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New estimates for elliptic equations and Hodge type systems

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Abstract. We establish new estimates for the Laplacian, the div-curl system, and more general Hodge systems in arbitrary dimension n, with data in L^1. We also present related results concerning differential forms with coefficients in the limiting Sobolev space $W^{1,n}$.

Keywords. Elliptic systems, data in L^1, div-curl, Hodge systems, limiting Sobolev spaces, differential forms, Littlewood–Paley decomposition, Ginzburg–Landau functional

1. Introduction

The starting point for this work is the following estimate from [5, Proposition 4] (proven for $n = 3$ but the argument generalizes).

Theorem 1. *Let Γ be a closed rectifiable curve in $\mathbb{R}^n$ with unit tangent vector $\vec{t}$ and let $Y \in C_0^\infty(\mathbb{R}^n)$. Then*

$$\left| \int_\Gamma Y\vec{t} \right| \le C_n |\Gamma| \, \|\nabla Y\|_n. \tag{1.1}$$

The proof in [5] relies on a Littlewood–Paley decomposition and the co-area formula; another proof was given recently by Van Schaftingen [13] which uses only the Morrey–Sobolev embedding in place of the Littlewood–Paley decomposition.

A more general form of Theorem 1 was given in [4, Theorem 1].

Theorem 1′. *For every $Y \in C_0^\infty(\mathbb{R}^n)$,*

$$\left| \int_{\mathbb{R}^n} Y\vec{f} \right| \le C_n \|\vec{f}\|_1 \|\nabla Y\|_n \quad \textit{for all } \vec{f} \in L^1_\#(\mathbb{R}^n, \mathbb{R}^n).$$

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Here

$$L^1_\#(\mathbb{R}^n,\mathbb{R}^n)=\{\vec f\in L^1(\mathbb{R}^n,\mathbb{R}^n)\mid \operatorname{div}\vec f=0\}.$$

Clearly Theorem 1′ implies Theorem 1 by taking $\vec f=\mathcal{H}_\Gamma\,\vec t$ where $\mathcal{H}_\Gamma$ is the one-dimensional Hausdorff measure on Γ. Conversely, one can deduce Theorem 1′ from Theorem 1 using Smirnov's theorem [10] on the integral representation of divergence-free vector fields. More precisely, every $\vec f\in L^1_\#$ may be written as a weak limit (in the sense of measures) of combinations of the form

$$\sum\alpha_i\frac{1}{|\Gamma_i|}\mathcal{H}_{\Gamma_i}\vec t_i$$

with $\alpha_i\ge 0$ and $\sum\alpha_i\le\|\vec f\|_1$.

A totally elementary direct proof of Theorem 1′ was given more recently by Van Schaftingen [14].

Observe that for $n=2$, Theorem 1′ is a trivial consequence of Nirenberg's inequality $\|\zeta\|_2\le C\|\nabla\zeta\|_1$.

The meaning of Theorem 1′ is that $L^1_\#\subset(W^{1,n})^*$, which has remarkable applications to linear elliptic PDE's. [Here $W^{1,n}$ denotes the completion of C_0^∞ for the norm $\|\nabla u\|_n$]. For example, consider the solution $\vec u=E*\vec f$, where $E(x)=c/|x|^{n-2}$, $n>2$, is the fundamental solution of $-\Delta$, of the equation

$$-\Delta\vec u=\vec f\quad\text{ in }\mathbb{R}^n. \tag{1.2}$$

We have

Theorem 2. *Let $\vec f\in L^1_\#(\mathbb{R}^n,\mathbb{R}^n)$ with $n>2$ and let $\vec u$ be the solution of* (1.2). *Then*

$$\|\nabla\vec u\|_{n/(n-1)}\le C_n\|\vec f\|_1 \tag{1.3}$$

and hence

$$\|\vec u\|_{n/(n-2)}\le C_n\|\vec f\|_1. \tag{1.4}$$

Let us remark that the analog of Theorem 2 for $n=2$ is

Theorem 3. *Let $\vec f\in L^1_\#(\mathbb{R}^2,\mathbb{R}^2)$. Then*

$$\|\nabla\vec u\|_2\le C\|\vec f\|_1 \tag{1.5}$$

and

$$\|\vec u\|_\infty\le C\|\vec f\|_1. \tag{1.6}$$

Indeed, write $\vec f=\nabla^\perp\zeta$ with $\|\nabla\zeta\|_1=\|\vec f\|_1$; thus $\nabla\vec u=\nabla\nabla^\perp(-\Delta)^{-1}\zeta$. Inequality (1.5) then follows from standard elliptic estimates and the inequality $\|\zeta\|_2\le C\|\nabla\zeta\|_1$. For inequality (1.6), write by partial integration

$$\left\|\vec f*\log\frac{1}{|x|}\right\|_\infty\le\left\||\zeta|*\frac{1}{|x|}\right\|_\infty$$

and integrate in polar coordinates:

$$\int \frac{|\zeta(x)|}{|x|}\,dx = \iint |\zeta(re^{i\theta})|\,dr\,d\theta \le \iint |\partial_r \zeta|\,r\,dr\,d\theta \le \int |\nabla \zeta|\,dx.$$

Remark 1. A 'natural' stronger inequality than (1.3) and (1.5), involving second order derivatives, would be

$$\|\nabla^2 \vec{u}\|_1 \le C\|\vec{f}\|_1. \tag{1.7}$$

This inequality however is easily seen to be false, at least in dimension $n \ge 3$. It is also false for $n = 2$, but the argument is more complicated (see Appendix).

In view of Van Schaftingen's argument in [14], Theorem 2 has now an elementary proof. Here is a generalized form of Theorem 2 which, so far, requires a much more involved argument.

Theorem 4. *Let $\vec{u}$ be the solution of* (1.2) *with*

$$\vec{f} = \vec{f}_0 + \sum \frac{\partial}{\partial x_i} \vec{f}_i \tag{1.8}$$

where $\vec{f}_0 \in L^1$, $\vec{f}_i \in L^{n/(n-1)}$ and $\operatorname{div} \vec{f} = 0$. *Then*

$$\|\nabla \vec{u}\|_{n/(n-1)} \le C_n \Big\{ \|\vec{f}_0\|_1 + \sum \|\vec{f}_i\|_{n/(n-1)} \Big\}. \tag{1.9}$$

Remark 2. Theorem 4 is equivalent to the following

Theorem 4′. *Let $\vec{f}_0 \in L^1$ and let $\vec{u}_0$ be the solution of*

$$-\Delta \vec{u}_0 = \vec{f}_0 \quad \text{in } \mathbb{R}^n,\ n \ge 2.$$

Assume $\operatorname{div} \vec{f}_0 \in W^{-2,n/(n-1)}$. *Then $\vec{u}_0 \in W^{1,n/(n-1)}$ and*

$$\|\nabla \vec{u}_0\|_{n/(n-1)} \le C\{\|\vec{f}_0\|_1 + \|\operatorname{div} \vec{f}_0\|_{-2,n/(n-1)}\}. \tag{1.10}$$

In other words, for every $\vec{f}_0 \in L^1$ with $\operatorname{div} \vec{f}_0 \in W^{-2,n/(n-1)}$,

$$\|\vec{f}_0\|_{-1,n/(n-1)} \le C\{\|\vec{f}_0\|_1 + \|\operatorname{div} \vec{f}_0\|_{-2,n/(n-1)}\}.$$

Indeed, set $\varphi = \operatorname{div} \vec{u}_0$, so that $-\Delta\varphi = \operatorname{div} \vec{f}_0$ and thus $\varphi \in L^{n/(n-1)}$. Let

$$\vec{f} = \vec{f}_0 + \operatorname{grad} \varphi.$$

Then

$$\operatorname{div} \vec{f} = \operatorname{div} \vec{f}_0 + \Delta\varphi = \operatorname{div} \vec{f}_0 - \operatorname{div} \vec{f}_0 = 0.$$

Applying Theorem 4 to $\vec{f}$ yields

$$\|\nabla \vec{u}\|_{n/(n-1)} \le C\{\|\vec{f}_0\|_1 + \|\varphi\|_{n/(n-1)}\}. \tag{1.11}$$

On the other hand,

$$-\Delta(\vec{u} - \vec{u}_0) = \vec{f} - \vec{f}_0 = \operatorname{grad} \varphi$$

and thus, by standard elliptic estimates,

$$\|\nabla(\vec{u} - \vec{u}_0)\|_{n/(n-1)} \leq C\|\varphi\|_{n/(n-1)}. \tag{1.12}$$

Combining (1.11) and (1.12) gives (1.10).

As we are going to see in Section 3, Theorem 4 is closely connected to a remarkable property concerning differential forms with coefficients in the critical Sobolev space $W^{1,n}$. It is slightly more convenient to work first on $\mathbb{T}^n$ instead of $\mathbb{R}^n$ and we will do so in the following. At the end of Section 2 and in Section 3 we will explain how to pass from $\mathbb{T}^n$ to $\mathbb{R}^n$ (see Remark 6).

We denote by $\Lambda^\ell \mathbb{T}^n$, $0 \leq \ell \leq n$, the space of ℓ-forms on $\mathbb{T}^n$, by $W^{1,n}(\Lambda^\ell \mathbb{T}^n)$, or simply $W^{1,n}(\Lambda^\ell)$, the ℓ-forms with coefficients in $W^{1,n}(\mathbb{T}^n)$, and by d the exterior differential operator (see e.g. [6] for the notations). One of the main results in our paper is

Theorem 5. *If $n \geq 2$ and $1 \leq \ell \leq n-1$ we have*

$$d[W^{1,n}(\Lambda^\ell)] = d[(W^{1,n} \cap L^\infty)(\Lambda^\ell)].$$

More precisely, given any $X \in W^{1,n}(\Lambda^\ell)$ there exists some $Y \in (W^{1,n} \cap L^\infty)(\Lambda^\ell)$ such that

$$dY = dX \tag{1.13}$$

and

$$\|\nabla Y\|_n + \|Y\|_\infty \leq C\|dX\|_n. \tag{1.14}$$

Notice that the conclusion obviously fails for $\ell = 0$: given a function $f \in W^{1,n}$ there need not exist a function $g \in L^\infty$ such that $\operatorname{grad}(f - g) = 0$.

In the extreme case $\ell = n-1$, Theorem 5 asserts that given any $\vec{X} \in W^{1,n}(\mathbb{T}^n, \mathbb{R}^n)$ there exists $\vec{Y} \in (W^{1,n} \cap L^\infty)(\mathbb{T}^n, \mathbb{R}^n)$ such that $\operatorname{div} \vec{Y} = \operatorname{div} \vec{X}$ with

$$\|\nabla \vec{Y}\|_n + \|\vec{Y}\|_\infty \leq C\|\operatorname{div} \vec{X}\|_n$$

or equivalently,

Corollary 6. *Given any $f \in L^n(\mathbb{T}^n, \mathbb{R})$ with $\int f = 0$ the equation*

$$\operatorname{div} \vec{Y} = f \tag{1.15}$$

admits a solution $\vec{Y} \in (W^{1,n} \cap L^\infty)(\mathbb{T}^n, \mathbb{R}^n)$ with

$$\|\nabla \vec{Y}\|_n + \|\vec{Y}\|_\infty \leq C\|f\|_n. \tag{1.16}$$

This case was already treated in [3]. As was pointed out in [3] this statement is equivalent via Hahn–Banach and duality to the estimate

$$\left\|\zeta - \fint \zeta\right\|_{n/(n-1)} \leq C\|\operatorname{grad} \zeta\|_{L^1 + W^{-1,n/(n-1)}} \quad \forall \zeta \in C^\infty(\mathbb{T}^n). \tag{1.17}$$

It was also proved in [3] that (surprisingly) the construction of some $\vec{Y}$ satisfying (1.15)–(1.16) cannot be linear. More precisely

Proposition 7. *There exists no bounded linear operator*

$$K : \left\{ f \in L^n \;\middle|\; \int f = 0 \right\} \to L^\infty$$

such that

$$\operatorname{div} Kf = f \quad \forall f.$$

The other extreme case, $\ell = 1$, in Theorem 5 corresponds to

Corollary 8. *Given any $\vec{X} \in W^{1,n}(\mathbb{T}^n, \mathbb{R}^n)$ there exist $\vec{Y} \in (W^{1,n} \cap L^\infty)(\mathbb{T}^n, \mathbb{R}^n)$ and $p \in W^{2,n}(\mathbb{T}^n, \mathbb{R})$ such that*

$$\vec{Y} - \vec{X} = \operatorname{grad} p \tag{1.18}$$

and

$$\|\nabla \vec{Y}\|_n + \left\| \vec{Y} - \fint \vec{Y} \right\|_\infty \le C \|\operatorname{curl} \vec{X}\|_n, \tag{1.19}$$

where $\operatorname{curl} \vec{X} = (\partial X_i / \partial x_j - \partial X_j / \partial x_i)$.

For example when $n = 3$, Corollary 8 takes the form

Corollary 8′. *Let $\vec{f} \in L^3(\mathbb{T}^3, \mathbb{R}^3)$ with $\operatorname{div} \vec{f} = 0$ and $\int \vec{f} = 0$. Then there exists $\vec{Y} \in (W^{1,3} \cap L^\infty)(\mathbb{T}^3, \mathbb{R}^3)$ such that*

$$\operatorname{curl} \vec{Y} = \vec{f} \quad \text{in } \mathbb{T}^3 \tag{1.20}$$

and

$$\|\nabla \vec{Y}\|_3 + \|\vec{Y}\|_\infty \le C \|\vec{f}\|_3. \tag{1.21}$$

Remark 3. Equation (1.20) is underdetermined. If we supplement it with the "canonical" condition

$$\operatorname{div} \vec{Y} = 0 \quad \text{in } \mathbb{T}^3 \tag{1.22}$$

the system (1.20)–(1.22) admits a unique (mod constants) solution which, in general, does **not** belong to L^∞.

Remark 4. One can ensure that $\vec{Y}$ obtained in Corollary 8 is moreover continuous. Details of this observation appear in [3] in the context of the div-equation (1.15).

We are going to prove in Section 3 that the construction of $\vec{Y}$ in Corollary 8 must also be nonlinear. More precisely:

Proposition 9. *There is no bounded linear operator $K : W^{1,n}(\Lambda^1) \to L^\infty(\Lambda^1)$ such that*

$$d(KX) = dX \quad \forall X \in W^{1,n}(\Lambda^1).$$

Theorem 5 is easily deduced from a considerably more general statement that has a number of other applications (as will be clear later on).

Theorem 10. *Let $S : \bigoplus_{s=1}^r W^{1,n}(\mathbb{T}^n) \to Y$ be a bounded operator into a Banach space Y with* **closed range**. *Assume further that for each $s = 1, \dots, r$ there is an index $i_s \in \{1, \dots, n\}$ such that*

$$\|S\vec{f}\| \le C \max_{1\le s\le r} \max_{i\ne i_s} \|\partial_i f_s\|_n. \tag{1.23}$$

Then, for all $\vec{f} \in \bigoplus_{s=1}^r W^{1,n}$, there is $\vec{g} \in \bigoplus_{s=1}^r (W^{1,n} \cap L^\infty)$ satisfying

$$S\vec{f} = S\vec{g} \tag{1.24}$$

and

$$\|\nabla \vec{g}\|_n + \|\vec{g}\|_\infty \le C\|S\vec{f}\| \le C'\|\nabla \vec{f}\|_n. \tag{1.25}$$

The proof of Theorem 10 depends on Theorem 11 which is the main analytical tool of the paper. It is an approximation result for $W^{1,n}$-functions on $\mathbb{T}^n$.

