \right) d[D\bar{\varphi}](y) \\
&= o_\varepsilon(1) + \int_\Omega \left(\int_{B_R(0)} \eta(y, x - \varepsilon y)\, \varepsilon\nabla\psi_\varepsilon(x - \varepsilon y)\, dy \right) d[D\bar{\varphi}](x).
\end{aligned}$$

Next, using (3.6), we get

$$\begin{aligned}
&\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\, dx = o_\varepsilon(1) \\
&- \int_\Omega \left(\int_{B_R(0)} \eta(y, x-\varepsilon y) \left(\int_{B_R(0)} \nabla_1\eta(z, x-\varepsilon y)\bar{\varphi}(x-\varepsilon y+\varepsilon z)\, dz \right) dy \right) d[D\bar{\varphi}](x) \\
&+ \varepsilon \int_\Omega \left(\int_{B_R(0)} \eta(y, x-\varepsilon y) \left(\int_{B_R(0)} \nabla_2\eta(z, x-\varepsilon y)\bar{\varphi}(x-\varepsilon y+\varepsilon z)\, dz \right) dy \right) d[D\bar{\varphi}](x).
\end{aligned} \tag{3.15}$$

As before, we have

$$\begin{aligned}
&\left| \varepsilon \int_\Omega \left(\int_{B_R(0)} \eta(y, x - \varepsilon y) \left(\int_{B_R(0)} \nabla_2\eta(z, x - \varepsilon y)\bar{\varphi}(x - \varepsilon y + \varepsilon z)\, dz \right) dy \right) d[D\bar{\varphi}](x) \right| \\
&\qquad \le \varepsilon C\|\bar{\varphi}\|_{\mathrm{BV}(\mathbb{R}^N)} = o_\varepsilon(1).
\end{aligned}$$

Therefore, by (3.15) we obtain

$$\begin{aligned}
&\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\, dx = o_\varepsilon(1) \\
&- \int_\Omega \left(\int_{B_R(0)} \eta(y, x - \varepsilon y) \left(\int_{B_R(0)} \nabla_1\eta(z, x - \varepsilon y)\bar{\varphi}(x - \varepsilon y + \varepsilon z)\, dz \right) dy \right) d[D\bar{\varphi}](x) \\
&= o_\varepsilon(1) - \int_\Omega \left(\int_{B_R(0)} \eta(y, x) \left(\int_{B_R(0)} \nabla_1\eta(z, x)\bar{\varphi}(x - \varepsilon y + \varepsilon z)\, dz \right) dy \right) d[D\bar{\varphi}](x),
\end{aligned}$$

where in the last equality we used the estimates

$$|\eta(z, x-\varepsilon y)-\eta(z, x)| \leq C\varepsilon|y| \quad \text{and} \quad |\nabla_1\eta(z, x-\varepsilon y)-\nabla_1\eta(z, x)| \leq C\varepsilon|y|.$$

Hence,

$$\begin{aligned}
&\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx \\
&= o_\varepsilon(1)-\int_\Omega\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B_R(0)}\nabla_1\eta(z,x)\bar\varphi(x-\varepsilon y+\varepsilon z)\,dz\right)dy\right)d[D\bar\varphi](x) \\
&= o_\varepsilon(1)-\int_\Omega\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B_{2R}(0)}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz\right)dy\right)d[D\bar\varphi](x).
\end{aligned} \tag{3.16}$$

Recall that $G_\varphi \subset \Omega$ is the set of approximate continuity of φ. For every x in G_φ we have

$$\lim_{\rho\to 0^+}\frac{1}{\rho^N}\int_{B_\rho(x)}|\bar\varphi(y)-\tilde{\bar\varphi}(x)|\,dy = 0.$$

Taking $\rho = 2R\varepsilon$ gives

$$\lim_{\varepsilon\to 0^+}\int_{B_{2R}(0)}|\bar\varphi(x+\varepsilon z)-\tilde{\bar\varphi}(x)|\,dz = 0 \quad \text{for } x \text{ in } G_\varphi. \tag{3.17}$$

Since $\mathcal{H}^{N-1}(\Omega\setminus(G_\varphi\cup J_\varphi)) = 0$ (see Theorem 2.1(i)) and the measure $[D\varphi]$ does not charge sets of $\mathcal{H}^{N-1}$-measure zero, we infer from (3.16) that

$$\begin{aligned}
&\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx = o_\varepsilon(1) \\
&\quad -\int_{J_\varphi}\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B_{2R}(0)}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz\right)dy\right)d[D\bar\varphi](x) \\
&\quad -\int_{G_\varphi}\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B_{2R}(0)}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz\right)dy\right)d[D\bar\varphi](x).
\end{aligned} \tag{3.18}$$

Using (3.17) we infer that for any $y \in \mathbb{R}^N$ and for every x in $G_{\bar\varphi}$ we have

$$\lim_{\varepsilon\to 0^+}\int_{B_{2R}(0)}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz = \tilde{\bar\varphi}(x)\int_{\mathbb{R}^N}\nabla_1\eta(y+z,x)\,dz = 0.$$

Therefore,

$$\lim_{\varepsilon\to 0^+}\int_{G_\varphi}\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B_{2R}(0)}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz\right)dy\right)d[D\bar\varphi](x) = 0.$$

From (3.18) we then deduce that

$$\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx = o_\varepsilon(1) - \int_{J_\varphi}\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B_{2R}(0)}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz\right)dy\right)d[D\bar\varphi](x). \tag{3.19}$$

By the definition of J_φ (see (2.6)), we have

$$\left.\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}|\varphi(x+\varepsilon z)-\varphi^+(x)|\,dz=0\\ \lim_{\varepsilon\to 0^+}\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}|\varphi(x+\varepsilon z)-\varphi^-(x)|\,dz=0\end{aligned}\right.\quad\text{for } x\in J_\varphi. \tag{3.20}$$

For every $x \in J_\varphi$ and every $y \in B_R(0)$ we have

$$\begin{aligned}&\int_{B_{2R}(0)}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz\\ &\quad=\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz+\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz.\end{aligned}$$

Using (3.20) we obtain, for every $x \in J_\varphi$ and every $y \in B_R(0)$,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz&=\varphi^+(x)\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\nabla_1\eta(y+z,x)\,dz,\\ \lim_{\varepsilon\to 0^+}\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}\nabla_1\eta(y+z,x)\bar\varphi(x+\varepsilon z)\,dz&=\varphi^-(x)\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}\nabla_1\eta(y+z,x)\,dz.\end{aligned}$$

Therefore, using (3.19), we infer that

$$\begin{aligned}&\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx = o_\varepsilon(1)\\ &\quad-\int_{J_\varphi}\varphi^+(x)\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\nabla_z\eta(y+z,x)\,dz\right)dy\right)d[D\bar\varphi](x)\\ &\quad-\int_{J_\varphi}\varphi^-(x)\left(\int_{B_R(0)}\eta(y,x)\left(\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}\nabla_z\eta(y+z,x)\,dz\right)dy\right)d[D\bar\varphi](x).\end{aligned} \tag{3.21}$$

Recalling the definitions (2.3)–(2.5), we have

$$\begin{aligned}\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\nabla_z\eta(y+z,x)\,dz&=\int_{H^N_+(0,\boldsymbol{\nu}(x))}\nabla_z\eta(y+z,x)\,dz\\ &=-\boldsymbol{\nu}(x)\int_{H_{\boldsymbol{\nu}(x)}}\eta(y+s,x)\,d\mathcal{H}^{N-1}(s)\\ &=-\boldsymbol{\nu}(x)\int_{H_{\boldsymbol{\nu}(x)}}\eta((y\cdot\boldsymbol{\nu}(x))\boldsymbol{\nu}(x)+s,x)\,d\mathcal{H}^{N-1}(s).\end{aligned} \tag{3.22}$$

In a similar way we obtain

$$\int_{B_{2R}^-(0,\boldsymbol{\nu}(x))} \nabla_z \eta(y+z,x)\,dz = \boldsymbol{\nu}(x)\int_{H_{\boldsymbol{\nu}(x)}} \eta((y\cdot\boldsymbol{\nu}(x))\boldsymbol{\nu}(x)+s,x)\,d\mathcal{H}^{N-1}(s). \quad (3.23)$$

Combining (3.21) with (3.22)–(3.23) we get

$$\int_\Omega \varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx = o_\varepsilon(1) + \int_{J_\varphi}\left(\int_{\mathbb{R}^N}\eta(y,x)\int_{H_{\boldsymbol{\nu}(x)}}\eta((y\cdot\boldsymbol{\nu}(x))\boldsymbol{\nu}(x)+s,x)\,d\mathcal{H}^{N-1}(s)\,dy\right) \times(\varphi^+(x)-\varphi^-(x))\boldsymbol{\nu}(x)\,d[D\bar{\varphi}](x). \quad (3.24)$$

Writing $y = t\boldsymbol{\nu}(x)+h$ with $t = y\cdot\boldsymbol{\nu}(x)$ and $h\in H_{\boldsymbol{\nu}(x)}$ yields

$$\begin{aligned}\int_{\mathbb{R}^N}\eta(y,x)\biggl(\int_{H_{\boldsymbol{\nu}(x)}}\eta((y\cdot\boldsymbol{\nu}(x))\boldsymbol{\nu}(x)+s,x)\,d\mathcal{H}^{N-1}(s)\biggr)dy \\ = \int_{\mathbb{R}}\left(\int_{H_{\boldsymbol{\nu}(x)}}\eta(t\boldsymbol{\nu}(x)+h,x)\biggl(\int_{H_{\boldsymbol{\nu}(x)}}\eta(t\boldsymbol{\nu}(x)+s,x)\,d\mathcal{H}^{N-1}(s)\biggr)d\mathcal{H}^{N-1}(h)\right)dt \\ = \int_{\mathbb{R}}\left(\int_{H_{\boldsymbol{\nu}(x)}}\eta(t\boldsymbol{\nu}(x)+h,x)\,d\mathcal{H}^{N-1}(h)\right)^2 dt = \int_{-\infty}^{\infty}p^2(t,x)\,dt \qquad (3.25)\end{aligned}$$

(see (3.5)). Using the equality

$$[D\varphi]\llcorner J_\varphi = (\varphi^+-\varphi^-)\boldsymbol{\nu}(x)\mathcal{H}^{N-1}\llcorner J_\varphi \quad \text{(see Theorem 2.2)}$$

together with (3.25) in (3.24) gives

$$\begin{aligned}\int_\Omega\varepsilon|\nabla\psi_\varepsilon(x)|^2\,dx &= o_\varepsilon(1)+\int_{J_\varphi}\left(\int_{-\infty}^{\infty}p^2(t,x)\,dt\right)(\varphi^+(x)-\varphi^-(x))\boldsymbol{\nu}(x)\,d[D\bar{\varphi}](x) \\ &= o_\varepsilon(1)+\int_{J_\varphi}\left(\int_{-\infty}^{\infty}p^2(t,x)\,dt\right)(\varphi^+(x)-\varphi^-(x))^2\,d\mathcal{H}^{N-1}(x),\end{aligned}$$

and (3.4) follows. □

Consider a function $\varphi = (\varphi_1,\dots,\varphi_k) \in BV(\Omega,\mathbb{R}^k)\cap L^\infty(\Omega,\mathbb{R}^k)$ together with its extension $\bar{\varphi}\in BV(\mathbb{R}^N,\mathbb{R}^k)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^k)$ such that $\bar{\varphi}=\varphi$ a.e. in Ω, $\operatorname{supp}\bar{\varphi}$ is compact and $\|D\bar{\varphi}\|(\partial\Omega)=0$. Consider also a function $\eta\in\mathcal{V}$ (see Definition 3.1) together with an $R>0$ satisfying (3.1). For any $\varepsilon>0$ and any $j=1,\dots,k$ define a function $\psi_{j,\varepsilon}:\mathbb{R}^N\to\mathbb{R}$ by

$$\psi_{j,\varepsilon}(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\eta\biggl(\frac{y-x}{\varepsilon},x\biggr)\bar{\varphi}_j(y)\,dy = \int_{\mathbb{R}^N}\eta(z,x)\bar{\varphi}_j(x+\varepsilon z)\,dz, \qquad \forall x\in\mathbb{R}^N. \quad (3.26)$$

Set also $\psi_\varepsilon := (\psi_{1,\varepsilon}, \dots, \psi_{k,\varepsilon}) \in C^1(\mathbb{R}^N, \mathbb{R}^k)$. By Lemma 3.1 we have $\int_\Omega |\psi_\varepsilon(x) - \varphi(x)|\, dx = O(\varepsilon)$ and by Proposition 3.1,

$$\lim_{\varepsilon\to 0} \int_\Omega \varepsilon |\nabla \psi_\varepsilon(x)|^2\, dx = \int_{J_\varphi} |\varphi^+(x) - \varphi^-(x)|^2 \left\{ \int_{\mathbb{R}} p^2(t,x)\, dt \right\} d\mathcal{H}^{N-1}(x),$$

where $p(t,x)$ is defined in (3.5).

Proposition 3.2. *Let $W : \mathbb{R}^k \times \mathbb{R}^q \to \mathbb{R}$ be a C^2 function satisfying*

$$\nabla_a W(a,b) = 0 \quad \textit{whenever} \quad W(a,b) = 0, \ \textit{for some } a \in \mathbb{R}^k \textit{ and } b \in \mathbb{R}^q. \tag{3.27}$$

Consider $u \in \mathrm{BV}(\Omega, \mathbb{R}^q) \cap L^\infty(\Omega, \mathbb{R}^q)$ and $\varphi \in \mathrm{BV}(\Omega, \mathbb{R}^k) \cap L^\infty(\Omega, \mathbb{R}^k)$ satisfying $W(\varphi(x), u(x)) = 0$ a.e. in Ω. Then

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_\Omega \frac{1}{\varepsilon} W(\psi_\varepsilon(x), u(x))\, dx \\ &= \int_{J_\varphi} \left\{ \int_{-\infty}^0 W(\gamma(t,x), u^+(x))\, dt + \int_0^\infty W(\gamma(t,x), u^-(x))\, dt \right\} d\mathcal{H}^{N-1}(x), \end{aligned} \tag{3.28}$$

where

$$\gamma(t,x) = \varphi^-(x) \int_{-\infty}^t p(s,x)\, ds + \varphi^+(x) \int_t^\infty p(s,x)\, ds, \tag{3.29}$$

and it is assumed that the orientation of J_u coincides $\mathcal{H}^{N-1}$-a.e. with the orientation of J_φ on $J_u \cap J_\varphi$.

