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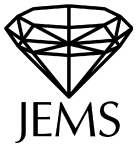

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Enveloping algebras of Slodowy slices and the Joseph ideal

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Abstract. Let G be a simple algebraic group over an algebraically closed field $\Bbbk$ of characteristic 0, and $\mathfrak{g} = \operatorname{Lie} G$. Let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with e being a long root vector in $\mathfrak{g}$. Let $(\cdot, \cdot)$ be the G-invariant bilinear form on $\mathfrak{g}$ with $(e, f) = 1$ and let $\chi \in \mathfrak{g}^*$ be such that $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$. Let $\mathcal{S}$ be the Slodowy slice at e through the adjoint orbit of e and let H be the enveloping algebra of $\mathcal{S}$; see [31]. In this article we give an explicit presentation of H by generators and relations. As a consequence we deduce that H contains an ideal of codimension 1 which is unique if $\mathfrak{g}$ is not of type A. Applying Skryabin's equivalence of categories we then construct an explicit Whittaker model for the Joseph ideal of $U(\mathfrak{g})$. Inspired by Joseph's Preparation Theorem we prove that there exists a homeomorphism between the primitive spectrum of H and the spectrum of all primitive ideals of infinite codimension in $U(\mathfrak{g})$ which respects Goldie rank and Gelfand–Kirillov dimension. We study highest weight modules for the algebra H and apply earlier results of Miličić–Soergel and Backelin to express the composition multiplicities of the Verma modules for H in terms of some inverse parabolic Kazhdan–Lusztig polynomials. Our results confirm in the minimal nilpotent case the de Vos–van Driel conjecture on composition multiplicities of Verma modules for finite $\mathcal{W}$-algebras. We also obtain some general results on the enveloping algebras of Slodowy slices and determine the associated varieties of related primitive ideals of $U(\mathfrak{g})$. A sequel to this paper will treat modular aspects of this theory.

1. Introduction

1.1. Let $\Bbbk$ be an algebraically closed field of characteristic 0 and let G be a simple algebraic group over $\Bbbk$. Let $\mathfrak{g} = \operatorname{Lie} G$ and let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $(\cdot, \cdot)$ be the G-invariant bilinear form on $\mathfrak{g}$ with $(e, f) = 1$ and define $\chi = \chi_e \in \mathfrak{g}^*$ by setting $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$. Let $\mathcal{O}_\chi$ denote the coadjoint orbit of χ.

Let $\mathcal{S}_e = e + \operatorname{Ker} \operatorname{ad} f$ be the Slodowy slice at e through the adjoint orbit of e and let H_χ be the enveloping algebra of $\mathcal{S}_e$; see [31, 10, 5]. Recall that $H_\chi = \operatorname{End}_{\mathfrak{g}}(Q_\chi)^{\mathrm{op}}$ where Q_χ is a generalised Gelfand–Graev module for $U(\mathfrak{g})$ associated with the $\mathfrak{sl}_2$-triple (e, h, f); see [22, 29]. The module Q_χ is induced from a one-dimensional module $\Bbbk_\chi$ over a nilpotent subalgebra $\mathfrak{m}_\chi$ of $\mathfrak{g}$ such that $\dim \mathfrak{m}_\chi = \frac{1}{2} \dim \mathcal{O}_\chi$. The subalgebra $\mathfrak{m}_\chi$ is $(\operatorname{ad} h)$-stable, all weights of $\operatorname{ad} h$ on $\mathfrak{m}_\chi$ are negative, and χ vanishes on the derived

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subalgebra of $\mathfrak{m}_\chi$. The action of $\mathfrak{m}_\chi$ on $\Bbbk_\chi = \Bbbk 1_\chi$ is given by $x(1_\chi) = \chi(x)1_\chi$ for all $x \in \mathfrak{m}_\chi$; see 2.1 for more details.

Let $\mathfrak{z}_\chi$ denote the stabiliser of χ in $\mathfrak{g}$. Clearly, $\mathfrak{z}_\chi$ coincides with the centraliser $\mathfrak{c}_\mathfrak{g}(e)$ of e in $\mathfrak{g}$. The subalgebra $\mathfrak{z}_\chi$ is $(\operatorname{ad} h)$-stable and it follows from the $\mathfrak{sl}_2$-theory that all weights of $\operatorname{ad} h$ on $\mathfrak{z}_\chi$ are nonnegative integers. Let $x_1, \dots, x_r$ be a basis of $\mathfrak{z}_\chi$ such that $[h, x_i] = n_i x_i$ for some $n_i \in \mathbb{Z}_+$. By [31, Theorem 4.6], to each basis vector x_i one can attach an element $\Theta_{x_i} \in H_\chi$ in such a way that the monomials $\Theta_{x_1}^{i_1} \cdots \Theta_{x_r}^{i_r}$ with $(i_1, \dots, i_r) \in \mathbb{Z}_+^r$ form a basis of H_χ over $\Bbbk$. We say that the monomial $\Theta_{x_1}^{a_1} \cdots \Theta_{x_r}^{a_r}$ has *Kazhdan degree* $\sum_{i=1}^r a_i(n_i+2)$ and denote by H_χ^k the span of all monomials as above of Kazhdan degree $\le k$. According to [31, (4.6)], we then have $H_\chi = \bigcup_{k\ge 0} H_\chi^k$ and $H_\chi^i \cdot H_\chi^j \subseteq H_\chi^{i+j}$ for all $i, j \in \mathbb{Z}_+$. In other words, $\{H_\chi^k \mid k \in \mathbb{Z}_+\}$ is an increasing filtration of the algebra H_χ. We call it the *Kazhdan filtration* of H_χ. The corresponding graded algebra $\operatorname{gr} H_\chi$ is a polynomial algebra in $\operatorname{gr}\Theta_{x_1}, \dots, \operatorname{gr}\Theta_{x_r}$ which identifies naturally with the function algebra $\Bbbk[\mathcal{S}_e]$ on the special transverse slice $\mathcal{S}_e = e + \operatorname{Ker}\operatorname{ad} f$ endowed with its Slodowy grading.

According to [31, Theorem 4.6(iv)],

$$[\Theta_{x_i}, \Theta_{x_j}] = \Theta_{x_j} \circ \Theta_{x_i} - \Theta_{x_i} \circ \Theta_{x_j} \equiv \Theta_{[x_i,x_j]} + q_{ij}(\Theta_1, \dots, \Theta_r) \ (\operatorname{mod} H_\chi^{n_i+n_j})$$

where q_{ij} is a polynomial in r variables with initial form of total degree ≥ 2. Using this result we prove in 2.3 that there exists an associative $\Bbbk[t]$-algebra $\mathcal{H}_\chi$ free as a module over $\Bbbk[t]$ and such that

$$\mathcal{H}_\chi/(t-\lambda)\mathcal{H}_\chi \cong \begin{cases} H_\chi & \text{if } \lambda \neq 0, \\ U(\mathfrak{z}_\chi) & \text{if } \lambda = 0 \end{cases}$$

as $\Bbbk$-algebras. Thus H_χ is a deformation of the universal enveloping algebra $U(\mathfrak{z}_\chi)$.

We have a certain degree of freedom in our choice of PBW generators Θ_{x_i}. In 2.2 we show that they can be chosen such that the map $\Theta_{x_i} \mapsto (-1)^{n_i}\Theta_{x_i}$, $1 \le i \le r$, extends to an automorphism of the algebra H_χ. This automorphism, denoted by σ, will play an important rôle later on.

1.2. Given $x \in \mathfrak{g}$ we denote by $Z_G(x)$ the centraliser of x in G. It is well-known that $C(e) := Z_G(h) \cap Z_G(e)$ is a Levi subgroup of $Z_G(e)$ and the centraliser $Z_G(e)$ decomposes as a semidirect product of $C(e)$ and the unipotent radical $R_u(Z_G(e))$. For $i \in \mathbb{Z}_+$ put $\mathfrak{z}_\chi(i) = \{x \in \mathfrak{z}_\chi \mid [h, x] = ix\}$. It is well-known that $\mathfrak{z}_\chi(0) = \operatorname{Lie} C(e)$. Clearly, the group $C(e)$ preserves each subspace $\mathfrak{z}_\chi(i)$ of $\mathfrak{z}_\chi$.

In [10], Gan and Ginzburg have found a different realisation of the algebra H_χ which enables one to observe that the reductive group $C(e)$ acts on H_χ as algebra automorphisms. Moreover, this action of $C(e)$ preserves the Kazhdan filtration of H_χ; see 2.1 for more details. In Section 2 we show that there exists an injective $C(e)$-equivariant linear map $\Theta\colon \mathfrak{z}_\chi \to H_\chi$, $x \mapsto \Theta_x$, whose image generates H_χ as an algebra, such that $\operatorname{gr}\Theta(\mathfrak{z}_\chi) \cong \mathfrak{z}_\chi$ as graded $C(e)$-modules. The subspace $\Theta(\mathfrak{z}_\chi(0))$ can be chosen to be a

Lie subalgebra of H_χ with respect to the commutator product in H_χ. It follows that the map Θ can be selected in such a way that

$$[\Theta_x, \Theta_y] = \Theta_{[x,y]} \quad (\forall x \in \mathfrak{z}_\chi(0),\ \forall y \in \mathfrak{z}_\chi).$$

This is a consequence of Lemma 2.4 which states, in particular, that the Lie algebra homomorphism $\mathrm{ad} \circ \Theta\colon \mathfrak{z}_\chi(0) \to \mathrm{Der}(H_\chi)$ coincides with the differential of the locally finite (rational) action of $C(e)$ on H_χ. Combined with Lemma 2.1 and Weyl's theorem on complete reducibility this implies that every two-sided ideal of H_χ is σ-stable; see Corollary 2.1.

1.3. For $\chi = (e, \cdot)$ we let $\mathcal{C}_\chi$ denote the category of all $\mathfrak{g}$-modules on which $x - \chi(x)$ acts locally nilpotently for all $x \in \mathfrak{m}_\chi$. Given a $\mathfrak{g}$-module M we set

$$\mathrm{Wh}(M) := \{m \in M \mid x.m = \chi(x)m\ (\forall x \in \mathfrak{m}_\chi)\}.$$

It should be mentioned here that the algebra H_χ acts on $\mathrm{Wh}(M)$ via a canonical isomorphism $H_\chi \cong (U(\mathfrak{g})/N_\chi)^{\mathrm{ad}\,\mathfrak{m}_\chi}$ where N_χ denotes the left ideal of $U(\mathfrak{g})$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}_\chi$. In the Appendix to [31], Skryabin proved that the functors $V \rightsquigarrow Q_\chi \otimes_{H_\chi} V$ and $M \rightsquigarrow \mathrm{Wh}(M)$ are mutually inverse equivalences between the category of all H_χ-modules and the category $\mathcal{C}_\chi$; see also [10, Theorem 6.1].

Skryabin's equivalence implies that for any irreducible H_χ-module V the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ is a primitive ideal of $U(\mathfrak{g})$. By the Irreducibility Theorem, the associated variety $\mathcal{VA}(\mathcal{I})$ of any primitive ideal $\mathcal{I}$ of $U(\mathfrak{g})$ is the closure of a nilpotent orbit in $\mathfrak{g}^*$. Generalising a classical result of Kostant on Whittaker modules we show in Section 3 that for any irreducible H_χ-module V the associated variety of $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)$ contains the coadjoint orbit $\mathcal{O}_\chi$. In the most interesting case where V is a finite-dimensional irreducible H_χ-module we prove that

$$\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} V)) = \overline{\mathcal{O}}_\chi \quad \text{and} \quad \mathrm{Dim}(Q_\chi \otimes_{H_\chi} V) = \tfrac{1}{2} \dim \overline{\mathcal{O}}_\chi$$

where $\mathrm{Dim}(M)$ is the Gelfand–Kirillov dimension of a finitely generated $U(\mathfrak{g})$-module M. In particular, this implies that for any irreducible finite-dimensional H_χ-module V the irreducible $U(\mathfrak{g})$-module $Q_\chi \otimes_{H_\chi} V$ is holonomic.

1.4. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a basis of simple roots in Φ with the elements in Π numbered as in [4], and let Φ^+ be the positive system of Φ relative to Π. If $\mathfrak{g}$ is not of type A or C, there is a unique long root in Π linked with the lowest root $-\widetilde{\alpha}$ on the extended Dynkin diagram of $\mathfrak{g}$; we call it β. For $\mathfrak{g}$ of type A_n and C_n we set $\beta = \alpha_n$. Choose root vectors $e_\beta, e_{-\beta} \in \mathfrak{g}$ corresponding to roots β and $-\beta$ such that $(e_\beta, [e_\beta, e_{-\beta}], e_{-\beta})$ is an $\mathfrak{sl}_2$-triple and put $h_\beta = [e_\beta, e_{-\beta}]$.

In this article we investigate the algebra H_χ in the case where $(e, h, f) = (e_\beta, h_\beta, e_{-\beta})$. Then $\mathcal{O}_\chi = \mathcal{O}_{\min}$, the minimal nonzero nilpotent orbit in $\mathfrak{g}^*$. We let H denote the minimal nilpotent algebra H_χ. One of our main objectives is to give a presentation of H by generators and relations.

The action of the inner derivation $\operatorname{ad} h$ gives $\mathfrak{g}$ a short $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2), \qquad \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$$

with $\mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and $\mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$ being Heisenberg Lie algebras. One knows of course that $\mathfrak{g}(\pm 2)$ is spanned by $e_{\pm\beta}$, that $\mathfrak{z}_\chi(i) = \mathfrak{g}(i)$ for $i = 1, 2$, and that $\mathfrak{z}_\chi(0)$ coincides with the image of the Lie algebra homomorphism

$$\sharp\colon \mathfrak{g}(0) \to \mathfrak{g}(0), \qquad x \mapsto x - \tfrac{1}{2}(x, h)h,$$

whose kernel $\Bbbk h$ is a central ideal of $\mathfrak{g}(0)$. The graded component $\mathfrak{g}(-1)$ has a basis $z_1, \dots, z_s, z_{s+1}, \dots, z_{2s}$ such that the z_i's with $1 \le i \le s$ (resp. $s+1 \le i \le 2s$) are root vectors for $\mathfrak{h}$ corresponding to negative (resp. positive) roots, and

$$[z_i, z_j] = [z_{i+s}, z_{j+s}] = 0, \qquad [z_{i+s}, z_j] = \delta_{ij} f \qquad (1 \le i, j \le s).$$

Moreover, in the present case we can choose $\mathfrak{m}_\chi$ to be the span of f and the z_i's with $s+1 \le i \le 2s$, an abelian subalgebra of $\mathfrak{g}$ of dimension $s+1 = \frac{1}{2}\dim \mathcal{O}_{\min}$. We set $z_i^* := z_{i+s}$ for $1 \le i \le s$ and $z_i^* := -z_{i-s}$ for $s+1 \le i \le 2s$.

Let C denote the Casimir element of $U(\mathfrak{g})$ corresponding to the bilinear form $(\cdot, \cdot)$. This form is nondegenerate on $\mathfrak{z}_\chi(0)$, hence we can find bases $\{a_i\}$ and $\{b_i\}$ of $\mathfrak{z}_\chi(0)$ such that $(a_i, b_j) = \delta_{ij}$. Set $\Theta_{\mathrm{Cas}} := \sum_i \Theta_{a_i}\Theta_{b_i}$, a central element of the associative subalgebra of H generated by the Lie algebra $\Theta(\mathfrak{z}_\chi(0))$. Obviously, we can regard C as a central element of H.

By a well-known result of Joseph, outside type A the universal enveloping algebra $U(\mathfrak{g})$ contains a unique completely prime primitive ideal whose associated variety is $\overline{\mathcal{O}}_{\min}$; see [14]. This ideal, often denoted $\mathcal{J}_0$, is known as the Joseph ideal of $U(\mathfrak{g})$.

We are finally in a position to formulate one of the main results of this article:

Theorem 1.1. *The algebra H is generated by the Casimir element C and the subspaces $\Theta(\mathfrak{z}_\chi(i))$ for $i = 0, 1$, subject to the following relations:*

(i) $[\Theta_x, \Theta_y] = \Theta_{[x,y]}$ *for all $x, y \in \mathfrak{z}_\chi(0)$;*
(ii) $[\Theta_x, \Theta_u] = \Theta_{[x,u]}$ *for all $x \in \mathfrak{z}_\chi(0)$ and $u \in \mathfrak{z}_\chi(1)$;*
(iii) *C is central in H;*
(iv) $[\Theta_u, \Theta_v] = \frac{1}{2}(f, [u, v])(C - \Theta_{\mathrm{Cas}} - c_0) + \frac{1}{2}\sum_{i=1}^{2s}(\Theta_{[u,z_i]^\sharp}\Theta_{[v,z_i^*]^\sharp} + \Theta_{[v,z_i^*]^\sharp}\Theta_{[u,z_i]^\sharp})$ *for all $u, v \in \mathfrak{z}_\chi(1)$, where c_0 is a constant depending on $\mathfrak{g}$.*

If $\mathfrak{g}$ is not of type A *then c_0 is the eigenvalue of C on the primitive quotient $U(\mathfrak{g})/\mathcal{J}_0$. If $\mathfrak{g}$ is of type* A_n, *$n \ge 2$, then $c_0 = -n(n+1)/4$. If $\mathfrak{g}$ is of type* A_1 *then $H = \Bbbk[C]$.*

We start proving this theorem in Section 2 where we show that (i) and (ii) hold in H for a suitable choice of $\Theta\colon \mathfrak{z}_\chi \to H$. In Section 4 we determine all of the quadratic relation (iv) except the elusive constant c_0.

1.5. We first computed c_0 by brute force, but later it turned out that there was a much better way to do it, based on a certain refined version of Joseph's Preparation Theorem.

This theorem, which we prove in Section 5 in our special case, enables us to link the primitive ideals of H directly with primitive ideals of $U(\mathfrak{g})$.

Let Δ denote the automorphism of the polynomial algebra $\Bbbk[h]$ such that $\Delta(h) = h + 1$. Let $\langle\Delta\rangle$ stand for the cyclic subgroup of $\mathrm{Aut}(\Bbbk[h])$ generated by Δ. The skew group algebra $\Bbbk[h] * \langle\Delta\rangle$ has $\{h^i\Delta^j \mid i \in \mathbb{Z}_+,\ j \in \mathbb{Z}\}$ as a $\Bbbk$-basis and multiplication in $\Bbbk[h] * \langle\Delta\rangle$ has the property that $\Delta \cdot h = (h+1) \cdot \Delta$.

Let $\mathbf{A}_e$ denote the Weyl algebra with standard generators $z_1, \ldots, z_s, \partial_1, \ldots, \partial_s$, so that $[\partial_i, z_j] = \delta_{ij}$ for $1 \le i, j \le s$. Let $\mathcal{A}_e := (\Bbbk[h] * \langle\Delta\rangle) \otimes \mathbf{A}_e$, a simple Noetherian algebra over $\Bbbk$, and identify $\Bbbk[h] * \langle\Delta\rangle$ and $\mathbf{A}_e$ with subalgebras of $\mathcal{A}_e$. Define an involution $\tau \in \mathrm{Aut}(\mathcal{A}_e)$ by setting

$$\tau(z_i) = -z_i, \quad \tau(\partial_i) = -\partial_i, \quad \tau(h) = h, \quad \tau(\Delta^k) = (-1)^k\Delta^k \quad (1 \le i \le s,\ k \in \mathbb{Z}).$$

Then $\tau \otimes \sigma$ is an automorphism of order two of the associative algebra $\mathcal{A}_e \otimes H$.

Let $U(\mathfrak{g})_f$ denote the localisation of $U(\mathfrak{g})$ with respect to the Ore set $\{f^i \mid i \in \mathbb{Z}_+\}$. By mapping $U(\mathfrak{g})_f$ into the endomorphism algebra of the induced module Q_χ we are able to identify $U(\mathfrak{g})_f$ with a subalgebra of $\mathcal{A}_e \otimes H$. More precisely, we prove that

$$U(\mathfrak{g})_f = (\mathcal{A}_e \otimes H)^{\tau\otimes\sigma} = \mathcal{A}_e^\tau \otimes H_+ \oplus \mathcal{A}_e^\tau\Delta \otimes H_-$$

where $H_\pm = \{x \in H \mid \sigma(x) = \pm x\}$. As mentioned in 1.1 every two-sided ideal I of H is stable under the involution $\sigma \in \mathrm{Aut}(H)$. Hence $I = I_+ \oplus I_-$ where $I_\pm = I \cap H_\pm$. We identify $U(\mathfrak{g})$ with a subalgebra of $U(\mathfrak{g})_f$ and set

$$\widetilde{I} := U(\mathfrak{g}) \cap (\mathcal{A}_e^\tau \otimes I_+ \oplus \mathcal{A}_e^\tau\Delta \otimes I_-).$$

Then $\widetilde{I}$ is a two-sided ideal of $U(\mathfrak{g})$. By Corollary 5.1(vi), the centre of H identifies canonically with $Z(\mathfrak{g})$, the centre of $U(\mathfrak{g})$. Let $\mathcal{X} = \mathrm{Prim}\, U(\mathfrak{g})$ and let $\mathcal{X}_{\mathrm{inf}}$ be the set of all primitive ideals of infinite codimension in $\mathcal{X}$. Given a prime Noetherian ring R we let $\mathrm{rk}(R)$ denote the Goldie rank of R.

Theorem 1.2. *Take* $\mathrm{Prim}\, H$ *with the Jacobson topology and take* $\mathcal{X}_{\mathrm{inf}}$ *with the topology induced by the Jacobson topology of* $\mathcal{X}$. *Then the following hold:*

(i) *The map* $I \mapsto \widetilde{I}$ *induces a homeomorphism* $\varkappa\colon \mathrm{Prim}\, H \xrightarrow{\sim} \mathcal{X}_{\mathrm{inf}}$.
(ii) *For any* $I \in \mathrm{Prim}\, H$ *we have* $\mathrm{Dim}(U(\mathfrak{g})/\widetilde{I}) = \mathrm{Dim}(H/I) + \dim \overline{\mathcal{O}}_{\min}$.
(iii) *If* $I = \mathrm{Ann}_H V$ *where* V *is a finite-dimensional irreducible* H*-module, then* $\widetilde{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V)$ *and* $\mathrm{rk}(U(\mathfrak{g})/\widetilde{I}) = \dim \mathrm{Wh}(Q_\chi \otimes_H V) = \dim V$.
(iv) *For any* $\mathcal{I} \in \mathcal{X}$ *with* $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$ *there is a finite-dimensional irreducible* H*-module* V *such that* $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V)$.
(v) *Let* V_1 *and* V_2 *be two finite-dimensional irreducible* H*-modules. Then* $V_1 \cong V_2$ *as* H*-modules if and only if* $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_1) = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_2)$.
(vi) *A prime ideal* I *of* H *is primitive if and only if* $I \cap Z(H)$ *is a maximal ideal of* $Z(H)$.

It follows from Theorem 1.2 that for any homomorphism $\eta\colon Z(\mathfrak{g}) \to \Bbbk$ the map $\varkappa$ induces a bijection between the isoclasses of finite-dimensional irreducible H-modules with central character η and the primitive ideals $\mathcal{I} \in \mathcal{X}$ such that $\mathcal{I} \cap Z(\mathfrak{g}) = \mathrm{Ker}\,\eta$ and

$\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$ (recall that $Z(\mathfrak{g}) = Z(H)$). This result indicates that for any nilpotent χ it should be possible to interpret the number of isoclasses of irreducible finite-dimensional H_χ-modules with a fixed central character as the dimension of a cell representation of the integral Weyl group of the character. In type A this agrees with recent results of Brundan–Kleshchev.

Theorem 1.2(iii) relates the dimensions of irreducible finite-dimensional H-modules with Goldie-rank polynomials. We explore this in 6.4 to obtain dimension formulae for all irreducible finite-dimensional representations of H for $\mathfrak{g}$ of type C_n and G_2. It is quite possible that *all* Goldie-rank polynomials, properly scaled, will appear in dimension formulae for "nonrestricted Weyl modules" over Lie algebras of reductive groups in characteristic p (we recall that in characteristic p a truncated version of H_χ is Morita equivalent to the reduced enveloping algebra $U_\chi(\mathfrak{g})$; see [31, (2.3), (2.6)]).

Theorem 1.2(vi) says that H satisfies the Dixmier–Mœglin–Rentschler equivalence. Again it is possible that this holds for any algebra H_χ.

1.6. In the last section of this article we introduce and study highest weight modules for the algebra H. Let $\Phi_e = \{\alpha \in \Phi \mid \alpha(h) = 0 \text{ or } 1\}$, and put $\Phi_e^\pm = \Phi_e \cap \Phi^\pm$ where $\Phi^- = -\Phi^+$. For $i = 0, 1$ put $\Phi_{e,i}^\pm = \{\alpha \in \Phi_e^\pm \mid \alpha(h) = i\}$. Note that $\mathfrak{z}_\chi$ is spanned by $\mathfrak{h}_e := \mathfrak{h} \cap \mathfrak{g}(0)^\sharp$, by root vectors e_α with $\alpha \in \Phi_e$, and by e. Let $h_1, \dots, h_{l-1}$ be a basis of $\mathfrak{h}_e$, and let $\mathfrak{n}^\pm(i)$ be the span of all e_α with $\alpha \in \Phi_{e,i}^\pm$. Clearly, $\mathfrak{n}^+(0)$ and $\mathfrak{n}^-(0)$ are maximal nilpotent subalgebras of $\mathfrak{g}(0)^\sharp$. Let $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$ be bases of $\mathfrak{n}^+(0)$ and $\mathfrak{n}^-(0)$ consisting of root vectors for $\mathfrak{h}$. For $1 \le i \le s$ let γ_i (resp. γ_i^*) denote the root of z_i (resp. z_i^*), and put $u_i = [e, z_i]$, $u_i^* = [e, z_i^*]$. Then $\{u_1, \dots, u_s, u_1^*, \dots, u_s^*\}$ is a $\Bbbk$-basis of $\mathfrak{z}_\chi(1)$.

In general, H is unlikely to possess a triangular decomposition similar to that of $U(\mathfrak{g})$. Nevertheless, one can still define Verma modules and highest weight modules for H. Given $\lambda \in \mathfrak{h}_e^*$ and $c \in \Bbbk$ we denote by $J_{\lambda,c}$ the linear span in H of all

$$\prod_{i=1}^{t} \Theta_{y_i}^{l_i} \cdot \prod_{i=1}^{s} \Theta_{u_i}^{m_i} \cdot \prod_{i=1}^{\ell-1} (\Theta_{h_i} - \lambda(h_i))^{n_i} \cdot (C - c)^{n_\ell} \cdot \prod_{i=1}^{s} \Theta_{u_i^*}^{r_i} \cdot \prod_{i=1}^{t} \Theta_{x_i}^{q_i}$$

with $\sum_{i=1}^{\ell} n_i + \sum_{i=1}^{t} r_i + \sum_{i=1}^{s} q_i > 0$. Using Theorem 1.1 we show in 7.1 that $J_{\lambda,c}$ is a left ideal of H. We call the H-module $Z_H(\lambda, c) := H/J_{\lambda,c}$ the *Verma module of level* c corresponding to λ. By the above, $Z_H(\lambda, c)$ has a nice PBW basis. In 7.2 we show that $Z_H(\lambda, c)$ contains a unique maximal submodule which we denote $Z_H^{\max}(\lambda, c)$. Thus to every $(\lambda, c) \in \mathfrak{h}_e^* \times \Bbbk$ there corresponds an irreducible highest weight H-module $L_H(\lambda, c) := Z_H(\lambda, c)/Z_H^{\max}(\lambda, c)$. It is fairly easy to show that $L_H(\lambda, c) \cong L_H(\lambda', c')$ if and only if $(\lambda, c) = (\lambda', c')$ and that any irreducible finite-dimensional H-module is isomorphic to exactly one of $L_H(\lambda, c)$ with λ satisfying a natural integrality condition.

