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Invariant densities for random β-expansions

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Abstract. Let $\beta > 1$ be a non-integer. We consider expansions of the form $\sum_{i=1}^{\infty} d_i/\beta^i$, where the digits $(d_i)_{i\geq 1}$ are generated by means of a Borel map K_β defined on $\{0,1\}^{\mathbb{N}} \times [0, \lfloor\beta\rfloor/(\beta-1)]$. We show existence and uniqueness of a K_β-invariant probability measure, absolutely continuous with respect to $m_p \otimes \lambda$, where m_p is the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ with parameter p $(0 < p < 1)$ and λ is the normalized Lebesgue measure on $[0, \lfloor\beta\rfloor/(\beta-1)]$. Furthermore, this measure is of the form $m_p \otimes \mu_{\beta,p}$, where $\mu_{\beta,p}$ is equivalent to λ. We prove that the measure of maximal entropy and $m_p \otimes \lambda$ are mutually singular. In case the number 1 has a finite greedy expansion with positive coefficients, the measure $m_p \otimes \mu_{\beta,p}$ is Markov. In the last section we answer a question concerning the number of universal expansions, a notion introduced in [EK].

Keywords. Greedy expansions, lazy expansions, absolutely continuous invariant measures, measures of maximal entropy, Markov chains, universal expansions

1. Introduction

Let $\beta > 1$ be a non-integer, and denote by $\lfloor\beta\rfloor$ the integer part of β. In this paper we consider expansions of numbers x in $J_\beta := [0, \lfloor\beta\rfloor/(\beta-1)]$ of the form

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}$$

with $a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$, $i \in \mathbb{N}$. We shall refer to expansions of this form as (β-)*expansions* or *expansions in base* β. The largest expansion in lexicographical order of a number $x \in J_\beta$ is the *greedy expansion* of x ([P], [R1], [R2]), and the smallest is the *lazy expansion* of x ([JS], [EJK], [DK1]). The greedy expansion is obtained by iterating the *greedy*

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transformation $T_\beta : J_\beta \to J_\beta$, defined by

$$T_\beta(x) = \beta x - d \quad \text{for } x \in C(d),$$

where

$$C(j) = \left[\frac{j}{\beta}, \frac{j+1}{\beta}\right), \quad j \in \{0, \dots, \lfloor\beta\rfloor - 1\},$$

and

$$C(\lfloor\beta\rfloor) = \left[\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta - 1}\right].$$

The greedy expansion of $x \in J_\beta$ is given by $x = \sum_{i=1}^{\infty} d_i(x)/\beta^i$, where $d_i(x) = d$ if and only if $T_\beta^{i-1}(x) \in C(d)$. Let $\ell : J_\beta \to J_\beta$ be given by

$$\ell(x) = \frac{\lfloor\beta\rfloor}{\beta - 1} - x.$$

Then the *lazy transformation* $L_\beta : J_\beta \to J_\beta$ is defined by

$$L_\beta(x) = \beta x - d \quad \text{for } x \in \Delta(d) = \ell(C(\lfloor\beta\rfloor - d)),\ d \in \{0, \dots, \lfloor\beta\rfloor\}.$$

The lazy expansion of $x \in J_\beta$ is given by $x = \sum_{i=1}^{\infty} \tilde{d}_i(x)/\beta^i$, where $\tilde{d}_i(x) = d$ if and only if $L_\beta^{i-1}(x) \in \Delta(d)$.

We denote by μ_β the extended T_β-invariant *Parry measure* (see [P], [G]) on J_β which is absolutely continuous with respect to Lebesgue measure, and with density

$$h_\beta(x) = \begin{cases} \dfrac{1}{F(\beta)} \displaystyle\sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T_\beta^n(1))}(x), & 0 \le x < 1, \\ 0, & 1 \le x \le \lfloor\beta\rfloor/(\beta - 1), \end{cases}$$

where $F(\beta)$ is the normalizing constant. Define the *lazy measure* ρ_β on J_β by setting $\rho_\beta = \mu_\beta \circ \ell^{-1}$. It is easy to see ([DK1]) that ℓ is a continuous isomorphism between $(J_\beta, \mu_\beta, T_\beta)$ and $(J_\beta, \rho_\beta, L_\beta)$.

In order to produce other expansions in a dynamical way, a new transformation K_β was introduced in [DK2]. The expansions generated by iterating this map are random mixtures of greedy and lazy expansions. This is done by superimposing the greedy map and the corresponding lazy map on J_β. In this way one obtains $\lfloor\beta\rfloor$ intervals on which the greedy map and the lazy map differ. These intervals are given by

$$S_k = \left[\frac{k}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta - 1)} + \frac{k-1}{\beta}\right], \quad k = 1, \dots, \lfloor\beta\rfloor,$$

which one refers to as *switch regions*. On S_k, the greedy map assigns the digit k, while the lazy map assigns the digit $k - 1$. Outside these switch regions both maps are identical, and hence they assign the same digits. Now define other expansions in base β by randomizing the choice of the map used in the switch regions. So, whenever x belongs to a switch region, flip a coin to decide which map will be applied to x, and hence which digit will be assigned. To be more precise, partition the interval J_β into switch regions S_k and *equality regions* E_k, where

$$E_k = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}, \frac{k+1}{\beta}\right), \quad k = 1, \ldots, \lfloor\beta\rfloor - 1,$$
$$E_0 = \left[0, \frac{1}{\beta}\right) \quad \text{and} \quad E_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor - 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right].$$

Let

$$S = \bigcup_{k=1}^{\lfloor\beta\rfloor} S_k \quad \text{and} \quad E = \bigcup_{k=0}^{\lfloor\beta\rfloor} E_k,$$

and consider $\Omega = \{0,1\}^{\mathbb{N}}$ with product σ-algebra $\mathcal{A}$. Let $\sigma : \Omega \to \Omega$ be the left shift, and define $K_\beta : \Omega \times J_\beta \to \Omega \times J_\beta$ by

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - k), & x \in E_k,\ k = 0, 1, \ldots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - k), & x \in S_k \text{ and } \omega_1 = 1,\ k = 1, \ldots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - k + 1), & x \in S_k \text{ and } \omega_1 = 0,\ k = 1, \ldots, \lfloor\beta\rfloor. \end{cases}$$

The elements of Ω represent the coin tosses ('heads' $= 1$ and 'tails' $= 0$) used every time the orbit $\{K_\beta^n(\omega, x) : n \geq 0\}$ hits $\Omega \times S$. Let

$$d_1 = d_1(\omega, x) = \begin{cases} k & \text{if } x \in E_k,\ k = 0, 1, \ldots, \lfloor\beta\rfloor, \\ & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_k,\ k = 1, \ldots, \lfloor\beta\rfloor, \\ k - 1 & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_k,\ k = 1, \ldots, \lfloor\beta\rfloor. \end{cases}$$

Then

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - d_1) & \text{if } x \in E, \\ (\sigma(\omega), \beta x - d_1) & \text{if } x \in S. \end{cases}$$

Set $d_n = d_n(\omega, x) = d_1(K_\beta^{n-1}(\omega, x))$, and let $\pi_2 : \Omega \times J_\beta \to J_\beta$ be the canonical projection onto the second coordinate. Then

$$\pi_2(K_\beta^n(\omega, x)) = \beta^n x - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n,$$

and rewriting yields

$$x = \frac{d_1}{\beta} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi_2(K_\beta^n(\omega, x))}{\beta^n}.$$

This shows that for all $\omega \in \Omega$ and for all $x \in J_\beta$ one has

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i}.$$

The random procedure just described shows that to each $\omega \in \Omega$ corresponds an algorithm that produces an expansion in base β. Furthermore, if we identify the point (ω, x) with

$(\omega, (d_1(\omega, x), d_2(\omega, x), \ldots))$, then the action of K_β on the second coordinate corresponds to the left shift.

