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# Vortex collisions and energy-dissipation rates in the Ginzburg–Landau heat flow

## Part II: The dynamics

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**Abstract.** We deduce from the first part of this paper [S1] estimates on the energy-dissipation rates for solutions of the Ginzburg–Landau heat flow, which allow us to study various phenomena occurring in this flow, including vortex collisions; they allow in particular extending the dynamical law of vortices past collision times.

**Keywords.** Ginzburg–Landau equation, Ginzburg–Landau vortices, vortex dynamics, vortex collisions

### 1. Introduction and statement of the main results

#### 1.1. Presentation of the problem

We recall from [S1] that we are interested in the following parabolic Ginzburg–Landau equation in 2 dimensions, in the asymptotic limit  $\varepsilon \rightarrow 0$ :

$$\begin{cases} \frac{\partial_t u}{|\log \varepsilon|} = \Delta u + \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \times \mathbb{R}_+, \\ u(\cdot, 0) = u_\varepsilon^0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a two-dimensional domain, assumed to be smooth, bounded and simply connected, and where  $u$  is a *complex-valued* function, assumed to satisfy either one of the boundary conditions

$$u = g \quad \text{on } \partial\Omega \quad (1.2)$$

with  $g$  a fixed regular map from  $\Omega$  to  $\mathbb{S}^1$ , in which case we also assume that  $\Omega$  is strictly starshaped with respect to a point; or

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (1.3)$$

in which case no further assumption is made.

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The Ginzburg–Landau heat flow is an  $L^2$  gradient flow (or steepest descent) for the Ginzburg–Landau functional

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left( |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} \right). \quad (1.4)$$

For the motivations to study this equation, and the notion of *vortices*, we refer to the first part of this paper [S1]. It was shown by Bethuel–Brezis–Hélein in [BBH] that under the assumption

$$E_\varepsilon(u) \leq C|\log \varepsilon|, \quad (1.5)$$

minimizers (respectively critical points) of  $E_\varepsilon$  have a bounded number of vortices which converge, as  $\varepsilon \rightarrow 0$ , to a finite set of points which are minimizers (respectively critical points) of an explicit finite-dimensional function called *renormalized energy*, and denoted by  $W$ . They proved the crucial relation that if  $u_\varepsilon$  has  $n$  limiting vortices at  $p_1, \dots, p_n$ , of degrees  $D_j$ , then

$$E_\varepsilon(u_\varepsilon) \geq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1, \dots, p_n) + n\gamma + o(1), \quad (1.6)$$

where  $\gamma$  is a universal constant introduced in [BBH] (it is the energy of the profile of the 1-vortex solution in the plane). The main term in  $W_{\mathbf{D}}$ ,

$$-\pi \sum_{i,j:i \neq j} D_i D_j \log |p_i - p_j|, \quad (1.7)$$

contains the interaction between the vortices and indicates that vortices of opposite sign attract each other, while vortices of same sign repel each other.

The dynamics of the vortices under the heat flow (1.1) has also been studied, and it was established by Lin [Li] and Jerrard–Soner [JS1] (see also Spirn [Sp] for the equation with magnetic field), that, as could be expected, the limiting vortices  $p_i$  of the solutions of (1.1) evolve (in that time scale) according to the gradient of the renormalized energy  $W$ , i.e. according to the set of ODE's

$$\begin{cases} \frac{dp_i}{dt} = -\frac{1}{\pi} \nabla_i W_{\mathbf{D}}(p_1, \dots, p_n)(t), \\ p_i(0) = p_i^0. \end{cases} \quad (1.8)$$

This was established under the following set of restrictions:

1. The initial vortices all have degree  $\pm 1$  and are well separated.
2. The initial data is assumed to be “well prepared”, i.e.

$$E_\varepsilon(u_\varepsilon^0) \leq \pi n |\log \varepsilon| + C, \quad (1.9)$$

where  $n$  is the number of initial vortices.

3. There are no collisions (or we work until the first collision time under the law (1.8)).
4. The vortices cannot exit  $\Omega$ .

A different proof via a  $\Gamma$ -convergence or energy-based method was later given by Sandier–Serfaty in [SS2], under the same conditions and the slightly stronger “very well prepared” assumption

$$E_\varepsilon(u_\varepsilon^0) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + o(1). \quad (1.10)$$

All these results were valid only up to collision time under the law (1.8); but, if there are initially vortices of opposite degrees, then this law does generically generate collisions (see the form (1.7)). Collisions create a problem in the analysis because the “well preparedness” breaks down during a collision. They are probably one of the most interesting phenomena in these Ginzburg–Landau dynamics, and Ginzburg–Landau itself is one of the simplest models in which collisions of vortices can be studied.

In this paper, we are interested in giving results relaxing the assumptions above. The main objectives of this work are to study collisions, which were not well understood, to determine how and how fast the energy dissipates during such collisions, to give a dynamical law after blow-up, and to see how the dynamical law of the vortices can be continued/extended after collisions. We also show how the well prepared assumption can be weakened, and relax the separation hypothesis, for example dealing with the possible separation of two +1 vortices which are initially very close. We use our study of Part I [S1] of “pathological situations”, i.e. those for which we have a group of vortices which are far from the others, of degrees  $d_i$  with  $(\sum_i d_i)^2 \neq \sum_i d_i^2$  in the group, which we called an “unbalanced cluster of vortices”.

While this work was being completed, very similar issues were addressed by Bethuel, Orlandi and Smets in [BOS1] and later on, their study was completed in [BOS2] (see also the more recent paper [BOS3]). Prior to all these works, the only partial result on collisions was the paper of Bauman–Chen–Phillips–Sternberg [BCPS], where they studied the situation in the whole plane with quite rigid conditions at infinity.

The first paper of Bethuel–Orlandi–Smets [BOS1] gives a geometric measure-theoretical description of the limiting vortex trajectories under very general assumptions (a simple bound  $E_\varepsilon(u_\varepsilon) \leq C |\log \varepsilon|$ ), including possible splittings, collisions and recombinations, and results of annihilation in the case of collisions with total degree 0; it also exhibits a phenomenon of “phase-vortex interaction” occurring (only) in infinite samples (their setup is the whole plane), which can create a drift of the vortices. Their later paper contains some results more similar to the present paper, it shows that the limiting trajectories of the vortices are rectifiable, and derives their limiting motion law past collision times, via the “balanced” property and a quantization of the energy like the one we mentioned in Part I [S1] (see relation (1.31) here).

Some of the main differences between our work and theirs is that we handle boundary conditions in bounded domains, and that our method, inspired by [SS2], is rather energy-based than PDE-based: it relies on examining the energy-dissipation rates through the study made in Part I [S1] of the perturbed Ginzburg–Landau equation

$$\begin{cases} \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) = f_\varepsilon & \text{in } \Omega, \\ u = g \text{ (resp. } \frac{\partial u}{\partial \nu} = 0) & \text{on } \partial\Omega. \end{cases} \quad (1.11)$$

under the hypotheses  $E_\varepsilon(u_\varepsilon) \leq M|\log \varepsilon|$ ,  $|u_\varepsilon| \leq 1$ , and  $|\nabla u_\varepsilon| \leq M/\varepsilon$ ; and on characterizing precisely the value of the energy and location of the zeroes of  $u_\varepsilon$  during the dynamics.

1.2. Methodology

Let us first recall the method of [SS2]. It is written as an abstract scheme, which we will not fully quote here (referring to [SS2]), but rather we describe here its implementation for Ginzburg–Landau.

The idea is to use the fact that, given a priori the number of limiting vortices  $n$ , and their degrees  $D_i = \pm 1$ ,  $E_\varepsilon - \pi n|\log \varepsilon|$   $\Gamma$ -converges to  $W$ , combined with some additional estimates on the  $C^1$  structure of the energy landscape. For the meaning of  $\Gamma$ -convergence, we need to specify a sense of convergence: we say  $u_\varepsilon \xrightarrow{S} (p_1, \dots, p_n)$  if

$$\mu_\varepsilon := \text{curl}(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i}, \tag{1.12}$$

where  $(\cdot, \cdot)$  denotes, here and throughout, the scalar product in  $\mathbb{C}$  identified with  $\mathbb{R}^2$ . This is the convergence of the Jacobian determinant, or vorticity of  $u_\varepsilon$  (exactly as in fluid mechanics); its role in Ginzburg–Landau has been first emphasized by Jerrard–Soner [JS2] and Alberti [Al] (see also [SS3]) and has been commonly used since then. It allows one to trace down the vortices and find the limiting vortices  $p_i$ . The best compactness for  $\mu_\varepsilon$  one obtains is in a weak norm: in the dual of  $C_c^{0,\gamma}(\Omega)$ , but this is not important here. Observe that the  $p_i$ 's are the limits as  $\varepsilon \rightarrow 0$  of the zeroes of  $u_\varepsilon$ , not the zeroes themselves, and the degrees are the limits of the total degrees of the zeroes converging to each  $p_i$ .

The equation (1.1) can be seen as the gradient flow

$$\partial_t u_\varepsilon = -\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \tag{1.13}$$

where  $\nabla_{X_\varepsilon} E_\varepsilon$  denotes the gradient of  $E_\varepsilon$  with respect to the Hilbert structure

$$\|\cdot\|_{X_\varepsilon}^2 = \frac{1}{|\log \varepsilon|} \|\cdot\|_{L^2(\Omega)}^2.$$

With these notations, if  $u_\varepsilon$  is a solution of the flow (1.1), the energy-dissipation rate is

$$\begin{aligned} -\frac{d}{dt} E_\varepsilon(u_\varepsilon(x, t)) &= -\langle \partial_t u_\varepsilon, \nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon) \rangle_{X_\varepsilon} \\ &= \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 = \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \end{aligned} \tag{1.14}$$

$$= \frac{1}{2} \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 + \frac{1}{2} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2. \tag{1.15}$$

The main idea of [SS2] is to write this energy dissipation as (1.15) and to prove two additional relations.

The first relation (which we recalled in Part I) was: if  $\text{curl}(iu_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i}$  as  $\varepsilon \rightarrow 0$ , then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\log \varepsilon| \left| \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \right|^2 &= \lim_{\varepsilon \rightarrow 0} \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2 \\ &\geq \frac{1}{\pi} \sum_i |\nabla_i W_{\mathbf{D}}(p_1, \dots, p_n)|^2, \end{aligned} \tag{1.16}$$

which relates the slope of the energy at a configuration to the slope of the renormalized energy at the underlying vortices.

The other relation is that, under the assumption  $E_\varepsilon(u_\varepsilon) \leq \pi n |\log \varepsilon| + O(1)$ , if for every  $t \in [0, T]$ ,  $\mu_\varepsilon(t) = \text{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i(t)}$  then

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega \times [0, T]} |\partial_t u_\varepsilon|^2 = \lim_{\varepsilon \rightarrow 0} \int_0^T \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 \geq \pi \sum_{i=1}^n \int_0^T |d_t p_i|^2 dt. \tag{1.17}$$

This lower bound is sharp and relates the kinetic energy  $\int_{\Omega} |\partial_t u|^2$  to the velocity of the underlying vortices. It comes as a corollary of a more general result called the “product estimate”, valid for any configuration (not necessarily solving (1.1)), proved in [SS1], which we use again several times in this paper, and whose time-dependent version is ( $\mathcal{M}(\Omega)$  denotes the space of bounded Radon measures on  $\Omega$ ):

**Theorem 1 (“Product estimate”, time-dependent version, see [SS1]).** *Let  $u_\varepsilon(x, t)$  be defined over  $\Omega \times [0, T]$  and be such that*

$$\begin{cases} \forall t \in [0, T], & E_\varepsilon(u_\varepsilon(t)) \leq C |\log \varepsilon|, \\ \int_{\Omega \times [0, T]} |\partial_t u_\varepsilon|^2 \leq C |\log \varepsilon|. \end{cases} \tag{1.18}$$

Then,  $V_\varepsilon$  being defined by

$$V_\varepsilon = (\partial_2(iu_\varepsilon, \partial_t u_\varepsilon) - \partial_t(iu_\varepsilon, \partial_2 u_\varepsilon), -\partial_1(iu_\varepsilon, \partial_t u_\varepsilon) + \partial_t(iu_\varepsilon, \partial_1 u_\varepsilon)) \tag{1.19}$$

there exist  $\mu \in L^\infty([0, T], \mathcal{M}(\Omega))$  of the form

$$\mu(t) = 2\pi \sum_i D_i(t) \delta_{p_i(t)}, \quad d_i(t) \in \mathbb{Z},$$

and  $V \in L^2([0, T], \mathcal{M}(\Omega))$  such that, up to a subsequence,

$$\begin{aligned} \mu_\varepsilon &\rightharpoonup \mu && \text{in } (C_c^{0,\gamma}([0, T] \times \Omega))', \quad \forall \gamma > 0, \\ V_\varepsilon &\rightharpoonup V && \text{in } (C_c^{0,\gamma}([0, T] \times \Omega))', \quad \forall \gamma > 0, \end{aligned}$$

with

$$\partial_t \mu + \text{div } V = 0. \tag{1.20}$$

Moreover, for any  $X \in C_c^0([0, T] \times \Omega, \mathbb{R}^2)$  and  $f \in C_c^0([0, T] \times \Omega)$ , we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|^2} \int_{\Omega \times [0, T]} |X \cdot \nabla u_\varepsilon|^2 \int_{\Omega \times [0, T]} f^2 |\partial_t u_\varepsilon|^2 \geq \frac{1}{4} \left| \int_{\Omega \times [0, T]} V \cdot f X \right|^2. \tag{1.21}$$

Observe that taking  $|X| \leq 1$  and  $|f| \leq 1$ , for solutions of (1.1) assuming  $\int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq C|\log \varepsilon|$ , we have

$$|\log \varepsilon|^{-1} \int_{\Omega \times [0, T]} |X \cdot \nabla u_{\varepsilon}|^2 \leq CT,$$

while  $|\log \varepsilon|^{-1} \int_{\Omega \times [0, T]} |\partial_t u_{\varepsilon}|^2 = E_{\varepsilon}(u_{\varepsilon}(0)) - E_{\varepsilon}(u_{\varepsilon}(T))$ , thus the relation (1.21) states essentially that for such solutions,

$$|p_i(T) - p_i(0)|^2 \leq CT(E_{\varepsilon}(u_{\varepsilon}(0)) - E_{\varepsilon}(u_{\varepsilon}(T))) \tag{1.22}$$

or in condensed notation

$$|\Delta p|^2 \leq C \Delta T \Delta E \tag{1.23}$$

relating  $\Delta p$ , the difference in position  $p$ ;  $\Delta T$ , the difference in time; and  $\Delta E$ , the difference in energy. This crucial relation reflects of course the parabolic scaling and will be used to bound collision times from below.

The product estimate also allowed us to define limiting continuous (in fact  $C^{0,1/2}$ ) vortex trajectories as follows.

**Proposition 1.1 (Vortex trajectories, see [SS2]).** *Let  $u_{\varepsilon}(x, t)$  be defined over  $\Omega \times \mathbb{R}_+$  and such that (1.18) holds with  $T = +\infty$  (in particular these hold for  $u_{\varepsilon}$  solving (1.1) with (1.27) holding). Then, after extraction of a subsequence, there exist points  $p_i(t)$  and integers  $D_i(t) \in \mathbb{Z}$  and  $n(t) \in \mathbb{N}$  such that*

$$\operatorname{curl}(iu_{\varepsilon}, \nabla u_{\varepsilon})(t) \rightharpoonup \mu(t) = 2\pi \sum_{i=1}^{n(t)} D_i(t) \delta_{p_i(t)} \quad \text{as } \varepsilon \rightarrow 0,$$

moreover  $t \mapsto \langle \zeta, \mu(t) \rangle \in H^1((0, \infty))$  for every  $\zeta \in C_c^1(\Omega)$ . If in addition, for a given  $\tau$ ,  $\sum_{i=1}^{n(t)} |D_i(t)| \leq \sum_{i=1}^{n(\tau)} |D_i(\tau)|$  for every  $t \geq \tau$ , and  $D_i(\tau) = \pm 1$  with the  $p_i(\tau)$  distinct, then there exists  $T_* > \tau$  such that for every  $t \in [\tau, T_*)$ ,

$$\mu(t) = 2\pi \sum_{i=1}^{n(\tau)} D_i(\tau) \delta_{p_i(t)}$$

where the  $p_i(t)$  are distinct points and  $p_i \in H^1((\tau, T_*), \Omega)$ . Moreover, if  $T_* < \infty$  then

$$\lim_{t \rightarrow T_*^-} \min_{i \neq j} \left( \min |p_i(t) - p_j(t)|, \min_i \operatorname{dist}(p_i(t), \partial\Omega) \right) = 0.$$

Returning to our two relations (1.16) and (1.17), once they are proved, we combine them with (1.15), and integrate in time, which leads to

$$\begin{aligned} E_{\varepsilon}(u_{\varepsilon}(0)) - E_{\varepsilon}(u_{\varepsilon}(T)) &\geq \frac{1}{2} \|\partial_t u_{\varepsilon}\|_{X_{\varepsilon}}^2 + \|\nabla_{X_{\varepsilon}} E_{\varepsilon}(u_{\varepsilon})\|_{X_{\varepsilon}}^2 \\ &\geq \frac{1}{2} \sum_i \int_0^T \pi \left( |d_t p_i|^2 + \frac{1}{\pi} |\nabla_i W_{\mathbf{D}}(p)|^2 \right) + o(1) \\ &\geq \sum_i \int_0^T \langle -d_t p_i, \nabla_i W_{\mathbf{D}}(p) \rangle + o(1) \\ &\geq W_{\mathbf{D}}(p_1(0), \dots, p_n(0)) - W_{\mathbf{D}}(p_1(T), \dots, p_n(T)) + o(1). \end{aligned} \tag{1.24}$$

When  $u_\varepsilon(0)$  is “very well prepared”, this implies that  $E_\varepsilon(u_\varepsilon(T)) \leq \pi n|\log \varepsilon| + W_{\mathbf{D}}(p_i(T)) + n\gamma + o(1)$ . But the  $\Gamma$ -convergence of  $E_\varepsilon$  yields the opposite inequality, hence there is equality, in particular equality in the Cauchy–Schwarz relation (1.24) which allows us to retrieve the dynamical law  $\partial_t p_i = -\pi^{-1}\nabla_i W_{\mathbf{D}}(p_1, \dots, p_n)$ , as long as the number of vortices remains fixed. An important fact which follows is that “very well preparedness” is preserved through the flow, i.e. we always have  $E_\varepsilon(u_\varepsilon(t)) = \pi n|\log \varepsilon| + W_{\mathbf{D}}(p_1(t), \dots, p_n(t)) + n\gamma + o(1)$ .

Part of what we do here is to prove that this scheme can be carried out even after blow-up in space-time, allowing us to treat the situation when vortices are at a distance  $l \ll 1$ , as long as  $l$  is not too small. This will be the object of Theorems 4 and 6.