Proof. We may assume without loss of generality that u is Borel measurable on Ω. For any $j = 1, \dots, k$, any $t \in (0,1]$ and any $x \in \mathbb{R}^N$ we have

$$\begin{aligned} \frac{d(\psi_{j,t\varepsilon}(x))}{dt} &= \frac{d}{dt}\left(\frac{1}{t^N \varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y-x}{t\varepsilon}, x \right) \bar{\varphi}_j(y)\, dy \right) \\ &= -\frac{1}{t^{N+1}\varepsilon^N} \int_{\mathbb{R}^N} \left\{ \nabla_1 \eta\left(\frac{y-x}{t\varepsilon}, x \right) \cdot \frac{y-x}{t\varepsilon} + N\eta\left(\frac{y-x}{t\varepsilon}, x \right) \right\} \bar{\varphi}_j(y)\, dy \\ &= -\frac{1}{t^N \varepsilon^{N-1}} \int_{\mathbb{R}^N} \mathrm{div}_y \left\{ \eta\left(\frac{y-x}{t\varepsilon}, x \right) \frac{y-x}{t\varepsilon} \right\} \bar{\varphi}_j(y)\, dy \\ &= \frac{1}{t^N \varepsilon^{N-1}} \int_{\mathbb{R}^N} \eta\left(\frac{y-x}{t\varepsilon}, x \right) \frac{y-x}{t\varepsilon}\, d[D\bar{\varphi}_j](y). \end{aligned} \tag{3.30}$$

Therefore,

$$\begin{aligned} \frac{dW(\psi_{t\varepsilon}(x), u(x))}{dt} &= \sum_{j=1}^k \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x), u(x)) \cdot \frac{d(\psi_{j,\varepsilon t}(x))}{dt} \\ = \varepsilon \sum_{j=1}^k \frac{\partial W}{\partial a_j}&(\psi_{t\varepsilon}(x), u(x)) \left(\frac{1}{t^N \varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y-x}{t\varepsilon}, x \right) \frac{y-x}{t\varepsilon}\, d[D\bar{\varphi}_j](y) \right), \end{aligned} \tag{3.31}$$

where $\partial W(a,b)/\partial a_j$ is the j-th coordinate of the partial gradient $\nabla_a W(a,b)$. For any $\rho \in (0,1)$ we have, by (3.31),

$$
\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon}\{W(\psi_\varepsilon(x),u(x)) - W(\psi_{\rho\varepsilon}(x),u(x))\}\,dx = \int_\Omega \frac{1}{\varepsilon}\left(\int_\rho^1 \frac{d\,W(\psi_{t\varepsilon}(x),u(x))}{dt}\right)dx\\
&= \int_\Omega \left\{\int_\rho^1 \sum_{j=1}^k \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x),u(x))\left(\frac{1}{t^N\varepsilon^N}\int_{\mathbb{R}^N}\eta\left(\frac{y-x}{t\varepsilon},x\right)\frac{y-x}{t\varepsilon}\,d[D\bar\varphi_j](y)\right)dt\right\}dx\\
&= \int_\rho^1 \left\{\int_\Omega \sum_{j=1}^k \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x),u(x))\left(\frac{1}{t^N\varepsilon^N}\int_{\mathbb{R}^N}\eta\left(\frac{y-x}{t\varepsilon},x\right)\frac{y-x}{t\varepsilon}\,d[D\bar\varphi_j](y)\right)dx\right\}dt\\
&= \int_\rho^1 \left\{\sum_{j=1}^k \int_{\mathbb{R}^N}\left(\frac{1}{t^N\varepsilon^N}\right.\right.\\
&\quad\left.\left.\times\int_{\Omega\cap B_{Rt\varepsilon}(y)}\frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x),u(x))\eta\left(\frac{y-x}{t\varepsilon},x\right)\frac{y-x}{t\varepsilon}\,dx\right)d[D\bar\varphi_j](y)\right\}dt. \qquad (3.32)
\end{aligned}
$$

From our assumptions on W it follows that there exists a constant $C > 0$, independent of ρ, such that $|\frac{\partial W}{\partial a_j}(\psi_\rho(x),u(x))| \le C$ for every $\rho > 0$ and every j. Therefore, letting ρ tend to zero in (3.32) and using Lemma 3.1, we get

$$
\begin{aligned}
\int_\Omega \frac{1}{\varepsilon}W(\psi_\varepsilon(x),u(x))\,dx &= \int_0^1 \left\{\sum_{j=1}^k \int_{\mathbb{R}^N}\left(\frac{1}{t^N\varepsilon^N}\right.\right.\\
&\quad\left.\left.\times\int_{\Omega\cap B_{Rt\varepsilon}(y)}\frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x),u(x))\eta\left(\frac{y-x}{t\varepsilon},x\right)\frac{y-x}{t\varepsilon}\,dx\right)d[D\bar\varphi_j](y)\right\}dt. \qquad (3.33)
\end{aligned}
$$

Let $\bar u \in \mathrm{BV}(\mathbb{R}^N,\mathbb{R}^q)\cap L^\infty(\mathbb{R}^N,\mathbb{R}^q)$ be an extension of u from Ω to $\mathbb{R}^N$. As in the proof of Proposition 3.1 (see (3.9)–(3.13)), we have

$$
\begin{aligned}
&\Bigg|\int_0^1\left\{\int_\Omega\left(\frac{1}{(t\varepsilon)^N}\int_{B_{Rt\varepsilon}(y)}\frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x),\bar u(x))\eta\left(\frac{y-x}{t\varepsilon},x\right)\frac{y-x}{t\varepsilon}\,dx\right)d[D\bar\varphi_j](y)\right\}dt\\
&-\int_0^1\left\{\int_{\mathbb{R}^N}\left(\frac{1}{(t\varepsilon)^N}\int_{\Omega\cap B_{Rt\varepsilon}(y)}\frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x),\bar u(x))\eta\left(\frac{y-x}{t\varepsilon},x\right)\frac{y-x}{t\varepsilon}\,dx\right)d[D\bar\varphi_j](y)\right\}dt\Bigg|\\
&\qquad\le C\int_{(\partial\Omega)_{R\varepsilon}} d|D\bar\varphi_j|(y) = C\|D\bar\varphi_j\|_{(\partial\Omega)_{R\varepsilon}} = o_\varepsilon(1), \qquad (3.34)
\end{aligned}
$$

recalling that $(\partial\Omega)_d = \{x \in \mathbb{R}^N : \mathrm{dist}(x,\partial\Omega) < d\}$. In the last equality in (3.34) we used the fact that $\|D\bar\varphi_j\|_{\partial\Omega} = 0$. Therefore, by (3.33) we obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon}W(\psi_\varepsilon(x),u(x))\,dx = o_\varepsilon(1)\\
&+\sum_{j=1}^k\int_0^1\int_\Omega\left(\frac{1}{(t\varepsilon)^N}\int_{B_{Rt\varepsilon}(y)}\eta\left(\frac{y-x}{t\varepsilon},x\right)\frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x),\bar u(x))\frac{y-x}{t\varepsilon}\,dx\right)d[D\bar\varphi_j](y)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= o_\varepsilon(1) \\
&+ \int_0^1 \sum_{j=1}^k \left(\int_\Omega \left\{ \int_{B_R(0)} \eta(z, y - \varepsilon t z) \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(y - \varepsilon t z), \bar{u}(y - \varepsilon t z)) z \, dz \right\} d[D\varphi_j](y) \right) dt \\
&= o_\varepsilon(1) \\
&\quad + \int_0^1 \sum_{j=1}^k \left(\int_\Omega \left\{ \int_{B_R(0)} \eta(z, x) \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x - \varepsilon t z), \bar{u}(x - \varepsilon t z)) z \, dz \right\} d[D\varphi_j](x) \right) dt,
\end{aligned} \tag{3.35}
$$

where in the last equality we used the estimate

$$|\eta(z, x - \varepsilon t z) - \eta(z, x)| \le C\varepsilon t |z|.$$

As before, for each j we have

$$
\begin{aligned}
&\int_\Omega \left\{ \int_{B_R(0)} \eta(z, x) \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x - \varepsilon t z), \bar{u}(x - \varepsilon t z)) z \, dz \right\} d[D\varphi_j](x) \\
&\quad = \int_{J_\varphi} \left\{ \int_{B_R(0)} \eta(z, x) \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x - \varepsilon t z), \bar{u}(x - \varepsilon t z)) z \, dz \right\} d[D\bar{\varphi}_j](x) \\
&\quad\quad + \int_{G_\varphi} \left\{ \int_{B_R(0)} \eta(z, x) \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x - \varepsilon t z), \bar{u}(x - \varepsilon t z)) z \, dz \right\} d[D\bar{\varphi}_j](x).
\end{aligned} \tag{3.36}
$$

By (3.27), $\partial W(\tilde{\varphi}(x), \tilde{u}(x))/\partial a_j = 0$ for $x \in G_\varphi \cap G_u$. Similarly, $\partial W(\tilde{\varphi}(x), u^+(x))/\partial a_j = 0$ and $\partial W(\tilde{\varphi}(x), u^-(x))/\partial a_j = 0$ for $x \in G_\varphi \cap J_u$. Therefore, the last integral in (3.36) tends to 0 as $\varepsilon \to 0$, for any $t \in (0, 1]$. By (3.35)–(3.36) we get

$$
\begin{aligned}
&\int_\Omega \frac{1}{\varepsilon} W(\psi_\varepsilon(x), u(x)) \, dx = o_\varepsilon(1) \\
&+ \int_0^1 \sum_{j=1}^k \left(\int_{J_\varphi} \left\{ \int_{B_R(0)} \eta(z, x) \frac{\partial W}{\partial a_j}(\psi_{t\varepsilon}(x - \varepsilon t z), \bar{u}(x - \varepsilon t z)) z \, dz \right\} d[D\varphi_j](x) \right) dt.
\end{aligned} \tag{3.37}
$$

For any $1 \le j \le k$, $\rho \in (0, 1)$, $x \in J_\varphi$ and $z \in B_R(0)$, we have

$$
\begin{aligned}
\psi_{j,\rho}(x - \rho z) &= \int_{\mathbb{R}^N} \eta(y, x - \rho z) \bar{\varphi}_j(x + \rho(y - z)) \, dy \\
&= \int_{B_{2R}(0)} \eta(y + z, x - \rho z) \bar{\varphi}_j(x + \rho y) \, dy \\
&= \int_{B^+_{2R}(0, \boldsymbol{\nu}(x))} \eta(y + z, x - \rho z) \bar{\varphi}_j(x + \rho y) \, dy \\
&\quad + \int_{B^-_{2R}(0, \boldsymbol{\nu}(x))} \eta(y + z, x - \rho z) \bar{\varphi}_j(x + \rho y) \, dy.
\end{aligned} \tag{3.38}
$$

Then, using (3.20) we infer that for every $x \in J_\varphi$ and every $z \in B_R(0)$ we have

$$\lim_{\rho\to 0^+}\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\eta(y+z,x-\rho z)\bar{\varphi}_j(x+\rho y)\,dy = \varphi^+_j(x)\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\eta(y+z,x)\,dy,$$
$$\lim_{\rho\to 0^+}\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}\eta(y+z,x-\rho z)\bar{\varphi}_j(x+\rho y)\,dy = \varphi^-_j(x)\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}\eta(y+z,x)\,dy. \tag{3.39}$$

Note that

$$\begin{aligned}\int_{B^+_{2R}(0,\boldsymbol{\nu}(x))}\eta(y+z,x)\,dy &= \int_{H^N_+(0,\boldsymbol{\nu}(x))}\eta(y+z,x)\,dy = \int_{H^N_+(z,\boldsymbol{\nu}(x))}\eta(y,x)\,dy\\ &= \int_{\boldsymbol{\nu}(x)\cdot z}^{\infty}\left(\int_{H_{\boldsymbol{\nu}(x)}}\eta(t\boldsymbol{\nu}(x)+y,x)\,d\mathcal{H}^{N-1}(y)\right)dt\\ &= \int_{\boldsymbol{\nu}(x)\cdot z}^{\infty}p(t,x)\,dt\end{aligned} \tag{3.40}$$

(see (3.5)). Similarly, we obtain

$$\int_{B^-_{2R}(0,\boldsymbol{\nu}(x))}\eta(y+z,x)\,dy = \int_{-\infty}^{\boldsymbol{\nu}(x)\cdot z}p(t,x)\,dt. \tag{3.41}$$

Combining (3.38)–(3.41) we deduce that for every $x \in J_\varphi$ and every $z \in B_R(0)$ we have (see (3.29))

$$\begin{aligned}\lim_{\rho\to 0+}\psi_{j,\rho}(x-\rho z) &= \varphi^+_j(x)\int_{\boldsymbol{\nu}(x)\cdot z}^{\infty}p(t,x)\,dt + \varphi^-_j(x)\int_{-\infty}^{\boldsymbol{\nu}(x)\cdot z}p(t,x)\,dt\\ &= \gamma_j(\boldsymbol{\nu}(x)\cdot z,x).\end{aligned} \tag{3.42}$$

Using (3.42) in (3.37) we obtain

$$\begin{aligned}&\int_\Omega \frac{1}{\varepsilon}W(\psi_\varepsilon(x),u(x))\,dx = o_\varepsilon(1)\\ &\quad+\int_0^1\sum_{j=1}^k\left(\int_{J_\varphi}\left\{\int_{B_R(0)}\eta(z,x)\frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z,x),\bar{u}(x-\varepsilon tz))z\,dz\right\}d[D\varphi_j](x)\right)dt\\ &= o_\varepsilon(1)\\ &\quad+\int_0^1\sum_{j=1}^k\left(\int_{J_\varphi}\left\{\int_{B^+_R(0,\boldsymbol{\nu}(x))}\eta(z,x)\right.\right.\\ &\qquad\qquad\left.\left.\times\frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z,x),\bar{u}(x-\varepsilon tz))z\,dz\right\}d[D\varphi_j](x)\right)dt\\ &\quad+\int_0^1\sum_{j=1}^k\left(\int_{J_\varphi}\left\{\int_{B^-_R(0,\boldsymbol{\nu}(x))}\eta(z,x)\right.\right.\\ &\qquad\qquad\left.\left.\times\frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z,x),\bar{u}(x-\varepsilon tz))z\,dz\right\}d[D\varphi_j](x)\right)dt.\end{aligned}$$

By the analogue of (3.20) for u we infer that

$$
\begin{aligned}
\int_\Omega \frac{1}{\varepsilon} W(\psi_\varepsilon(x), u(x))\, dx &= o_\varepsilon(1) \\
&+ \sum_{j=1}^k \left(\int_{J_\varphi} \left\{ \int_{B_R^+(0,\boldsymbol{\nu}(x))} \eta(z,x) \frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z, x), u^-(x)) z\, dz \right\} d[D\varphi_j](x) \right) \\
&+ \sum_{j=1}^k \left(\int_{J_\varphi} \left\{ \int_{B_R^-(0,\boldsymbol{\nu}(x))} \eta(z,x) \frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z, x), u^+(x)) z\, dz \right\} d[D\varphi_j](x) \right).
\end{aligned} \tag{3.43}
$$

Here we used the assumption that the orientation of J_u coincides $\mathcal{H}^{N-1}$-a.e. with the orientation of J_φ on $J_u \cap J_\varphi$. For each j we have

$$
\begin{aligned}
&\int_{B_R^+(0,\boldsymbol{\nu}(x))} \eta(z,x) \frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z, x), u^-(x)) z\, dz \\
&\quad = \int_{H_+^N(0,\boldsymbol{\nu}(x))} \eta(z,x) \frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z, x), u^-(x)) z\, dz \\
&\quad = \int_0^\infty \frac{\partial W}{\partial a_j}(\gamma(\tau,x), u^-(x)) \left\{ \int_{H_{\boldsymbol{\nu}(x)}} \eta(\tau\boldsymbol{\nu}(x)+y, x)(\tau\boldsymbol{\nu}(x)+y)\, d\mathcal{H}^{N-1}(y) \right\} d\tau \\
&\quad = \boldsymbol{\nu}(x) \int_0^\infty \frac{\partial W}{\partial a_j}(\gamma(\tau,x), u^-(x)) p(\tau,x)\tau\, d\tau + \boldsymbol{\beta}_x
\end{aligned} \tag{3.44}
$$

(see (3.5)), with $\boldsymbol{\beta}_x \in H_{\boldsymbol{\nu}(x)}$ (i.e. $\boldsymbol{\beta}_x \perp \boldsymbol{\nu}(x)$). In the same way we have

$$
\begin{aligned}
&\int_{B_R^-(0,\boldsymbol{\nu}(x))} \eta(z,x) \frac{\partial W}{\partial a_j}(\gamma(\boldsymbol{\nu}(x)\cdot z, x), u^+(x)) z\, dz \\
&\qquad\qquad = \boldsymbol{\nu}(x) \int_{-\infty}^0 \frac{\partial W}{\partial a_j}(\gamma(\tau,x), u^+(x)) p(\tau,x)\tau\, d\tau + \boldsymbol{\alpha}_x,
\end{aligned} \tag{3.45}
$$

with $\boldsymbol{\alpha}_x \in H_{\boldsymbol{\nu}(x)}$. Since $[D\varphi_j] \llcorner J_\varphi = (\varphi_j^+ - \varphi_j^-)\boldsymbol{\nu}(x)\mathcal{H}^{N-1} \llcorner J_\varphi$, we infer from (3.43)–(3.45) that

$$
\begin{aligned}
\int_\Omega \frac{1}{\varepsilon} W(\psi_\varepsilon(x), u(x))\, dx &= o_\varepsilon(1) \\
&+ \int_{J_\varphi} \left\{ \int_0^\infty \sum_{j=1}^k \frac{\partial W}{\partial a_j}(\gamma(\tau,x), u^-(x))(\varphi_j^+(x) - \varphi_j^-(x)) p(\tau,x)\tau\, d\tau \right\} d\mathcal{H}^{N-1}(x) \\
&+ \int_{J_\varphi} \left\{ \int_{-\infty}^0 \sum_{j=1}^k \frac{\partial W}{\partial a_j}(\gamma(\tau,x), u^+(x))(\varphi_j^+(x) - \varphi_j^-(x)) p(\tau,x)\tau\, d\tau \right\} d\mathcal{H}^{N-1}(x).
\end{aligned} \tag{3.46}
$$

Next, since $\frac{\partial\gamma_j(\tau,x)}{\partial\tau} = -(\varphi_j^+(x) - \varphi_j^-(x))p(\tau,x)$, we have

$$\begin{aligned}
&\int_0^\infty \sum_{j=1}^k \frac{\partial W}{\partial a_j}(\gamma(\tau,x),u^-(x))(\varphi_j^+(x)-\varphi_j^-(x))p(\tau,x)\tau\,d\tau \\
&\qquad\qquad\qquad\qquad = \int_0^\infty W(\gamma(\tau,x),u^-(x))\,d\tau, \\
&\int_{-\infty}^0 \sum_{j=1}^k \frac{\partial W}{\partial a_j}(\gamma(\tau,x),u^+(x))(\varphi_j^+(x)-\varphi_j^-(x))p(\tau,x)\tau\,d\tau \\
&\qquad\qquad\qquad\qquad = \int_{-\infty}^0 W(\gamma(\tau,x),u^+(x))\,d\tau.
\end{aligned} \tag{3.47}$$

Plugging (3.47) in (3.46) gives the desired result (3.28). □

The next theorem is a direct consequence of Propositions 3.1 and 3.2.