Let $<_{\text{lex}}$ and $\leq_{\text{lex}}$ denote the lexicographical ordering on both Ω and $\{0, \ldots, \lfloor\beta\rfloor\}^{\mathbb{N}}$. We recall from [DdV] the following basic properties of random β-expansions.

Theorem 1. *Suppose $\omega, \omega' \in \Omega$ are such that $\omega <_{\text{lex}} \omega'$. Then*

$$(d_1(\omega, x), d_2(\omega, x), \ldots) \leq_{\text{lex}} (d_1(\omega', x), d_2(\omega', x), \ldots).$$

Theorem 2. *Let $x \in J_\beta$ and let $x = \sum_{i=1}^{\infty} a_i/\beta^i$ with $a_i \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$ be an expansion of x in base β. Then there exists an $\omega \in \Omega$ such that $a_i = d_i(\omega, x)$ for all $i \geq 1$.*

In [DdV] it is shown that there exists a unique measure of maximal entropy ν_β for the map K_β. It is the main goal of this paper to investigate the relationship between this measure and the measure $m_p \otimes \lambda$, where λ is the normalized Lebesgue measure on J_β and m_p is the Bernoulli measure on Ω with parameter p $(0 < p < 1)$:

$$m_p(\{\omega_1 = i_1, \ldots, \omega_n = i_n\}) = p^{\sum_{j=1}^{n} i_j}(1-p)^{n-\sum_{j=1}^{n} i_j}, \quad i_1, \ldots, i_n \in \{0, 1\}.$$

In this paper, the parameter $p \in (0, 1)$ is fixed but arbitrary, unless stated otherwise. In order to prove that the measures ν_β and $m_p \otimes \lambda$ are mutually singular, we introduce in the next section another K_β-invariant probability measure. It is a product measure $m_p \otimes \mu_{\beta,p}$ and we show in Section 3 that K_β is ergodic with respect to it. Furthermore, the measures $m_p \otimes \lambda$ and $m_p \otimes \mu_{\beta,p}$ are shown to be equivalent. These facts enable us to conclude that the measures ν_β and $m_p \otimes \lambda$ are mutually singular. Moreover, it follows that $m_p \otimes \mu_{\beta,p}$ is the unique absolutely continuous K_β-invariant probability measure with respect to $m_p \otimes \lambda$. The measure $\mu_{\beta,p}$ satisfies the important relationship

$$\mu_{\beta,p} = p \cdot \mu_{\beta,p} \circ T_\beta^{-1} + (1-p) \cdot \mu_{\beta,p} \circ L_\beta^{-1}.$$

In Section 4 we show that if 1 has a finite greedy expansion with positive coefficients, then the measure $m_p \otimes \mu_{\beta,p}$ is Markov, and we determine the measure $\mu_{\beta,p}$ explicitly. In Section 5 we discuss some open problems. As an application of some of the results in this paper, we also show that for λ-a.e. $x \in J_\beta$, there exist $2^{\aleph_0}$ so-called universal expansions of x in base β.

2. The skew product transformation R_β

Define the *skew product* transformation R_β on $\Omega \times J_\beta$ as follows:

$$R_\beta(\omega, x) = \begin{cases} (\sigma(\omega), T_\beta x) & \text{if } \omega_1 = 1, \\ (\sigma(\omega), L_\beta x) & \text{if } \omega_1 = 0. \end{cases}$$

On the set $\Omega \times J_\beta$, we consider the σ-algebra $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{A}$ is the product σ-algebra on Ω and $\mathcal{B}$ is the Borel σ-algebra on J_β. Let μ be an arbitrary probability measure on J_β. The following result shows that a product measure of the form $m_p \otimes \mu$ is K_β-invariant if and only if it is R_β-invariant.

Lemma 1. *$m_p \otimes \mu \circ K_\beta^{-1} = m_p \otimes \mu \circ R_\beta^{-1} = m_p \otimes \nu$, where*

$$\nu = p \cdot \mu \circ T_\beta^{-1} + (1-p) \cdot \mu \circ L_\beta^{-1}.$$

Proof. Denote by C an arbitrary cylinder in Ω and let $[a, b]$ be an interval in J_β. It suffices to verify that the measures coincide on sets of the form $C \times [a, b]$, because the collection of these sets forms a generating π-system. Furthermore, let $[i, C] = \{\omega_1 = i\} \cap \sigma^{-1}(C)$ for $i = 0, 1$. Note that $E \cap T_\beta^{-1}[a, b] = E \cap L_\beta^{-1}[a, b]$, and that

$$K_\beta^{-1}(C \times [a, b]) = C \times (E \cap T_\beta^{-1}[a, b]) \cup [0, C] \times (S \cap L_\beta^{-1}[a, b]) \\ \cup [1, C] \times (S \cap T_\beta^{-1}[a, b]).$$

Hence,

$$\begin{aligned} m_p \otimes \mu \circ K_\beta^{-1}(C \times [a, b]) &= p \cdot m_p(C) \cdot \mu(T_\beta^{-1}[a, b]) \\ &\quad + (1-p) \cdot m_p(C) \cdot \mu(L_\beta^{-1}[a, b]) \\ &= m_p \otimes \nu(C \times [a, b]). \end{aligned}$$

On the other hand,

$$R_\beta^{-1}(C \times [a, b]) = [0, C] \times L_\beta^{-1}[a, b] \cup [1, C] \times T_\beta^{-1}[a, b],$$

and the result follows. □

Let $\mathfrak{D} = \mathfrak{D}(J_\beta, \mathcal{B}, \lambda)$ denote the space of probability density functions on J_β with respect to λ. A measurable transformation $T : J_\beta \to J_\beta$ is called *nonsingular* if $\lambda(T^{-1}B) = 0$ whenever $\lambda(B) = 0$.

If μ is absolutely continuous with respect to λ with probability density $f = d\mu/d\lambda$ and if T is a nonsingular transformation, then $\mu \circ T^{-1}$ is absolutely continuous with respect to λ with probability density $P_T f$ (say). Equivalently, the Frobenius–Perron operator $P_T : \mathfrak{D} \to \mathfrak{D}$ is defined as a linear operator such that for $f \in \mathfrak{D}$, $P_T f$ is the function for which

$$\int_B P_T f \, d\lambda = \int_{T^{-1}B} f \, d\lambda \quad \text{for all } B \in \mathcal{B}.$$

Existence and uniqueness (λ-a.e.) follow from the Radon–Nikodým theorem. A nonsingular transformation $T : J_\beta \to J_\beta$ is said to be a *Lasota–Yorke type map* (L-Y map) if T is piecewise monotone and C^2. Piecewise monotone and C^2 means that there exists a partition $\mathcal{P} = \{[a_{i-1}, a_i] : i = 1, \ldots, k\}$ such that for each $i = 1, \ldots, k$, the restriction of T to (a_{i-1}, a_i) is monotone and extends to a C^2 map on $[a_{i-1}, a_i]$. For such a transformation the Frobenius–Perron operator can be computed explicitly (see [BG, p. 86]) by the formula

$$P_T f(x) = \sum_{T(y)=x} \frac{f(y)}{|T'(y)|}. \tag{1}$$

If, in addition, $|T'(x)| \geq \alpha > 1$ for each $x \in (a_{i-1}, a_i)$, $i = 1, \ldots, k$, then we say that T is a *piecewise expanding L-Y map*. Let $T_1, \ldots, T_n$ be L-Y maps on J_β with common partition of joint monotonicity $\mathcal{P} = \{[a_{i-1}, a_i] : i = 1, \ldots, k\}$. For $f \in \mathfrak{D}$, define $Pf = \sum_{i=1}^{n} p_i \cdot P_{T_i} f$, where $(p_1, \ldots, p_n)$ is a probability vector. We recall the following important theorem, due to Pelikan [Pel]. For more results concerning invariant densities of L-Y maps see [LY], [LiY], [Pel].