To determine the composition multiplicities of the Verma modules $Z_H(\lambda, c)$ we link the latter with $\mathfrak{g}$-modules obtained by parabolic induction from Whittaker modules for $\mathfrak{sl}_2$. Let $\mathfrak{s}_\beta = \Bbbk e_\beta \oplus \Bbbk h_\beta \oplus \Bbbk f_\beta$ and put

$$\mathfrak{p}_\beta := \mathfrak{s}_\beta + \mathfrak{h} + \sum_{\alpha \in \Phi^+} \Bbbk e_\alpha, \qquad \mathfrak{n}_\beta := \sum_{\alpha \in \Phi^+ \setminus \{\beta\}} \Bbbk e_\alpha, \qquad \widetilde{\mathfrak{s}}_\beta := \mathfrak{h}_e \oplus \mathfrak{s}_\beta.$$

Let $C_\beta = ef + fe + \frac{1}{2}h^2 = 2ef + \frac{1}{2}h^2 - h$, a central element of $U(\widetilde{\mathfrak{s}}_\beta)$. Given $\lambda \in \mathfrak{h}_e^*$ and $c \in \Bbbk$ we denote by $I_\beta(\lambda, c)$ the left ideal of $U(\mathfrak{p}_\beta)$ generated by $f - 1$, $C_\beta - c$, all $h - \lambda(h)$ with $h \in \mathfrak{h}_e$, and all e_γ with $\gamma \in \Phi^+ \setminus \{\beta\}$. Let $Y(\lambda, c) := U(\mathfrak{p}_\beta)/I_\beta(\lambda, c)$, a $\mathfrak{p}_\beta$-module with the trivial action of $\mathfrak{n}_\beta$. Note that $Y(\lambda, c)$ is a Whittaker module for the Levi subalgebra $\widetilde{\mathfrak{s}}_\beta$. Now define

$$M(\lambda, c) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\beta)} Y(\lambda, c).$$

Recall that each z_i^* with $i \leq s$ is a root vector corresponding to $\gamma_i^* = -\beta - \gamma_i \in \Phi^+$. Let $\delta = \frac{1}{2}(\gamma_1^* + \cdots + \gamma_s^*)$ and $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Since the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}_e$ is nondegenerate, for any $\eta \in \mathfrak{h}_e^*$ there is a unique $t_\eta \in \mathfrak{h}_e$ such that $\varphi = (t_\eta, \cdot)$. Hence $(\cdot, \cdot)$ induces a bilinear form on $\mathfrak{h}_e^*$ via $(\mu, \nu) := (t_\mu, t_\nu)$ for all $\mu, \nu \in \mathfrak{h}_e^*$. Given a linear function $\varphi \in \mathfrak{h}^*$ we denote by $\bar{\varphi}$ the restriction of φ to $\mathfrak{h}_e$.

Theorem 1.3. *Each $\mathfrak{g}$-module $M(\lambda, c)$ is an object of the category $\mathcal{C}_\chi$. Furthermore,* $\mathrm{Wh}(M(\lambda, c)) \cong Z_H(\lambda + \bar{\delta}, c + (\lambda + 2\bar{\rho}, \lambda))$ *as H-modules.*

Combined with Skryabin's equivalence and the main results of Miličić–Soergel [28] and Backelin [1], Theorem 1.3 shows that the composition multiplicities of the Verma modules $Z_H(\lambda, c)$ can be computed with the help of certain parabolic Kazhdan–Lusztig polynomials. This confirms in the minimal nilpotent case the Kazhdan–Lusztig conjecture for finite $\mathcal{W}$-algebras formulated by de Vos and van Driel in [8]; see Remark 7.1 for more details.

Apart from its relevance to the theory of primitive ideals this work is a contribution to the rapidly growing theory of $\mathcal{W}$-algebras. *Finite* $\mathcal{W}$-algebras are attached to nilpotent elements of finite-dimensional simple Lie algebras via quantum Hamiltonian reduction. All finite $\mathcal{W}$-algebras of type A were recently described by J. Brundan and A. Kleshchev [5] who identified them with shifted truncated Yangians. It seems likely that their results can be extended to some nilpotent elements in Lie algebras of types B, C and D. Hidden Yangian symmetry of finite $\mathcal{W}$-algebras of type A was first discovered, in some special cases, by E. Ragoucy and P. Sorba [33].

Affine counterparts of finite $\mathcal{W}$-algebras have been studied even more intensively. It should be mentioned here that V. G. Kac and M. Wakimoto described *minimal nilpotent* superconformal algebras in the context of vertex operators and quantum reduction; see [20] and the references therein. It would be interesting to compare the algebras H of this paper with quasiclassical limits of vertex algebras of Kac–Wakimoto.

2. Structural features of the algebras H_χ

2.1. In this section we assume that e is an arbitrary nilpotent element in $\mathfrak{g}$. Decompose $\mathfrak{g}$ into the weight spaces relative to $\mathrm{ad}\, h$ giving a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$. Let χ be as in 1.1 and denote by $\mathfrak{z}_\chi$ the centraliser of χ in $\mathfrak{g}$. It is well-known that $\mathfrak{z}_\chi = \mathfrak{c}_\mathfrak{g}(e)$ is a graded subalgebra of $\mathfrak{p}_e := \bigoplus_{i \geq 0} \mathfrak{g}(i)$, that is, $\mathfrak{z}_\chi = \bigoplus_{i \geq 0} \mathfrak{z}_\chi(i)$ where $\mathfrak{z}_\chi(i) = \mathfrak{z}_\chi \cap \mathfrak{g}(i)$. Choose a $\Bbbk$-basis $x_1, \ldots, x_m$ of the parabolic subalgebra $\mathfrak{p}_e$ with $x_i \in \mathfrak{g}(n_i)$ such that $x_1, \ldots, x_r$ span $\mathfrak{z}_\chi$. Let $\mathcal{O} = \mathcal{O}_\chi$ and let d denote half of the dimension of $\mathcal{O}$.

Define the skew-symmetric bilinear form $\langle \cdot , \cdot \rangle$ on the subspace $\mathfrak{g}(-1)$ by setting $\langle x, y \rangle = (e, [x, y])$ for all $x, y \in \mathfrak{g}(-1)$. As $\mathfrak{z}_\chi \subset \mathfrak{p}_e$, this form is nondegenerate. Choose a basis $z_1, \dots, z_s, z_{s+1}, \dots, z_{2s}$ of $\mathfrak{g}(-1)$ such that

$$\langle z_{i+s}, z_j \rangle = \delta_{ij}, \quad \langle z_i, z_j \rangle = \langle z_{i+s}, z_{j+s} \rangle = 0 \quad (1 \le i, j \le r)$$

and denote by $\mathfrak{g}(-1)^0$ the linear span of $z_{s+1}, \dots, z_{2s}$. Let $\mathfrak{m}_\chi = \mathfrak{g}(-1)^0 \oplus \sum_{i \le 2} \mathfrak{g}(i)$, a nilpotent subalgebra of $\mathfrak{g}$ of dimension d; see [31] for example. Since χ vanishes on the derived subalgebra of $\mathfrak{m}_\chi$ the ideal N_χ of $U(\mathfrak{m}_\chi)$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}_\chi$ has codimension one in the enveloping algebra $U(\mathfrak{m}_\chi)$. Let $\Bbbk_\chi = U(\mathfrak{m}_\chi)/N_\chi$, a one-dimensional left $U(\mathfrak{m}_\chi)$-module, and let 1_χ stand for the image of 1 in $\Bbbk_\chi$. We denote by Q_χ the induced $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{m}_\chi)} \Bbbk_\chi$ and set

$$H_\chi := \mathrm{End}_{\mathfrak{g}}(Q_\chi)^{\mathrm{op}}.$$

It is proved in [31] that the algebra H_χ is a filtered deformation of the graded coordinate ring $\Bbbk[\mathcal{S}_e]$.

In what follows we will rely on a different realisation of H_χ found by W. L. Gan and Ginzburg [10]. Let $\mathfrak{n}_\chi = \bigoplus_{i \le -1} \mathfrak{g}(i)$ and $\mathfrak{n}'_\chi = \bigoplus_{i \le -2} \mathfrak{g}(i)$. Clearly, $\mathfrak{n}_\chi$ and $\mathfrak{n}'_\chi$ are nilpotent subalgebras of $\mathfrak{g}$ and $\mathfrak{n}'_\chi$ is an ideal of $\mathfrak{n}_\chi$. Since $\mathfrak{n}'_\chi \subseteq \mathfrak{m}_\chi$, we may view $\Bbbk_\chi$ as an $\mathfrak{n}'_\chi$-module. Let $\widehat{Q}_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}'_\chi)} \Bbbk_\chi$, an induced $\mathfrak{g}$-module and the quotient of $U(\mathfrak{g})$ by the left ideal $\mathcal{I}_\chi$ generated by all $x - \chi(x)$ with $x \in \mathfrak{n}'_\chi$. The representation of $U(\mathfrak{g})$ in $\mathrm{End}(Q)$ will be denoted by $\widehat{\rho}_\chi$. Since χ vanishes on $[\mathfrak{n}_\chi, \mathfrak{n}'_\chi] \subseteq \bigoplus_{i \le -3} \mathfrak{g}(i)$, the left ideal $\mathcal{I}_\chi$ is stable under the adjoint action of $\mathfrak{n}_\chi$ on $U(\mathfrak{g})$. Therefore, $\mathrm{ad}\, \mathfrak{n}_\chi$ acts on $\widehat{Q}_\chi$. The fixed point space $\widehat{Q}_\chi^{\mathrm{ad}\, \mathfrak{n}_\chi}$ carries a natural algebra structure given by $(x + \mathcal{I}_\chi)(y + \mathcal{I}_\chi) = xy + \mathcal{I}_\chi$ for all $x + \mathcal{I}_\chi, y + \mathcal{I}_\chi \in \widehat{Q}_\chi$; see [10, p. 244] for more details. We furnish $Q_\chi^{\mathrm{ad}\, \mathfrak{m}_\chi}$ and $\widehat{Q}_\chi^{\mathrm{ad}\, \mathfrak{n}'_\chi}$ with algebra structures in a similar fashion. It is well known (and easily seen) that

$$H_\chi \cong Q_\chi^{\mathrm{ad}\, \mathfrak{m}_\chi} \quad \text{and} \quad \mathrm{End}_{\mathfrak{g}}(\widehat{Q}_\chi)^{\mathrm{op}} \cong \widehat{Q}_\chi^{\mathrm{ad}\, \mathfrak{n}'_\chi}$$

as algebras. As $\mathfrak{n}'_\chi \subseteq \mathfrak{m}_\chi$, there is a natural $\mathfrak{g}$-module epimorphism $\widehat{Q}_\chi \twoheadrightarrow Q_\chi$. As $\mathfrak{m}_\chi \subseteq \mathfrak{n}_\chi$, it induces an algebra map $\eta\colon \widehat{Q}_\chi^{\mathrm{ad}\, \mathfrak{n}_\chi} \to H_\chi$. By [10, Theorem 4.1], η is an isomorphism of algebras. Henceforth we will make no distinction between H_χ and $\widehat{Q}_\chi^{\mathrm{ad}\, \mathfrak{n}_\chi}$ and view the latter as a subalgebra of $\mathrm{End}_{\mathfrak{g}}(\widehat{Q})^{\mathrm{op}}$.

Given $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^{2s}$ we set $x^{\mathbf{a}} z^{\mathbf{b}} := x_1^{a_1} \cdots x_m^{a_m} z_1^{b_1} \cdots z_{2s}^{b_{2s}}$. By the PBW theorem, the monomials $x^{\mathbf{a}} z^{\mathbf{b}} \otimes 1_\chi$ with $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^{2s}$ form a $\Bbbk$-basis of $\widehat{Q}_\chi$. For $k \in \mathbb{Z}_+$ we denote by $\widehat{Q}_\chi^k$ the linear span of all $x^{\mathbf{a}} z^{\mathbf{b}} \otimes 1_\chi$ with

$$|(\mathbf{a}, \mathbf{b})|_e := \sum_{i=1}^{m} a_i (n_i + 2) + \sum_{i=1}^{2s} b_i \le k. \tag{2.1.1}$$

We let H_χ^k denote the subspace of H_χ consisting of all $h \in H_\chi$ with $h(1_\chi) \in \widehat{Q}_\chi^k$. By [31] and [10], the subspaces $\{H_\chi^k \mid k \in \mathbb{Z}_+\}$ form an increasing filtration of the algebra

H_χ and the corresponding graded algebra $\operatorname{gr} H_\chi$ is isomorphic to a polynomial ring in r variables with free homogeneous generators of degree $n_1+2,\dots,n_r+2$. The elements x in $\widehat{Q}_\chi^k \setminus \widehat{Q}_\chi^{k-1}$ and $H_\chi^k \setminus H_\chi^{k-1}$ are said to have *Kazhdan degree* k, written $\deg_e(x)=k$. It is immediate from [31, Theorem 4.6] that in our present realisation the algebra H_χ has a distinguished generating set $\Theta_1,\dots,\Theta_r$ such that $\operatorname{gr}\Theta_1,\dots,\operatorname{gr}\Theta_r$ generate $\operatorname{gr} H_\chi$ and

$$\Theta_k(1_\chi)=\Big(x_k+\sum_{0<|(\mathbf{i},\mathbf{j})|_e\le n_k+2}\lambda_{\mathbf{i},\mathbf{j}}^k x^{\mathbf{i}}z^{\mathbf{j}}\Big)\otimes 1_\chi,\qquad 1\le k\le r, \tag{2.1.2}$$

where $\lambda_{\mathbf{i},\mathbf{j}}^k\in\Bbbk$ and $\lambda_{\mathbf{i},\mathbf{j}}^k=0$ if either $|(\mathbf{i},\mathbf{j})|_e=n_k+2$ and $|\mathbf{i}|+|\mathbf{j}|=1$, or $\mathbf{i}\ne\mathbf{0}$, $\mathbf{j}=\mathbf{0}$, and $i_j=0$ for $j>r$. The monomials $\Theta_1^{a_1}\cdots\Theta_r^{a_r}$ with $(a_1,\dots,a_r)\in\mathbb{Z}_+^r$ form a PBW basis of H_χ.

2.2. Given a subset X of $\mathfrak{g}$ we denote by $Z_G(X)$ the closed subgroup of G consisting of all $g\in G$ with $(\operatorname{Ad} g)(x)=x$ for all $x\in X$. Let P_e denote the parabolic subgroup of G with $\operatorname{Lie} P_e=\mathfrak{p}_e$. There exists a 1-parameter subgroup $\lambda_e\colon\Bbbk^\times\to G$ optimal for the G-unstable vector e and such that:

- $(\operatorname{Ad}\lambda_e(t))_{|\mathfrak{g}(i)}=t^i\,\mathrm{id}$ for all $t\in\Bbbk^\times$ and $i\in\mathbb{Z}$;
- $Z_G(e)\subset P(e)$, $R_u(Z_G(e))\subset R_u(P_e)$, $Z_G(e)=(Z_G(e)\cap Z_G(\lambda_e))R_u(Z_G(e))$;
- $C(e):=Z_G(e)\cap Z_G(\lambda_e)$ is a reductive group, and $\operatorname{Lie} C(e)=\mathfrak{z}_\chi(0)$;

see [6, Chapter 5] and [32]. Let $\operatorname{Ad} C(e)$ denote the image of $C(e)$ in the adjoint group $\operatorname{Ad} G=(\operatorname{Aut}\mathfrak{g})^\circ$. Put $\sigma=\operatorname{Ad}\lambda_e(-1)$, an element of order ≤ 2 in $\operatorname{Ad} G$. Clearly, σ lies in the centre of $\operatorname{Ad} C(e)$ and $\sigma(x)=(-1)^i x$ for all $x\in\mathfrak{g}(i)$ and $i\in\mathbb{Z}$.

Lemma 2.1. *The element σ belongs to any maximal torus of* $\operatorname{Ad} C(e)$.

Proof. Let T_0 be a maximal torus of $\operatorname{Ad} C(e)$, $\widetilde{T}_0$ the inverse image of T_0 in G, and $L=Z_G(\widetilde{T}_0)$. Then L is a Levi subgroup of G and e is a distinguished nilpotent element in $\mathfrak{l}=\operatorname{Lie} L$. The construction in [32] shows that all weights of $\operatorname{Ad}\lambda_e(\Bbbk^\times)$ on $\mathfrak{l}$ are even. Then σ acts trivially on $\mathfrak{l}$, yielding $\mathfrak{g}^{\sigma t}\supseteq\mathfrak{l}$ for all $t\in T_0$. As $\Bbbk$ is infinite, there is $t_0\in T_0$ such that $\mathfrak{g}^{\sigma t_0}=\mathfrak{l}$. Let $\mathcal{C}$ denote the conjugacy class of the image of σt_0 in the component group $Z_{\operatorname{Ad} G}(e)/Z_{\operatorname{Ad} G}(e)^\circ\cong(\operatorname{Ad} C(e))/(\operatorname{Ad} C(e))^\circ$. As $\operatorname{Ad} G$ is a group of adjoint type, the G-conjugacy class of the pair (L,e) corresponds under Sommers' bijection to the G-conjugacy class of the pair $(e,\mathcal{C})$; see [35, 27, 32]. As L is a Levi subgroup in G, the cited references also show that $\mathcal{C}=\{1\}$. But $T_0\subseteq(\operatorname{Ad} C(e))^\circ$ and $t_0\in T_0$. So we get $\sigma\in Z((\operatorname{Ad} C(e))^\circ)$. As $(\operatorname{Ad} C(e))^\circ$ is a reductive group, the torus T_0 is self-centralising in $(\operatorname{Ad} C(e))^\circ$. Hence $\sigma\in T_0$, completing the proof. □

We now fix a maximal torus T_e in $\operatorname{Ad} C(e)$ and assume (without loss of generality) that all z_i with $i\le 2s$ and x_j with $j\le m$ are weight vectors with respect to T_e. By Lemma 2.1, $\sigma\in T_e$. Note that $C(e)$ preserves both $\mathfrak{n}'_\chi$ and $\operatorname{Ker}\chi$. Since $C(e)$ acts on $U(\mathfrak{g})$ as algebra automorphisms, it preserves the left ideal $\mathfrak{I}_\chi$ and thus acts on $\widehat{Q}_\chi$. This action is compatible with that of $\mathfrak{g}$, i.e.

$$g\circ\widehat{\rho}_\chi(x)\circ g^{-1}=\widehat{\rho}_\chi((\operatorname{Ad} g)(x))\quad(\forall g\in C(e),\ x\in\mathfrak{g}). \tag{2.2.1}$$

Since $C(e)$ preserves $\mathfrak{n}_\chi$ too, it acts on $H_\chi = \widehat{Q}_\chi^{\mathrm{ad}\,\mathfrak{n}_\chi}$ as algebra automorphisms. Since $g(1_\chi) = 1_\chi$ for all $g \in C(e)$, the action of $C(e)$ on $\widehat{Q}_\chi$ and H_χ is filtration preserving, hence locally finite. Since $Z(G)$ acts trivially on $U(\mathfrak{g})$, there is a natural action of $\mathrm{Ad}\,C(e)$ on $\widehat{Q}_\chi$ and H_χ. It should be mentioned that

$$\sigma(x^{\mathbf{a}}z^{\mathbf{b}} \otimes 1_\chi) = (-1)^{|(\mathbf{a},\mathbf{b})|_e} x^{\mathbf{a}}z^{\mathbf{b}} \otimes 1_\chi \tag{2.2.2}$$

for all $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^{2s}$.

Lemma 2.2. *Each generator $\Theta_k \in H_\chi$ can be chosen to be a weight vector for T_e of the same weight as x_k.*

Proof. Let γ_k denote the T_e-weight of x_k. If $\gamma_k \neq 0$, we assume without loss of generality that $\lambda^k_{\mathbf{0},\mathbf{0}} = 0$. Let $t \in T_e$ and $\Theta_t := t(\Theta_k) - \gamma_k(t)\Theta_k$, an element in H_χ. Since all $x^{\mathbf{i}}z^{\mathbf{j}} \otimes 1_\chi \in \widehat{Q}_\chi$ are weight vectors for T_e, we deduce from (2.1.2) and (2.2.1) that $\Theta_t(1_\chi)$ is a linear combination of $x^{\mathbf{a}}z^{\mathbf{b}} \otimes 1_\chi$ with either $\mathbf{b} \neq \mathbf{0}$ or $a_j \neq 0$ for some $j > r$. Then $\Theta_t = 0$ for all $t \in T_e$, by [31, Lemma 4.5], and the result follows. □

2.3. We now consider the linear map $\Theta\colon \mathfrak{z}_\chi \to H_\chi$, $x \mapsto \Theta_x$, such that $\Theta_{x_i} = \Theta_i$ for all i. Thanks to Lemma 2.2, Θ is an injective homomorphism of T_e-modules. Although Θ is not a Lie algebra homomorphism, in general, it follows from [31, Theorem 4.6(iv)] and (2.2.2) and Lemma 2.2 that

$$[\Theta_{x_i}, \Theta_{x_j}] \equiv \Theta_{[x_i,x_j]} + q_{ij}(\Theta_1, \dots, \Theta_r) \pmod{H_\chi^{n_i+n_j}} \tag{2.3.1}$$

where q_{ij} is a polynomial in r variables with initial form of total degree ≥ 2.

Remark 2.1. As $C(e)$ is a reductive group, each $C(e)$-module H_χ^k is completely reducible. From this it follows that there exists a unitriangular polynomial substitution

$$F\colon (\Theta_1, \dots, \Theta_r) \mapsto (F_1(\Theta_1, \dots, \Theta_r), \dots, F_r(\Theta_1, \dots, \Theta_r))$$

which satisfies the following conditions:

- $\deg_e F(\Theta_i) = \deg_e \Theta_i = n_i + 2$ for all $i \leq r$;
- the linear map $\Theta_F\colon \mathfrak{z}_\chi \to H_\chi$ with $\Theta_F(x_i) = F(\Theta_i)$ for all i is an injective homomorphism of $C(e)$-modules, and $\mathfrak{z}_\chi \cong \mathrm{gr}\,\Theta_F(\mathfrak{z}_\chi)$ as graded $C(e)$-modules;
- an analogue of (2.3.1) holds for $F(\Theta_1), \dots, F(\Theta_r)$, and the subspaces $\Theta_F(\mathfrak{z}_\chi)$ and $\mathrm{gr}\,\Theta_F(\mathfrak{z}_\chi)$ generate the algebras H_χ and $\mathrm{gr}\,H_\chi$, respectively.

Proposition 2.1. *There exists an associative $\Bbbk[t]$-algebra $\mathcal{H}_\chi$ free as a module over $\Bbbk[t]$ and such that*

$$\mathcal{H}_\chi/(t-\lambda)\mathcal{H}_\chi \cong \begin{cases} H_\chi & \text{if } \lambda \neq 0, \\ U(\mathfrak{z}_\chi) & \text{if } \lambda = 0 \end{cases}$$

as $\Bbbk$-algebras. In other words, the enveloping algebra $U(\mathfrak{z}_\chi)$ is a contraction of H_χ.

Proof. Consider the algebra $H(R) = R \otimes H_\chi$ over the ring of Laurent polynomials $R = k[t, t^{-1}]$ obtained from H_χ by extension of scalars, and identify H_χ with the subspace $\Bbbk \otimes H_\chi$ of $H(R)$. Define an invertible R-linear transformation π on $H(R)$ by setting

$$\pi(\Theta_1^{k_1} \cdots \Theta_r^{k_r}) = t^{n_1 k_1 + \cdots + n_r k_r} \Theta_1^{k_1} \cdots \Theta_r^{k_r} \quad \forall (k_1, \ldots, k_r) \in \mathbb{Z}_+^r$$

and extending to $H(R)$ by R-linearity. We view π as an isomorphism from $H(R)$ onto a new R-algebra $H(R, \pi)$ with underlying R-module $R \otimes H_\chi$ and with associative product given by $(x \cdot y)_\pi := \pi^{-1}(\pi(x) \cdot \pi(y))$ for all $x, y \in R \otimes H_\chi$. We denote by $\mathcal{H}_\chi$ the free $\Bbbk[t]$-submodule of $H(R, \pi)$ generated by $\Theta_1^{a_1} \cdots \Theta_r^{a_r}$ with $(a_1, \ldots, a_r) \in \mathbb{Z}_+^r$. It follows from (2.1.1) and (2.1.2) that

$$\deg_e(\Theta_1^{k_1} \cdots \Theta_r^{k_r}) = \sum_{i=1}^r n_i k_i + 2 \sum_{i=1}^r k_i.$$

In view of (2.3.1) this yields

$$(\Theta_i \cdot \Theta_j - \Theta_j \cdot \Theta_i)_\pi = \pi^{-1}(t^{n_i + n_j}[\Theta_i, \Theta_j]) \equiv \Theta_{[x_i, x_j]} \,(\mathrm{mod}\, t\mathcal{H}_\chi)$$

(since the initial form of q_{ij} has total degree ≥ 2 and $\deg_e q_{ij}(\Theta_1, \ldots, \Theta_r) = n_i + n_j + 2$ if $q_{ij} \neq 0$). Using induction on the Kazhdan degree of $\Theta_1^{k_1} \cdots \Theta_r^{k_r}$ and the commutativity of $\mathrm{gr}\, H_\chi$ we now deduce that $(\Theta_i \cdot \mathcal{H}_\chi)_\pi \subseteq \mathcal{H}_\chi$ for all i. So $\mathcal{H}_\chi$ is a $\Bbbk[t]$-subalgebra of $H(R, \pi)$.

If $\lambda \neq 0$ then the homomorphism $\Bbbk[t] \to k$ taking t to λ extends to a homomorphism $R \to k$. The isomorphism π^{-1} injects $(t - \lambda)H(R, \pi)$ onto $(t - \lambda)H(R)$. Because $\mathcal{H}_\chi \cap (t - \lambda)H(R, \pi) = (t - \lambda)\mathcal{H}_\chi$ and $H_\chi \cap (t - \lambda)H(R) = 0$, we have

$$\mathcal{H}_\chi/(t - \lambda)\mathcal{H}_\chi \cong H(R, \pi)/(t - \lambda)H(R, \pi) \cong H(R)/(t - \lambda)H(R) \cong H_\chi,$$

by the theorem on isomorphism. Now put $\overline{\mathcal{H}}_\chi := \mathcal{H}_\chi/t\mathcal{H}_\chi$ and identify the generators $\Theta_i = \Theta_{x_i}$ of $\mathcal{H}_\chi$ with their images in $\overline{\mathcal{H}}_\chi$. It is immediate from our earlier remarks that these images satisfy the relations $[\Theta_{x_i}, \Theta_{x_j}] = \Theta_{[x_i, x_j]}$ for all i, j. By the universality property of the enveloping algebra $U(\mathfrak{z}_\chi)$, there exists an algebra homomorphism $\phi\colon U(\mathfrak{z}_\chi) \twoheadrightarrow \overline{\mathcal{H}}_\chi$ with $\phi(x_i) = \Theta_i$ for all i. Since $\mathcal{H}_\chi$ is a free $\Bbbk[t]$-module, the monomials $\Theta_1^{a_1} \cdots \Theta_r^{a_r}$ with $(a_1, \ldots, a_r) \in \mathbb{Z}_+^r$ are linearly independent in $\overline{\mathcal{H}}_\chi$. As a consequence, ϕ is an isomorphism. □

2.4. Let $\mathbf{A}_e$ denote the associative algebra over $\Bbbk$ generated by $z_1, \ldots, z_s, z_{s+1}, \ldots, z_{2s}$ subject to the relations $[z_{i+s}, z_j] = \delta_{ij}$ and $[z_i, z_j] = [z_{i+s}, z_{j+s}] = 0$ where $1 \leq i, j \leq s$. Clearly, $\mathbf{A}_e \cong \mathbf{A}_s(\Bbbk)$, the sth Weyl algebra over $\Bbbk$. If $s = 0$ then $\mathbf{A}_e = \Bbbk$.