Let us now turn to the other part of the approach. Vortices colliding correspond to the more general fact that several vortices converge to the same limit as  $\varepsilon \rightarrow 0$ , with possible (but not necessarily) limiting degree 0. When vortices are well separated, then time needs to be accelerated as in (1.1) in order to see vortex motion, as first observed in [RS]. But this is not true when vortices become very close, because formally the phase excess  $\varphi$  of the solution  $u_\varepsilon$  then decays according to an accelerated heat equation  $\partial_t \varphi / |\log \varepsilon| = \Delta \varphi$ , as pointed out in [Li, JS1, BOS1], thus in the faster time scale  $1/|\log \varepsilon|$ , while the other remote vortices should not move. The task will thus consist in retrieving these phenomena quantitatively.

For solutions of the gradient flow, we have seen that the energy-dissipation rate is

$$-\frac{d}{dt} E_\varepsilon(u_\varepsilon(t)) = \|\partial_t u_\varepsilon\|_{X_\varepsilon}^2 = \|\nabla_{X_\varepsilon} E_\varepsilon(u_\varepsilon)\|_{X_\varepsilon}^2. \tag{1.25}$$

If we write for simplicity that (1.11) holds, with  $f_\varepsilon = \partial_t u_\varepsilon / |\log \varepsilon|$ , we have  $f_\varepsilon = -(1/|\log \varepsilon|)\nabla_{X_\varepsilon} E_\varepsilon(u)$  in the previous notations, and

$$\|\nabla_{X_\varepsilon} E_\varepsilon(u)\|_{X_\varepsilon}^2 = |\log \varepsilon| \|f_\varepsilon\|_{L^2(\Omega)}^2. \tag{1.26}$$

Combining this with (1.25), we see that knowing  $\|f_\varepsilon\|_{L^2}$  gives the energy-dissipation rate (in time), or rather  $-\frac{1}{|\log \varepsilon|} \frac{d}{dt} E_\varepsilon$ . If  $\|f_\varepsilon\|_{L^2}$  is large, then the energy dissipates fast, thus decreasing to a point which allows ruling out certain configurations (for example if  $E_\varepsilon$  decreases so much that  $E_\varepsilon \leq C$  then there can be no more vortices). On the other hand, if  $f_\varepsilon$  is small, then the behavior of vortices can be controlled through the results obtained in Part I [S1]. The idea is thus to use this alternative in a *quantitative* way, in order to obtain information on vortex collisions or other pathological situations.

Let us recall one of the main results of Part I (see Theorem 1 in [S1]): assuming that  $u_\varepsilon$  solves (1.11) and under the additional hypotheses

$$E_\varepsilon(u_\varepsilon) \leq M|\log \varepsilon|, \tag{1.27}$$

$$|u_\varepsilon| \leq 1, \quad |\nabla u_\varepsilon| \leq M/\varepsilon, \tag{1.28}$$

$$\|f_\varepsilon\|_{L^2(\Omega)}^2 \leq 1/\varepsilon^\beta \quad \text{for some } \beta < 2, \tag{1.29}$$

we can find what we called a “good collection” of vortices and degrees  $(a_i, d_i)$  of  $u_\varepsilon$ , and we have

$$\forall \alpha < 1, \quad \alpha\pi \sum_{i=1}^n d_i^2 \leq \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} + C|\log \varepsilon|^{7/2} \varepsilon^{1-\alpha} \|f_\varepsilon\|_{L^2(\Omega)} + o(1), \tag{1.30}$$

and

$$\begin{aligned} o(1) &\leq E_\varepsilon(u_\varepsilon) - \left( \pi \sum_{i=1}^n d_i^2 \log \frac{1}{\varepsilon} + W_{\mathbf{d}}(a_1, \dots, a_n) + \sum_{i=1}^n \gamma(V_i) \right) \\ &\leq C \|f_\varepsilon\|_{L^2(\Omega)}^2 + o(1), \end{aligned} \tag{1.31}$$

where  $\gamma(V_i)$  are constants depending on the  $d_i$ 's and equal to  $\gamma$  when  $d_i = \pm 1$ .

This allows us to deduce two important ingredients: an upper bound on the number of actual zeroes of  $u_\varepsilon$  from (1.30), and a *differential inequality on the energy* through (1.31), which is optimal, and allows retrieving the fast parabolic scaling.

### 1.3. Main results on the dynamics

Several of our results give information on the vortices of the solutions  $u_\varepsilon$  at the  $\varepsilon$ -level, giving asymptotic time scales of collisions and of energy dissipation. This is of course a little more precise than just characterizing the trajectories of the limiting vortices, which we do in Theorem 5. We also derive the dynamical law after blow-up (at any not too small scale) during collisions, which is also more precise.

The first application of the theorems proved in [S1] consists in showing that the ‘‘very well prepared assumption’’ that was used in [SS2] is not restrictive since ‘‘well prepared’’ data become instantaneously (i.e. in  $o(1)$  time) ‘‘very well prepared’’, by fast dissipation of the energy excess obtained in (1.31). In fact, we can further relax the well prepared assumption through the following.

**Theorem 2 (Instantaneous ‘‘very well preparedness’’).** *Assume that  $u_\varepsilon$  is a solution of (1.1) such that (1.28) holds and*

$$\operatorname{curl}(iu_\varepsilon^0, \nabla u_\varepsilon^0) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \quad \text{as } \varepsilon \rightarrow 0, \tag{1.32}$$

where  $D_i = \pm 1$  and the  $p_i^0$  are distinct points, and

$$E_\varepsilon(u_\varepsilon^0) \leq \pi n |\log \varepsilon| + \frac{|\log \varepsilon|}{(\log |\log \varepsilon|)^\beta} \tag{1.33}$$

for some  $\beta > 1$ . Then there exists a time  $T_\varepsilon \leq (C \log |\log \varepsilon|)/|\log \varepsilon|$  such that for every  $t_\varepsilon \in [0, T_\varepsilon]$ , we have

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \quad \text{as } \varepsilon \rightarrow 0, \tag{1.34}$$

and

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + o(1). \tag{1.35}$$

That is, under these weaker assumptions (an energy excess  $\gg 1$  is allowed in (1.33)), in a time  $T_\varepsilon = o(1)$ , the initial vortices have not moved, and  $u_\varepsilon$  has become well prepared, i.e. all excess energy has dissipated; one can then apply the previous results [JS1, Li, SS2] starting at time  $T_\varepsilon$  and retrieve the same dynamical law (1.8).

As an application of Theorem 2 of [S1], we get the next result, which allows continuing the dynamics after vortex collisions. Let us assume that we are in the following generic case:  $u_\varepsilon$  has a dipole of vortices of degree  $\pm 1$  colliding, i.e. at a distance  $l \ll 1$  (as  $\varepsilon \rightarrow 0$ ) from each other and converging to a point  $p_{\text{dip}}$  as  $\varepsilon \rightarrow 0$ , and  $n$  other vortices of degree  $\pm 1$ , converging to distinct points  $p_1, \dots, p_n$ , distinct from  $p_{\text{dip}}$ . This situation implies that

$$\operatorname{curl}(iu_\varepsilon^0, \nabla u_\varepsilon^0) \rightarrow 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \quad \text{as } \varepsilon \rightarrow 0, \quad (1.36)$$

with  $D_i = \pm 1$ . We may also assume that there exists  $p_\varepsilon \rightarrow p_{\text{dip}}$  such that, considering  $\bar{u}_\varepsilon(x) = u_\varepsilon(p_\varepsilon + lx, 0)$  we have

$$\operatorname{curl}(i\bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) \rightarrow 2\pi(\delta_{b_+} - \delta_{b_-}) \quad \text{as } \varepsilon \rightarrow 0 \quad (1.37)$$

where  $|b_+ - b_-| = 1$ .

We may also assume that this situation is inherited from a well prepared data at a previous time, so we may assume that  $u_\varepsilon$  is well prepared with respect to these vortices, i.e.  $E_\varepsilon(u_\varepsilon) \leq \pi n |\log \varepsilon| + 2\pi \log(l/\varepsilon) + O(1)$ .

**Theorem 3 (Collisions).** *Let  $u_\varepsilon$  be a solution of (1.1) such that at time 0, (1.28) and (1.36)–(1.37) hold, and*

$$E_\varepsilon(u_\varepsilon^0) \leq \pi n |\log \varepsilon| + 2\pi \log \frac{l}{\varepsilon} + O(1),$$

with  $l = o(1)$ . Then there exists a first time  $T_1 \leq C_1 l^2 + C_2 |\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}} = o(1)$  for which  $u_\varepsilon(T_1)$  has exactly  $n$  zeroes (i.e. the dipole has collided). If  $l \geq \varepsilon^\beta$  with  $\beta < 1$ , then also  $T_1 \geq C_3 l^2$ . Moreover, there exists a time  $T_2 \leq T_1 + (C_4 \log |\log \varepsilon|)/|\log \varepsilon| = o(1)$  such that for every  $t_\varepsilon \leq T_2$ , we have

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightarrow 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \quad \text{as } \varepsilon \rightarrow 0 \quad (1.38)$$

and

$$E_\varepsilon(u_\varepsilon(T_2)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + o(1). \quad (1.39)$$

The relation (1.38) indicates that the vortices not involved in the collision have not moved during the time  $T_2 = o(1)$ , and (1.39) that  $u_\varepsilon$  has become well prepared again relative to those vortices within that time. Thus all excess energy carried by the colliding vortices has dissipated in  $o(1)$  time, and the previously known results apply after that time  $T_2$ , i.e. one may continue and retrieve the dynamical law with the remaining vortices. Moreover, our result shows that the actual collision of the zeroes should happen in  $O(l^2)$  time (the lower bound on the collision time is simply provided by an appropriate version of the “product estimate” Theorem 1 or (1.23)), in agreement with the expectation that the distance between colliding vortices decreases like  $\sqrt{T_* - t}$ , if they interact according to the expected law

$$\frac{da_i}{dt} = \frac{1}{\pi} \frac{a_j - a_i}{|a_j - a_i|^2},$$

while the leftover energy excess dissipates in  $(\log |\log \varepsilon|)/|\log \varepsilon|$  time, in agreement with the time scaling of the equation. A further justification is given by the result of Theorem 4 below.

Analogous results can be derived from Theorems 1 and 2 of [S1] for other “bad” situations when vortices accumulate in an unbalanced cluster, i.e. with  $\sum_i d_i^2 \neq (\sum_i d_i)^2$ , for example two repulsing +1 starting at a distance  $l$ . These results are given in Sections 3.2 and 3.3.

The next result consists in analyzing the vortex collisions or vortex separation by blow-up, in order to retrieve some dynamical law. Thanks to Theorem 1 of [S1], which allows us to control errors, the analysis of [SS2] which we presented above carries through after blow-up, as long as the blow-up scale  $l$  satisfies  $\log^4 l \leq O(|\log \varepsilon|)$ . We assume that, blowing up around  $p^\varepsilon$ , we see blown-up limit vortices  $b_k$ , and give the dynamical law of the  $b_k$ 's. This is the result, where for simplicity of statement we assume there is a unique point  $p$  of accumulation of the vortices (a more general result is given later in the paper, see Theorem 6). Observe it is valid for any number of vortices and any interaction (attractive or repulsive).

**Theorem 4 (Exact dynamical law after blow-up).** *Assume  $u_\varepsilon$  is a solution to (1.1) with (1.27) and (1.28). Assume  $l = o(1)$  with  $\log^4 l \leq C|\log \varepsilon|$ , and the points  $p^\varepsilon \rightarrow p$  are such that, defining  $\bar{u}_\varepsilon(x, t) = u_\varepsilon(p^\varepsilon + lx, l^2t)$ , we have*

$$\operatorname{curl}(i\bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)(0) \rightarrow 2\pi \sum_{k=1}^n D_k \delta_{b_k^0} \quad \text{as } \varepsilon \rightarrow 0 \tag{1.40}$$

with  $D_k = \pm 1$ , and assume

$$E_\varepsilon(u_\varepsilon^0) \leq \pi n |\log \varepsilon| + W_{\sum_k \mathbf{D}_k}(p) - \pi \sum_{k, k': k \neq k'} D_k D_{k'} \log(l|b_k^0 - b_{k'}^0|) + n\gamma + r_\varepsilon \tag{1.41}$$

with either  $r_\varepsilon = o(1)$  or  $r_\varepsilon \leq l^2 |\log \varepsilon| / (\log |\log \varepsilon|)^\beta$  for some  $\beta > 1$ . Then there exist  $H^1((0, T^*))$  trajectories  $b_k(t)$  such that, for every  $t \in [0, T^*)$ ,

$$\operatorname{curl}(i\bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon)(t) \rightarrow 2\pi \sum_{k=1}^n D_k \delta_{b_k(t)} \quad \text{as } \varepsilon \rightarrow 0$$

where  $b_k$  solves the dynamical law

$$\begin{cases} \frac{db_k}{dt} = -\frac{1}{\pi} \sum_{k' \neq k} D_{k'} D_k \frac{b_{k'} - b_k}{|b_{k'} - b_k|^2}, \\ b_k(0) = b_k^0, \end{cases} \tag{1.42}$$

and  $T^*$  is the first collision time under this law. Moreover, for every  $t \in (0, T^*)$ , we have

$$\begin{aligned} E_\varepsilon(u_\varepsilon(l^2t)) &\leq \pi n |\log \varepsilon| + W_{\sum_k \mathbf{D}_k}(p) - \pi \sum_{k, k'} D_k D_{k'} \log(l|b_k(t) - b_{k'}(t)|) + n\gamma + o(1) \end{aligned} \tag{1.43}$$

as  $\varepsilon \rightarrow 0$ .

1.4. The dynamical law of the limiting vortices

Combining easily the results of the previous theorems, we can extend the dynamical law of the limiting vortices (1.8) past collision times, provided there are only “simple” or “dual” collisions.

**Definition 1.** In Proposition 1.1, we say the collision(s) at time  $T_*$  are simple if for every  $i$ ,  $\text{Card}\{j \neq i : \lim_{t \rightarrow T_*^-} |p_j(t) - p_i(t)| = 0\} = 1$  or 0 if  $p_i(t) \rightarrow \partial\Omega$ .

We now state the dynamical law, assuming for simplicity that we are in the case of the Dirichlet boundary condition (which allows us to rule out the case of vortices exiting  $\Omega$ ). The terminology follows that of Proposition 1.1, and the statement is meant to be applied iteratively to  $k = 0, 1, 2, \dots$

**Theorem 5 (Global in time dynamical law).** Let  $u_\varepsilon$  solve (1.1) with Dirichlet boundary condition and be such that (1.28), (1.32) and (1.33) hold. Setting  $T_0 = 0$ , there exist collision times  $0 < T_1 < T_2 < \dots \leq \infty$  such that if either  $k = 0$  or the collisions at times  $T_1, \dots, T_k$  are simple then, denoting by  $p_i^k$  the distinct points in  $\Omega$  and by  $D_i^k = \pm 1$  the integers such that

$$\mu(t) \rightarrow 2\pi \sum_{i=1}^{n_k} D_i^k \delta_{p_i^k} \quad \text{as } t \rightarrow T_k^-,$$

we have

$$\forall t \in [T_k, T_{k+1}) \quad \text{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightarrow \mu(t) = 2\pi \sum_{i=1}^{n_k} D_i^k \delta_{p_i(t)} \quad \text{as } \varepsilon \rightarrow 0$$

where the  $p_i(t)$  solve the initial value problem

$$\begin{cases} \frac{dp_i}{dt} = -\frac{1}{\pi} \nabla_i W_{\mathbf{D}}(p_1, \dots, p_{n_k})(t), \\ p_i(T_k) = p_i^k, \end{cases} \tag{1.44}$$

and  $T_{k+1} \leq \infty$  is the first collision time under this law. Moreover, for every  $t \in (T_k, T_{k+1})$ ,

$$E_\varepsilon(u_\varepsilon(t)) = \pi n_k |\log \varepsilon| + W_{\mathbf{D}}(p_1(t), \dots, p_{n_k}(t)) + o(1) \quad \text{as } \varepsilon \rightarrow 0. \tag{1.45}$$

Finally,  $n_k \leq n_{k-1} - 2$ , hence the number of simple collisions is bounded by  $n_0/2 = n/2$ .

We may sum this theorem up by the following principle: If  $u_\varepsilon$  is a solution of (1.1) such that (1.28), (1.32) and (1.33) hold, then, as long as there are only simple (and not multiple) collisions, the dynamical law of its vortices is given by (1.8), where, when two vortices collide, they should be erased from the list, and the law (1.8) should afterwards be understood as the law with the remaining vortices.

Let us finally point out that in the course of the paper, we also prove general and sharp lower bounds for the Ginzburg–Landau energy in terms of vortices (see Section 4.2).

### 1.5. Perspectives

As we mentioned, one cannot rule out, even though they are not generic, the possibility of multiple collisions under the law (1.8), i.e. of more than two vortices meeting at the same time and place, with mutual distances of the same order. One would first need to classify all the types of collisions that are possible under (1.8). Of particular difficulty is the case of collisions of a group of “balanced” vortices with  $\sum_i d_i^2 = (\sum_i d_i)^2$ , because this does not seem to dissipate any energy. This may be related to the conjecture of Ovchinnikov–Sigal [OS] of existence of nonradial solutions of Ginzburg–Landau in the whole plane, that is, with several vortices satisfying  $\sum_i d_i^2 = (\sum_i d_i)^2$ . The other (i.e. unbalanced) collisions can be treated in the same way as here for dual collisions. (The paper [BOS3] derives the limiting dynamical law for all types of collisions.)

We have not written down every possible result that can be obtained through our method but rather we have tried to treat the most striking cases, and explain in the course of the paper how to generalize to other situations. In contrast to [BOS1, BOS2], our study does not really allow relaxing further the prepared assumption (1.33) into (1.27) nor relaxing the hypothesis  $D_i = \pm 1$ , because under the only hypothesis  $E_\varepsilon(u_\varepsilon^0) \leq C|\log \varepsilon|$ , the hypothesis (1.32) allows for substructures of vortices converging to each  $p_i^0$ . However, Theorems 4 and 6 give an example of how to deal with such cases (see Remark 5.3). Also, in very short time we must have  $\|\partial_t u_\varepsilon / |\log \varepsilon|\|_{L^2}^2 = \|f_\varepsilon\|_{L^2}^2 \leq C/\varepsilon^\beta$ ,  $\beta < 2$ , and then these substructures of vortices are well defined (see Proposition 2.2 in Part I) and satisfy (1.31). The first difficulty here is to prove that when the small vortex structures form, the  $p_i^0$  do not move (this should be done as in Theorems 3, 4 and 6), the second more delicate one is to understand what happens to zeroes of degree  $\neq \pm 1$  (we know that configurations with vortices of degree  $> 1$  can be stationary even though not stable; on the other hand, once we know that a vortex of degree  $> 1$  has split into several vortices, we can use our method as in Section 3.2). Then these clusters of vortices should interact according to, typically, Theorem 4 or 6. The closest vortices, at distance  $l$ , should collide (or separate) first, in  $O(l^2)$  time, while the others do not move in that time scale; then the closest vortices among those left should interact, etc., until, after an  $o(1)$  time there should only be vortices at finite distances left, probably near each  $p_i^0$  if  $D_i = \pm 1$ —but not necessarily otherwise—and the configuration should become “very well prepared” according to (1.31).