Theorem 3. *Suppose that for all $x \in J_\beta \setminus \{a_0, \ldots, a_k\}$, $\sum_{i=1}^{n} p_i/|T_i'(x)| \leq \gamma < 1$. Then for all $f \in \mathfrak{D}$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*$$

exists in $L_1(J_\beta, \lambda)$. Furthermore, $Pf^* = f^*$ and one can choose f^* to be of bounded variation.

Since T_β and L_β are both piecewise expanding L-Y maps, it follows at once from Theorem 3 that for all $f \in \mathfrak{D}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*$$

exists in $L_1(J_\beta, \lambda)$, where

$$Pf = p \cdot P_{T_\beta} f + (1-p) \cdot P_{L_\beta} f.$$

Define for $f \in \mathfrak{D}$ the probability measure μ_f by

$$\mu_f(B) = \int_B f \, d\lambda \qquad [B \in \mathcal{B}].$$

Observe that $Pf = f$ if and only if

$$\mu_f = p \cdot \mu_f \circ T_\beta^{-1} + (1-p) \cdot \mu_f \circ L_\beta^{-1},$$

i.e., if and only if $m_p \otimes \mu_f$ is R_β-invariant (cf. Lemma 1).

Let $\mathbf{1}$ denote the constant function equal to 1 on J_β and consider the function $\mathbf{1}^*$, given by

$$\mathbf{1}^* = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j \mathbf{1} \quad \text{in } L_1(J_\beta, \lambda).$$

We shall assume that the function $\mathbf{1}^*$ is of bounded variation. Note that this is possible by Theorem 3. It follows easily from the definition of bounded variation that the left- and right-hand limits of $\mathbf{1}^*$ at every point $x \in J_\beta$ exist and that the function $\mathbf{1}^*$ is continuous except maybe at countably many points. Now we modify the function $\mathbf{1}^*$ in such a way that it becomes lower semicontinuous. Replace $\mathbf{1}^*(x)$ at every discontinuity point x in the interior of J_β by setting

$$\mathbf{1}^*(x) = \min\{\mathbf{1}^*(x^-), \mathbf{1}^*(x^+)\}$$

and replace $\mathbf{1}^*(x)$ by its left- or right-hand limit if x is an endpoint of J_β. In the remainder of this section we work with this modified version of $\mathbf{1}^*$ which we denote again by $\mathbf{1}^*$. In the next theorem, we show that this function is bounded below by a positive constant $d > 0$, everywhere on J_β.

Theorem 4. *The skew product transformation R_β is ergodic with respect to the measure $m_p \otimes \mu_{\mathbf{1}^*}$. Furthermore, the measures $m_p \otimes \mu_{\mathbf{1}^*}$ and $m_p \otimes \lambda$ are equivalent and the density $\mathbf{1}^*$ is bounded below by a positive constant d, everywhere on J_β.*

Proof. Since $P\mathbf{1}^* = \mathbf{1}^*$, it follows from Lemma 1 that the measure $m_p \otimes \mu_{\mathbf{1}^*}$ is R_β-invariant. It is well known that the greedy transformation T_β is ergodic with respect to its unique absolutely continuous invariant measure, which is the Parry measure μ_β (see Section 1). Similarly, the lazy transformation is ergodic with respect to its unique absolutely continuous invariant measure. This implies [Pel, Corollary 7] that the skew product transformation R_β is ergodic with respect to $m_p \otimes \mu_{\mathbf{1}^*}$. Since the random Frobenius–Perron operator P is integral preserving with respect to λ, we have

$$\int_{J_\beta} \mathbf{1}^* d\lambda = 1.$$

In particular, there exists a point x_0 in the interior of J_β for which $\mathbf{1}^*(x_0) > 0$. By lower semicontinuity of $\mathbf{1}^*$, there exist an open interval $(a, b) \subset J_\beta$ and a constant $c > 0$ such that $\mathbf{1}^*(x) > c$ for each $x \in (a, b)$. Rewriting (1) one gets, for λ-a.e. x,

$$P_{T_\beta} f(x) = \frac{1}{\beta} \sum_{T_\beta y = x} f(y), \qquad P_{L_\beta} f(x) = \frac{1}{\beta} \sum_{L_\beta y = x} f(y) \tag{2}$$

(see also [P, Theorem 1]), and thus

$$\mathbf{1}^*(x) = \frac{p}{\beta} \sum_{T_\beta y = x} \mathbf{1}^*(y) + \frac{1-p}{\beta} \sum_{L_\beta y = x} \mathbf{1}^*(y).$$

Hence, for λ-a.e. $x \in T_\beta(a, b)$, we have

$$\mathbf{1}^*(x) > \frac{pc}{\beta}.$$

By induction, for each n and for λ-a.e. $x \in T_\beta^n(a, b)$, we have

$$\mathbf{1}^*(x) > \frac{p^n c}{\beta^n}.$$

It is easy to verify that there exist a number $\delta > 0$ and a positive integer n such that

$$T_\beta^n(a, b) \supset [z, z + \delta),$$

where z is a discontinuity point of T_β. Hence,

$$T_\beta^{n+1}(a, b) \supset [0, \beta\delta).$$

Moreover, there exists a positive integer m such that

$$L_\beta^m([0, \beta\delta)) = J_\beta.$$

Using the same argument as before, we conclude that for λ-a.e. $x \in J_\beta$,

$$\mathbf{1}^*(x) > d := \frac{p^{n+1}(1-p)^m c}{\beta^{n+m+1}}.$$

Hence, the function $\mathbf{1}^*$ is larger than or equal to d at every continuity point of $\mathbf{1}^*$. Due to our modification of $\mathbf{1}^*$ at discontinuity points, the function $\mathbf{1}^*$ is everywhere larger than or equal to d. The equivalence of $m_p \otimes \mu_{\mathbf{1}^*}$ and $m_p \otimes \lambda$ is an immediate consequence. □

Since any invariant probability measure absolutely continuous with respect to an ergodic invariant probability measure coincides with this measure, we deduce from Theorems 3 and 4 that for all $f \in \mathfrak{D}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = \mathbf{1}^* \quad \text{in } L_1(J_\beta, \lambda).$$

Remarks 1.

(i) From now on we write $\mu_{\beta,p}$ instead of $\mu_{\mathbf{1}^*}$, since the measure depends on both β and p. It is the unique probability measure, absolutely continuous with respect to λ, satisfying the relationship

$$\mu_{\beta,p} = p \cdot \mu_{\beta,p} \circ T_\beta^{-1} + (1-p) \cdot \mu_{\beta,p} \circ L_\beta^{-1}. \tag{3}$$

(ii) Recall that $\ell : J_\beta \to J_\beta$ given by $\ell(x) = \lfloor\beta\rfloor/(\beta - 1) - x$ satisfies $T_\beta \circ \ell = \ell \circ L_\beta$. It follows from the previous remark that $\mu_{\beta,p} \circ \ell^{-1} = \mu_{\beta,1-p}$. In particular, we see that the invariant density $\mathbf{1}^*$ is symmetric on J_β if $p = 1/2$.