Theorem 3.1. *Let W, u and φ be as in Proposition* 3.2. *For any $\eta \in \mathcal{V}$ let $\psi_\varepsilon = (\psi_{1,\varepsilon},\dots,\psi_{k,\varepsilon})$ be defined by* (3.26). *Then*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\{\varepsilon\int_\Omega |\nabla\psi_\varepsilon|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(\psi_\varepsilon,u)\,dx\right\} \\
&\quad = \int_{J_\varphi} |\varphi^+(x)-\varphi^-(x)|^2\left\{\int_{\mathbb{R}} p^2(t,x)\,dt\right\} d\mathcal{H}^{N-1}(x) \\
&\qquad + \int_{J_\varphi}\left\{\int_{-\infty}^0 W(\gamma(t,x),u^+(x))\,dt + \int_0^\infty W(\gamma(t,x),u^-(x))\,dt\right\} d\mathcal{H}^{N-1}(x),
\end{aligned} \tag{3.48}$$

where p and γ are defined in (3.5) *and* (3.29) *respectively, and it is assumed that the orientation of J_u coincides $\mathcal{H}^{N-1}$-a.e. with the orientation of J_φ on $J_u \cap J_\varphi$.*

Next we turn to the minimization problem of the term on the r.h.s. of (3.48), over all kernels $\eta \in \mathcal{V}$. We shall need the following lemma.

Lemma 3.3. *Let $F, G \in C^2(\mathbb{R}^k,\mathbb{R})$ satisfy $F(x), G(x) \geq 0$ for all $x \in \mathbb{R}^k$, and $F(\boldsymbol{a}) = G(\boldsymbol{b}) = 0$ for some $\boldsymbol{a} \neq \boldsymbol{b}$ in $\mathbb{R}^k$. Set*

$$\mathcal{P} = \left\{p \in L_c^\infty(\mathbb{R},\mathbb{R}) : \int_{\mathbb{R}} p(t)\,dt = 1\right\}$$

and let $U : \mathcal{P} \to \mathbb{R}$ be defined by

$$\begin{aligned}
U(p) = |\boldsymbol{b}-\boldsymbol{a}|^2\int_{-\infty}^\infty p^2(t)\,dt &+ \int_{-\infty}^0 F\left(\boldsymbol{b}\int_{-\infty}^t p(s)\,ds + \boldsymbol{a}\int_t^\infty p(s)\,ds\right)dt \\
&+ \int_0^\infty G\left(\boldsymbol{b}\int_{-\infty}^t p(s)\,ds + \boldsymbol{a}\int_t^\infty p(s)\,ds\right)dt.
\end{aligned}$$

Then

$$\inf_{p\in\mathcal{P}} U(p) = I$$
$$:= 2|\boldsymbol{b}-\boldsymbol{a}| \inf_{\tau\in[0,1]} \left\{ \int_0^{\tau} \sqrt{F(s\boldsymbol{b}+(1-s)\boldsymbol{a})}\, ds + \int_{\tau}^{1} \sqrt{G(s\boldsymbol{b}+(1-s)\boldsymbol{a})}\, ds \right\}. \tag{3.49}$$

Proof. For any $p \in \mathcal{P}$, by the inequality $a^2 + b^2 \geq 2ab$ we have

$$\begin{aligned} U(p) &\geq 2|\boldsymbol{b}-\boldsymbol{a}| \left| \int_{-\infty}^{0} p(t) \sqrt{F\left(\boldsymbol{b}\int_{-\infty}^{t} p(s)\, ds + \boldsymbol{a}\int_{t}^{\infty} p(s)\, ds\right)}\, dt \right| \\ &\quad + 2|\boldsymbol{b}-\boldsymbol{a}| \left| \int_{0}^{\infty} p(t) \sqrt{G\left(\boldsymbol{b}\int_{-\infty}^{t} p(s)\, ds + \boldsymbol{a}\int_{t}^{\infty} p(s)\, ds\right)}\, dt \right| \\ &= 2|\boldsymbol{b}-\boldsymbol{a}| \left\{ \left| \int_0^{d_p} \sqrt{F(s\boldsymbol{b}+(1-s)\boldsymbol{a})}\, ds \right| + \left| \int_{d_p}^{1} \sqrt{G(s\boldsymbol{b}+(1-s)\boldsymbol{a})}\, ds \right| \right\}, \end{aligned}$$

with $d_p := \int_{-\infty}^{0} p(s)\, ds$. Therefore,

$$\inf_{p\in\mathcal{P}} U(p) \geq I. \tag{3.50}$$

Now, let $0 \leq \tau_0 \leq 1$ satisfy

$$\int_0^{\tau_0} \sqrt{F(s\boldsymbol{b}+(1-s)\boldsymbol{a})}\, ds + \int_{\tau_0}^{1} \sqrt{G(s\boldsymbol{b}+(1-s)\boldsymbol{a})}\, ds = I.$$

For every $n \geq 1$ define $\tau_n \in C(\mathbb{R}, \mathbb{R})$ as the solution of

$$\begin{cases} \tau_n'(t) = \dfrac{1}{|\boldsymbol{b}-\boldsymbol{a}|} \sqrt{F(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})} + \dfrac{1}{n}\tau_n(t) \quad \forall t \in (-\infty, 0), \\ \tau_n'(t) = \dfrac{1}{|\boldsymbol{b}-\boldsymbol{a}|} \sqrt{G(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})} + \dfrac{1}{n}(1-\tau_n(t)) \quad \forall t \in (0, \infty), \\ \tau_n(0) = \tau_0. \end{cases} \tag{3.51}$$

Then $\tau_n \in \mathrm{Lip}(\mathbb{R})$, τ_n is increasing on $\mathbb{R}$, $\lim_{t\to-\infty} \tau_n(t) = 0$ and $\lim_{t\to\infty} \tau_n(t) = 1$. Moreover,

$$0 \leq \tau_n(t) \leq \tau_0 e^{t/n} \quad \forall t \leq 0 \quad \text{and} \quad 0 \leq 1-\tau_n(t) \leq (1-\tau_0)e^{-t/n} \quad \forall t \geq 0. \tag{3.52}$$

Setting $p_n(t) := \tau_n'(t)$, we have $\int_{\mathbb{R}} p_n(t)\, dt = 1$ and

$$\begin{aligned} U(p_n) &= |\boldsymbol{b}-\boldsymbol{a}|^2 \int_{-\infty}^{\infty} (\tau_n'(t))^2 dt \\ &\quad + \int_{-\infty}^{0} F(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})\, dt + \int_0^{\infty} G(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a}))\, dt. \end{aligned}$$

Therefore, by (3.51) we have

$$
\begin{aligned}
U(p_n) &= \int_{-\infty}^{0} \tau_n'(t)|\boldsymbol{b}-\boldsymbol{a}|\left(\sqrt{F(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})}+\frac{1}{n}\tau_n(t)|\boldsymbol{b}-\boldsymbol{a}|\right)dt \\
&\quad+\int_{0}^{\infty} \tau_n'(t)|\boldsymbol{b}-\boldsymbol{a}|\left(\sqrt{G(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})}+\frac{1}{n}(1-\tau_n(t))|\boldsymbol{b}-\boldsymbol{a}|\right)dt \\
&\quad+\int_{-\infty}^{0}\left(\tau_n'(t)|\boldsymbol{b}-\boldsymbol{a}|-\frac{1}{n}\tau_n(t)|\boldsymbol{b}-\boldsymbol{a}|\right)\sqrt{F(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})}\,dt \\
&\quad+\int_{0}^{\infty}\left(\tau_n'(t)|\boldsymbol{b}-\boldsymbol{a}|-\frac{1}{n}(1-\tau_n(t))|\boldsymbol{b}-\boldsymbol{a}|\right)\sqrt{G(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})}\,dt \\
&= I+\frac{1}{n}\int_{-\infty}^{0}\tau_n(t)|\boldsymbol{b}-\boldsymbol{a}|(\tau_n'(t)|\boldsymbol{b}-\boldsymbol{a}|-\sqrt{F(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})})\,dt \\
&\quad+\frac{1}{n}\int_{0}^{\infty}(1-\tau_n(t))|\boldsymbol{b}-\boldsymbol{a}|(\tau_n'(t)|\boldsymbol{b}-\boldsymbol{a}|-\sqrt{G(\tau_n(t)\boldsymbol{b}+(1-\tau_n(t))\boldsymbol{a})})\,dt \\
&= I+\frac{1}{n^2}|\boldsymbol{b}-\boldsymbol{a}|^2\left(\int_{-\infty}^{0}(\tau_n(t))^2dt+\int_{0}^{\infty}(1-\tau_n(t))^2dt\right)=I+O\left(\frac{1}{n}\right), \qquad (3.53)
\end{aligned}
$$

where in the last equality we used (3.52). Next, for $m > 0$ define a function $p^{\{n,m\}} \in \mathcal{P}$ by

$$
p^{\{n,m\}}(t) = \begin{cases} \dfrac{p_n(t)}{\int_{-m}^{m} p_n(s)\,ds}, & t \in [-m,m], \\ 0, & t \in \mathbb{R}\setminus[-m,m]. \end{cases}
$$

Using (3.52) we see easily that for every n,

$$
\lim_{m\to\infty} U(p^{\{n,m\}}) = U(p_n).
$$

From (3.53) and a diagonal argument it follows that there exists a sequence $\{\bar{p}_n\} \subset \mathcal{P}$ such that $\lim_{n\to\infty} U(\bar{p}_n) = I$. Combining it with (3.50) we are led to (3.49). □

Remark 3.1. From the above proof it also follows that

$$
\inf_{p\in\mathcal{P},\ p\geq 0} U(p) = I.
$$

Let φ, u and W be as in Theorem 3.1. Assume in addition that $W \geq 0$ and define $Y : \mathcal{V} \to \mathbb{R}$ by

$$
\begin{aligned}
Y(\eta) &= \int_{J_\varphi} |\varphi^+(x)-\varphi^-(x)|^2\left\{\int_{\mathbb{R}} p^2(t,x)\,dt\right\}d\mathcal{H}^{N-1}(x) \\
&+\int_{J_\varphi}\left\{\int_{-\infty}^{0} W(\gamma(t,x),u^+(x))\,dt+\int_{0}^{\infty} W(\gamma(t,x),u^-(x))\,dt\right\}d\mathcal{H}^{N-1}(x), \qquad (3.54)
\end{aligned}
$$

where p and γ are defined in (3.5) and (3.29), respectively.

Lemma 3.4. *We have*

$$\inf_{\eta\in\mathcal{V}} Y(\eta) = \int_{J_\varphi} \{\inf_{p\in\mathcal{P}} Q_x(p)\}\, d\mathcal{H}^{N-1}(x), \tag{3.55}$$

where

$$\begin{aligned} Q_x(p) = |\varphi^+(x) - \varphi^-(x)|^2 \int_{\mathbb{R}} p^2(t)\, dt \\ + \int_{-\infty}^0 W\Big(\varphi^-(x) \int_{-\infty}^t p(s)\, ds + \varphi^+(x) \int_t^\infty p(s)\, ds, u^+(x)\Big)\, dt \\ + \int_0^\infty W\Big(\varphi^-(x) \int_{-\infty}^t p(s)\, ds + \varphi^+(x) \int_t^\infty p(s)\, ds, u^-(x)\Big)\, dt. \end{aligned}$$

Proof. Since $Y(\eta)$ and $\inf_{p\in\mathcal{P}} Q_x(p)$ do not depend on the orientation of the vector $\boldsymbol{\nu}(x)$, we may assume that the orientation of J_φ is such that the function $\boldsymbol{\nu} : J_\varphi \to S^{N-1}$ is Borel measurable (see [2, Proposition 3.69]). Clearly

$$\inf_{\eta\in\mathcal{V}} Y(\eta) \geq \int_{J_\varphi} \{\inf_{p\in\mathcal{P}} Q_x(p)\}\, d\mathcal{H}^{N-1}(x), \tag{3.56}$$

so (3.55) will follow once we prove the reverse inequality to (3.56). In the proof we shall establish three claims.

Claim 1. *The function $\zeta(x) := \inf_{p\in\mathcal{P}} Q_x(p)$, $x \in J_\varphi$, is $\mathcal{H}^{N-1}$-measurable.*

Consider the countable subset $\mathcal{P}_r \subset \mathcal{P}$ defined by

$$\begin{aligned} \mathcal{P}_r := \{p \in \mathcal{P} : p(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_m t^m \text{ for } |t| \leq l \text{ and } p(t) = 0 \text{ for } |t| > l, \\ \text{for some } m, l \in \mathbb{N} \text{ and } \alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{Q}\}. \end{aligned}$$

Clearly, for any $p \in \mathcal{P}$ there exists a sequence $p_n \in \mathcal{P}_r$ and a number $M > 0$ such that $p_n \to p$ in $L^2(\mathbb{R}, \mathbb{R})$, supp $p \subset [-M, M]$ and supp $p_n \subset [-M, M]$ for all n. Therefore, if $Q_x(p) < t$ for some $t \in \mathbb{R}$ and $x \in J_\varphi$, then there exists $p_r \in \mathcal{P}_r$ such that $Q_x(p_r) < t$. Thus,

$$\{x \in J_\varphi : \zeta(x) < t\} = \bigcup_{p\in\mathcal{P}} \{x \in J_\varphi : Q_x(p) < t\} = \bigcup_{p\in\mathcal{P}_r} \{x \in J_\varphi : Q_x(p) < t\},$$

and the measurability of $\zeta(x)$ follows.