Let $i \mapsto i^*$ denote the involution on the set of indices $\{1, \ldots, s, s+1, \ldots, 2s\}$ such that $i^* = i + s$ for $i \leq s$ and $i^* = i - s$ for $i > s$. For $1 \leq i \leq 2s$ define $z_i^* := (-1)^{p(i)} z_{i^*}$ where

$$p(i) = \begin{cases} 0 & \text{if } i \leq s, \\ 1 & \text{if } i > s. \end{cases}$$

Note that $z_i = (-1)^{p(i^*)} z_{i^*}^*$ for $1 \leq i \leq 2s$ and $z_i^* = z_{i+s}$, $z_{i+s}^* = -z_i$ for $1 \leq i \leq s$. It is worth remarking that the following relation holds in $U(\mathfrak{g})$:

$$\sum_{i=1}^{2s} z_i z_i^* = -\sum_{i=1}^{2s} z_i^* z_i \equiv s \ (\mathrm{mod}\, \mathfrak{I}_\chi). \tag{2.4.1}$$

As the form $\langle \cdot\,,\, \cdot \rangle$ is $\mathfrak{z}_\chi(0)$-invariant and $\langle z_i^*, z_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq 2s$, for all $x \in \mathfrak{z}_\chi(0)$ we have

$$[x, z_k^*] = \sum_{i=1}^{2s} \langle z_k^*, [x, z_i^*] \rangle z_i = -\sum_{i=1}^{2s} \langle [x, z_i^*], z_k^* \rangle z_i. \tag{2.4.2}$$

Each $h \in H_\chi$ is determined by its effect on the canonical generator $1_\chi \in \widehat{Q}_\chi$. Since the vector $h(1_\chi)$ can be uniquely expressed as $h(1_\chi) = (\sum_{\mathbf{i} \in \mathbb{Z}_+^{2s}} u_{\mathbf{i}} \cdot z^{\mathbf{i}}) \otimes 1_\chi$ with $u_{\mathbf{i}} \in U(\mathfrak{p}_e)$, one obtains a natural linear injection

$$\widetilde{\mu} \colon H_\chi \to U(\mathfrak{p}_e) \otimes \mathbf{A}_e^{\mathrm{op}}, \qquad \widetilde{\mu}(h) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{2s}} u_{\mathbf{i}} \otimes z^{\mathbf{i}}. \tag{2.4.3}$$

As the form $\langle \cdot\,,\, \cdot \rangle$ is $C(e)$-invariant, the group $C(e)$ acts on $\mathbf{A}_e^{\mathrm{op}}$ as automorphisms. As $C(e)$ also acts on $U(\mathfrak{p}_e)$, it acts as automorphisms on the algebra $U(\mathfrak{p}_e) \otimes \mathbf{A}_e^{\mathrm{op}}$, via $g(u \otimes a) = g(u) \otimes g(a)$ with the obvious choices of g, u, a.

Proposition 2.2. *The map $\widetilde{\mu} \colon H_\chi \hookrightarrow U(\mathfrak{p}_e) \otimes \mathbf{A}_e^{\mathrm{op}}$ is a $C(e)$-equivariant algebra homomorphism.*

Proof. Let $\mathcal{Z}$ denote the linear span of all $z^{\mathbf{i}} \otimes 1_\chi$ with $\mathbf{i} \in \mathbb{Z}_+^{2s}$. We identify $\mathcal{Z}$ with the space of the left regular representation of $\mathbf{A}_e$ via $z^{\mathbf{i}} \otimes 1_\chi \mapsto z^{\mathbf{i}}$. Now $\widehat{\rho}_\chi$ induces a representation of $U(\mathfrak{n}_\chi)$ in $\mathrm{End}(\mathcal{Z})$, say ψ_0. Since $\mathfrak{g}(-1) \subset \mathfrak{n}_\chi$ and $\mathfrak{g}(i) \subset \mathrm{Ker}\, \chi$ for all $i \leq -3$, the definition of $\widehat{Q}_\chi^{\mathrm{ad}\, \mathfrak{n}_\chi}$ and induction on k show that

$$\widehat{\rho}_\chi(z_1 \cdots z_k)(h(1_\chi)) = \sum_{\mathbf{i} \in \mathbb{Z}_+^{2s}} u_{\mathbf{i}} \cdot \widehat{\rho}_\chi(z^{\mathbf{i}} \cdot z_1 \cdots z_k)(1_\chi)$$

for all $z_1, \ldots, z_k \in \mathfrak{g}(-1)$. Now let h' be another element in H_χ and suppose that $h'(1_\chi) = (\sum_{\mathbf{i} \in \mathbb{Z}_+^{2s}} u'_{\mathbf{i}} \cdot z^{\mathbf{i}}) \otimes 1_\chi$ where $u'_{\mathbf{i}} \in U(\mathfrak{p}_e)$. Then

$$\begin{aligned}(h \cdot h')(1_\chi) = h'(h(1_\chi)) &= \sum_{\mathbf{i}} \widehat{\rho}_\chi(u_{\mathbf{i}}) \cdot \widehat{\rho}_\chi(z^{\mathbf{i}})(h'(1_\chi)) \\ &= \sum_{\mathbf{i}} \sum_{\mathbf{j}} u_{\mathbf{i}} \cdot u'_{\mathbf{j}} \cdot \widehat{\rho}_\chi(z^{\mathbf{j}} \cdot z^{\mathbf{i}})(1_\chi).\end{aligned}$$

It remains to note that the map $z^{\mathbf{i}} \otimes 1_\chi \mapsto z^{\mathbf{i}}$ mentioned above identifies $\psi_0(U(\mathfrak{n}_\chi))$ with the image of $\mathbf{A}_e$ in its left regular representation. The $C(e)$-equivariance of $\widetilde{\mu}$ is immediate from the definitions. □

Remark 2.2. Composing $\widetilde{\mu}$ with the natural projection $U(\mathfrak{p}_e) \otimes \mathbf{A}_e^{\mathrm{op}} \twoheadrightarrow U(\mathfrak{g}(0)) \otimes \mathbf{A}_e^{\mathrm{op}}$ one obtains an algebra homomorphism

$$\mu\colon H_\chi \to U(\mathfrak{g}(0)) \otimes \mathbf{A}_e^{\mathrm{op}}$$

which will be referred to as the *Miura map*. In the special case where e is even this map has already appeared in [31, (7.1)] (note that for e even we have $\mathbf{A}_e^{\mathrm{op}} = \Bbbk$). It can be proved that the map μ is always injective (this will not be required in the present article).

The adjoint action of $\mathfrak{z}_\chi(0)$ on $\mathfrak{g}$ induces Lie algebra maps $\mathfrak{z}_\chi(0) \to \mathrm{Der}(\mathbf{A}_e^{\mathrm{op}})$, $\mathfrak{z}_\chi(0) \to \mathrm{Der}(U(\mathfrak{p}_e) \otimes \mathbf{A}_e^{\mathrm{op}})$ and $\mathfrak{z}_\chi(0) \to \mathrm{Der}(H_\chi)$ (of course, the same maps can be obtained by differentiating the respective actions of $C(e)$ on $\mathbf{A}_e^{\mathrm{op}}$, $U(\mathfrak{p}_e) \otimes \mathbf{A}_e^{\mathrm{op}}$ and H_χ). By abuse of notation, the image of $x \in \mathfrak{z}_\chi(0)$ under each of these maps will be denoted by $\operatorname{ad} x$.

2.5. In what follows we will need explicit formulae for the generators Θ_k of small Kazhdan degree. The reader will observe strong similarity between our formulae and the expressions for conserved fields of low conformal weight found by Kac and Wakimoto [20] in the context of vertex algebras and quantum reduction.

Lemma 2.3. *If $v \in \mathfrak{z}_\chi(0)$ then it can be assumed that*

$$\Theta_v(1_\chi) = \left(v + \frac{1}{2}\sum_{i=1}^{2s} z_i[v, z_i^*]\right) \otimes 1_\chi = \left(v + \frac{1}{2}\sum_{i=1}^{2s} [v, z_i^*]z_i\right) \otimes 1_\chi.$$

Proof. It follows from (2.1.2) and (2.2.2) that there exist a scalar β and a symmetric matrix $A = (\alpha_{ij})$ of order $2s$ such that

$$\Theta_v(1_\chi) = \left(v + \frac{1}{2}\sum_{i,j=1}^{2s} \alpha_{ij} z_i z_j + \beta\right) \otimes 1_\chi.$$

Since A is symmetric and $[z_k^*, v + \frac{1}{2}\sum_{i,j=1}^{2s} \alpha_{ij} z_i z_j + \beta] \in \mathfrak{I}_\chi$ for all k, it must be that $[v, z_k^*] = \sum_{j=1}^{2s} \alpha_{kj} z_j$. Therefore, after a proper adjustment of β we get

$$\Theta_v(1_\chi) = \left(v + \frac{1}{2}\sum_{i=1}^{2s} z_i[v, z_i^*]\right) \otimes 1_\chi.$$

Since $\mathfrak{I}_\chi$ is $(\operatorname{ad} v)$-stable, (2.4.1) yields $\sum_{i=1}^{2s} z_i[v, z_i^*] \equiv \sum_{i=1}^{2s} [v, z_i^*]z_i \pmod{\mathfrak{I}_\chi}$. This completes the proof. □

From now on we always assume that the generators Θ_v with $v \in \mathfrak{z}_\chi(0)$ are chosen in accordance with Lemma 2.3. This has the following advantage:

Lemma 2.4. *The restriction of Θ to $\mathfrak{z}_\chi(0)$ is a Lie algebra homomorphism, i.e.*

$$[\Theta_u, \Theta_v] = \Theta_{[u,v]} \quad (\forall u, v \in \mathfrak{z}_\chi(0)).$$

Moreover, the Lie algebra homomorphism $\mathrm{ad} \circ \Theta\colon \mathfrak{z}_\chi(0) \to \mathrm{Der}(H_\chi)$ *coincides with the differential of the rational action* $C(e) \to \mathrm{Aut}(H_\chi)$.

Proof. We are going to use the injective homomorphism $\widetilde{\mu}$ from 2.4. Let $x \in \mathfrak{z}_\chi(0)$. Computing in $\mathbf{A}_e^{\mathrm{op}}$ and applying (2.4.2) we get

$$\begin{aligned}\left[\frac{1}{2}\sum_{i=1}^{2s}[x, z_i^*]z_i, z\right] &= -\frac{1}{2}\sum_{i=1}^{2s}(\langle [x, z_i^*], z\rangle z_i + \langle z_i, z\rangle [x, z_i^*]) \\ &= \frac{1}{2}([x, z] + [x, z]) = [x, z] \end{aligned} \tag{2.5.1}$$

for all $z \in \mathfrak{g}(-1)$. Hence $\mathrm{ad}\, x = \mathrm{ad}(\frac{1}{2}\sum_{i=1}^{2s}[x, z_i^*]z_i)$ as derivations of $\mathbf{A}_e^{\mathrm{op}}$. Then

$$[\widetilde{\mu}(\Theta_x), \widetilde{\mu}(h)] = \left[\widetilde{\mu}(\Theta_x), \sum u_{\mathbf{i}} \otimes z^{\mathbf{i}}\right] = \sum([x, u_{\mathbf{i}}] \otimes z^{\mathbf{i}} + u_{\mathbf{i}} \otimes [x, z^{\mathbf{i}}]) = \widetilde{\mu}((\mathrm{ad}\, x)(h))$$

for all $h \in H_\chi$. As $\widetilde{\mu}$ is injective, it must be that $[\Theta_x, h] = (\mathrm{ad}\, x)(h)$, i.e. the adjoint action of $\Theta(\mathfrak{z}_\chi(0))$ coincides with the differential of the action of $C(e)$ on H_χ. Also,

$$\begin{aligned}\sum_{i=1}^{2s}[v, z_i^*][u, z_i] &\overset{(2.4.2)}{=\!=} -\sum_{i,j=1}^{2s}\langle z_j^*, [u, z_i^*]\rangle [v, z_i]z_j \\ &\overset{(2.4.2)}{=\!=} -\sum_{i,j=1}^{2s}\langle z_i^*, [u, z_j^*]\rangle [v, z_i]z_j = -\sum_{j=1}^{2s}[v, [u, z_j^*]]z_j \end{aligned} \tag{2.5.2}$$

as elements in $\mathbf{A}_e^{\mathrm{op}}$, for all $u, v \in \mathfrak{z}_\chi(0)$. It follows that

$$\begin{aligned}[\widetilde{\mu}(\Theta_u), \widetilde{\mu}(\Theta_v)] &\overset{(2.5.1)}{=\!=} [u, v] \otimes 1 + \frac{1}{2} \otimes \sum_{i=1}^{2s}([u, [v, z_i^*]]z_i + [v, z_i^*][u, z_i]) \\ &\overset{(2.5.2)}{=\!=} [u, v] \otimes 1 + \frac{1}{2} \otimes \sum_{i=1}^{2s}[u, [v, z_i^*]]z_i - \frac{1}{2} \otimes \sum_{i=1}^{2s}[v, [u, z_i^*]]z_i \\ &= [u, v] \otimes 1 + \frac{1}{2} \otimes \sum_{i=1}^{2s}[[u, v], z_i^*]z_i = \widetilde{\mu}(\Theta_{[u,v]}). \end{aligned}$$

But then $[\Theta_u, \Theta_v] = \Theta_{[u,v]}$ for all $u, v \in \mathfrak{z}_\chi(0)$, as stated. □

Corollary 2.1. *Any two-sided ideal of H_χ is σ-stable.*

Proof. Let I be a two-sided ideal of H_χ. Clearly, I is invariant under the adjoint action of $\Theta(\mathfrak{z}_\chi(0))$. By Lemma 2.4, I is then stable under the differential of the $C(e)$-action on H_χ. Since $C(e)^\circ$ is a connected reductive group and the action of $C(e)$ on H_χ is filtration preserving, Weyl's theorem on complete reducibility shows that all subspaces $I \cap H_\chi^k$ are $C(e)^\circ$-stable. Since $Z(G) \subseteq C(e)$ acts trivially on H_χ and $(\mathrm{Ad}\, C(e))^\circ$ coincides with the image of $C(e)^\circ$ in $\mathrm{Ad}\, C(e) \cong C(e)/Z(G)$, Lemma 2.1 shows that I is σ-stable, as claimed. □

Given n elements $x_1, x_2, x_3, \ldots, x_n$ in a Lie algebra we denote by $[x_1 x_2 x_3 \ldots x_n]$ the commutator $[\ldots[[x_1, x_2], x_3], \ldots, x_n]$.

Lemma 2.5. *If $v \in \mathfrak{z}_\chi(1)$ then the generator $\Theta_v \in H_\chi$ has the following property:*

$$\Theta_v(1_\chi) = \left(v + \sum_{i=1}^{2s} [v, z_i^*] z_i + \frac{1}{3} \sum_{i,j=1}^{2s} [v z_i^* z_j^*] z_j z_i + z_v \right) \otimes 1_\chi$$

where $z_v = \frac{1}{3} \sum_{i=1}^{2s} (\sum_{k=1}^{2s} \langle z_k, [v, [z_k^*, z_i^*]] \rangle) z_i$. Moreover,

$$[\Theta_u, \Theta_v] = \Theta_{[u,v]} \quad (\forall u \in \mathfrak{z}_\chi(0)).$$

Proof. Let $h_v = \sum_i [v, z_i^*] z_i + \frac{1}{3} \sum_{i,j} [v z_i^* z_j^*] z_j z_i + z_v$, an element in $U(\mathfrak{g})$. By anti-commutativity and the Jacobi identity, we have

$$[z_k^*, [v z_i^* z_j^*]] = [z_j^*, [v z_i^* z_k^*]] + [v [z_k^* z_j^*] z_i^*] + [v, [z_i^* [z_k^*, z_j^*]]].$$

Since $(e, [v, x]) = 0$ for all $x \in \mathfrak{g}$, this yields

$$\langle z_k^*, [v z_i^* z_j^*] \rangle = \langle z_j^*, [v z_i^* z_k^*] \rangle - \langle z_i^*, [v z_k^* z_j^*] \rangle + \langle z_i^*, [v z_j^* z_k^*] \rangle \tag{2.5.3}$$

where $1 \le i, j, k \le 2s$. Computing in $U(\mathfrak{g})$ modulo $\mathfrak{I}_\chi$ we now get

$$\begin{aligned}
\Big[z_k^*, \sum_{i,j} [v z_i^* z_j^*] z_j z_i \Big] &\equiv \sum_{ij} \langle z_k^*, [v z_i^* z_j^*] \rangle z_j z_i + \sum_i [v z_i^* z_k^*] z_i + \sum_i [v z_k^* z_i^*] z_i \\
&= \sum_{ij} (\langle z_j^*, [v z_i^* z_k^*] \rangle - \langle z_i^*, [v z_k^* z_j^*] \rangle + \langle z_i^*, [v z_j^* z_k^*] \rangle) z_j z_i \\
&\quad + \sum_i ([v z_i^* z_k^*] z_i + [v z_k^* z_i^*] z_i) \\
&\equiv \sum_i ([v z_i^* z_k^*] z_i - z_i [v z_k^* z_i^*] + z_i [v z_i^* z_k^*]) \\
&\equiv 3 \sum_i ([v z_i^* z_k^*] z_i - \tfrac{1}{3} \langle z_i, [v z_k^* z_i^*] \rangle + \tfrac{1}{3} \langle z_i, [v z_i^* z_k^*] \rangle) \\
&\equiv 3 \Big(\sum_i ([v z_i^* z_k^*] z_i \Big) - 3 [z_k^*, z_v].
\end{aligned}$$

As a consequence,

$$[z_k^*, h_v] \equiv [z_k^*, v] + \sum_i [z_k^*, [v, z_i^*]]z_i + [v, z_k^*] + [z_k^*, z_v]$$
$$+ \sum_i [v z_i^* z_k^*]z_i - [z_k^*, z_v] \equiv 0 \ (\mathrm{mod}\, \mathcal{I}_\chi)$$

for all k. It is easy to see that $[z, h_v] \equiv 0 \ (\mathrm{mod}\, \mathcal{I}_\chi)$ for all $z \in \mathfrak{n}'_\chi$. By Lemma 2.2, Θ_v is a (-1)-eigenvector for σ. In conjunction with (2.1.2), [31, Lemma 4.5], and the computation above this shows that $\Theta_v(1_\chi) = h_v \otimes 1_\chi$.

Now let u be any element in $\mathfrak{z}_\chi(0)$ and put $\Theta' := [\Theta_u, \Theta_v] - \Theta_{[u,v]}$. It is immediate from (2.3.1) that Θ' is a polynomial in Θ_{x_i} with $x_i \in \mathfrak{z}_\chi(0)$. Since $\sigma(\Theta') = -\Theta'$ by Lemma 2.2, this polynomial must be zero. So $[\Theta_u, \Theta_v] = \Theta_{[u,v]}$ necessarily holds, completing the proof. □

3. Associated varieties and Gelfand–Kirillov dimension

3.1. At present very little is known about finite-dimensional representations of the algebras H_χ. In view of Proposition 2.1 this can be partly explained by the lack of detailed information on the structure of the centraliser $\mathfrak{c}_\mathfrak{g}(e)$. Besides, if e is not even then there is no obvious reason for $H_\chi = H_{\chi_e}$ to possess such representations. On the other hand, the evidence collected so far suggests that each algebra H_χ has infinitely many isoclasses of finite-dimensional irreducible representations and dimension formulae for those have roughly the same format as the Weyl dimension formula for $H_0 = U(\mathfrak{g})$; see 6.4. It is therefore natural to ask:

Question 3.1. *Is it true that for any nonzero $h \in H_\chi$ there exists a finite-dimensional irreducible representation ρ of H_χ such that $\rho(h) \neq 0$?*

Let $\mathcal{C}_\chi$ denote the category of all $\mathfrak{g}$-modules on which $x - \chi(x)$ acts locally nilpotently for each $x \in \mathfrak{m}_\chi$. Given a $\mathfrak{g}$-module M we denote by $\mathrm{Wh}(M)$ the subspace of M consisting of all $m \in M$ such that $x.m = \chi(x)m$ for all $x \in \mathfrak{m}_\chi$. Of course, for $M \in \mathcal{C}_\chi$ we have $\mathrm{Wh}(M) = 0$ if and only if $M = 0$. Let H_χ-mod denote the category of all left H_χ-modules. In the Appendix to [31], Skryabin proved that the functor

$$H_\chi\text{-mod} \to \mathcal{C}_\chi, \qquad V \mapsto Q_\chi \otimes_{H_\chi} V, \tag{3.1.1}$$

is an equivalence of categories. The inverse equivalence is given by the functor

$$\mathcal{C}_\chi \to H_\chi\text{-mod}, \qquad M \mapsto \mathrm{Wh}(M); \tag{3.1.2}$$

see also [10, Sect. 6]. Skryabin's result implies that the $\mathfrak{g}$-module $Q_\chi \otimes_{H_\chi} V$ is simple if and only if so is the H_χ-module V. By the Irreducibility Theorem, the associated variety of the annihilator in $U(\mathfrak{g})$ of any simple $\mathfrak{g}$-module coincides with the closure of a nilpotent orbit in $\mathfrak{g}^*$; see [2, 18, 21, 36, 12]. Our goal in this section is to determine the associated varieties of the annihilators $\mathrm{Ann}_{U(\mathfrak{g})} M$ for all $M \in \mathcal{C}_\chi$ with $\dim \mathrm{Wh}(M) < \infty$. Such modules are in 1-1 correspondence with the finite-dimensional representations of H_χ.

3.2. We recall the definition of the Gelfand–Kirillov dimension of a finitely generated $U(\mathfrak{g})$-module M. Firstly note that there exists a finite-dimensional subspace $M_0 \subseteq M$ such that $M = \bigcup_{n\geq 0} U_n M_0$ where U_n stands for the nth component of the standard filtration of $U(\mathfrak{g})$. It is known that for all $n \gg 0$ the dimension of $U_n M_0$ is a polynomial in n. The *Gelfand–Kirillov dimension* of M, denoted $\mathrm{Dim}(M)$, is defined as the degree of this polynomial. The key point in this definition is that $\mathrm{Dim}(M)$ is independent of the choice of M_0; see [13, p. 134] for more details.

Now let I be a two-sided ideal of the universal enveloping algebra $U(\mathfrak{g})$. The subspaces $I_n := I \cap U_n$ with $n \in \mathbb{Z}_+$ form an increasing filtration of I satisfying $U_m I_n \subseteq I_{m+n}$ for all $m, n \in \mathbb{Z}_+$. The associated graded algebra $\mathrm{gr}\, I \hookrightarrow \mathrm{gr}\, U(\mathfrak{g}) \cong S(\mathfrak{g})$ is therefore identified with a homogeneous ideal of the symmetric algebra $S(\mathfrak{g})$ stable under the adjoint action of G. The *associated variety* $\mathcal{VA}(I)$ of the ideal I is defined as the maximal spectrum of the affine algebra $S(\mathfrak{g})/\mathrm{gr}\, I$. It is immediate from the definition that $\mathcal{VA}(I)$ is a Zariski closed, conical, G-invariant subset of $\mathrm{Max}\, S(\mathfrak{g}) = \mathfrak{g}^*$. For M as above we have

$$\dim \mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})} M) \leq 2\,\mathrm{Dim}(M); \tag{3.2.1}$$

see [13, (10.7) and (17.11)]. The $\mathfrak{g}$-module M is called *holonomic* if equality holds here, that is, $\dim \mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})} M) = 2\,\mathrm{Dim}(M)$.

3.3. We are now in a position to state and prove the main result of this section.

Theorem 3.1. *Let $M \in \mathcal{C}_\chi$ and $I = \mathrm{Ann}_{U(\mathfrak{g})} M$. Then the following hold:*

(i) $\mathcal{O}_\chi \subset \mathcal{VA}(I)$.

(ii) *If* $\dim \mathrm{Wh}(M) < \infty$, *then* $\mathrm{Dim}(M) = \frac{1}{2}\dim \mathcal{O}_\chi$ *and* $\mathcal{VA}(I)$ *coincides with the Zariski closure of* $\mathcal{O}_\chi$. *In particular, M is a holonomic $\mathfrak{g}$-module.*

Proof. (1) Let $\top$ denote the anti-involution of the algebra $U(\mathfrak{g})$ such that $x^\top = -x$ and $(uv)^\top = v^\top u^\top$ for all $x \in \mathfrak{g}$ and all $u, v \in U(\mathfrak{g})$. Let M^* denote the $\mathfrak{g}$-module dual to M. It is easy to see that $\mathrm{Ann}_{U(\mathfrak{g})} M^* = I^\top$. Since $\top$ preserves the standard filtration of $U(\mathfrak{g})$ and acts as a scalar operator on each factor space U_n/U_{n-1} we have $\mathrm{gr}\,\mathrm{Ann}_{U(\mathfrak{g})} M^* = \mathrm{gr}(I^\top) = \mathrm{gr}\, I$. Consequently, $\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})} M^*) = \mathcal{VA}(I)$.

Pick any nonzero $m \in \mathrm{Wh}(M)$ and view it as a linear function on M^* via $m(f) = f(m)$ for all $f \in M^*$. Then

$$m(x.f) = (x.f)(m) = -f(x.m) = -\chi(x) f(m) = -\chi(x) m(f) \quad (\forall x \in \mathfrak{m}_\chi).$$

This shows that m is a dual $(\mathfrak{m}_\chi, -\chi)$-Whittaker vector of the $\mathfrak{g}$-module M^*; see [24, p. 221]. Thanks to Matumoto's theorem [24] we are now able to deduce that the associated variety of $\mathrm{Ann}_{U(\mathfrak{g})} M^*$ contains $-\chi$. As $\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})} M^*) = \mathcal{VA}(I)$ is conical and G-stable this yields $\mathcal{O}_\chi \subset \mathcal{VA}(I)$, proving (i).

(2) From now on suppose that $M_0 := \mathrm{Wh}(M)$ is finite-dimensional. Let $2d = \dim \mathcal{O}_\chi$. By the $\mathfrak{sl}_2$-theory, $r = \dim \mathfrak{z}_\chi = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(-1) = \dim \mathfrak{g}(0) + 2s$. Hence $d = \dim \mathfrak{m}_\chi = \dim \mathfrak{p}_e - \dim \mathfrak{g}(0) - s = m - r + s$. Let $Y_1, \ldots, Y_d$ be a basis of $\mathfrak{m}_\chi$ with $Y_i \in \mathfrak{g}(-l_i - 2)$ for some $l_i \geq -1$ and choose $X_i \in \mathfrak{g}(l_i)$ with $1 \leq i \leq d$ such that

$\chi([Y_i, X_j]) = \delta_{ij}$. No generality will be lost by assuming further that $X_i = z_i$ for $i \leq s$ and $X_{s+j} = x_{r+j}$ for $1 \leq j \leq m-r$, where z_i and x_j are basis vectors introduced in 2.1. For $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}_+^d$ put

$$|\mathbf{a}| = \sum_{i=1}^{d} a_i, \quad \operatorname{wt} \mathbf{a} = -\sum_{i=1}^{s} a_i + \sum_{i=1}^{m-r} n_{r+i} a_{s+i}, \quad X^{\mathbf{a}} = X_1^{a_1} \cdots X_d^{a_d} \in U_{|\mathbf{a}|}.$$

(3) Let $\{m_i\}$ be a basis of M_0. Since the $U(\mathfrak{g})$-module Q_χ is generated by 1_χ it follows from (3.1.1) and (3.1.2) that the $U(\mathfrak{g})$-module M is generated by M_0. As explained in [31, p. 53] the vectors $X^{\mathbf{a}}(m_i)$ with $\mathbf{a} \in \mathbb{Z}_+^d$ and $i \leq \dim M_0$ are linearly independent. Therefore,

$$\dim U_n M_0 \geq (\dim M_0) \cdot \operatorname{Card}\{\mathbf{a} \in \mathbb{Z}_+^d \mid n \geq |\mathbf{a}|\} = (\dim M_0) \cdot \binom{n+d}{d}.$$

For all $n \gg 0$ the LHS is a polynomial in n of degree $\operatorname{Dim}(M)$, while the RHS is a polynomial in n of degree d. This yields $\operatorname{Dim}(M) \geq d$.