(iii) Let $T_1, \dots, T_n$ be piecewise expanding L-Y maps on J_β and let $(p_1, \dots, p_n)$ be a probability vector. Recently it has been shown by Boyarsky, Góra and Islam (see [BGI]) that functions $f \in \mathfrak{D}$ satisfying $f = Pf = \sum_{i=1}^n p_i \cdot P_{T_i} f$ are bounded below by a positive constant on their support (λ-a.e.). Hence, the fact that $\mathbf{1}^*$ is bounded below by a positive constant on J_β can also be deduced from their result combined with the equivalence of $m_p \otimes \lambda$ and $m_p \otimes \mu_{\beta,p}$.

(iv) It is well known that the Parry measure μ_β is the unique probability measure, absolutely continuous with respect to λ and satisfying equation (3) with $p = 1$. Note however that μ_β and λ are *not* equivalent on J_β. Similarly, the lazy measure ρ_β and λ are not equivalent. For this reason, we restrict ourselves to values of the parameter p in the open interval $(0, 1)$.

3. Main Theorem

It is the object of this section to show that the measure ν_β of maximal entropy for the map K_β and the measure $m_p \otimes \lambda$ are mutually singular.

Let $D = \{0, 1, \dots, \lfloor\beta\rfloor\}^{\mathbb{N}}$ be equipped with the product σ-algebra $\mathcal{D}$ and let σ' be the left shift on D. Define the function $\varphi : \Omega \times J_\beta \to D$ by

$$\varphi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \dots).$$

Clearly, φ is measurable and $\varphi \circ K_\beta = \sigma' \circ \varphi$. Furthermore, Theorem 2 implies that φ is surjective. Let

$$Z = \{(\omega, x) \in \Omega \times J_\beta : K_\beta^n(\omega, x) \in \Omega \times S \text{ for infinitely many } n \geq 0\},$$

$$D' = \left\{(a_1, a_2, \dots) \in D : \sum_{i=1}^{\infty} \frac{a_{j+i-1}}{\beta^i} \in S \text{ for infinitely many } j \geq 1\right\}.$$

Observe that $K_\beta^{-1}(Z) = Z$, $(\sigma')^{-1}(D') = D'$ and that the restriction $\varphi' : Z \to D'$ of φ to Z is a bimeasurable bijection. Let $\mathbb{P}$ denote the uniform product measure on D. We recall from [DdV] that the measure ν_β defined on $\mathcal{A} \otimes \mathcal{B}$ by $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$ is the unique K_β-invariant measure of maximal entropy $\log(1+\lfloor\beta\rfloor)$. It was also shown that the projection of ν_β on the second coordinate is an infinite convolution of Bernoulli measures (see [E1], [E2]). More precisely, consider the purely discrete probability measures $\{\delta_i\}_{i\geq 1}$ defined on J_β and determined by

$$\delta_i(\{k\beta^{-i}\}) = \frac{1}{\lfloor\beta\rfloor + 1} \quad \text{for } k = 0, 1, \dots, \lfloor\beta\rfloor.$$

Let δ_β be the corresponding infinite Bernoulli convolution, i.e.,

$$\delta_\beta = \lim_{n\to\infty} \delta_1 * \dots * \delta_n.$$

Then $\nu_\beta \circ \pi_2^{-1} = \delta_\beta$.

For $\omega \in \Omega$, let $\overline{\omega}$ be given by

$$\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \dots) = (1 - \omega_1, 1 - \omega_2, \dots).$$

Concerning the projection $\pi_1 : \Omega \times J_\beta \to \Omega$ of the measure ν_β on the first coordinate, we have the following lemma.

Lemma 2. *For $n \geq 1$ and $i_1, \dots, i_n \in \{0, 1\}$, we have*

$$\nu_\beta \circ \pi_1^{-1}(\{\omega_1 = i_1, \dots, \omega_n = i_n\}) = \nu_\beta \circ \pi_1^{-1}(\{\overline{\omega}_1 = i_1, \dots, \overline{\omega}_n = i_n\}).$$

Proof. Define the map $r : D \to D$ by

$$r(a_1, a_2, \dots) = (\lfloor\beta\rfloor - a_1, \lfloor\beta\rfloor - a_2, \dots).$$

It follows easily by induction that for $i \geq 1$ and $(\omega, x) \in \Omega \times J_\beta$,

$$d_i(\omega, x) = \lfloor\beta\rfloor - d_i(\overline{\omega}, \ell(x)).$$

Hence,

$$\varphi(\omega, x) = r \circ \varphi(\overline{\omega}, \ell(x)).$$

Since the map r is clearly invariant with respect to $\mathbb{P}$, the assertion follows. □

In particular, it follows from Lemma 2 that $\nu_\beta \circ \pi_1^{-1}(\{\omega_i = 1\}) = 1/2$ for all $i \geq 1$. However, in general, the measure $\nu_\beta \circ \pi_1^{-1}$ is *not* the uniform Bernoulli measure on $\{0,1\}^{\mathbb{N}}$. For instance, using the techniques in [DdV, Section 4], one easily shows that if the greedy expansion of 1 in base β satisfies $1 = 1/\beta + 1/\beta^3$, then $\nu_\beta \circ \pi_1^{-1}$ provides a counterexample. In the case that 1 has a finite greedy expansion with positive coefficients, it has been shown in [DdV, Theorem 8] that $\nu_\beta \circ \pi_1^{-1}$ *is* the uniform Bernoulli measure. The next lemma shows that the K_β-invariant measures ν_β and $m_p \otimes \mu_{\beta,p}$ are different.

Lemma 3. $\nu_\beta \neq m_p \otimes \mu_{\beta,p}$.

Proof. According to Theorem 4, there exists a constant $c > 0$ such that $\mathbf{1}^*(x) \geq c$ for all $x \in J_\beta$. Choose $n \in \mathbb{N}$ such that $1/\beta + 1/\beta^n \in S_1$. Now, suppose the converse is true, i.e., that the measures ν_β and $m_p \otimes \mu_{\beta,p}$ coincide. In particular, ν_β is a product measure and $\delta_\beta = \mu_{\beta,p}$.

On the one hand, we infer from Lemma 2 that

$$\nu_\beta\left(\{\omega_1 = 1\} \times J_\beta \,\middle|\, \Omega \times \left[\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^n}\right)\right) = \frac{1}{2}.$$

On the other hand, since the digits $(d_i)_{i\geq 1}$ form a uniform Bernoulli process under ν_β,

$$\begin{aligned}\nu_\beta\left(\{\omega_1 = 1\} \times J_\beta \,\middle|\, \Omega \times \left[\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^n}\right)\right) &= \nu_\beta\left(\{d_1 = 1\} \,\middle|\, \Omega \times \left[\frac{1}{\beta}, \frac{1}{\beta} + \frac{1}{\beta^n}\right)\right) \\ &= \frac{\nu_\beta(\{d_1 = 1, d_2 = 0, \ldots, d_n = 0, \sum_{i=1}^\infty d_{n+i}/\beta^i \in [0,1)\})}{\mu_{\beta,p}([1/\beta, 1/\beta + 1/\beta^n))} \\ &\leq \frac{1}{c}\left(\frac{\beta}{\lfloor\beta\rfloor + 1}\right)^n \delta_\beta([0,1)).\end{aligned}$$

Passing to the limit, we get a contradiction. □

Define the map $F : \Omega \times J_\beta \to D$ by

$$F(\omega, x) = (d_1(\omega, x), d_1(R_\beta(\omega, x)), d_1(R_\beta^2(\omega, x)), \ldots).$$

We have $\sum_{i=1}^\infty d_1(R_\beta^{i-1}(\omega, x))/\beta^i = x$ for all $(\omega, x) \in \Omega \times J_\beta$. Moreover, the map F is surjective and $\sigma' \circ F = F \circ R_\beta$. Hence F is a factor map and σ' is ergodic with respect to the measure $\rho = m_p \otimes \mu_{\beta,p} \circ F^{-1}$. Note, however, that F is not injective, even if we restrict it to the set for which R_β hits $\Omega \times S$ infinitely many times; this is due to the fact that in equality regions only one digit can be assigned. It follows from Theorem 4 and Birkhoff's ergodic theorem that ρ is concentrated on D'. Therefore, the measure ρ' defined on $\mathcal{A} \otimes \mathcal{B}$ by $\rho'(A) = \rho(\varphi(A \cap Z))$ is a K_β-invariant probability measure and K_β is ergodic with respect to ρ'.