Let $\mathcal{W}$ denote set of functions $p : \mathbb{R} \times J_\varphi \to \mathbb{R}$ satisfying the following conditions:

(i) p is Borel measurable,
(ii) p is bounded on $\mathbb{R} \times J_\varphi$,
(iii) there exists $M > 0$ such that $p(t, x) = 0$ for $|t| > M$ and any $x \in J_\varphi$,
(iv) $\int_{\mathbb{R}} p(t, x)\, dt = 1$ for all $x \in J_\varphi$.

Claim 2.

$$\int_{J_\varphi} \{\inf_{p\in\mathcal{P}} Q_x(p)\}\, d\mathcal{H}^{N-1}(x) = \inf_{p\in\mathcal{W}} \left\{ \int_{J_\varphi} Q_x(p(\cdot,x))\, d\mathcal{H}^{N-1}(x) \right\}. \tag{3.57}$$

Fix a bounded Borel measurable function $p \in \mathcal{P}$. Since $\varphi^+, \varphi^- \in L^\infty(J_\varphi)$ and $W(\varphi^\pm, u^\pm) = 0$ $\mathcal{H}^{N-1}$-a.e. on J_φ, using the Lipschitz property of W we deduce that

$$\frac{1}{|\varphi^+(x) - \varphi^-(x)|} Q_x(p) \leq L, \qquad \forall x \in J_\varphi, \tag{3.58}$$

where $L > 0$ does not depend on x. Let μ be the finite Borel regular measure on J_φ defined by

$$\mu = |\varphi^+ - \varphi^-|\, \mathcal{H}^{N-1} \llcorner J_\varphi. \tag{3.59}$$

Note that the $\mathcal{H}^{N-1}$ measure of J_φ may be infinite. Fix any $\varepsilon > 0$. By Lusin's theorem there exists a compact set $K \subset J_\varphi$ such that φ^+, φ^-, u^+, u^- and ζ are continuous functions on K and

$$\mu(J_\varphi \setminus K) \leq \frac{\varepsilon}{2L}. \tag{3.60}$$

Here ζ is the function from Claim 1. For any $x \in K$ there exists $p_x \in \mathcal{P}$ which is a bounded Borel measurable function on $\mathbb{R}$ and

$$Q_x(p_x) - \zeta(x) < \frac{|\varphi^+(x) - \varphi^-(x)|\varepsilon}{4 + 4\mu(J_\varphi)}. \tag{3.61}$$

Using the continuity of ζ, φ^+, φ^-, u^+ and u^- on K, we infer from (3.61) that for any $x \in K$ there exists $\delta_x > 0$ such that

$$Q_y(p_x) - \zeta(y) < \frac{|\varphi^+(y) - \varphi^-(y)|\varepsilon}{2 + 2\mu(J_\varphi)}, \qquad y \in K \cap B_{\delta_x}(x). \tag{3.62}$$

Since the set K is compact, there exist a finite number of points $x_1, \ldots, x_l \in K$ such that $K \subset \bigcup_{j=1}^{l} B_{\delta_{x_j}}(x_j)$. Define the function $\bar{p}(t,x)$ on $\mathbb{R} \times J_\varphi$ by

$$\bar{p}(t,x) = \begin{cases} p_{x_i}(t), & x \in (K \cap B_{\delta_{x_i}}(x)) \setminus \bigcup_{1\leq j\leq i-1} B_{\delta_{x_j}}(x_j),\ 1 \leq i \leq l, \\ p(t), & x \in J_\varphi \setminus K. \end{cases}$$

Clearly $\bar{p} \in \mathcal{W}$. From (3.62) and (3.58)–(3.60) we get

$$\int_{J_\varphi} Q_x(\bar{p}(\cdot,x))\, d\mathcal{H}^{N-1}(x) - \int_{J_\varphi} \zeta(x)\, d\mathcal{H}^{N-1}(x) < \varepsilon,$$

which implies (3.57) since $\varepsilon > 0$ is arbitrary.

Claim 3. *Let $p(t,x) \in \mathcal{W}$. Then there exists a sequence of functions $\eta_n \in \mathcal{V}$ (see Definition* 3.1*) such that the sequence $\{p_n\}$ of functions defined on $\mathbb{R} \times J_\varphi$ by*

$$p_n(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} \eta_n(t\boldsymbol{\nu}(x) + y, x)\, d\mathcal{H}^{N-1}(y),$$

has the following properties:

(i) *there exists* C_0 *such that* $|p_n(t,x)| \le C_0$ *for all* $x \in J_\varphi$ *and* $t \in \mathbb{R}$,
(ii) *there exists* $M > 0$ *such that* $p_n(t,x) = 0$ *for* $|t| > M$ *and any* $x \in J_\varphi$,
(iii) $\lim_{n\to\infty} \int_{J_\varphi} \int_{\mathbb{R}} |p_n(t,x) - p(t,x)|\,dt\,d\mu(x) = 0$, *where* μ *is defined in* (3.59).

Since $p \in \mathcal{W}$, there exists $M > 0$ such that $|p(t,x)| \le M$ for all x and t, and $p(t,x) = 0$ for $|t| > M$ and any $x \in J_\varphi$. Let $H : [0,\infty) \to [0,\infty)$ be any continuous function supported in $[0,M]$ satisfying $\int_{\mathbb{R}^{N-1}} H(|x|)\,dx = 1$ if $N > 1$ and $H(0) = 1$ if $N = 1$. Consider also a function $l \in C_c(\mathbb{R}^N, [0,\infty))$ satisfying $\int_{\mathbb{R}^N} l(z)\,dz = 1$ and $l(z) = 0$ for $|z| > M$. Let $\Omega' \subset \mathbb{R}^N$ be some open bounded domain such that $\overline{\Omega} \subset\subset \Omega'$. Define the function $\eta : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\eta(z,x) = \begin{cases} p(\boldsymbol{\nu}(x)\cdot z, x)H\left(\sqrt{|z|^2 - (\boldsymbol{\nu}(x)\cdot z)^2}\right), & (z,x) \in \mathbb{R}^N \times J_\varphi, \\ l(z), & (z,x) \in \mathbb{R}^N \times (\Omega' \setminus J_\varphi), \\ 0, & (z,x) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \Omega'). \end{cases}$$

Hence, η is Borel measurable. Moreover, there exists $\bar{M} > 0$ such that $|\eta(z,x)| \le \bar{M}$ for $(z,x) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\eta(z,x) = 0$ for $|z| > \bar{M}$ and any $x \in \mathbb{R}^N$, i.e., $\operatorname{supp}\eta \subset \overline{B}_{\bar{M}} \times \overline{\Omega'}$ is compact. We also have

$$\int_{\mathbb{R}^N} \eta(z,x)\,dz = 1, \qquad \forall x \in \Omega',$$

and

$$\int_{H_{\boldsymbol{\nu}(x)}} \eta(t\boldsymbol{\nu}(x) + y, x)\,d\mathcal{H}^{N-1}(y) = p(t,x), \qquad \forall x \in J_\varphi.$$

Let $\bar{\mu}$ be the measure on $\mathbb{R}^N$ satisfying, for every Borel set $A \subset \mathbb{R}^N$,

$$\bar{\mu}(A) = \|D\bar{\varphi}\|(A) + \mathcal{L}^N(A).$$

In particular, $\mu = \bar{\mu} \llcorner J_\varphi$. For every $0 < r < 1$, $z \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$ define

$$\eta_{0,r}(z,x) := \frac{1}{\mathcal{L}^N(B_r(z))} \int_{B_r(z)} \eta(y,x)\,dy = \frac{1}{\mathcal{L}^N(B_r(0))} \int_{B_r(0)} \eta(z+y,x)\,dy.$$

Then $\eta_{0,r}$ is a Borel measurable function which satisfies, for every $z_1, z_2 \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$,

$$|\eta_{0,r}(z_1,x) - \eta_{0,r}(z_2,x)| \le C_r|z_1 - z_2|,$$

where C_r depends only on r. We also have $|\eta_{0,r}(z,x)| \le \bar{M}$ for all z and x, and $\eta_{0,r}(z,x) = 0$ if either $|z| > \bar{M} + 1$ or $x \in \mathbb{R}^N \setminus \Omega'$. For each $x \in \Omega'$ we have

$$\begin{aligned} \int_{\mathbb{R}^N} \eta_{0,r}(z,x)\,dz &= \frac{1}{\mathcal{L}^N(B_r(0))} \int_{\mathbb{R}^N} \left(\int_{B_r(0)} \eta(z+y,x)\,dy \right) dz \\ &= \frac{1}{\mathcal{L}^N(B_r(0))} \int_{B_r(0)} \left(\int_{\mathbb{R}^N} \eta(z+y,x)\,dz \right) dy \\ &= \frac{1}{\mathcal{L}^N(B_r(0))} \int_{B_r(0)} \left(\int_{\mathbb{R}^N} \eta(z,x)\,dz \right) dy = 1. \end{aligned}$$

From Theorem 1 in [10, Section 1.7] we infer that for any fixed $x \in \mathbb{R}^N$ we have

$$\lim_{r\to 0^+} \eta_{0,r}(z,x) = \eta(z,x) \quad \text{for a.e. } z \in \mathbb{R}^N. \tag{3.63}$$

Take a sequence $r_n \downarrow 0$ and define a sequence $\{\bar\eta_n\}$ of functions on $\mathbb{R}^N \times \mathbb{R}^N$ by

$$\bar\eta_n(z,x) = \eta_{0,r_n}(z,x).$$

Clearly $|\bar\eta_n(z,x)| \le \bar M$ for all z and x, and $\bar\eta_n(z,x) = 0$ if either $|z| > \bar M + 1$ or $x \in \mathbb{R}^N \setminus \Omega'$. By (3.63) we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\bar\eta_n(z,x) - \eta(z,x)|\, dz\, d\bar\mu(x) = 0. \tag{3.64}$$

In addition, for all $x \in \mathbb{R}^N$ and $z_1, z_2 \in \mathbb{R}^N$ we have

$$|\bar\eta_n(z_1,x) - \bar\eta_n(z_2,x)| \le C_n\, |z_1 - z_2|, \tag{3.65}$$

where $C_n > 0$ depends only on n.

For every n and $\rho > 0$ define $\eta_{n,\rho} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\eta_{n,\rho}(z,x) = \frac{1}{\bar\mu(B_\rho(x))} \int_{B_\rho(x)} \bar\eta_n(z,y)\, d\bar\mu(y).$$

Consider a bounded open domain Ω'' such that $\Omega \subset\subset \Omega'' \subset\subset \Omega'$. Then, for $\rho > 0$ sufficiently small and any $x \in \Omega''$, we have

$$\int_{\mathbb{R}^N} \eta_{n,\rho}(z,x)\, dz = \frac{1}{\bar\mu(B_\rho(x))} \int_{B_\rho(x)} \int_{\mathbb{R}^N} \bar\eta_n(z,y)\, dz\, d\bar\mu(y) = 1.$$

Using (3.65) we obtain

$$|\eta_{n,\rho}(z_1,x_1) - \eta_{n,\rho}(z_2,x_2)| \le |\eta_{n,\rho}(z_1,x_1) - \eta_{n,\rho}(z_1,x_2)| + C_n|z_1 - z_2|. \tag{3.66}$$

But for every $z \in \mathbb{R}^N$ we have

$$\left| \int_{B_\rho(x_1)} \bar\eta_n(z,y)\, d\bar\mu(y) - \int_{B_\rho(x_2)} \bar\eta_n(z,y)\, d\bar\mu(y) \right| \\ \le \bar M \{\bar\mu(B_\rho(x_1) \setminus B_\rho(x_2)) + \bar\mu(B_\rho(x_1) \setminus B_\rho(x_2))\}. \tag{3.67}$$

Therefore, combining (3.66) with (3.67) we see that $\eta_{n,\rho}$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N$. Again, by Theorem 1 in [10, Section 1.7] we have, for any fixed $z \in \mathbb{R}$,

$$\lim_{\rho\to 0^+} \eta_{n,\rho}(z,x) = \bar\eta_n(z,x) \quad \text{for } \bar\mu\text{-almost every } x \in \mathbb{R}^N. \tag{3.68}$$

We also have $|\eta_{n,\rho}(z,x)| \le \bar{M}$ for all z and x, and $\eta_{n,\rho}(z,x) = 0$ for $|z| > \bar{M} + 1$ and any x. Moreover, there exists $\bar{M}_0 > 0$ such that for $\rho > 0$ sufficiently small and any n we have $\eta_{n,\rho}(z,x) = 0$ for $|x| > \bar{M}_0$ and every z. By (3.68) we obtain

$$\lim_{\rho\to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\eta_{n,\rho}(z,x) - \bar{\eta}_n(z,x)|\, dz\, d\bar{\mu}(x) = 0.$$

From (3.64) it follows that there exists a sequence $\rho_n \downarrow 0$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\eta_{n,\rho_n}(z,x) - \eta(z,x)|\, dz\, d\bar{\mu}(x) = 0,$$

and $\int_{\mathbb{R}^N} \eta_{n,\rho_n}(z,x)\, dz = 1$ for every n and each $x \in \overline{\Omega''}$. Put $\hat{\eta}_n(z,x) := \eta_{n,\rho_n}(z,x)$. Then $\hat{\eta}_n \in C_c(\mathbb{R}^N \times \mathbb{R}^N)$, $\int_{\mathbb{R}^N} \hat{\eta}_n(z,x)\, dz = 1$ for every $x \in \overline{\Omega''}$ and there exists $\tilde{M} > 0$, independent of n, such that $\hat{\eta}_n(z,x) = 0$ for $|z| > \tilde{M}$ and $|\hat{\eta}_n(z,x)| \le \tilde{M}$ for all $(z,x) \in \mathbb{R}^N \times \mathbb{R}^N$. Furthermore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\hat{\eta}_n(z,x) - \eta(z,x)|\, dz\, d\bar{\mu}(x) = 0,$$

which implies that

$$\lim_{n\to\infty} \int_{J_\varphi} \int_{\mathbb{R}} |\hat{p}_n(t,x) - p(t,x)|\, dt\, d\mu(x) = 0, \tag{3.69}$$

where

$$\hat{p}_n(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} \hat{\eta}_n(t\boldsymbol{\nu}(x) + y, x)\, d\mathcal{H}^{N-1}(y).$$

Next, let $\omega : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfy $\omega \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\omega \ge 0$ and $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega = 1$. For any $0 < \varepsilon < 1$ define

$$\begin{aligned}(\eta_n)_\varepsilon(z,x) &= \frac{1}{\varepsilon^{2N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega\left(\frac{y_1 - z}{\varepsilon}, \frac{y_2 - x}{\varepsilon}\right) \hat{\eta}_n(y_1, y_2)\, dy_1\, dy_2 \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega(\zeta_1, \zeta_2) \hat{\eta}_n(z + \varepsilon\zeta_1, x + \varepsilon\zeta_2)\, d\zeta_1\, d\zeta_2.\end{aligned}$$

Then $(\eta_n)_\varepsilon \in C_c^2(\mathbb{R}^N \times \mathbb{R}^N)$ and there exists $\hat{M} > 0$, independent of n and ε, such that for every $0 < \varepsilon < 1$ and every n we have $(\eta_n)_\varepsilon(z,x) = 0$ for $|z| > \hat{M}$ and $|(\eta_n)_\varepsilon(z,x)| \le \hat{M}$ for all $(z,x) \in \mathbb{R}^N \times \mathbb{R}^N$. Moreover, for sufficiently small $\varepsilon > 0$ we have, for every $x \in \Omega$,

$$\begin{aligned}\int_{\mathbb{R}^N} (\eta_n)_\varepsilon(z,x)\, dz &= \frac{1}{\varepsilon^{2N}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega(y_1/\varepsilon, y_2/\varepsilon) \hat{\eta}_n(z + y_1, x + y_2)\, dy_1\, dy_2 \right) dz \\ &= \frac{1}{\varepsilon^{2N}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \omega(y_1/\varepsilon, y_2/\varepsilon) \left(\int_{\mathbb{R}^N} \hat{\eta}_n(z + y_1, x + y_2)\, dz \right) dy_1\, dy_2 \\ &= 1.\end{aligned}$$

Therefore, $(\eta_n)_\varepsilon \in \mathcal{V}$ for $\varepsilon > 0$ sufficiently small. For every $t \in \mathbb{R}$ and every $x \in J_\varphi$ set

$$(\hat{p}_n)_\varepsilon(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} (\eta_n)_\varepsilon(t\boldsymbol{\nu}(x) + y, x)\, d\mathcal{H}^{N-1}(y).$$

Since $(\eta_n)_\varepsilon \to \eta_n$ uniformly on $\mathbb{R}^N \times \mathbb{R}^N$ as ε goes to 0, we have

$$\lim_{\varepsilon\to 0^+} \int_{J_\varphi}\int_{\mathbb{R}} |(\hat{p}_n)_\varepsilon(t,x) - \hat{p}_n(t,x)|\, dt\, d\mu(x) = 0.$$

Therefore, by (3.69) there exists a sequence $\varepsilon_n \downarrow 0$ such that $(\eta_n)_{\varepsilon_n} \in \mathcal{V}$ and

$$\lim_{n\to\infty} \int_{J_\varphi}\int_{\mathbb{R}} |(\hat{p}_n)_{\varepsilon_n}(t,x) - p(t,x)|\, dt\, d\mu(x) = 0.$$

Set $\eta_n(z,x) := (\eta_n)_{\varepsilon_n}(z,x) \in \mathcal{V}$, and let $p_n(t,x)$ be defined on $\mathbb{R} \times J_\varphi$ by

$$p_n(t,x) = \int_{H_{\boldsymbol{\nu}(x)}} \eta_n(t\boldsymbol{\nu}(x) + y, x)\, d\mathcal{H}^{N-1}(y).$$

There exists C_0 such that $|p_n(t,x)| \leq C_0$ for all $x \in J_\varphi$ and $t \in \mathbb{R}$, and there exists $\hat{C}$ such that $p_n(t,x) = 0$ for $|t| > \hat{C}$ and any $x \in J_\varphi$. Moreover,

$$\lim_{n\to\infty} \int_{J_\varphi}\int_{\mathbb{R}} |p_n(t,x) - p(t,x)|\, dt\, d\mu(x) = 0, \tag{3.70}$$

so that Claim 3 follows.