(4) Now put $N = \max\{n \in \mathbb{Z} \mid \mathfrak{g}(n) \neq 0\}$ and let $M_{d,j}$ denote the subspace of M spanned by all vectors $X^{\mathbf{a}}(m_i)$ with $|\mathbf{a}| \leq j$. We claim that

$$U_k M_0 \subseteq M_{d,(N+2)k} \quad (\forall k \in \mathbb{Z}_+).$$

For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_+^r$ set

$$\operatorname{wt} \mathbf{k} = \sum_{i=1}^{r} n_i k_i, \quad x^{\mathbf{k}} = x_1^{k_1} \cdots x_r^{k_r} \in U(\mathfrak{g}), \quad \Theta^{\mathbf{k}} = \Theta_1^{k_1} \cdots \Theta_r^{k_r} \in H_\chi.$$

Note that $\operatorname{wt} \mathbf{k} \geq 0$. Given $\mathbf{a} \in \mathbb{Z}_+^d$ and $\mathbf{b} \in \mathbb{Z}_+^r$ put $|(\mathbf{a}; \mathbf{b})|_e := \operatorname{wt} \mathbf{a} + \operatorname{wt} \mathbf{b} + 2|\mathbf{a}| + 2|\mathbf{b}|$. Using the formula in [31, p. 27] and the isomorphism $M \cong Q_\chi \otimes_{H_\chi} M_0$ it is easy to observe that

$$X^{\mathbf{a}} x^{\mathbf{b}}(m_i) = \Big(X^{\mathbf{a}} \Theta^{\mathbf{b}} + \sum_{|(\mathbf{i};\mathbf{j})|_e = |(\mathbf{a};\mathbf{b})|_e, |\mathbf{i}|+|\mathbf{j}| > |\mathbf{a}|+|\mathbf{b}|} \mu_{\mathbf{i},\mathbf{j}} X^{\mathbf{i}} \Theta^{\mathbf{j}} + \sum_{|(\mathbf{i};\mathbf{j})|_e < |(\mathbf{a};\mathbf{b})|_e} \mu_{\mathbf{i},\mathbf{j}} X^{\mathbf{i}} \Theta^{\mathbf{j}} \Big)(m_i)$$

for some $\mu_{\mathbf{i},\mathbf{j}} \in \Bbbk$. As the subspace $U_k M_0$ is spanned by all $X^{\mathbf{a}} x^{\mathbf{b}}(m_i)$ with $|\mathbf{a}| + |\mathbf{b}| \leq k$, it is contained in the span of all $X^{\mathbf{i}}(m_j)$ such that

$$\begin{aligned} \operatorname{wt} \mathbf{i} \leq |(\mathbf{i}; \mathbf{j})|_e &\leq \max_{|\mathbf{a}|+|\mathbf{b}| \leq k} |(\mathbf{a}; \mathbf{b})|_e \leq 2k + \max_{|\mathbf{a}|+|\mathbf{b}| \leq k} (\operatorname{wt} \mathbf{a} + \operatorname{wt} \mathbf{b}) \\ &\leq 2k + \max_{|\mathbf{a}|+|\mathbf{b}| \leq k} (N|\mathbf{a}| + N|\mathbf{b}|) \leq (N+2)k. \end{aligned}$$

The claim follows. Since all vectors $X^{\mathbf{i}}(m_j)$ are linearly independent, we derive

$$\begin{aligned} \dim U_k M_0 &\leq \dim M_{d,(N+2)k} = (\dim M_0) \cdot \operatorname{Card}\{\mathbf{i} \in \mathbb{Z}_+^d \mid (N+2)k \geq |\mathbf{i}|\} \\ &= (\dim M_0) \cdot \binom{Nk + 2k + d}{d}. \end{aligned}$$

Since the RHS is a polynomial in k of degree d, we get $\mathrm{Dim}(M) \leq d$. In conjunction with part (3) this shows that $\mathrm{Dim}(M) = d = (\dim \mathcal{O}_\chi)/2$.

(5) Since the H_χ-module $\mathrm{Wh}(M)$ is finite-dimensional, Skryabin's equivalence of categories described in 3.1 implies that the $\mathfrak{g}$-module M has a composition series $M = M_1 \supset \cdots \supset M_l \supset M_{l+1} = 0$ such that $M_i/M_{i+1} \in \mathcal{C}_\chi$ and $\dim \mathrm{Wh}(M_i/M_{i+1}) < \infty$ for all $i \leq l$. Set $J_i := \mathrm{Ann}_{U(\mathfrak{g})}(M_i/M_{i+1})$. It is immediate from the discussion in [14, (17.7)] that

$$\sqrt{\mathrm{gr}\, I} = \bigcap_{i=1}^{l} \sqrt{\mathrm{gr}\, J_i}. \tag{3.3.1}$$

On the other hand, it follows from (3.2.1) and parts (1) and (4) of this proof that for any $\mathfrak{g}$-module $N \in \mathcal{C}_\chi$ with $\dim \mathrm{Wh}(N) < \infty$ one has

$$\dim \mathcal{O}_\chi \leq \dim \mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})} N) \leq 2\,\mathrm{Dim}(N) = \dim \mathcal{O}_\chi.$$

In conjunction with the Irreducibility Theorem mentioned in 3.1 this shows that $\mathcal{VA}(M_i/M_{i+1})$ coincides with the Zariski closure $\overline{\mathcal{O}}_\chi$ for all $i \leq l$. But then (3.3.1) yields $\mathcal{VA}(I) = \overline{\mathcal{O}}_\chi$, completing the proof. □

3.4. Recall that a two-sided ideal I of $U(\mathfrak{g})$ is called *completely prime* (respectively, *primitive*) if $U(\mathfrak{g})/I$ is a domain (respectively, if I is the annihilator of a simple $\mathfrak{g}$-module). For $n \in \mathbb{Z}_+$ the set $Y_n := \{\psi \in \mathfrak{g}^* \mid \dim(\mathrm{Ad}^* G)\psi = n\}$ is locally closed in the Zariski topology of $\mathfrak{g}^*$. A (locally closed) subset of $\mathfrak{g}^*$ is called a *sheet* if it coincides with an irreducible component of one of the locally closed sets Y_n. It is well-known that each sheet is G-invariant and contains a unique nilpotent coadjoint orbit (such an orbit may lie in several sheets, however).

Conjecture 3.1. Let e be an arbitrary nilpotent element in $\mathfrak{g}$ and let $\chi = \chi_e$ be the corresponding linear function on $\mathfrak{g}$.

1. The algebra H_χ contains an ideal of codimension 1.
2. The ideals of codimension 1 in H_χ are finite in number if and only if $\mathcal{O}_\chi$ is a sheet in $\mathfrak{g}^*$.
3. For any ideal I of codimension 1 in H_χ the ideal $\widetilde{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_{H_\chi} H_\chi/I)$ of $U(\mathfrak{g})$ is completely prime.

Our last conjecture provides a hypothetical converse to Theorem 3.1(ii). It indicates that each category $\mathcal{C}_\chi$ is potentially very important for the theory of primitive ideals.

Conjecture 3.2. Let χ be as above and let I be a primitive ideal of $U(\mathfrak{g})$ whose associated variety equals $\overline{\mathcal{O}}_\chi$. Then there exists a simple $\mathfrak{g}$-module $M \in \mathcal{C}_\chi$ with $\dim \mathrm{Wh}(M) < \infty$ such that $I = \mathrm{Ann}_{U(\mathfrak{g})} M$.

4. Minimal nilpotent algebras: a quadratic relation

4.1. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Let $\{e_\alpha \mid \alpha \in \Phi\} \cup \{h_\alpha \mid \alpha \in \Phi\}$ be a Chevalley system in $\mathfrak{g}$ with each triple $(e_\alpha, h_\alpha, e_{-\alpha})$ being an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be a basis of simple roots in Φ with the elements in Π numbered as in [4], and let $\{\varpi_1, \dots, \varpi_\ell\}$ be the corresponding system of fundamental weights in $\mathfrak{h}^*$. Let Φ^+ and Φ^- be the positive and the negative system of Φ relative to Π, respectively, and let P denote the lattice of integral weights in $\mathfrak{h}^*$. As usual, given $\lambda, \mu \in P$ we write $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a sum of positive roots. Let $P^+ = \{\sum_i a_i \varpi_i \mid i \in \mathbb{Z}_+\}$, the set of dominant weights, and $\rho = \varpi_1 + \cdots + \varpi_\ell = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, the half-sum of positive roots. Let W be the Weyl group of the root system Φ; it is generated by reflections s_α with $\alpha \in \Phi$. The *dot action* of W on $\mathfrak{h}^*$ is defined by setting $w.\lambda = w(\lambda + \rho) - \rho$ for all $w \in W$ and $\lambda \in \mathfrak{h}^*$.

If $\mathfrak{g}$ is not of type A or C, there is a unique long root in Π linked with the lowest root $-\widetilde{\alpha} \in \Phi^-$ on the extended Dynkin diagram of $\mathfrak{g}$; we call it β. For $\mathfrak{g}$ of type A_n and C_n put $\beta = \alpha_n$. In this paper, we will be mostly concerned with the $\mathfrak{sl}_2$-triple $(e, h, f) = (e_\beta, h_\beta, e_{-\beta})$. Recall that the invariant form $(\cdot, \cdot)$ on $\mathfrak{g}$ has the property that $(e, f) = 1$. This entails $(h, h) = 2$. It is well-known that the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}$ is nondegenerate and induces a W-invariant scalar product on the $\mathbb{Q}$-span of P in $\mathfrak{h}^*$. More precisely, for all $\lambda, \mu \in \mathfrak{h}^*$ we have $(\lambda, \mu) = (t_\lambda, t_\mu)$ where $t_\lambda, t_\mu \in \mathfrak{h}$ are such that $\lambda = (t_\lambda, \cdot)$ and $\mu = (t_\mu, \cdot)$. Put $\langle \lambda, \alpha \rangle = 2(\lambda, \alpha)/(\alpha, \alpha)$ for all $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Phi$. Since $(\cdot, \cdot)$ is a multiple of the Killing form of $\mathfrak{g}$, there is a constant $c \in \Bbbk^\times$ such that $\beta(x) = c(h_\beta, x)$ for all $x \in \mathfrak{h}$. The equality $\beta(h_\beta) = 2 = (h_\beta, h_\beta)$ now shows that $c = 1$ and $t_\beta = h_\beta$. Hence $(\gamma, \gamma) = 2$ for all long roots $\gamma \in \Phi$.

From now on we assume that $\chi = (e, \cdot)$ where $e = e_\beta$. It is well-known that the adjoint action of $h = h_\beta$ on $\mathfrak{g}$ gives rise to a short $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$$

with $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$ for all $i \in \mathbb{Z}$ and with $\mathfrak{g}(1) \oplus \mathfrak{g}(2) \cong \mathfrak{g}(-1) \oplus \mathfrak{g}(-2)$ isomorphic to a Heisenberg Lie algebra. We also have $\mathfrak{g}(2) = \Bbbk e$, $\mathfrak{g}(-2) = \Bbbk f$, and $\mathfrak{g}(0) = \mathfrak{c}_\mathfrak{g}(h)$. The $\mathfrak{sl}_2$-theory implies that

$$\mathfrak{c}_\mathfrak{g}(e) = \mathfrak{g}(0)^\sharp \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$$

where $\mathfrak{g}(0)^\sharp = \{x \in \mathfrak{g}(0) \mid [x, e] = 0\}$. More importantly for our later deliberations, $\mathfrak{g}(0)^\sharp$ is the orthogonal complement to $\Bbbk h$ in $\mathfrak{g}(0)$ and hence coincides with the image of the Lie algebra endomorphism

$$\sharp\colon \mathfrak{g}(0) \to \mathfrak{g}(0), \qquad x \mapsto x - \tfrac{1}{2}(x, h)h.$$

In particular, $\mathfrak{g}(0)^\sharp$ is an ideal of codimension 1 in the Levi subalgebra $\mathfrak{g}(0)$. It is well-known that outside type A the centre of $\mathfrak{g}(0)$ coincides with $\Bbbk h$ and $\mathfrak{g}(1)$ is an irreducible $\operatorname{ad}\mathfrak{g}(0)^\sharp$-module. As a consequence, if $\mathfrak{g}$ is not of type A, then $\mathfrak{g}(0)^\sharp = [\mathfrak{g}(0), \mathfrak{g}(0)]$ is the only ideal of codimension 1 in $\mathfrak{g}(0)$ and $\mathfrak{c}_\mathfrak{g}(e) = [\mathfrak{c}_\mathfrak{g}(e), \mathfrak{c}_\mathfrak{g}(e)]$ is a perfect Lie algebra.

Note that in the present case $\mathfrak{g}(0)^\sharp = \operatorname{Lie} C(e)$. Put $\mathfrak{h}_e := \mathfrak{h} \cap \mathfrak{c}_\mathfrak{g}(e)$, a Cartan subalgebra in $\mathfrak{g}(0)^\sharp$. It can be assumed without loss of generality that $\mathfrak{h}_e = \operatorname{Lie} T_e$ where T_e is as in Section 2. We can choose $z_1, \dots, z_s, z_{s+1}, \dots, z_{2s}$ to be root vectors for $\mathfrak{h}$. Moreover, we can (and will) assume that the z_i's with $1 \le i \le s$ are root vectors in $\mathfrak{g}(-1)$ corresponding to *negative* roots $\gamma_i \in \Phi^-$. Then each z_i^* with $1 \le i \le s$ is a root vector in $\mathfrak{g}(-1)$ corresponding to $\gamma_i^* := -\beta - \gamma_i \in \Phi^+$.

4.2. Set $H := H_\chi$ and identify H with $\widetilde{\mu}(H)$. Given $a \in \mathfrak{z}_\chi(0)$ and $w \in \mathfrak{z}_\chi(1)$ we define the following elements of $\mathbf{A}_e$:

$$\psi_a := \frac{1}{2}\sum_{i=1}^{2s} [a, z_i^*]z_i, \qquad \varphi_w := \frac{1}{3}\sum_{i,j=1}^{2s} [wz_i^*z_j^*]z_j z_i + z_w.$$

Recall that $\sum_{i=1}^{2s}[w, z_i^*]z_i = -\sum_{i=1}^{2s}[w, z_i]z_i^*$. The computation used in the proof of Lemma 2.5 shows that

$$[z_k^*, \varphi_w] = \sum_{i=1}^{2s} [wz_i^*z_k^*]z_i. \tag{4.2.1}$$

Notice that in $U(\mathfrak{p}_e) \otimes \mathbf{A}_e^{\mathrm{op}}$ we have $\sum_{i=1}^{2s}[w, z_i^*] \otimes z_i = -\sum_{i=1}^{2s}[w, z_i] \otimes z_i^*$, and

$$[a \otimes f, b \otimes g] = [a, b] \otimes gf - ba \otimes [f, g] \quad (\forall a, b \in U(\mathfrak{p}_e),\ \forall f, g \in \mathbf{A}_e).$$

Keeping this in mind it is straightforward to see that for all $u, v \in \mathfrak{z}_\chi(1)$,

$$\begin{aligned}
[\Theta_u, \Theta_v] &= \Big[u \otimes 1 + \sum_{i=1}^{2s} [u, z_i^*] \otimes z_i + 1 \otimes \varphi_u,\ v \otimes 1 + \sum_{i=1}^{2s} [v, z_i^*] \otimes z_i + 1 \otimes \varphi_v\Big] \\
&= [u, v] \otimes 1 + \sum_{i=1}^{2s} [[u, v], z_i^*] \otimes z_i + \sum_{i,j=1}^{2s} [u, z_i] \otimes [z_i^*, \varphi_v] \\
&\quad - \sum_{i,j=1}^{2s} [v, z_i] \otimes [z_i^*, \varphi_u] + \sum_{i,j=1}^{2s} [[u, z_i^*], [v, z_j^*]] \otimes z_j z_i \\
&\quad + \sum_{i=1}^{2s} [v, z_i^*][u, z_i] \otimes 1 - 1 \otimes [\varphi_u, \varphi_v].
\end{aligned} \tag{4.2.2}$$

In view of (4.2.1) and (2.5.3) we have

$$\begin{aligned}
\sum_{i=1}^{2s} [x, z_i] \otimes [z_i^*, \varphi_y] &= \sum_{i,j=1}^{2s} [x, z_i] \otimes [yz_j^*z_i^*]z_j = \sum_{i,j,k} [x, z_i] \otimes \langle z_k^*, [yz_j^*z_i^*]\rangle z_k z_j \\
&= \sum_{i,j,k} [x, \langle z_i^*, [yz_j^*z_k^*]\rangle z_i] \otimes z_k z_j + \sum_{i,j,k} [x, z_i] \otimes z_k \langle z_j^*, [y, [z_i^*, z_k^*]]\rangle z_j \\
&= \sum_{i,j=1}^{2s} [x, [yz_j^*z_i^*]] \otimes z_i z_j + \sum_{i=1}^{2s} [x, z_i^*] \otimes z_i [y, f]
\end{aligned} \tag{4.2.3}$$

for all $x, y \in \mathfrak{z}_\chi(1)$. Combining this with (4.2.2) and taking into account that

$$\sum_{i,j=1}^{2s}[[u,z_i^*],[v,z_j^*]]\otimes z_iz_j=\sum_{i,j=1}^{2s}[[u,z_i^*],[v,z_j^*]]\otimes z_jz_i-\sum_{i=1}^{2s}[[u,z_i],[v,z_j^*]],$$

we now obtain

$$\begin{aligned}[\Theta_u,\Theta_v]&=[u,v]\otimes 1+\sum_{i=1}^{2s}[[u,v],z_i^*]\otimes z_i-\sum_{i,j=1}^{2s}[v,[uz_j^*z_i^*]]\otimes z_iz_j\\&\quad-\sum_{i=1}^{2s}[v,z_i^*]\otimes z_i[u,f]+\sum_{i,j=1}^{2s}[u,[vz_j^*z_i^*]]\otimes z_iz_j+\sum_{i=1}^{2s}[u,z_i^*]\otimes z_i[v,f]\\&\quad+\sum_{i,j=1}^{2s}[[u,z_i^*],[v,z_j^*]]\otimes z_jz_i+\sum_{i=1}^{2s}[v,z_i^*][u,z_i]\otimes 1-1\otimes[\varphi_u,\varphi_v]\\&=(f,[u,v])\Big(e\otimes 1+\sum_{i=1}^{2s}[e,z_i^*]\otimes z_i+\sum_{i,j=1}^{2s}[ez_i^*z_j^*]\otimes z_jz_i\Big)\\&\quad-\sum_{i,j=1}^{2s}[[u,z_i^*],[v,z_j^*]]\otimes z_jz_i+\sum_{i=1}^{2s}([u,z_i^*]\otimes z_i[v,f]-[v,z_i^*]\otimes z_i[u,f])\\&\quad+\sum_{i=1}^{2s}[u,z_i][v,z_i^*]\otimes 1-1\otimes[\varphi_u,\varphi_v].\end{aligned}\tag{4.2.4}$$

Next observe that $[a,z_i^*]^\sharp=[a,z_i]-\frac{1}{2}(h,[a,z_i^*])h$, and

$$\sum_{i=1}^{2s}(h,[a,z_i^*])z_i=\sum_{i=1}^{2s}(e,[f,[a,z_i^*]])z_i=\sum_{i=1}^{2s}\langle z_i^*,[a,f]\rangle z_i=[a,f]$$

for all $a\in\mathfrak{z}_\chi(1)$. Since $\sum_{i=1}^{2s}z_i^*z_i=s$ (as elements in $\mathbf{A}_e$), it follows from (4.2.3) that

$$\begin{aligned}&\sum_{i,j=1}^{2s}[x,z_i]^\sharp\otimes[[y,z_i^*]^\sharp,z_j^*]z_j=\sum_{i,j=1}^{2s}[x,z_i]^\sharp\otimes[yz_i^*z_j^*]z_j+\frac{s}{2}\sum_{i=1}^{2s}(h,[y,z_i^*])[x,z_i]^\sharp\otimes 1\\&=\sum_{i,j=1}^{2s}[x,z_i]\otimes[yz_i^*z_j^*]z_j-\frac{1}{2}\sum_{i,j=1}^{2s}(h,[x,z_i^*])h\otimes[yz_i^*z_j^*]z_j+\frac{s}{2}[x,[y,f]]^\sharp\otimes 1\\&=\frac{s}{2}[x,[y,f]]^\sharp\otimes 1+\sum_{i,j=1}^{2s}[x,z_i]\otimes[yz_j^*z_i^*]z_i+\sum_{i=1}^{2s}[x,z_i^*]\otimes[y,f]z_i\\&\quad+\frac{1}{2}\sum_{i=1}^{2s}h\otimes[[y,[x,f]],z_j^*]z_j=\frac{s}{2}[x,[y,f]]^\sharp\otimes 1+\frac{1}{2}\sum_{i=1}^{2s}h\otimes[[y,[x,f]],z_i^*]z_i\\&\quad+\sum_{i,j=1}^{2s}[x,[yz_j^*z_i^*]]\otimes z_iz_j+\sum_{i=1}^{2s}[x,z_i^*]\otimes z_i[y,f]+\sum_{i=1}^{2s}[x,z_i^*]\otimes[y,f]z_i\end{aligned}$$

for all $x, y \in \mathfrak{z}_\chi(1)$. But then

$$\sum_{i,j=1}^{2s} [x, z_i]^\sharp \otimes [[y, z_i^*]^\sharp, z_j^*]z_j + \sum_{i,j=1}^{2s} [y, z_i^*]^\sharp \otimes [[x, z_i]^\sharp, z_j^*]z_j$$
$$= \sum_{i,j=1}^{2s} [x, z_i]^\sharp \otimes [[y, z_i^*]^\sharp, z_j^*]z_j - \sum_{i,j=1}^{2s} [y, z_i]^\sharp \otimes [[x, z_i^*]^\sharp, z_j^*]z_j$$
$$= (f, [x, y])\Big(\frac{s}{2}h^\sharp \otimes 1 - \frac{1}{2}h \otimes \sum_{i=1}^{2s} [h, z_i]^* z_i + \sum_{i,j=1}^{2s} [ez_i^* z_j^*] \otimes z_j z_i\Big)$$
$$-2\sum_{i,j=1}^{2s} [[x, z_i^*], [y, z_j^*]] \otimes z_j z_i - \sum_{i=1}^{2s} [[x, z_i], [y, z_i^*]] \otimes 1$$
$$+2\sum_{i=1}^{2s}([x, z_i^*] \otimes z_i[y, f] - [y, z_i^*] \otimes z_i[x, f]) - [[x, y], f] \otimes 1. \qquad (4.2.5)$$

We used the fact that

$$\sum_{i=1}^{2s} [x, z_i^*] \otimes [[y, f], z_i] = \sum_{i=1}^{2s} [x, z_i] \otimes \langle z_i^*, [y, f]\rangle = -[x, [y, f]] \otimes 1.$$

To ease notation, set

$$A(u, v) := [\Theta_u, \Theta_v] - \frac{1}{2}\sum_{i=1}^{2s}(\Theta_{[u,z_i]^\sharp}\Theta_{[v,z_i^*]^\sharp} + \Theta_{[v,z_i^*]^\sharp}\Theta_{[u,z_i]^\sharp}).$$

Note that

$$\sum_{i=1}^{2s}(\Theta_{[u,z_i]^\sharp}\Theta_{[v,z_i^*]^\sharp} + \Theta_{[v,z_i^*]^\sharp}\Theta_{[u,z_i]^\sharp}) = \sum_{i=1}^{2s}([u, z_i]^\sharp[v, z_i^*]^\sharp + [v, z_i^*]^\sharp[u, z_i]^\sharp) \otimes 1$$
$$+ \sum_{i,j=1}^{2s} [u, z_i]^\sharp \otimes [[v, z_i^*]^\sharp, z_j^*]z_j z_i + \sum_{i,j=1}^{2s} [v, z_i^*]^\sharp \otimes [[u, z_i]^\sharp, z_j^*]z_j z_i + 1 \otimes \psi(u, v),$$

where $\psi(u, v) = \frac{1}{4}\sum_{i=1}^{2s}(\psi_{[v,z_i^*]^\sharp}\psi_{[u,z_i]^\sharp} + \psi_{[u,z_i]^\sharp}\psi_{[v,z_i^*]^\sharp}) \in \mathbf{A}_e$. Since

$$\sum_{i=1}^{2s}(x, z_i^*)z_i = -\sum_{i=1}^{2s}(x, z_i)z_i^* = [x, f] \quad (\forall x \in \mathfrak{z}_\chi(1)),$$

we have

$$\sum_{i=1}^{2s}([u, z_i]^\sharp[v, z_i^*]^\sharp + [v, z_i^*]^\sharp[u, z_i]^\sharp) = \sum_{i=1}^{2s}([u, z_i][v, z_i^*] + [v, z_i^*][u, z_i])$$
$$-\sum_{i=1}^{2s}((u, z_i)h[v, z_i^*] + (v, z_i^*)h[u, z_i] - \tfrac{1}{4}(u, z_i)(v, z_i^*)h^2 - \tfrac{1}{4}(v, z_i^*)(u, z_i)h^2)$$
$$= -\tfrac{1}{2}(e, [u, v])h^2 + 2\sum_{i=1}^{2s}[u, z_i][v, z_i^*] - \sum_{i=1}^{2s}[[u, z_i], [v, z_i^*]].$$

Since $h^\sharp = 0$ and $h \otimes \sum_{i=1}^{2s}[h, z_i^*]z_i = -sh \otimes 1$, we now combine the above with (4.2.4) and (4.2.5) to deduce that

$$A(u,v) = \tfrac{1}{2}(e,[u,v])\Big(\frac{4e+h^2-(s+2)h}{2} \otimes 1 + 2\sum_{i=1}^{2s}[e, z_i^*] \otimes z_i + \sum_{i,j=1}^{2s}[ez_i^*z_j^*] \otimes z_j z_i\Big) + 1 \otimes ([\varphi_v, \varphi_u] - \tfrac{1}{2}\psi(u,v)). \tag{4.2.6}$$

4.3. Recall that $\mathfrak{z}_\chi(0) = \mathfrak{g}(0)^\sharp$ is an ideal of codimension 1 in the Levi subalgebra $\mathfrak{g}(0) = \mathfrak{c}_\mathfrak{g}(h)$ of $\mathfrak{g}$. Let $\{a_i\}$ and $\{b_i\}$ be dual bases of $\mathfrak{z}_\chi(0)$ with respect to the restriction of $(\cdot,\cdot)$ to $\mathfrak{z}_\chi(0)$, and let $C_0 = \sum_i a_i b_i$ be the corresponding Casimir element of $U(\mathfrak{z}_\chi(0))$. Set $\Theta_{\mathrm{Cas}} := \sum_i \Theta_{a_i}\Theta_{b_i}$, an element of H. Although Θ_{Cas} is not central in H, Lemma 2.4 shows that it commutes with all operators Θ_x for $x \in \mathfrak{z}_\chi(0)$. Since the skew-symmetric form $\langle\cdot,\cdot\rangle$ is invariant under $\mathfrak{z}_\chi(0)$ and $\Bbbk h$ is orthogonal to $\mathfrak{g}(0)^\sharp$ with respect to $(\cdot,\cdot)$, we have

$$\begin{aligned}\sum_{i,j=1}^{2s}[ez_i^*z_j^*]^\sharp \otimes z_j z_i &= \sum_{i,j,k}(b_k,[ez_i^*z_j^*]^\sharp])a_k \otimes z_j z_i = \sum_{i,j,k}(b_k,[ez_i^*z_j^*]])a_k \otimes z_j z_i\\ &= \sum_{i,j,k}([b_k z_j^* z_i^*], e)a_k \otimes z_j z_i = -\sum_{i,j,k}\langle z_i^*, [b_k z_j^*]\rangle a_k \otimes z_j z_i\\ &= -\sum_{i,j,k}\langle z_j^*, [b_k z_i^*]\rangle a_k \otimes z_j z_i = -\sum_{k,i} a_k \otimes [b_k, z_i^*]z_i.\end{aligned}$$

Interchanging the rôles of $\{a_i\}$ and $\{b_i\}$ we now obtain

$$\sum_{i,j=1}^{2s}[ez_i^*z_j^*]^\sharp \otimes z_j z_i = -\sum_{k,i} b_k \otimes [a_k, z_i^*]z_i.$$

Next observe that

$$\begin{aligned}\sum_{i,j=1}^{2s}[ez_i^*z_j^*]^\sharp \otimes z_j z_i &= \sum_{i,j=1}^{2s}[ez_i^*z_j^*] \otimes z_j z_i - \frac{1}{2}\sum_{i,j=1}^{2s}(h,[ez_i^*z_j^*])h \otimes z_j z_i\\ &= \sum_{i,j=1}^{2s}[ez_i^*z_j^*] \otimes z_j z_i + \frac{1}{2}\sum_{i,j=1}^{2s}([[h,z_j],z_i^*],e)h \otimes z_j^* z_i\\ &= \sum_{i,j=1}^{2s}[ez_i^*z_j^*] \otimes z_j z_i + \frac{sh}{2} \otimes 1.\end{aligned}$$

It follows that

$$\sum_{i,j=1}^{2s}[ez_i^*z_j^*] \otimes z_j z_i = -\frac{sh}{2} \otimes 1 - \frac{1}{2}\Big(\sum_{i,j} a_i \otimes [b_i, z_j^*]z_j + \sum_{i,j} b_i \otimes [a_i, z_j^*]z_j\Big).$$

As a result,

$$\Theta_{\mathrm{Cas}} = \sum_i a_i b_i \otimes 1 + \frac{1}{2}\sum_{i,j}(a_i \otimes [b_i, z_j^*]z_j + b_i \otimes [a_i, z_j^*]z_j) + 1 \otimes c_0'$$
$$= \Big(-\frac{sh}{2} + \sum_i a_i b_i\Big) \otimes 1 - \sum_{i,j=1}^{2s} [ez_i^* z_j^*] \otimes z_j z_i + 1 \otimes c_0', \tag{4.3.1}$$

where $c_0' = \frac{1}{4}\sum_{i,j,k}[b_i, z_j^*]z_j[a_i, z_k^*]z_k \in \mathbf{A}_e$.