A delicate open problem would be to completely release the assumption (1.27) and thus the upper bound on the number of vortices.

Finally, it would be interesting to study the law (1.44) and see in particular if the following results hold: in the Dirichlet case, after a finite time (independent of  $\varepsilon$ ), there are  $d = \deg g \geq 0$  vortices of degree 1 left; in the Neumann case, after a finite time, there are no vortices left in  $\Omega$ .

## 2. First applications to the energy dissipation

We start by presenting the most direct applications of the “static” results of Part I. They rely mainly on studying the energy decay through a simple differential inequality. We

always assume that  $u_\varepsilon$  solves (1.1) with Dirichlet or Neumann boundary conditions, with  $E_\varepsilon(u_\varepsilon^0) \leq M|\log \varepsilon|$ ,  $|u_\varepsilon^0| \leq 1$  and  $|\nabla u_\varepsilon^0| \leq M/\varepsilon$ . We recall that the existence and uniqueness of the solution of (1.1) is known, and that standard estimates prove that the above estimates on  $u_\varepsilon^0$  remain true at later times, with constants independent of  $t$ . Thus the results of Part I, where the error terms only depend on these constants, can be applied, and yield errors independent of time.

2.1. A clearing-out lemma

We start with a first simple result, because it gives the model for the other proofs; it is a sort of clearing-out result (here we use this terminology borrowed from the literature—e.g. Iلمانen’s paper on Allen–Cahn—in a loose sense meaning disappearance of all vortices and excess energy), saying that if initially there is little energy (less than what is needed to create a vortex), then the solution is completely cleaned up in very short time. This may happen for instance with an initial dipole of vortices of degree  $\pm 1$  at distance  $l \leq \varepsilon^\gamma$ ,  $\gamma > 1/2$ , initially, which can be constructed to have an energy  $\leq 2\pi \log(l/\varepsilon) \leq 2\pi(1 - \gamma)|\log \varepsilon|$ . The result corresponds to the energy decay of the phase excess through the accelerated heat equation.

We recall the definition of  $W_0$  was

$$W_0 = \int_{\Omega} |\nabla \Phi|^2 \tag{2.1}$$

where  $\Phi = 0$  in the Neumann case, and  $\Phi$  is a harmonic function with  $\partial\Phi/\partial\nu = (ig, \partial g/\partial\tau)$  on  $\partial\Omega$  in the Dirichlet case.

**Proposition 2.1 (Clearing-out lemma).** *Let  $u_\varepsilon$  be a solution of (1.1) with Dirichlet or Neumann boundary condition, such that*

$$E_\varepsilon(u_\varepsilon^0) \leq \eta|\log \varepsilon|$$

with  $\eta < \pi$  (this is possible only if  $\deg g = 0$  in the Dirichlet case). Then

1. For any  $\gamma < 2 - \eta/\pi$  in the Dirichlet case, resp.  $\gamma < 2 - 2\eta/\pi$  in the Neumann case, there exists a time  $T_1 \leq \varepsilon^\gamma$  such that  $\|1 - |u_\varepsilon(T_1)|\|_{L^\infty(\Omega)} = o(1)$ .
2. There exists a time  $T_2 \leq (C \log |\log \varepsilon|)/|\log \varepsilon|$  such that for all  $t_\varepsilon \geq T_2$ ,  $\|1 - |u_\varepsilon(t_\varepsilon)|\|_{L^\infty(\Omega)} = o(1)$ , and

$$E_\varepsilon(u_\varepsilon(t_\varepsilon)) \leq W_0 + o(1).$$

*Proof.* First, recall that the energy decreases in time so we always have  $E_\varepsilon(u_\varepsilon(t)) \leq \eta|\log \varepsilon|$  for  $t \geq 0$ . Moreover, writing  $f_\varepsilon = \partial_t u_\varepsilon/|\log \varepsilon|$ , we have

$$|\log \varepsilon| \int_0^t \|f_\varepsilon\|_{L^2(\Omega)}^2 ds = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) \leq \eta|\log \varepsilon|.$$

Hence, by a mean-value argument, we deduce that, for  $\gamma < 2$ , there exists a time  $T_1 \leq \varepsilon^\gamma$  ( $T_1$  depending on  $\varepsilon$ ) such that

$$\|f_\varepsilon\|_{L^2}^2 \leq \eta\varepsilon^{-\gamma}. \tag{2.2}$$

At time  $T_1$ , Proposition 2.2 in [S1] applies, and yields vortices  $(a_i, d_i)$ . Moreover, (1.30) holds, thus

$$\alpha\pi \sum_i d_i^2 \leq \eta + C|\log \varepsilon|^{7/2}\varepsilon^{1-\alpha-\gamma/2} + o(1).$$

Taking  $\alpha > \eta/(2\pi)$  and  $\gamma < 2 - 2\alpha$ , we find that as  $\varepsilon$  gets small enough,  $\sum_i d_i^2 < 2$ , hence  $\sum d_i^2 = 0$  or  $1$ . But the  $d_i$ 's given by Proposition 2.2 in [S1] are all nonzero, hence we deduce that either the set of  $a_i$ 's is empty, and then  $\|1 - |u_\varepsilon(T_1)|\|_{L^\infty(\Omega)} = o(1)$ ; or there is only one  $a_i$  with degree  $+1$  or  $-1$ . This implies that the total degree of  $u_\varepsilon$  in  $\Omega$  is  $\pm 1$ , which is impossible in the Dirichlet case. In the Neumann case, if there is such a vortex  $a_i$ , using Lemma 3.3 in [S1] and examining closely the form of  $W_{\mathbf{a}}$ , we can show that the energy is bounded from below by  $\pi \log(l/\varepsilon) + O(1)$  where  $l$  is  $\text{dist}(a_i, \partial\Omega)$ . This contradicts  $E_\varepsilon(u_\varepsilon) \leq \eta|\log \varepsilon|$  unless  $\text{dist}(a_i, \partial\Omega) \leq \varepsilon^\mu$  for some  $\mu \geq 1 - \eta/\pi$ . But then the second assertion of Theorem 2 in [S1] would give  $\|f_\varepsilon\|_{L^2}^2 \geq C/(|\log \varepsilon|^2\varepsilon^{2\mu})$ . When  $\gamma < 2 - 2\eta/\pi < 2\mu$ , this contradicts (2.2).

This proves that the only possible case is that the set of  $a_i$ 's is empty at time  $T_1$ , and thus the desired property holds at time  $T_1$ .

Let us prove the second property. At any time  $t \geq 0$ , either  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \geq \eta|\log \varepsilon|$  in which case  $E_\varepsilon(u_\varepsilon(t)) \leq \|f_\varepsilon\|_{L^2(\Omega)}^2$ , or  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \leq \eta|\log \varepsilon|$ . In the latter case, Proposition 2.2 in [S1] applies and gives vortices  $(a_i, d_i)$ , and (1.30) yields, for every  $\alpha < 1$ ,

$$\alpha\pi \sum_i d_i^2 \leq \eta + o(1).$$

Since  $\eta < \pi$ , this implies that  $\sum_i d_i^2 < 1$  if  $\varepsilon$  is small enough, hence (using again the fact that the  $d_i$ 's are nonzero integers) the set of vortices  $a_i$  is empty. Applying (1.31), i.e. Theorem 1 of [S1], then yields

$$E_\varepsilon(u_\varepsilon(t)) \leq W_0 + C\|f_\varepsilon\|_{L^2(\Omega)}^2 + o(1),$$

where the constant and the  $o(1)$  only depend on the a priori estimates on  $u_\varepsilon$ , hence not on  $t$ . Changing  $C$  if necessary, this means that in all cases, for every  $t \geq 0$ ,

$$E_\varepsilon(u_\varepsilon(t)) \leq W_0 + C\|f_\varepsilon\|_{L^2(\Omega)}^2 + o(1). \tag{2.3}$$

On the other hand, we have  $dE_\varepsilon/dt = -|\log \varepsilon| \|f_\varepsilon\|_{L^2(\Omega)}^2$ , hence we may write

$$E_\varepsilon(u_\varepsilon(t)) \leq W_0 + o(1) - \frac{C}{|\log \varepsilon|} \frac{dE_\varepsilon(u_\varepsilon(t))}{dt}. \tag{2.4}$$

Solving this ordinary differential inequality, we find

$$E_\varepsilon(u_\varepsilon(t)) \leq W_0 + o(1) + (E_\varepsilon(u_\varepsilon^0) - W_0 + o(1))e^{-t|\log \varepsilon|/C}. \tag{2.5}$$

Therefore, if  $t \geq (c \log |\log \varepsilon|)/|\log \varepsilon|$  with  $c$  suitably chosen, we have  $e^{-t|\log \varepsilon|/C} \leq |\log \varepsilon|^{-2}$  and thus from (2.5), using (1.27), we obtain

$$E_\varepsilon(u_\varepsilon(t)) \leq W_0 + o(1).$$

On the other hand, it is not difficult to check that

$$\int_\Omega \frac{(1 - |u_\varepsilon|^2)^2}{\varepsilon^2} \leq C(E_\varepsilon(u_\varepsilon) - W_0),$$

hence  $\int_\Omega (1 - |u_\varepsilon|^2)^2/\varepsilon^2 = o(1)$ , and since  $|\nabla u_\varepsilon| \leq C/\varepsilon$ , this implies by standard arguments that  $|u_\varepsilon| \geq 1 - o(1)$  at any time  $t \geq (c \log |\log \varepsilon|)/|\log \varepsilon|$ , hence the result.  $\square$

### 2.2. Proof of Theorem 2

In this subsection, we prove Theorem 2 which shows that, under some weaker assumptions, solutions become “very well prepared” in short time.

We start with a lemma which will be used several times, and whose proof is very similar to that of Proposition 2.1. It asserts that, under a weak condition on the initial energy, solutions become very well prepared in  $O((\log |\log \varepsilon|)/|\log \varepsilon|)$  time if we know that their vortices do not move during that time.

**Lemma 2.1 (Instantaneous very well preparedness provided vortices do not move).**

Let  $u_\varepsilon$  be a solution of (1.1) with Dirichlet or Neumann boundary condition, and (1.28). There exists a time  $T_\varepsilon = (M \log |\log \varepsilon|)/|\log \varepsilon|$  such that if

$$\forall t_\varepsilon \in [0, T_\varepsilon], \quad \text{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightarrow 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \tag{2.6}$$

where the  $p_i^0$ 's are distinct points in  $\Omega$  and  $D_i = \pm 1$ , and

$$E_\varepsilon(u_\varepsilon^0) \leq \pi(n + \eta)|\log \varepsilon| \tag{2.7}$$

for some  $\eta < 1$ , then for every  $t_\varepsilon \leq T_\varepsilon$ ,

$$E_\varepsilon(u_\varepsilon(t_\varepsilon)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + C|\log \varepsilon|e^{-t_\varepsilon|\log \varepsilon|/C} + o(1), \tag{2.8}$$

in particular, if  $M$  is large enough,

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + o(1). \tag{2.9}$$

*Proof.* The strategy is as in the previous proof. For each time, either

$$\|f_\varepsilon\|_{L^2(\Omega)}^2 = \left\| \frac{\partial_t u_\varepsilon}{|\log \varepsilon|} \right\|_{L^2(\Omega)}^2 \gg |\log \varepsilon|,$$

in which case we automatically have

$$E_\varepsilon(u_\varepsilon(t)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + C \|f_\varepsilon\|_{L^2(\Omega)}^2 + o(1), \quad (2.10)$$

or  $\|f_\varepsilon\|_{L^2(\Omega)}^2 = O(|\log \varepsilon|)$ . In that second case, Proposition 2.2 in [S1] applies, giving vortices  $(a_i, d_i)$ , and we may apply (1.30). Combining it with the bound (2.7) on the energy, valid for all times, we find, for every  $\alpha < 1$ ,

$$\alpha \pi \sum_i d_i^2 \leq \pi(n + \eta) + o(1).$$

Taking  $\alpha$  large enough, and using the fact that the  $d_i$ 's are integers, we find

$$\sum_i d_i^2 \leq n.$$

Therefore, the number of  $a_i$ 's is bounded by  $n$ , with equality if and only if there are  $n$  points with  $d_i = \pm 1$  for each  $i$ . On the other hand, for every  $t \in [0, T_\varepsilon]$ , we have (2.6), which implies that there exists at least one  $a_i$  converging to each  $p_k^0$ . Combining this with the above, there can only be one  $a_i$  converging to each  $p_i^0$ , with degree  $d_i = D_i = \pm 1$ . But Theorem 1 of [S1] applies at that time, thus from (1.31), we have

$$E_\varepsilon(u_\varepsilon(t)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(a_1, \dots, a_n) + n\gamma + C \|f_\varepsilon\|_{L^2(\Omega)}^2 + o(1).$$

Combining this with the above convergence of the  $a_i$ 's, we find that (2.10) holds in this case as well. So for every  $t \in [0, T_\varepsilon]$ , we have

$$E_\varepsilon(u_\varepsilon(t)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma - \frac{C}{|\log \varepsilon|} \frac{dE_\varepsilon(u_\varepsilon(t))}{dt} + o(1). \quad (2.11)$$

Solving this differential inequality as in (2.5), we find

$$\begin{aligned} E_\varepsilon(u_\varepsilon(t)) &\leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma \\ &\quad + e^{-t|\log \varepsilon|/C} (E_\varepsilon(u_\varepsilon(0)) - \pi n |\log \varepsilon| - W_{\mathbf{D}}(p_1^0, \dots, p_n^0) - n\gamma + o(1)) + o(1) \\ &\leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + C |\log \varepsilon| e^{-t|\log \varepsilon|/C} + o(1). \end{aligned} \quad (2.12)$$

We see that choosing  $T_\varepsilon = (C \log |\log \varepsilon|)/|\log \varepsilon|$  with  $C$  large enough, we get (2.9).  $\square$

In order to prove Theorem 2, it remains to prove that (2.6) holds, i.e. that the vortices do not move in time  $T_\varepsilon$ . This will follow from a suitable application of the product estimate Theorem 1 (see also (1.23)).

We now assume the hypotheses of Theorem 2 are satisfied. By standard lower bounds (for example (4.6) in Proposition 4.3 below), there exists a constant  $K$  (depending on the  $p_i^0$ 's) such that for any  $u_\varepsilon$  such that (1.32) holds with  $D_i = \pm 1$ , we have

$$E_\varepsilon(u_\varepsilon) \geq \pi n |\log \varepsilon| - K. \quad (2.13)$$

Let us now assume by contradiction that there exists  $T_\varepsilon = O((\log |\log \varepsilon|)/|\log \varepsilon|)$  such that

$$\begin{aligned} |\log \varepsilon| \int_0^{T_\varepsilon} \|f_\varepsilon\|_{L^2(\Omega)}^2(t) dt &= \frac{1}{|\log \varepsilon|} \int_{\Omega \times [0, T_\varepsilon]} |\partial_t u_\varepsilon|^2 \\ &= E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T_\varepsilon)) = \frac{|\log \varepsilon|}{\log |\log \varepsilon|^\beta} + K + 1. \end{aligned}$$

Thus,

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) = E_\varepsilon(u_\varepsilon(0)) - \frac{|\log \varepsilon|}{\log |\log \varepsilon|^\beta} - K - 1 \leq \pi n |\log \varepsilon| - K - 1. \tag{2.14}$$

Rescaling in time, and considering  $w_\varepsilon(x, t) = u_\varepsilon(x, T_\varepsilon t)$ , we have

$$\frac{1}{T_\varepsilon |\log \varepsilon|} \int_{\Omega \times [0, 1]} |\partial_t w_\varepsilon|^2 = \frac{|\log \varepsilon|}{\log |\log \varepsilon|^\beta} + K + 1. \tag{2.15}$$

Applying Theorem 1, we find that for every test function  $f$  compactly supported in  $[0, 1]$  such that  $|f| \leq 1$ , and every test vector field  $X$  compactly supported in  $\Omega \times [0, 1]$ , we have

$$\begin{aligned} \left| \int_{\Omega \times [0, 1]} V \cdot f X \right|^2 &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|^2} \left( \int_{\Omega \times [0, 1]} |X \cdot \nabla w_\varepsilon|^2 \int_{\Omega \times [0, 1]} f^2 |\partial_t w_\varepsilon|^2 \right) \\ &\leq \liminf_{\varepsilon \rightarrow 0} C T_\varepsilon \frac{|\log \varepsilon|}{\log |\log \varepsilon|^\beta} = 0 \end{aligned} \tag{2.16}$$

where  $V$  is the limiting velocity associated to the vortices of  $w_\varepsilon$ . Here, we have used the upper bound on the energy, giving  $\int_\Omega |\nabla w_\varepsilon|^2 \leq C |\log \varepsilon|$ , and (2.15). But we have  $T_\varepsilon ((\log \varepsilon)/(\log |\log \varepsilon|)^\beta) = o(1)$  because  $\beta > 1$  and  $T_\varepsilon \leq (\log |\log \varepsilon|)/|\log \varepsilon|$ , hence we deduce  $V = 0$ , or in other words  $\text{curl}(i w_\varepsilon, \nabla w_\varepsilon)(t) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i^0}$  for every  $t \in [0, 1]$ . This means that the vortices of  $u_\varepsilon$  do not move in  $[0, T_\varepsilon]$ , hence we must have  $E_\varepsilon(u_\varepsilon(T_\varepsilon)) \geq \pi n |\log \varepsilon| - K$ , a contradiction with (2.14). This implies that for every  $T_\varepsilon \leq O((\log |\log \varepsilon|)/|\log \varepsilon|)$ , we have

$$\begin{aligned} |\log \varepsilon| \int_0^{T_\varepsilon} \|f_\varepsilon\|_{L^2(\Omega)}^2(t) dt &= \frac{1}{|\log \varepsilon|} \int_{\Omega \times [0, T_\varepsilon]} |\partial_t u_\varepsilon|^2 = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T_\varepsilon)) \\ &< \frac{|\log \varepsilon|}{\log |\log \varepsilon|^\beta} + K + 1. \end{aligned}$$

Arguing as above, we deduce that  $\text{curl}(i w_\varepsilon, \nabla w_\varepsilon)(t) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i^0}$  for every  $t \in [0, 1]$ , thus, after rescaling, that

$$\text{curl}(i u_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_i D_i \delta_{p_i^0}$$

for every  $t_\varepsilon \leq O((\log |\log \varepsilon|)/|\log \varepsilon|)$ . Then, Lemma 2.1 applies, and proves Theorem 2.