Theorem 11. *Given $\delta > 0$, there is C_δ such that the following holds. Let $f \in W^{1,n}(\mathbb{T}^n)$. Then there is $F \in W^{1,n} \cap L^\infty$ satisfying*

$$\|F\|_{1,n} + \|F\|_\infty \le C_\delta \|f\|_{1,n}, \tag{1.26}$$

$$\sum_{1\le i\le n-1} \|\partial_i (f - F)\|_n \le \delta \|f\|_{1,n}. \tag{1.27}$$

Theorems 10 and 11 are proved in Section 2. In Section 3 we discuss Theorem 5 and its variant on $\mathbb{R}^n$ (instead of $\mathbb{T}^n$). We will explain the connections between Theorem 4 and the special case $\ell = 1$ of Theorem 5 (i.e., Corollary 8). We will present further applications to Hodge systems. Here are some typical examples in 3-d.

Corollary 12. *Consider the system*

$$\operatorname{curl} \vec{Z} = \vec{f} \quad \text{in } \mathbb{T}^3, \tag{1.28}$$

$$\operatorname{div} \vec{Z} = 0 \quad \text{in } \mathbb{T}^3, \tag{1.29}$$

$$\int_{\mathbb{T}^3} \vec{Z} = 0. \tag{1.30}$$

Then for every $\vec{f} \in L^1 + W^{-1,3/2}$ with $\operatorname{div} \vec{f} = 0$ *and $\int \vec{f} = 0$, the unique solution $\vec{Z}$ of* (1.28)–(1.30) *satisfies*

$$\|\vec{Z}\|_{3/2} \le C\|\vec{f}\|_{L^1 + W^{-1,3/2}}. \tag{1.31}$$

Remark 5. Note that curl and div do not play a symmetric role; a similar conclusion for the system

$$\operatorname{curl} \vec{Z} = 0 \quad \text{in } \mathbb{T}^3, \tag{1.32}$$

$$\operatorname{div} \vec{Z} = g \quad \text{in } \mathbb{T}^3, \tag{1.33}$$

$$\int_{\mathbb{T}^3} \vec{Z} = 0, \tag{1.34}$$

fails even for $g \in L^1$ (with $\int g = 0$). Indeed the solution of (1.32)–(1.34) is given by $\vec{Z} = \operatorname{grad} \Delta^{-1} g$, and $\vec{Z} \notin L^{3/2}$ when $g = \delta + C$.

Standard Hodge theory gives, for any $\vec{f} \in L^3(\mathbb{T}^3, \mathbb{R}^3)$ with $\int \vec{f} = 0$, a unique decomposition

$$\vec{f} = \vec{g} + \operatorname{grad} p$$

with $\vec{g} \in L^3$, $\operatorname{div} \vec{g} = 0$ and $p \in W^{1,3}$. Combining this with Corollary 8′ yields

Corollary 13. *Any $\vec{f} \in L^3(\mathbb{T}^3, \mathbb{R}^3)$ with $\int \vec{f} = 0$ admits a (nonunique) decomposition*

$$\vec{f} = \operatorname{curl} \vec{Y} + \operatorname{grad} p \tag{1.35}$$

with

$$\|\nabla \vec{Y}\|_3 + \|\vec{Y}\|_\infty \le C \|\vec{f}\|_3. \tag{1.36}$$

In Section 3 we will discuss variants and higher dimensional generalizations of Corollaries 12 and 13.

As an application of Theorem 5, we present in Section 4 a proof of the endpoint regularity result for Ginzburg–Landau minimizers due to Bethuel, Orlandi and Smets [1] (see the comments in Section 4 on the background).

In Section 5, further applications of Theorem 10 are given. Firstly we obtain the following generalization of Theorem 2, which answers a question raised in [15, Open Problem 2].

Corollary 14. *Let $\vec{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfy the differential relation*

$$\sum_{i=1}^{n} \partial_i^{(\ell)} f_i = 0 \quad \textit{(in the distributional sense)}$$

with $\ell \ge 1$ an arbitrary integer. Then the solution $\vec{u}$ of (1.2) *satisfies*

$$\|\nabla \vec{u}\|_{n/(n-1)} \le C \|\vec{f}\|_1.$$

Thus Theorem 2 corresponds to the case $\ell = 1$.

Secondly, we establish certain estimates for linear elliptic systems of first order generalizing the classical Korn inequality as extended by M. Strauss [11] to the case $p = 1$ (see also R. Temam [12, Theorem 1.2]):

$$\|\vec{u}\|_{n/(n-1)} \le C \sum_{i,j=1}^{n} \|\partial_i u_j + \partial_j u_i\|_1$$

where $\vec{u} = (u_1, \dots, u_n)$ is a vector field on $\mathbb{R}^n$.

In the Appendix, we show the failure of inequality (1.7) for $\vec{f} \in L^1_{\#}(\mathbb{R}^n, \mathbb{R}^n)$, $n \ge 2$.

Most of the results in Sections 1–3 of this paper were announced in [4].

2. The main tool. Proofs of Theorems 10 and 11

Our primary goal in this section is to prove Theorem 11. But we will first explain how to deduce Theorem 10 from Theorem 11. We will then prove Lemma 1 below which is the main technical tool and which clearly implies Theorem 11. At the end of this section we will discuss some variants involving boundary conditions.

Proof of Theorem 10 assuming Theorem 11. Since S has closed range, there is a constant A such that if $y \in \operatorname{Im} S \subset Y$, then $y = S\vec{f}$ with

$$\|\vec{f}\|_{1,n} \leq A\|y\|. \tag{2.1}$$

Apply now Theorem 11 to each coordinate $f_s \in W^{1,n}(\mathbb{T}^n)$ of $\vec{f} = (f_1, \dots, f_r)$, where we take x_{i_s} as the 'exceptional variable'. This gives $g_s \in W^{1,n} \cap L^\infty$ satisfying

$$\|g_s\|_{1,n} + \|g_s\|_\infty \leq C_\delta \|f_s\|_{1,n} \overset{(2.1)}{\leq} C_\delta A\|y\| \tag{2.2}$$

and

$$\sum_{i \neq i_s} \|\partial_i (f_s - g_s)\|_n \leq \delta \|f_s\|_{1,n} \leq \delta A\|y\|. \tag{2.3}$$

Let $\vec{g} = (g_1, \dots, g_r) \in \bigoplus_{s=1}^r (W^{1,n} \cap L^\infty)$. From (1.23) and (2.3),

$$\|y - S\vec{g}\| = \|S(\vec{f} - \vec{g})\| \leq CA\delta\|y\| \leq \frac{1}{2}\|y\| \tag{2.4}$$

if we let $\delta = 1/2CA$.

Theorem 10 follows by standard iterations as in the classical proof of the Open Mapping Principle.

We now turn to the proof of Theorem 11. Theorem 11 strengthens a similar result obtained in [3] where (1.27) is replaced by the weaker statement

$$\|\partial_i (f - F)\|_n \leq \delta \|f\|_{1,n} \tag{2.5}$$

where $i = 1, \dots, n$ is a single index preliminary chosen (and F dependent on i). The argument in [3] does not seem to give (1.27) in a straightforward way. The proof of Theorem 11 given below is based on a similar approach, but presents additional technical complications.

Theorem 11 is clearly a consequence of

Lemma 1. *If $f \in W^{1,n}(\mathbb{T}^n)$ with $\|f\|_{1,n} < c_n < 1$, then there is F satisfying*

$$\|F\|_\infty \leq C_\delta, \tag{2.6}$$

$$\|F\|_{1,n} \leq C_\delta \|f\|_{1,n}, \tag{2.7}$$

$$\sum_{1 \leq i \leq n-1} \|\partial_i (f - F)\|_n \leq \delta \|f\|_{1,n} + C_\delta \|f\|_{1,n}^2. \tag{2.8}$$

Proof. For the sake of notational simplicity, we take $n = 3$, the general case being completely similar.

Let $f = \sum_{j=0}^{\infty} \Delta_j f$ be a Littlewood–Paley decomposition. We assume $\|f\|_{1,3} < 10^{-3}$. Fix a large integer $R > 0$. Partitioning $\mathbb{Z}_+$ into R cosets $\{R\mathbb{Z}_+ + q\}$, $q = 0, 1, \dots, R-1$, we may assume

$$f = \sum \Delta_j f, \qquad |j_1 - j_2| \geq R \text{ for } j_1 \neq j_2, \tag{2.9}$$

provided the bound (2.6) is multiplied by R.

Define

$$\varphi_j(\theta) = e^{-2^j\|\theta\|} \qquad \text{for } \theta \in \mathbb{T}. \tag{2.10}$$

Letting $\sigma < R$ be another large integer, set

$$\omega_j(x) = \sup_y [|\Delta_j f|(y_1, y_2, y_3)\varphi_j(x_1 - y_1)\varphi_{j-\sigma}(x_2 - y_2)\varphi_{j-\sigma}(x_3 - y_3)]. \tag{2.11}$$

Thus clearly

$$|\Delta_j f| \leq \omega_j \quad \text{and} \quad \|\omega_j\|_\infty = \|\Delta_j f\|_\infty < \frac{1}{100}$$

and

$$|\nabla \omega_j| \leq 2^j \omega_j, \qquad |\overset{(2,3)}{\nabla} \omega_j| \leq 2^{j-\sigma}\omega_j.$$

Let K_j be the trapezoidal Fourier multiplier satisfying

$$\hat{K}_j = 1 \quad \text{on } [-2^j, 2^j], \quad \operatorname{supp} \hat{K}_j \subset [-2^{j+1}, 2^{j+1}], \qquad |K_j| \leq 3F_j$$

with F_j the Fejér kernel. Decompose

$$\Delta_j f = g_j + h_j$$

with

$$g_j = \{\Delta_j f \cdot \chi_{[\omega_j \leq \sum_{k<j} 2^{k-j}\omega_k]}\} * K_j^{\otimes}, \tag{2.12}$$

$$h_j = \{\Delta_j f \cdot \chi_{[\omega_j > \sum_{k<j} 2^{k-j}\omega_k]}\} * K_j^{\otimes}. \tag{2.13}$$

Recall that all indices are restricted to $R \cdot \mathbb{Z}_+$. Here we have denoted $K_j^{\otimes}(x) = K_j(x_1)K_j(x_2)K_j(x_3)$. For notational simplicity, we denote in what follows $K_j^{\otimes}$ (resp. $F_j^{\otimes}$) also by K_j (resp. F_j), with now $|K_j| \leq 27F_j$.

In order to construct F, we treat $\{g_j\}$ and $\{h_j\}$ separately.

Sequence $\{g_j\}$**.** It follows from (2.11), (2.12) that

$$|g_j| \leq 27 \sum_{k<j} 2^{k-j}\omega_k * F_j \equiv G_j < 1.$$

Thus $|g_j| + (1 - G_j) \leq 1$ and the functions

$$\tilde{g}_j = g_j \prod_{j'>j} (1 - G_{j'}) \tag{2.14}$$

satisfy $\sum |\tilde{g}_j| \le 1$. Write

$$\sum (g_j - \tilde{g}_j) = \sum H_{j'} G_{j'},$$

where

$$H_{j'} = g_{j'-1} + (1 - G_{j'-1}) g_{j'-2} + (1 - G_{j'-1})(1 - G_{j'-2}) g_{j'-3} + \cdots$$

satisfies

$$|H_{j'}| \le 1.$$

Since supp $\widehat{H_j G_j} \subset [-2^{j+2}, 2^{j+2}]$, we have

$$\Big\| \sum (g_j - \tilde{g}_j) \Big\|_{W^{1,3}} \lesssim \sum_{s \ge 0} \Big\| \Big(\sum_j |P_{j-s}[\nabla(H_j G_j)]|^2 \Big)^{1/2} \Big\|_3 \tag{2.15}$$

where P_j is a Fourier projection on $|\xi| \sim 2^j$.

Fixing s, decompose $G_j = G_j^{(1)} + G_j^{(2)}$ where

$$G_j^{(1)} = 27 \sum_{j - \bar{s} < k < j} 2^{k-j} (\omega_k * K_j)$$

and $\bar{s}$ depends on s in a way to be specified.

We estimate the contribution of $G_j^{(1)}$ in (2.15):

$$\begin{aligned} \Big\| \Big(\sum_j 4^{j-s} |H_j G_j^{(1)}|^2 \Big)^{1/2} \Big\|_3 &\le 2^{-s} \Big\| \Big(\sum_j 4^j |G_j^{(1)}|^2 \Big)^{1/2} \Big\|_3 \\ &\le C 2^{-s} \sum_{R \le t < \bar{s}} 2^{-t} \Big\| \Big(\sum_j 4^j (\omega_{j-t} * F_j)^2 \Big)^{1/2} \Big\|_3 \\ &\le C 2^{-s} \bar{s} \Big\| \Big(\sum_j 4^j \omega_j^2 \Big)^{1/2} \Big\|_3 . \end{aligned} \tag{2.16}$$

The contribution of $G_j^{(2)}$ in (2.15) is estimated by

$$\begin{aligned} \Big\| \Big(\sum_j |P_{j-s}[\nabla(H_j G_j^{(2)})]|^2 \Big)^{1/2} \Big\|_3 &\le \Big\| \Big(\sum_j |\nabla H_j|^2 |G_j^{(2)}|^2 \Big)^{1/2} \Big\|_3 \\ &\quad + \Big\| \Big(\sum_j |H_j|^2 |\nabla G_j^{(2)}|^2 \Big)^{1/2} \Big\|_3 = (2.17) + (2.18). \end{aligned}$$

Here

$$\begin{aligned} (2.18) &\le \Big\| \Big(\sum_j |\nabla G_j^{(2)}|^2 \Big)^{1/2} \Big\|_3 \le C \sum_{t > \bar{s} \vee R} 2^{-t} \Big\| \Big(\sum_j |\nabla \omega_{j-t}|^2 \Big)^{1/2} \Big\|_3 \\ &\le C 2^{-(\bar{s} \vee R)} \Big\| \Big(\sum_j 4^j \omega_j^2 \Big)^{1/2} \Big\|_3 . \end{aligned} \tag{2.19}$$

To estimate (2.17), write

$$|\nabla H_j| \le \sum_{j'<j}(|\nabla g_{j'}| + |\nabla G_{j'}|) \le \sum_{j'<j}|\nabla g_{j'}| + \sum_{k<j'<j} 2^{k-j'}2^k\tilde{\omega}_k$$
$$\le \sum_{j'<j}|\nabla g_{j'}| + \sum_{j'<j} 2^{j'}\tilde{\omega}_{j'}$$

where $\tilde{\omega}$ denotes the Hardy–Littlewood maximal function of ω. Hence

$$(2.17) \lesssim \sum_{\ell\ge 0}\Big\|\Big(\sum_j |\nabla g_{j-\ell}|^2 |G_j^{(2)}|^2\Big)^{1/2}\Big\|_3 + \sum_{\ell\ge 0}\Big\|\Big(\sum_j 4^{j-\ell}\tilde{\omega}_{j-\ell}^2 |G_j^{(2)}|^2\Big)^{1/2}\Big\|_3$$
$$= (2.20) + (2.21),$$

where

$$(2.20) \le \sum_{\ell\ge 0}\sum_{t>\bar{s}} 2^{-t}\Big\|\Big(\sum_j |\nabla g_{j-\ell}|^2\tilde{\omega}_{j-t}^2\Big)^{1/2}\Big\|_3$$
$$\lesssim \sum_{\ell\ge 0}\sum_{t>\bar{s}} 2^{-t}\Big\|\Big(\sum_j 4^{j-\ell}\tilde{\omega}_{j-\ell}^2\,\tilde{\omega}_{j-t}^2\Big)^{1/2}\Big\|_3.$$