4.4. Let C denote the Casimir element of $U(\mathfrak{g})$ corresponding to the invariant form $(\cdot,\cdot)$. Clearly, C induces a $\mathfrak{g}$-endomorphism of $\widehat{Q}_\chi$, and hence can be viewed as a central element of H. To determine $\widetilde{\mu}(C)$ we first observe that the (ordered) sets

$$\{e, h, f\} \cup \{a_i\} \cup \{[e, z_i^*] \mid 1 \le i \le 2s\} \cup \{z_i \mid 1 \le i \le 2s\}$$

and

$$\{f, h/2, e\} \cup \{b_i\} \cup \{z_i \mid 1 \le i \le 2s\} \cup \{[e, z_i^*] \mid 1 \le i \le 2s\}$$

are dual bases of $\mathfrak{g}$ with respect to $(\cdot,\cdot)$. Since

$$\sum_{i=1}^{2s}[z_i, [e, z_i^*]] = \sum_{i=1}^{2s}([[z_i, e], z_i^*] + [e, [z_i, z_i^*]]) = -\sum_{i=1}^{2s}([[z_i^*, e], z_i] + [e, [z_i^*, z_i]]),$$

we have $\sum_{i=1}^{2s}[z_i, [e, z_i^*]] = -sh$. As $\widehat{\rho}_\chi(f)1_\chi = (e, f)1_\chi = 1_\chi$, it follows that

$$C(1_\chi) = \Big(2e + \frac{h^2}{2} - h + \sum_i a_i b_i + 2\sum_{i=1}^{2s}[e, z_i^*]z_i + \sum_{i=1}^{2s}[z_i, [e, z_i^*]]\Big) \otimes 1_\chi$$
$$= \Big(2e + \frac{h^2}{2} - (s+1)h + \sum_i a_i b_i + 2\sum_{i=1}^{2s}[e, z_i^*]z_i\Big) \otimes 1_\chi.$$

As a consequence,

$$\widetilde{\mu}(C) = \Big(2e + \frac{h^2}{2} - (s+1)h + \sum_i a_i b_i\Big) \otimes 1 + 2\sum_{i=1}^{2s}[e, z_i^*] \otimes z_i. \tag{4.4.1}$$

In view of (4.3.1) and the convention of 4.1 this yields

$$C - \Theta_{\mathrm{Cas}} = \Big(2e + \frac{h^2 - (s+2)h}{2}\Big) \otimes 1$$
$$+ 2\sum_{i=1}^{2s}[e, z_i^*] \otimes z_i + \sum_{i,j=1}^{2s}[ez_i^* z_j^*] \otimes z_j z_i - 1 \otimes c_0'. \tag{4.4.2}$$

Proposition 4.1. *Let $u, v \in \mathfrak{z}_\chi(1)$. Then the following relation holds in H:*

$$[\Theta_u, \Theta_v] = \frac{1}{2}(f,[u,v])(C - \Theta_{\mathrm{Cas}} - c_0) + \frac{1}{2}\sum_{i=1}^{2s}(\Theta_{[u,z_i]^\sharp}\Theta_{[v,z_i^*]^\sharp} + \Theta_{[v,z_i^*]^\sharp}\Theta_{[u,z_i]^\sharp}),$$

where $c_0 \in \Bbbk$ is a constant depending on $\mathfrak{g}$.

Proof. Set $B(u,v) := A(u,v) - \frac{1}{2}(f,[u,v])(C - \Theta_{\mathrm{Cas}})$, an element in H. From (4.2.6) and (4.4.2) we deduce that $\widetilde{\mu}(B(u,v)) = 1 \otimes b(u,v)$ for some $b(u,v) \in \mathbf{A}_e$. In conjunction with [31, Lemma 4.5] this shows that $b(u,v) \in \Bbbk$ for all $u, v \in \mathfrak{z}_\chi(1)$. In other words, $b\colon \mathfrak{z}_\chi(1) \times \mathfrak{z}_\chi(1) \to \Bbbk$, $(u,v) \mapsto b(u,v)$, is a bilinear form on $\mathfrak{z}_\chi(1)$. In view of Lemma 2.5, the $\mathfrak{z}_\chi(0)$-invariance of $\langle\cdot,\cdot\rangle$, and the Jacobi identity, this form is invariant under $\mathfrak{z}_\chi(0)$. On the other hand, it is well-known (and easily seen) that if $\mathfrak{g}$ is not of type A, then $\mathfrak{z}_\chi(1)$ is an irreducible $\mathfrak{z}_\chi(0)$-module, and if $\mathfrak{g}$ is of type A and $\mathfrak{z}_\chi(1) \neq 0$, then $\mathfrak{z}_\chi(1) \cong M \oplus M^*$ where M is an irreducible $\mathfrak{z}_\chi(0)$-module such that $M \ncong M^*$. Therefore, in all cases $b = c_0(f,[\cdot,\cdot])$ for some $c_0 \in \Bbbk$. This completes the proof. □

Let $\{x_1, \dots, x_n\}$ and $\{u_1, \dots, u_{2s}\}$ be bases of $\mathfrak{z}_\chi(0)$ and $\mathfrak{z}_\chi(1)$, respectively, and let H^+ denote the $\Bbbk$-span of

$$\left\{\Theta_{x_1}^{i_1}\cdots\Theta_{x_n}^{i_n}\cdot\Theta_{u_1}^{j_1}\cdots\Theta_{u_{2s}}^{j_{2s}}\cdot(C-c_0)^l \;\middle|\; \sum i_k + \sum j_k + l \geq 1\right\},$$

a subspace of codimension 1 in H; see [31, Theorem 4.6(ii)].

Corollary 4.1. *The subspace H^+ is a two-sided ideal of the minimal nilpotent algebra H. If $\mathfrak{g}$ is not of type* A*, then H^+ is the only ideal of codimension* 1 *in H.*

Proof. We need to show that $h \cdot h' \in H^+$ for all $h, h' \in H^+$. Since $C - c_0 \in Z(H)$, it suffices to show that $\Theta_x \cdot H^+ \subset H^+$ for all $x \in \mathfrak{z}_\chi(0) \cup \mathfrak{z}_\chi(1)$. Using Lemma 2.4 it is easy to observe that the span of all $\Theta_{x_1}^{i_1}\cdots\Theta_{x_n}^{i_n}$ with $\sum_{k=1}^n i_k \geq 1$ is stable under the left multiplications by Θ_x with $x \in \mathfrak{z}_\chi(0)$. Thus we may assume that $x \in \mathfrak{z}_\chi(1)$. Lemma 2.5 (in conjunction with [31, Theorem 4.6(ii)]) now provides a further reduction: it suffices to show that $\Theta_x \cdot (\Theta_{u_{j_1}}\cdots\Theta_{u_{j_N}}) \in H^+$ for all $x \in \mathfrak{z}_\chi(1)$ and all $j_1, \dots, j_N \in \{1, \dots, 2s\}$. This follows from Proposition 4.1 and Lemma 2.5 by induction on N.

Suppose $\mathfrak{g}$ is not of type A. Then only one node of the extended Dynkin diagram of $\mathfrak{g}$ is linked with the node corresponding to $-\widetilde{\alpha}$. From this it is immediate that the derived subalgebra of $\mathfrak{c}_\mathfrak{g}(h_{\widetilde{\alpha}})$ is semisimple and has codimension 1 in $\mathfrak{c}_\mathfrak{g}(h_{\widetilde{\alpha}})$. Since the roots β and $-\widetilde{\alpha}$ lie in the same W-orbit, the subalgebras $\mathfrak{c}_\mathfrak{g}(h_{\widetilde{\alpha}})$ and $\mathfrak{g}(0) = \mathfrak{c}_\mathfrak{g}(h)$ are conjugate under $\mathrm{Ad}\, G$. It follows that $\mathfrak{z}_\chi(0) = [\mathfrak{g}(0), \mathfrak{g}(0)]$ is semisimple.

Let I be any ideal of codimension 1 in H. Then $xy - yx \in I$ for all $x, y \in H$. Since $\mathfrak{z}_\chi(0)$ is semisimple, we have $\mathfrak{z}_\chi(0) = [\mathfrak{z}_\chi(0), \mathfrak{z}_\chi(0)]$. In view of Lemma 2.4 this implies that $\Theta_x \in I$ for all $x \in \mathfrak{z}_\chi(0)$. We have already mentioned that in the present case $\mathfrak{z}_\chi(1)$ is an irreducible $\mathfrak{z}_\chi(0)$-module. So $\mathfrak{z}_\chi(1) = [\mathfrak{z}_\chi(0), \mathfrak{z}_\chi(1)]$, yielding $\Theta_u \in I$ for all $u \in \mathfrak{z}_\chi(1)$; see Lemma 2.5. Since I is a subalgebra of H containing $\mathfrak{z}_\chi(0) \cup \mathfrak{z}_\chi(1)$, Proposition 4.1 entails $C - c_0 \in I$. As a consequence, $H^+ \subseteq I$. Since H_χ^+ has codimension 1 in H, we conclude that $I = H^+$, as desired. □

5. Primitive ideals and Goldie rank

5.1. We retain the assumptions of the previous section and denote by $\mathcal{C}$ the category $\mathcal{C}_\chi$ for $\chi = (e_\beta, \cdot)$. According to (3.1), given $M \in \mathcal{C}$ one has $M \cong Q_\chi \otimes_H M_0$ where $M_0 = \mathrm{Wh}(M)$. Let $\{m_i \mid i \in I\}$ be a basis of M_0. It is immediate from [31, p. 52] that the vectors

$$\{h^l z_1^{i_1} \cdots z_s^{i_s} \otimes m_j \mid j \in I; l, i_i, \ldots, i_s \in \mathbb{Z}_+\}$$

form a basis of $Q_\chi \otimes_H M_0$ over $\Bbbk$. We can thus identify M and $\Bbbk[h, z_1, \ldots, z_s] \otimes M_0$ as vector spaces over $\Bbbk$. Recall that $[z_i, z_j] = 0$ and $[h, z_i] = -z_i$ for all $i, j \leq s$.

Let Δ denote the automorphism of the polynomial algebra $\Bbbk[h]$ such that $\Delta(h) = h + 1$. Clearly, $\Delta^{-1}(h) = h - 1$. Let $\langle\Delta\rangle$ stand for the cyclic subgroup of $\mathrm{Aut}(\Bbbk[h])$ generated by Δ. The skew group algebra $\Bbbk[h] * \langle\Delta\rangle$ has the set $\{h^i \Delta^j \mid i \in \mathbb{Z}_+, j \in \mathbb{Z}\}$ as a $\Bbbk$-basis and the following relations hold:

$$\Delta^i \cdot h^k = (h + i)^k \cdot \Delta^i \quad (i \in \mathbb{Z},\ k \in \mathbb{Z}_+).$$

Let $\mathcal{A}_e := (\Bbbk[h] * \langle\Delta\rangle) \otimes \mathbf{A}_e$, an associative algebra over $\Bbbk$, and identify the Weyl algebra $\mathbf{A}_e$ and the skew group algebra $\Bbbk[h] * \langle\Delta\rangle$ with the subalgebras $\Bbbk \otimes \mathbf{A}_e$ and $(\Bbbk[h] * \langle\Delta\rangle) \otimes \Bbbk$ of $\mathcal{A}_e$, respectively. Define an involution $\tau \in \mathrm{Aut}(\mathcal{A}_e)$ by setting

$$\tau(z_i) = -z_i, \quad \tau(\partial_i) = -\partial_i, \quad \tau(h) = h, \quad \tau(\Delta^k) = (-1)^k \Delta^k \quad (1 \leq i \leq s,\ k \in \mathbb{Z}).$$

The polynomial algebra $\Bbbk[h, z_1, \ldots, z_s] = \Bbbk[h][z_1, \ldots, z_s]$ has a natural $\mathcal{A}_e$-module structure such that $h.f(h, z_1, \ldots, z_s) = hf(h, z_1, \ldots, z_s)$ and $\Delta^k.f(h, z_1, \ldots, z_s) = f(h+k, z_1, \ldots, z_s)$. As this $\mathcal{A}_e$-module is faithful, we will identify $\mathcal{A}_e$ with a subalgebra of $\mathrm{End}(\Bbbk[h, z_1, \ldots, z_s])$. Since $\Delta^{2k} \in \mathcal{A}_e^\tau$ for all $k \in \mathbb{Z}$ and $\Delta \otimes z_i, \Delta \otimes \partial_i \in \mathcal{A}_e^\tau$ for all $i \leq s$, it is easy to see that $\Bbbk[h, z_1, \ldots, z_s]$ remains irreducible when restricted to the fixed point algebra $\mathcal{A}_e^\tau$.

Let I be any two-sided ideal of H. Since I is σ-stable by Corollary 2.1, the ideal $\mathcal{A}_e \otimes I$ of the algebra $\mathcal{A}_e \otimes H$ is invariant under the involution $\tau \otimes \sigma$ of $\mathcal{A}_e \otimes H$. Hence $\tau \otimes \sigma$ indices an automorphism of order two on the algebra $\mathcal{A}_e \otimes (H/I)$. We mention for further references that $(\mathcal{A}_e \otimes (H/I))^{\tau\otimes\sigma}$ contains the images in $\mathcal{A}_e \otimes (H/I)$ of $\mathcal{A}_e^\tau \otimes \Bbbk$, $1 \otimes \Theta_a$, and $\Delta^{-1} \otimes \Theta_u$ for all $a \in \mathfrak{z}_\chi(0)$ and $u \in \mathfrak{z}_\chi(1)$.

5.2. Recall that in our present setting the element f is a root vector for $\mathfrak{h}$ corresponding to $-\beta$. Since $\mathrm{ad}\, f$ is locally nilpotent on $U(\mathfrak{g})$, the set $S_f := \{f^i \mid i \in \mathbb{Z}_+\}$ is an Ore set in $U(\mathfrak{g})$; see [13, (11.2)]. We denote by $U(\mathfrak{g})_f$ the localization $S_f^{-1}U(\mathfrak{g})$. Since f commutes with the z_i's and $fh^m = (h+2)^m f$ for all $m \in \mathbb{Z}_+$, it follows that f acts on $M = \Bbbk[h, z_1, \ldots, z_s] \otimes M_0$ as $\Delta^2 \otimes \mathrm{id}_{M_0}$ (one should keep in mind that $\chi(f) = 1$). In particular, f acts invertibly on M. From this it follows that the localization $S_f^{-1}M$ can be identified with M; see [13, (11.4)]. As a result, the action of $U(\mathfrak{g})$ on M extends uniquely to a representation of $U(\mathfrak{g})_f$ in $\mathrm{End}\, M$. We note for completeness that the enveloping algebra $U(\mathfrak{g})$ is canonically identified with a subalgebra of $U(\mathfrak{g})_f$.

Theorem 5.1. *Let $M \in \mathcal{C}$ and identify M with $\Bbbk[h, z_1, \dots, z_s] \otimes M_0$ where $M_0 = \mathrm{Wh}(M)$. Let $\tilde{\rho}\colon U(\mathfrak{g})_f \to \mathrm{End}\, M$ and $\rho\colon H \to \mathrm{End}\, M_0$ be the representations of $U(\mathfrak{g})_f$ and H induced by the action of $\mathfrak{g}$ on M. Then the following hold:*

(i) $(\Delta \otimes \mathrm{id}_{M_0}) \circ \tilde{\rho}(U(\mathfrak{g})_f) \circ (\Delta \otimes \mathrm{id}_{M_0})^{-1} = \tilde{\rho}(U(\mathfrak{g})_f)$;
(ii) $\tilde{\rho}(f) = \Delta^2 \otimes \mathrm{id}_{M_0}$;
(iii) $\mathcal{A}_e \otimes \rho(H) = \tilde{\rho}(U(\mathfrak{g})_f) \oplus \tilde{\rho}(U(\mathfrak{g})_f)(\Delta \otimes \mathrm{id}_{M_0})$;
(iv) $\tilde{\rho}(U(\mathfrak{g})_f) = (\mathcal{A}_e \otimes \rho(H))^{\tau\otimes\sigma}$.

Proof. Put $\mathrm{id} = \mathrm{id}_{M_0}$. We have already mentioned that $\tilde{\rho}(f) = \Delta^2 \otimes \mathrm{id}$, showing (ii). Since $z_i h^m = (h+1)^m z_i$ for all $m \in \mathbb{Z}_+$ and $i \le 2s$, it follows that

$$\tilde{\rho}(h) = h \otimes \mathrm{id}, \quad \tilde{\rho}(z_i) = (\Delta \circ z_i) \otimes \mathrm{id}, \quad \tilde{\rho}(z_{i+s}) = (\Delta \circ \partial_i) \otimes \mathrm{id} \quad (1 \le i \le s).$$

All these are in $\mathcal{A}_e^{\tau} \otimes \mathrm{id} \subset (\mathcal{A}_e \otimes \rho(H))^{\tau\otimes\sigma}$. Now let $a \in \mathfrak{z}_\chi(0)$ and write $[a, z_i] = \sum_{i=1}^{2s} \mu_{ij} z_j$ for $1 \le i \le 2s$, where $\mu_{ij} \in \Bbbk$. Note that for $1 \le i, j \le s$ we have $[a z_i z_j] = \mu_{i,j+s} f$, and

$$\begin{aligned} a z_1^{k_1} \cdots z_s^{k_s} = {} & \sum_{i=1}^{s} k_i z_1^{k_1} \cdots z_i^{k_i - 1} \cdots z_s^{k_s} [a, z_i] + \frac{k_i(k_i-1)}{2} \sum_{i=1}^{s} z_1^{k_1} \cdots z_i^{k_i-2} \cdots z_s^{k_s} [a z_i^2] \\ & + k_i k_j \sum_{1 \le i < j \le s} z_1^{k_1} \cdots z_i^{k_i - 1} \cdots z_j^{k_j - 1} \cdots z_s^{k_s} [a z_i z_j] + z_1^{k_1} \cdots z_s^{k_s} a \end{aligned}$$

for all $k_i \in \mathbb{Z}_+$. Since $\langle \cdot, \cdot \rangle$ is $\mathfrak{z}_\chi(0)$-invariant, it must be that $\mu_{i+s,j+s} = -\mu_{ji}$, $\mu_{i,j+s} = \mu_{j,i+s}$, and $\mu_{i+s,j} = \mu_{j+s,i}$ where $1 \le i, j \le s$. In view of Lemma 2.3 and the fact that $z_{i+s} \in \mathfrak{m}_\chi$ for all $i \le s$, we must have $\tilde{\rho}(a)(1 \otimes m) = 1 \otimes \rho(\Theta_a)(m) - \frac{1}{2} \sum_{i=1}^{s} [a, z_{i+s}] z_i \otimes 1$ for all $m \in M_0$. In conjunction with our earlier remarks (and the fact that $[a, h] = 0$) this yields

$$\begin{aligned} \tilde{\rho}(a) = {} & 1 \otimes \Big(\rho(\Theta_a) + \sum_{i=1}^{s} \mu_{ii} \Big) + \Big(\sum_{i,j=1}^{s} \mu_{ij} z_j \partial_i \Big) \otimes \mathrm{id} \\ & + \Big(\sum_{i=1}^{s} \frac{\mu_{i,i+s}}{2} \partial_i^2 + \sum_{1 \le i < j \le s} \mu_{i,j+s} \partial_i \partial_j \Big) \otimes \mathrm{id} \\ & - \Big(\sum_{i=1}^{s} \frac{\mu_{i,i+s}}{2} z_i^2 + \sum_{1 \le i < j \le s} \mu_{i,j+s} z_i z_j \Big) \otimes \mathrm{id}. \end{aligned} \tag{5.2.1}$$

This shows that $\tilde{\rho}(a)$ commutes with $\Delta \otimes \mathrm{id}$ and lies in $(\mathcal{A}_e \otimes \rho(H))^{\tau\otimes\sigma}$. Since all operators $(\Delta^2 \circ z_i z_j) \otimes \mathrm{id}$, $(\Delta^2 \circ \partial_i \partial_j) \otimes \mathrm{id}$, and $(\Delta^2 \circ z_j \partial_i) \otimes \mathrm{id}$ are in $\tilde{\rho}(U(\mathfrak{n}_\chi))$ by our remarks earlier in the proof, and since $\tilde{\rho}(f^{-1}) = \Delta^{-2} \otimes \mathrm{id}$, it also follows that $1 \otimes \rho(\Theta_a)$ is in $\tilde{\rho}(U(\mathfrak{g})_f)$ for all $a \in \mathfrak{z}_\chi(0)$.

Now let $u \in \mathfrak{z}_\chi(1)$. First note that $[u, z_i] = [u, z_i]^\sharp + \frac{1}{2}(u, z_i)h$ and $uh^m = (h-1)^m u$ for all $i \le s$ and $m \in \mathbb{Z}_+$. Next observe that

$$\begin{aligned} uz_1^{k_1}\cdots z_s^{k_s} &= \sum_{i=1}^{s} z_1^{k_1}\cdots z_i^{k_i-1}\cdots z_s^{k_s}[u, z_i] + \sum_{i=1}^{s} \frac{k_i(k_i-1)}{2} z_1^{k_1}\cdots z_i^{k_i-2}\cdots z_s^{k_s}[uz_i^2] \\ &\quad + \sum_{1\le i<j\le s} k_i k_j z_1^{k_1}\cdots z_i^{k_i-1}\cdots z_j^{k_j-1}\cdots z_s^{k_s}[uz_iz_j] \\ &\quad + \sum_{1\le i<j<l\le s} k_i k_j k_l z_1^{k_1}\cdots z_i^{k_i-1}\cdots z_j^{k_j-1}\cdots z_l^{k_l-1}\cdots z_s^{k_s}[uz_iz_jz_l] \\ &\quad + \sum_{1\le i<j\le s} \frac{k_i(k_i-1)}{2} k_j z_1^{k_1}\cdots z_i^{k_i-2}\cdots z_j^{k_j-1}\cdots z_s^{k_s}[uz_i^2z_j] \\ &\quad + \sum_{1\le i<j\le s} k_i \frac{k_j(k_j-1)}{2} z_1^{k_1}\cdots z_i^{k_i-1}\cdots z_j^{k_j-2}\cdots z_s^{k_s}[uz_iz_j^2] \\ &\quad + \sum_{i=1}^{s} \frac{k_i(k_i-1)(k_i-2)}{6} z_1^{k_1}\cdots z_i^{k_i-3}\cdots z_s^{k_s}[uz_i^3] + z_1^{k_1}\cdots z_s^{k_s}u, \end{aligned}$$

and

$$\begin{aligned} z_1^{k_1}\cdots z_s^{k_s}h &= hz_1^{k_1}\cdots z_s^{k_s} - [h, z_1^{k_1}\cdots z_s^{k_s}] = hz_1^{k_1}\cdots z_s^{k_s} + (k_1+\cdots+k_s)z_1^{k_1}\cdots z_s^{k_s} \\ &= \Big(h + \sum_{i=1}^{s} z_i\partial_i\Big)(z_1^{k_1}\cdots z_s^{k_s}). \end{aligned}$$

This shows that for all $m \in M_0$ we have

$$\begin{aligned} &\sum_{i=1}^{s} z_1^{k_1}\cdots z_i^{k_i-1}\cdots z_s^{k_s}\tilde\rho([u, z_i])(1\otimes m) \\ &= \bigg(\Big(h + \sum_{i=1}^{s} z_i\partial_i\Big)\circ\Big(\sum_{i=1}^{s}\frac{(u, z_i)}{2}\partial_i\Big) + \sum_{i=1}^{s}\partial_i\otimes\rho(\Theta_{[u,z_i]^\sharp})\bigg)(z_1^{k_1}\cdots z_s^{k_s}\otimes m) \\ &\quad -\frac{1}{2}\sum_{i,j=1}^{s} z_1^{k_1}\cdots z_i^{k_i-1}\cdots z_s^{k_s}\tilde\rho([[u, z_i]^\sharp, z_{j+s}]z_j)(1\otimes m). \end{aligned}$$

In view of Lemma 2.5 we now get

$$\begin{aligned} z_1^{k_1}\cdots z_s^{k_s}\tilde\rho(u)(1\otimes m) &= (\mathrm{id}\otimes\rho(\Theta_u))(z_1^{k_1}\cdots z_s^{k_s}\otimes m) \\ &\quad - z_1^{k_1}\cdots z_s^{k_s}\bigg(\sum_{i=1}^{s} z_i\tilde\rho([u, z_{i+s}])(1\otimes m) + \frac{1}{3}\sum_{i,j=1}^{s}\tilde\rho([uz_{i+s}z_{j+s}])(z_jz_i\otimes m) \\ &\qquad + \frac{2}{3}\sum_{i=1}^{s}\tilde\rho([uz_{i+s}z_i])(1\otimes m) + \tilde\rho(z_u)(1\otimes m)\bigg). \end{aligned}$$

Together with the above remarks this says that $\tilde{\rho}(u)$ is a linear combination of the following operators:

$$\Big(h+\sum_{i=1}^{s} z_i\partial_i\Big)\circ\Big(\sum_{i=1}^{s}\frac{(u,z_i)}{2}\partial_i\Big)\circ\Delta^{-1}\otimes\mathrm{id},\ \sum_{i=1}^{s}(\partial_i\circ\Delta^{-1})\otimes\rho(\Theta_{[u,z_i]^\sharp}),$$

$$\Big(h+\sum_{i=1}^{s} z_i\partial_i\Big)\circ\Big(\sum_{i=1}^{s}\frac{(u,z_{i+s})}{2}z_i\Big)\circ\Delta^{-1}\otimes\mathrm{id},\ \sum_{i=1}^{s}(z_i\circ\Delta^{-1})\otimes\rho(\Theta_{[u,z_{i+s}]^\sharp}),$$

$$(\partial_i\partial_j\partial_k\circ\Delta^{-1})\otimes\mathrm{id},\ (z_iz_jz_k\circ\Delta^{-1})\otimes\mathrm{id},\ (z_iz_j\partial_k\circ\Delta^{-1})\otimes\mathrm{id},$$

$$(z_k\partial_i\partial_j\circ\Delta^{-1})\otimes\mathrm{id},\ (\partial_i\circ\Delta^{-1})\otimes\mathrm{id},\ (z_i\circ\Delta^{-1})\otimes\mathrm{id},\ \Delta^{-1}\otimes\rho(\Theta_u),$$

where $i,\ j,\ k\le s$. But then $\tilde{\rho}(u)\in(\mathcal{A}_e\otimes\rho(H))^{\sigma\otimes\tau}$ and $(\Delta\otimes\mathrm{id})\tilde{\rho}(u)(\Delta\otimes\mathrm{id})^{-1}-\tilde{\rho}(u)\in\mathcal{A}_e^{\tau}\otimes\mathrm{id}$. As $\mathcal{A}_e^{\tau}\otimes\mathrm{id}\subset\tilde{\rho}(U(\mathfrak{g})_f)$ by our earlier remarks, this yields

$$(\Delta\otimes\mathrm{id})\tilde{\rho}(x)(\Delta\otimes\mathrm{id})^{-1}\in\tilde{\rho}(U(\mathfrak{g})_f)\quad(\forall x\in\textstyle\bigcup_{i\le1}\mathfrak{g}(i)).$$

Since $\mathfrak{g}(2)=[\mathfrak{g}(1),\mathfrak{g}(1)]$, we obtain (i) and the inclusion $\tilde{\rho}(U(\mathfrak{g})_f)\subseteq(\mathcal{A}_e\otimes\rho(H))^{\sigma\otimes\tau}$. Notice that $\Delta^{-1}\otimes\rho(\Theta_u)\in\mathcal{A}_e^{\tau}\otimes\mathrm{id}+\tilde{\rho}(U(\mathfrak{g})_f)$ for all $u\in\mathfrak{z}_\chi(1)$ and $1\otimes\rho(\Theta_a)\in\mathcal{A}_e^{\tau}\otimes\mathrm{id}+\tilde{\rho}(U(\mathfrak{g})_f)$ for all $a\in\mathfrak{z}_\chi(0)$; see (5.2.1). Since the algebra H is generated by the elements Θ_x with $x\in\mathfrak{z}_\chi(0)\cup\mathfrak{z}_\chi(1)$, by Proposition 4.1, we get

$$1\otimes\rho(H)\subset\tilde{\rho}(U(\mathfrak{g})_f)+\tilde{\rho}(U(\mathfrak{g})_f)(\Delta\otimes\mathrm{id}).$$

Since $\mathcal{A}_e=\mathcal{A}_e^{\tau}+\mathcal{A}_e^{\tau}\Delta$ and $\tilde{\rho}(U(\mathfrak{g})_f)\subseteq(\mathcal{A}_e\otimes\rho(H))^{\sigma\otimes\tau}$ we derive (iii). Then (iv) follows, completing the proof. □

5.3. By [31, (6.2)], the restriction of the representation $\tilde{\rho}_\chi: U(\mathfrak{g})\to\mathrm{End}(Q_\chi)$ to the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is injective. Since any nonzero two-sided ideal of $U(\mathfrak{g})$ intersects $Z(\mathfrak{g})$ by [9, (4.2.2)], it follows that $\tilde{\rho}_\chi$ is a faithful representation of $U(\mathfrak{g})$. Note that $Q_\chi\in\mathcal{C}$ and $\mathrm{Wh}(Q_\chi)$ is canonically identified with H (viewed as the left regular H-module). By 5.1, $Q_\chi\cong Q_\chi\otimes_H H$ is then identified with $\mathcal{A}_e\otimes H$ as vector spaces over $\Bbbk$. Since $\tilde{\rho}$ is faithful, it extends to a faithful representation of $U(\mathfrak{g})_f$ in $\mathrm{End}(Q_\chi)$ (one should take into account that $\tilde{\rho}_\chi(f)$ is invertible). Since $\mathrm{Wh}(Q_\chi)$ is identified with the left regular H-module, Theorem 5.1(iv) yields $\tilde{\rho}_\chi(U(\mathfrak{g})_f)\subseteq\mathcal{A}_e\otimes H$. Applying Theorem 5.1 in this situation we now obtain:

Corollary 5.1. *Set $\widehat{\Delta}=\Delta\otimes\mathrm{id}_H$ and identify $U(\mathfrak{g})_f$ with its image in $\mathcal{A}_e\otimes H$. Then the following hold:*

(i) $\widehat{\Delta}U(\mathfrak{g})_f\widehat{\Delta}^{-1}=U(\mathfrak{g})_f$;
(ii) $f=\widehat{\Delta}^2$;
(iii) $\mathcal{A}_e\otimes H=U(\mathfrak{g})_f\oplus U(\mathfrak{g})_f\widehat{\Delta}$;
(iv) $U(\mathfrak{g})_f=(\mathcal{A}_e\otimes H)^{\tau\otimes\sigma}$.
(v) $(\mathcal{A}_e\otimes H)^{\mathfrak{n}_\chi}=\Bbbk[\Delta,\Delta^{-1}]\otimes H=(U(\mathfrak{g})^{\mathfrak{n}_\chi})_f\oplus(U(\mathfrak{g})^{\mathfrak{n}_\chi})_f\widehat{\Delta}$;
(vi) $Z(H)\cong Z(U(\mathfrak{g})_f)\cong Z(\mathfrak{g})$.