### 3. Energy clearing-out during collisions

In this section, we examine how the energy excess dissipates rapidly during collisions or separation of vortices. Starting with collisions, for simplicity, we consider the generic case of  $n$  isolated vortices of degree  $\pm 1$ , plus a dipole of two vortices of opposite degree  $\pm 1$  colliding. We may also assume that this configuration is inherited from a previous evolution and thus that the configuration is “well prepared” with respect to these vortices, i.e.

$$E_\varepsilon(u_\varepsilon(0)) \leq \pi n |\log \varepsilon| + 2\pi \log \frac{l(0)}{\varepsilon} + O(1) \tag{3.1}$$

where  $l(0)$  is the initial (small) distance between the two vortices of the dipole.

In a later section, we will show the exact dynamical law of such vortices, Theorem 4, valid as long as  $l \geq 1/|\log \varepsilon|^2$  for example. So we may restrict to the situation where  $l(0) \leq 1/|\log \varepsilon|^2$ .

#### 3.1. Motion of the energy-concentration points

We first wish to show that the collision of the two vortices, even though they carry excess energy which dissipates, does not trigger any motion of the other vortices. This requires examining the evolution of the energy-density space-distribution. This is the only point where the method is not purely energetic, and uses the equation (1.1).

We denote by

$$e_\varepsilon(u) = \frac{1}{2} |\nabla u|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2}$$

the energy density. The following result is standard (see for example [Li, JS1, BOS1]):

**Lemma 3.1.** *Let  $u_\varepsilon$  be a solution of (1.1) and  $\chi$  be a  $C^2$  function in  $\Omega$ , constant in a neighborhood of  $\partial\Omega$ . Then*

$$\frac{d}{dt} \int_\Omega \chi e_\varepsilon(u_\varepsilon(t)) = - \int_\Omega \chi \frac{|\partial_t u_\varepsilon|^2}{|\log \varepsilon|} - |\log \varepsilon| \int_\Omega \sum_{i,j=1,2} (\partial_i \partial_j \chi) T_{ij} \tag{3.2}$$

where  $T_{ij}$  denotes the “stress-energy tensor” with coefficients  $T_{ij} = e_\varepsilon(u) \delta_{ij} - (\partial_i u, \partial_j u)$  as in [S1].

*Proof.* A direct calculation yields

$$\partial_t e_\varepsilon(u(x, t)) = \operatorname{div}(\partial_t u, \nabla u) - \left( \partial_t u, \left( \Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) \right) \right) = \operatorname{div}(\partial_t u, \nabla u) - \frac{|\partial_t u|^2}{|\log \varepsilon|}$$

using (1.1). On the other hand, as seen in [S1, eq. (2.3)], with another direct computation, we have

$$\sum_i \partial_i T_{ij} = - \left( \partial_j u, \Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) \right) = - \left( \partial_j u, \frac{\partial_t u}{|\log \varepsilon|} \right).$$

We also observe that  $(\partial_t, u \frac{\partial u}{\partial \nu}) = 0$  on  $\partial\Omega$  in view of the boundary conditions (Dirichlet or Neumann). Combining these relations, and using several integrations by parts, we are led to (3.2).  $\square$

We deduce the following lemma, which states that if the energy of a solution concentrates at initial time only at a finite number of isolated points  $p_1, \dots, p_n$ , then these points do not move in time  $\leq 1/|\log \varepsilon|$ .

Let  $x_1^\varepsilon, \dots, x_n^\varepsilon$  be points such that there exists  $\rho > 0$  independent of  $\varepsilon$  such that  $\min_{i \neq j} |x_i - x_j| > 4\rho$  and  $\min_i \text{dist}(x_i, \partial\Omega) > 4\rho$ . Let us construct a function  $\chi$  such that

$$\begin{cases} \chi \equiv 1 & \text{in } \Omega \setminus \bigcup_i B(x_i, 2\rho), \\ \chi = |x - x_i|^2 & \text{in } B(x_i, \rho), \\ \chi \geq \rho^2 & \text{in } \Omega \setminus \bigcup_i B(x_i, \rho), \\ \chi \in C^2(\Omega). \end{cases} \tag{3.3}$$

**Lemma 3.2.** *Let  $u_\varepsilon$  be a solution of (1.1), and let  $x_1^\varepsilon, \dots, x_n^\varepsilon$  and  $\chi$  be as above. Then for any  $t \geq 0$ ,*

$$\int_\Omega \chi e_\varepsilon(u_\varepsilon(t)) \leq e^{ct|\log \varepsilon|} \int_\Omega \chi e_\varepsilon(u_\varepsilon(0)), \tag{3.4}$$

where the constant  $c$  depends only on  $\rho$ .

*Proof.* We apply Lemma 3.1 with this  $\chi \geq 0$ . First, we use the property of  $|x|^2$  with respect to (3.2), as observed by De Giorgi and used in [So, RS] among others: observing that  $\partial_i \partial_j |x - x_0|^2 = 2\delta_{ij}$ , we find that in  $B(x_k, \rho)$ ,

$$\sum_{i,j} (\partial_i \partial_j |x - x_k|^2) T_{ij} = 2(T_{11} + T_{22}) = \frac{(1 - |u|^2)^2}{\varepsilon^2} \geq 0.$$

Therefore, the contributions in  $\bigcup_i B(x_k, \rho)$  of the right-hand side of (3.2) are nonpositive, and we can write

$$\partial_t \int_\Omega \chi e_\varepsilon(u, t) \leq |\log \varepsilon| \int_{\bigcup_k (B(x_k, 2\rho) \setminus B(x_k, \rho))} \sum_{i,j} \partial_i \partial_j \chi T_{ij}.$$

Observing that  $D^2\chi$  is bounded, and  $\chi \geq \rho^2$  in  $\bigcup_k (B(x_k, 2\rho) \setminus B(x_k, \rho))$ , we may write  $|D^2\chi| \leq C_\rho \chi$  where the constant depends on  $\rho$ . Using in addition the observation that pointwise,  $|T_{ij}| \leq e_\varepsilon(u)$ , we are led (changing  $C_\rho$  if necessary) to

$$\partial_t \int_\Omega \chi e_\varepsilon(u(t)) \leq |\log \varepsilon| \int_{\bigcup_k (B(x_k, 2\rho) \setminus B(x_k, \rho))} C_\rho \chi e_\varepsilon(u(t)) \leq C_\rho |\log \varepsilon| \int_\Omega \chi e_\varepsilon(u(t)).$$

We deduce by Gronwall’s lemma that (3.4) holds.  $\square$

This allows us to deduce

**Proposition 3.1.** *Let  $u_\varepsilon$  be a solution of (1.1) such that*

$$\operatorname{curl}(iu_\varepsilon^0, \nabla u_\varepsilon^0) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \tag{3.5}$$

with  $D_i = \pm 1$ . Assume that there exist  $p_\varepsilon \rightarrow p_{\text{dip}}$  as  $\varepsilon \rightarrow 0$ , with  $p_{\text{dip}}$  distinct from  $\{p_1^0, \dots, p_n^0\}$  and  $l(0) \leq 1/|\log \varepsilon|$ , such that, considering  $\bar{u}_\varepsilon(x) = u_\varepsilon^0(p_\varepsilon + l(0)x)$ , we have

$$\operatorname{curl}(i\bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) \rightharpoonup 2\pi(\delta_{b_+} - \delta_{b_-}) \tag{3.6}$$

where  $b_+$  and  $b_-$  are two points in  $\mathbb{R}^2$  at distance 1 from each other. Assume also that

$$E_\varepsilon(u_\varepsilon^0) \leq \pi n |\log \varepsilon| + 2\pi \log \frac{l(0)}{\varepsilon} + C. \tag{3.7}$$

Then, if  $T_\varepsilon = (\eta \log |\log \varepsilon|)/|\log \varepsilon|$  with  $\eta$  a small enough constant, we have

$$\forall t_\varepsilon \in [0, T_\varepsilon], \quad \operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \tag{3.8}$$

and

$$E_\varepsilon(u_\varepsilon(T_\varepsilon)) \leq \pi n |\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + o(1). \tag{3.9}$$

We can observe right away that this proposition says that in  $O((\log |\log \varepsilon|)/|\log \varepsilon|)$  time, the solution above becomes “very well prepared” with respect to its vortices  $p_1^0, \dots, p_n^0$ , thus the dipole and its energy have completely disappeared in that short time, without affecting the other vortices.

*Proof.* We start by applying the lower bounds obtained through the ball-construction method of Jerrard/Sandier (see for example [SS4, main theorem of Chapter 3]). Before we apply the result, we consider a constant  $\rho > 0$  small enough such that  $\Omega_1 := \bigcup_i B(p_i^0, \rho)$  and  $\Omega_2 := B(p_{\text{dip}}, \rho)$  are disjoint. We then apply the main theorem of Chapter 3 of [SS4] in  $\Omega_1$  and  $\Omega_2$  successively. In  $\Omega_1$ , we apply it with a final radius  $1/|\log \varepsilon|$ . It yields the existence of a finite collection  $\mathcal{B}$  of disjoint closed balls which cover all the zeroes of  $u_\varepsilon^0$  in  $\Omega_1$ , such that the sum of their radii is smaller than  $1/|\log \varepsilon|$  and for every  $B \in \mathcal{B}$ ,

$$\int_B e_\varepsilon(u_\varepsilon^0) \geq \pi |d_B| (|\log \varepsilon| - C \log |\log \varepsilon|)$$

where  $d_B = \deg(u_\varepsilon^0, \partial B)$  if  $B \subset \Omega_1$ , and 0 otherwise. In view of the hypotheses (3.5) and (3.6),  $u_\varepsilon^0$  has at least one zero of nonzero degree converging to each  $p_i^0$ . Since the  $B \in \mathcal{B}$  cover these zeroes, we can deduce that for each  $p_i^0$ , there exists a ball  $B \in \mathcal{B}$  whose center converges to  $p_i^0$ , and such that  $|d_B| \neq 0$ , hence  $|d_B| \geq 1$ . Let us call it  $B_i$  and denote  $x_i$  its center. We have

$$\int_{B_i} e_\varepsilon(u_\varepsilon^0) \geq \pi |\log \varepsilon| - C \log |\log \varepsilon|. \tag{3.10}$$

Similarly, we apply the method in  $\Omega_2 = B(p_{\text{dip}}, \rho)$  with final radius  $l(0)/|\log \varepsilon|$ . Since  $u_\varepsilon^0$  has at least two zeroes of nonzero degree converging to  $p_{\text{dip}}$ , and since the radii are  $\leq l(0)/|\log \varepsilon| \ll l(0)$ , there exist at least two balls with nonzero degree, at distance  $\leq l(0)$  from each other, converging to  $p_{\text{dip}}$  as  $\varepsilon \rightarrow 0$ . They can be included in a larger ball  $B_{\text{dip}}$  of radius  $\leq 2l(0)$ , centered at  $x_{\text{dip}}$ , and such that

$$\int_{B_{\text{dip}}} e_\varepsilon(u_\varepsilon^0) \geq 2\pi \log \frac{l(0)}{\varepsilon} - C \log |\log \varepsilon|. \tag{3.11}$$

We keep this set of balls and discard the others. Combining (3.10), (3.11) and (3.7), we find that

$$\int_{\Omega \setminus (\cup_{i=1}^n B_i \cup B_{\text{dip}})} e_\varepsilon(u_\varepsilon^0) \leq C \log |\log \varepsilon|, \tag{3.12}$$

and

$$\int_{\Omega_2} e_\varepsilon(u_\varepsilon^0) \leq 2\pi \log \frac{l(0)}{\varepsilon} + C \log |\log \varepsilon|. \tag{3.13}$$

Moreover, since the radii are bounded by  $\max(l(0)/|\log \varepsilon|, 1/|\log \varepsilon|) \leq 1/|\log \varepsilon|$ , we have

$$\int_{B_i} |x - x_i|^2 e_\varepsilon(u_\varepsilon^0) \leq \frac{1}{|\log \varepsilon|^2} E_\varepsilon(u_\varepsilon^0) = o(1),$$

and similarly  $\int_{B_{\text{dip}}} |x - x_{\text{dip}}|^2 e_\varepsilon(u_\varepsilon^0) = o(1)$ . Constructing  $\chi$  associated to the points  $x_1, \dots, x_n, x_{\text{dip}}$ , as in (3.3), we deduce from this and (3.12) that

$$\int_{\Omega} \chi e_\varepsilon(u_\varepsilon^0) \leq C \log |\log \varepsilon|.$$

Applying Lemma 3.2, we deduce that for any  $t \geq 0$ ,

$$\int_{\Omega} \chi e_\varepsilon(u_\varepsilon(t)) \leq C e^{ct|\log \varepsilon|} \log |\log \varepsilon|.$$

If  $t_\varepsilon \leq T_\varepsilon = (\eta \log |\log \varepsilon|)/|\log \varepsilon|$ , with  $\eta < 1/(2c)$ , we find

$$\int_{\Omega} \chi e_\varepsilon(u_\varepsilon(t_\varepsilon)) \leq C |\log \varepsilon|^{1/2} \log |\log \varepsilon|. \tag{3.14}$$

This suffices to ensure that (3.5) holds. Indeed, if not then, by continuity of the zeroes of  $u_\varepsilon$  in time, this would imply that for some  $t_\varepsilon \leq T_\varepsilon$ ,  $u_\varepsilon$  has a cluster of zeroes of nonzero total degree, at a distance from the  $x_i$ 's bounded below by a constant independent of  $\varepsilon$ . By the same argument we used above (using lower bounds given by the ball construction), we would get a lower bound contradicting (3.14). Thus (3.5) holds. We shall prove (3.9) after the next proposition.  $\square$

By using the same type of arguments as for Proposition 2.1 and Lemma 2.1, i.e. a differential inequality, combined with Theorem 2 of [S1], we now deduce an upper bound on the time of collision of the vortices, characterized by the fact that  $|u_\varepsilon| \geq 1/2$  in a neighborhood of the collision point. The fact that  $u_\varepsilon^0$  has a dipole at distance  $l(0)$  will only be characterized through the hypothesis on the energy.

**Proposition 3.2 (Upper bound on the collision time).** *Under the same hypotheses as in Proposition 3.1, there exists a time*

$$T'_\varepsilon \leq Cl(0)^2 + C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}$$

such that  $u_\varepsilon(T'_\varepsilon)$  has exactly  $n$  zeroes (given by Proposition 2.2 in [S1]) of degree  $D_i$ .

*Proof.* Let  $T_\varepsilon$  be given by Proposition 3.1 and  $S_\varepsilon$  denote the set of times  $\leq T_\varepsilon$  at which  $\|f_\varepsilon\|_{L^2(\Omega)}^2 = \|\partial_t u_\varepsilon / |\log \varepsilon|\|_{L^2(\Omega)}^2 \geq 1/\varepsilon^\beta$  for some  $\beta < 2$ . Observe that since

$$|\log \varepsilon| \int_0^t \|f_\varepsilon\|_{L^2}^2 = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) \leq C|\log \varepsilon|, \tag{3.15}$$

we have  $\text{meas}(S_\varepsilon) \leq C\varepsilon^\beta$ .

When  $t \notin S_\varepsilon$ , we have  $\|f_\varepsilon\|_{L^2}^2 \leq 1/\varepsilon^\beta$ , thus Proposition 2.2 in [S1] applies, yielding vortices  $(a_i, d_i)$  for which (1.30) holds, hence

$$\alpha\pi \sum_i d_i^2 \leq (n+2)\pi + C|\log \varepsilon|^{7/2} \varepsilon^{1-\alpha-\beta/2},$$

and we may choose  $\frac{n+2}{n+3} < \alpha < 1$  and  $\beta < 2 - 2\alpha$  to get

$$\sum_i d_i^2 < n + 3 + o(1).$$

This gives an upper bound on the possible number of zeroes of  $u_\varepsilon$ : they are fewer than  $n + 2$ . Since there is at least one zero converging to each  $p_i^0$ , this means that there are at most two extra vortices. Moreover, comparing (3.10) with relation (2.33) in [S1], we have  $\sum_{i: a_i \notin \Omega_1} d_i^2 = o(1)$ , hence all extra vortices are at a distance bounded below from the  $p_i^0$ 's. If there are no extra vortices, that is what we want. If there were only one extra vortex, then, since it would have nonzero degree, (3.5) would be contradicted. We are thus left with the case of two vortices, far away from the  $p_i^0$ . Therefore, the sum of their degrees must be 0, otherwise they would add an extra contribution to (3.5). We may denote by  $l(t)$  their distance, and using lower bounds of Lemma 3.3 in [S1] or arguing as in the proof of Proposition 3.1 (using the lower bounds of [SS4] but with final radii 1 in  $\Omega_1$  and  $l(t)/2$  in  $\Omega_2$ ), we have

$$E_\varepsilon(u_\varepsilon(t)) \geq \pi n |\log \varepsilon| + 2\pi \log \frac{l(t)}{\varepsilon} - C. \tag{3.16}$$

Comparing with (3.7) we must have  $\log l(t) \leq \log l(0) + C$ , hence  $l(t) \leq Cl(0)$ , and thus  $l(t) = o(1)$ . Therefore, these two vortices form an unbalanced cluster of vortices at scale  $l(t)$ . If  $l(t) \gg \varepsilon\sqrt{|\log \varepsilon|}$ , then Theorem 2 of [S1] applies and implies that

$$\|f_\varepsilon\|_{L^2(\Omega)}^2 \geq \min\left(\frac{C}{l(t)^2 |\log \varepsilon|}, \frac{C}{l(t)^2 \log^2 \frac{1}{l(t)}}\right). \tag{3.17}$$

If  $l(t) = O(\varepsilon\sqrt{|\log \varepsilon|})$  then the two vortices also form an unbalanced cluster at scale  $\varepsilon|\log \varepsilon|$  and we may also apply Theorem 2 of [S1] to conclude that  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \geq 1/(\varepsilon^2|\log \varepsilon|^4)$ .