Distinguishing the contributions $\sum_{\ell>t>\bar{s}} = (2.22)$ and $\sum_{t>\bar{s},\,\ell\le t} = (2.23)$, we estimate

$$(2.22) \le (\sup_j \|\tilde{\omega}_j\|_\infty)\sum_{\ell>t>\bar{s}} 2^{-\ell}\Big\|\Big(\sum_j 4^j\tilde{\omega}_j^2\Big)^{1/2}\Big\|_3$$
$$\lesssim (\sup_j \|\Delta_j f\|_\infty)\Big\|\Big(\sum_j 4^j\omega_j^2\Big)^{1/2}\Big\|_3 \cdot \Big(\sum_{\ell>\bar{s}}(\ell-\bar{s})2^{-\ell}\Big)$$
$$\lesssim 2^{-\bar{s}}\|f\|_{1,3}\,\Big\|\Big(\sum_j 4^j\omega_j^2\Big)^{1/2}\Big\|_3 \tag{2.24}$$

and similarly

$$(2.23) \lesssim (1+\bar{s})2^{-\bar{s}}\|f\|_{1,3}\Big\|\Big(\sum_j 4^j\omega_j^2\Big)^{1/2}\Big\|_3. \tag{2.25}$$

Also

$$(2.21) \le \sum_{\ell\ge 0}\sum_{t>\bar{s}} 2^{-t}\Big\|\Big(\sum_j 4^{j-\ell}\tilde{\omega}_{j-\ell}^2\tilde{\omega}_{j-t}^2\Big)^{1/2}\Big\|_3$$
$$\lesssim (1+\bar{s})2^{-\bar{s}}\|f\|_{1,3}\Big\|\Big(\sum_j 4^j\omega_j^2\Big)^{1/2}\Big\|_3^2. \tag{2.26}$$

Hence

$$(2.17) \lesssim (1+\bar{s})2^{-\bar{s}}\|f\|_{1,3}\Big\|\Big(\sum_j 4^j\omega_j^2\Big)^{1/2}\Big\|_3. \tag{2.27}$$

It remains to bound $\|(\sum_j 4^j \omega_j^2)^{1/2}\|_3$. Recalling (2.11), we have

$$\omega_j(x) \lesssim \sup_{r_1,r_2,r_3\in\mathbb{Z}_+} e^{-r_1-2^{-\sigma}(r_2+r_3)}(|\Delta_j f| * F_j)(x_1 + r_1 2^{-j}, x_2 + r_2 2^{-j}, x_3 + r_3 2^{-j}). \tag{2.28}$$

Therefore

$$\begin{aligned}\Big\|\Big(\sum_j 4^j \omega_j^2\Big)^{1/2}\Big\|_3 &\lesssim \sum_{r_1,r_2,r_3} e^{-r_1-2^{-\sigma}(r_2+r_3)} \Big\|\Big(\sum_j 4^j (|\Delta_j f| * F_j)^2(x + \vec{r}\cdot 2^{-j})\Big)^{1/2}\Big\|_3 \\ &\lesssim \sum_{\vec{r}} 2^{-r_1-2^{-\sigma}(r_2+r_3)} \log|\vec{r}| \cdot \|f\|_{1,3} \\ &\lesssim \sigma 4^\sigma \|f\|_{1,3}.\end{aligned} \tag{2.29}$$

Collecting estimates (2.16), (2.19), (2.27), (2.29) implies

$$\begin{aligned}&\Big\|\Big(\sum_j |P_{j-s}[\nabla(H_j G_j)]|^2\Big)^{1/2}\Big\|_3 \\ &\qquad \le C(\bar{s}2^{-s} + 2^{-(\bar{s}\vee R)})\sigma 4^\sigma \|f\|_{1,3} + C(1+\bar{s})2^{-\bar{s}}\sigma 4^\sigma \|f\|_{1,3}^2.\end{aligned} \tag{2.30}$$

For $s \le R$, take $\bar{s} = 0$, i.e. drop $G_j^{(1)}$. For $s > R$, take $\bar{s} = s$. Performing the s-summation in (2.15) using estimate (2.30) gives

$$\Big\|\sum(g_j - \tilde{g}_j)\Big\|_{1,3} \le CR2^{-R}\sigma 4^\sigma \|f\|_{1,3} + R\sigma 4^\sigma \|f\|_{1,3}^2. \tag{2.31}$$

Sequence $\{h_j\}$**.** This is the crucial part of our analysis. Consider further bump functions ψ_j on $\mathbb{T}$ such that

$$\begin{cases} 0 \le \psi_j \le 1, \quad \operatorname{supp}\psi_j \subset [-2^{-j}, 2^{-j}], \\ \psi_j(0) = 1, \quad |\psi_j'| \lesssim 2^j. \end{cases} \tag{2.32}$$

It follows from the definition of h_j in (2.13) that

$$|h_j| \le 27(\omega_j \chi_{[\omega_j > \sum_{k<j} 2^{k-j}\omega_k]}) * F_j \le 27(u_j * F_j) \equiv U_j \tag{2.33}$$

upon defining

$$u_j(x) = \sup[(\omega_j \chi_{[\omega_j > \sum_{k<j} 2^{k-j}\omega_k]})(y)\psi_j(x_1 - y_1)\psi_{j-\sigma}(x_2 - y_2)\psi_{j-\sigma}(x_3 - y_3)]. \tag{2.34}$$

Observe first, from (2.10), (2.11), that

$$\omega_j(x_1 + y_1, x_2 + y_2, x_3 + y_3) \le e^3\omega_j(x) \quad \text{ if } |y_1| < 2^{-j} \text{ and } |y_2|, |y_3| < 2^{\sigma - j}.$$

Therefore

$$u_j \le 25\omega_j \chi_{[\omega_j > 10^{-3}\sum_{k<j} 2^{k-j}\omega_k]}. \tag{2.35}$$

Also, by (2.34),

$$|\nabla u_j| \lesssim 2^j \omega_j \chi_{[\omega_j > 10^{-3}\sum_{k<j} 2^{k-j}\omega_k]}, \tag{2.36}$$

$$|\overset{(2,3)}{\nabla} u_j| \lesssim 2^{j-\sigma} \omega_j \chi_{[\omega_j > 10^{-3}\sum_{k<j} 2^{k-j}\omega_k]}. \tag{2.37}$$

Define then again

$$\tilde{h}_j = h_j \prod_{j'>j} (1 - U_{j'}) \tag{2.38}$$

so that $\sum |\tilde{h}_j| \leq 1$. We have

$$\sum (h_j - \tilde{h}_j) = \sum U_j V_j$$

with

$$V_j = h_{j-1} + (1 - U_{j-1}) h_{j-2} + (1 - U_{j-1})(1 - U_{j-2}) h_{j-3} + \cdots, \qquad |V_j| \leq 1.$$

We estimate

$$\Big\| \overset{(2,3)}{\nabla} \Big[\sum (h_j - \tilde{h}_j) \Big] \Big\|_3 \leq \Big\| \sum |V_j| \, |\overset{(2,3)}{\nabla} U_j| \Big\|_3 + \Big\| \sum |\nabla V_j| U_j \Big\|_3 = (2.39) + (2.40).$$

From (2.33), (2.37) we obtain

$$\begin{aligned} (2.39) &\lesssim 2^{-\sigma} \Big\| \sum_j (2^j \omega_j \chi_{[2^j \omega_j > 10^{-3} \sum_{k<j} 2^k \omega_k]}) * F_j \Big\|_3 \\ &\lesssim 2^{-\sigma} \Big\| \sum_j 2^j \omega_j \chi_{[2^j \omega_j > 10^{-3} \sum_{k<j} 2^k \omega_k]} \Big\|_3 \\ &\lesssim 2^{-\sigma} \|\max_j 2^j \omega_j \|_3 \lesssim 2^{-\sigma} \Big(\sum_j 8^j \|\omega_j\|_3^3 \Big)^{1/3}. \end{aligned} \tag{2.41}$$

From (2.28), we may clearly estimate

$$\|\omega_j\|_3^3 \lesssim \Big(\sum_{r_1, r_2, r_3} e^{-3(r_1 + 2^{-\sigma}(r_2 + r_3))} \Big) \|\Delta_j f\|_3^3 \lesssim 4^\sigma \|\Delta_j f\|_3^3 \tag{2.42}$$

so that

$$(2.39) \lesssim 2^{-\sigma/3} \Big(\sum_j 8^j \|\Delta_j f\|_3^3 \Big)^{1/3} \lesssim 2^{-\sigma/3} \|f\|_{1,3}. \tag{2.43}$$

This estimate is a key point in our approach. It also follows from the preceding that

$$\Big\| \sum |V_j| \, |\nabla U_j| \Big\|_3 \lesssim 4^{\sigma/3} \|f\|_{1,3}. \tag{2.44}$$

To estimate (2.40), note that

$$|\nabla V_j| \leq \sum_{j'<j} (|\nabla h_{j'}| + |\nabla U_{j'}|) \lesssim \sum_{j'<j} 2^{j'} \tilde{\omega}_{j'}.$$

Thus

$$\begin{aligned} (2.40) &\lesssim \sum_{t>0} 2^{-t} \Big\| \sum_j 2^j \tilde{\omega}_{j-t} U_j \Big\|_3 \lesssim (\sup_j \|\omega_j\|_\infty) \Big\| \sum_j 2^j (u_j * F_j) \Big\|_3 \\ &\lesssim (\sup_j \|\Delta_j f\|_\infty) \Big\| \sum_j 2^j \omega_j \chi_{[2^j \omega_j > 10^{-3} \sum_{k<j} 2^k \omega_k]} \Big\|_3 \qquad \text{(by (2.35))} \\ &\lesssim 4^{\sigma/3} \|f\|_{1,3}^2. \end{aligned} \tag{2.45}$$

This completes the analysis. Define

$$F = \sum (\tilde{g}_j + \tilde{h}_j) \tag{2.46}$$

satisfying $\|F\|_\infty \le 2$ and from (2.31), (2.43), (2.45),

$$\begin{aligned} \|\overset{(2,3)}{\nabla}(f - F)\|_3 &\le \Big\|\sum_j (g_j - \tilde{g}_j)\Big\|_{1,3} + \Big\|\overset{(2,3)}{\nabla}\Big[\sum_j (h_j - \tilde{h}_j)\Big]\Big\|_3 \\ &\le C(R2^{-R}\sigma 4^\sigma + 2^{-\sigma/3})\|f\|_{1,3} + (R\sigma 4^\sigma + 4^{\sigma/3})\|f\|_{1,3}^2 \end{aligned} \tag{2.47}$$

and from (2.31), (2.44), (2.45),

$$\|F\|_{1,3} \le \|f\|_{1,3} + C(R2^{-R}\sigma 4^\sigma + 4^{\sigma/3})\|f\|_{1,3} + (R\sigma 4^\sigma + 4^{\sigma/3})\|f\|_{1,3}^2. \tag{2.48}$$

Recall that since we restricted j to a progression $R\mathbb{Z}_+ + q$ $(0 \le q < R)$, these bounds need to be multiplied by R. Taking $\sigma = R/4$, this implies the existence of a function F satisfying

$$\|F\|_\infty \le 2R, \tag{2.49}$$

$$\|F\|_{1,3} \le 2^R\|f\|_{1,3} + 2^R\|f\|_{1,3}^2 \le 2^{R+1}\|f\|_{1,3}, \tag{2.50}$$

$$\|\overset{(2,3)}{\nabla}(f - F)\|_3 \le 2^{-R/13}\|f\|_{1,3} + 2^R\|f\|_{1,3}^2. \tag{2.51}$$

This proves Lemma 1 with $\delta = 2^{-R/13}$, $C_\delta = 2^{R+1}$.

Remark 6. Here is a variant of the previous Theorems 10 and 11.

Corollary 15. *The statements of Theorem 11 and hence 10 remain valid if $\mathbb{T}^n$ is replaced by a cube $Q = (0, a)^n$ and $W^{1,n}(\mathbb{T}^n)$ replaced by $W^{1,n}(Q)$ or $W_0^{1,n}(Q)$. They also remain valid if $W^{1,n}(\mathbb{T}^n)$ is replaced by $W^{1,n}(\mathbb{R}^n)$.*

Proof. We start with $W^{1,n}(Q)$. If $f \in W^{1,n}(Q)$, it can be extended to a function $\tilde{f} \in W_0^{1,n}(\tilde{Q})$ where $\tilde{Q} \supset Q$ is a larger cube. This $\tilde{f}$ may be viewed as a periodic function to which previous results apply and the conclusion follows by restriction to Q. Next let $f \in W_0^{1,n}(Q)$, $Q = (0, 1)^n$. Extend f to $\mathbb{R}^n$ by the usual anti-symmetrization and periodization. Thus f may be seen as a restriction of a function $\tilde{f}$ which is periodic and odd in each variable. Let $\tilde{F}$ be the associated function given by Theorem 11. Assume for simplicity that $n = 2$ (the general case is similar). Set

$$F(x_1, x_2) = \tfrac{1}{4}(\tilde{F}(x_1, x_2) - \tilde{F}(x_1, -x_2) - \tilde{F}(-x_1, x_2) + \tilde{F}(-x_1, -x_2)).$$

Then $F|_Q$ is in $W_0^{1,n}(Q)$ and has all the required properties.

3. Proofs of Theorems 4, 5 and Proposition 9. Applications to div-curl and Hodge systems

We start with

Proof of Theorem 5. We apply Theorem 10 to $S = d : W^{1,n}(\Lambda^\ell) \to L^n(\Lambda^{\ell+1})$, $0 < \ell \le n - 1$. Since $\ell > 0$ condition (1.23) is satisfied. For example when $n = 3$ and $\ell = 1$ we have

$$\|SX\|_3 \le \sum_{s=1}^{3} \sum_{i \neq s} \|\partial_i X_s\|_3 \quad \forall X \in W^{1,n}(\Lambda^1).$$

On the other hand, S has closed range in $L^n(\Lambda^{\ell+1})$. More precisely,

- if $0 < \ell \le n-2$, then $R(S) = \{\omega \in L^n(\Lambda^{\ell+1}) \mid d\omega = 0 \text{ and } \int_{\mathbb{T}^n} \omega = 0\}$,
- if $\ell = n-1$, then $R(S) = \{\omega \in L^n(\mathbb{T}^n) \mid \int_{\mathbb{T}^n} \omega = 0\}$.

We may also state variants of Theorem 5 when $\mathbb{T}^n$ is replaced by $M = (0,1)^n$ or $\mathbb{R}^n$.

Theorem 5′. *Assume $M = (0,1)^n$ or $M = \mathbb{R}^n$ with $n \ge 2$, and $1 \le \ell \le n-1$. Then*

$$d[W^{1,n}(\Lambda^\ell M)] = d[(W^{1,n} \cap L^\infty)(\Lambda^\ell M)].$$

More precisely, given any $X \in W^{1,n}(\Lambda^\ell M)$ there exists some $Y \in (W^{1,n} \cap L^\infty)(\Lambda^\ell M)$ such that

$$dY = dX \tag{3.1}$$

and

$$\|\nabla Y\|_n + \|Y\|_\infty \le C\|dX\|_n. \tag{3.2}$$

Proof. Apply the variant of Theorem 10 stated as Corollary 15. Once more S has closed range:

- if $0 < \ell \le n-2$, then $R(S) = \{\omega \in L^n(\Lambda^{\ell+1} M) \mid d\omega = 0\}$,
- if $\ell = n-1$, then $R(S) = L^n(M)$.