We are now ready to complete the proof of Lemma 3.4. Consider a function $p \in \mathcal{W}$ and let $\eta_n \in \mathcal{V}$ and p_n be the corresponding functions as given by Claim 3. We have

$$\begin{aligned}
&Y(\eta_n) - \int_{J_\varphi} Q_x(p(\cdot,x))\, d\mathcal{H}^{N-1}(x) = \int_{J_\varphi} |\varphi^+ - \varphi^-|^2 \left\{ \int_{\mathbb{R}} (p_n^2 - p^2)\, dt \right\} d\mathcal{H}^{N-1} \\
&\quad + \int_{J_\varphi} \left(\int_{-\infty}^0 \left\{ W\left((\varphi^- - \varphi^+) \int_{-\infty}^t p_n(s,\cdot)\, ds + \varphi^+, u^+ \right) \right.\right. \\
&\quad \left.\left. - W\left((\varphi^- - \varphi^+) \int_{-\infty}^t p(s,\cdot)\, ds + \varphi^+, u^+ \right) \right\} dt \right) d\mathcal{H}^{N-1} \\
&\quad + \int_{J_\varphi} \left(\int_0^{\infty} \left\{ W\left((\varphi^+ - \varphi^-) \int_t^{\infty} p_n(s,\cdot)\, ds + \varphi^-, u^- \right) \right.\right. \\
&\quad \left.\left. - W\left((\varphi^+ - \varphi^-) \int_t^{\infty} p(s,\cdot)\, ds + \varphi^-, u^- \right) \right\} dt \right) d\mathcal{H}^{N-1} \\
&\leq C \int_{J_\varphi} \left(\int_{-c}^{c} \left\{ |p_n(t,\cdot) - p(t,\cdot)| + \int_{-c}^{c} |p_n(s,\cdot) - p(s,\cdot)| ds \right\} dt \right) |\varphi^+ - \varphi^-| d\mathcal{H}^{N-1}
\end{aligned} \tag{3.71}$$

for some constants $c, C > 0$. Therefore, using (3.70) we deduce from (3.71) that

$$\overline{\lim_{n\to\infty}}\, Y(\eta_n) \le \int_{J_\varphi} Q_x(p(\cdot, x))\, d\mathcal{H}^{N-1}(x).$$

Since this holds for every $p \in \mathcal{W}$, we get

$$\inf_{\eta\in\mathcal{V}} Y(\eta) = \inf_{p\in\mathcal{W}} \left\{ \int_{J_\varphi} Q_x(p(\cdot, x))\, d\mathcal{H}^{N-1}(x) \right\}. \tag{3.72}$$

Combining (3.72) with Claim 2 we obtain the desired result (3.55). □

Combining the results of Lemmas 3.4 and 3.3 we deduce the following.

Theorem 3.2. *Let W, u and φ be as in Theorem* 3.1. *Assume in addition that $W \ge 0$. Let $Y : \mathcal{V} \to \mathbb{R}$ be defined as in* (3.54). *Then*

$$\begin{aligned}\inf_{\eta\in\mathcal{V}} Y(\eta) &= \mathcal{J}_0(\varphi)\\ &:= 2\int_{J_\varphi} |\varphi^+(x) - \varphi^-(x)| \inf_{\tau\in[0,1]} \left\{ \int_0^\tau \sqrt{W(s\varphi^-(x) + (1-s)\varphi^+(x), u^+(x))}\, ds \right.\\ &\quad \left. + \int_\tau^1 \sqrt{W(s\varphi^-(x) + (1-s)\varphi^+(x), u^-(x))}\, ds \right\} d\mathcal{H}^{N-1}(x).\end{aligned}$$

Remark 3.2. Using Remark 3.1, and slightly modifying the arguments of Lemma 3.4, we infer that also $\inf_{\eta\in\mathcal{V},\, \eta\ge 0} Y(\eta) = \mathcal{J}_0(\varphi)$.

Definition 3.2. *The set $\mathcal{V}_r$ is the subset of $\mathcal{V}$ consisting of all $\eta \in \mathcal{V}$ for which there exist an open set $G = G_\eta \subset \mathbb{R}^N$ satisfying $\partial\Omega \subset G$ and a function $l = l_\eta : [0, \infty) \to \mathbb{R}$ such that $\eta(z, x) = l(|z|)$ for all $x \in G$ and $z \in \mathbb{R}^N$.*

Lemma 3.5. *For any $\eta \in \mathcal{V}$ there exists a sequence $\eta^{(n)} \in \mathcal{V}_r$ such that*

$$\lim_{n\to\infty} Y(\eta^{(n)}) = Y(\eta).$$

Moreover, if $\eta \ge 0$ then we can also take $\eta^{(n)} \ge 0$.

Proof. Let $\eta \in \mathcal{V}$. Consider a sequence of open sets $U_n \subset \mathbb{R}^N$ such that $U_{n+1} \subset U_n$ for all n and $\bigcap_{n=1}^\infty U_n = \partial\Omega$. For each n, let $\zeta_{1,n}, \zeta_{2,n} \in C_c^\infty(\mathbb{R}^N, [0,1])$ be such that $\operatorname{supp}\zeta_{1,n} \subset\subset \Omega$, $\operatorname{supp}\zeta_{2,n} \subset\subset U_n$ and $\zeta_{1,n}(x) + \zeta_{2,n}(x) = 1$ for every $x \in \overline{\Omega}$. Fix a nonnegative radial C^2 function h with compact support satisfying $\int_{\mathbb{R}^N} h(|z|)\, dz = 1$. Define $\eta^{(n)} \in \mathcal{V}_r$ by

$$\eta_{(n)}(z, x) = \zeta_{1,n}(x)\eta(z, x) + \zeta_{2,n}(x)h(|z|).$$

Then $\eta^{(n)} \in \mathcal{V}_r$ and

$$\lim_{n\to\infty} |Y(\eta_{(n)}) - Y(\eta)| \le C \lim_{n\to\infty} \int_{J_\varphi \cap U_n} |\varphi^+(x) - \varphi^-(x)|\, d\mathcal{H}^{N-1}(x) = 0. \qquad □$$

From Lemma 3.5, Theorem 3.2 and Remark 3.2 we deduce the following.

Corollary 3.1. $\inf_{\eta\in\mathcal{V}_r,\, \eta\ge 0} Y(\eta) = \mathcal{J}_0(\varphi)$.

4. The upper bound for the second order problem

In this section we prove the main results of this paper for the second order problem, Theorem 1.1 and Corollary 1.1, using the results of Section 3.

Definition 4.1. *For a domain $\Omega \subset \mathbb{R}^N$ we define*

$$\mathrm{BVG}(\Omega, \mathbb{R}) := \{v \in W^{1,\infty}(\Omega, \mathbb{R}) : \nabla v \in BV(\Omega, \mathbb{R}^N)\}.$$

Definition 4.2. *We say that the domain $\Omega \subset \mathbb{R}^N$ is an* extension domain of second order *if for any $v \in \mathrm{BVG}(\Omega, \mathbb{R})$ there exists $\bar{v} \in \mathrm{BVG}(\mathbb{R}^N, \mathbb{R})$ such that $\bar{v}(x) = v(x)$ for any $x \in \Omega$ and $\|D(\nabla \bar{v})\|(\partial\Omega) = 0$.*

Definition 4.3. *We say that the bounded domain $\Omega \subset \mathbb{R}^N$ is a* BVG-domain *if for each point $x \in \partial\Omega$ there exist $r, \delta > 0$ and a mapping $\gamma \in \mathrm{BVG}(\mathbb{R}^{N-1}, \mathbb{R})$ such that—upon rotating and relabeling the coordinate axes if necessary—we have*

$$\Omega \cap C(x, r, \delta) = \{y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : \ y_1 > \gamma(y')\} \cap C(x, r, \delta),$$

where $C(x, r, \delta) := \{y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : \ |y_1 - x_1| < \delta, \ |y' - x'| < r\}$. In other words, near x, $\partial\Omega$ is the graph of a BVG*-function.*

The proof of the following proposition is given in the Appendix.

Proposition 4.1. *Any bounded* BVG*-domain is an extension domain of second order.*

Let Ω be a bounded BVG-domain. Consider $v \in \mathrm{BVG}(\Omega, \mathbb{R})$ and $\eta \in \mathcal{V}$ (see Definition 3.1). By Proposition 4.1, we may extend v to $\bar{v} \in \mathrm{BVG}(\mathbb{R}^N, \mathbb{R})$ such that $\bar{v} = v$ a.e. in Ω, supp $\bar{v}$ is compact and $\|D(\nabla \bar{v})\|(\partial\Omega) = 0$. For any $\varepsilon > 0$ and $x \in \mathbb{R}^N$ set

$$v_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} \eta\left(\frac{y - x}{\varepsilon}, x\right) \bar{v}(y)\, dy = \int_{\mathbb{R}^N} \eta(z, x) \bar{v}(x + \varepsilon z)\, dz. \tag{4.1}$$

Then $v_\varepsilon \in C_c^2(\mathbb{R}^N, \mathbb{R})$ and $\lim_{\varepsilon \to 0^+} v_\varepsilon = v$ in $W^{1,p}(\Omega)$ for every $p \geq 1$. Next we prove:

Proposition 4.2. *Let Ω be a bounded* BVG*-domain and let $F \in C^2(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^q)$ satisfy $F \geq 0$. Let $f \in BV(\Omega, \mathbb{R}^q) \cap L^\infty(\Omega, \mathbb{R}^q)$ and $v \in \mathrm{BVG}(\Omega, \mathbb{R})$ be such that $F(\nabla v(x), v(x), f(x)) = 0$ a.e. in Ω. Then*

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left\{\varepsilon \int_\Omega |\nabla^2 v_\varepsilon(x)|^2\, dx + \frac{1}{\varepsilon} \int_\Omega F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x))\, dx\right\} \\
&= \int_{J_{\nabla v}} |\nabla v^+(x) - \nabla v^-(x)|^2 \left\{\int_{\mathbb{R}} p^2(t, x)\, dt\right\} d\mathcal{H}^{N-1}(x) \\
&+ \int_{J_{\nabla v}} \left\{\int_{-\infty}^0 F\left(\nabla v^-(x) \int_{-\infty}^t p(s, x)\, ds + \nabla v^+(x) \int_t^\infty p(s, x)\, ds, v(x), f^+(x)\right) dt\right. \\
&+ \int_0^\infty F\left(\nabla v^-(x) \int_{-\infty}^t p(s, x)\, ds\right. \\
&\qquad\qquad \left.\left. + \nabla v^+(x) \int_t^\infty p(s, x)\, ds, v(x), f^-(x)\right) dt\right\} d\mathcal{H}^{N-1}(x),
\end{aligned} \tag{4.2}$$

where $p(t,x)$ is defined in (3.5) *(with $\boldsymbol{\nu}(x)$ denoting the orientation vector of $J_{\nabla v}$) and we assume that the orientation of J_f coincides $\mathcal{H}^{N-1}$-a.e. with the orientation of $J_{\nabla v}$ on $J_f \cap J_{\nabla v}$.*

Proof. Put $\varphi(x) := \nabla v(x)$ and $\bar\varphi(x) := \nabla \bar v(x)$. Then $\varphi \in \mathrm{BV}(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$, $\bar\varphi \in \mathrm{BV}(\mathbb{R}^N,\mathbb{R}^N) \cap L^\infty_c(\mathbb{R}^N,\mathbb{R}^N)$, $\bar\varphi = \varphi$ on Ω and $\|D\bar\varphi\|(\partial\Omega) = 0$. Define

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \eta\left(\frac{y-x}{\varepsilon}, x\right)\bar\varphi(y)\,dy = \int_{\mathbb{R}^N} \eta(z,x)\bar\varphi(x+\varepsilon z)\,dz.$$

Proposition 3.1 gives

$$\lim_{\varepsilon\to 0} \varepsilon \int_\Omega |\nabla\psi_\varepsilon(x)|^2\,dx = \int_{J_\varphi} |\varphi^+(x) - \varphi^-(x)|^2 \left\{\int_{\mathbb{R}} p^2(t,x)\,dt\right\} d\mathcal{H}^{N-1}(x). \tag{4.3}$$

From our assumptions on $F \in C^2$ it follows that if $F(a,b,c) = 0$ then $\nabla_a F(a,b,c) = 0$ and $\partial_b F(a,b,c) = 0$. Applying Proposition 3.2, using the continuity of v, yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega F(\psi_\varepsilon(x), v_\varepsilon(x), f(x))\,dx \\
&\quad = \int_{J_\varphi} \left\{ \int_{-\infty}^0 F\left(\varphi^-(x)\int_{-\infty}^t p(s,x)\,ds + \varphi^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^+(x)\right) dt \right. \\
&\qquad + \int_0^\infty F\left(\varphi^-(x)\int_{-\infty}^t p(s,x)\,ds \right. \\
&\qquad\qquad \left.\left. + \varphi^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^-(x)\right) dt \right\} d\mathcal{H}^{N-1}(x).
\end{aligned} \tag{4.4}$$