Proof. Parts (i)–(iv) follow from Theorem 5.1 and the discussion above. For (v), note that $(\Delta\circ z_i)\otimes 1, (\Delta\circ\partial_i)\otimes 1 \in \mathfrak{n}_\chi$ for all $i \le s$ (see the beginning of the proof of Theorem 5.1). From this it is immediate that $(\mathcal{A}_e\otimes H)^{\mathfrak{n}_\chi} = \Bbbk[\Delta,\Delta^{-1}]\otimes H$. Since $f\in Z(U(\mathfrak{g})^{\mathfrak{n}_\chi})$ we have $(U(\mathfrak{g})_f)^{\mathfrak{n}_\chi} = (U(\mathfrak{g})^{\mathfrak{n}_\chi})_f$, hence the result.

For (vi), we first recall that the action of $1\otimes\sigma$ on $\mathcal{A}_e\otimes H$ is induced by the adjoint action of $1\otimes\Theta(\mathfrak{z}_\chi(0))\subset 1\otimes H$. Consequently,

$$Z(H)\cong Z(\mathcal{A}_e\otimes H) = \Bbbk\otimes(H^\sigma\cap Z(H)) = (Z(\mathcal{A}_e\otimes H))^{\tau\otimes\sigma}\subseteq Z(U(\mathfrak{g})_f),$$

by (iv). On the other hand, $Z(U(\mathfrak{g})_f)\subseteq Z((U(\mathfrak{g})^{\mathfrak{n}_\chi})_f)\subseteq\Bbbk[\Delta,\Delta^{-1}]\otimes Z(H)$, by (v). Since $[h,\Delta]=-\Delta$, it must be that $Z(U(\mathfrak{g})_f)\subseteq\Bbbk\otimes Z(H) = Z(\mathcal{A}_e\otimes H)$. Therefore, $Z(H)\cong Z(U(\mathfrak{g})_f)$.

It remains to show that $Z(U(\mathfrak{g})_f) = Z(\mathfrak{g})$. This is easy and must be well-known, but we could not find a good reference. One can argue as follows: Let $K(\mathfrak{g}) = \operatorname{Fract}U(\mathfrak{g})$ be the Lie field of $\mathfrak{g}$. By [9, (4.3.2)], the centre of $K(\mathfrak{g})$ coincides with $\operatorname{Fract}Z(\mathfrak{g})$. Let $z\in Z(U(\mathfrak{g})_f)$. Then $z = f^{-d}c$ for some $c\in U(\mathfrak{g})$ and $d\in\mathbb{Z}_+$. Since $Z(U(\mathfrak{g})_f)\subset Z(K(\mathfrak{g}))$, there are $a,b\in Z(\mathfrak{g})$ such that $af^d = bc$. By a classical result of Kostant, $U(\mathfrak{g})\cong Z(\mathfrak{g})\otimes\mathsf{H}(\mathfrak{g})$ as $Z(\mathfrak{g})$-modules, where $\mathsf{H}(\mathfrak{g})$ denotes the subspace of $U(\mathfrak{g})$ spanned by the powers of the nilpotent elements of $\mathfrak{g}$; see [9, (8.2.4) and (8.5.5)]. Choose a basis $\{u_i\}$ in $\mathsf{H}(\mathfrak{g})$ with $u_1 = f^d$. Write $c=\sum_i z_iu_i$ with $z_i\in Z(\mathfrak{g})$. Then $a = bz_1$ and $z_i = 0$ for $i\ne 1$, forcing $z = f^{-d}c = f^{-d}z_1f^d = z_1\in Z(\mathfrak{g})$. The result follows. □

5.4. Recall that a two-sided ideal I of an associative ring R is called *prime* if $I\ne R$ and for any two two-sided ideals J_1, J_2 the inclusion $J_1J_2\subseteq I$ implies that either $J_1\subseteq I$ or $J_2\subseteq I$. We let $\operatorname{Spec}R$ denote the set of all prime ideals of R. Any primitive ideal of R is prime. The ring R is termed *prime* if $(0)\in\operatorname{Spec}R$.

According to [9, (3.6.17)], if R is Noetherian and S is an Ore set in R, then the mapping $I\mapsto S^{-1}I$ sets up a bijection between the subset $(\operatorname{Spec}R)_S := \{I\in\operatorname{Spec}R \mid I\cap S = \emptyset\}$ of $\operatorname{Spec}R$ and $\operatorname{Spec}S^{-1}R$. On the other hand, the complement to $(\operatorname{Spec}U(\mathfrak{g}))_{S_f} = \{I\in\operatorname{Spec}U(\mathfrak{g}) \mid f^n\notin I \text{ for all } n\in\mathbb{Z}_+\}$ in $\operatorname{Spec}U(\mathfrak{g})$ consists of all prime ideals of finite codimension in $U(\mathfrak{g})$; see [13, Lemma 13.17] for example. Thus the mapping $I\mapsto S_f^{-1}I$ is a bijection between the set of all prime ideals of infinite codimension in $U(\mathfrak{g})$ and the set of all prime ideals of $U(\mathfrak{g})_f$.

By Goldie's theorem, the set S of all regular elements of a prime Noetherian ring R is an Ore set in R and the localisation $\mathcal{Q}(R) := S^{-1}R$ is isomorphic to the matrix algebra $\operatorname{Mat}_n(K)$ over a noncommutative field K. Both K and n can be described intrinsically, hence are uniquely determined by R. They are called the *Goldie field* and the *Goldie rank* of R, respectively. We write $n = \operatorname{rk}(R)$.

5.5. Corollary 5.1(iv) allows us to identify $U(\mathfrak{g})_f$ with the subalgebra $(\mathcal{A}_e\otimes H)^{\tau\otimes\sigma}$ of $\mathcal{A}_e\otimes H$. Since $\mathcal{A}_e = \mathcal{A}_e^\tau\oplus\mathcal{A}_e^\tau\Delta$ and $\tau(\Delta) = -\Delta$, we obtain the decomposition

$$U(\mathfrak{g})_f = \mathcal{A}_e^\tau\otimes H_+\oplus\mathcal{A}_e^\tau\Delta\otimes H_-$$

where $H_+ = \{x\in H \mid \sigma(x) = x\}$ and $H_- = \{x\in H\mid\sigma(x) = -x\}$. Let I be any two-sided ideal of H. We let $\operatorname{Dim}(H/I)$ denote the Gelfand–Kirillov dimension of the

factor algebra H/I. Since I is σ-stable (Corollary 2.1), we have $I = I_+ \oplus I_-$ where $I_\pm = I \cap H_\pm$. Put

$$\widetilde{I}_f := \mathcal{A}_e^\tau \otimes I_+ \oplus \mathcal{A}_e^\tau \Delta \otimes I_- \quad \text{and} \quad \widetilde{I} := \widetilde{I}_f \cap U(\mathfrak{g}). \tag{5.5.1}$$

Clearly, $\widetilde{I}_f$ and $\widetilde{I}$ are two-sided ideals of $U(\mathfrak{g})_f$ and $U(\mathfrak{g})$, respectively. Let $\mathcal{O}_{\min}$ denote the minimal nilpotent orbit $(\mathrm{Ad}^* G) \cdot \chi$ in $\mathfrak{g}^*$. Recall that

$$\dim \mathcal{O}_{\min} = 2 \dim \mathfrak{m}_\chi = 2(s+1).$$

Theorem 5.2. *The following are true:*

(i) *The map $I \mapsto \widetilde{I}_f$ sets up a bijection between the set of all two-sided ideals of H and the set of all two-sided ideals of $U(\mathfrak{g})_f$. For any two-sided ideal I of H one has*

$$\mathrm{Dim}(U(\mathfrak{g})_f/\widetilde{I}_f) = \mathrm{Dim}(H/I) + \dim \mathcal{O}_{\min}.$$

(ii) *The map $I \mapsto \widetilde{I}$ is a bijection between* $\mathrm{Spec}H$ *and the set of all prime ideals of infinite codimension in $U(\mathfrak{g})$. Furthermore,*

$$\mathrm{Dim}(U(\mathfrak{g})/\widetilde{I}) = \mathrm{Dim}(H/I) + \dim \mathcal{O}_{\min} \quad (\forall I \in \mathrm{Spec}\, H).$$

Proof. (a) Let Ξ denote the set of all quadruples $(m, n, \mathbf{i}, \mathbf{j})$ with $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, and $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^s$. Order the elements in Ξ lexicographically. Given $\xi = (m, n, \mathbf{i}, \mathbf{j}) \in \Xi$ define $a_\xi \in \mathcal{A}_e$ by setting $a_\xi = h^m \Delta^n z_1^{i_1} \cdots z_s^{j_s} \partial_1^{j_1} \cdots \partial_s^{j_s}$, where $\mathbf{i} = (i_1, \dots, i_s)$ and $\mathbf{j} = (j_1, \dots, j_s)$. Any nonzero $x \in \mathcal{A}_e \otimes H$ can be written uniquely as $x = \sum_{\xi \in \Xi(x)} a_\xi \otimes h_\xi$ for some nonzero $h_\xi \in H$. Here $\Xi(x)$ is a finite subset of Ξ depending on x.

Let $\mathcal{I}$ be any two-sided ideal of $U(\mathfrak{g})_f = (\mathcal{A}_e \otimes H)^{\tau \otimes \sigma}$. Recall that the action of σ on H is induced by the adjoint action of $\Theta(\mathfrak{z}_\chi(0))$. Since $1 \otimes \Theta(\mathfrak{z}_\chi(0)) \subset (\mathcal{A}_e \otimes H)^{\tau \otimes \sigma}$, the ideal $\mathcal{I}$ is stable under the involution $1 \otimes \sigma$ of $\mathcal{A}_e \otimes H$. It follows that $\mathcal{I} = \mathcal{I}_+ \oplus \mathcal{I}_-$ where $\mathcal{I}_\pm = \{x \in \mathcal{I} \mid (1 \otimes \sigma)(x) = \pm x\}$. Let $x \in \mathcal{I}_+ \cup \mathcal{I}_-$ and let $\xi_0 = (m_0, n_0, \mathbf{a}, \mathbf{b})$ be the maximal element in $\Xi(x)$. Then there exists a polynomial $f_x(t) \in \Bbbk[t]$ such that

$$f_x(\mathrm{ad}\, h) \circ \Big(\prod_{i=1}^s (\mathrm{ad}\, \Delta \otimes \partial_i)^{a_i} \circ \prod_{i=1}^s (\mathrm{ad}\, \Delta \otimes z_i)^{b_i} \Big) \circ (\mathrm{ad}\, \widehat{\Delta}^{2m_0})(x) \in \Bbbk^\times \Delta^{N+n_0} \otimes h_{\xi_0}$$

where $N = 2m_0 + \sum_{i=1}^s (a_i + b_i)$. As $\widehat{\Delta}^2$ is invertible, it follows that $\Delta^{\varepsilon(x)} \otimes h_{\xi_0} \in \mathcal{I}_\pm$, where $\varepsilon(x) = 0$ if $x \in \mathcal{I}_+$ and $\varepsilon(x) = 1$ if $x \in \mathcal{I}_-$. But then $a_{\xi_0} \Delta^{-\varepsilon(x)} \otimes 1 \in \mathcal{A}_e^\tau \otimes \Bbbk \subset (\mathcal{A}_e \otimes H)^{\tau \otimes \sigma}$ yielding $a_{\xi_0} \otimes h_{\xi_0} \in \mathcal{I}_\pm$. Continuing this process one eventually observes that $a_\xi \otimes h_\xi \in \mathcal{I}_\pm$ for all $\xi \in \Xi(x)$. This implies that there is a graded subspace $I = I_+ \oplus I_-$ in $H = H_+ \oplus H_-$ such that $\mathcal{I} = \mathcal{A}_e^\tau \otimes I_+ \oplus \mathcal{A}_e^\tau \Delta \otimes I_-$. Since $1 \otimes H_+ \cup \Delta \otimes H_- \subset (\mathcal{A}_e \otimes H)^{\tau \otimes \sigma}$ and $\widehat{\Delta}^2$ is invertible, it follows that I is a two-sided ideal of H. As a result, $\mathcal{I} = \widetilde{I}_f$, showing that the map $I \mapsto \widetilde{I}_f$ is surjective. The injectivity of this map is obvious.

(b) Let $\Xi_{\geq 0}$ be the subset of Ξ consisting of all $(m, n, \mathbf{i}, \mathbf{j})$ with $n \geq 0$, and let $\mathcal{A}'_e$ denote the $\Bbbk$-span of all a_ξ with $\xi \in \Xi_{\geq 0}$. Clearly, $\mathcal{A}'_e$ is a τ-invariant subalgebra of $\mathcal{A}_e$. For $l \in \mathbb{Z}_+$ we let $\mathcal{A}'_{e,l}$ denote the $\Bbbk$-span of all a_ξ with $\xi = (m, n, \mathbf{i}, \mathbf{j}) \in \Xi_{\geq 0}$ such that

$m + n + \sum_{k=1}^{s}(i_k + j_k) \leq l$. It is easy to see that $\{\mathcal{A}'_{e,l} \mid l \in \mathbb{Z}_+\}$ is an increasing τ-invariant filtration in $\mathcal{A}'_e$ and the corresponding graded algebra $\operatorname{gr}\mathcal{A}'_e$ is isomorphic to the graded polynomial algebra $\Bbbk[X_1, \ldots, X_{2s+2}]$ with all X_i having degree 1. Note that $\operatorname{ad}(\Delta^2)$ is locally nilpotent on $\mathcal{A}'_e$ and the algebra $\mathcal{A}_e$ identifies with the localisation $(\mathcal{A}'_e)_{\Delta^2}$.

Let I be a two-sided ideal of H. It follows from part (a) of this proof that the two-sided ideal $\widetilde{I}_f \oplus \widetilde{I}_f\widehat{\Delta}$ of $\mathcal{A}_e \otimes H$ coincides with $\mathcal{A}_e \otimes I$, so that $\widetilde{I}_f = (\mathcal{A}_e \otimes I)^{\tau\otimes\sigma}$. Therefore, the involution $\tau \otimes \sigma$ acts on the algebra $\mathcal{A}_e \otimes (H/I) \cong (\mathcal{A}_e \otimes H)/(\mathcal{A}_e \otimes I)$ in such a way that the quotient $U(\mathfrak{g})_f/\widetilde{I}_f$ identifies with $(\mathcal{A}_e \otimes (H/I))^{\tau\otimes\sigma}$. Since $f = \widehat{\Delta}^2$, it is straightforward to see that

$$\begin{aligned}(\mathcal{A}_e \otimes (H/I))^{\tau\otimes\sigma} &\cong ((\mathcal{A}'_e)_{\Delta^2} \otimes (H/I))^{\tau\otimes\sigma} \cong ((\mathcal{A}'_e \otimes (H/I))_{\bar{f}})^{\tau\otimes\sigma}\\ &\cong ((\mathcal{A}'_e \otimes (H/I))^{\tau\otimes\sigma})_{\bar{f}},\end{aligned}$$

where $\bar{f}$ denotes the image of f in $\mathcal{A}_e \otimes (H/I)$. In view of [3, (6.3)] we then have

$$\operatorname{Dim}(U(\mathfrak{g})_f/\widetilde{I}_f) = \operatorname{Dim}((\mathcal{A}'_e \otimes (H/I))^{\tau\otimes\sigma}; \tag{5.5.2}$$

see also [13, (11A.2)]. The Kazhdan filtration $\{H^k \mid k \in \mathbb{Z}_+\}$ gives rise to the natural filtration $\{(H/I)^k = H^k/(H^k \cap I) \mid k \in \mathbb{Z}_+\}$ of the algebra H/I. Thanks to [31, Theorem 4.6(iii)], $\operatorname{gr} H$ is a Noetherian commutative $\Bbbk$-algebra. Hence so is the corresponding graded algebra $\operatorname{gr}(H/I) \cong \operatorname{gr} H/\operatorname{gr} I$. Since σ preserves both I and the Kazhdan filtration of H, it induces an automorphism of the graded algebra $\operatorname{gr}(H/I)$.

Next we observe that the subspaces

$$\left\{(\mathcal{A}'_e \otimes (H/I))_k = \sum_{i+j\leq k} \mathcal{A}'_{e,i} \otimes (H/I)^j \;\middle|\; k \in \mathbb{Z}_+\right\}$$

form an increasing filtration of the algebra $\mathcal{A}'_e \otimes (H/I)$ such that

$$\operatorname{gr}(\mathcal{A}'_e \otimes (H/I)) \cong \operatorname{gr}(\mathcal{A}'_e) \otimes \operatorname{gr}(H/I) \cong \Bbbk[X_1, \ldots, X_{2s+2}] \otimes \operatorname{gr}(H/I).$$

By construction, the involution $\tau \otimes \sigma$ acts on the graded algebra $\operatorname{gr}(\mathcal{A}'_e \otimes (H/I))$ and

$$(\operatorname{gr}(\mathcal{A}'_e \otimes (H/I)))^{\tau\otimes\sigma} \cong \operatorname{gr}((\mathcal{A}'_e \otimes (H/I))^{\tau\otimes\sigma})$$

as graded algebras. Since the morphism

$$\operatorname{Spec}\operatorname{gr}(\mathcal{A}'_e \otimes (H/I)) \to \operatorname{Spec}(\operatorname{gr}(\mathcal{A}'_e \otimes (H/I)))^{\tau\otimes\sigma}$$

induced by inclusion $(\operatorname{gr}(\mathcal{A}'_e \otimes (H/I)))^{\tau\otimes\sigma} \hookrightarrow \operatorname{gr}(\mathcal{A}'_e \otimes (H/I))$ is finite, the Noetherian $\Bbbk$-algebras $(\operatorname{gr}(\mathcal{A}'_e \otimes (H/I)))^{\tau\otimes\sigma}$ and $\operatorname{gr}(\mathcal{A}'_e) \otimes \operatorname{gr}(H/I)$ have the same Krull dimension. Since the Krull dimensions of the graded algebras $(\operatorname{gr}(\mathcal{A}'_e \otimes (H/I)))^{\tau\otimes\sigma}$ and $\operatorname{gr}(H/I)$ coincide with the degrees of their respective Hilbert polynomials, we derive that

$$\operatorname{Dim}\operatorname{gr}((\mathcal{A}'_e \otimes (H/I))^{\tau\otimes\sigma}) = 2(s+1) + \operatorname{Dim}\operatorname{gr}(H/I).$$

On the other hand, it follows from [25, Proposition 8.6.6] that $\operatorname{Dim}\operatorname{gr}(H/I) = \operatorname{Dim}(H/I)$ and $\operatorname{Dim}\operatorname{gr}((\mathcal{A}'_e \otimes (H/I))^{\tau\otimes\sigma}) = \operatorname{Dim}((\mathcal{A}'_e \otimes (H/I))^{\tau\otimes\sigma})$. Combining this with (5.5.2) we get (i).

(c) Let $I \in \operatorname{Spec} H$ and suppose $\mathcal{J}_1\mathcal{J}_2 \subseteq \widetilde{I}_f$ for some two-sided ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ of $U(\mathfrak{g})_f = (\mathcal{A}_e \otimes H)^{\tau\otimes\sigma}$. By part (a), there exist two-sided ideals J_1 and J_2 of H such that $\mathcal{J}_i = \mathcal{A}_e^\tau \otimes J_{i,+} \oplus \mathcal{A}_e^\tau\Delta \otimes J_{i,-}$, $i = 1, 2$. Since $\widehat{\Delta}^2$ is invertible, it is easy to see that $J_1J_2 \subseteq I$. Then either $J_1 \subseteq I$ or $J_2 \subseteq I$, and hence either $\mathcal{J}_1 \subseteq \widetilde{I}_f$ or $\mathcal{J}_2 \subseteq \widetilde{I}_f$. As a result, $\widetilde{I}_f \in \operatorname{Spec} U(\mathfrak{g})_f$ and $\widetilde{I} = \widetilde{I}_f \cap U(\mathfrak{g}) \in \operatorname{Spec} U(\mathfrak{g})$.

Conversely, let $\mathcal{I}$ be a prime ideal of infinite codimension in $U(\mathfrak{g})$. The discussion at the beginning of this subsection shows that $S_f^{-1}\mathcal{I} \in \operatorname{Spec} U(\mathfrak{g})_f$. By part (a) of this proof, there is a two-sided ideal I of H such that $S_f^{-1}\mathcal{I} = \mathcal{A}_e^\tau \otimes I_+ \oplus \mathcal{A}_e^\tau\Delta \otimes I_-$. Clearly, $\mathcal{I} = \widetilde{I}_f \cap U(\mathfrak{g}) = \widetilde{I}$. If $PQ \subseteq I$ for some two-sided ideals P, Q of H, then $\widetilde{P}_f\widetilde{Q}_f \subseteq S_f^{-1}\mathcal{I}$, forcing either $\widetilde{P}_f \subseteq S_f^{-1}\mathcal{I}$ or $\widetilde{Q}_f \subseteq S_f^{-1}\mathcal{I}$. As $S_f^{-1}\mathcal{I} = \widetilde{I}_f$, we infer that either $P \subseteq I$ or $Q \subseteq I$. Therefore, $I \in \operatorname{Spec} H$.

Now let $\bar{f}$ be the image of f in $U(\mathfrak{g})/\widetilde{I}$ where $I \in \operatorname{Spec} H$. Since $U(\mathfrak{g})_f/\widetilde{I}_f \cong (U(\mathfrak{g})/\widetilde{I})_{\bar{f}}$ by the exactness of localisation, it follows from [3, (6.3)] and (i) that $\operatorname{Dim}(U(\mathfrak{g})/\widetilde{I}) = \operatorname{Dim}(H/I) + \dim \mathcal{O}_{\min}$. □

5.6. Given a ring R we let $\operatorname{Prim} R$ denote the primitive spectrum of R, the set of all primitive ideals of R taken with the Jacobson topology. Set $\mathcal{X} := \operatorname{Prim} U(\mathfrak{g})$ and denote by $\mathcal{X}_{\mathrm{fin}}$ the set of all primitive ideals of finite codimension in $U(\mathfrak{g})$. Using the highest weight theory and [9, (2.5.6), (3.2.3)] it is easy to observe that $\mathcal{X}_{\mathrm{fin}}$ is a countable dense subset of $\mathcal{X}$. The topology of $\mathcal{X}$ induces that on the complement $\mathcal{X}_{\mathrm{inf}} := \mathcal{X} \setminus \mathcal{X}_{\mathrm{fin}}$.

Recall that $\operatorname{Prim} H$ is a subset of $\operatorname{Spec} H$. By Theorem 5.2(ii), the map $I \mapsto \widetilde{I}$ given by (5.5.1) sets up a bijection between $\operatorname{Spec} H$ and the set of all prime ideals of infinite codimension in $U(\mathfrak{g})$. Identify $Z(\mathfrak{g})$ with $Z(H)$ according to Corollary 5.1(vi).

Theorem 5.3. *The following are true:*

(1) *The map $I \mapsto \widetilde{I}$ takes* $\operatorname{Prim} H$ *onto $\mathcal{X}_{\mathrm{inf}}$ and induces a homeomorphism of topological spaces* $\operatorname{Prim} H \xrightarrow{\sim} \mathcal{X}_{\mathrm{inf}}$.
(2) *Let V be a finite-dimensional irreducible H-module and $I = \operatorname{Ann}_H V$. Then $\widetilde{I} = \operatorname{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V)$ and $\operatorname{rk}(U(\mathfrak{g})/\widetilde{I}) = \dim \operatorname{Wh}(Q_\chi \otimes_H V) = \dim V$.*
(3) *Let $\mathcal{I}$ be a primitive ideal of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$. Then there is a finite-dimensional irreducible H-module E such that $\mathcal{I} = \operatorname{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H E)$.*
(4) *Let V_1 and V_2 be two finite-dimensional irreducible H-modules. Then $V_1 \cong V_2$ as H-modules if and only if $\operatorname{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_1) = \operatorname{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_2)$.*
(5) *For any algebra homomorphism $\eta\colon Z(\mathfrak{g}) \to \Bbbk$ there is a bijection between the isoclasses of finite-dimensional irreducible H-modules with central character η and the primitive ideals $\mathcal{I}$ of $U(\mathfrak{g})$ with $\mathcal{I} \cap Z(\mathfrak{g}) = \operatorname{Ker}\eta$ and $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$.*
(6) *A prime ideal I of H is primitive if and only if $I \cap Z(H)$ is a maximal ideal of $Z(H)$.*

Proof. (a) Let $\{J^\alpha \mid \alpha \in A\}$ be a set of two-sided ideals of H, and $J = \bigcap_{\alpha\in A} J^\alpha$. Since any two-sided ideal of H is σ-stable, we have

$$J_\pm = \{x \pm \sigma(x) \mid x \in J\} = \bigcap_{\alpha\in A} \{x \pm \sigma(x) \mid x \in J^\alpha\} = \bigcap_{\alpha\in A} J^\alpha_\pm.$$

Using (5.5.1) it is now easy to deduce that $\widetilde{J}_f = \bigcap_{\alpha\in A} \widetilde{J}^\alpha_f$ and $\widetilde{J} = \bigcap_{\alpha\in A} \widetilde{J}^\alpha$. Arguing similarly and using Theorem 5.2 one also observes that given $I, J \in \operatorname{Spec} H$ one has $I \subsetneq J$ if and only if $\widetilde{I} \subsetneq \widetilde{J}$.