Since we always have  $l(t) \geq \varepsilon$ , we may always write, for  $t \notin S_\varepsilon$ ,

$$\|f_\varepsilon\|_{L^2(\Omega)}^2 \geq \frac{C}{l(t)^2|\log \varepsilon|^4}. \quad (3.18)$$

To summarize, in all generality, we can write (3.18), and if  $l(t) \geq e^{-\sqrt{|\log \varepsilon|}}$ , we can write  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \geq C/(l(t)^2|\log \varepsilon|)$ . Let  $S'_\varepsilon$  be the set of times for which  $l(t) \leq e^{-\sqrt{|\log \varepsilon|}}$ . Since (3.15) and (3.18) hold, we have

$$|S'_\varepsilon| \leq C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}.$$

We may now map  $\mathbb{R}_+ \setminus (S_\varepsilon \cup S'_\varepsilon)$  to  $\mathbb{R}_+$  by a mapping  $R_\varepsilon$  which takes out the times in  $S_\varepsilon \cup S'_\varepsilon$  and translates otherwise, thus which shifts every time by at most  $|S_\varepsilon| + |S'_\varepsilon| \leq C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}$ . Considering  $F(t) = E_\varepsilon(u_\varepsilon(R_\varepsilon^{-1}(t)))$  and  $L(t) = l(R_\varepsilon^{-1}(t))$ , we find from (3.15) and (3.17) that

$$F(0) - F(t) \geq \int_0^t \frac{C}{L(s)^2} ds.$$

On the other hand,

$$F(0) - F(t) = E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(R_\varepsilon^{-1}(t))) \leq 2\pi \left( \log \frac{l(0)}{\varepsilon} - \log \frac{L(t)}{\varepsilon} \right) - C$$

from (3.16) and (3.7). Setting  $M(t) = \int_0^t (C/L(s)^2) ds$ , we have

$$2\pi \log \frac{l(0)}{L(t)} - C \geq M(t); \quad (3.19)$$

but since  $M'(t) = C/L(t)^2$ , we may write

$$\pi \log \frac{l(0)^2}{C} + \pi \log M'(t) \geq M(t), \quad (3.20)$$

which transforms into

$$e^{M(t)/\pi} \leq \frac{l(0)^2}{C} M'(t).$$

Integrating, we find

$$e^{-M(t)/\pi} \leq 1 - \frac{t}{Cl(0)^2} \quad (3.21)$$

for some constant  $C$ . If  $t \geq Cl(0)^2$ , we find  $e^{-M(t)/\pi} \leq 0$ , hence a contradiction. We deduce that the set of times for which we cannot say that  $u_\varepsilon$  has exactly  $n$  zeroes has measure less than  $T'_\varepsilon = |S_\varepsilon| + |S'_\varepsilon| + Cl(0)^2 \leq Cl(0)^2 + C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}$ , which implies the result.  $\square$

*Proof of (3.9).* From (3.15), the set of times for which  $\|f_\varepsilon\|_{L^2(\Omega)} \geq \eta|\log \varepsilon|$  has measure  $O(1/|\log \varepsilon|)$ . Hence, with the previous Proposition 3.2, the set of times such that either  $\|f_\varepsilon\|_{L^2(\Omega)} \geq \eta|\log \varepsilon|$  or  $u_\varepsilon$  does not have exactly  $n$  zeroes of degree  $D_i$  has a measure less than  $Cl(0)^2 + C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}} + C/|\log \varepsilon|$ . We deduce that there exists a time  $T_\varepsilon'' \leq Cl(0)^2 + C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}} + C/|\log \varepsilon|$  for which  $u_\varepsilon$  has exactly  $n$  zeroes and  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \leq \eta|\log \varepsilon|$ . In view of (3.5) the zeroes of  $u_\varepsilon(T_\varepsilon'')$  converge to the  $p_i^0$ , and thus we may write, by (1.31),

$$E_\varepsilon(u_\varepsilon(T_\varepsilon'')) \leq \pi n|\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + n\gamma + C\eta|\log \varepsilon| + o(1).$$

Choosing  $\eta$  small enough so that  $C\eta < \pi$ , we find that Lemma 2.1 applies, and thus after a time  $\leq T_\varepsilon'' + O((\log |\log \varepsilon|)/|\log \varepsilon|)$ , (3.9) holds.

A second possible proof is the following: We claim that, for every  $t \leq T_\varepsilon$ ,

$$E_\varepsilon(u_\varepsilon(t)) \leq \pi n|\log \varepsilon| + W_{\mathbf{D}}(p_1^0, \dots, p_n^0) + C\|f_\varepsilon\|_{L^2(\Omega)}^2 + o(1). \tag{3.22}$$

If  $\|f_\varepsilon\|_{L^2}^2 \gg |\log \varepsilon|$  then this is trivially true. If not, then  $\|f_\varepsilon\|_{L^2}^2 = O(|\log \varepsilon|)$ . Returning to the proof of Proposition 3.2, we find that in that case either  $u_\varepsilon(t)$  has exactly  $n$  zeroes of degrees  $D_i$ , in which case (3.22) follows from (1.31), or  $u_\varepsilon(t)$  has two extra vortices, at distance  $l(t)$ , with  $l(t) \leq Cl(0) \ll 1/|\log \varepsilon|$ ; plugging into (3.17) we find  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \gg |\log \varepsilon|$ , a contradiction. Hence, in all cases, (3.22) holds. We may finish as in Lemma 2.1, find that (2.8) holds, from which (3.9) follows.  $\square$

To prove Theorem 3, it only remains to prove the lower bounds on the collision time  $T_1$ , which will be done in Lemma 5.1, and to see what happens when  $l(0) \geq 1/|\log \varepsilon|^2$ , which will be done in Theorem 6 (see the note after Theorem 6).

### 3.2. Time of separation of two vortices

Let us see another example of application, this time for the separation of two vortices of degree  $+1$ . We consider the simplest case where there are initially two vortices of degree  $+1$  at small distance  $l(0)$  from each other, and we assume that initially  $E_\varepsilon(u_\varepsilon^0) \leq 2\pi \log(1/(l(0)\varepsilon)) + C$ . The case where there are other well separated vortices in the sample can be treated as well, as in Theorem 3.

**Proposition 3.3.** *Let  $u_\varepsilon$  be a solution of (1.1) with Dirichlet boundary condition of degree 2. Assume that at time 0,*

$$E_\varepsilon(u_\varepsilon^0) \leq 2\pi \log \frac{1}{l(0)\varepsilon} + C, \tag{3.23}$$

with  $l_0 > l(0) \geq \varepsilon^\beta$ ,  $\beta < 1$ . Then, for every  $l \geq 2l(0)$ , there exists a time

$$T_\varepsilon \leq l^2 \log \frac{l}{l(0)} + |\log \varepsilon|^2 e^{-2\sqrt{|\log \varepsilon|}}$$

for which  $u_\varepsilon$  has two vortices of degree  $+1$ , at distance  $\geq l$ . If in addition, there exists a point  $p_\varepsilon$  such that, considering  $\bar{u}_\varepsilon(x) = u_\varepsilon^0(p_\varepsilon + l(0)x)$ , we have

$$\operatorname{curl}(i\bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon) \rightarrow 2\pi(\delta_{b_1} + \delta_{b_2}) \quad \text{as } \varepsilon \rightarrow 0, \tag{3.24}$$

where  $b_1$  and  $b_2$  are two points in  $\mathbb{R}^2$  at distance 1 from each other; then we must have  $T_\varepsilon \geq Cl(0)^2$ .

*Proof.* Let us argue as before, and let  $S_\varepsilon$  be the set of times for which  $\|f_\varepsilon\|_{L^2}^2 \geq 1/\varepsilon^\gamma$ , for some  $\gamma < 2$ . As previously  $|S_\varepsilon| \leq \varepsilon^\gamma$ . On the other hand, for  $t \notin S_\varepsilon$ , we have  $\|f_\varepsilon\|_{L^2}^2 \leq \varepsilon^{-\gamma}$ , hence Proposition 2.2 in [S1] applies and yields vortices  $(a_i, d_i)$  with

$$\alpha\pi \sum_i d_i^2 \leq 2\pi(1 + \beta) + C|\log \varepsilon|^{7/2}\varepsilon^{1-\alpha-\gamma/2}.$$

We may choose  $\gamma$  and  $\alpha$  such that  $(2 + 2\beta)/4 < \alpha < 1 - \gamma/2$  to get

$$\sum_i d_i^2 < 4$$

for  $\varepsilon$  small enough. Knowing that  $\sum_i d_i = 2$  and  $\sum_i d_i^2 \leq 3$ , the only possibility is to have two vortices of degree  $+1$ . Denoting by  $l(t)$  their distance, we easily check that

$$E_\varepsilon(u(t)) \geq 2\pi \log \frac{1}{l(t)\varepsilon} - C. \tag{3.25}$$

On the other hand, the two vortices form an unbalanced cluster at scale  $l(t)$  so Theorem 2 of [S1] yields, if  $l(t) \gg \varepsilon\sqrt{|\log \varepsilon|}$ ,

$$\|f_\varepsilon\|_{L^2}^2 \geq \min\left(\frac{C}{l(t)^2|\log \varepsilon|}, \frac{C}{l(t)^2 \log^2 \frac{1}{l(t)}}\right),$$

and if  $l(t)$  is smaller, we still have a cluster at scale  $\varepsilon|\log \varepsilon|$ . In all cases we have

$$|\log \varepsilon| \|f_\varepsilon\|_{L^2}^2 \geq \frac{C}{|\log \varepsilon|^4 l(t)^2}.$$

Let  $S'_\varepsilon$  be the set of times for which  $l(t) \leq e^{-\sqrt{|\log \varepsilon|}}$ . We have

$$|S'_\varepsilon| \leq C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}.$$

For  $t \notin S'_\varepsilon$ , comparing (3.25) and (3.23), we have

$$2\pi \log \frac{l(t)}{l(0)} + C \geq \int_0^t \frac{C}{l(s)^2} ds. \tag{3.26}$$

Now, assume by contradiction that  $l(t) \leq l$ . Then we must have

$$2\pi \log \frac{l}{l(0)} + C \geq \frac{Ct}{l^2}$$

and  $t \leq Cl^2(\log(l/l(0)) + 1)$ . Adding the times when  $t \in S_\varepsilon \cup S'_\varepsilon$ , we find there must exist some  $t \leq Cl^2 \log(l/l(0)) + C|\log \varepsilon|^4 e^{-2\sqrt{|\log \varepsilon|}}$  for which  $l(t) \geq l$ . The other assertion can be obtained exactly as in Lemma 5.1.  $\square$

**Remark 3.1.** Observe that again we only need to consider small  $l$ 's here, because otherwise, the dynamics is given by Theorem 6.

### 3.3. Exit through the boundary

The situation of vortices exiting through the boundary can only happen for the Neumann boundary condition, and is in fact very similar to the case of colliding vortices, since it can be viewed as the collision of a vortex with its “image vortex”, the vortex of opposite degree reflected through the boundary. Assume for example that initially a solution  $u_\varepsilon$  of (1.1) has a vortex converging as  $\varepsilon \rightarrow 0$  to a point  $p \in \partial\Omega$ , and that  $\partial\Omega$  is locally flat near  $p$ . Then  $\Omega$  and  $u_\varepsilon$  can be reflected around this piece of boundary, leading to a double domain with a colliding dipole. The case of a nonflat boundary requires adjustments, but the idea is the same. Therefore, we shall not treat the exit case in detail, but mention that exactly the analogous results to Theorem 3 could be obtained.

## 4. Applications to lower bounds

### 4.1. Time estimates through blow-up

In this subsection, we rescale the “product-estimate” Theorem 1, in order to bound from above the movement of the vortices (or bound from below the time it takes them to collide according to (1.23)). This will allow retrieving the vortex dynamics after blow-up.

First, we write a blown-up version of Theorem 1.

**Proposition 4.1.** *Let  $R$  be a constant. Let  $l \rightarrow 0$  as  $\varepsilon \rightarrow 0$  with  $l \geq \varepsilon^\beta$  for some  $\beta < 1$ , and let  $\eta = \varepsilon/l$ . Let  $u_\varepsilon(x, t)$  be defined over  $[0, l^2T] \times B(p_\varepsilon, Rl)$  such that*

$$\forall t \in [0, l^2T], \quad E_\varepsilon(u_\varepsilon(t), B(p_\varepsilon, Rl)) \leq C|\log \eta|, \tag{4.1}$$

$$\int_{B(p_\varepsilon, Rl) \times [0, l^2T]} |\partial_t u_\varepsilon|^2 \leq C|\log \eta|. \tag{4.2}$$

Consider  $\bar{u}_\varepsilon(x, t) = u_\varepsilon(p_\varepsilon + lx, l^2t)$  defined in  $[0, T] \times B(0, R)$ . Then, up to a subsequence, for every  $t \in [0, T]$ ,

$$\text{curl}(i\bar{u}_\varepsilon, \nabla\bar{u}_\varepsilon) \rightharpoonup \mu(t) \quad \text{in } (C_c^{0,\gamma}(B(0, R)))^*, \quad \forall \gamma > 0,$$

where  $\mu(t)$  is of the form

$$2\pi \sum_i D_i(t) \delta_{b_i(t)}, \quad D_i(t) \in \mathbb{Z}.$$

Moreover, there exists a vector-valued measure  $V$  such that  $\partial_t \mu + \operatorname{div} V = 0$ ; and, for every  $X \in C_c^0([0, T] \times B(0, R), \mathbb{R}^2)$  and  $f \in C_c^0([0, T] \times B(0, R))$ ,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \eta|^2} \int_{B(0, R) \times [0, T]} |X \cdot \nabla \bar{u}_\varepsilon|^2 \int_{B(0, R) \times [0, T]} f^2 |\partial_t \bar{u}_\varepsilon|^2 \\ \geq \frac{1}{4} \left| \int_{B(0, R) \times [0, T]} V \cdot f X \right|^2. \end{aligned} \quad (4.3)$$

We deduce the existence of vortex trajectories at that scale, analogous to Propositions 1.1 (as in Proposition 3.2 and 3.3 of [SS2] and Corollary 7 if [SS1]).

**Proposition 4.2.** *Let  $u_\varepsilon$  satisfy the same hypotheses as the previous proposition. If  $D_i(0) = \pm 1$ , the  $b_i(0)$  are distinct and  $\sum_i |D_i(t)| \leq \sum_i |D_i(0)|$  for every  $t \in [0, T]$ , then there exists  $T^* < T$  and  $n = n(0)$  functions  $b_i(t) \in H^1((0, T^*), \mathbb{R}^2)$  such that for all  $t \in [0, T^*)$ , the points  $b_i(t)$  are distinct and  $\mu(t) = 2\pi \sum_i D_i(0) \delta_{b_i(t)}$ . Moreover, if  $T^* < T$ , then as  $t \rightarrow T^*$ , either one  $b_i(t)$  tends to  $\partial B(0, R)$  or there exist  $i \neq j$  such that  $b_i(t)$  and  $b_j(t)$  tend to the same point.*

*If in addition  $\int_{B(0, R)} |\nabla \bar{u}_\varepsilon|^2 \leq 2\pi n |\log \eta| (1 + o(1))$  for all  $t \in [t_1, t_2] \subset [0, T^*)$ , then*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \eta|} \int_{B(0, R) \times [t_1, t_2]} |\partial_t \bar{u}_\varepsilon|^2 \geq \pi \sum_i \int_{t_1}^{t_2} |\partial_t b_i|^2 dt. \quad (4.4)$$

#### 4.2. Applications to lower bounds

This section is a little detour out of the question of Ginzburg–Landau dynamics into the question of sharp lower bounds for the Ginzburg–Landau energy. Thanks to the time-dependent approach, we can obtain in a simple manner very general lower bounds for the energy, improving that of Lemma 3.3 in [S1] (which required  $\|f_\varepsilon\| \leq C/\varepsilon^\gamma$  so that we have vortex small balls given by Proposition 2.2 in [S1]).

The idea is to flow the configuration for a very short time according to (1.1). This decreases the energy and smooths out small irregularities. It yields an alternative to a discrete parabolic regularization as done in [AB].

**Proposition 4.3.** *Assume  $u_\varepsilon$  is such that (1.2) or (1.3) hold with (1.27) and (1.28). Then, up to taking a subsequence, we may assume that there exist distinct points  $p_j$  and integers  $\mathcal{D}_j$  such that*

$$\operatorname{curl}(i u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup 2\pi \sum_{j=1}^n \mathcal{D}_j \delta_{p_j} \quad \text{as } \varepsilon \rightarrow 0. \quad (4.5)$$

Moreover,

$$E_\varepsilon(u_\varepsilon) \geq \pi \sum_j |\mathcal{D}_j| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) + \left( \sum_{i=1}^n |\mathcal{D}_i| \right) \gamma + o(1). \quad (4.6)$$

Suppose there exist a bounded number of points  $p_j^\varepsilon \rightarrow p_j$  (where the  $p_j$ 's are the ones above plus possibly some additional ones with  $\mathcal{D}_j = 0$ ) and  $l_j = o(1)$  with  $|\log l_j| \ll |\log \varepsilon|$  such that, setting  $\bar{u}_{\varepsilon j} = u_\varepsilon(p_j^\varepsilon + l_j x)$  we have

$$\operatorname{curl}(i\bar{u}_{\varepsilon j}, \nabla \bar{u}_{\varepsilon j}) \rightarrow 2\pi \sum_{k=1}^m D_{j,k} \delta_{b_{j,k}} \tag{4.7}$$

with  $D_{j,k} \in \mathbb{Z}$ ,  $\sum_{k=1}^m D_{j,k} = \mathcal{D}_j$ . Then

$$E_\varepsilon(u_\varepsilon) \geq \pi \sum_{j,k} |D_{j,k}| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) - \pi \sum_j \sum_{k,k'} D_{j,k} D_{j,k'} \log(l_j |b_{j,k} - b_{j,k'}|) + \sum_{j,k} |D_{j,k}| \gamma + o(1). \tag{4.8}$$

*Proof.* The fact that we may assume (4.5) follows again from the compactness of the Jacobians  $\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)$ .

Let us write  $u_\varepsilon^0$  for  $u_\varepsilon$ , and denote by  $u_\varepsilon(x, t)$  the solution of the Cauchy problem (1.1) with initial data  $u_\varepsilon^0$  at time 0. We have  $\int_0^T |\partial_t u_\varepsilon|^2 = |\log \varepsilon| (E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(T))) \leq C |\log \varepsilon|^2$ . Therefore, by a mean-value argument, there exists  $T_\varepsilon \leq 1/|\log \varepsilon|^2$  such that  $\int_\Omega |\partial_t u_\varepsilon|^2(T_\varepsilon) \leq C |\log \varepsilon|^4$ . So  $u_\varepsilon(T_\varepsilon)$  solves (1.11) with  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \leq C |\log \varepsilon|^2$ . On the other hand, from Theorem 1, arguing as in the proof of Lemma 2.1 for example, since  $T_\varepsilon \ll 1/|\log \varepsilon|$ , we have

$$\operatorname{curl}(iu_\varepsilon, \nabla u_\varepsilon)(T_\varepsilon) \rightarrow 2\pi \sum_j \mathcal{D}_j \delta_{p_j} \quad \text{as } \varepsilon \rightarrow 0,$$

i.e. the limiting vortices  $p_j$  have not moved. Moreover, since the parabolic flow decreases the energy, we have

$$E_\varepsilon(u_\varepsilon^0) \geq E_\varepsilon(u_\varepsilon(T_\varepsilon)).$$

Therefore, in order to bound  $E_\varepsilon(u_\varepsilon^0)$  from below, we may replace it with  $u_\varepsilon(T_\varepsilon)$ , which has the same vortices in the sense of (4.5) and satisfies (1.11) with  $\|f_\varepsilon\|_{L^2(\Omega)} \leq C |\log \varepsilon|$ . We denote again by  $u_\varepsilon$  the map obtained after replacement. Since we wish to prove (4.6), we may always assume that

$$E_\varepsilon(u_\varepsilon) \leq \pi \sum_j |\mathcal{D}_j| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) + \sum_{i=1}^n |\mathcal{D}_i| \gamma, \tag{4.9}$$

otherwise the result is true.