Lemma 4. $\rho' = m_p \otimes \mu_{\beta,p}$.

Proof. Let

$$A_{00} = \{\omega_1 = 0\} \times S_1, \quad A_{\lfloor\beta\rfloor 1} = \{\omega_1 = 1\} \times S_{\lfloor\beta\rfloor},$$
$$A_{02} = \Omega \times E_0, \quad A_{\lfloor\beta\rfloor 2} = \Omega \times E_{\lfloor\beta\rfloor},$$

and

$$A_{i0} = \{\omega_1 = 0\} \times S_{i+1},$$
$$A_{i1} = \{\omega_1 = 1\} \times S_i,$$
$$A_{i2} = \Omega \times E_i,$$

for $1 \le i \le \lfloor\beta\rfloor - 1$. Note that for all $0 \le i \le \lfloor\beta\rfloor$, $\varphi^{-1}(\{d_1 = i\})$ is the union of the sets A_{ij}. It is enough to show that $\rho' = m_p \otimes \mu_{\beta,p}$ on sets of the form

$$\varphi^{-1}(\{d_1 = i_1, \dots, d_n = i_n\}), \quad i_1, \dots, i_n \in \{0, \dots, \lfloor\beta\rfloor\}.$$

Now,

$$\varphi^{-1}(\{d_1 = i_1, \dots, d_n = i_n\}) = \bigcup_{j_1,\dots,j_n} A_{i_1 j_1} \cap \dots \cap K_\beta^{-n+1} A_{i_n j_n},$$

where the union is taken over all $j_1, \dots, j_n$ for which the sets $A_{i_1 j_1}, \dots, A_{i_n j_n}$ are defined. Hence, it is enough to show that

$$\rho'(A_{i_1 j_1} \cap \dots \cap K_\beta^{-n+1} A_{i_n j_n}) = m_p \otimes \mu_{\beta,p}(A_{i_1 j_1} \cap \dots \cap K_\beta^{-n+1} A_{i_n j_n}).$$

It is easy to see that $A_{i_1 j_1} \cap \dots \cap K_\beta^{-n+1} A_{i_n j_n}$ is a product set. Denote its projection on the second coordinate by $V_{i_1 j_1 \dots i_n j_n}$. Define

$$\mathcal{U} = \{(0,0), (\lfloor\beta\rfloor, 1)\} \cup \{(i,j) : 1 \le i \le \lfloor\beta\rfloor - 1, j \in \{0,1\}\}$$

and

$$\{\ell_1, \dots, \ell_L\} = \{\ell : (i_\ell, j_\ell) \in \mathcal{U}\} \subset \{1, \dots, n\}, \quad \ell_1 < \dots < \ell_L.$$

Then

$$A_{i_1 j_1} \cap \dots \cap K_\beta^{-n+1} A_{i_n j_n} = \{\omega_1 = j_{\ell_1}, \dots, \omega_L = j_{\ell_L}\} \times V_{i_1 j_1 \dots i_n j_n}. \tag{4}$$

Note that for all $x \in V_{i_1 j_1 \dots i_n j_n}$,

$$F^{-1} \circ \varphi(\{\omega_1 = j_{\ell_1}, \dots, \omega_L = j_{\ell_L}\} \times \{x\}) = \{\omega_{\ell_1} = j_{\ell_1}, \dots, \omega_{\ell_L} = j_{\ell_L}\} \times \{x\}.$$

Therefore,

$$F^{-1} \circ \varphi(A_{i_1 j_1} \cap \dots \cap K_\beta^{-n+1} A_{i_n j_n}) = \{\omega_{\ell_1} = j_{\ell_1}, \dots, \omega_{\ell_L} = j_{\ell_L}\} \times V_{i_1 j_1 \dots i_n j_n}. \tag{5}$$

The assertion follows immediately from (4) and (5). □

From Theorem 4, Lemmas 3 and 4, and the ergodicity of K_β with respect to ρ' and ν_β, we arrive at the following theorem.

Theorem 5. *The measures ν_β and $m_p \otimes \lambda$ are mutually singular.*

Remark 2. If $\beta \in (1,2)$ is a Pisot number, the mutual singularity of ν_β and $m_p \otimes \lambda$ is a simple consequence of the fact that in this case δ_β and λ are mutually singular (see [E1], [E2]).

4. Finite greedy expansion of 1 with positive coefficients, and the Markov property of the random β-expansion

In this section we assume that the greedy expansion of 1 in base β satisfies $1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n$ with $b_i \geq 1$ for $i = 1, \ldots, n$ and $n \geq 2$ (note that $\lfloor\beta\rfloor = b_1$). It has been shown in [DdV] that in this case the dynamics of K_β can be identified with a subshift of finite type with an irreducible adjacency matrix.

We exhibit the measure $m_p \otimes \mu_{\beta,p}$ obtained in the previous section explicitly. Moreover, it turns out that K_β is exact with respect to $m_p \otimes \mu_{\beta,p}$. The mutual singularity of ν_β and $m_p \otimes \lambda$, i.e., Theorem 5, will be derived by elementary means, independent of the results established in the previous sections.

The analysis of the case $\beta^2 = b_1\beta + 1$ needs some adjustments. For this reason, we assume here that $\beta^2 \neq b_1\beta + 1$, and refer the reader to [DdV, Remarks 6(2)] for the appropriate modifications needed for the case $\beta^2 = b_1\beta + 1$. We first briefly recall some results obtained in [DdV].

We begin with a proposition which plays a crucial role in finding the Markov partition describing the dynamics of K_β.

Proposition 1. *Suppose* 1 *has a finite greedy expansion of the form* $1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n$. *If* $b_j \geq 1$ *for* $1 \leq j \leq n$, *then*

(i) $T_\beta^i 1 = L_\beta^i 1 \in E_{b_{i+1}}, \quad 0 \leq i \leq n-2.$

(ii) $T_\beta^{n-1} 1 = L_\beta^{n-1} 1 = \dfrac{b_n}{\beta} \in S_{b_n}, \quad T_\beta^n 1 = 0, \quad \textit{and} \quad L_\beta^n 1 = 1.$

(iii) $T_\beta^i\left(\dfrac{b_1}{\beta-1} - 1\right) = L_\beta^i\left(\dfrac{b_1}{\beta-1} - 1\right) \in E_{b_1 - b_{i+1}}, \quad 0 \leq i \leq n-2.$

(iv) $T_\beta^{n-1}\left(\dfrac{b_1}{\beta-1} - 1\right) = L_\beta^{n-1}\left(\dfrac{b_1}{\beta-1} - 1\right) = \dfrac{b_1}{\beta(\beta-1)} + \dfrac{b_1 - b_n}{\beta} \in S_{b_1 - b_n + 1},$
$T_\beta^n\left(\dfrac{b_1}{\beta-1} - 1\right) = \dfrac{b_1}{\beta-1} - 1, \quad \textit{and} \quad L_\beta^n\left(\dfrac{b_1}{\beta-1} - 1\right) = \dfrac{b_1}{\beta-1}.$

To find the Markov chain behind the map K_β, one starts by refining the partition

$$\mathcal{E} = \{E_0, S_1, E_1, \ldots, S_{b_1}, E_{b_1}\}$$

of $[0, b_1/(\beta-1)]$, using the orbits of 1 and $b_1/(\beta-1) - 1$ under the transformation T_β. We place the endpoints of $\mathcal{E}$ together with $T_\beta^i 1$, $T_\beta^i(b_1/(\beta-1) - 1)$, $i = 0, \ldots, n-2$, in increasing order. We use these points to form a new partition $\mathcal{C}$ which is a refinement of $\mathcal{E}$, consisting of intervals. We write $\mathcal{C}$ as

$$\mathcal{C} = \{C_0, C_1, \ldots, C_L\}.$$

We choose $\mathcal{C}$ to satisfy the following. For $0 \leq i \leq n-2$,

- $T_\beta^i 1 \in C_j$ if and only if $T_\beta^i 1$ is a left endpoint of C_j,
- $T_\beta^i(b_1/(\beta-1) - 1) \in C_j$ if and only if $T_\beta^i(b_1/(\beta-1) - 1)$ is a right endpoint of C_j.