(b) Let $I \in \operatorname{Prim} H$. Then $I = \operatorname{Ann}_H M_0$ for some irreducible H-module M_0. Let $M = Q \otimes_H M_0$ and identify M with $\Bbbk[h, z_1, \ldots, z_s] \otimes M_0$; see 5.1 for details. Theorem 5.2 shows that $\operatorname{Ann}_{U(\mathfrak{g})} M = \widetilde{J}$ for some $J \in \operatorname{Spec} H$, while Theorem 5.1 yields $\widetilde{J}_f = \widetilde{I}_f$. Then $I = J$ in view of Theorem 5.2(i), forcing $\widetilde{I} = \operatorname{Ann}_{U(\mathfrak{g})} M$. As a consequence, the map $I \mapsto \widetilde{I}$ takes $\operatorname{Prim} H$ into $\mathcal{X}_{\mathrm{inf}}$.

(c) Now suppose that $\widetilde{I} \in \mathcal{X}_{\mathrm{inf}}$ for some two-sided ideal I of H. By Theorem 5.2(ii), $I \in \operatorname{Spec} H$. Let $\mathfrak{I} = \{J \in \operatorname{Spec} H \mid J \supsetneq I\}$, and $J_0 = \bigcap_{J\in\mathfrak{I}} J$. Our discussion in part (a) in conjunction with Theorem 5.2(ii) implies that $\widetilde{\mathfrak{I}} := \{\widetilde{J} \mid J \in \mathfrak{I}\}$ coincides with the set of all prime ideals of $U(\mathfrak{g})$ containing $\widetilde{I}$ properly, and $\widetilde{J_0} = \bigcap_{\mathcal{I}\in\widetilde{\mathfrak{I}}} \mathcal{I}$. Since $\widetilde{I}$ is a primitive ideal, [9, (8.5.7)] applies, yielding $\widetilde{J_0} \supsetneq \widetilde{I}$. But then $J_0 \supsetneq I$ by our concluding remark in part (a).

(d) We claim that the prime ideal I from part (c) is the intersection of some primitive ideals of H. To see this one can mimic the proof of Proposition 3.1.15 in [9] which deals with enveloping algebras but applies to a larger class of filtered rings. For the reader's convenience we include the argument which goes back to Duflo. As in [9] we put $B := H/I$, let X be a variable, and set $C := B \otimes \Bbbk[X]$. The Kazhdan filtration of H induces a filtration of B, which in turn gives rise to a filtration of C. Since $\operatorname{gr} H$ is finitely generated and commutative, by [31, Theorem 4.6(iii)], so are $\operatorname{gr} B$ and $\operatorname{gr} C$. Let $a \in J(B)$ where $J(B)$ is the Jacobson radical of B.

Suppose $C(1 - aX) \neq C$. By Zorn's lemma, there exists a maximal left ideal of C containing $C(1 - aX)$, say L. Put $M := C/L$, a simple C-module, and let m_0 denote the image of $1 \in C$ in M. Then $m_0 \neq 0$ and $(1-aX)m_0 = 0$. Let x and a_M denote the images of X and a in $\operatorname{End} M$. Since $X \in Z(C)$, we have $x \in \operatorname{End}_C M$. Since $\operatorname{gr} C$ is finitely generated and commutative, x is invertible in $\operatorname{End}_C M$ and algebraic over $\Bbbk$, by Quillen's lemma. Put $y := x^{-1}$. Then $x = p(y)$ for some $p \in \Bbbk[X]$, and $a_M(m_0) = y(m_0)$. Therefore, $(1 - ap(a))m_0 = (1 - yp(y))(m_0) = 0$. On the other hand, $ap(a) \in J(B)$, hence $1 - ap(a)$ is invertible in C; see [9, (3.1.12)] for instance. This contradiction shows that $C(1 - aX) = C$.

As a consequence, $(a_0 + a_1X + \cdots + a_nX^n)(1 - aX) = 1$ for some $a_i \in B$. Easy induction on i gives $a_i = a^i$ for $0 \leq i \leq n$. Then $a^{n+1} = 0$, showing that all elements in $J(B)$ are nilpotent. As (0) is a prime ideal of $B = H/I$, it follows from [9, (3.1.14)] that $\bigcap_{J\in\operatorname{Prim} B} J = J(B) = 0$. But then I is the intersection of some primitive ideals of H, as claimed.

If $I \notin \operatorname{Prim} H$, then all primitive ideals of H containing I lie in $\mathfrak{I}$. However, this is impossible as $J_0 \supsetneq I$; see part (c). Thus I must be a primitive ideal of H, showing that the map $I \mapsto \widetilde{I}$ induces a bijection between $\operatorname{Prim} H$ and $\mathcal{X}_{\mathrm{inf}}$; call it $\varkappa$.

Let $\mathcal{Y}$ be a closed set in $\operatorname{Prim} H$. Then there is a two-sided ideal $I_{\mathcal{Y}}$ of H such that $\mathcal{Y} = \{J \in \operatorname{Prim} H \mid J \supseteq I_{\mathcal{Y}}\}$; see [9, (3.2.3)]. Our earlier remarks in the proof show that $\varkappa(\mathcal{Y}) = \{\mathcal{I} \in \mathcal{X}_{\mathrm{inf}} \mid \mathcal{I} \supseteq \widetilde{I_{\mathcal{Y}}}\}$. Therefore, $\varkappa\colon \operatorname{Prim} H \to \mathcal{X}_{\mathrm{inf}}$ is a closed map.

Recall that the topology on $\mathcal{X}_{\mathrm{inf}}$ is induced by the Jacobson topology on $\operatorname{Prim} U(\mathfrak{g})$. From [9, (3.1.10)] it follows that $\operatorname{Prim} U(\mathfrak{g})$ is a Zariski space, that is, any closed set in $\operatorname{Prim} U(\mathfrak{g})$ is a finite union of irreducible closed sets. But then $\mathcal{X}_{\mathrm{inf}}$ is a Zariski space as well. Let $\widetilde{\mathcal{Y}}$ be an irreducible closed set in $\mathcal{X}_{\mathrm{inf}}$. Then there is an $\mathcal{I} \in \operatorname{Spec} U(\mathfrak{g})$ such that $\widetilde{\mathcal{Y}} = \{\mathcal{J} \in \mathcal{X}_{\mathrm{inf}} \mid \mathcal{J} \supseteq \mathcal{I}\}$; see [9, (3.2.5)]. By Theorem 5.2(ii), $\mathcal{I} = \widetilde{I}$ for some $I \in \operatorname{Spec} H$. Furthermore, $\varkappa^{-1}(\widetilde{\mathcal{Y}}) = \{J \in \operatorname{Prim} H \mid J \supseteq I\}$ by our remarks earlier in the proof. From this it is immediate that $\varkappa^{-1}\colon \mathcal{X}_{\mathrm{inf}} \to \operatorname{Prim} H$ is a closed map too, proving (1).

(e) Let V be a finite-dimensional irreducible H-module and $I = \operatorname{Ann}_H V$, a primitive ideal of finite codimension in H. By part (b) of this proof, $\widetilde{I} = \operatorname{Ann}_{U(\mathfrak{g})} \widetilde{V}$ where $\widetilde{V} = Q_\chi \otimes_H V$. As before, we identify $\widetilde{V}$ with $\Bbbk[h, z_1, \dots, z_s] \otimes V$; see 5.1. Then $\widetilde{\mathcal{B}}_0 := U(\mathfrak{g})_f/\widetilde{I}_f$ identifies with a subalgebra of $\widetilde{\mathcal{B}} := \mathcal{A}_e \otimes (H/I)$. More precisely, from part (b) of the proof of Theorem 5.2 we know that $\tau \otimes \sigma$ acts on $\widetilde{\mathcal{B}}$, and $\widetilde{\mathcal{B}}_0 = \widetilde{\mathcal{B}}^{\tau\otimes\sigma}$.

Since $U(\mathfrak{g})$ is a completely reducible $\operatorname{ad}\mathfrak{g}$-module, $(U(\mathfrak{g})/\widetilde{I})^{\mathfrak{n}_\chi} \cong U(\mathfrak{g})^{\mathfrak{n}_\chi}/\widetilde{I}^{\mathfrak{n}_\chi}$; see [13, (3.2)] for example. Since f is central in $U(\mathfrak{g})^{\mathfrak{n}_\chi}$, we then have

$$\mathcal{B}_0 := \widetilde{\mathcal{B}}_0^{\mathfrak{n}_\chi} \cong ((U(\mathfrak{g})/\widetilde{I})_{\bar f})^{\mathfrak{n}_\chi} \cong (U(\mathfrak{g})^{\mathfrak{n}_\chi})_f/(U(\mathfrak{g})^{\mathfrak{n}_\chi})_f \cap \widetilde{I}_f,$$

where $\bar f$ is the image of f in $U(\mathfrak{g})/\widetilde{I}$. Put $\bar H = H/I$ and $\mathcal{B} = \widetilde{\mathcal{B}}^{\mathfrak{n}_\chi}$. Corollary 5.1(iii) implies that $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_0 \oplus \widetilde{\mathcal{B}}_0 t$ where t stands for the image of $\widehat{\Delta}$ in $\widetilde{\mathcal{B}}$. Since $\widehat{\Delta}$ commutes with $\mathfrak{n}_\chi$, it must be that $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_0 t$. In conjunction with Corollary 5.1(v) this shows that the natural map $(\mathcal{A}_e \otimes H)^{\mathfrak{n}_\chi} \to \mathcal{B}$ is surjective and $\mathcal{B} = \Bbbk[t, t^{-1}] \otimes \bar H$ as algebras, where $\Bbbk[t, t^{-1}]$ is the Laurent polynomial ring in t over $\Bbbk$. As $\bar H \cong \operatorname{End}_\Bbbk V$ is a prime ring, so too is $\mathcal{B} = \bar H[t, t^{-1}]$.

Since $\widetilde{I}$ is a prime ideal of $U(\mathfrak{g})$, the ring $(U(\mathfrak{g})/\widetilde{I})^{\mathfrak{n}_\chi}$ is prime with $\operatorname{rk}(U(\mathfrak{g})/I)^{\mathfrak{n}_\chi} = \operatorname{rk}(U(\mathfrak{g})/\widetilde{I})$; see [13, (13.10)]. Applying [9, (3.6.15)] we derive that the ring $\mathcal{B}_0 = ((U(\mathfrak{g})/I)^{\mathfrak{n}_\chi})_{\bar f}$ is prime with $\operatorname{rk}(\mathcal{B}_0) = \operatorname{rk}(U(\mathfrak{g})/\widetilde{I})$. On the other hand, it follows from [34] that $\operatorname{rk}(\Bbbk[t] \otimes \bar H) = \operatorname{rk}(\bar H)$. As $\Bbbk[t, t^{-1}] \otimes \bar H$ is a localisation of $\Bbbk[t] \otimes \bar H$ with respect to the Ore set $\{t^i \mid i \in \mathbb{Z}_+\}$, we have

$$\dim V = \operatorname{rk}(\bar H) = \operatorname{rk}(\Bbbk[t, t^{-1}] \otimes \bar H) = \operatorname{rk}(\mathcal{B});$$

see [25, Lemma 2.2.12] for example. We thus need to show that $\operatorname{rk}(\mathcal{B}) = \operatorname{rk}(\mathcal{B}_0)$.

It should be mentioned that our present setting (involving a quadratic extension of rings) resembles that of [16, (6.5)] where the above equality has been claimed in a more general situation. However, the proof of Lemma 6.5 in [16] is based on a faulty argument. Moreover, an example involving the first Weyl field shows that under the assumptions of [16, (6.5)] the Goldie ranks of $\mathcal{B}$ and $\mathcal{B}_0$ can be different (I am thankful to Anthony

Joseph for this clarification). We are lucky here because in the present case one can use another argument to prove the required equality of Goldie ranks.

Put $K := \Bbbk(t)$, $K_0 := \Bbbk(t^2)$, and $S := \Bbbk[t^2] \setminus \{0\}$, a central Ore set in $\mathcal{B}_0$ and $\mathcal{B}$. Put $B := S^{-1}\mathcal{B}$ and $B_0 := S^{-1}\mathcal{B}_0$. Since $\mathcal{B} = \bar{H}[t, t^{-1}]$ and $S^{-1}\Bbbk[t, t^{-1}] = K$, we have $B \cong K \otimes_{\Bbbk} \bar{H} \cong \mathrm{Mat}_n(K)$ where $n = \dim V$. Since B is a simple Artinian ring, all regular elements of B are invertible. Since S consists of regular elements of $\mathcal{B}$, the universality property of quotient rings yields $B \cong \mathcal{Q}(\mathcal{B})$. In particular, K is the Goldie field of $\mathcal{B}$ (this argument provides another proof for the equality $\mathrm{rk}(\mathcal{B}) = \dim V$). As in [16, (6.5)], we regard B as a Galois extension of B_0. The involution $\tau \otimes \sigma$ induces a K_0-automorphism of B, call it ι. It is easy to see that $\iota(t) = -t$, $B = B_0 \oplus B_0 t$, and $B_0 = B^{\iota}$. Thus ι can be viewed as the generator of the Galois group $\mathrm{Gal}(K/K_0)$.

Note that $Z(B_0) = Z(B)^{\iota} = K^{\iota} = K_0$ and $B \cong B_0 \otimes_{K_0} K$ as K-algebras. Since $\mathcal{B}_0$ is a prime ring, so too is B_0; see [9, (3.6.15)]. The preceding remark then shows that B_0 is a simple algebra finite-dimensional over its centre K_0. Since $\Bbbk$ is algebraically closed, $K_0 \cong \Bbbk(t)$ is a C_1-field, by Tsen's theorem. Therefore, $B_0 \cong \mathrm{Mat}_m(K_0)$ as K_0-algebras. As in the previous paragraph one observes that $B_0 \cong \mathcal{Q}(\mathcal{B}_0)$ and K_0 is the Goldie field of $\mathcal{B}_0$. Since $B \cong B_0 \otimes_{K_0} K$, one has $m = n$, proving (2).

(f) Let $\mathcal{I}$ be a primitive ideal of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$. Then $\mathcal{I} = \widetilde{I}$ for some $I \in \mathrm{Prim}\, H$, by part (1) of this theorem. Thanks to Theorem 5.2(ii), $\mathrm{Dim}(H/I) = 0$. Hence H/I is finite-dimensional over k; see [25, (8.1.17)] for example. Since H/I is a prime ring, it must be that $H/I \cong \mathrm{End}(E)$ for some finite-dimensional H-module E. As H/I is simple, $I = \mathrm{Ann}_H E$. But then $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H E)$ by part (b) of this proof, as stated in (3).

Now let V_1 and V_2 be two finite-dimensional irreducible H-modules, and set $I_i := \mathrm{Ann}_H V_i$, $i = 1, 2$. If $V_1 \cong V_2$ as H-modules then, of course, $I_1 = I_2$. Hence $\widetilde{I_1} = \widetilde{I_2}$, yielding $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_1) = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_2)$, again by part (b). Conversely, if $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_1) = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V_2)$, then $I_1 = I_2$ in view of Theorem 5.2(ii) and part (b). So $H/I_1 = H/I_2 \cong \mathrm{Mat}_m(\Bbbk)$ for some m. It is now straightforward to see that $V_1 \cong V_2$ as H-modules, giving (4).

Fix an algebra homomorphism $\eta\colon Z(\mathfrak{g}) \to \Bbbk$ and identify $Z(\mathfrak{g})$ with $Z(H)$; see Corollary 5.1(vi). If V is a finite-dimensional H-module with central character η and $I = \mathrm{Ann}_H V$, then $I \cap Z(H) = \mathrm{Ker}\,\eta$ is a maximal ideal of $Z(H)$, by Schur's lemma. Hence $\mathrm{Ker}\,\eta = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H V) \cap Z(\mathfrak{g}) = \widetilde{I} \cap Z(\mathfrak{g})$. Thanks to Theorem 5.2(ii) we also have $\mathcal{VA}(\widetilde{I}) = \overline{\mathcal{O}}_{\min}$. Now let $\mathcal{I} \in \mathcal{X}$ be such that $Z(\mathfrak{g}) \cap \mathcal{I} = \mathrm{Ker}\,\eta$ and $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$. By parts (3) and (4) of this theorem, $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(Q_\chi \otimes_H E)$ for some finite-dimensional irreducible H-module E, which is uniquely determined up to isomorphism. Since $\mathrm{Ker}\,\eta \subset \mathcal{I}$ and $Z(\mathfrak{g}) = Z(H)$, the H-module E has central character η. We obtain (5).

(g) Let $I \in \mathrm{Spec}\, H$ and suppose that $I \cap Z(H)$ is a maximal ideal of $Z(H)$. By Theorem 5.2(ii), $\widetilde{I} \in \mathrm{Spec}\, U(\mathfrak{g})$. As explained in the proof of Corollary 5.1(vi),

$$Z(H) = Z(\mathcal{A}_e \otimes H) = \Bbbk \otimes (H^{\sigma} \cap Z(H)) = (Z(\mathcal{A}_e \otimes H))^{\tau\otimes\sigma} \subseteq Z(U(\mathfrak{g})_f) = Z(\mathfrak{g}).$$

It follows that $I \cap Z(H) \subseteq (\Bbbk \otimes I_+) \cap Z(\mathfrak{g}) \subseteq \widetilde{I}_f \cap Z(\mathfrak{g}) = \widetilde{I} \cap Z(\mathfrak{g})$. As $Z(\mathfrak{g}) = Z(H)$, we deduce that $\widetilde{I} \cap Z(\mathfrak{g})$ is a maximal ideal of $Z(\mathfrak{g})$. But then $\widetilde{I} \in \mathcal{X}$; see [9, (8.5.7)]. Since $\widetilde{I}$ has infinite codimension in $U(\mathfrak{g})$, part (1) of this theorem implies that $I = \varkappa^{-1}(\widetilde{I}) \in \operatorname{Prim} H$. Finally, suppose $I \in \operatorname{Prim} H$ and let M be an irreducible H-module such that $I = \operatorname{Ann}_H M$. Since $\operatorname{gr} H$ is finitely generated and commutative, Quillen's lemma shows that $Z(H)$ acts on M as scalar operators. Consequently, $I \cap Z(H)$ is a maximal ideal of $Z(H)$. The proof of the theorem is now complete. □

5.7. Before finishing this section off we wish to discuss a possible extension of the above results to the case of a general algebra H_χ. Let e be any nilpotent element in $\mathfrak{g}$ and $\chi = \chi_e \in \mathfrak{g}^*$. Put $\mathcal{O} = \mathcal{O}_\chi$ and denote by $\mathcal{X}_\mathcal{O}$ the set of all primitive ideals $\mathcal{I}$ of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathcal{I}) \supset \mathcal{O}$. Take $\operatorname{Prim} H_\chi$ with the Jacobson topology and $\mathcal{X}_\mathcal{O}$ with the topology induced by that of $\mathcal{X}$.

Question 5.1. Are the following true?

1. The centre of H_χ coincides with the image of $Z(\mathfrak{g})$ in H_χ.[1]
2. There exists a homeomorphism $\varkappa\colon \operatorname{Prim} H_\chi \to \mathcal{X}_\mathcal{O}$ such that:
   (a) $\operatorname{Dim}(U(\mathfrak{g})/\varkappa(I)) = \operatorname{Dim}(H/I) + \dim \mathcal{O}$ for all $I \in \operatorname{Prim} H_\chi$;
   (b) $\operatorname{rk}(U(\mathfrak{g})/\varkappa(I)) = \sqrt{\dim_\Bbbk (H/I)}$ for all $I \in \operatorname{Prim} H_\chi$ with $\operatorname{codim}_{H_\chi} I < \infty$.
3. For every character η of $Z(H_\chi) = Z(\mathfrak{g})$ the map $\varkappa$ induces a bijection between the isoclasses of finite-dimensional H_χ-modules with central character η and the primitive ideals $\mathcal{I}$ of $U(\mathfrak{g})$ with $\mathcal{I} \cap Z(\mathfrak{g}) = \operatorname{Ker} \eta$ and $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}$.

6. The Joseph ideal and a presentation of H

6.1. In his seminal work [14] Joseph has discovered that outside type A the enveloping algebra $U(\mathfrak{g})$ has a unique completely prime primitive ideal whose associated variety is $\overline{\mathcal{O}}_{\min}$. This ideal is denoted $\mathcal{J}_0$ and referred to as the *Joseph ideal* of $U(\mathfrak{g})$. For $\mathfrak{g}$ of type A the completely prime primitive ideals of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$ form a single family parametrised by the elements of $\Bbbk$ (this will be explained in more detail in the course of proving Theorem 6.1).

[1] At the Oberwolfach meeting on enveloping algebras in March 2005 Victor Ginzburg has explained to me that this is a consequence of the finiteness of the number of symplectic leaves of $\mathcal{S}_e$ contained in the fibres of the morphism $f\colon \mathcal{S}_e \to \mathfrak{g}/\!\!/G$ iduced by the adjoint quotient map of $\mathfrak{g}$. Each homogeneous element $z \in \operatorname{gr} Z(H_\chi)$ lies in the Poisson centre of $\operatorname{gr} H_\chi = \Bbbk[\mathcal{S}_e]$, hence reduces to scalars on all symplectic leaves of $\mathcal{S}_e$. The Poisson structure on $\mathcal{S}_e$ induced by multiplication in H_χ is determined in [10, (3.2), (5.5)]. By [31, (5.4), (6.3)], all scheme-theoretic fibres of f are reduced and irreducible, and $\operatorname{gr} H_\chi$ is a flat module over $\operatorname{gr} Z(\mathfrak{g})$. These results are needed to carry out Ginzburg's argument: Since each fibre of f contains a Zariski dense symplectic leaf of $\mathcal{S}_e$, the regular function z is constant on each fibre of f. The flatness of the $\operatorname{gr} Z(\mathfrak{g})$-module $\operatorname{gr} H_\chi$ along with the fact that all scheme-theoretic fibres of f are reduced then yields $z \in \operatorname{gr} Z(\mathfrak{g})$, implying $Z(H_\chi) = Z(\mathfrak{g})$.

The Joseph ideal is prominent in several areas of representation theory, especially in the theory of minimal representations of p-adic groups. Different realisations of $\mathcal{J}_0$ can be found in the literature for various types of $\mathfrak{g}$ but most of them are ad hoc. This seems almost inevitable as outside type A the orbit $\mathcal{O}_{\min}$ is *rigid*, that is, forms a single sheet in $\mathfrak{g}^*$. Hence $\mathcal{J}_0$ cannot be obtained by parabolic induction from a primitive ideal of a proper Levi subalgebra of $\mathfrak{g}$, the only 'regular' way so far to obtain primitive ideals.

It was noticed by Savin (in a letter to Vogan) that Joseph's original proof of the uniqueness of $\mathcal{J}_0$ was incomplete. This was recently fixed by W. T. Gan and Savin with the assistance of Wallach; see [11]. The argument in [11] relies on some invariant theory and earlier results of Garfinkle. We shall see in a moment that the existence and uniqueness of $\mathcal{J}_0$ follow readily from our results; see also Remark 6.4.

6.2. Retain the assumptions and conventions of Sections 4 and 5. Set $\Bbbk_0 := H/H^+$. Since $\Bbbk_0$ is an irreducible H-module, so is the $\mathfrak{g}$-module $Q_{\chi,0} := Q_\chi \otimes_H \Bbbk_0$. So $J_0 := \mathrm{Ann}_{U(\mathfrak{g})} Q_{\chi,0}$ is a primitive ideal of $U(\mathfrak{g})$.

Proposition 6.1. *The ideal J_0 is completely prime and $\mathcal{VA}(J_0) = \overline{\mathcal{O}}_{\min}$. If $\mathfrak{g}$ is not of type* A *then J_0 is the only primitive ideal of $U(\mathfrak{g})$ with these properties, and hence J_0 is the Joseph ideal in this case.*

Proof. Theorem 5.3(2) shows that $\mathrm{rk}(U(\mathfrak{g})/J_0) = \dim_{\Bbbk} \Bbbk_0 = 1$. Hence $\mathcal{Q}(U(\mathfrak{g})/J_0)$ is a division ring. Then $U(\mathfrak{g})/J_0$ is a domain, that is, J_0 is completely prime. Theorem 3.1 gives $\mathcal{VA}(J_0) = \overline{\mathcal{O}}_{\min}$. Now suppose $\mathfrak{g}$ is not of type A. Then Corollary 4.1 implies that H has a unique one-dimensional representation. In view of Theorem 5.3 this means that $U(\mathfrak{g})$ has a *unique* completely prime primitive ideal whose associated variety is $\overline{\mathcal{O}}_{\min}$. So $J_0 = \mathcal{J}_0$ in this case. □

Remark 6.1. The existence part of our proof is hardly shorter than Joseph's original proof in [14] as it relies on the brute force computations of Section 4. However, there is a slightly different proof of the uniqueness of $\mathcal{J}_0$ which eludes Section 4 completely. We just sketch the argument leaving the details to the interested reader: If I is an ideal of codimension 1 in H then $[H, H] \subset I$. Since outside type A the Lie algebra $\mathfrak{z}_\chi(0)$ is semisimple, we have $\Theta(\mathfrak{z}_\chi(i)) \subset I$ for $i = 1, 2$. Also, $C - \mu \in I$ for some $\mu \in \Bbbk$. Using Proposition 2.1 it is not hard to observe that μ is independent of I. Therefore, H cannot afford more than one ideal of codimension 1. The rest of the proof is unchanged.

6.3. In [14], Joseph has also computed the *infinitesimal character* of $\mathcal{J}_0$, that is, the algebra homomorphism $Z(\mathfrak{g}) \to \Bbbk$ through which the centre $Z(\mathfrak{g})$ acts on the primitive quotient $U(\mathfrak{g})/\mathcal{J}_0$. We are going to use his result to determine the elusive constant c_0 from Proposition 4.1.