Next, we compute

$$\begin{aligned}
\nabla v_\varepsilon(x) &= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \nabla_x\left\{\eta\left(\frac{y-x}{\varepsilon}, x\right)\right\}\bar v(y)\,dy \\
&= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \left\{-\frac{1}{\varepsilon}\nabla_1\eta\left(\frac{y-x}{\varepsilon}, x\right) + \nabla_2\eta\left(\frac{y-x}{\varepsilon}, x\right)\right\}\bar v(y)\,dy \\
&= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \left\{-\nabla_y\eta\left(\frac{y-x}{\varepsilon}, x\right)\right\}\bar v(y)\,dy + \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \nabla_2\eta\left(\frac{y-x}{\varepsilon}, x\right)\bar v(y)\,dy \\
&= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \eta\left(\frac{y-x}{\varepsilon}, x\right)\nabla\bar v(y)\,dy + \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \nabla_2\eta\left(\frac{y-x}{\varepsilon}, x\right)\bar v(y)\,dy \\
&= \int_{\mathbb{R}^N} \eta(z,x)\nabla\bar v(x+\varepsilon z)\,dz + \int_{\mathbb{R}^N} \nabla_2\eta(z,x)\bar v(x+\varepsilon z)\,dz \\
&= \psi_\varepsilon(x) + \int_{\mathbb{R}^N} \nabla_2\eta(z,x)\bar v(x+\varepsilon z)\,dz.
\end{aligned} \tag{4.5}$$

Differentiating one more time gives

$$
\begin{aligned}
\nabla^2 v_\varepsilon(x) &= \nabla\psi_\varepsilon(x) + \nabla\left(\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\bar{v}(y)\,dy\right)\\
&= \nabla\psi_\varepsilon(x) + \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla_x\left\{\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\right\}\bar{v}(y)\,dy\\
&= \nabla\psi_\varepsilon(x) + \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\left(-\frac{1}{\varepsilon}\nabla_1\left\{\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\right\} + \nabla_2\left\{\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\right\}\right)\bar{v}(y)\,dy\\
&= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\left(-\nabla_y\left\{\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\right\}\right)\bar{v}(y)\,dy + \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla_2\left\{\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)\right\}\bar{v}(y)\,dy\\
&\quad + \nabla\psi_\varepsilon(x)\\
&= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla\bar{v}(y)\otimes\nabla_2\eta\left(\frac{y-x}{\varepsilon},x\right)dy + \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\nabla_2^2\eta\left(\frac{y-x}{\varepsilon},x\right)\bar{v}(y)\,dy + \nabla\psi_\varepsilon(x)\\
&= \nabla\psi_\varepsilon(x) + \int_{\mathbb{R}^N}\bar{\varphi}(x+\varepsilon z)\otimes\nabla_2\eta(z,x)\,dz + \int_{\mathbb{R}^N}\nabla_2^2\eta(z,x)\bar{v}(x+\varepsilon z)\,dz. \qquad (4.6)
\end{aligned}
$$

Therefore, $|\nabla^2 v_\varepsilon(x) - \nabla\psi_\varepsilon(x)| \le C$ for some constant $C > 0$. Hence,

$$
\begin{aligned}
\left||\nabla^2 v_\varepsilon(x)|^2 - |\nabla\psi_\varepsilon(x)|^2\right| &\le |\nabla^2 v_\varepsilon(x) - \nabla\psi_\varepsilon(x)|\cdot(|\nabla^2 v_\varepsilon(x)| + |\nabla\psi_\varepsilon(x)|)\\
&\le C(2|\nabla\psi_\varepsilon(x)| + C). \qquad (4.7)
\end{aligned}
$$

From (4.3) and (4.7) we get

$$
\lim_{\varepsilon\to 0}\varepsilon\int_\Omega|\nabla^2 v_\varepsilon(x)|^2\,dx = \int_{J_\varphi}|\varphi^+(x)-\varphi^-(x)|^2\left\{\int_{\mathbb{R}}p^2(t,x)\,dt\right\}d\mathcal{H}^{N-1}(x). \qquad (4.8)
$$

For each $x\in\Omega$ we have, using (4.5),

$$
\begin{aligned}
&F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x)) - F(\psi_\varepsilon(x), v_\varepsilon(x), f(x))\\
&= \left(\int_{\mathbb{R}^N}\nabla_2\eta(z,x)\bar{v}(x+\varepsilon z)\,dz\right)\cdot\int_0^1\nabla_1 F((1-t)\psi_\varepsilon(x) + t\nabla v_\varepsilon(x), v_\varepsilon(x), f(x))\,dt\\
&= \left(\int_{\mathbb{R}^N}\nabla_2\eta(z,x)(\bar{v}(x+\varepsilon z) - \bar{v}(x))\,dz\right)\\
&\quad\times\int_0^1\nabla_1 F\left(\psi_\varepsilon(x) + t\int_{\mathbb{R}^N}\nabla_2\eta(z,x)\bar{v}(x+\varepsilon z)\,dz, v_\varepsilon(x), f(x)\right)dt.
\end{aligned}
$$

In the last equality we used the fact that $\int_{\mathbb{R}^N}\nabla_x\eta(z,x)\,dz = 0$ for every $x\in\Omega$. Since $\bar{v}$ is a Lipschitz function, we have

$$
\left|\int_{\mathbb{R}^N}\nabla_2\eta(z,x)(\bar{v}(x+\varepsilon z) - \bar{v}(x))\,dz\right| \le C\varepsilon
$$

for some constant $C > 0$. Therefore,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega |F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x)) - F(\psi_\varepsilon(x), v_\varepsilon(x), f(x))|\,dx$$
$$\le \lim_{\varepsilon\to 0} C\int_0^1\int_\Omega \left|\nabla_1 F\left(\psi_\varepsilon + t\int_{\mathbb{R}^N}\nabla_2\eta(z,x)(\bar{v}(x+\varepsilon z)-\bar{v}(x))\,dz, v_\varepsilon(x), f(x)\right)\right| dx\,dt$$
$$= 0, \qquad (4.9)$$

where we used the assumption $\nabla_1 F(\varphi, v, f) = 0$ a.e. Combining (4.9) with (4.4) we are led to

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_\Omega F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x))\,dx$$
$$= \int_{J_\varphi}\left\{\int_{-\infty}^0 F\left(\varphi^-(x)\int_{-\infty}^t p(s,x)\,ds + \varphi^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^+(x)\right)dt\right.$$
$$\left. + \int_0^\infty F\left(\varphi^-(x)\int_{-\infty}^t p(s,x)\,ds + \varphi^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^-(x)\right)dt\right\} d\mathcal{H}^{N-1}(x).$$
$$(4.10)$$

The desired result (4.2) follows from (4.10) and (4.8). □

Next we define the distance function to $\partial\Omega$ by

$$d(x) = \inf\{|x - z| : z \in \partial\Omega\}, \qquad \forall x \in \mathbb{R}^N. \qquad (4.11)$$

For any $\beta > 0$ set

$$\Omega_\beta := \{x \in \Omega : d(x) < \beta\} \quad \text{and} \quad \Sigma_\beta := \{x \in \Omega : d(x) = \beta\}. \qquad (4.12)$$

The proof of the following technical lemma is given in the Appendix.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2. If $\varphi \in \mathrm{BV}(\Omega, \mathbb{R}^k)$ satisfies $T\varphi = 0$ on $\partial\Omega$, where T is the trace operator, then*

$$\lim_{\rho\to 0^+}\frac{1}{\rho}\int_{\Omega_\rho}|\varphi(x)|\,dx = 0. \qquad (4.13)$$

If $v \in \mathrm{BVG}(\Omega, \mathbb{R})$ satisfies $v = 0$ on $\partial\Omega$ and $T\nabla v = 0$ on $\partial\Omega$ then

$$\lim_{\rho\to 0^+}\frac{1}{\rho^2}\int_{\Omega_\rho}|v(x)|\,dx = 0. \qquad (4.14)$$

Next we prove an analogous result to Proposition 4.2 for the case where a boundary condition is given.

Proposition 4.3. *Let Ω be a bounded C^2 domain. Let $F \in C^2(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^q)$, $f \in \mathrm{BV}(\Omega, \mathbb{R}^q) \cap L^\infty(\Omega, \mathbb{R}^q)$ and $v \in \mathrm{BVG}(\Omega, \mathbb{R})$ be as in Proposition* 4.2. *Suppose also that there exists a function $h \in C^2(\mathbb{R}^N)$ which satisfies the boundary conditions:*

$$h = v \quad \text{and} \quad \nabla h = T\nabla v \quad \text{on } \partial\Omega.$$

Then, for any $\eta \in \mathcal{V}_r$ (see Definition 3.2*) there exists a family $\{\bar{v}_\varepsilon\}_{0<\varepsilon<1} \subset C^2(\mathbb{R}^N)$ of functions that satisfy the same boundary conditions, $\bar{v}_\varepsilon = v$ and $\nabla\bar{v}_\varepsilon = T\nabla v$ on $\partial\Omega$, such that*

$$\lim_{\varepsilon\to 0^+} \bar{v}_\varepsilon = v \quad \text{in } W^{1,p}(\Omega),\ \forall p \in [1,\infty),$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \left\{\varepsilon \int_\Omega |\nabla^2 \bar{v}_\varepsilon(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega F(\nabla\bar{v}_\varepsilon(x), \bar{v}_\varepsilon(x), f(x))\,dx\right\} \\
&= \int_{J_{\nabla v}} |\nabla v^+(x) - \nabla v^-(x)|^2 \left\{\int_{\mathbb{R}} p^2(t,x)\,dt\right\} d\mathcal{H}^{N-1}(x) \\
&\quad + \int_{J_{\nabla v}} \left\{\int_{-\infty}^0 F\left(\nabla v^-(x)\int_{-\infty}^t p(s,x)\,ds + \nabla v^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^+(x)\right) dt\right. \\
&\quad + \int_0^\infty F\left(\nabla v^-(x)\int_{-\infty}^t p(s,x)\,ds\right. \\
&\qquad\qquad \left.\left. + \nabla v^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^-(x)\right) dt\right\} d\mathcal{H}^{N-1}(x),
\end{aligned}$$

where $p(t,x)$ is defined in (3.5)*, with $\boldsymbol{\nu}(x)$ denoting the orientation vector of $J_{\nabla v}$, and we assume that the orientation of J_f coincides $\mathcal{H}^{N-1}$-a.e. with the orientation of $J_{\nabla v}$ on $J_f \cap J_{\nabla v}$.*

Proof. Let $v_\varepsilon \in C^2(\mathbb{R}^N)$ be defined by (4.1). Then

$$\lim_{\varepsilon\to 0^+} v_\varepsilon = v \quad \text{in } W^{1,p}(\Omega),\ \forall p \in [1,\infty),$$

and by Proposition 4.2 we also have (4.2). So we only need to slightly modify v_ε in order that it satisfies the same boundary conditions as v.

Fix some function $\omega \in C^2(\mathbb{R})$ satisfying $0 \le \omega(t) \le 1$ for all $t \in \mathbb{R}$, $\omega(t) = 0$ for $t \ge 3/4$ and $\omega(t) = 1$ for $t \le 1/2$. For every $0 < \varepsilon < 1$ and every $x \in \mathbb{R}^N$ define

$$\bar{v}_\varepsilon(x) := v_\varepsilon(x) + (h(x) - v_\varepsilon(x))\omega(d(x)/\varepsilon),$$

where $d(x)$ is defined in (4.11). Then $\bar{v}_\varepsilon \in C^2(\mathbb{R}^N)$ for $0 < \varepsilon < \beta_0$ (see the proof of Lemma 4.1 in the Appendix) and we have $\bar{v}_\varepsilon(x) = v(x)$ and $\nabla\bar{v}_\varepsilon(x) = T\nabla v(x)$ for $x \in \partial\Omega$. Therefore, in view of (4.2) we only need to prove that

$$\lim_{\varepsilon\to 0^+} (\bar{v}_\varepsilon - v_\varepsilon) = 0 \quad \text{in } W^{1,p}(\Omega)$$

and

$$\lim_{\varepsilon\to 0^+}\left\{\varepsilon\int_\Omega (|\nabla^2\bar{v}_\varepsilon(x)|^2 - |\nabla^2 v_\varepsilon(x)|^2)\,dx \right.$$
$$\left. + \frac{1}{\varepsilon}\int_\Omega (F(\nabla\bar{v}_\varepsilon(x), \bar{v}_\varepsilon(x), f(x)) - F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x)))\,dx\right\} = 0. \qquad (4.15)$$

For any $\varepsilon > 0$ and any $0 < t \le 1$ we have, by the same computation as in (3.30),

$$\begin{aligned}\frac{d(v_{\varepsilon t}(x))}{dt} &= \frac{d}{dt}\left(\frac{1}{t^N\varepsilon^N}\int_{\mathbb{R}^N}\eta\left(\frac{y-x}{t\varepsilon}, x\right)\bar{v}(y)\,dy\right)\\ &= \frac{\varepsilon}{t^N\varepsilon^N}\int_{\mathbb{R}^N}\eta\left(\frac{y-x}{t\varepsilon}, x\right)\frac{y-x}{t\varepsilon}\cdot\nabla\bar{v}(y)\,dy\\ &= \varepsilon\int_{\mathbb{R}^N}(\eta(y,x)y)\cdot\nabla\bar{v}(x+\varepsilon t y)\,dy.\end{aligned}$$

Therefore, for $\varepsilon > 0$ and $x \in \Omega$ we have

$$v_\varepsilon(x) - v(x) = \varepsilon\int_0^1\left\{\int_{\mathbb{R}^N}(\eta(y,x)\,y)\cdot\nabla\bar{v}(x+\varepsilon t y)\,dy\right\}dt. \qquad (4.16)$$

In particular, since $\nabla v \in L^\infty$, it follows that

$$\frac{1}{\varepsilon}|v_\varepsilon(x) - v(x)| \le C, \quad \forall x\in\Omega,\ \forall\varepsilon > 0, \qquad (4.17)$$

for some constant $C > 0$. Since $\eta \in \mathcal{V}_r$ there exist a sufficiently large $R > 0$ and a sufficiently small $\beta > 0$ such that:

(i) $\eta(z,x) = 0$ for $|z| \ge R$ and every x,
(ii) $\eta(z,x) = l(|z|)$ for $x \in \Omega_\beta$ (see (4.12)).