Theorem 5″. *Assume $M = (0,1)^n$. Then for $n \ge 2$ and $1 \le \ell \le n-1$,*

$$d[W_0^{1,n}(\Lambda^\ell M)] = d[(W_0^{1,n} \cap L^\infty)(\Lambda^\ell M)].$$

More precisely, given any $X \in W_0^{1,n}(\Lambda^\ell M)$ there exists some $Y \in (W_0^{1,n} \cap L^\infty)(\Lambda^\ell M)$ such that

$$dY = dX \tag{3.1}$$

and

$$\|\nabla Y\|_n + \|Y\|_\infty \le C\|dX\|_n. \tag{3.2}$$

Proof. Following the same argument as above it remains to verify that $S = d : W_0^{1,n}(\Lambda^\ell) \to L^n(\Lambda^\ell)$ has closed range. It is well known that $d[W_T^{1,p}(\Lambda^\ell)]$ is closed in $L^p(\Lambda^{\ell+1})$ for any $1 < p < \infty$, where $W_T^{1,p}(\Lambda^\ell)$ denotes the ℓ-forms with vanishing tangential part on ∂M (see [6]). To complete the proof it suffices to establish

Lemma 2. *Given any $1 < p < \infty$ and $1 \le \ell \le n-1$, we have*

$$d[W_T^{1,p}(\Lambda^\ell M)] = d[W_0^{1,p}(\Lambda^\ell M)].$$

Proof. Given any $\omega \in W^{1,p}_T(\Lambda^\ell M)$, we will construct some $\eta \in W^{2,p}(\Lambda^{\ell-1}M)$ such that

$$\eta = 0 \quad \text{and} \quad \omega + d\eta = 0 \quad \text{on } \partial M.$$

We start with the case $\ell = 1$ which is quite transparent. We are given $\omega \in W^{1,p}_T(\Lambda^1 M)$, i.e., $\omega = \vec{X} \in W^{1,p}(M, \mathbb{R}^n)$ is a vector field such that its tangential component vanishes on ∂M. We look for a function $\eta \in W^{2,p}(M, \mathbb{R}) = W^{2,p}(\Lambda^0 M)$ such that

$$\eta = 0 \quad \text{and} \quad \vec{X} \cdot \vec{\nu} + \frac{\partial \eta}{\partial \nu} = 0 \quad \text{on } \partial M,$$

where $\vec{\nu}$ denotes the normal to ∂M. The existence of η follows from a general result of Lions and Magenes [9] asserting that the map $\psi \mapsto (\psi|_{\partial M}, \frac{\partial \psi}{\partial \nu}|_{\partial M})$ maps $W^{2,p}(M)$ onto $W^{2-1/p,p}(\partial M) \times W^{1-1/p,p}(\partial M)$. Observe that $\vec{X} \cdot \vec{\nu} \in W^{1-1/p,p}(\partial M)$. (The additional difficulties arising from the corners of M can be handled as in [3].)

We now assume that $\ell \geq 2$. Since $\omega_T = 0$ (by assumption) and $(d\eta)_T = 0$ (because we look for $\eta = 0$ on ∂M), we have

$$\omega_T + (d\eta)_T = 0.$$

Therefore it suffices to achieve

$$\eta = 0 \quad \text{and} \quad (d\eta)_N = -\omega_N \quad \text{on } \partial M.$$

In local coordinates near a point of ∂M we assume that x_n is the normal direction and set $y = x_n$. Write

$$\omega_N = \sum_{1 \leq i_1 < \cdots < i_{\ell-1} < n} \omega_{i_1,\ldots,i_{\ell-1}}\, dx_{i_1} \wedge \cdots \wedge dx_{i_{\ell-1}} \wedge dy$$

and

$$\eta = \sum_{1 \leq i_1 < \cdots < i_{\ell-1} < n} \eta_{i_1,\ldots,i_{\ell-1}}\, dx_{i_1} \wedge \cdots \wedge dx_{i_{\ell-1}} + \sum_{1 \leq j_1 < \cdots < j_{\ell-2} < n} \eta_{j_1,\ldots,j_{\ell-2},n}\, dx_{j_1} \wedge \cdots \wedge dx_{j_{\ell-2}} \wedge dy.$$

Using the fact that $\eta = 0$ on ∂M, we have, on ∂M,

$$(d\eta)_N = \sum_{1 \leq i_1 < \cdots < i_{\ell-1} < n} \frac{\partial \eta_{i_1,\ldots,i_{\ell-1}}}{\partial y}\, dx_{i_1} \wedge \cdots \wedge dx_{i_{\ell-1}} \wedge dy.$$

We are thus led to find η satisfying $\eta = 0$ on ∂M and

$$\frac{\partial \eta_{i_1,\ldots,i_{\ell-1}}}{\partial y} = -\omega_{i_1,\ldots,i_{\ell-1}} \quad \text{on } \partial M.$$

The existence of η follows again from the result of Lions and Magenes [9].

Remark 7. With the help of Theorem 5″ we may now state a slightly sharper version of Theorem 1′:

Theorem 1″. *For every* $\vec{X} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} \vec{X} \cdot \vec{f} \right| \leq C \|\vec{f}\|_1 \|\operatorname{curl} \vec{X}\|_n \quad \text{for all } \vec{f} \in L^1_\#(\mathbb{R}^n, \mathbb{R}^n),$$

where $\operatorname{curl} \vec{X} = (\partial X_i / \partial x_j - \partial X_j / \partial x_i)$.

Proof. Let M be a large cube containing $\operatorname{supp} \vec{X}$. We may view $\vec{X}$ as an element of $W_0^{1,n}(\Lambda^1 M)$. By Theorem 5″ there exists $Y \in (W_0^{1,n} \cap L^\infty)(\Lambda^1 M)$ such that $dY = dX$ and

$$\|Y\|_\infty \leq C \|dX\|_n = C \|\operatorname{curl} \vec{X}\|_n.$$

Hence $\vec{Y} - \vec{X} = \operatorname{grad} p$ for some $p \in (W^{2,n} \cap W^{1,\infty})(M)$. Moreover $\operatorname{grad} p = 0$ on ∂M; thus p is constant on ∂M and we may assume that $p = 0$ on ∂M. We have

$$\int_M \vec{X} \cdot \vec{f} = \int_M (\vec{Y} + \operatorname{grad} p) \cdot \vec{f} = \int_M \vec{Y} \cdot \vec{f},$$

since $\operatorname{div} f = 0$ and $p = 0$ on ∂M. Hence

$$\left| \int \vec{X} \cdot \vec{f} \right| = \left| \int \vec{Y} \cdot \vec{f} \right| \leq \|\vec{f}\|_1 \|\vec{Y}\|_\infty \leq C \|\vec{f}\|_1 \|\operatorname{curl} \vec{X}\|_n.$$

We now turn to

Proof of Theorem 4. Let $\vec{f}$ be given by (1.8). In view of standard elliptic estimates it suffices to prove that $\vec{f} \in W^{-1,n/(n-1)}(\mathbb{R}^n, \mathbb{R}^n)$ with

$$\|\vec{f}\|_{-1,n/(n-1)} \leq C \left\{ \|\vec{f}_0\|_1 + \sum \|\vec{f}_i\|_{n/(n-1)} \right\}. \tag{3.3}$$

Let thus $\vec{X} \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\vec{X}\|_{1,n} \leq 1$. We may assume that $\vec{X}$ is smooth and has compact support, say $\operatorname{supp} X \subset Q$.

According to Theorem 5″ there is some $\vec{Y} \in (W^{1,n} \cap L^\infty)(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{supp} Y \subset Q$ and $\|\vec{Y}\|_{1,n} + \|\vec{Y}\|_\infty \leq C$, such that $dY = dX$. Hence $\vec{X} - \vec{Y} = \operatorname{grad} p$ and since $\operatorname{div} \vec{f} = 0$,

$$\begin{aligned} |\langle \vec{X}, \vec{f} \rangle| = |\langle \vec{Y}, f \rangle| &\leq |\langle \vec{Y}, \vec{f}_0 \rangle| + \sum \left| \left\langle \frac{\partial \vec{Y}}{\partial x_i}, \vec{f}_i \right\rangle \right| \\ &\leq \|\vec{Y}\|_\infty \|\vec{f}_0\|_1 + \|\vec{Y}\|_{1,n} \sum \|\vec{f}_i\|_{n/(n-1)} \leq C \left\{ \|\vec{f}_0\|_1 + \sum_i \|\vec{f}_i\|_{n/(n-1)} \right\}, \end{aligned}$$

which is the desired estimate (3.3).

Remark 8. In fact, Theorem 4 and Theorem 5′ (with $\ell = 1$ and $M = \mathbb{R}^n$) are equivalent. Here is a proof of the implication Theorem 4 ⇒ Theorem 5′. Fix $\vec{X} \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$;

we have to find $\vec{Y}$ satisfying (3.1) and (3.2). We are going to define a linear functional on $(L^1+W^{-1,n/(n-1)})_\#$. Given $\vec{f} \in (L^1+W^{-1,n/(n-1)})_\#$, let $\vec{u}$ be the solution of (1.2) given by Theorem 4. Set

$$T(\vec{f}) = \int \sum \frac{\partial \vec{X}}{\partial x_i} \frac{\partial \vec{u}}{\partial x_i}.$$

By Theorem 4 we have

$$|T(\vec{f})| \le C\|\vec{X}\|_{1,n}\|\vec{f}\|_{L^1+W^{-1,n/(n-1)}}.$$

Applying Hahn–Banach we may extend T to a continuous linear functional $\tilde{T}$ on all of $L^1+W^{-1,n/(n-1)}$, with $\|\tilde{T}\| \le C\|\vec{X}\|_{1,n}$. Hence there is some $\vec{Y} \in W^{1,n} \cap L^\infty$ satisfying (3.2) and moreover

$$\int \vec{Y} \cdot \vec{f} = T(\vec{f}) = \int \sum \frac{\partial \vec{X}}{\partial x_i} \frac{\partial u}{\partial x_i} = \int \vec{X} \cdot \vec{f} \quad \forall \vec{f} \in (L^1 + W^{-1,n/(n-1)})_\#.$$

Thus (3.1) holds.

Similarly, the weaker version, Theorem 2, of Theorem 4 corresponds to a weaker form of Theorem 5′ asserting only that given $\vec{X} \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$, there exists some $\vec{Y} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\vec{Y} - \vec{X} = \operatorname{grad} p$ and $\|\vec{Y}\|_\infty \le C\|\vec{X}\|_{1,n}$. Hence this weaker statement admits an elementary proof à la Van Schaftingen [14].

The above construction of $\vec{Y}$ (starting from $\vec{X}$) relies on Hahn–Banach and need not be linear in $\vec{X}$. In fact, we now prove Proposition 9 which asserts that the construction **must** be nonlinear. For simplicity we return to the case $M = \mathbb{T}^n$.

Proof of Proposition 9. Assume, by contradiction, that there exists a bounded linear operator $K : W^{1,n}(\Lambda^1\mathbb{T}^n) \to L^\infty(\Lambda^1\mathbb{T}^n)$ such that

$$d(KX) = dX \quad \forall X \in W^{1,n}(\Lambda^1).$$

When $n = 2$ this is impossible from the div-case proved in [3] and recalled as Proposition 7. Assume $n \ge 3$. We are going to construct a bounded linear operator

$$\tilde{K} : W^{1,n}(\Lambda^{n-1}) \to L^\infty(\Lambda^{n-1}) \tag{3.4}$$

such that

$$d(\tilde{K}\omega) = d\omega \quad \forall \omega \in W^{1,n}(\Lambda^{n-1}) \tag{3.5}$$

and this again contradicts the div-case (Proposition 7). Given $\omega \in W^{1,n}(\Lambda^{n-1})$ write

$$\omega = \alpha_1 \widehat{dx_1} \wedge dx_2 \wedge \cdots \wedge dx_n + \alpha_2\, dx_1 \wedge \widehat{dx_2} \wedge \cdots \wedge dx_n + \cdots + \alpha_n\, dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_n}.$$

Applying the operator K to the 1-form $X = \alpha_j\, dx_i - \alpha_i\, dx_j$, $i \ne j$, and writing $d(KX) = dX$ we obtain in particular some functions $\beta^{ij}, \gamma^{ij} \in L^\infty(\mathbb{T}^n)$ such that

$$\frac{\partial}{\partial x_i}(\alpha_i - \beta^{ij}) + \frac{\partial}{\partial x_j}(\alpha_j - \gamma^{ij}) = 0. \tag{3.6}$$

Moreover β^{ij} and γ^{ij} depend linearly on α_i and α_j; thus they define bounded linear operators from $W^{1,n}$ into L^∞. Adding all the equations (3.6) for $i \neq j$ we obtain

$$(n-1)\,\mathrm{div}(\vec{\alpha} - \vec{\sigma}) = 0$$

for some $\vec{\sigma} = \tilde{K}(\vec{\alpha}) = \tilde{K}(\omega)$ where $\tilde{K}$ is a bounded linear operator satisfying (3.4) and (3.5). Impossible by Proposition 7.

We now turn to div-curl and Hodge systems. We start with

Proof of Corollary 12. Using the formula

$$\text{curl curl} = -\Delta + \text{grad div} \tag{3.7}$$

we see that the solution $\vec{Z}$ of (1.28)–(1.30) is given by

$$\vec{Z} = \mathrm{curl}\,(-\Delta)^{-1}\vec{f}$$

where $(-\Delta)^{-1}$ is the inverse of $-\Delta$ on $\mathbb{T}^3$. We may then apply Theorem 4 (or rather its variant on $\mathbb{T}^3$ instead of $\mathbb{R}^3$) to conclude that $\vec{Z} \in L^{3/2}$ with the corresponding estimate.

In connection with Corollary 12, let us mention an open problem. Consider the div-curl system (1.28)–(1.30) with $\vec{f} \in L^1(\mathbb{T}^3)$, div $\vec{f} = 0$ and $\int \vec{f} = 0$. We know that the solution $\vec{Z}$ belongs to $L^{3/2}$ and that $\vec{Z}$ does **not** belong to $W^{1,1}$ (see Remark 1 and the Appendix).

Open Problem 1. Is it true that $\vec{Z}$ belongs to the Lorentz space $L(3/2, 1)$? In particular, is it true that $\vec{Z}(x)/|x-a| \in L^1$ for every $a \in \mathbb{T}^3$?

When $M = \mathbb{T}^n$ or $M = \mathbb{R}^n$ recall the classical Hodge decomposition. Any $\omega \in L^n(\Lambda^\ell M)$, $1 \le \ell \le n-1$, (with $\int \omega = 0$ if $M = \mathbb{T}^n$) may be written as

$$\omega = d\alpha + d^*\beta \tag{3.8}$$

with $\alpha \in W^{1,n}(\Lambda^{\ell-1}M)$ and $\beta \in W^{1,n}(\Lambda^{\ell+1}M)$. Here $d^* = (-1)^{n\ell+1} {*d*}$ where $*$ denotes the Hodge $*$-operator $\Lambda^\ell M \to \Lambda^{n-\ell}M$. In addition one can choose α and β satisfying the bounds

$$\|\alpha\|_{1,n} + \|\beta\|_{1,n} \le C\|\omega\|_n.$$

Combining this with Theorem 5 (when $M = \mathbb{T}^n$) or Theorem 5$'$ (when $M = \mathbb{R}^n$) we may improve the conclusion.