Since  $\|f_\varepsilon\|_{L^2(\Omega)} \leq C |\log \varepsilon|$ ,  $u_\varepsilon$  satisfies (1.11), (1.29), and the results of Proposition 2.2 in [S1], in particular this yields the  $(a_i, d_i)$ 's with

$$\sum_i d_i^2 \leq \frac{E_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} + o(1),$$

from (1.30). From (4.9), we deduce  $\sum_i d_i^2 \leq \sum_j |\mathcal{D}_j|$ . But, since  $\sum_{i: a_i \rightarrow p_j} d_i = \mathcal{D}_j$ , this implies that the  $d_i$ 's are all  $\pm 1$ , every  $a_i$  converges to one of the  $p_j$ 's (recall  $d_i \neq 0$ ), and that for each  $j$ , the degrees  $d_i$  associated to  $a_i \rightarrow p_j$  all have the sign of  $\mathcal{D}_j$ , hence  $\mathcal{D}_j \neq 0$ .

We may find balls  $B(p_j, \rho)$  with  $\rho \ll 1$  converging to 0 slower than the distance of the  $a_i$ 's to the  $p_j$ 's. This ensures that the hypotheses of Lemma 3.4 of [S1] hold for these balls, and (4.6) is a direct consequence of Lemma 3.4 in [S1].

For the second part of the proposition, we follow the same reasoning. Arguing as above, let us flow  $u_\varepsilon$  according to (1.1). By a mean-value argument, as before, letting  $l = \min_j l_j$ , we may find  $T_\varepsilon \leq 1/|\log \varepsilon|^2$  such that  $u_\varepsilon(T_\varepsilon l^2)$  solves (1.11) with

$$\int_\Omega |f_\varepsilon|^2 \leq C \frac{|\log \varepsilon|^4}{l^2} \tag{4.10}$$

and in view of the hypotheses on  $l_j$ , this ensures that (1.29) is satisfied at time  $l^2 T_\varepsilon$ . On the other hand, considering  $\bar{u}_{\varepsilon j} = u(p_j^\varepsilon + l_j x, l_j^2 t)$ , we have  $\text{curl}(i\bar{u}_{\varepsilon j}, \nabla \bar{u}_{\varepsilon j})(T_\varepsilon) \rightarrow 2\pi \sum_k D_{j,k} \delta_{b_{j,k}}$ , i.e. the  $b_{j,k}$ 's have not moved in that time, according to Proposition 4.1 (see (4.3)). Since the energy decreases in time, this means that we can assume that  $u_\varepsilon$  is such that (1.11)–(1.29) hold. We may also assume that

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &\leq \pi \sum_{j,k} |D_{j,k}| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) \\ &\quad - \pi \sum_j \sum_{k,k'} D_{j,k} D_{j,k'} \log(l_j |b_{j,k} - b_{j,k'}|) + \sum_{j,k} |D_{j,k}| \gamma \end{aligned}$$

as otherwise the desired result is true. Since  $|\log l_j| \ll |\log \varepsilon|$ , this implies

$$E_\varepsilon(u_\varepsilon) \leq \pi \sum_{j,k} |D_{j,k}| |\log \varepsilon| (1 + o(1)). \tag{4.11}$$

Applying Proposition 2.2 in [S1], we find a bounded collection of  $(a_i, d_i)$ . Combining (4.11) with (1.30) and (4.10) yields

$$\begin{aligned} \alpha \sum_i d_i^2 &\leq \sum_{j,k} |D_{j,k}| + \varepsilon^{1-\alpha} |\log \varepsilon|^{7/2} \|f_\varepsilon\|_{L^2(\Omega)} + o(1) \\ &\leq \sum_{j,k} |D_{j,k}| + \varepsilon^{1-\alpha} |\log \varepsilon|^{7/2} \frac{|\log \varepsilon|^2}{l} + o(1). \end{aligned} \tag{4.12}$$

Using the fact that  $|\log l_j| \ll |\log \varepsilon|$ , i.e.  $l \geq \varepsilon^{c_\varepsilon}$  with  $c_\varepsilon \rightarrow 0$ , and taking  $\alpha$  close to 1, we find, since the  $d_i$ 's and  $D_{j,k}$ 's are integers, that  $\sum_i d_i^2 \leq \sum_{j,k} |D_{j,k}|$ . This implies that each  $d_i = \pm 1$  and has the sign of the corresponding  $D_{j,k}$ . Moreover, this also implies that (since  $d_i \neq 0$ ) all the  $a_i$ 's are close to the  $p_j^\varepsilon + l_j b_{j,k}$ . Let us now consider the  $y_{j,k} = p_j^\varepsilon + l_j b_{j,k}$ . Then all the  $a_i$ 's remain inside the  $B(y_{j,k}, \rho l_j)$  for some  $\rho \ll 1$ . We may also choose  $\rho$  large enough so that the hypotheses of Lemma 3.4 in [S1] are satisfied for these balls. The total degree on each ball is  $D_{j,k}$  and using that result, we easily deduce (4.8).  $\square$

**Remark 4.1.** If  $|\log l_j| \ll |\log \varepsilon|$  is not satisfied but we still have  $l_j \geq \varepsilon^\beta$  for some  $\beta < 1$ , then we may still get analogous results from Lemma 3.4 in [S1].

**5. Exact dynamical laws —Theorems 4 and 5**

5.1. Statement of the result

Given points  $b_k$  and integers  $D_k$ , we introduce

$$\bar{W}(b_1, \dots, b_m) = -\pi \sum_{i,j:i \neq j} D_i D_j \log |b_i - b_j|. \tag{5.1}$$

Observe that

$$\nabla_k \bar{W}(b_1, \dots, b_m) = -\pi \sum_{i \neq k} D_i D_k \frac{b_i - b_k}{|b_i - b_k|^2}. \tag{5.2}$$

**Remark 5.1.** It would be interesting to prove that if  $\sum_{i \neq k} D_i D_k \neq 0$  then  $\nabla \bar{W}(b_i) \neq 0$ .

Our main result of this section is

**Theorem 6.** Assume  $u_\varepsilon$  is a solution to (1.1), with (1.27) and (1.28). Assume  $l = o(1)$  with  $\log^4 l \leq C|\log \varepsilon|$ , and the points  $p_j^\varepsilon \rightarrow p_j$ ,  $j \in [1, n]$ , are such that, defining  $\bar{u}_{\varepsilon j}(x, t) = u_\varepsilon(p_j^\varepsilon + lx, l^2t)$ , we have

$$\text{curl}(i\bar{u}_{\varepsilon j}, \nabla \bar{u}_{\varepsilon j})(0) \rightarrow 2\pi \sum_{k=1}^m D_{j,k} \delta_{b_{j,k}^0} \quad \text{as } \varepsilon \rightarrow 0, \tag{5.3}$$

with  $D_{j,k} = \pm 1$ ,  $\sum_k D_{j,k} = D_j$ , and assume

$$E_\varepsilon(u_\varepsilon^0) \leq \pi \sum_{j,k} |D_{j,k}| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) - \pi \sum_j \sum_{k,k': k \neq k'} D_{j,k} D_{j,k'} \log(l|b_{j,k}^0 - b_{j,k'}^0|) + \sum_{j,k} |D_{j,k}| \gamma + r_\varepsilon \tag{5.4}$$

with either  $r_\varepsilon = o(1)$  or  $r_\varepsilon \leq l^2 |\log \varepsilon| / (\log |\log \varepsilon|)^\beta$  with  $\beta > 1$ . Then there exist  $H^1((0, T^*))$  trajectories  $b_{j,k}(t)$  such that for every  $t \in [0, T^*)$ ,

$$\text{curl}(i\bar{u}_{\varepsilon j}, \nabla \bar{u}_{\varepsilon j})(t) \rightarrow 2\pi \sum_k D_{j,k} \delta_{b_{j,k}(t)} \quad \text{as } \varepsilon \rightarrow 0$$

where  $b_{j,k}$  solves the dynamical law

$$\begin{cases} \frac{db_{j,k}}{dt} = -\frac{1}{\pi} \sum_{k' \neq k} D_{j,k'} D_{j,k} \frac{b_{j,k'} - b_{j,k}}{|b_{j,k'} - b_{j,k}|^2} = -\frac{1}{\pi} \nabla_k \bar{W}(b_{j,i}), \\ b_{j,k}(0) = b_{j,k}^0, \end{cases} \tag{5.5}$$

and  $T^*$  is the first collision time under this law. Moreover, for every time  $t \in (0, T^*)$ ,

$$E_\varepsilon(u_\varepsilon(l^2t)) \leq \pi \sum_{j,k} |D_{j,k}| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) - \pi \sum_j \sum_{k,k'} D_{j,k} D_{j,k'} \log(l|b_{j,k}(t) - b_{j,k'}(t)|) + \sum_{j,k} |D_{j,k}| \gamma + o(1), \tag{5.6}$$

as  $\varepsilon \rightarrow 0$ .

**Remark 5.2.** Observe that this result includes the possibility of only one or several vortices at distance  $\ll l$  from  $p_j^\varepsilon$ , in which case there is only one  $b_{j,1}$  equal to the origin, which does not move in this time scale, according to (5.5). This allows us to treat, among others, the case of one dipole colliding while other vortices remain fixed, just as in Theorem 3.

*End of proof of Theorem 3.* To complete the proof of Theorem 3, it remains to consider the case  $1/|\log \varepsilon|^2 \leq l = l(0) = o(1)$ , which can be treated by bridging with Theorem 6. To prove that Theorem 3 also holds in this case, it suffices to show the existence of a time  $T_\varepsilon \leq Cl(0)^2$  at which the hypotheses of Theorem 3 are satisfied (taking the new initial time to be  $T_\varepsilon$ ) with vortices at distance  $1/|\log \varepsilon|^2$ , and that

$$\forall t \in [0, T_\varepsilon], \quad \text{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightarrow 2\pi \sum_{i=1}^n D_i \delta_{p_i^0} \quad \text{as } \varepsilon \rightarrow 0. \tag{5.7}$$

Let us thus start with  $u_\varepsilon$  satisfying the hypotheses of Theorem 3 at time 0, with  $l(0) \geq 1/|\log \varepsilon|^2$ . It also satisfies the hypotheses of Theorem 6 if we take  $l = l(0)$  and the points  $p_j$  to be  $p_1^0, \dots, p_n^0, p_{\text{dip}}$  (with the notations of Section 3). We also have  $b_{1,1}^0 = \dots = b_{n,1}^0 = 0$  while  $b_{n+1,1} = b_+$  and  $b_{n+1,2} = b_-$ . Applying Theorem 6, we obtain the dynamical law of the  $b_{j,k}(t)$ : the  $b_{j,1}(t)$  are fixed for  $j = 1, \dots, n$ , that is, the points  $p_i^0$  do not move in  $O(l(0)^2)$  time, which will prove that for all  $t \leq Cl(0)^2$ , we have  $\text{curl}(iu, \nabla u)(t) \rightarrow 2\pi \sum_{i=1}^n D_i \delta_{p_i^0}$ , that is, (5.7) holds. It remains to prove the existence of  $T_\varepsilon$ . Examining the dynamical law (5.5) for the dipole after space-time rescaling

$$\frac{db_+(t)}{dt} = \frac{1}{\pi} \frac{b_- - b_+}{|b_- - b_+|^2}$$

(and the symmetric law for  $b_-$ ) we see that

$$\frac{d}{dt}|b_+ - b_-|(t) = -\frac{2}{\pi} \frac{1}{|b_+ - b_-|(t)},$$

so we easily find that

$$|b_+ - b_-|(t) = \sqrt{|b_+ - b_-|(0)^2 - 4t/\pi}. \tag{5.8}$$

Now we saw in Section 3 (see the proof of Proposition 3.2 which still applies here) that for all times except a set of measure  $\varepsilon^\beta$ ,  $u_\varepsilon(t)$  has vortices given by Proposition 2.2 in [S1],

and has exactly  $n$  of them converging to each  $p_j^0$ , plus two (the dipole) at distance  $o(l(0))$  from  $p_{\text{dip}} + l(0)b_{\pm}(t)$ . Therefore, in the original space-time, the distance between the vortices of the dipole is  $l(0)\sqrt{l(0)^2 - 4t/(\pi l(0)^2)} + o(l(0))$ . Thus, at time  $t_1 = 3\pi l(0)^2/16$ , the vortices of the dipole are at a distance  $l_1 = l(0)/2 + o(l(0)) < \frac{3}{4}l(0)$ . Moreover, at  $t_1$ , the configuration is well prepared because (5.6) holds. The hypotheses of Theorem 6 are satisfied again at initial time  $t_1$  with scale  $l_1$ . Applying Theorem 6 with this new scale, we find a time  $t_2 = t_1 + 3\pi l_1^2/16$  at which the distance between the vortices of the dipole is  $l_2 = l_1/2 + o(l_1) < \frac{3}{4}l_1$ . We may iterate this process and find times

$$t_k = \frac{3\pi}{16} \left( l(0)^2 + l_1^2 + \dots + l_k^2 \right) \tag{5.9}$$

at which the distance between the vortices is  $< \frac{3}{4}l_k$  with

$$l_k < \frac{3}{4}l_{k-1}. \tag{5.10}$$

This reasoning applies as long as  $\log^4 l_k \leq C|\log \varepsilon|$ , hence we may apply it until final  $l_K \leq 1/|\log \varepsilon|^2$ . Combining (5.9) and (5.10), we find that  $t_K \leq Cl(0)^2$ . Adding if necessary the times for which Proposition 2.2 in [S1] does not apply, we find that in time  $T_\varepsilon \leq \varepsilon^\beta + t_K \leq C\varepsilon^\beta + Cl(0)^2$ , we have a dipole at distance  $\leq 1/|\log \varepsilon|^2$  with (5.6) holding. We have also seen that (5.7) holds. Therefore, all the hypotheses of Theorem 3 hold at that new initial time and the proof of Theorem 3 under the restriction  $l(0) \leq 1/|\log \varepsilon|^2$  can be used to finish the general proof.  $\square$

5.2. Proof of Theorem 5

The existence of collision times follows from Proposition 1.1. Notice also from the form of  $W$  that in the Dirichlet case, no vortex can exit from  $\Omega$  under the law (1.8). Also with any boundary condition, no pairs of vortices of degree  $+1$  (or  $-1$ ) can collide under the law (1.8).

Using Theorem 2 (which yields a time  $T_\varepsilon$ ) and then applying the result of [Li, JS1, SS2] to the solution  $u_\varepsilon(x, t + T_\varepsilon)$  on  $\Omega \times \mathbb{R}_+$ , we find that

$$\text{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t + T_\varepsilon) \rightharpoonup 2\pi \sum_{i=1}^n D_i \delta_{p_i(t)} \quad \text{as } \varepsilon \rightarrow 0, \forall t \in [0, T_*],$$

where the  $p_i$ 's solve (1.8) and  $T_*$  is the first collision time under (1.8). Moreover, since (1.34) holds for every  $t_\varepsilon \in [0, T_\varepsilon]$  and since the  $p_i(t)$  are continuous in time, we deduce that

$$\text{curl}(iu_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup \mu(t) = 2\pi \sum_{i=1}^n D_i \delta_{p_i(t)} \quad \forall t \in [0, T_*]. \tag{5.11}$$

Arguing as in [SS2], the first collision time  $T_*$  is also equal to  $T_1$ , the first collision time of the trajectories in the sense of Proposition 1.1. At time  $T_1$ , there exist one or several pairs of vortices colliding at different places in  $\Omega$ . Assume for simplicity that there is only one pair, say  $|p_1(t) - p_2(t)| \rightarrow 0$  as  $t \rightarrow T_1^-$ . We must have  $D_1 = -D_2$  (otherwise it

would contradict the dynamical law after the blow-up, Theorem 6). We deduce

$$\mu(t) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i^1} \quad \text{as } t \rightarrow T_1^-, \tag{5.12}$$

where  $p_i^1 = \lim_{t \rightarrow T_1^-} p_i(t)$  for  $i = 3, \dots, n$  are distinct points.

By a mean-value argument combining (1.25) and (1.27), we may find a positive  $\tau_\varepsilon \rightarrow 0$  such that at time  $T_1 - \tau_\varepsilon$  we have  $\|f_\varepsilon\|_{L^2(\Omega)}^2 = O(1)$  (where  $f_\varepsilon$  denotes  $\partial_t u_\varepsilon / |\log \varepsilon|$ ). Applying then (1.30), we find that the vortices  $(a_i, d_i)$  of  $u_\varepsilon(T_1 - \tau_\varepsilon)$  given by Proposition 2.2 of [S1] satisfy  $\sum_i d_i^2 \leq n$ . It is then easy to check that there is one vortex  $a_i$  of degree  $D_i$  converging to the respective  $p_i^1$  as  $\varepsilon \rightarrow 0$  for  $i = 3, \dots, n$ ; and two vortices of opposite degrees  $a_1, a_2$ , at distances  $o(1)$  respectively to  $p_1(T_1 - \tau_\varepsilon)$  and  $p_2(T_2 - \tau_\varepsilon)$ , hence at a distance  $l_\varepsilon = o(1)$  from each other. Defining  $v_\varepsilon(t) = u_\varepsilon(t + T_1 - \tau_\varepsilon)$ , we deduce that  $v_\varepsilon$  satisfies the hypotheses (1.36)–(1.37) of Theorem 3. Moreover, from (1.31) (see Theorem 1 in [S1]) and the bound on  $\|f_\varepsilon\|_{L^2}$ , in view of the expression of  $W$ , we have  $E_\varepsilon(v_\varepsilon(0)) \leq \pi(n - 2)|\log \varepsilon| + 2\pi \log(l_\varepsilon/\varepsilon) + O(1)$ . Therefore, we may apply Theorem 3 to  $v_\varepsilon$ , and we deduce the existence of a time  $\tau'_\varepsilon = o(1)$  such that

$$\text{curl}(i v_\varepsilon, \nabla v_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i^1} \quad \forall t_\varepsilon \in [0, \tau'_\varepsilon]$$

and

$$E_\varepsilon(v_\varepsilon(\tau'_\varepsilon)) \leq \pi(n - 2)|\log \varepsilon| + W_{\mathbf{D}}(p_3^1, \dots, p_n^1) + (n - 2)\gamma + o(1).$$

We deduce

$$\text{curl}(i u_\varepsilon, \nabla u_\varepsilon)(t_\varepsilon) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i} \quad \forall t_\varepsilon \in (T_1 - \tau_\varepsilon, T_1 - \tau_\varepsilon + \tau'_\varepsilon) \quad \text{as } \varepsilon \rightarrow 0. \tag{5.13}$$

We may now apply the result of [Li, JS1, SS2] to  $v_\varepsilon$  starting at time  $\tau'_\varepsilon$  and find that

$$\forall t \in [0, T_*), \quad \text{curl}(i v_\varepsilon, \nabla v_\varepsilon)(t + \tau'_\varepsilon) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i(t)}$$

where  $p_i(t)$  solves

$$\begin{cases} \frac{dp_i}{dt} = -\frac{1}{\pi} \nabla_i W_{\mathbf{D}}(p_3, \dots, p_n)(t), \\ p_i(0) = p_i^1, \quad i = 3, \dots, n, \end{cases} \tag{5.14}$$

until the first collision time  $T_*$  under this law. Combining with (5.13) and using the continuity of the  $p_i$ 's, we find

$$\text{curl}(i u_\varepsilon, \nabla u_\varepsilon)(t) \rightharpoonup 2\pi \sum_{i=3}^n D_i \delta_{p_i^1(t)}$$

for every  $t \in [T_1, T_2)$  where  $T_2 = T_1 + T_*$  is the second collision time. The relation (1.45) follows easily from the analysis of [SS2] for example.