Thus,

$$\int_{\mathbb{R}^N}\eta(y,x)y\,dy = 0 \quad \text{for } x\in\Omega_\beta.$$

For all $\varepsilon > 0$ and almost every $x \in \Omega_\beta$ we have, by (4.16),

$$v_\varepsilon(x) - v(x) = \varepsilon\int_0^1\left\{\int_{B_R(0)}(\eta(y,x)\,y)\cdot(\nabla\bar{v}(x+\varepsilon t y) - \nabla\bar{v}(x))\,dy\right\}dt. \qquad (4.18)$$

Since $\nabla\bar{v} \in \mathrm{BV}$, by [2, Exercise 3.3] we have

$$\int_{\Omega_\varepsilon}|\nabla\bar{v}(x+ty) - \nabla\bar{v}(x)|\,dx \le t|y|\cdot\|D(\nabla\bar{v})\|((\partial\Omega)_{\varepsilon+tR}),$$
$$\forall t\in(0,1],\ \forall y\in B_R(0), \qquad (4.19)$$

where $(\partial\Omega)_l = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \partial\Omega) < l\}$ (see (3.11)). Therefore, from (4.18) we get, for every $0 < \varepsilon < \beta$,

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |v_\varepsilon(x) - v(x)|\,dx \\
&\leq \frac{1}{\varepsilon}\int_0^1 \left\{ \int_{B_R(0)} |\eta(y,x)|\cdot|y| \left(\int_{\Omega_\varepsilon} |\nabla\bar{v}(x+\varepsilon t y) - \nabla\bar{v}(x)|\,dx \right) dy \right\} dt \\
&\leq \bar{C}\|D(\nabla\bar{v})\|((\partial\Omega)_{\varepsilon+\varepsilon R}),
\end{aligned} \tag{4.20}$$

where $\bar{C} > 0$ is independent of ε. Since $\|D(\nabla\bar{v})\|(\partial\Omega) = 0$, we deduce from (4.20) and (4.17) that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |v_\varepsilon(x) - v(x)|\,dx = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} |v_\varepsilon(x) - v(x)|^2\,dx = 0. \tag{4.21}$$

By Lemma 4.1 we get

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |v(x) - h(x)|\,dx = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v(x) - \nabla h(x)|\,dx = 0. \tag{4.22}$$

Since v and h are Lipschitz functions and $v = h$ on $\partial\Omega$, we also have $|v(x) - h(x)| \leq C\varepsilon$ for $x \in \Omega_\varepsilon$. Therefore, from the first equation in (4.22) we obtain

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} |v(x) - h(x)|^2\,dx = 0.$$

Combining it with (4.21) and (4.22) we infer that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |v_\varepsilon(x) - h(x)|\,dx = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} |v_\varepsilon(x) - h(x)|^2\,dx = 0. \tag{4.23}$$

As in (4.5), we have

$$\nabla v_\varepsilon(x) = \int_{\mathbb{R}^N} \eta(y,x)\nabla\bar{v}(x+\varepsilon y)\,dy + \int_{\mathbb{R}^N} \nabla_x\eta(y,x)\bar{v}(x+\varepsilon y)\,dy. \tag{4.24}$$

Since $\int_{\mathbb{R}^N} \nabla_x\eta(y,x) = 0$ for $x \in \Omega$, we of course also have

$$\nabla v(x) = \int_{\mathbb{R}^N} \eta(y,x)\nabla\bar{v}(x)\,dy + \int_{\mathbb{R}^N} \nabla_x\eta(y,x)\bar{v}(x)\,dy, \qquad \forall x \in \Omega. \tag{4.25}$$

By (4.24)–(4.25), (4.17) and (4.19) we get

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x) - \nabla v(x)|\,dx &\leq \frac{1}{\varepsilon}\int_{B_R(0)}\int_{\Omega_\varepsilon} |\eta(y,x)|\cdot|\nabla\bar{v}(x+\varepsilon y) - \nabla v(x)|\,dx\,dy \\
&\quad + \frac{1}{\varepsilon}\int_{B_R(0)}\int_{\Omega_\varepsilon} |\nabla_x\eta(y,x)|\cdot|\bar{v}(x+\varepsilon y) - v(x)|\,dx\,dy \\
&\leq \tilde{C}(\|D(\nabla\bar{v})\|((\partial\Omega)_{\varepsilon+\varepsilon R}) + \mathcal{L}^N(\Omega_\varepsilon)),
\end{aligned}$$

where $\tilde{C}$ is independent of ε. Therefore, since $\|D(\nabla\bar{v})\|(\partial\Omega) = 0$ and $|\nabla v_\varepsilon(x) - \nabla v(x)|$ is bounded, we obtain

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x) - \nabla v(x)|\,dx = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x) - \nabla v(x)|^2\,dx = 0. \tag{4.26}$$

Combining (4.26) with (4.22) gives

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x) - \nabla h(x)|\,dx = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x) - \nabla h(x)|^2\,dx = 0. \tag{4.27}$$

As in (4.6) we have

$$\begin{aligned}\nabla^2 v_\varepsilon(x) = \nabla\Big(\int_{\mathbb{R}^N} \eta(y,x)\nabla\bar{v}(x+\varepsilon y)\,dy\Big)\\ + \int_{\mathbb{R}^N} \nabla\bar{v}(x+\varepsilon y)\otimes\nabla_x\eta(y,x)\,dy + \int_{\mathbb{R}^N} \nabla_x^2\eta(y,x)\,\bar{v}(x+\varepsilon y)\,dy.\end{aligned} \tag{4.28}$$

But

$$\begin{aligned}\nabla\Big(\int_{\mathbb{R}^N} \eta(y,x)\nabla\bar{v}(x+\varepsilon y)\,dy\Big) &= \nabla\Big(\frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} \eta\Big(\frac{y-x}{\varepsilon},x\Big)\nabla\bar{v}(y)\,dy\Big)\\ &= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N}\Big\{-\frac{1}{\varepsilon}\nabla_1\eta\Big(\frac{y-x}{\varepsilon},x\Big) + \nabla_2\eta\Big(\frac{y-x}{\varepsilon},x\Big)\Big\}\otimes\nabla\bar{v}(y)\,dy\\ &= \int_{\mathbb{R}^N}\Big\{-\frac{1}{\varepsilon}\nabla_y\eta(y,x) + \nabla_x\eta(y,x)\Big\}\otimes\nabla\bar{v}(x+\varepsilon y)\,dy.\end{aligned}$$

Therefore, by (4.28) we get

$$\begin{aligned}\nabla^2 v_\varepsilon(x) = &-\frac{1}{\varepsilon}\int_{\mathbb{R}^N} \nabla_y\eta(y,x)\otimes\nabla\bar{v}(x+\varepsilon y)\,dy\\ &+ \int_{\mathbb{R}^N} (\nabla_x\eta(y,x)\otimes\nabla\bar{v}(x+\varepsilon y) + \nabla\bar{v}(x+\varepsilon y)\otimes\nabla_x\eta(y,x))\,dy\\ &+ \int_{\mathbb{R}^N} \nabla_x^2\eta(y,x)\bar{v}(x+\varepsilon y)\,dy.\end{aligned}$$

Hence, there exist constants C_1, C_2 such that

$$\int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon(x)|^2\,dx \le \frac{C_1}{\varepsilon^2}\int_{B_R(0)}\int_{\Omega_\varepsilon} |\nabla_y\eta(y,x)|\cdot|\nabla\bar{v}(x+\varepsilon y) - \nabla\bar{v}(x)|\,dx\,dy + C_2.$$

Applying (4.19) we obtain

$$\varepsilon\int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon(x)|^2\,dx \le \bar{C}_1\|D(\nabla\bar{v})\|((\partial\Omega)_{\varepsilon+\varepsilon R}) + C_2\varepsilon. \tag{4.29}$$

Since $\|D(\nabla\bar{v})\|(\partial\Omega) = 0$ it follows from (4.29) that $\lim_{\varepsilon\to 0} \varepsilon \int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon(x)|^2\,dx = 0$, which clearly implies that

$$\lim_{\varepsilon\to 0^+} \varepsilon \int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon(x) - \nabla^2 h(x)|^2\,dx = 0. \tag{4.30}$$

Next, we estimate

$$\begin{aligned}
&\varepsilon \int_{\Omega} |\nabla^2\{(h(x) - v_\varepsilon(x))\omega(d(x)/\varepsilon)\}|^2\,dx \\
&\leq C\bigg(\varepsilon \int_{\Omega_\varepsilon} |\nabla^2 v_\varepsilon(x) - \nabla^2 h(x)|^2\,dx \\
&\qquad + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x) - \nabla h(x)|^2\,dx + \frac{1}{\varepsilon^3}\int_{\Omega_\varepsilon} |v_\varepsilon(x) - h(x)|^2\,dx\bigg),
\end{aligned}$$

where $C > 0$ is a constant, independent of ε. By (4.23), (4.27) and (4.30) we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \varepsilon \int_{\Omega} |\nabla^2\{\bar{v}_\varepsilon(x) - v_\varepsilon(x)\}|^2\,dx &= \lim_{\varepsilon\to 0^+} \varepsilon \int_{\Omega_\varepsilon} |\nabla^2\{(h(x) - v_\varepsilon(x))\omega(d(x)/\varepsilon)\}|^2\,dx \\
&= 0.
\end{aligned} \tag{4.31}$$

We also have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla\{(h(x) - v_\varepsilon(x))\omega(d(x)/\varepsilon)\}|\,dx \\
&\qquad \leq C_0\bigg(\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x) - \nabla h(x)|\,dx + \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |v_\varepsilon(x) - h(x)|\,dx\bigg),
\end{aligned}$$

where $C_0 > 0$ is a constant independent of ε. Therefore, from (4.23) and (4.27) we get

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega} |\nabla\{\bar{v}_\varepsilon(x) - v_\varepsilon(x)\}|\,dx &= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} |\nabla\{(h(x) - v_\varepsilon(x))\omega(d(x)/\varepsilon)\}|\,dx \\
&= 0.
\end{aligned} \tag{4.32}$$

By (4.23) we also have

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\int_{\Omega} |\bar{v}_\varepsilon(x) - v_\varepsilon(x)|\,dx = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |(h(x) - v_\varepsilon(x))\omega(d(x)/\varepsilon)|\,dx = 0. \tag{4.33}$$

Using (4.17) we obtain

$$\begin{aligned}
&|\nabla\{\bar{v}_\varepsilon(x) - v_\varepsilon(x)\}| = |\nabla\{(h(x) - v_\varepsilon(x))\omega(d(x)/\varepsilon)\}| \\
&\quad \leq C\bigg\{|\nabla v_\varepsilon(x) - \nabla h(x)| + \frac{1}{\varepsilon}|v_\varepsilon(x) - v(x)| + \frac{1}{\varepsilon}|v(x) - h(x)|\bigg\} \leq C_1, \qquad \forall x \in \Omega_\varepsilon,
\end{aligned}$$

where $C_1 > 0$ is a constant independent of x and ε. Since $\nabla\{\bar{v}_\varepsilon(x) - v_\varepsilon(x)\} = 0$ for every $x \in \Omega \setminus \Omega_\varepsilon$, we infer from (4.32) and (4.33) that

$$\bar{v}_\varepsilon - v_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in } W^{1,p}(\Omega),\ \forall p \in [1,\infty).$$

Since by Proposition 4.2, $\varepsilon \int_\Omega |\nabla^2 v_\varepsilon(x)|^2 \le C$, we have

$$\begin{aligned}
&\bigg|\varepsilon \int_\Omega (|\nabla^2 \bar{v}_\varepsilon(x)|^2 - |\nabla^2 v_\varepsilon(x)|^2)\,dx \\
&\qquad + \frac{1}{\varepsilon}\int_\Omega (F(\nabla\bar{v}_\varepsilon(x), \bar{v}_\varepsilon(x), f(x)) - F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x)))\,dx\bigg| \\
&\quad \le \bar{C}\bigg(\varepsilon\int_\Omega |\nabla^2\{\bar{v}_\varepsilon(x) - v_\varepsilon(x)\}|^2\,dx + \sqrt{\varepsilon\int_\Omega |\nabla^2\{\bar{v}_\varepsilon(x) - v_\varepsilon(x)\}|^2\,dx} \\
&\qquad + \frac{1}{\varepsilon}\int_\Omega |\nabla\{\bar{v}_\varepsilon(x) - v_\varepsilon(x)\}|\,dx + \frac{1}{\varepsilon}\int_\Omega |\bar{v}_\varepsilon(x) - v_\varepsilon(x)|\,dx\bigg),
\end{aligned}$$

and the desired result (4.15) follows from (4.31)–(4.33). □

Remark 4.1. From the proof of Proposition 4.3 it can be easily seen that if we add the assumptions $\eta, v \ge 0$ then $\bar{v}_\varepsilon$ satisfies $\bar{v}_\varepsilon(x) \ge 0$ in Ω.

We are now in a position to present the proof of our main result, Theorem 1.1.

Proof of Theorem 1.1. We shall prove assertion (i) under the weaker assumption that Ω is a bounded BVG-domain. For any $\eta \in \mathcal{V}$ define, analogously to (3.54),

$$\begin{aligned}
\widetilde{Y}(\eta) &= \int_{J_{\nabla v}} \left\{\int_{\mathbb{R}} |\nabla v^+(x) - \nabla v^-(x)|^2 \cdot p^2(t,x)\,dt\right\} d\mathcal{H}^{N-1}(x) \\
&+ \int_{J_{\nabla v}} \bigg\{\int_{-\infty}^0 F\bigg(\nabla v^-(x)\int_{-\infty}^t p(s,x)\,ds + \nabla v^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^+(x)\bigg)\,dt \\
&+ \int_0^\infty F\bigg(\nabla v^-(x)\int_{-\infty}^t p(s,x)\,ds \\
&\qquad + \nabla v^+(x)\int_t^\infty p(s,x)\,ds, v(x), f^-(x)\bigg)\,dt\bigg\}\,d\mathcal{H}^{N-1}(x),
\end{aligned}$$

where $p(t,x)$ is defined in (3.5). By Corollary 3.1 there exists a sequence $\eta_n \in \mathcal{V}_r$ such that $\eta_n \ge 0$ and

$$0 \le \widetilde{Y}(\eta_n) - \widetilde{\mathcal{J}}_0(v) \le 1/n, \qquad \forall n, \tag{4.34}$$

where

$$\begin{aligned}
\widetilde{\mathcal{J}}_0(v) &:= 2\int_{J_{\nabla v}} |\nabla v^+(x) - \nabla v^-(x)| \\
&\quad \times \inf_{\tau\in[0,1]}\bigg\{\int_0^\tau \sqrt{F(s\nabla v^-(x) + (1-s)\nabla v^+(x), v(x), f^+(x))}\,ds \\
&\quad + \int_\tau^1 \sqrt{F(s\nabla v^-(x) + (1-s)\nabla v^+(x), v(x), f^-(x))}\,ds\bigg\}\,d\mathcal{H}^{N-1}(x).
\end{aligned}$$

By Propositions 4.2 and 4.3, for each n and for every $0 < \varepsilon < 1$ there exists $v_{\varepsilon,n} \in C^2(\mathbb{R}^N)$ such that

$$\lim_{\varepsilon\to 0^+} v_{\varepsilon,n} = v \quad \text{in } W^{1,p}(\Omega),\ \forall p \in [1,\infty),$$

and

$$\lim_{\varepsilon\to 0}\left\{\varepsilon\int_\Omega |\nabla^2 v_{\varepsilon,n}(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega F(\nabla v_{\varepsilon,n}(x), v_{\varepsilon,n}(x), f(x))\,dx\right\} = \widetilde{Y}(\eta_n).$$

Furthermore, under the assumptions of (ii), we can choose $v_{\varepsilon,n}$ to satisfy the boundary conditions, $v_{\varepsilon,n} = v$ and $\nabla v_{\varepsilon,n} = T\nabla v$ on $\partial\Omega$. If $v \geq 0$ in Ω then by Remark 4.1, we also have $v_{\varepsilon,n} \geq 0$ in Ω.

Next, we define a positive sequence $\{\varepsilon_n\}_{n=0}^\infty$ as follows. Set $\varepsilon_0 = 1$. Assuming that ε_{n-1} was already defined, we choose $0 < \varepsilon_n < \min\{\varepsilon_{n-1}, 1/n\}$ such that for every $0 < \varepsilon < \varepsilon_n$ we have

$$\int_\Omega |\nabla v_{\varepsilon,n}(x) - \nabla v(x)|^p\,dx + \int_\Omega |v_{\varepsilon,n}(x) - v(x)|^p\,dx < 1/n$$

and

$$\left|\varepsilon\int_\Omega |\nabla^2 v_{\varepsilon,n}(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega F(\nabla v_{\varepsilon,n}(x), v_{\varepsilon,n}(x), f(x))\,dx - \widetilde{Y}(\eta_n)\right| < \frac{1}{n}.$$

So we have a disjoint union $\bigcup_{n=0}^\infty [\varepsilon_{n+1}, \varepsilon_n) = (0,1)$. In order to define v_ε for all $\varepsilon \in (0,1)$ we argue as follows. For any $\varepsilon \in (0,1)$ let k be the unique integer such that $\varepsilon \in [\varepsilon_{k+1}, \varepsilon_k)$, and then define $v_\varepsilon(x) = v_{\varepsilon,k}(x)$. Then, by (4.34), for any $n \geq 1$ we have, for all $\varepsilon < \varepsilon_n$,

$$\int_\Omega |\nabla v_\varepsilon(x) - \nabla v(x)|^p\,dx + \int_\Omega |v_\varepsilon(x) - v(x)|^p\,dx < 1/n$$

and

$$\left|\varepsilon\int_\Omega |\nabla^2 v_\varepsilon(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega F(\nabla v_\varepsilon(x), v_\varepsilon(x), f(x))\,dx - \widetilde{\mathcal{J}}_0(v)\right| < \frac{2}{n},$$

and the result follows. □

Appendix

This appendix is devoted to the proof of two technical results, Proposition 4.1 and Lemma 4.1, that were used in Section 4.