Note that this choice is possible, because the points $T_\beta^i 1, T_\beta^i(b_1/(\beta-1)-1)$ for $0 \le i \le n-2$ are all different. From the dynamics of K_β on this refinement, one reads the following properties of $\mathcal{C}$.

p1. $C_0 = [0, b_1/(\beta-1)-1]$ and $C_L = [1, b_1/(\beta-1)]$.
p2. For $i = 0, 1, \dots, b_1$, E_i can be written as a finite disjoint union of the form $E_i = \bigcup_{j \in M_i} C_j$ with $M_0, M_1, \dots, M_{b_1}$ disjoint subsets of $\{0, 1, \dots, L\}$. Further, the number of elements in M_i equals the number of elements in M_{b_1-i}.
p3. For each S_i there is exactly one $j \in \{0, 1, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k$ such that $S_i = C_j$.
p4. If $C_j \subset E_i$, then $T_\beta(C_j) = L_\beta(C_j)$ is a finite disjoint union of elements of $\mathcal{C}$, say $T_\beta(C_j) = C_{i_1} \cup \dots \cup C_{i_l}$. Since $\ell(C_j) = C_{L-j} \subset E_{b_1-i}$, it follows that $T_\beta(C_{L-j}) = C_{L-i_1} \cup \dots \cup C_{L-i_l}$.
p5. If $C_j = S_i$, then $T_\beta(C_j) = C_0$ and $L_\beta(C_j) = C_L$.

To define the underlying subshift of finite type associated with the map K_β, we consider the $(L+1) \times (L+1)$ matrix $A = (a_{i,j})$ with entries in $\{0, 1\}$ defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } i \in \bigcup_{k=0}^{b_1} M_k \text{ and } \lambda(C_j \cap T_\beta(C_i)) = \lambda(C_j), \\ 0 & \text{if } i \in \bigcup_{k=0}^{b_1} M_k \text{ and } C_i \cap T_\beta^{-1} C_j = \emptyset, \\ 1 & \text{if } i \in \{0, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = 0, L, \\ 0 & \text{if } i \in \{0, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j \ne 0, L. \end{cases}$$

Let Y denote the topological Markov chain (or the subshift of finite type) determined by the matrix A. That is, $Y = \{y = (y_i) \in \{0, 1, \dots, L\}^{\mathbb{N}} : a_{y_i, y_{i+1}} = 1\}$. We let σ_Y be the left shift on Y. For ease of notation, we denote by $s_1, \dots, s_{b_1}$ the states $j \in \{0, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k$ corresponding to the switch regions $S_1, \dots, S_{b_1}$ respectively.

To each $y \in Y$, we associate a sequence $(e_i) \in \{0, 1, \dots, b_1\}^{\mathbb{N}}$ and a point $x \in [0, b_1/(\beta-1)]$ as follows. Let

$$e_j = \begin{cases} i & \text{if } y_j \in M_i, \\ i & \text{if } y_j = s_i \text{ and } y_{j+1} = 0, \\ i-1 & \text{if } y_j = s_i \text{ and } y_{j+1} = L. \end{cases} \tag{6}$$

Now set

$$x = \sum_{j=1}^{\infty} \frac{e_j}{\beta^j}. \tag{7}$$

Our aim is to define a map $\psi : Y \to \Omega \times [0, b_1/(\beta-1)]$ that intertwines the actions of K_β and σ_Y. Given $y \in Y$, equations (6) and (7) describe what the second coordinate of ψ should be. In order to be able to associate an $\omega \in \Omega$, one needs that $y_i \in \{s_1, \dots, s_{b_1}\}$ infinitely often. For this reason it is not possible to define ψ on all of Y, but only on an invariant subset. To be more precise, let

$$Y' = \{y = (y_1, y_2, \dots) \in Y : y_i \in \{s_1, \dots, s_{b_1}\} \text{ for infinitely many } i\text{'s}\}.$$

Define $\psi : Y' \to \Omega \times [0, b_1/(\beta - 1)]$ as follows. Let $y = (y_1, y_2, \dots) \in Y'$, and define x as in (7). To define a point $\omega \in \Omega$ corresponding to y, we first locate the indices $n_i = n_i(y)$ where the realization y of the Markov chain is in state s_r for some $r \in \{1, \dots, b_1\}$. That is, let $n_1 < n_2 < \cdots$ be the indices such that $y_{n_i} = s_r$ for some $r = 1, \dots, b_1$. Define

$$\omega_j = \begin{cases} 1 & \text{if } y_{n_j+1} = 0, \\ 0 & \text{if } y_{n_j+1} = L. \end{cases}$$

Now set $\psi(y) = (\omega, x)$.

The following two lemmas reflect the fact that the dynamics of K_β is essentially the same as that of the Markov chain Y.

Lemma 5. *Let $y \in Y'$ be such that $\psi(y) = (\omega, x)$. Then:*

(i) $y_1 = k$ *for some* $k \in \bigcup_{i=0}^{b_1} M_i \Rightarrow x \in C_k$.
(ii) $y_1 = s_i, y_2 = 0 \Rightarrow x \in S_i$ *and* $\omega_1 = 1$ *for* $i = 1, \dots, b_1$.
(iii) $y_1 = s_i, y_2 = L \Rightarrow x \in S_i$ *and* $\omega_1 = 0$ *for* $i = 1, \dots, b_1$.

Lemma 6. *For $y \in Y'$, we have*

$$\psi \circ \sigma_Y(y) = K_\beta \circ \psi(y).$$

We now consider on Y the Markov measure $Q_{\beta,p}$ with transition matrix $P = (p_{i,j})$, given by

$$p_{i,j} = \begin{cases} \lambda(C_i \cap T_\beta^{-1} C_j)/\lambda(C_i) & \text{if } i \in \bigcup_{k=0}^{b_1} M_k, \\ p & \text{if } i \in \{0, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = 0, \\ 1 - p & \text{if } i \in \{0, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = L, \\ 0 & \text{if } i \in \{0, \dots, L\} \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L, \end{cases}$$

and initial distribution the corresponding stationary distribution π.