Corollary 16. *Assume $n \ge 3$ and $1 \le \ell \le n-2$. Then any $\omega \in L^n(\Lambda^\ell M)$ (with $\int \omega = 0$ when $M = \mathbb{T}^n$) may be written as*

$$\omega = d\alpha + d^*\beta$$

with $\alpha \in W^{1,n}(\Lambda^{\ell-1}M)$, $\beta \in (W^{1,n} \cap L^\infty)(\Lambda^{\ell+1}M)$, and

$$\|\alpha\|_{1,n} + \|\beta\|_{1,n} + \|\beta\|_\infty \le C\|\omega\|_n. \tag{3.9}$$

If $n \geq 4$ and $2 \leq \ell \leq n-2$, then any $\omega \in L^n(\Lambda^\ell M)$ (with $\int \omega = 0$ when $M = \mathbb{T}^n$) may be written as

$$\omega = d\alpha + d^*\beta$$

with $\alpha \in (W^{1,n} \cap L^\infty)(\Lambda^{\ell-1} M)$ and $\beta \in (W^{1,n} \cap L^\infty)(\Lambda^{\ell+1} M)$ and

$$\|\alpha\|_{1,n} + \|\alpha\|_\infty + \|\beta\|_{1,n} + \|\beta\|_\infty \leq C\|\omega\|_n. \tag{3.10}$$

In order to apply Theorem 5 to the β-term, we need thus to assume that $n - \ell - 1 > 0$, i.e., $\ell \leq n-2$. Similarly for the α-term we need $\ell - 1 > 0$, i.e., $\ell \geq 2$.

Corollary 17. *Assume $n \geq 4$ and $2 \leq \ell \leq n-2$. Then for every $X \in W^{1,1}(\Lambda^\ell \mathbb{R}^n)$ we have*

$$\|X\|_{n/(n-1)} \leq C\{\|dX\|_{L^1 + W^{-1,n/(n-1)}} + \|d^*X\|_{L^1 + W^{-1,n/(n-1)}}\} \tag{3.11}$$

and in particular

$$\|X\|_{n/(n-1)} \leq C(\|dX\|_1 + \|d^*X\|_1). \tag{3.12}$$

Proof. If $\omega \in L^n(\Lambda^\ell \mathbb{R}^n)$ we may write $\omega = d\alpha + d^*\beta$ with α, β satisfying (3.10). Then

$$|\langle X, \omega\rangle| = |\langle d^*X, \alpha\rangle + \langle dX, \beta\rangle| \leq C\{\|d^*X\|_{L^1 + W^{-1,n/(n-1)}} + \|dX\|_{L^1 + W^{-1,n/(n-1)}}\}\|\omega\|_n.$$

Remark 9. The weaker assertion (3.12) of Corollary 17 was obtained independently by Lanzani and Stein [7] with an elementary approach in the spirit of [14].

Remark 10. Notice that Corollary 17 does not imply anything for $n = 3$. Indeed (3.12) does not hold in the div-curl setting as was already pointed out in Remark 5.

Next, we present another example on $M = (0,1)^n$ involving a boundary condition. It will be used in the context of Ginzburg–Landau minimizers (as discussed in the next section).

Corollary 18. *Assume $n \geq 3$ and $M = (0,1)^n$. Then any $X \in L^n(\Lambda^1 M) = L^n(M, \mathbb{R}^n)$ may be written as*

$$X = d\phi + d^*k$$

for some $\phi \in W_0^{1,n}(\Lambda^0 M) = W_0^{1,n}(M, \mathbb{R})$ and some $k \in (W^{1,n} \cap L^\infty)(\Lambda^2 M)$ satisfying

$$\|\phi\|_{1,n} + \|k\|_{1,n} + \|k\|_\infty \leq C\|X\|_n.$$

Proof. By standard Hodge theory we may write $X = d\phi + d^*\beta$ for some $\phi \in W_0^{1,n}(\Lambda^0 M)$ and some $\beta \in W^{1,n}(\Lambda^2 M)$ with control of norms. The additional information comes from Theorem 5′ which applies since $n - 2 > 0$.

We also have a ‘dual’ form:

Corollary 18′. *Assume $n \geq 3$ and $M = (0,1)^n$. Then any $X \in L^n(\Lambda^1 M) = L^n(X, \mathbb{R}^n)$ may be written as*

$$X = d\phi + d^*k$$

for some $\phi \in W^{1,n}(\Lambda^0 M) = W^{1,n}(M, \mathbb{R})$ and some $k \in (W_0^{1,n} \cap L^\infty)(\Lambda^2 M)$ satisfying

$$\|\phi\|_{1,n} + \|k\|_{1,n} + \|k\|_\infty \leq C\|X\|_n.$$

Proof. By standard Hodge theory we may write $X = d\phi + d^*\beta$ for some $\phi \in W^{1,n}(\Lambda^0 M)$ and some $\beta \in W_N^{1,n}(\Lambda^2 M)$. Then $*\beta \in W_T^{1,n}(\Lambda^{n-2}M)$ and we may apply Lemma 2, together with Theorem 5$''$ for $\ell = n-2$, to conclude that $d^*\beta = d^*k$ for some $k \in (W_0^{1,n} \cap L^\infty)(\Lambda^2 M)$.

Remark 11. Instead of the limiting Sobolev space $W^{1,n}$, one can consider similar questions in the fractional Sobolev space $W^{s,p}$ with $sp = n,\ p > 1$. For example, as an analog of Theorem 1 we have

$$\left| \int_\Gamma Y\vec{t} \right| \leq C|\Gamma|\, \|Y\|_{s,p}$$

and hence

$$\left| \int_{\mathbb{R}^n} Y\vec{f} \right| \leq C\|\vec{f}\|_1 \|Y\|_{s,p}$$

for every $\vec{f} \in L^1_\#(\mathbb{R}^n, \mathbb{R}^n)$ and $Y \in C_0^\infty(\mathbb{R}^n)$. This can be proved by the same argument as in [4], [5], or by the argument due to Van Schaftingen [13], [14].

Turning to differential forms, this shows that if $X \in W^{s,p}(\Lambda^1)$, there exists some $Y \in L^\infty(\Lambda^1)$ such that $dX = dY$. Here we do not claim that this Y is in $W^{s,p}$. In fact we leave it as

Open Problem 2. Let $0 < s < n$, $s \neq 1$, $p = n/s$ and $1 \leq \ell \leq n-1$. Is it true that given $X \in W^{s,p}(\Lambda^\ell)$, there is $Y \in (W^{s,p} \cap L^\infty)(\Lambda^\ell)$ such that $dX = dY$? The question can be asked in particular when s is an integer, for example $s = 2$, and $p = n/2$.

Remark 12. It is easily verified that Theorem 1 fails if in (1.1) we replace $\|\nabla Y\|_n$ by $\|Y\|_{\mathrm{BMO}}$. In fact, a natural seminorm on $C_0^\infty(\mathbb{R}^n)$ can be defined considering

$$\langle Y \rangle = \sup \frac{1}{|\Gamma|} \left| \int_\Gamma Y\vec{t} \right|$$

with the sup taken over all closed rectifiable curves Γ. It has been shown by Van Schaftingen [16] that

$$\|Y\|_{\mathrm{BMO}} \leq C\langle Y \rangle$$

(and the corresponding embedding is strict).

Finally, we express some of the above results as simple but general estimates for ℓ-forms.

Let $M \subset \mathbb{R}^n$ be a smooth, compact, orientable, ℓ-dimensional manifold without boundary, $1 \leq \ell \leq n-1$. Let $\omega \in C_0^\infty(\Lambda^\ell \mathbb{R}^n)$. Recall that the quantity $\int_M \omega$ is well defined.

Corollary 19. *We have*

$$\left| \int_M \omega \right| \leq C|M|\, \|d\omega\|_n$$

where C depends only on n.

Proof. Let Q be a cube containing $M \cup \operatorname{supp} \omega$. By Theorem 5″ there exists some $\tilde{\omega} \in (W_0^{1,n} \cap L^\infty)(\Lambda^\ell Q)$ such that

$$d\tilde{\omega} = d\omega \quad \text{and} \quad \|\tilde{\omega}\|_\infty \leq C\|d\omega\|_n,$$

where C is independent of the size of Q (by scale invariance). Then

$$\int_M \omega = \int_M \tilde{\omega},$$

since $\omega - \tilde{\omega} = d\xi$ for some $\xi \in W^{2,n}(\Lambda^{\ell+1} Q)$ and $\int_M d\xi = 0$. Hence

$$\left| \int_M \omega \right| \leq |M| \, \|\tilde{\omega}\|_\infty \leq C|M| \, \|d\omega\|_n.$$

Next, we work on $M = \mathbb{T}^n$ (for simplicity). Let $X, \omega \in C^\infty(\Lambda^\ell M)$ with $1 \leq \ell \leq n-1$. Recall the standard definition

$$\langle X, \omega \rangle = \int_M X \wedge (*\omega).$$

Corollary 20. *Assume* $\int_M X = 0$. *Then*

$$|\langle X, \omega \rangle| \leq C\{\|\omega\|_1 \|dX\|_n + \|d^*\omega\|_{-2,n/(n-1)} \|\nabla X\|_n\}, \tag{3.13}$$

with C *depending only on* n.

Proof. By Theorem 5 there exists some $Y \in (W^{1,n} \cap L^\infty)(\Lambda^\ell M)$ such that

$$dY = dX, \tag{3.14}$$

$$\int_M Y = 0, \tag{3.15}$$

$$\|\nabla Y\|_n + \|Y\|_\infty \leq C\|dX\|_n. \tag{3.16}$$

Since $d(Y - X) = 0$ and $\int_M (Y - X) = 0$ we may solve the elliptic system

$$d\xi = X - Y, \qquad d^*\xi = 0$$

with the estimate

$$\|\xi\|_{2,n} \leq C\|d^*(X - Y)\|_n \leq C\|\nabla X\|_n. \tag{3.17}$$

Then

$$\langle X, \omega \rangle = \langle X - Y, \omega \rangle + \langle Y, \omega \rangle = \langle d\xi, \omega \rangle + \langle Y, \omega \rangle = \langle \xi, d^*\omega \rangle + \langle Y, \omega \rangle.$$

Therefore

$$|\langle X, \omega \rangle| \leq \|\xi\|_{2,n} \|d^*\omega\|_{-2,n/(n-1)} + \|Y\|_\infty \|\omega\|_1.$$

Combining this with (3.16) and (3.17) yields the desired estimate (3.13).

Remark 13. One cannot replace $\|\nabla X\|_n$ by $\|d^* X\|_n$ in (3.13). Indeed, if we could, we would infer that

$$|\langle X, \omega\rangle| \le C\|\omega\|_1 \|dX\|_n$$

whenever $d^* X = 0$. Consequently,

$$\|X\|_\infty \le C\|dX\|_n$$

for every X with $d^* X = 0$. But such an estimate fails: it suffices to find a $\xi \in W^{2,n}$ with $\xi \notin W^{1,\infty}$ and to choose $X = d^*\xi$.

Similarly, there is failure of the estimate

$$|\langle X, \omega\rangle| \le C\{\|d\omega\|_{-1,1}\|dX\|_n + \|d^*\omega\|_{-2,n/(n-1)}\|\nabla X\|_n\}. \tag{3.18}$$

Indeed, (3.18) would imply

$$\|dY\|_{n/(n-1)} \le C\|\Delta Y\|_1 \tag{3.19}$$

for every $Y \in C^\infty(\Lambda^\ell M)$. To see this let $h \in L^n(\Lambda^{\ell+1})$ with $\int h = 0$. Using Hodge write

$$h = d\alpha + d^*\beta$$

with $\|d\alpha\|_n \le C\|h\|_n$. Then

$$\langle h, dY\rangle = \langle d\alpha, dY\rangle = \langle \alpha, d^* dY\rangle.$$

Applying (3.18) to $X = \alpha$ and $\omega = d^* dY$ would give

$$\begin{aligned}|\langle h, dY\rangle| &\le C\|dd^* dY\|_{-1,1}\|d\alpha\|_n \le C\|dd^* dY\|_{-1,1}\|h\|_n \\ &= C\|\Delta dY\|_{-1,1}\|h\|_n \le C\|\Delta Y\|_1\|h\|_n.\end{aligned}$$

Thus (3.19) would hold. From (3.19) we would deduce that $d\Delta^{-1}$ is a bounded operator from $\{f \in L^1 \mid \int f = 0\}$ into $L^{n/(n-1)}$ and by duality that $d^*\Delta^{-1}$ is bounded from $\{f \in L^n \mid \int f = 0\}$ into L^∞. But we already observed that this is not true.

4. Consequences for Ginzburg–Landau minimizers

Let $M \subset \mathbb{R}^n$ $(n \ge 3)$ be a domain, say a cube for simplicity. For $\varepsilon > 0$, the Ginzburg–Landau functional $E_\varepsilon(u)$ is defined by

$$E_\varepsilon(u) = \frac{1}{2}\int_M |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_M (|u|^2 - 1)^2. \tag{4.1}$$

Theorem 21. *Let $g \in H^{1/2}(\partial M, S^1)$. Then*

(a) *If u_ε is a minimizer of E_ε subject to the Dirichlet boundary condition $u_\varepsilon|_{\partial M} = g$, then*

$$\|\nabla u_\varepsilon\|_{n/(n-1)} \le C(g) \quad \text{as } \varepsilon \to 0. \tag{4.2}$$

(b) *In particular, any weak limit point u_* of $\{u_\varepsilon\}$ belongs to $W^{1,n/(n-1)}(M)$.*

It was shown in [5] that $\|\nabla u_\varepsilon\|_p \le C_p(g)$ for $p < n/(n-1)$ (results for special g displaying only finitely many point singularities of a certain type were obtained earlier by Lin and Rivière [8]).

For $n = 3$, Theorem 21(b) was first obtained in [5]. Theorem 21(a) is due to Bethuel, Orlandi and Smets [2]. Below we present a proof, based on Corollary 18, that is conceptually particularly pleasing.

Consider first a larger cube Q such that $M \subset \mathring{Q}$. Using the fact that $u_\varepsilon|_{\partial M} = g \in H^{1/2}(\partial M, S^1)$, we may construct an extension $\tilde{u}$ of u_ε to Q satisfying

$$|\tilde{u}| \le 1, \tag{4.3}$$

$$E_\varepsilon(\tilde{u}; Q) \le C \log(1/\varepsilon), \tag{4.4}$$

$$\|\tilde{u}\|_{W^{1,p}(Q\setminus M)} \le C_p(g) \quad \text{for all } p < 2 \tag{4.5}$$

(see [5, Lemma 30]).

Next, we apply to the function $\tilde{u}$ on Q the following approximation result due to Bethuel, Orlandi and Smets [2] (with roots in the work of Jerrard and Sooner).