The case of more than one collision pair can be treated similarly, observing that just as for Theorems 3 and 6 (applying the method of Proposition 3.1), collisions centered at distinct points in  $\Omega$  do not interfere with one another. Moreover the number of vortices decreases by at least 2 during each collision. The proof can then be iterated at the next collision time  $T_2$ , under the assumption of simple collisions. This completes the proof of Theorem 5.

5.3. Proof of Theorem 6

Before we prove this theorem, we will state a few propositions.

**Proposition 5.1.** *Let  $u_\varepsilon$  be such that (1.27) holds. Assume that for each  $j$  the points  $p_j^\varepsilon$  are such that, defining  $\bar{u}_{\varepsilon j}(x) = u_\varepsilon(p_j^\varepsilon + lx)$ , we have*

$$\operatorname{curl}(i\bar{u}_j, \nabla\bar{u}_j) \rightarrow 2\pi \sum_k D_{j,k} \delta_{b_{j,k}} \quad \text{as } \varepsilon \rightarrow 0 \tag{5.15}$$

with  $D_{j,k} = \pm 1$ ,  $\sum_k D_{j,k} = \mathcal{D}_j$ , and for every constant  $R$ ,

$$E_\varepsilon(u_\varepsilon, B(p_j^\varepsilon, lR)) \leq \pi \sum_k |D_{j,k}| |\log \varepsilon| (1 + o(1)) \tag{5.16}$$

and

$$\log^4 l = O(|\log \varepsilon|). \tag{5.17}$$

Then

$$\lim_{\varepsilon \rightarrow 0} \left( |\log \varepsilon| l^2 \int_\Omega \left| \Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) \right|^2 \right) \geq \frac{1}{\pi} \sum_j \|\nabla \bar{W}(b_{j,k})\|^2 + o_R(1), \tag{5.18}$$

where  $o_R(1) \rightarrow 0$  as  $R \rightarrow \infty$ .

This proposition will be proved further below. We also need a result which is an analogue of Theorem 2 after blow-up.

**Proposition 5.2.** *Under the hypotheses of Theorem 6 without the hypothesis (5.17), there exists  $T_\varepsilon \leq \log(l^2 |\log \varepsilon|) / (l^2 |\log \varepsilon|)$  with  $T_\varepsilon = o(1)$ , and such that for every  $t_\varepsilon \in [0, T_\varepsilon]$ ,*

$$\operatorname{curl}(i\bar{u}_j, \nabla\bar{u}_j)(t_\varepsilon) \rightarrow 2\pi \sum_k D_{j,k} \delta_{b_{j,k}^0}, \tag{5.19}$$

and

$$\begin{aligned} E_\varepsilon(u_\varepsilon(l^2 T_\varepsilon)) &\leq \sum_{j,k} |D_{j,k}| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) \\ &\quad - \pi \sum_j \sum_{k,k': k \neq k'} D_{j,k} D_{j,k'} \log(l|b_{j,k}^0 - b_{j,k'}^0|) + \sum_{j,k} |D_{j,k}| \gamma + o(1). \end{aligned} \tag{5.20}$$

Moreover, there exist  $T_0$  and  $C$  independent of  $\varepsilon$  such that for every  $R$ ,

$$\frac{1}{|\log \eta|} \int_{B(0,R) \times [T_\varepsilon, T_0]} |\partial_t \bar{u}_j|^2 \leq C \tag{5.21}$$

and thus the results of Propositions 4.1 and 4.2 apply, giving  $H^1$  trajectories  $b_{j,k}(t)$  for  $t \in [0, T_0)$  (before collision time).

*Proof.* Let us start with the first assertion, the existence of  $T_\varepsilon$ . If  $l^2|\log \varepsilon|/(\log |\log \varepsilon|)^\beta = o(1)$  then one should take  $r_\varepsilon = o(1)$  and  $T_\varepsilon = 0$  and there is nothing to prove. We can thus focus on  $l^2 \geq C(\log |\log \varepsilon|)^\beta/|\log \varepsilon|$ , which implies that  $T_\varepsilon = o(1)$  in all cases, and  $|\log l| \ll |\log \varepsilon|$ .

We can easily show an analogue of Lemma 2.1: there exists a time  $T_\varepsilon = \log(l^2|\log \varepsilon|)/l^2|\log \varepsilon|$  such that if for all  $t \in [0, T_\varepsilon]$ , and for all  $j$ ,  $\text{curl}(i\bar{u}_{\varepsilon j}, \nabla \bar{u}_{\varepsilon j})(t) \rightarrow 2\pi \sum_k D_{j,k} \delta_{b_{j,k}^0}$ , then (5.20) holds. The proof is along exactly the same lines as that of Lemma 2.1. We show that the hypothesis (5.4) combined with (1.30) implies that for most times  $t_\varepsilon \leq l^2 T_\varepsilon$ ,  $u_\varepsilon(t_\varepsilon)$  has exactly one vortex of degree  $D_{j,k}$  converging (after rescaling) to each  $b_{j,k}^0$ . Then, (1.31), i.e. Theorem 1 of [S1], and a differential inequality lead to

$$E_\varepsilon(u_\varepsilon(t)) \leq \pi \sum_j \sum_k |D_{j,k}| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) + \pi \sum_j \sum_{k,k': k \neq k'} D_{j,k} D_{j,k'} \log(l|b_{j,k}^0 - b_{j,k'}^0|) + \sum_{j,k} |D_{j,k}| \gamma + C e^{-t|\log \varepsilon|} r_\varepsilon.$$

Taking  $T_\varepsilon = \log(l^2|\log \varepsilon|)/l^2|\log \varepsilon|$ , in view of the bound on  $r_\varepsilon$ , we find that (5.20) holds, provided the vortices have not moved.

To prove that the vortices  $b_{j,k}$  have not moved in that time, we argue as in the proof of Theorem 2, and use the product estimate as given in Proposition 4.1.

Assuming that there exists  $T_\varepsilon \leq \log(l^2|\log \varepsilon|)/l^2|\log \varepsilon|$  for which

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 T_\varepsilon)) = r_\varepsilon + K. \tag{5.22}$$

For each  $j$ , define  $w(x, t) = \bar{u}_{\varepsilon j}(x, T_\varepsilon t) = u_\varepsilon(p_j^\varepsilon + lx, l^2 T_\varepsilon t)$ . Since

$$\begin{aligned} \frac{1}{|\log \varepsilon|} \int_{B(0,R) \times [0,1]} |\partial_t w|^2 &= \frac{T_\varepsilon}{|\log \varepsilon|} \int_{B(p_j^\varepsilon, lR) \times [0, l^2 T_\varepsilon]} |\partial_t u_\varepsilon|^2 \leq T_\varepsilon (E_\varepsilon(u_\varepsilon^0) - E_\varepsilon(u_\varepsilon(l^2 T_\varepsilon))), \end{aligned}$$

letting  $V$  be the vortex velocity associated to  $w$ , we deduce that for every compactly supported  $X$  and  $|f| \leq 1$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}^2 \times [0,1]} V \cdot f X \right|^2 &\leq 4 \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|^2} \int_{\mathbb{R}^2 \times [0,1]} |X \cdot \nabla w|^2 \int_{\mathbb{R}^2 \times [0,1]} f^2 |\partial_t w|^2 \\ &\leq \liminf_{\varepsilon \rightarrow 0} C T_\varepsilon (r_\varepsilon + K) \leq C \frac{\log(l^2|\log \varepsilon|)}{l^2|\log \varepsilon|} \left( \frac{l^2|\log \varepsilon|}{(\log |\log \varepsilon|)^\beta} + K \right) \\ &\leq \frac{\log |\log \varepsilon|}{(\log |\log \varepsilon|)^\beta} = o(1). \end{aligned} \tag{5.23}$$

We deduce  $V = 0$ , and thus, the vortices of  $w$  do not move in time 1, i.e. the vortices of  $\bar{u}_{\varepsilon j}$  do not move in time  $T_\varepsilon$ , which implies, from Proposition 4.3 (whose hypotheses are satisfied), the lower bound

$$E_\varepsilon(u_\varepsilon(l^2 T_\varepsilon)) \geq \pi \sum_{j,k} |D_{j,k}| |\log \varepsilon| + W_{\mathcal{D}}(p_1, \dots, p_n) + \pi \sum_j \sum_{k,k': k \neq k'} D_{j,k} D_{j,k'} \log(l|b_{j,k}^0 - b_{j,k'}^0|) + \sum_{j,k} |D_{j,k}| \gamma + o(1), \quad (5.24)$$

a contradiction with (5.22) if  $K$  is chosen large enough. We deduce that for every  $t \leq \log(l^2 |\log \varepsilon|) / |\log \varepsilon|$  we have  $E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(t)) \leq r_\varepsilon + K$ , and following the same reasoning, that the vortices of  $\bar{u}_{\varepsilon j}$  do not move in time  $t/l^2$ . This proves the first part of the proposition.

For the second part, the reasoning is the same. First, we may start from the new initial time  $l^2 T_\varepsilon$  and assume that the solution is very well prepared originally, i.e. that (5.20) holds. Assume by contradiction that there exists  $\tau_\varepsilon \ll l^2$  such that

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(\tau_\varepsilon)) = 1. \quad (5.25)$$

Arguing as above (replacing  $T_\varepsilon$  by  $\tau_\varepsilon$  and  $r_\varepsilon$  by 1) we find that  $V = 0$  in (5.23), and thus the vortices of  $\bar{u}_{\varepsilon j}$  have not moved in time  $\tau_\varepsilon/l^2$ , a contradiction between (5.25), (5.20) holding at time 0 and the lower bound (4.8). Thus, we deduce that there exists a constant  $T_0$  independent of  $\varepsilon$  such that  $E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 T_0)) \leq 1$ , from which (5.21) follows.  $\square$

We now show the lower bound on the collision time, which was left to prove from Section 3 to complete the proof of Theorem 3.

**Lemma 5.1 (Lower bound on collision time).** *Under the hypotheses of Theorem 3, letting  $T_1$  be the first time such that  $u_\varepsilon(x, t)$  has only  $n$  zeroes, we have  $T_1 \geq C_1 l^2$  for some constant  $C_1 > 0$ .*

*Proof.* We may consider  $\bar{u}_\varepsilon(x, t) = u_\varepsilon(p_\varepsilon + lx, l^2 t)$ . Arguing exactly as in the previous proof, we can show that there exists a constant  $T_0 > 0$  independent of  $\varepsilon$  such that

$$C \geq E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 T_0)) = \frac{1}{|\log \varepsilon|} \int_0^{l^2 T_0} \int_\Omega |\partial_t u_\varepsilon|^2 \geq \frac{1}{|\log \varepsilon|} \int_0^{T_0} \int_{B(0,R)} |\partial_t \bar{u}_\varepsilon|^2$$

and thus Proposition 4.2 applies, giving  $H^1$  trajectories  $b_+(t)$  and  $b_-(t)$  for the vortices of  $\bar{u}_\varepsilon$ . Since  $|b_+(0) - b_-(0)| = 1$ , by continuity  $|b_+(t) - b_-(t)| \geq 1/2$  in some time interval  $[0, C_1]$ , which implies that  $u_\varepsilon$  does have two zeroes near  $p_\varepsilon$  in the time interval  $[0, C_1 l^2]$ , hence  $n + 2$  zeroes in all, implying  $T_1 \geq C_1 l^2$ .  $\square$

*Proof of Theorem 6.* Under the hypotheses of Theorem 6, Proposition 5.2 applies. It first proves that we can reduce to the case of very well prepared data, i.e. the case where (5.20) holds, since  $T_\varepsilon = o(1)$ . It also proves that Propositions 4.1 and 4.2 apply on some interval

$[0, T_0]$  (or  $[0, l^2 T_0]$  in original time), giving that for each  $j$ , the  $b_{j,k}(t)$ 's move continuously and remain distinct until collision, while the  $p_j$ 's do not move in that time scale. Moreover, we can check through lower bounds that at each time  $t \geq 0$ , the hypothesis (5.16) of Proposition 5.1 holds and we may apply it to  $u_\varepsilon(t)$ .

We then follow the scheme of [SS2], as presented in the introduction (see for example (1.24)) and write

$$\begin{aligned} E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 t)) &= \frac{1}{|\log \varepsilon|} \int_{\Omega \times [0, l^2 t]} |\partial_t u_\varepsilon|^2 \\ &= \frac{1}{2} \int_{\Omega \times [0, l^2 t]} \frac{1}{|\log \varepsilon|} |\partial_t u_\varepsilon|^2 + \frac{1}{2} \int_{\Omega \times [0, l^2 t]} \left( |\log \varepsilon| |\Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2} (1 - |u_\varepsilon|^2)|^2 \right). \end{aligned}$$

Now given  $R$ , since  $l = o(1)$ , the  $B(p_j^\varepsilon, Rl)$  are disjoint balls for  $\varepsilon$  small enough, hence, after a change of scale, we may write

$$\begin{aligned} E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 t)) &\geq \sum_j \left( \frac{1}{|\log \varepsilon|} \int_{B(0, R) \times [0, t]} \frac{1}{2} |\partial_t \bar{u}_{\varepsilon j}|^2 \right. \\ &\quad \left. + \frac{1}{2} \int_{B(p_j^\varepsilon, Rl) \times [0, t]} \left( l^2 |\log \varepsilon| |\Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2} (1 - |u_\varepsilon|^2)|^2 (l^2 s) \right) \right). \end{aligned}$$

Using the fact that  $|\log \varepsilon| \sim |\log \eta|$  and plugging in (4.4) for the first part and (5.18) for the second, we are led to

$$\begin{aligned} E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 t)) &\geq \sum_j \left( \int_0^t \left( \frac{\pi}{2} \sum_k \left( |d_t b_{j,k}|^2 + \frac{1}{2\pi} \|\nabla \bar{W}(b_{j,k}(s))\|^2 \right) + o_R(1) \right) ds + o(1) \right). \end{aligned} \tag{5.26}$$

Using the crucial Cauchy–Schwarz argument of [SS2], this becomes

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 t)) \geq - \sum_j \int_0^t \sum_k d_t b_{j,k} \cdot \nabla_k \bar{W}(b_{j,k}(s)) + o_R(1) + o(1),$$

hence (taking the limit  $R \rightarrow \infty$ )

$$\lim_{\varepsilon \rightarrow 0} (E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2 t))) \geq \sum_j \sum_k (\bar{W}(b_{j,k}(0)) - \bar{W}(b_{j,k}(t))) \tag{5.27}$$

with equality if and only if for every  $j$ ,  $\partial_t b_{j,k} = -\frac{1}{\pi} \nabla_k \bar{W}(b_{j,i})$  for every  $k$ . But, in view of Proposition 4.3, we must have

$$\begin{aligned} E_\varepsilon(u_\varepsilon(l^2 t)) &\geq \pi \sum_j \sum_k |D_{j,k}| |\log \varepsilon| - \pi \sum_j \sum_{k,k'} D_{j,k} D_{j,k'} \log l \\ &\quad + \sum_j \sum_k \bar{W}(b_{j,k}(t)) + C + o(1) \end{aligned}$$

where the constant  $C$  depends only on the  $p_j$ 's and the set of degrees  $D_{j,k}$ , all constant during the motion. Examining the hypothesis at initial time  $t = 0$ , we see that

$$E_\varepsilon(u_\varepsilon(0)) - E_\varepsilon(u_\varepsilon(l^2t)) \leq \sum_j \sum_k (\overline{W}(b_{j,k}(0)) - \overline{W}(b_{j,k}(t))) + o(1). \tag{5.28}$$

Hence there has to be equality in (5.27) and we conclude that (5.5) holds.  $\square$

**Remark 5.3.** Of course, this can be generalized to multiple scales. Here we have zoomed up at the scale  $l$ , but one should first zoom up at the smallest scale when we see distinct vortices, and rescale time by that  $l^2$ . In that time scale, the other vortices do not move, just like the  $p_j$ 's do not move, only the vortices at small distances from the others move, etc.

5.4. Proof of Proposition 5.1

We assume for simplicity that  $p_j^\varepsilon$  is the origin. We recall that  $\eta = \varepsilon/l$  and that  $|\log l| \ll |\log \varepsilon|$  so that  $|\log \eta| \sim |\log \varepsilon|$ . First, we may assume that

$$\|f_\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{C}{l^2|\log \varepsilon|}, \tag{5.29}$$

otherwise the result stated is true.

Then Proposition 2.2 in [S1] applies and gives vortex points  $a_i$ . For each  $j$ , let us consider the  $a_i$ 's which are at distance  $O(l)$  from  $p_j^\varepsilon$ , and consider their blown-up points  $\bar{a}_i = (a_i - p_j^\varepsilon)/l$ . We may find a constant  $K$  such that, for  $R$  arbitrarily large,  $B(p_j^\varepsilon, 2Rl) \setminus B(p_j^\varepsilon, Kl)$  does not contain any of these  $\bar{a}_i$ 's. Moreover, we claim that the  $\bar{a}_i$ 's converge, up to a subsequence, to some points, which are the  $b_{j,k}$ 's of (5.15). Indeed, if not, then there would be some subset of them converging to another point, with total degree 0 (otherwise it would appear on the right-hand side of (5.15)). But they would then form an unbalanced cluster of vortices at original scale  $\ll l$ . From Theorem 2 of [S1] we would deduce  $\|f_\varepsilon\|_{L^2(\Omega)}^2 \gg 1/(l^2|\log \varepsilon|)$ , contradicting (5.29).