Theorem 6. *$Q_{\beta,p} \circ \psi^{-1}$ is a product measure of the form $m_p \otimes \mu$.*

Proof. Define the measure μ on $[0, b_1/(\beta - 1)]$ by

$$\mu(B) = \sum_{j=0}^{L} \frac{\lambda(B \cap C_j)}{\lambda(C_j)} \cdot \pi(j) \quad [B \in \mathcal{B}].$$

Define the Markov partition $\mathcal{P}_0$ of $\Omega \times [0, b_1/(\beta - 1)]$ by

$$\mathcal{P}_0 = \left\{ \Omega \times C_j : j \in \bigcup_{k=0}^{b_1} M_k \right\} \cup \{\{\omega_1 = i\} \times S_j : i = 0, 1,\ j = 1, \dots, b_1\},$$

and let $\mathcal{P}_n = \mathcal{P}_0 \vee K_\beta^{-1} \mathcal{P}_0 \vee \cdots \vee K_\beta^{-n} \mathcal{P}_0$. It is straightforward to see that the inverse images of elements in $\mathcal{P}_n$ under ψ are cylinders in Y and that for each element $P \in \mathcal{P}_n$, $m_p \otimes \mu(P) = Q_{\beta,p} \circ \psi^{-1}(P)$. It follows that $Q_{\beta,p} \circ \psi^{-1} = m_p \otimes \mu$. □

Since P is an irreducible transition matrix, σ_Y is ergodic with respect to $Q_{\beta,p}$ and $\pi(i) > 0$ for all $i \in \{0, \dots, L\}$. It follows from Lemma 6 that K_β is ergodic with respect to $m_p \otimes \mu$. Furthermore, it is immediately seen from the definition that μ is equivalent to λ. Hence, the measure $Q_{\beta,p} \circ \psi^{-1}$ is equivalent to $m_p \otimes \lambda$.

Proposition 2. *The map K_β is exact with respect to $m_p \otimes \mu_{\beta,p}$. Moreover, $\mu = \mu_{\beta,p}$.*

Proof. It follows from Lemma 1 and Remarks 1(i) that $\mu = \mu_{\beta,p}$. Since the transition matrix P is also aperiodic, σ_Y is exact with respect to $Q_{\beta,p}$. It follows from Lemma 6 that K_β is exact with respect to $m_p \otimes \mu_{\beta,p}$. □

It also follows from the above proposition that the density $\mathbf{1}^*$ assumes the constant value $\pi(j)/\lambda(C_j)$ on the interval C_j, $j \in \{0, \dots, L\}$.

Example 1. Let $\beta = G = \frac{1}{2}(1 + \sqrt{5})$ and let $g = G - 1 = \frac{1}{2}(\sqrt{5} - 1)$. Note that $1 = 1/\beta + 1/\beta^2$. In this case, we let $\mathcal{C} = \mathcal{E}$, since 1 and $1/(\beta - 1) - 1$ are already endpoints of intervals in $\mathcal{E}$. Using the techniques in this section it is easily verified that the dynamical system $(\Omega \times J_\beta, \mathcal{A} \otimes \mathcal{B}, m_p \otimes \mu_{\beta,p}, K_\beta)$ is measurably isomorphic to the Markov chain with transition matrix P, given by

$$P = \begin{pmatrix} g & g^2 & 0 \\ p & 0 & 1 - p \\ 0 & g^2 & g \end{pmatrix},$$

and stationary distribution π determined by $\pi P = \pi$.

It remains to prove that $Q_{\beta,p} \circ \psi^{-1}$ and ν_β are mutually singular. Since K_β is ergodic with respect to both measures, it suffices to show that the measures do not coincide.

Lemma 7. $\nu_\beta \neq Q_{\beta,p} \circ \psi^{-1}$.

Proof. We distinguish between the cases $p = 1/2$ and $p \neq 1/2$.

Suppose $p = 1/2$. On the one hand, for all $i \in \{1, \dots, \lfloor\beta\rfloor\}$ we have

$$\frac{i}{\beta} + \sum_{i=2}^{\infty} \frac{d_i}{\beta^i} \in S_i \;\Leftrightarrow\; \sum_{i=1}^{\infty} \frac{d_{i+1}}{\beta^i} \in C_0,$$

$$\frac{i-1}{\beta} + \sum_{i=2}^{\infty} \frac{d_i}{\beta^i} \in S_i \;\Leftrightarrow\; \sum_{i=1}^{\infty} \frac{d_{i+1}}{\beta^i} \in C_L.$$

Using the fact that the digits $(d_i)_{i \geq 1}$ form a uniform Bernoulli process under ν_β, a simple calculation yields

$$\nu_\beta(\Omega \times S) = \frac{\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1} \cdot \nu_\beta(\Omega \times C_0) + \frac{\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1} \cdot \nu_\beta(\Omega \times C_L).$$

Since $\nu_\beta(\Omega \times C_0) = \nu_\beta(\Omega \times C_L)$, it follows that

$$\frac{\nu_\beta(\Omega \times S)}{\nu_\beta(\Omega \times C_0)} = \frac{2\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1}.$$

On the other hand, it follows from $\pi P = \pi$ that

$$\pi(0) = \frac{1}{\beta}\pi(0) + \frac{1}{2}(\pi(s_1) + \cdots + \pi(s_{b_1})).$$

Rewriting one gets

$$\frac{\pi(s_1) + \cdots + \pi(s_{b_1})}{\pi(0)} = \frac{Q_{\beta,p} \circ \psi^{-1}(\Omega \times S)}{Q_{\beta,p} \circ \psi^{-1}(\Omega \times C_0)} = \frac{2(\beta - 1)}{\beta}.$$

However,

$$\frac{2(\beta - 1)}{\beta} \neq \frac{2\lfloor\beta\rfloor}{\lfloor\beta\rfloor + 1}$$

for all non-integer β, in particular for the β's under consideration.

Suppose $p \neq 1/2$. In this case, the assertion follows from the fact that the projection of ν_β on the first coordinate is the uniform Bernoulli measure on $\{0, 1\}^{\mathbb{N}}$ [DdV, Theorem 8]. Note that this result is applicable because 1 has a finite greedy expansion with positive coefficients. □

The mutual singularity of ν_β and $m_p \otimes \lambda$ follows as before.

5. Open problems and final remarks

1. We have not been able to find an explicit formula for $\mathbf{1}^*$. Recall that the Parry density $h_\beta = P_{T_\beta} h_\beta$ is given by

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T_\beta^n(1)} \frac{1}{\beta^n}$$

(see Section 1). We expect that the density $\mathbf{1}^*$ can be expressed in a similar way, but now the random orbits of 1 as well as the random orbits of the complementary point $\lfloor\beta\rfloor/(\beta - 1) - 1$ are involved. Let us consider an example.

Example 2. Let $p = 1/2$ and $\beta = 3/2$. Note that in this case $\lfloor\beta\rfloor/(\beta - 1) - 1 = 1$. Rewriting (2) one gets

$$P_{T_\beta} f(x) = \frac{1}{\beta} \sum_{i=0}^{1} f\left(\frac{x+i}{\beta}\right) \cdot 1_{[0,1)}(x) + \frac{1}{\beta} f\left(\frac{x+1}{\beta}\right) \cdot 1_{[1,2]}(x),$$

$$P_{L_\beta} f(x) = \frac{1}{\beta} f\left(\frac{x}{\beta}\right) \cdot 1_{[0,1]}(x) + \frac{1}{\beta} \sum_{i=0}^{1} f\left(\frac{x+i}{\beta}\right) \cdot 1_{(1,2]}(x).$$

It is easy to verify that $\mathbf{1} \in \mathfrak{D}$ satisfies $P\mathbf{1} = \mathbf{1}$, hence $\mathbf{1}^* = \mathbf{1}$. It follows that $m_{1/2} \otimes \lambda$ is $K_{3/2}$-invariant.

2. We have not been able to give an explicit formula for $h_{m_p \otimes \mu_{\beta,p}}(K_\beta)$. However, in the special case that $\beta^2 = b_1\beta + 1$, the entropy is already calculated in [DK2]:

$$h_{m_p \otimes \mu_{\beta,p}}(K_\beta) = \log\beta - \frac{b_1}{1+\beta^2}(p\log p + (1-p)\log(1-p)).$$

Since in this case $\pi(s_i) = \frac{1}{1+\beta^2}$, $i = 1, \dots, b_1$, it follows that

$$h_{m_p \otimes \mu_{\beta,p}}(K_\beta) = \log\beta - \mu_{\beta,p}(S)(p\log p + (1-p)\log(1-p)).$$

One might conjecture that this formula holds in general.