Proposition 22 ([2]). *Let u on Q satisfy $E_\varepsilon(u) \le C \log(1/\varepsilon)$. Then there is v satisfying*

$$|v| \le 1, \tag{4.6}$$

$$E_\varepsilon(v) \le C E_\varepsilon(u), \tag{4.7}$$

$$\|Jv\|_{L^1(Q)} \le C \frac{E_\varepsilon(u)}{\log(1/\varepsilon)} \le C, \tag{4.8}$$

where

$$Jv = d(v \wedge dv) = \sum_{i<j} (\partial_i v \times \partial_j v)\, dx_i \wedge dx_j \tag{4.9}$$

denotes the Jacobian, and

$$\|u - v\|_{L^2(Q)} \le \varepsilon^\alpha \quad \textit{(for some constant } \alpha > 0\textit{)}. \tag{4.10}$$

Sketch of the $W^{1,n/(n-1)}$-regularity property. It suffices to show that

$$\|u \wedge du\|_{L^{n/(n-1)}(M)} \le C \tag{4.11}$$

(cf. [5]). By duality, we need to control $\langle u \wedge du, X\rangle$ with $X \in L^n(\Lambda^1 M)$, $\|X\|_n \le 1$. According to Corollary 18,

$$X = d\phi + d^*k \tag{4.12}$$

where

$$\phi \in W_0^{1,n}(M), \qquad \|\phi\|_{1,n} \le C, \tag{4.13}$$

$$k \in (W^{1,n} \wedge L^\infty)(\Lambda^2 M), \qquad \|k\|_{1,n} + \|k\|_\infty \le C. \tag{4.14}$$

Thus

$$\langle u \wedge du, X\rangle = \langle u \wedge du, d\phi\rangle + \langle u \wedge du, d^*k\rangle.$$

Since $\phi = 0$ on ∂M,

$$\langle u \wedge du, d\phi\rangle = \int_M (\operatorname{Im} \bar{u}\Delta u)\phi = 0 \tag{4.15}$$

because u satisfies the Ginzburg–Landau equation

$$-\Delta u = \frac{1}{\varepsilon^2}(1 - |u|^2)u. \tag{4.16}$$

It remains to control $\langle u \wedge du, d^*k\rangle$.

Let $\tilde{k}$ be an extension of k to $\mathbb{R}^n$ such that $\operatorname{supp}\tilde{k} \subset Q$ and

$$\|\tilde{k}\|_{W^{1,n}} + \|\tilde{k}\|_\infty \le C(\|k\|_{W^{1,n}(M)} + \|k\|_\infty) \le C. \tag{4.17}$$

Then

$$|\langle u \wedge du, d^*k\rangle| \le |\langle \tilde{u} \wedge d\tilde{u}, d^*\tilde{k}\rangle| + \int_{Q\setminus M} |\nabla \tilde{u}|\,|\nabla \tilde{k}|.$$

By (4.5),

$$\int_{Q\setminus M} |\nabla \tilde{u}|\,|\nabla \tilde{k}| \le \|\tilde{u}\|_{W^{1,n/(n-1)}(Q\setminus M)}\|\tilde{k}\|_{W^{1,n}(Q)} \le C.$$

Next write

$$|\langle \tilde{u} \wedge d\tilde{u}, d^*\tilde{k}\rangle| = |\langle J\tilde{u}, \tilde{k}\rangle| \le |\langle Jv, \tilde{k}\rangle| + |\langle J\tilde{u} - Jv, \tilde{k}\rangle| = (4.18) + (4.19)$$

with v taken according to Proposition 22.

Estimate (4.18) from (4.8) and (4.17):

$$|\langle Jv, \tilde{k}\rangle| \le \|Jv\|_{L^1(Q)}\|\tilde{k}\|_{L^\infty(Q)} \le C.$$

Write

$$J\tilde{u} - Jv = d((\tilde{u} - v) \wedge d(\tilde{u} + v)),$$

hence

$$\begin{aligned}(4.19) &= |\langle (\tilde{u} - v) \wedge d(\tilde{u} + v), d^*\tilde{k}\rangle| \le \|\tilde{u} - v\|_{2n/(n-2)}\|\tilde{u} + v\|_{W^{1,2}}\|\tilde{k}\|_{W^{1,n}} \\ &\le C\|\tilde{u} - v\|_2^{1-2/n}[E_\varepsilon(\tilde{u}) + E_\varepsilon(v)]^{1/2} \\ &\le C(\log(1/\varepsilon))^{1/2}\varepsilon^{\alpha(1-2/n)} \qquad \text{(by (4.4), (4.7), (4.10)).}\end{aligned}$$

This completes the argument.

Remark 14. If $n = 2$, the conclusion of Theorem 21 is well known to fail and the estimate

$$\|u_\varepsilon\|_{W^{1,p}} \le c_p(g) \quad \text{for } p < 2 \tag{4.20}$$

is the optimal regularity result here.

5. Some other applications

Let j be an integer and $1 \le p < \infty$. As usual the Sobolev space $W^{j,p}(\mathbb{T}^n)$ is equipped with the norm

$$\|\varphi\|_{W^{j,p}(\mathbb{T}^n)} = \sum_{|\alpha|\le j} \|D^\alpha \varphi\|_{L^p(\mathbb{T}^n)}$$

and its dual space $W^{-j,p'}(\mathbb{T}^n)$ is equipped with its dual norm.

Theorem 23. *Let $\mathcal{X} \subset L^2(\mathbb{T}^n, \mathbb{R}^r)$ be an invariant function space and assume that the orthogonal projection P on $\mathcal{X}$ satisfies*

$$\|Pf\|_p \le C_p \sum_{s=1}^{r} \sum_{i \neq i_s} \|R_i f_s\|_p \quad (\textit{for all } 1 < p < \infty) \tag{5.1}$$

for some choice of indices $i_s \in \{1, \dots, n\}$ $(1 \le s \le r)$, where R_i denotes the i-th Riesz transform. Then, for every $u \in W^{-1,n/(n-1)}(\mathbb{T}^n, \mathbb{R}^r)$,

$$\|u\|_{W^{-1,n/(n-1)}} \le C(\|u\|_{L^1} + \mathrm{dist}(u, \mathcal{X})) \tag{5.2}$$

where dist *denotes the distance in $W^{-1,n/(n-1)}$.*

Proof. It follows in particular from (5.1) that the projection P is bounded on L^n and hence the operator $S = P \circ (-\Delta)^{1/2} : W^{1,n} \to L^n$ has closed range. Moreover

$$\|Sf\|_n \le C \sum_{s=1}^{r} \sum_{i \neq i_s} \|R_i((-\Delta)^{1/2} f_s)\|_n = c \sum_{s=1}^{r} \sum_{i \neq i_s} \|\partial_i f_s\|_n.$$

Thus Theorem 10 applies.

Let $f \in W^{1,n}(\mathbb{T}^n, \mathbb{R}^r)$ with $\|f\|_{1,n} \le 1$. By Theorem 10, there is $g \in (W^{1,n} \cap L^\infty)(\mathbb{T}^n, \mathbb{R}^r)$ such that $f - g \in$ Ker S. But since P is invariant, also $S = (-\Delta)^{1/2} \circ P$, hence $P(f - g) = 0$. Therefore, if $v \in \mathcal{X}$ then

$$|\langle v, f\rangle| = |\langle v, g\rangle| \le \|v\|_{L^1 + W^{-1,n/(n-1)}}(\|g\|_{L^\infty} + \|g\|_{W^{1,n}}).$$

Thus, for all $v \in \mathcal{X}$,

$$\|v\|_{W^{-1,n/(n-1)}} \le C\|v\|_{L^1 + W^{-1,n/(n-1)}}. \tag{5.3}$$

Write, for $u \in W^{-1,n/(n-1)}$ and $v \in \mathcal{X}$,

$$\begin{aligned}\|u\|_{W^{-1,n/(n-1)}} &\le \|u - v\|_{W^{-1,n/(n-1)}} + \|v\|_{W^{-1,n/(n-1)}} \\ &\overset{(5.3)}{\le} \|u - v\|_{W^{-1,n/(n-1)}} + C\|v\|_{L^1 + W^{-1,n/(n-1)}} \\ &\le \|u - v\|_{W^{-1,n/(n-1)}} + C(\|v - u\|_{W^{-1,n/(n-1)}} + \|u\|_{L^1}). \end{aligned} \tag{5.4}$$

Taking the infimum in (5.4) over $v \in \mathcal{X}$ yields (5.2).

Let $\ell \ge 1$ be an integer. Set

$$Au = \sum_{i=1}^{n} \partial_i^{(\ell)} u_i, \quad u = (u_1, \dots, u_n),$$

so that A may be viewed as a bounded operator from $E = W^{-1,n/(n-1)}(\mathbb{T}^n, \mathbb{R}^n)$ into $F = W^{-(\ell+1),n/(n-1)}(\mathbb{T}^n, \mathbb{R})$. It is also convenient to consider the unbounded operator

$$A_0 : D(A_0) \subset L^1(\mathbb{T}^n, \mathbb{R}^n) \to F, \quad A_0 = A,$$

with domain

$$D(A_0) = \{u \in L^1 \mid Au \in F \text{ in the sense of } \mathcal{D}'(\mathbb{T}^n)\}.$$

Corollary 24. *We have $D(A_0) \subset E$ and, for every $u \in D(A_0)$,*

$$\|u\|_E \le C(\|u\|_{L^1} + \|Au\|_F). \tag{5.5}$$

Proof. Consider the invariant space

$$\mathcal{X} = \{u \in L^2(\mathbb{T}^n, \mathbb{R}^n) \mid Au = 0 \text{ in } \mathcal{D}'(\mathbb{T}^n)\}.$$

The original projection P on $\mathcal{X}$ is given by

$$\begin{aligned}\widehat{Pu}(\xi) &= \left\{\hat{u}_i(\xi) - \sum_{j=1}^n \frac{\xi_i^\ell \xi_j^\ell}{\sum_k \xi_k^{2\ell}} \hat{u}_j(\xi)\right\}_{i=1,\dots,n} \\ &= \left\{\frac{\sum_{k\ne i} \xi_k^{2\ell}}{\sum_k \xi_k^{2\ell}} \hat{u}_i(\xi) - \sum_{j\ne i} \frac{\xi_i^\ell \xi_j^\ell}{\sum_k \xi_k^{2\ell}} \hat{u}_j(\xi)\right\}_{i=1,\dots,n}.\end{aligned} \tag{5.6}$$

Write

$$\frac{\xi_j^{2\ell}}{\sum_k \xi_k^{2\ell}} \hat{\varphi}(\xi) = \frac{\xi_j^{2\ell-1}|\xi|}{\sum_k \xi_k^{2\ell}} \widehat{R_j\varphi}(\xi) \tag{5.7}$$

with R_j the j-th Riesz transformation and observe that the Fourier multiplier $\xi_j^{2\ell-1}|\xi|/\sum \xi_k^{2\ell}$ acts boundedly on L^p $(1 < p < \infty)$ (since it satisfies Hörmander's condition). Hence (5.6) shows that for $1 < p < \infty$,

$$\|Pu\|_p \le \sum_{i=1}^n \left\{\sum_{k\ne i} \|R_k u_i\|_p + \sum_{j\ne i} \|R_i u_j\|_p\right\} \lesssim \sum_{i=1}^n \sum_{j\ne i} \|R_j u_i\|_p. \tag{5.8}$$

Thus condition (5.1) holds with $i_s = s$ $(1 \le s \le n)$ and Theorem 23 applies.

Next we claim that, for the bounded operator $A : E \to F$,

$$R(A) \text{ is closed in } F. \tag{5.9}$$

More precisely, we have

$$R(A) = \left\{f \in F \;\middle|\; \int_{\mathbb{T}^n} f = 0\right\}.$$

Indeed, fix $f \in F$ with $\int f = 0$.

If $\ell = 2k$, take $u_i = \varphi$, $i = 1, \dots, n$, where φ is the solution of the elliptic equation

$$\sum_{i=1}^n \partial_i^{(2k)} \varphi = f \quad \text{in } \mathbb{T}^n.$$

Note that $\varphi \in W^{-1,n/(n-1)}$ by elliptic regularity. Thus $u \in E$ satisfies $Au = f$.

If $\ell = 2k+1$, take $u_i = \partial_i \psi$, $i = 1, \dots, n$, where ψ is the solution of the elliptic equation

$$\sum_{i=n}^{n} \partial_i^{(2k+2)} \psi = f \text{ in } \mathbb{T}^n.$$

Note that $\psi \in L^{n/(n-1)}$ by elliptic regularity. Thus $u \in E$ satisfies $Au = f$ and the proof of (5.9) is complete.

From (5.9) and standard functional analysis we know that

$$\operatorname{dist}_E(u, N(A)) \le C\|Au\|_F. \tag{5.10}$$

On the other hand, it is clear that

$$N(A) = \{u \in E \mid Au = 0\}$$

is the closure of $\mathcal{X}$ in E and thus

$$\operatorname{dist}_E(u, N(A)) = \operatorname{dist}_E(u, \mathcal{X}) \quad \forall u \in E. \tag{5.11}$$

Combining (5.2), (5.10) and (5.11) yields the desired conclusion (5.5).

Corollary 24 carries over if $\mathbb{T}^n$ is replaced by $\mathbb{R}^n$ provided we use the space $W^{j,p}(\mathbb{R}^n)$ defined as the completion of $C_0^\infty(\mathbb{R}^n)$ for the norm

$$\|\varphi\|_{W^{j,p}(\mathbb{R}^n)} = \sum_{|\alpha|=j} \|D^\alpha \varphi\|_p,$$

and its dual space $W^{-j,p'}(\mathbb{R}^n)$ is equipped with its dual norm. As above set

$$Au = \sum_{i=1}^{n} \partial_i^{(\ell)} u_i.$$

Corollary 24′. *Let $u \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ be such that $Au \in W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)$ in the sense of $\mathcal{D}'(\mathbb{R}^n)$. Then $u \in W^{-1,n/(n-1)}(\mathbb{R}^n, \mathbb{R}^n)$ and*

$$\|u\|_{W^{-1,n/(n-1)}} \le C(\|u\|_{L^1} + \|Au\|_{W^{-(\ell+1),n/(n-1)}}). \tag{5.12}$$

Proof. Set $Q = (-1/2, +1/2)^n$ and **fix** a cut-off function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \zeta \le 1$ and

$$\zeta(x) = 1 \quad \text{for } |x| \le 1/4. \tag{5.13}$$

Let $u \in L^1(\mathbb{R}^n, \mathbb{R}^n)$ with $Au \in W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)$ and let $\varphi \in C_0^\infty(\mathbb{R}^n)$. We claim that for every integer $k \ge 1$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} \zeta^2(x/k) u(x) \varphi(x)\, dx\right| \\ &\quad \le C(\|u\|_{L^1(\mathbb{R}^n)} + \|Au\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)} + o(1))(\|\nabla\varphi\|_{L^n(\mathbb{R}^n)} + o(1)),\end{aligned} \tag{5.14}$$

with $o(1) \to 0$ as $k \to \infty$. In (5.14), and in all the estimates below, the constant C may depend on ζ (but it is independent of u, φ and k). Passing to the limit in (5.14) yields

$$\left|\int_{\mathbb{R}^n} u\varphi\right| \le C(\|u\|_{L^1(\mathbb{R}^n)} + \|Au\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)})\|\nabla\varphi\|_{L^n(\mathbb{R}^n)},$$

which corresponds to the desired conclusion (5.12).