We may thus find a radius  $\rho > 0$  such that the  $B(b_{j,k}, \rho)$  are disjoint, and for small  $\varepsilon$ , the  $B(\bar{a}_i, R_\varepsilon \varepsilon/l)$  we consider are included in the  $B(b_{j,k}, \rho)$ , and we recall  $\sum_{i: \bar{a}_i \rightarrow b_{j,k}} d_i = D_{j,k} = \pm 1$ .

Let us define

$$\begin{cases} -\Delta \Phi_0 = 2\pi \sum_k D_{j,k} \delta_{b_{j,k}} & \text{in } B(0, R), \\ \Phi_0 = h & \text{on } \partial B(0, R), \end{cases} \tag{5.30}$$

where  $h$  will be specified later, and  $G$  by

$$\begin{cases} -\Delta_x G(x, y) = \delta_y & \text{in } B(0, R), \\ G(x, y) = \frac{h(x)}{2\pi \sum_k D_{j,k}} & \text{on } \partial B(0, R), \end{cases} \tag{5.31}$$

and  $S(x, y) = 2\pi G(x, y) + \log|x - y|$ . We have

$$\Phi_0(x) = 2\pi \sum_k D_{j,k} G(x, b_{j,k}) = \sum_k -D_{j,k} \log|x - b_{j,k}| + D_{j,k} S(x, b_{j,k}).$$

We introduce the renormalized energy relative to the ball  $B(0, R)$ :

$$\begin{aligned} \bar{W}_R(b_1, \dots, b_m) &= -\pi \sum_{k,k':k \neq k'} D_{j,k} D_{j,k'} \log|b_{j,k} - b_{j,k'}| \\ &\quad + 2\pi \sum_{k,k'} D_{j,k} D_{j,k'} S(b_{j,k}, b_{j,k'}) + \int_{\partial B(0,R)} h(x) \frac{\partial \Phi_0}{\partial \nu}. \end{aligned} \tag{5.32}$$

A direct calculation identical to the one done for Lemma 3.1 of [S1] shows that

$$\frac{1}{2} \int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, r)} |\nabla \Phi_0|^2 = \pi \sum_k D_{j,k}^2 \log \frac{1}{r} + \bar{W}_R(b_{j,k}) + o_r(1). \tag{5.33}$$

Let us now gather a few intermediate results.

**Lemma 5.2.** *We have*

$$\int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, \rho)} |\nabla \bar{\Phi} - \nabla \Phi_0|^2 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \tag{5.34}$$

where  $\Phi_0$  is defined as the solution of (5.30),  $\bar{\Phi}$  as  $\Phi(p_j^\varepsilon + lx)$  (where  $\Phi$  solves (3.5) in [S1]), and  $h$  is taken to be the uniform limit of  $\bar{\Phi} - (2\pi R)^{-1} \int_{\partial B(0,R)} \bar{\Phi}$  on  $\partial B(0, R)$ . Moreover,

$$\lim_{R \rightarrow \infty} \|\nabla \bar{W}_R(b_{j,1}, \dots, b_{j,m}) - \nabla \bar{W}(b_{j,1}, \dots, b_{j,m})\|_{L^\infty(B(0, K+1)^m)} = 0. \tag{5.35}$$

**Lemma 5.3.** *Let the  $b_{j,k}, D_{j,k}$  be as before, with  $D_{j,k} = \pm 1$  and (5.16). Then for any  $\rho$  such that the  $B(b_{j,k}, \rho)$  are disjoint and do not intersect  $\partial B(0, R)$ , we have*

$$\frac{1}{2} \int_{B(b_{j,k}, \rho)} |\nabla \bar{u}_{\varepsilon j}|^2 = \pi |\log \eta| (1 + o(1)), \tag{5.36}$$

$$\int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, \rho)} \left( |\nabla |\bar{u}_{\varepsilon j}||^2 + \frac{1}{2\eta^2} (1 - |\bar{u}_{\varepsilon j}|^2)^2 \right) = o(l^2 \log^2 l \|f_\varepsilon\|_{L^2(\Omega)}^2 + 1), \tag{5.37}$$

$$\int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, \rho)} |\nabla \bar{u}_\varepsilon - i \bar{u}_\varepsilon \nabla^\perp \Phi_0|^2 = o(l^2 \log^4 l \|f_\varepsilon\|_{L^2}^2 + 1). \tag{5.38}$$

Once we have these results, we can follow closely the proof of [SS2, Proposition 3.5]. For simplicity, we drop the subscripts  $j$ .

Through the change of scale we have

$$\begin{aligned} \bar{E}(\bar{u}_\varepsilon, B(0, R)) &:= \frac{1}{2} \int_{B(0,R)} \left( |\nabla \bar{u}_\varepsilon|^2 + \frac{(1 - |\bar{u}_\varepsilon|^2)^2}{2\eta^2} \right) \\ &= \frac{1}{2} \int_{B(p_j^\varepsilon, lR)} \left( |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} \right) \leq C |\log \varepsilon| \leq C |\log \eta|. \end{aligned} \tag{5.39}$$

By scaling, from (5.29), for every  $R > 0$ ,

$$\int_{B(0,R)} \left| \Delta \bar{u}_\varepsilon + \frac{\bar{u}_\varepsilon}{\eta^2} (1 - |\bar{u}_\varepsilon|^2) \right|^2 \leq \frac{C}{|\log \varepsilon|}, \tag{5.40}$$

and also  $o(1)l^2 \|f_\varepsilon\|_{L^2}^2 \log^4 l \leq o(1) \log^4 l / |\log \varepsilon| = o(1)$  from the assumption (5.17). Hence all the right-hand sides in (5.37) and (5.38) tend to 0 and we can rewrite this

$$\int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, \rho)} \left( |\nabla |\bar{u}_\varepsilon||^2 + \frac{1}{2\eta^2} (1 - |\bar{u}_\varepsilon|^2)^2 \right) = o(1), \tag{5.41}$$

$$\int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, \rho)} |\nabla \bar{u}_\varepsilon - i\bar{u}_\varepsilon \nabla^\perp \Phi_0|^2 = o(1). \tag{5.42}$$

The relation (5.40) is used to obtain, as in [SS2],  $(i\bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) \rightharpoonup \nabla^\perp \Phi_0 + \text{const}$ , and in view of (5.42), the constant vector is 0, that is,

$$(i\bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) \rightharpoonup \nabla^\perp \Phi_0. \tag{5.43}$$

As in [SS2], we consider a set of vectors  $(V_1, \dots, V_m) \in \mathbb{R}^2$ , and a family  $\chi_t$  of diffeomorphisms of  $B(0, R)$  preserving  $\partial B(0, R)$  and such that  $\chi_t(x) = x + tV_k$  in each  $B(b_{j,k}, \rho)$ . We also define  $b_{j,k}(t) = b_{j,k} + tV_{j,k}$ , and

$$\begin{cases} -\Delta \Phi_t = 2\pi \sum_{j,k} D_{j,k} \delta_{b_{j,k}(t)} & \text{in } B(0, R), \\ \partial \Phi_t / \partial \nu = \partial \Phi_0 / \partial \nu & \text{on } \partial B(0, R). \end{cases} \tag{5.44}$$

From  $\Phi_t$  we define  $\psi_t$  exactly as in [SS2, (3.24)], vanishing on  $\partial B(0, R)$ , and  $v_\varepsilon(\chi_t(x), t) = \bar{u}_\varepsilon(x) e^{i\psi_t}$ . Reproducing the proof of [SS2] shows that under the previous conditions (5.41)–(5.43), we have

$$\frac{d}{dt} \Big|_{t=0} \bar{E}(v_\varepsilon(x, t)) = \lim_{r \rightarrow 0} \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \int_{B(0,R) \setminus \bigcup_i B(b_{j,k}(t), r)} |\nabla \Phi_t|^2, \tag{5.45}$$

while

$$\frac{1}{|\log \eta|} \int_{B(0,R)} |\partial_t v_\varepsilon(0)|^2 = \pi \sum_k |V_k|^2 + o(1). \tag{5.46}$$

As in [SS2], this follows from (5.36) and [SS1, Corollary 4].

Next, we claim we have

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \int_{B(0,R) \setminus \bigcup_k B(b_{j,k}(t), r)} |\nabla \Phi_t|^2 &= \frac{d}{dt} \Big|_{t=0} \bar{W}_R(b_{j,k}(t)) \\ &= \sum_k \nabla_k \bar{W}_R(b_{j,k}) \cdot V_k \end{aligned} \tag{5.47}$$

(the proof can be reproduced from (3.39) of [SS2].) This will suffice to conclude that

$$|\log \eta| \int_{B(0,R)} \left| \Delta \bar{u}_\varepsilon + \frac{\bar{u}_\varepsilon}{\eta^2} (1 - |\bar{u}_\varepsilon|^2) \right|^2 \geq \frac{1}{\pi} \|\nabla \bar{W}_R(b_{j,k})\|^2 + o(1), \tag{5.48}$$

that is,

$$|\log \varepsilon| l^2 \int_{B(p_j^\varepsilon, Rl)} \left| \Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2} (1 - |u_\varepsilon|^2) \right|^2 \geq \frac{1}{\pi} \|\nabla \bar{W}_R(b_{j,k})\|^2 + o(1). \tag{5.49}$$

Indeed, this follows as in [SS2] by a simple Cauchy–Schwarz inequality: choosing  $V_k = \nabla_k \bar{W}_R(b_i)$ , since  $v_\varepsilon(x, t) = \bar{u}_\varepsilon(x)$  on  $\partial B(0, R)$  for each  $t$ , we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \bar{E}(v_\varepsilon(x, t)) &= \int_{B(0, R)} \partial_t v_\varepsilon(0) \left( \Delta \bar{u}_\varepsilon + \frac{\bar{u}_\varepsilon}{\eta^2} (1 - |\bar{u}_\varepsilon|^2) \right) \\ &\leq \left( \frac{1}{|\log \eta|} \int_{B(0, R)} |\partial_t v_\varepsilon(0)|^2 \right)^{1/2} \left( |\log \eta| \int_{B(0, R)} \left| -\Delta \bar{u}_\varepsilon + \frac{\bar{u}_\varepsilon}{\eta^2} (1 - |\bar{u}_\varepsilon|^2) \right|^2 \right)^{1/2} \\ &\leq \left( \pi \sum_k |\nabla_k \bar{W}_R(b_{j,k})|^2 + o(1) \right)^{1/2} \left( |\log \varepsilon| l^2 \int_{B(p_j^\varepsilon, Rl)} |f_\varepsilon|^2 \right)^{1/2}, \end{aligned} \tag{5.50}$$

using (5.46) and the choice of  $V_k$ . Inserting this and (5.47) into (5.45), we are led to (5.49). Adding up the relations (5.49) over  $j$  and using (5.35), we conclude that (5.18) holds.

**Remark 5.4.** This proof is the only place where the assumption (5.17) is needed.

5.5. Proof of the additional lemmas

*Proof of Lemma 5.2.* First, we recall that  $\bar{\Phi}$  satisfies  $-\Delta \bar{\Phi} = 2\pi \sum_i d_i \delta_{\bar{a}_i}$  with  $\sum_i d_i \delta_{\bar{a}_i} \rightharpoonup \sum_k D_{j,k} \delta_{b_{j,k}}$  in the weak sense of measures in  $B(0, R)$ . Moreover, since  $B(p_j^\varepsilon, 2Rl) \setminus B(p_j^\varepsilon, Kl)$  does not contain any vortex, examining (3.5) of [S1], as for (3.9) of [S1], we have

$$|\nabla \Phi| \leq \frac{C}{Rl} \quad \text{on } \partial B(p_j^\varepsilon, Rl), \tag{5.51}$$

$$|D^2 \Phi| \leq \frac{C}{R^2 l^2} \quad \text{on } \partial B(p_j^\varepsilon, Rl), \tag{5.52}$$

so  $|\nabla \bar{\Phi}| \leq C/R$  and  $|D^2 \bar{\Phi}| \leq C/R^2$  on  $\partial B(0, R)$ . Thus,  $\bar{\Phi} - (2\pi R)^{-1} \int_{\partial B(0, R)} \bar{\Phi}$  is uniformly bounded and equicontinuous on  $\partial B(0, R)$  and we may assume it converges uniformly to some  $h$  as  $\varepsilon \rightarrow 0$ . Moreover, returning to (3.5) of [S1], and since  $B(p_j^\varepsilon, 2Rl) \setminus B(p_j^\varepsilon, Kl)$  contains no  $a_i$ ,  $\Phi$  behaves more and more like a constant on  $\partial B(p_j^\varepsilon, Rl)$  as  $R$  becomes large. That is,  $h \rightarrow 0$  uniformly on  $\partial B(0, R)$  as  $R \rightarrow \infty$ .

On the other hand, since  $\bar{\Phi} - (2\pi R)^{-1} \int_{\partial B(0, R)} \bar{\Phi} - \Phi_0$  tends to 0 uniformly on  $\partial B(0, R)$  and its Laplacian tends to 0 in the weak sense of measures on  $B(0, R)$ , we conclude that the function converges to 0 uniformly on  $B(0, R)$  as  $\varepsilon \rightarrow 0$ . Then, using an integration by parts (and assuming  $(2\pi R)^{-1} \int_{\partial B(0, R)} \bar{\Phi} = 0$  for simplicity), we have

$$\begin{aligned} & \int_{B(0,R) \setminus \bigcup_k B(b_{j,k},\rho)} |\nabla \bar{\Phi} - \nabla \Phi_0|^2 \\ &= \left| \int_{\partial B(0,R)} (\bar{\Phi} - \Phi_0) \frac{\partial}{\partial \nu} (\bar{\Phi} - \Phi_0) - \sum_k \int_{\partial B(b_{j,k},\rho)} (\bar{\Phi} - \Phi_0) \frac{\partial}{\partial \nu} (\bar{\Phi} - \Phi_0) \right| \rightarrow 0 \end{aligned}$$

in view of the bounds on  $|\nabla \Phi|$  and  $|\nabla \Phi_0|$ . This proves (5.34).

Let us now prove (5.35). Observe that  $S(x, y) = 2\pi(G_0(x, y) + G_h(x)) + \log|x - y|$  where  $G$  is written as  $G_0 + G_h$ , with

$$\begin{cases} -\Delta_x G_0(x, y) = \delta_y & \text{in } B(0, R), \\ G_0(x, y) = 0 & \text{on } \partial B(0, R), \end{cases} \tag{5.53}$$

and

$$\begin{cases} -\Delta G_h = 0 & \text{in } B(0, R), \\ G_h = h(x) & \text{on } \partial B(0, R). \end{cases} \tag{5.54}$$

It is a standard fact that  $G_0$  is symmetric, i.e.  $G_0(x, y) = G_0(y, x)$ . In fact, there is an explicit expression (in complex coordinates)

$$G_0(x, y) = \frac{1}{2\pi} \log \left| \frac{R(x - y)}{R^2 - x\bar{y}} \right|. \tag{5.55}$$

Thus  $S(x, y)$  is the sum of a symmetric function and a function that depends only on  $x$ . Now,

$$\begin{aligned} \bar{W}_R(b_{j,k}) - \bar{W}(b_{j,k}) &= 2\pi \sum_{k'} D_{j,k} D_{j,k'} S(b_{j,k}, b_{j,k'}) \\ &\quad + \int_{\partial B(0,R)} h(x) \frac{\partial}{\partial \nu} (2\pi D_{j,k} G(x, b_{j,k})), \end{aligned}$$

hence

$$\begin{aligned} \nabla(\bar{W}_R(b_{j,k}) - \bar{W}(b_{j,k})) &= 2\pi \sum_{k'} D_{j,k} D_{j,k'} \left( \nabla_x S(b_{j,k}, b_{j,k'}) + \nabla_y S(b_{j,k}, b_{j,k'}) \right) \\ &\quad + \int_{\partial B(0,R)} h(x) \frac{\partial}{\partial \nu} (2\pi D_{j,k} \nabla_y G_0(x, b_{j,k})) \end{aligned}$$

Thus, to conclude that (5.35) holds, it suffices to check that  $\max_{x,y \in B(0,K+1)} |\nabla_x S|$  and  $\max_{x,y \in B(0,K+1)} |\nabla_y S|$  tend to 0 as  $R \rightarrow \infty$ , and that  $|(\partial/\partial \nu) \nabla_y G_0(x, b_{j,k})| \leq C/R^2$ . The second fact follows from the formula (5.55). For the first, use the fact that  $S(x, y) = 2\pi(G_0(x, y) + G_h(x, y)) + \log|x - y|$ . Now  $G_h$  is harmonic, with values  $h \rightarrow 0$  on  $\partial B(0, R)$ , hence tends to 0 in  $C^1(B(0, K + 1))$  as  $R \rightarrow \infty$ , by elliptic estimates. The remaining part of  $S$  is easy to handle.  $\square$

*Proof of Lemma 5.3.* By an application of the standard lower bounds, since all the vortices converge to the  $b_{j,k}$ 's with total degree  $D_{j,k} = \pm 1$ , we have

$$\frac{1}{2} \int_{B(b_{j,k}, \rho)} |\nabla \bar{u}_\varepsilon|^2 \geq \pi \log \frac{1}{\eta} (1 - o(1))$$

and since (5.16) holds, we must have (5.36).

On the other hand, Theorem 1 of [S1] yields

$$\int_{B(0,R)} \left( |\nabla |\bar{u}_\varepsilon||^2 + \frac{(1 - |\bar{u}_\varepsilon|^2)^2}{\eta^2} \right) = o(l^2 \log^2 l \|f_\varepsilon\|_{L^2(\Omega)}^2 + 1)$$

and

$$\int_{B(0,R)} |\nabla \bar{\psi}|^2 = o(l^2 \log^4 l \|f_\varepsilon\|_{L^2(\Omega)}^2 + 1), \quad (5.56)$$

where  $\bar{\psi}$  is the blow-up of  $\psi$ . This proves (5.37). In addition, we can easily check that  $|\nabla \bar{u}_\varepsilon - i \bar{u}_\varepsilon \nabla^\perp \Phi_0|^2 = |\nabla |\bar{u}_\varepsilon||^2 + |\bar{u}_\varepsilon|^2 |\nabla \bar{\psi} + \nabla^\perp \bar{\Phi} - \nabla^\perp \Phi_0|^2$ , hence

$$\begin{aligned} \int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, \rho)} |\nabla \bar{u}_\varepsilon - i \bar{u}_\varepsilon \nabla^\perp \Phi_0|^2 \\ \leq \int_{B(0,R) \setminus \bigcup_k B(b_{j,k}, \rho)} |\nabla |\bar{u}_\varepsilon||^2 + 2 |\nabla \bar{\psi}|^2 + 2 |\nabla (\bar{\Phi} - \Phi_0)|^2 \end{aligned}$$

and in view of (5.34) and (5.56), (5.38) follows.  $\square$

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