Proof of Proposition 4.1. Let Ω be a bounded BVG-domain and let $v : \Omega \to \mathbb{R}$ be a BVG-function. Fix a point $x \in \partial\Omega$. From Definition 4.3 it follows that there exist $r > 0$,

$\delta > 0$ and a mapping $\gamma \in \mathrm{BVG}(\mathbb{R}^{N-1}, \mathbb{R})$ such that—upon rotating and relabeling the coordinate axes if necessary—we have

$$\Omega \cap C(x, r, \delta) = \{y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : y_1 > \gamma(y')\} \cap C(x, r, \delta),$$

where $C(x, r, \delta) := \{y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1} : |y_1 - x_1| < \delta,\ |y' - x'| < r\}$. Moreover, we may choose r small enough to ensure that $|\gamma(y') - \gamma(x')| \le \delta/4$ for $y' \in B_r(x')$. Obviously $\gamma(x') = x_1$.

Since $v \in W^{1,\infty}(\Omega)$, we know in particular that $v \in \mathrm{Lip}(C(x, r, \delta) \cap \Omega)$, so it can be extended to $\overline{C(x, r, \delta)} \cap \overline{\Omega}$ by continuity. Next we extend v to $C(x, r, \delta/2)$ using a *higher order reflection* (see [9, Section 5.4]). Define $\bar{v} : C(x, r, \delta/2) \to \mathbb{R}$ by

$$\bar{v}(y) := \begin{cases} v(y_1, y') & \text{if } y_1 \ge \gamma(y'), \\ 4v\big(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1, y'\big) - 3v(2\gamma(y') - y_1, y') & \text{if } y_1 < \gamma(y'). \end{cases}$$

Clearly $\bar{v} \in W^{1,\infty}(C(x, r, \delta/2))$. First, we compute $\partial_{y_1}\bar{v}$. We have

$$\partial_{y_1}\bar{v}(y) = \begin{cases} \partial_{y_1} v(y_1, y') & \text{if } y_1 > \gamma(y'), \\ -2\partial_{y_1} v\big(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1, y'\big) + 3\partial_{y_1} v(2\gamma(y') - y_1, y') & \text{if } y_1 < \gamma(y'). \end{cases} \tag{A.1}$$

For $\nabla_{y'}\bar{v}$ we have

$$\nabla_{y'}\bar{v}(y) = \begin{cases} \nabla_{y'} v(y_1, y') & \text{if } y_1 > \gamma(y'), \\ 4\nabla_{y'} v\big(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1, y'\big) - 3\nabla_{y'} v(2\gamma(y') - y_1, y') & \\ +6\nabla_{y'}\gamma(y')\big(\partial_{y_1} v\big(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1, y'\big) - \partial_{y_1} v(2\gamma(y') - y_1, y')\big) & \text{if } y_1 < \gamma(y'). \end{cases} \tag{A.2}$$

Therefore, $\nabla\bar{v} \in \mathrm{BV}(C(x, r, \delta/2))$ and it follows that $\bar{v} \in \mathrm{BVG}(C(x, r, \delta/2))$.

Let $(a(z), b(z))$ denote the trace of $\nabla v = (\partial_{y_1} v, \nabla_{y'} v)$ on

$$\Gamma := \{(z_1, z') \in C(x, r, \delta/2) : z_1 = \gamma(z')\}.$$

By [2, Theorem 3.87] for $\mathcal{H}^{N-1}$-almost every z in Γ we have

$$\lim_{\rho \to 0^+} \frac{1}{\rho^N} \int_{B_\rho(z) \cap \Omega} |\partial_{y_1} v(y) - a(z')| dy = \lim_{\rho \to 0^+} \frac{1}{\rho^N} \int_{B_\rho(z) \cap \Omega} |\nabla_{y'} v(y) - b(z')| dy = 0. \tag{A.3}$$

Consider $z' \in B_r(x')$ and let $z = (\gamma(z'), z')$. Since γ is Lipschitz, for every small $\rho > 0$ we have, for $y = (y_1, y') \in B_\rho(z)$,

$$\left(\left(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1\right) - \gamma(z')\right)^2 + (y' - z')^2 \le C((y_1 - \gamma(z'))^2 + (y' - z')^2) \le C^2\rho^2,$$
$$((2\gamma(y') - y_1) - \gamma(z'))^2 + (y' - z')^2 \le C((y_1 - \gamma(z'))^2 + (y' - z')^2) \le C^2\rho^2,$$

for some constant C. Therefore, changing variables in the first integral in (A.3) gives, for any sufficiently small $\rho > 0$,

$$\int_{B_\rho(z)\setminus\Omega} \left|\partial_{y_1} v\left(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1, y'\right) - a(z')\right| dy \leq 2\int_{B_{C\rho}(z)\cap\Omega} |\partial_{y_1} v(y_1, y') - a(z')|\, dy,$$
$$\int_{B_\rho(z)\setminus\Omega} |\partial_{y_1} v(2\gamma(y') - y_1, y') - a(z')|\, dy \leq \int_{B_{C\rho}(z)\cap\Omega} |\partial_{y_1} v(y_1, y') - a(z')|\, dy. \tag{A.4}$$

Using (A.4) in (A.3) yields, for $\mathcal{H}^{N-1}$-almost every z in Γ,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)\setminus\Omega} \left|\partial_{y_1} v\left(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1, y'\right) - a(z')\right| dy = 0,$$
$$\lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)\setminus\Omega} |\partial_{y_1} v(2\gamma(y') - y_1, y') - a(z')|\, dy = 0. \tag{A.5}$$

By the same method,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)\setminus\Omega} \left|\nabla_{y'} v\left(\frac{3}{2}\gamma(y') - \frac{1}{2}y_1, y'\right) - b(z')\right| dy = 0,$$
$$\lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)\setminus\Omega} |\nabla_{y'} v(2\gamma(y') - y_1, y') - b(z')|\, dy = 0. \tag{A.6}$$

Combining (A.5)–(A.6) with (A.1)–(A.2) yields, for $\mathcal{H}^{N-1}$-almost every z,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)\setminus\Omega} |\partial_{y_1}\bar{v}(y) - a(z')|\, dy = \lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)\setminus\Omega} |\nabla_{y'}\bar{v}(y) - b(z')|\, dy = 0. \tag{A.7}$$

By (A.7) and (A.3) we obtain, for $\mathcal{H}^{N-1}$-a.e. z in Γ,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)} |\partial_{y_1}\bar{v}(y) - a(z')|\, dy = \lim_{\rho\to 0^+} \frac{1}{\rho^N}\int_{B_\rho(z)} |\nabla_{y'}\bar{v}(y) - b(z')|\, dy = 0,$$

i.e., for $\mathcal{H}^{N-1}$-almost every z in Γ, $\nabla\bar{v}$ is approximately continuous at the point z. So far we have proved that $\bar{v} \in \mathrm{BVG}(C(x, r, \delta/2))$, $\bar{v} = v$ on $C(x, r, \delta/2)\cap\Omega$ and $\|D(\nabla\bar{v})\|(\Gamma) = 0$.

From the above it follows that for each $x \in \partial\Omega$ there exists an open neighborhood U_x of x and a function $\bar{v}_x \in \mathrm{BVG}(U_x)$ such that $\bar{v}_x(y) = v(y)$ for every $y \in U_x \cap \Omega$ and $\|D(\nabla\bar{v}_x)\|(\partial\Omega\cap U_x) = 0$. Since $\partial\Omega$ is compact, there exists a finite collection of such sets $\{U_{x_j}\}_{j=1}^m$ such that $\partial\Omega \subset \bigcup_{j=1}^m U_{x_j}$. Write $U_j = U_{x_j}$ and $\bar{v}_j = \bar{v}_{x_j}$ for $j = 1, \ldots, m$, and set also $U_0 = \Omega$ and $\bar{v}_0 = v$. Since $\overline{\Omega} \subset\subset \bigcup_{j=0}^m U_j$, there exists a corresponding partition of unity, i.e., $m + 1$ functions $\zeta_j \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ for $0 \leq j \leq m$ such that $\operatorname{supp}\zeta_j \subset U_j$, $0 \leq \zeta_j \leq 1$ and $\sum_{j=0}^d \zeta_j(y) = 1$ for every $y \in \overline{\Omega}$. Define

$$\bar{v}(y) := \sum_{j=0}^d \zeta_j(y)\bar{v}_j(y), \qquad \forall y \in \mathbb{R}^N.$$

We then have $\bar{v} = v$ on Ω, $\bar{v} \in \mathrm{BVG}(\mathbb{R}^N)$ and $\|D(\nabla\bar{v})\|(\partial\Omega) = 0$, as required. □

Proof of Lemma 4.1. Since Ω is of class C^2, there exists $\beta_0 > 0$ such that the distance function to $\partial\Omega$, $d(x)$, is in $C^2(\Omega_{\beta_0})$, and for every $x \in \Omega_{\beta_0}$ there exists a unique nearest point projection $\sigma(x) \in \partial\Omega$ (see [11, Sec. 14.6]). The mapping $\sigma : \Omega_{\beta_0} \to \partial\Omega$ is of class C^1, and it follows that for each $\beta \in (0, \beta_0)$ the mapping $\sigma_\beta := \sigma|_{\Sigma_\beta} : \Sigma_\beta \to \partial\Omega$ (see (4.12)) is a C^1 diffeomorphism. Its inverse, $\sigma_\beta^{-1} : \partial\Omega \to \Sigma_\beta$, which is also a C^1 diffeomorphism, is given by

$$\sigma_\beta^{-1}(y) = y - \beta \boldsymbol{n}(y), \qquad \forall y \in \partial\Omega,$$

where $\boldsymbol{n}(y)$ denotes the external normal to $\partial\Omega$ at the point y. Furthermore, the Jacobian J_β of σ_β^{-1} satisfies

$$|J_\beta(y) - 1| \le \bar{c}\beta, \qquad \forall y \in \Omega,\ \forall \beta \in (0, \beta_0), \tag{A.8}$$

for some constant $\bar{c}$. We may choose β_0 small enough that $1 - \bar{c}\beta_0 > 0$.

Next we fix some $\rho \in (0, \beta_0)$. Since $\varphi \in \mathrm{BV}(\Omega, \mathbb{R}^k)$, there exists a sequence $\{\varphi_n\} \subset C^1(\mathbb{R}^N, \mathbb{R}^k)$ such that

$$\lim_{n\to\infty} \int_{\Omega_\rho} |\varphi_n(x) - \varphi(x)|\, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\Omega_\rho} |\nabla\varphi_n(x)|\, dx = \|D\varphi\|(\Omega_\rho). \tag{A.9}$$

Since $T\varphi = 0$ on $\partial\Omega$, we also have

$$\lim_{n\to\infty} \int_{\partial\Omega} |\varphi_n(y)|\, d\mathcal{H}^{N-1} = 0. \tag{A.10}$$

Applying the coarea formula gives

$$\begin{aligned}
\int_{\Omega_\rho} |\varphi_n(x)|\, dx &= \int_0^\rho \int_{\Sigma_t} |\varphi_n(x)|\, d\mathcal{H}^{N-1}(x)\, dt \\
&= \int_0^\rho \int_{\partial\Omega} |\varphi_n(y - t\boldsymbol{n}(y))|\, J_t(y)\, d\mathcal{H}^{N-1}(y)\, dt \\
&\le (1 + \bar{c}\beta_0) \int_0^\rho \int_{\partial\Omega} |\varphi_n(y - t\boldsymbol{n}(y))|\, d\mathcal{H}^{N-1}(y)\, dt \\
&= (1 + \bar{c}\beta_0)\rho \int_0^1 \int_{\partial\Omega} |\varphi_n(y - t\rho\boldsymbol{n}(y))|\, d\mathcal{H}^{N-1}(y)\, dt.
\end{aligned} \tag{A.11}$$

Using

$$\varphi_n(y - t\boldsymbol{n}(y)) = \varphi_n(y) - \int_0^t \nabla\varphi_n(y - s\boldsymbol{n}(y)) \cdot \boldsymbol{n}(y)\, ds$$

in (A.11) yields

$$\begin{aligned}
\frac{1}{\rho} \int_{\Omega_\rho} |\varphi_n(x)|\, dx &\le (1 + \bar{c}\beta_0) \int_{\partial\Omega} |\varphi_n(y)|\, d\mathcal{H}^{N-1}(y) \\
&\quad + (1 + \bar{c}\beta_0) \int_0^1 \int_{\partial\Omega} \left(\int_0^{t\rho} |\nabla\varphi_n(y - s\boldsymbol{n}(y))| ds \right) d\mathcal{H}^{N-1}(y)\, dt.
\end{aligned} \tag{A.12}$$

But, by (A.8), for every $t \in (0,1)$ we have

$$\begin{aligned}\int_{\partial\Omega}\left(\int_0^{t\rho}|\nabla\varphi_n(y-s\boldsymbol{n}(y))|ds\right)d\mathcal{H}^{N-1}(y)\\ &\leq \frac{1}{1-\bar{c}\beta_0}\int_0^{t\rho}\int_{\partial\Omega}|\nabla\varphi_n(y-s\boldsymbol{n}(y))|J_s(y)\,d\mathcal{H}^{N-1}(y)\,ds\\ &=\frac{1}{1-\bar{c}\beta_0}\int_{\Omega_{t\rho}}|\nabla\varphi_n(x)|\,dx.\end{aligned}$$

Therefore, by (A.12) we infer, for any $t \in (0,1)$, that

$$\begin{aligned}\frac{1}{\rho}\int_{\Omega_\rho}|\varphi_n(x)|\,dx &\leq (1+\bar{c}\beta_0)\int_{\partial\Omega}|\varphi_n(y)|\,d\mathcal{H}^{N-1}(y)\\ &\quad+\frac{1+\bar{c}\beta_0}{1-\bar{c}\beta_0}\int_0^1\left(\int_{\Omega_{t\rho}}|\nabla\varphi_n(x)|\,dx\right)dt\\ &\leq (1+\bar{c}\beta_0)\int_{\partial\Omega}|\varphi_n(y)|\,d\mathcal{H}^{N-1}(y)+\frac{1+\bar{c}\beta_0}{1-\bar{c}\beta_0}\left(\int_{\Omega_\rho}|\nabla\varphi_n(x)|\,dx\right).\end{aligned} \tag{A.13}$$

Combining (A.9) with (A.10) and (A.13) gives

$$\frac{1}{\rho}\int_{\Omega_\rho}|\varphi(x)|\,dx \leq \frac{1+\bar{c}\beta_0}{1-\bar{c}\beta_0}\,\|D\varphi\|(\Omega_\rho), \qquad \forall\rho\in(0,\beta_0). \tag{A.14}$$

Since $\bigcap_{\rho>0}\Omega_\rho=\emptyset$, we deduce (4.13) by passing to the limit in (A.14).

Next we turn to the proof of (4.14). Let $v \in \mathrm{BVG}(\Omega)$ satisfy $v=0$ and $T\nabla v=0$ on $\partial\Omega$. Since $v\in\mathrm{Lip}(\overline{\Omega})$ and $v=0$ on $\partial\Omega$, from (A.14) we obtain

$$\frac{1}{\rho}\int_{\Omega_\rho}|v(x)|\,dx \leq \frac{1+\bar{c}\beta_0}{1-\bar{c}\beta_0}\int_{\Omega_\rho}|\nabla v(x)|\,dx, \qquad \forall\rho\in(0,\beta_0). \tag{A.15}$$

But since $\nabla v\in \mathrm{BV}(\Omega)$ and $T\nabla v=0$ on $\partial\Omega$, (4.13) applied to $\varphi=\nabla v$ gives

$$\lim_{\rho\to0^+}\frac{1}{\rho}\int_{\Omega_\rho}|\nabla v(x)|\,dx=0. \tag{A.16}$$

Finally, (4.14) follows from (A.16) and (A.15). □

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