We have

$$\int_{\mathbb{R}^n} \zeta^2(x/k)u(x)\varphi(x)\,dx = k^n \int_Q \zeta^2(y)u_k(y)\varphi_k(y)\,dy \tag{5.15}$$

where $u_k(y) = u(ky)$ and $\varphi_k(y) = \varphi(ky)$. Applying the periodic case (Corollary 24) to the functions ζu_k and $\zeta\varphi_k$ on $\mathbb{T}^n = Q$ we find

$$\left|\int_Q \zeta^2 u_k\varphi_k\right| \le C(\|\zeta u_k\|_{L^1(Q)} + \|A(\zeta u_k)\|_{W^{-(\ell+1),n/(n-1)}(Q)})\|\zeta\varphi_k\|_{W^{1,n}(Q)}. \tag{5.16}$$

Clearly

$$\|\zeta u_k\|_{L^1(Q)} \le \frac{1}{k^n}\|u\|_{L^1(\mathbb{R}^n)} \tag{5.17}$$

and

$$\|\zeta\varphi_k\|_{W^{1,n}(Q)} \le C\left(\|\nabla\varphi\|_{L^n(\mathbb{R}^n)} + \frac{1}{k}\|\varphi\|_{L^n(\mathbb{R}^n)}\right). \tag{5.18}$$

Next we claim that $A(\zeta u_k) \in W^{-(\ell+1),n/(n-1)}(\mathbb{T}^n)$ and

$$\|A(\zeta u_k)\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{T}^n)} \le \frac{1}{k^n}(\|Au\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)} + o(1). \tag{5.19}$$

Combining (5.15)–(5.19) gives (5.14). Therefore it remains to prove (5.19).

With obvious notation write

$$A(\zeta u_k) = \zeta Au_k + \sum_{\substack{|\alpha|+|\beta|=\ell \\ |\beta|\ge 1}} c_{\alpha,\beta} D^\alpha u_k D^\beta \zeta. \tag{5.20}$$

Note that for $\psi \in C^\infty(\bar{Q})$,

$$\begin{aligned}\left|\int_Q \zeta(Au_k)\psi\right| &= \frac{k^\ell}{k^n}\left|\int_{\mathbb{R}^n}(Au(y))(\zeta\psi)(y/k)\,dy\right| \\ &\le \frac{1}{k^n}\|Au\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)}\|D^{\ell+1}(\zeta\psi)\|_{L^n(\mathbb{R}^n)} \\ &\le \frac{C}{k^n}\|Au\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)}\|\psi\|_{W^{\ell+1,n}(\mathbb{T}^n)},\end{aligned}$$

and therefore

$$\|\zeta Au_k\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{T}^n)} \le \frac{C}{k^n}\|Au\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{R}^n)}. \tag{5.21}$$

Finally, for $|\alpha| + |\beta| = \ell$, $|\beta| \geq 1$, and $\psi \in C^\infty(\bar{Q})$, we have

$$\left|\int_Q (D^\alpha u_k)(D^\beta \zeta)\psi\right| = \left|\int_Q u_k D^\alpha((D^\beta \zeta)\psi)\right|$$
$$\leq C \int\limits_{\substack{y\in Q\\ |y|\geq 1/4}} |u_k(y)| \sum_{|\gamma|\leq \ell-1} |D^\gamma \psi(y)|\, dy, \tag{5.22}$$

since $|\beta| \geq 1$ and $\zeta(y) = 1$ for $|y| \leq 1/4$.

On the other hand, by Sobolev, $W^{2,n}(Q) \subset L^\infty(Q)$ and thus, for $|\gamma| \leq \ell - 1$,

$$\|D^\gamma \psi\|_{L^\infty(Q)} \leq C\|\psi\|_{W^{\ell+1,n}(Q)}. \tag{5.23}$$

From (5.22) and (5.23) we deduce that

$$\|(D^\alpha u_k)(D^\beta \zeta)\|_{W^{-(\ell+1),n/(n-1)}(\mathbb{T}^n)}$$
$$\leq C \int\limits_{\substack{y\in Q\\ |y|\geq 1/4}} |u_k(y)|\, dy = \frac{C}{k^n} \int\limits_{|x|\geq k/4} |u(x)|\, dx. \tag{5.24}$$

Combining (5.20), (5.21) and (5.24) yields (5.19).

Next, returning to Theorem 11 and considering functions on $\mathbb{R}^n$ (rather than $\mathbb{T}^n$; see Remark 6), notice that by a linear change of variables, condition (1.27) may be replaced by

$$\|A(\nabla(f - F))\|_n \leq \delta\|f\|_{1,n} \tag{5.25}$$

where A is any given $n \times n$ matrix of zero determinant (we are considering the $\mathbb{R}^n$-setting here to allow the coordinate change).

Hence, Theorem 10 may be restated as follows:

Theorem 10′. *Let $S : W^{1,n}(\mathbb{R}^n, \mathbb{R}^r) \to Y$ be a bounded operator with closed range. Assume $A^{(s)}$ $(1 \leq s \leq r)$ are singular $n \times n$ matrices such that*

$$\|S\vec{f}\| \leq C \max_{1\leq s\leq r} \|A^{(s)}(\nabla f_s)\|_n. \tag{5.26}$$

Then, for any $\vec{f} \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^r)$, there is $\vec{g} \in (W^{1,n} \cap L^\infty)(\mathbb{R}^n, \mathbb{R}^r)$ such that

$$\|\vec{g}\|_{1,n} + \|\vec{g}\|_\infty \leq C\|\vec{f}\|_{1,n} \tag{5.27}$$

and

$$\vec{f} - \vec{g} \in \operatorname{Ker} S. \tag{5.28}$$

Theorem 25. *Assume $\mathcal{L} = (L^{(s)})_{1\leq s\leq r} \subset \mathbb{R}^{n\times n}$ satisfies*

$$\max_s |\langle L^{(s)}\xi, \eta\rangle| \neq 0 \quad \text{if } \xi, \eta \in \mathbb{R}^n\backslash\{0\}, \tag{5.29}$$

$$\det L^{(s)} = 0 \quad \textit{for each } s = 1, \dots, r. \tag{5.30}$$

Define $L^{(s)}(D)\vec{u} = \sum_{i,j=1}^{n} L_{ij}^{(s)} \partial_j u_i$. *Then*

$$\|\vec{u}\|_{n/(n-1)} \leq C \max_s \|L^{(s)}(D)\vec{u}\|_1. \tag{5.31}$$

Proof. It follows from the ellipticity condition (5.29) that the operator

$$\mathcal{L} : L^{n/(n-1)}(\mathbb{R}^n, \mathbb{R}^n) \to \bigoplus_{s=1}^{r} W^{-1,n/(n-1)}(\mathbb{R}^n, \mathbb{R}) : \vec{u} \mapsto (L^{(s)}(D)\vec{u})_{s=1,\dots,r}$$

satisfies

$$\|\vec{u}\|_{n/(n-1)} \sim \|\mathcal{L}\vec{u}\|_{W^{-1,n/(n-1)}} = \sum_{s=1}^{r} \|L^{(s)}(D)\vec{u}\|_{-1,n/(n-1)}. \tag{5.32}$$

Hence the adjoint operator

$$S = \mathcal{L}^* : \bigoplus_{s=1}^{r} W^{1,n}(\mathbb{R}^n, \mathbb{R}) \to L^n(\mathbb{R}^n, \mathbb{R}^n)$$

is onto and satisfies

$$\|S\vec{f}\|_n = \sum_{i=1}^{n} \Big\| \sum_{s=1}^{r} \sum_{j=1}^{n} L_{ij}^{(s)}(\partial_j f_s) \Big\|_n \leq \sum_{s=1}^{r} \|L^{(s)}(\nabla f_s)\|_n. \tag{5.33}$$

By (5.30), the matrices $L^{(s)}$ are singular so that (5.26) holds with $A^{(s)} = L^{(s)}$. Therefore, given $\vec{f} \in W^{1,n}(\mathbb{R}^n; \mathbb{R}^r)$, $\|\vec{f}\|_{1,n} \leq 1$, there is $\vec{g} \in W^{1,n} \cap L^\infty$ with $\|\vec{g}\|_{1,n} + \|\vec{g}\|_\infty < C$ such that $S\vec{f} = S\vec{g}$.

Returning to (5.32) and proceeding by duality, write

$$\begin{aligned}\Big|\sum_s \langle L^{(s)}(D)\vec{u}, f_s\rangle\Big| &= \Big|\Big\langle \vec{u}, \sum_s L^{(s)}(\nabla f_s)\Big\rangle\Big| = |\langle \vec{u}, S\vec{f}\rangle| = |\langle \vec{u}, S\vec{g}\rangle| = \Big|\sum_s \langle L^{(s)}(D)\vec{u}, g_s\rangle\Big| \\ &\leq \sum_s \|L^{(s)}(D)\vec{u}\|_1 \|g_s\|_\infty \leq C \max_s \|L^{(s)}(D)\vec{u}\|_1,\end{aligned}$$

proving (5.31).

Remark 15. Obviously condition (5.32) may be reformulated by requiring that the linear subspace $[L^{(s)}; s = 1, \dots, r]$ of $\mathbb{R}^{n\times n}$, generated by $\mathcal{L}$, is also generated by its singular elements.

Theorem 25 implies in particular Korn's inequalities in plasticity theory (see [11], [12]).

Corollary 26. *One has the inequality*

$$\|\vec{u}\|_{n/(n-1)} \leq C \sum_{i,j=1}^{n} \|\partial_i u_j + \partial_j u_i\|_1 \tag{5.34}$$

for $\vec{u} = (u_1, \dots, u_n)$ *on* $\mathbb{R}^n$.

Proof. Let $\mathcal{L} = \{e_i \otimes e_j + e_j \otimes e_i \mid 1 \leq i, j \leq n\}$, thus $[\mathcal{L}] =$ symmetric $n \times n$ matrices. Condition (5.29) clearly holds. Obviously (5.30) holds if $n \geq 3$. For $n = 2$, observe that

$$[\mathcal{L}] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right]$$

and apply the previous remark.

Remark 16. In Corollary 26, $\dim [\mathcal{L}] = n + n(n-1)/2 = n(n+1)/2$. It was already pointed out in [11] that, for $n \geq 3$, the result is not optimal, in the sense that there is a system $(L^{(s)})_{1 \leq s \leq 2n-1}$ of $n \times n$ matrices satisfying (5.29) and (5.30). Following an earlier idea of D. G. de Figueiredo, M. Strauss [11] constructed such a family consisting of n matrices of rank 1 and $n-1$ matrices of rank $n-1$. A different family composed of $2n-1$ matrices of rank 1 can be obtained using a simple observation communicated to us by J. Van Schaftingen. Let $r = 2n-1$. Choose vectors $(v_i)_{1 \leq i \leq r}$ in $\mathbb{R}^n$ such that every subset of n vectors is a basis for $\mathbb{R}^n$. Define $L^{(i)} = v_i \otimes v_i$. Assume $\xi, \eta \in \mathbb{R}^n$ are such that

$$\langle L^{(i)} \xi, \eta \rangle = 0 \quad \forall i = 1, \ldots, r.$$

Then $(v_i \cdot \xi)(v_i \cdot \eta) = 0$ for all i. Letting $I = \{i \mid v_i \cdot \xi = 0\}$ and $J = \{i \mid v_i \cdot \eta = 0\}$ we have $I \cup J = \{1, \ldots, r\}$ and therefore card $I \geq n$ or card $J \geq n$. In the first case $\xi = 0$ and in the second case $\eta = 0$.

Open Problem 3. What is the smallest $r = r(n)$ for which there is a system $(L^{(s)})_{1 \leq s \leq r} \subset \mathbb{R}^{n \times n}$ satisfying the assumptions of Theorem 25?

Obviously $r(n) \geq n+1$ and from the preceding $r(n) \leq 2n-1$.

Open Problem 4. When is a subspace of $\mathbb{R}^{n \times n}$ generated by its singular elements?

Appendix: Proof of Remark 1

Our purpose is to show that the inequality

$$\|(-\Delta)^{-1} \vec{f}\|_{W^{2,1}} \leq C \|\vec{f}\|_1 \tag{1}$$

fails also for $\vec{f} \in L^1_\#(\mathbb{R}^n, \mathbb{R}^n)$.

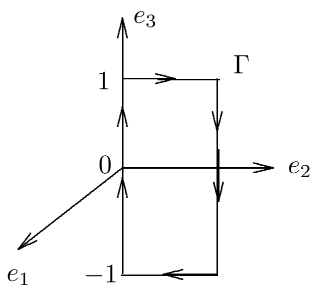
By Smirnov's result cited earlier, this statement is equivalent to disproving that

$$\|(-\Delta)^{-1}(\mathcal{H}_\Gamma \vec{t})\|_{W^{2,1}} < C \tag{2}$$

holds, whenever Γ is a closed rectifiable curve in $\mathbb{R}^n$ of length $|\Gamma| = 1$ and $\vec{t}$ the unit tangent vector to Γ.

As mentioned in the Introduction, for $n \geq 3$ this is quite easily seen.

Let $n = 3$ and take Γ to be any simple closed curve containing the segment $(0, 0, x_3)$, $|x_3| \leq 1$. For example,



Let $\vec{u}$ be the solution of $-\Delta\vec{u} = \mathcal{H}_\Gamma\vec{t}$. On a neighborhood of 0, we get

$$\vec{u} = C(\log(1/r))\vec{e}_3 + \vec{R} \tag{3}$$

with $\vec{R}$ smooth and $r^2 = x_1^2 + x_2^2$. Recall that $\log(1/r) \notin W^{2,1}(\mathbb{R}^2)$ and hence $\vec{u} \notin W^{2,1}(\mathbb{R}^3)$.

Consider now the case $n = 2$. Producing a counterexample seems less obvious and requires curves Γ with a more complicated structure.

Notice that if Γ is smooth with nonvanishing curvature and $m(\xi)$ a 0-order even Fourier multiplier, then by the stationary phase principle

$$(\vec{t}\mathcal{H}_\Gamma)^\wedge(\xi) \cdot m(\xi) = (\tilde{m} \cdot \vec{t}\mathcal{H}_\Gamma)^\wedge(\xi) + O(|\xi|^{-3/2}) \tag{4}$$

as $|\xi| \to \infty$, where $\tilde{m}$ is the function on Γ defined by $\tilde{m}(x) = m(\zeta_x)$ where ζ_x is the normal vector to Γ at x.

Returning to (2), apply (4) with $m(\xi)$ one of the multipliers

$$\frac{\xi_1^2}{\xi_1^2 + \xi_2^2}, \quad \frac{\xi_2^2}{\xi_1^2 + \xi_2^2}, \quad \frac{\xi_1\xi_1}{\xi_1^2 + \xi_2^2}.$$

Since $\tilde{m}$ is a bounded density on Γ, it follows in particular from (4) that

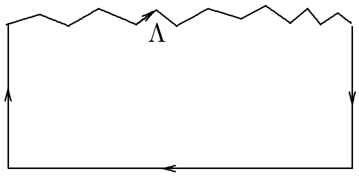
$$\partial^{(2)}[(-\Delta)^{-1}(\mathcal{H}_\Gamma\vec{t})] \in L^\infty(\mathcal{H}_\Gamma) + L^2 \tag{5}$$

and hence a bounded measure.

We produce a counterexample to (2) using a rectifiable curve Γ with a multi-scale structure.

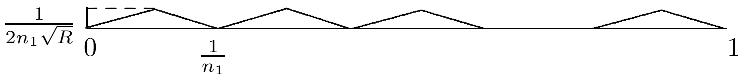
Fix a large integer R. Let $n_1 \ll \cdots \ll n_R$ be a sequence of integers that are very lacunary (the precise conditions will become clear later on).