3. Fix $p \in (0,1)$. It is a direct consequence of Birkhoff's ergodic theorem, Theorem 4 and the ergodicity of K_β with respect to $m_p \otimes \mu_{\beta,p}$ that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{\Omega\times S}(K_\beta^i(\omega, x)) = \mu_{\beta,p}(S) > 0. \tag{8}$$

In particular, we infer from (8) that the set

$$G = \{x \in J_\beta : x \text{ has a unique expansion in base } \beta\}$$

has Lebesgue measure zero, since $K_\beta^n(\omega, x) \in \Omega \times E$ for all $(\omega, x) \in \Omega \times G$ and all $n \geq 0$. Let $T_0 = L_\beta$, $T_1 = T_\beta$, and let

$$N = \bigcup_{n=1}^{\infty} \{x \in J_\beta : T_{u_1} \circ \cdots \circ T_{u_n} x \in G \text{ for some } u_1, \dots, u_n \in \{0,1\}\}.$$

Since the greedy map and the lazy map are nonsingular, $\lambda(N) = 0$. Note that $\Omega \times J_\beta \backslash N \subset Z$ and that for $x \in J_\beta \setminus N$, different elements of Ω give rise to different expansions of x in base β. We conclude that for λ-a.e. $x \in J_\beta$, there exist $2^{\aleph_0}$ expansions of x in base β. For a more elementary proof of this fact in case $\beta \in (1,2)$, we refer to [S1].

4. Erdős and Komornik introduced in [EK] the notion of universal expansions. They called an expansion $(d_1, d_2, \dots)$ in base β of some $x \in J_\beta$ *universal* if for each (finite) block $b_1 \dots b_n$ consisting of digits in the set $\{0, \dots, \lfloor\beta\rfloor\}$, there exists an index $k \geq 1$ such that $d_k \dots d_{k+n-1} = b_1 \dots b_n$. They proved that there exists a number $\beta_0 \in (1,2)$ such that for each $\beta \in (1, \beta_0)$, *every* $x \in (0, 1/(\beta-1))$ has a universal expansion in base β. Subsequently, Sidorov proved in [S2] that for a given $\beta \in (1,2)$ and for λ-a.e. $x \in J_\beta$, there exists a universal expansion of x in base β. We now strengthen his result and the conclusion of the preceding remark by the following theorem.

Theorem 7. *For any non-integer $\beta > 1$, and for λ-a.e. $x \in J_\beta$, there exist $2^{\aleph_0}$ universal expansions of x in base β.*

In order to prove Theorem 7 we need the following lemma.

Lemma 8. *Let $\beta > 1$ be a non-integer and let $p \in (0,1)$. Then, for $n \geq 1$ and $i_1, \ldots, i_n \in \{0, \ldots, \lfloor\beta\rfloor\}$, we have*

$$m_p \otimes \mu_{\beta,p}(\{d_1 = i_1, \ldots, d_n = i_n\}) > 0.$$

Proof. By Theorem 4, it suffices to show that

$$m_p \otimes \lambda(\{d_1 = i_1, \ldots, d_n = i_n\}) > 0.$$

It is easy to verify that there exists a sequence $(j_1, j_2, \ldots) \in D$, starting with $i_1 \ldots i_n$, such that the numbers $x_1, \ldots, x_n$, given by

$$x_r = \sum_{i=1}^{\infty} \frac{j_{i+r-1}}{\beta^i}, \quad r = 1, \ldots, n,$$

are elements of $J_\beta \setminus \partial(S)$, where $\partial(S)$ denotes the boundary of S. For $m \geq 1$, consider the set

$$I_m = \left[\sum_{i=1}^{n+m} \frac{j_i}{\beta^i}, \sum_{i=1}^{n+m} \frac{j_i}{\beta^i} + \sum_{i=n+m+1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^i} \right].$$

Let $y \in I_m$ and let $(a_1, a_2, \ldots)$ be an expansion of y, starting with $j_1 \ldots j_{n+m}$. Define

$$y_r = \sum_{i=1}^{\infty} \frac{a_{i+r-1}}{\beta^i}, \quad r = 1, \ldots, n.$$

Choose m large enough, so that for each $r = 1, \ldots, n$, x_r and y_r are elements of the same equal or switch region, regardless of the values of the digits a_ℓ, $\ell > n + m$, and hence regardless of the chosen element $y \in I_m$. Note that this is possible because $x_r \notin \partial(S)$ for $r = 1, \ldots, n$. Denote the set of indices $r \in \{1, \ldots, n\}$ for which $x_r \in S$ by $\{\ell_1, \ldots, \ell_L\}$. Then, for suitably chosen $u_1, \ldots, u_L \in \{0, 1\}$, we have

$$\{\omega_1 = u_1, \ldots, \omega_L = u_L\} \times I_m \subset \{d_1 = i_1, \ldots, d_n = i_n\}$$

and the conclusion follows. □

Proof of Theorem 7. Fix $p \in (0,1)$ and let $b_1 \ldots b_n$ be an arbitrary block. Using Birkhoff's ergodic theorem, Theorem 4, Lemma 8 and the ergodicity of K_β with respect to $m_p \otimes \mu_{\beta,p}$, we may conclude that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the block $b_1 \ldots b_n$ occurs in

$$(d_1(\omega, x), d_2(\omega, x), \ldots) \tag{9}$$

with positive limiting frequency $m_p \otimes \mu_{\beta,p}(\{d_1 = b_1, \ldots, d_n = b_n\})$. In particular, for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the block $b_1 \ldots b_n$ occurs in (9). Since there are only countably many blocks, we deduce that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the expansion (9) is universal in base β. An application of Fubini's theorem shows that there exists a Borel set $B \subset J_\beta \setminus N$ of full Lebesgue measure and there exist sets $A_x \in \mathcal{A}$ with $m_p(A_x) = 1$ $(x \in B)$ such that for all $x \in B$ and $(\omega, x) \in A_x \times \{x\}$, the expansion (9) is universal in base β. Since the sets A_x have necessarily the cardinality of the continuum and since different elements of Ω give rise to different expansions of x in base β for any $x \in J_\beta \setminus N$, the assertion follows. □

5. An expansion $(a_1, a_2, \ldots)$ in base β of some number $x \in J_\beta$ is called *normal* if each block $i_1 \ldots i_n$ with digits in $\{0, \ldots, \lfloor\beta\rfloor\}$ occurs in $(a_1, a_2, \ldots)$ with limiting frequency $(\lfloor\beta\rfloor + 1)^{-n}$. Note that a normal expansion is in particular universal.

Fix $p \in (0, 1)$. Since $\nu_\beta \neq m_p \otimes \mu_{\beta,p}$ and since both measures ν_β and $m_p \otimes \mu_{\beta,p}$ are concentrated on Z, there exists a block $i_1 \ldots i_n$ such that

$$m_p \otimes \mu_{\beta,p}(\{d_1 = i_1, \ldots, d_n = i_n\}) \neq (\lfloor\beta\rfloor + 1)^{-n}.$$

Hence, for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the expansion (9) is universal but *not* normal. On the other hand, Sidorov proved in [S2] that there exists a Borel set $V \subset (1, 2)$ of full Lebesgue measure such that for each $\beta \in V$ and for λ-a.e. $x \in J_\beta$, there exists a normal expansion of x in base β.

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