Γ will be obtained as a polygonal line Λ joining $(0,0)$ to $(1,0)$ which we close by adding the segments $[(1,0),(1,-1)] \cup [(1,-1),(0,-1)] \cup [(0,-1),(0,0)]$:



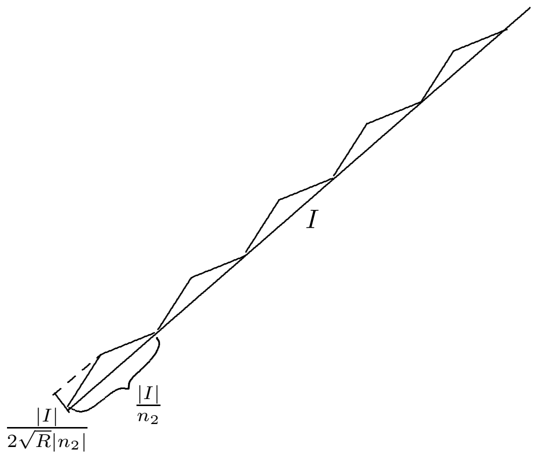
(only Λ is relevant for our purpose).

Next we specify Λ. Let Λ_0 be the segment $[(0,0),(1,0)]$. We take Λ_1 to be a 'saw-tooth' perturbation of Λ_0 with n_1 teeth and inclination $1/\sqrt{R}$:



Thus Λ_1 is a polygonal line consisting of $2n_1$ segments.

To obtain Λ_2, perturb each segment I of Λ_1 by a saw-tooth line with n_2 teeth and again relative inclination $1/\sqrt{R}$ (with respect to I):



The continuation of the process is clear and we let $\Lambda = \Lambda_R$.

Obviously

$$|\Lambda_1| = 2n_1\sqrt{\left(\frac{1}{2n_1}\right)^2 + \left(\frac{1}{2n_1\sqrt{R}}\right)^2} = \sqrt{1+\frac{1}{R}},$$

$$|\Lambda_2| = \left(1+\frac{1}{R}\right)^{1/2} |\Lambda_1|,$$

$$|\Lambda_R| = \left(1+\frac{1}{R}\right)^{R/2} < e. \tag{6}$$

Notice also that, from the construction, the Hausdorff distance satisfies

$$d(\Lambda_{s-1}, \Lambda_s) \lesssim \frac{1}{b_{s-1} n_s \sqrt{R}} \tag{7}$$

where $b_s = 2^s n_1 \cdots n_s$ is the number of segments $I_{s,\alpha}$ of Λ_s. These segments are of equal length $|I_{s,\alpha}| \sim 1/b_s$.

Our next claim is that

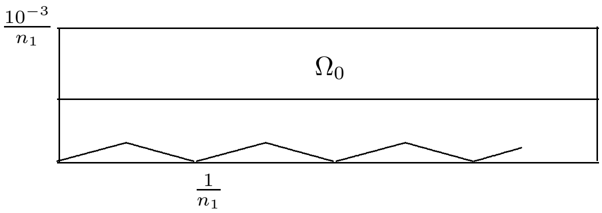
$$\|(-\Delta)^{-1} \mathcal{H}_\Gamma \vec{t}\|_{W^{2,1}} \gtrsim \sqrt{R}, \tag{8}$$

This contribution will be obtained near Λ and hence (8) amounts to

$$\|(-\Delta)^{-1} \mathcal{H}_{\Lambda_R} \vec{t}\|_{W^{2,1}(\text{near } \Lambda_R)} \gtrsim \sqrt{R}, \tag{8$'$}$$

Let us next construct a sequence of disjoint regions $\Omega_0, \Omega_1, \ldots, \Omega_{R-1}$ that in some sense will 'shadow' $\Lambda_0, \Lambda_1, \ldots, \Lambda_{R-1}$.

Let $\Omega_0 = \{x \in \mathbb{R}^2 \mid 10^{-3}/2n_1 < \text{dist}(x, \Lambda_0) < 10^{-3}/n_1\}$:



and in general for $s < R$,

$$\Omega_s = \left\{ x \in \mathbb{R}^2 \;\middle|\; \frac{10^{-3}}{2n_{s+1} b_s} < \text{dist}(x, \Lambda_s) < \frac{10^{-3}}{n_{s+1} b_s} \right\}.$$

Hence, if $s > s'$, by (7),

$$\begin{aligned}
\text{dist}\,(\Omega_s, \Omega_{s'}) &\geq \text{dist}\,(\Lambda_s, \Omega_{s'}) - \frac{10^{-3}}{n_{s+1} b_s} \\
&\geq \text{dist}\,(\Lambda_{s'}, \Omega_{s'}) - d(\Lambda_s, \Lambda_{s'}) - \frac{10^{-3}}{n_{s+1} b_s} \\
&\geq \frac{10^{-3}}{2n_{s'+1} b_{s'}} - (d(\Lambda_s, \Lambda_{s-1}) + \cdots + d(\Lambda_{s'+1}, \Lambda_{s'})) - \frac{10^{-3}}{n_{s+1} b_s} \\
&\geq \frac{10^{-3}}{2n_{s'+1} b_{s'}} - \frac{1}{\sqrt{R}} \left(\frac{1}{n_{s'+1} b_{s'}} + \cdots + \frac{1}{n_s b_{s-1}} \right) - \frac{10^{-3}}{n_{s+1} b_s} \geq \frac{10^{-3}}{3n_{s'+1} b_{s'}}
\end{aligned}$$

and the Ω_s are disjoint.

Returning to (8$'$), write

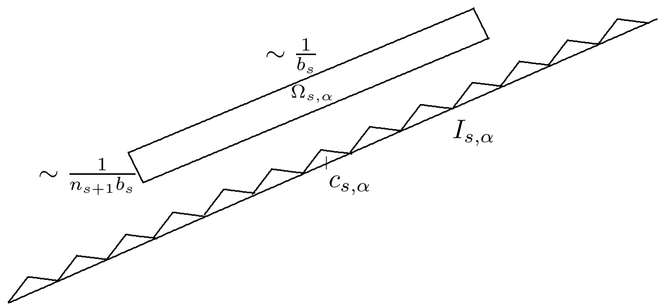
$$\|(-\Delta)^{-1} (\mathcal{H}_{\Lambda_R} \vec{t})\|_{W^{2,1}} \geq \sum_{s=1}^{R} \|(-\Delta)^{-1} (\mathcal{H}_{\Lambda_R} \vec{t})\|_{W^{2,1}(\Omega_s)}. \tag{9}$$

Decompose further Ω_s into the b_s rectangular regions $\Omega_{s,\alpha}$ parallel to $I_{s,\alpha}$ of length $|I_{s,\alpha}| \sim 1/b_s$ and width $\sim 1/n_{s+1}b_s$.

Let $\Omega'_{s,\alpha} \subset \Omega_{s,\alpha}$ be the sub-rectangle projecting onto a $\frac{1}{4}|I_{s,\alpha}|$-neighborhood of the center $c_{s,\alpha}$ of $I_{s,\alpha}$. Write

$$\|(-\Delta)^{-1}(\mathcal{H}_{\Lambda_R}\vec{t})\|_{W^{2,1}(\Omega_s)} \geq \sum_{\alpha=1}^{b_s} \|(-\Delta)^{-1}(\mathcal{H}_{\Lambda_R}\vec{t})\|_{W^{2,1}(\Omega'_{s,\alpha})}. \tag{10}$$

Next, we analyze further $(-\Delta)^{-1}(\mathcal{H}_{\Lambda_R}\vec{t})$ on $\Omega'_{s,\alpha}$ for a fixed α.



First, we restrict $\mathcal{H}_{\Lambda_R}\vec{t}$ to a neighborhood $B(c_{s,\alpha}, |I_{s,\alpha}|/2) = B_{s,\alpha}$ in the α-summand of (10).

Indeed, for $x \in \Omega'_{s,\alpha}$ one has

$$|\partial^{(2)}(-\Delta)^{-1}(\mathcal{H}_{\Lambda_R \setminus B_{s,\alpha}}\vec{t})(x)| \lesssim \left\| \partial^{(2)}\left(\log \frac{1}{|x|}\right)\right\|_{L^\infty(|x|>\frac{1}{4}|I_{s,\alpha}|)} \lesssim b_s^2$$

and hence

$$\|\partial^{(2)}(-\Delta)^{-1}\,(\mathcal{H}_{\Lambda_R \setminus B_{s,\alpha}}\vec{t})\|_{L^1(\Omega'_{s,\alpha})} \lesssim \frac{1}{n_{s+1}}. \tag{11}$$

Summing (11) over $\alpha = 1, \dots, b_s$ gives the contribution of at most

$$\frac{b_s}{n_{s+1}} < \frac{1}{R} \tag{12}$$

provided we take n_{s+1} large enough.

Thus in (10), we may replace the α-summand by

$$\|(-\Delta)^{-1}(\mathcal{H}_{\Lambda_R \cap B_{s,\alpha}}\vec{t})\|_{W^{2,1}(\Omega'_{s,\alpha})}. \tag{13}$$

Next, we replace Λ_R by Λ_{s+1} in (13). Taking $x \in \Omega'_{s,\alpha}$, it follows from the construction of the polygonal lines Λ_s that

$$|(\partial^2(-\Delta)^{-1})[(\mathcal{H}_{\Lambda_R\cap B_{s,\alpha}}\vec{t}) - (\mathcal{H}_{\Lambda_{s+1}\cap B_{s,\alpha}}\vec{t})](x)| \lesssim \left\|\partial^2\left(\log\frac{1}{|x|}\right)\right\|_{\mathrm{Lip}(|x|\gtrsim 1/b_{s+1})}\frac{1}{b_{s+1}n_{s+2}}|I_{s,\alpha}| \lesssim b_{s+1}^3\frac{1}{b_{s+1}n_{s+2}}\frac{1}{b_s} \lesssim \frac{b_s n_{s+1}^2}{n_{s+2}}.$$

Hence

$$\|\partial^2(-\Delta)^{-1}[(\mathcal{H}_{\Lambda_R\cap B_{s,\alpha}}\vec{t}) - (\mathcal{H}_{\Lambda_{s+1}\cap B_{s,\alpha}}\vec{t})]\|_{L^1(\Omega'_{s,\alpha})} \lesssim \frac{1}{n_{s+1}b_s^2}\frac{b_s n_{s+1}^2}{n_{s+2}} = \frac{n_{s+1}}{b_s n_{s+2}} \tag{14}$$

and summing over $\alpha = 1, \dots, b_s$ gives the contribution $\frac{n_{s+1}}{n_{s+2}} < \frac{1}{R}$.

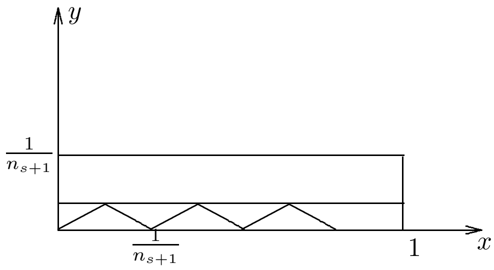
Therefore (13) can further be replaced by

$$\|(-\Delta)^{-1}(\mathcal{H}_{\Lambda_{s+1}\cap B_{s,\alpha}}\vec{t})\|_{W^{2,1}(\Omega'_{s,\alpha})}. \tag{15}$$

Clearly, (15) is independent of α and performing an affine transformation with expansion factor $\sim b_s$, we see that

$$(15) \sim \frac{1}{b_s}\|(-\Delta)^{-1}\mathcal{H}_\Sigma\vec{t}\|_{W^{2,1}([1/4,3/4]\times[10^{-3}/2n_{s+1},10^{-3}/n_{s+1}])} \tag{16}$$

where Σ is a saw-tooth polygonal line along $\vec{e}_1$ with n_{s+1} teeth and inclination $1/\sqrt{R}$.



Consider in (16) the coordinate t_y of $\vec{t}$ given by

$$t_y = \frac{1}{\sqrt{R}}\,\mathrm{sign}\sin 2\pi n_{s+1}x \tag{17}$$

and the contribution

$$\frac{1}{b_s\sqrt{R}}\|\partial_{xy}^2(-\Delta)^{-1}[(\mathrm{sign}\sin 2\pi n_{s+1}x)\mathcal{H}_\Sigma]\|_{L^1([1/4,3/4]\times[10^{-3}/2n_{s+1},10^{-3}/n_{s+1}])}. \tag{18}$$

Next, replace $\mathcal{H}_\Sigma$ by $|\Sigma|\cdot\mathcal{H}_{[0,1]\vec{e}_1}$ projecting on the x-axis. Clearly

$$\begin{aligned}
&\|\partial^2_{xy}(-\Delta)^{-1}[(\operatorname{sign}\sin 2\pi n_{s+1}x)\mathcal{H}_\Sigma]\|_{L^1(\cdots)} \\
&\qquad = |\Sigma|\|\partial^2_{xy}(-\Delta)^{-1}[(\operatorname{sign}\sin 2\pi n_{s+1}x)\mathcal{H}_{[0,1]\vec{e}_1}\|_{L^1(\cdots)} \qquad (19)\\
&\qquad\quad + O\left\{\frac{1}{\sqrt{R}n_{s+1}}\left\|\partial_y\left[\frac{xy}{(x^2+y^2)^2}\right]\right\|_{L^1(|y|>10^{-3}/3n_{s+1})}\right\} \qquad (20)
\end{aligned}$$

and

$$(20) \lesssim \frac{1}{\sqrt{R}}. \tag{21}$$

By partial integration

$$\begin{aligned}
(19) &\sim \left\| y\left[\int_0^1 \frac{x-x'}{((x-x')^2+y^2)^2}(\operatorname{sign}\sin 2\pi n_{s+1}x')\,dx'\right]\right\|_{L^1(\cdots)}\\
&= \left\| y\int \frac{1}{(x-x')^2+y^2}\Big(\sum_{j=1}^{2n_{s+1}}(-1)^j\delta_{j/2n_{s+1}}\Big)(dx')\right\|_{L^1(\cdots)}\\
&\qquad\qquad (\delta_t = \text{Dirac measure at } t\in\mathbb{R})\\
&= \left\|\sum_{j=1}^{2n_{s+1}}(-1)^j\frac{y}{(x-\frac{j}{2n_{s+1}})^2+y^2}\right\|_{L^1([1/4,3/4]\times[10^{-3}/2n_{s+1},10^{-3}/n_{s+1}])} \gtrsim 1. \qquad (22)
\end{aligned}$$

Summarizing, it follows that

$$(18) \gtrsim \frac{1}{b_s\sqrt{R}},$$

hence

$$\begin{aligned}
(13),(15) &\gtrsim \frac{1}{b_s\sqrt{R}},\\
(10) &\gtrsim \frac{1}{\sqrt{R}},\\
(9) &\gtrsim \sqrt{R},
\end{aligned}$$

providing the lower bound (8′).

Remark A1. Another way of stating the failure of (2) for $n = 2$ is to say that if u solves

$$-\Delta u = \chi_\Omega \tag{23}$$

where Ω has Γ as boundary, then its characteristic function, χ_Ω, is a BV function and u fails to have $\partial^{(3)}u$ bounded as measure.

Therefore the same conclusion holds in any dimension n. Consequently, letting $n = 3$ say, (1) fails also on the ‘smaller’ class of $\vec{f} \in L^1(\mathbb{R}^3, \mathbb{R}^3)$ for which curl $\vec{f} = 0$.

Remark A2. Returning to equation (23), let us observe that if Ω is a circle, then it is true (and somewhat surprising) that $\partial^{(3)}u$ is a bounded measure (as is checked easily by explicit computation). From this, one deduces that the equation $-\Delta u = f$ with f **radial** and BV has its solution u with $\partial^{(3)}u$ a measure.

More generally, assume for instance that Ω has smooth boundary $\partial\Omega$ with nonvanishing curvature. Then again the solution u of (23) is such that $\partial^{(3)}u$ is a bounded measure.

This is a consequence of (5). (But the construction shows that this may fail if Ω is only Lipschitz.)

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