Theorem 6.1. *In the notation of Section* 4, *the algebra H is generated by the Casimir element C and the subspaces $\Theta(\mathfrak{z}_\chi(i))$ for $i = 0, 1$, subject to the following relations:*

(i) $[\Theta_x, \Theta_y] = \Theta_{[x,y]}$ *for all $x, y \in \mathfrak{z}_\chi(0)$;*

(ii) $[\Theta_x, \Theta_u] = \Theta_{[x,u]}$ *for all* $x \in \mathfrak{z}_\chi(0)$ *and* $u \in \mathfrak{z}_\chi(1)$*;*

(iii) C *is central in* H*;*

(iv) $[\Theta_u, \Theta_v] = \frac{1}{2}(f,[u,v])(C - \Theta_{\mathrm{Cas}} - c_0) + \frac{1}{2}\sum_{i=1}^{2s}(\Theta_{[u,z_i]^\sharp}\Theta_{[v,z_i^*]^\sharp} + \Theta_{[v,z_i^*]^\sharp}\Theta_{[u,z_i]^\sharp})$ *for all* $u, v \in \mathfrak{z}_\chi(1)$*, where* $\Theta_{\mathrm{Cas}} = \sum_i \Theta_{a_i}\Theta_{b_i}$ *is a Casimir element of the Lie algebra* $\Theta(\mathfrak{z}_\chi(0))$ *and the constant* $-c_0$ *is given in the table below:*

Type	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$-c_0$	$\frac{n(n+1)}{4}$	$\frac{(2n+1)(2n-3)}{4}$	$\frac{n(2n+1)}{8}$	$n(n-2)$	36	84	240	$\frac{39}{2}$	$\frac{28}{9}$

If $\mathfrak{g}$ *is not of type* A *then* c_0 *is the eigenvalue of* C *on the primitive quotient* $U(\mathfrak{g})/\mathcal{J}_0$*. If* $\mathfrak{g}$ *is of type* A_1 *then* $\mathfrak{z}_\chi(0) = \mathfrak{z}_\chi(1) = 0$ *and* $H = \Bbbk[C]$*.*

Proof. First we determine c_0. Recall from 4.1 that $(\gamma,\gamma) = 2$. Therefore, if $\mathfrak{g}$ is not of type C_n or G_2, then the scalar product $(\cdot,\cdot)$ on the $\mathbb{Q}$-span of P in $\mathfrak{h}^*$ coincides with the scalar product $(\cdot|\cdot)$ from Bourbaki's tables in [4]. In the remaining two cases, $(\cdot,\cdot) = \frac{1}{2}(\cdot|\cdot)$ for $\mathfrak{g}$ of type C_n, and $(\cdot,\cdot) = \frac{1}{3}(\cdot|\cdot)$ for $\mathfrak{g}$ of type G_2. Recall that for any $\lambda \in \mathfrak{h}^*$ the eigenvalue of the Casimir element C on the irreducible highest weight module $L(\lambda)$ equals $(\lambda, \lambda + 2\rho)$.

(a) Suppose $\mathfrak{g}$ is not of type A. In [14, p. 15], Joseph has found an irreducible highest weight module $L(\lambda_0)$ with $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda_0) = \mathcal{J}_0$. It is immediate from the definition of H^+ that C acts on $\Bbbk_0 = H/H^+$ as scalar c_0. But then $C_{|\mathcal{Q}_0} = c_0\,\mathrm{id}$. Proposition 6.1 now shows that C acts as c_0 id on the primitive quotient $U(\mathfrak{g})/\mathcal{J}_0$. In view of our remarks above this yields $c_0 = (\lambda_0, \lambda_0 + 2\rho)$.

If $\mathfrak{g}$ is of type E, then $\lambda_0 = -\varpi_4$; see [14]. Using parts (VI) and (VII) of Tables V–VII in [4] one finds that $c_0 = -240$ for $\mathfrak{g}$ of type E_8, $c_0 = -84$ for $\mathfrak{g}$ of type E_7, and $c_0 = -36$ for $\mathfrak{g}$ of type E_6. If $\mathfrak{g}$ is of type D_n, $n \geq 4$, then $\lambda_0 = -\varpi_{n-2}$. Parts (VI) and (VII) of Table V in [4] yield $(\varpi_{n-2}, \varpi_{n-2}) = n - 2$ and $(\varpi_{n-2}, 2\rho) = n^2 - n - 2$. Therefore, $c_0 = -n(n-2)$ in this case.

If $\mathfrak{g}$ is of type B_n, $n \geq 3$, then $\lambda_0 = -\frac{1}{2}(\varpi_{n-2} + \varpi_{n-1})$. Form Table II in [4] we get $(\lambda_0, \lambda_0) = (4n-7)/4$ and $(\lambda_0, 2\rho) = -n^2 + \frac{5}{2}$. Then $c_0 = -(n^2 - n - \frac{3}{4}) = -(2n+1)(2n-3)/4$. If $\mathfrak{g}$ is of type C_n, $n \geq 2$, then $\lambda_0 = -\frac{1}{2}\varpi_n$. Table III in [4] yields $(\lambda_0 \,|\, \lambda_0) = n/4$ and $(\lambda_0 \,|\, 2\rho) = -n(n+1)/2$. Consequently, $(\lambda_0 \,|\, \lambda_0 + 2\rho) = -n(2n+1)/4$ and $c_0 = \frac{1}{2}(\lambda_0 \,|\, \lambda_0 + 2\rho) = -n(2n+1)/8$. If $\mathfrak{g}$ is of type G_2, then $\lambda_0 = -\frac{2}{3}\varpi_2 = -\frac{2}{3}\widetilde{\alpha}$. Hence $(\lambda_0 \,|\, \lambda_0) = \frac{8}{3}$ and

$$(\lambda_0 \,|\, 2\rho) = (-\tfrac{2}{3}\varpi_2 \,|\, 10\alpha_1 + 6\alpha_2) = -4(\varpi_2 \,|\, \alpha_2) = -2(\alpha_2 \,|\, \alpha_2)\langle \varpi_2, \alpha_2\rangle = -12;$$

see [4, Table IX]. Therefore, $(\lambda_0 \,|\, \mu_0 + 2\rho) = -\frac{28}{3}$ and $c_0 = \frac{1}{3}(\lambda_0 \,|\, \lambda_0 + 2\rho) = -\frac{28}{9}$. If $\mathfrak{g}$ is of type F_4, then $\lambda_0 = -\frac{1}{2}(\varpi_1 + \varpi_2)$. Using Table VIII in [4] we get $(\lambda_0 \,|\, \lambda_0) = \frac{7}{2}$ and $(\lambda_0 \,|\, 2\rho) = -23$. Hence $c_0 = (\lambda_0, \lambda_0 + 2\rho) = (\lambda_0 \,|\, \lambda_0 + 2\rho) = -\frac{39}{2}$.

(b) Now suppose $\mathfrak{g}$ is of type A_n, $n \geq 2$. This case is more subtle because here we have an infinite family of completely prime ideals in $\mathcal{X}$ sharing the same associated variety $\overline{\mathcal{O}}_{\min}$.

In order to determine c_0 in the present case we shall have to locate a special member of this family. Theorem 5.3 will play a crucial rôle here.

Let $\mathfrak{p}_1$ be the standard parabolic subalgebra of $\mathfrak{g}$ whose Levi subalgebra is generated by $\mathfrak{h}$ and all $e_{\pm\alpha_i}$ with $2 \leq i \leq n$. For any $t \in \Bbbk$ the linear function $t\varpi_1$ vanishes on all h_{α_i} with $i \geq 2$, hence extends uniquely to a one-dimensional representation of $\mathfrak{p}_1$. Let $\Bbbk_t$ denote the corresponding one-dimensional $\mathfrak{p}_1$-module, and put $I(\mathfrak{p}_1, t) := \mathrm{Ann}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} \Bbbk_t)$. Although some of the induced $\mathfrak{g}$-modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} \Bbbk_t$ are reducible, it follows from [9, (8.5.7)] and Conze's theorem [7] that all two-sided ideals $I(\mathfrak{p}_1, t)$ are primitive and completely prime. It is not hard to check that $t_1\varpi_1$ and $t_2\varpi_1$ are conjugate under the dot action of the Weyl group $W \cong \mathfrak{S}_{n+1}$ if and only if $t_1 = t_2$. Consequently, all members of the family $\mathfrak{I}_\Bbbk := \{I(\mathfrak{p}_1, t) \mid t \in \Bbbk\}$ have pairwise distinct infinitesimal characters. In view of [13, (17.17)] they share the same associated variety $\overline{\mathcal{O}}_{\min}$.

Let $\mathcal{I} \in \mathcal{X}$ be a completely prime ideal with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$. It follows from the main result of Mœglin in [30] that there exist a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a one-dimensional representation $f\colon \mathfrak{p} \to \Bbbk$ such that $\mathcal{I} = \mathrm{Ann}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \Bbbk_f)$. In conjunction with [13, (17.17), (15.27)] this yields $\mathcal{I} \in \mathfrak{I}_\Bbbk$.

Let I_0 be the two-sided ideal of H generated by $[H, H]$. In order to describe the one-dimensional representations of H we have to take a close look at the commutative $\Bbbk$-algebra $H^{\mathrm{ab}} := H/I_0$. Given $x \in H$ we denote by $\bar{x}$ the image of x in H^{ab}. We may assume that $\mathfrak{g} = \mathfrak{sl}_{n+1}(\Bbbk)$ and $\mathfrak{h}$ is the subalgebra of all diagonal matrices in $\mathfrak{g}$. Let $\{e_{ij} \mid 1 \leq i, j \leq n+1\}$ be the matrix units in $\mathfrak{gl}_{n+1}(\Bbbk)$. We may also assume that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $e_{\alpha_i} = e_{i,i+1}$ for $1 \leq i \leq n$; see Table I in [4]. Then $e = e_{n,n+1}$, $h = e_{nn} - e_{n+1,n+1}$, and $f = e_{n+1,n}$ by our conventions in 4.1. No generality will be lost by assuming that $z_i = e_{n+1,i}$ and $z_i^* = -e_{in}$ for $1 \leq i \leq n-1$ (notice that $s = n-1$ in the present case). It is straightforward to see that the centre of the subalgebra $\mathfrak{g}(0)^\sharp = \mathfrak{z}_\chi(0)$ is spanned by the element $z := e_{nn} + e_{n+1,n+1} - \frac{2}{n+1} I_{n+1}$. Therefore, $\mathfrak{z}_\chi(0) = \Bbbk z \oplus \mathfrak{z}_\chi(0)'$ where $\mathfrak{z}_\chi(0)' = [\mathfrak{z}_\chi(0), \mathfrak{z}_\chi(0)]$. Since $\mathfrak{z}_\chi(1)$ has no zero weight relative to $\mathfrak{h}_e = \mathfrak{h} \cap \mathfrak{z}_\chi(0)$, this implies that the $\Bbbk$-algebra H^{ab} is generated by $\bar{C}$ and $\bar{\Theta}_z$.

Set $u = e_{1,n+1}$ and $v = -e_{n,1}$. We have $[u, z_i]^\sharp \in [\mathfrak{z}_\chi(0), \mathfrak{z}_\chi(0)]$ for $2 \leq i \leq s$ and $[u, z_i^*] = 0$ for $1 \leq i \leq s$. Also, $[u, z_1]^\sharp = (e_{11} - e_{n+1,n+1}) - \frac{1}{2}h = e_{11} - \frac{1}{2}(e_{nn} + e_{n+1,n+1})$. Likewise, $[v, z_1^*]^\sharp = (e_{nn} - e_{11}) - \frac{1}{2}h = -e_{11} + \frac{1}{2}(e_{nn} + e_{n+1,n+1})$. Since $(z, z) = 2 - \frac{8}{n+1} + \frac{4}{n+1} = \frac{2(n-1)}{n+1}$, we can take $a_1 = z$, $b_1 = \frac{n+1}{2(n-1)} z$, and $a_i, b_i \in \mathfrak{z}_\chi(0)'$ for $i > 1$. Next observe that $(z, e_{11} - \frac{1}{2}(e_{nn} + e_{n+1,n+1})) = -1$. As $z \perp \mathfrak{z}_\chi(0)'$, it follows that $e_{11} - \frac{1}{2}(e_{nn} + e_{n+1,n+1})$ is congruent to $-\frac{1}{(z,z)} z$ modulo $\mathfrak{z}_\chi(0)'$. As $(f, [u, v]) = 1$, Proposition 4.1 now yields

$$\frac{1}{2}\left(\bar{C} - c_0 - \frac{n+1}{2(n-1)}\bar{\Theta}_z^2\right) + \frac{1}{2} \cdot (-2) \cdot \frac{(n+1)^2}{4(n-1)^2}\bar{\Theta}_z^2 = 0.$$

As a consequence, the following relation holds in H^{ab}:

$$\bar{\Theta}_z^2 = \frac{(n-1)^2}{n(n+1)}(\bar{C} - c_0). \tag{6.3.1}$$

Since $n \geq 2$, this shows that H^{ab} is a homomorphic image of the polynomial algebra $\Bbbk[X]$. On the other hand, Theorem 5.3 in conjunction with our earlier remarks entails that the one-dimensional representations of H^{ab} are in 1-1 correspondence with the elements in $\mathfrak{I}_{\Bbbk}$. But then the maximal spectrum $\mathrm{Max}(H^{\mathrm{ab}})$ of H^{ab} is infinite, forcing $H^{\mathrm{ab}} \cong \Bbbk[X,Y]/(X^2 - \frac{(n-1)^2}{n(n+1)}Y)$ (under the algebra map $X \mapsto \bar{\Theta}_z$, $Y \mapsto \bar{C} - c_0$).

For $c \in \Bbbk$ we let $\mathrm{Max}_c(H^{\mathrm{ab}})$ (resp. $\mathfrak{I}_{\Bbbk,c}$) denote the set of all I in $\mathrm{Max}(H^{\mathrm{ab}})$ (resp. $I(\mathfrak{p}_1, t)$ in $\mathfrak{I}_{\Bbbk}$) containing $\bar{C} - c$ (resp. $C - c$). It is immediate from (6.3.1) that

$$|\mathrm{Max}_c(H^{\mathrm{ab}})| = \begin{cases} 2 & \text{when } c \neq c_0, \\ 1 & \text{when } c = c_0. \end{cases}$$

Theorem 5.3 implies that for any $c \in \Bbbk$ the map $\varkappa\colon \mathrm{Prim}\, H \to \mathfrak{X}_{\mathrm{inf}},\ I \mapsto \widetilde{I}$, induces a bijection between $\mathrm{Max}_c(H^{\mathrm{ab}})$ and $\mathfrak{I}_{\Bbbk,c}$. It is well-known that C acts on the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} \Bbbk_t$ as $(t\varpi_1, t\varpi_1 + 2\rho)\mathrm{id}$. So $I(\mathfrak{p}_1, t)$ contains $C - (t\varpi_1, t\varpi_1 + 2\rho)$. It is immediate from [4, Table I] that $(t\varpi_1, t\varpi_1 + 2\rho) = \frac{n}{n+1}t^2 + nt$. The equation $\frac{n}{n+1}t^2 + nt - c = 0$ has two distinct roots if and only if $n^2 + \frac{4nc}{n+1} \neq 0$. Therefore,

$$|\mathfrak{I}_{\Bbbk,c}| = \begin{cases} 2 & \text{when } c \neq -n(n+1)/4, \\ 1 & \text{when } c = -n(n+1)/4. \end{cases}$$

But then $\mathrm{Max}_{c_0}(H^{\mathrm{ab}})$ must be mapped onto $\mathfrak{I}_{\Bbbk,-n(n+1)/4}$, forcing $c_0 = -n(n+1)/4$.

(c) Now let $\widehat{H}$ be the associative $\Bbbk$-algebra generated by an element $\widehat{C}$ and isomorphic copies $\widehat{\Theta}(\mathfrak{z}_\chi(i))$ of the subspaces $\mathfrak{z}_\chi(i)$ with $i = 1, 2$, subject to the relations (i)–(iv) from the formulation of this theorem. Define an increasing filtration in $\widehat{H}$ by giving $\widehat{C}$ filtration degree 4, by assigning to all nonzero elements of $\widehat{\Theta}(\mathfrak{z}_\chi(i))$ filtration degree $i+2$, and by extending to $\widehat{H}$ algebraically.

Choose bases $x_1, \dots, x_q$ and $y_1, \dots, y_{2s}$ in $\mathfrak{z}_\chi(0)$ and $\mathfrak{z}_\chi(1)$, respectively, and set $X_i = \widehat{\Theta}_{x_i}$ for $1 \leq i \leq q$ and $Y_i = \widehat{\Theta}_{y_i}$ for $1 \leq i \leq 2s$. Let $\widehat{H}'$ be the $\Bbbk$-span of all monomials $m(\mathbf{a}, \mathbf{b}, l) := X_1^{a_1} \cdots X_q^{a_q} \cdot Y_1^{b_1} \cdots Y_{2s}^{b_{2s}} \cdot \widehat{C}^l$ with $a_i, b_j, l \in \mathbb{Z}_+$. Note that $m(\mathbf{a}, \mathbf{b}, l)$ has filtration degree $2\sum a_i + 3\sum b_j + 4l$. Using the relations (i)–(iv) and induction on the filtration degree of $m(\mathbf{a}, \mathbf{b}, l)$ it is straightforward to see that $\widehat{H}'$ is a two-sided ideal of $\widehat{H}$. Since $1 \in \widehat{H}'$ it must be that $\widehat{H} = \widehat{H}'$.

It follows from Proposition 4.1 and Lemmas 2.4 and 2.5 that there is a surjective algebra homomorphism $f\colon \widehat{H} \to H$ such that $f(\widehat{C}) = C$, $f(X_i) = \Theta_{x_i}$ for $i \leq q$, and $f(Y_i) = \Theta_{y_i}$ for $i \leq 2s$. Since the vectors $f(m(\mathbf{a}, \mathbf{b}, l))$ are linearly independent in H, by [31, Theorem 4.6(ii)], the equality $\widehat{H} = \widehat{H}'$ shows that f is injective. But then $\widehat{H} \cong H$, and our proof is complete. □

Remark 6.2. We have originally computed the infinitesimal character of J_0 by using a direct approach in the spirit of Section 4; this was done before we established a link between H and $\mathcal{J}_0$. Having established that link we discovered that outside type A our result was consistent with [14, p. 15]. In type A we have found two different proofs yielding the same result. This eventually convinced us that the quadratic relation of Theorem 6.1 was

correct. Our computations are rather lengthy, especially in type C, and will not be presented here.

Remark 6.3. Suppose $\mathfrak{g}$ is of type C_2. Then $\mathfrak{z}_\chi(0) \cong \mathfrak{sl}_2$ and $\mathfrak{z}_\chi(1)$ is an irreducible 2-dimensional $\mathfrak{z}_\chi(0)$-module. In this case Theorem 6.1 shows that the algebra H is generated by six elements e, h, f, u, v, c, subject to the following relations:

1. (e, h, f) is an $\mathfrak{sl}_2$-triple relative to the commutator product in H;
2. $[e, u] = 0 = [f, v],\ [e, v] = u,\ [f, u] = v,\ [h, u] = u,\ [h, v] = -v$;
3. $[u, v] = ef + fe + \frac{1}{2}h^2 - \frac{1}{2}c - \frac{5}{8}$;
4. c is central in H.

For any $t \in \Bbbk$ the factor algebra $H_t := H/(c - t)$ is isomorphic to one of the deformed symplectic oscillator algebras H_f studied by Khare in [23] (in type C_n, $n \geq 3$, the defining relations of the algebras $H/(C - t)$ differ from those of H_f). Arguing as in the proof of Theorem 6.4 in [31] one can observe that the image of $Z(\mathfrak{g})$ in H_t is isomorphic to a polynomial algebra in one variable. It is likely that the centre of H_t is generated by that image. It would be very interesting to describe the Goldie field of H in the present case. In view of Corollary 5.1(iv) this might help to resolve the Gelfand–Kirillov conjecture for $\mathfrak{g} = \mathfrak{sp}_4(\Bbbk)$.

Remark 6.4 (A. Joseph)**.** Assume $\mathfrak{g}$ is not of type A. The argument below gives a short proof of the uniqueness of $\mathcal{J}_0$ relying only on the information available at the time when [14] was written. Let (e, h, f) be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with e being a highest root vector. Let $\mathfrak{d} = \mathrm{Ker}(\mathrm{ad}\, h - \mathrm{id}) \oplus \mathrm{Ker}(\mathrm{ad}\, h - 2\mathrm{id})$, a Heisenberg Lie subalgebra of $\mathfrak{g}$, and $\mathfrak{r} = \Bbbk h \oplus \mathfrak{d}$. Let J be a completely prime primitive ideal of $U(\mathfrak{g})$ such that $\mathrm{Dim}(U(\mathfrak{g})/J) = \dim \mathcal{O}_{\min}$. Since $\mathrm{ad}\, e$ is nilpotent, $U(\mathfrak{g})/J$ embeds into its localisation U at e, which contains the localisation A of $U(\mathfrak{r})$ at e. Let Z denote the centraliser of A in U. It follows from [14, Lemma 4.1] that A is a localised Weyl algebra with $\mathrm{Dim}(A) = \dim \mathcal{O}_{\min}$. Clearly, Z inherits a filtration from $U(\mathfrak{g})$ such that $\mathrm{gr}\, Z$ is commutative. Since $\mathfrak{d}$ is the nilradical of a parabolic subalgebra of $\mathfrak{g}$, Hadziev's theorem shows that the algebra $\mathrm{gr}\, Z$ is finitely generated (one also needs the fact that $\mathrm{ad}\, h$ is semisimple). Since A is central simple, the multiplication map $Z \otimes A \to U$ is injective. Since both $\mathrm{gr}\, Z$ and $\mathrm{gr}\, A$ are commutative and finitely generated, we have $\mathrm{Dim}(Z \otimes A) = \mathrm{Dim}(Z) + \mathrm{Dim}(A)$. As $\mathrm{Dim}(U) = \mathrm{Dim}(A)$, we get $\mathrm{Dim}(Z) = 0$. Hence Z is algebraic over $\Bbbk$. Since U is a domain, we now obtain $Z = \Bbbk$. Since $\mathfrak{d}$ consists of nilpotent elements of $\mathfrak{g}$, Taylor's lemma proved in [15] implies that for any h-weight vector $u \in U$ there is an h-weight vector $a \in A$ of the same weight as u such that $u - a$ commutes with the image of $\mathfrak{d}$ in U (a preprint version of [15] was available since 1973 and is quoted in [14]). Taking u to be the image of f in U we get $e(f - a) \in Z = \Bbbk$. But then $f \in A$ and so $U = A$, by the simplicity of $\mathfrak{g}$. Now apply [14, Theorem 4.3] to deduce the equality $J = \mathcal{J}_0$.

Lemma 6.1. *If $\mathfrak{g}$ is of type G_2 then the algebra H admits a 2-dimensional irreducible representation ρ such that $\rho(C) = -\frac{16}{9}\,\mathrm{id}$ and $\rho(\Theta_u) = 0$ for all $u \in \mathfrak{z}_\chi(1)$.*

Proof. For any $\alpha = m\alpha_1 + n\alpha_2 \in \Phi^+$ we set $e_{m,n} = e_\alpha$, $h_{m,n} = h_\alpha$, and $f_{m,n} = e_{-\alpha}$. Recall from (4.1) that $\beta = \alpha_2$. It is easy to see that in the present case $\mathfrak{z}_\chi(0) \cong \mathfrak{sl}_2$

and $\mathfrak{z}_\chi(1)$ is an irreducible 4-dimensional $\mathfrak{z}_\chi(0)$-module. Furthermore, $\mathfrak{z}_\chi(0) = \Bbbk e_{2,1} \oplus \Bbbk h_{2,1} \oplus \Bbbk f_{2,1}$. We can assume, after possibly an admissible sign change, that $z_1 = f_{3,2}$, $z_1^* = e_{3,1}$, $z_2 = f_{1,1}$, and $z_2^* = \frac{1}{3}e_{1,0}$. Then $[e_{3,1}, f_{3,2}] = \frac{1}{3}[e_{1,0}, f_{1,1}] = f_{0,1} = f$. Put $u_1^* = e_{3,2}$, $u_1 = -f_{3,1}$, $u_2^* = e_{1,1}$, and $u_2 = af_{1,0}$, where $a \in \Bbbk^\times$. Clearly, $\mathfrak{z}_\chi(1)$ is spanned by the u_1, u_2, u_1^*, u_2^*. Since

$$\begin{aligned}([e_{3,2}, -f_{3,1}], [f_{3,2}, e_{3,1}]) &= (e_{3,2}, [f_{3,2}, h_{3,1}]) = -(e_{3,2}, [h_{3,1}, f_{3,2}])\\ &= (e_{3,2}, f_{3,2}) = 1,\end{aligned}$$

there is $a \in \Bbbk^\times$ such that $[u_i, u_j] = [u_i^*, u_j^*] = 0$ and $[u_i^*, u_j] = -\delta_{ij}e$ for $1 \le i, j \le 2$.

Let $\{E, H, F\}$ be the standard basis of $\mathfrak{sl}_2(\Bbbk)$ and $\mathcal{E} = \Theta(\mathfrak{z}_\chi(0)) \cup \Theta(\mathfrak{z}_\chi(1)) \cup \{C\}$. Let $\rho\colon \mathcal{E} \to \mathrm{Mat}_2(\Bbbk)$ be such that $\rho(\Theta_{xe_{2,1}+yh_{2,1}+zf_{2,1}}) = xE + yH + zF$, $\rho(\Theta_u) = 0$, and $\rho(C) = -\frac{16}{9}I_2$ for all $x, y, z \in \Bbbk$ and $u \in \mathfrak{z}_\chi(1)$. We claim that the elements from $\rho(\mathcal{E})$ satisfy the relations (i)–(iv) of Theorem 6.1. Since the relations (i)–(iii) are satisfied for obvious reasons, we just need to check the quadratic relation (iv).

Since $2\alpha_1+\alpha_2$ is a short root, we have $(h_{2,1}, h_{2,1}) = 6$ and $(e_{2,1}, f_{2,1}) = 3$. Therefore, $C_0 = \frac{1}{3}(e_{2,1}f_{2,1} + f_{2,1}e_{2,1} + \frac{1}{2}h_{2,1}^2)$. Note that $EF + FE + \frac{1}{2}H^2 = \frac{3}{2}I_2$. Since $c_0 = -\frac{28}{9}$ in the present case, we have to show that

$$\sum_{i=1}^{4}(\rho(\Theta_{[u,z_i]^\sharp})\rho(\Theta_{[v,z_i^*]^\sharp}) + \rho(\Theta_{[v,z_i^*]^\sharp})\rho(\Theta_{[u,z_i]^\sharp})) = -\tfrac{5}{6}(f, [u, v])I_2 \quad (6.3.2)$$

for all $u, v \in \mathfrak{z}_\chi(1)$. When u and v run through the set $\{u_1, u_2, u_1^*, u_2^*\}$, the LHS of (6.3.2) is always a linear combination of matrices $XY + YX$ with $X, Y \in \mathfrak{sl}_2(\Bbbk)$. Since all such matrices are multiples of I_2, the LHS of (6.3.2) equals $g(u, v)I_2$ for some skew-symmetric bilinear form g on $\mathfrak{z}_\chi(1)$. Using the relations (i) and (ii) of Theorem 6.1 it is easy to observe that this form is $\mathfrak{z}_\chi(0)$-invariant. As $\mathfrak{z}_\chi(1)$ is an irreducible $\mathfrak{z}_\chi(0)$-module, there is a scalar $c \in \Bbbk$ such that $g(u, v) = c(f, [u, v])$ for all $u, v \in \mathfrak{z}_\chi(1)$. Thus we need to check that $c = -\frac{5}{6}$.

Note that $[u_1^*, z_i^*] = [u_1, z_i] = 0$ for $i = 1, 2$. Also, $[u_1^*, z_2]^\sharp = [u_1^*, z_2]$ and $[u_1, z_2^*]^\sharp = [u_1, z_2^*]$. It follows that for $i = 1, 2$,

$$\begin{aligned}([u_1^*, z_i], [u_1, z_i^*]) &= (u_1^*, [u_1, [z_i, z_i^*]]) = -(u_1^*, [u_1, f]) = -([u_1^*, u_1], f)\\ &= (e, f) = 1.\end{aligned}$$

As $(u_1^*, z_1) = -(u_1, z_1^*) = 1$, we have $[u_1^*, z_1]^\sharp = [u_1^*, z_1] - \frac{1}{2}h$ and $[u_1, z_1^*]^\sharp = [u_1, z_1^*] + \frac{1}{2}h$. Consequently,

$$\begin{aligned}([u_1^*, z_1]^\sharp, [u_1, z_1^*]^\sharp) &= ([u_1^*, z_1], [u_1, z_1^*]) + \tfrac{1}{2}([u_1^*, z_1], h) - \tfrac{1}{2}(h, [u_1, z_1^*]) - \tfrac{1}{4}(h, h)\\ &= 1 + \tfrac{1}{2}(u_1^*, z_1) - \tfrac{1}{2}(u_1, z_1^*) - \tfrac{1}{2} = 2 - \tfrac{1}{2} = \tfrac{3}{2}.\end{aligned}$$

Since $[u_1^*, z_1]^\sharp$, $[u_1^*, z_2]$, $[u_1, z_1^*]^\sharp$, and $[u_1, z_2^*]$ are multiples of $h_{2,1}$, $e_{2,1}$, $h_{2,1}$, and $f_{2,1}$, respectively, and $(e_{2,1}, f_{2,1}) = \frac{1}{2}(h_{2,1}, h_{2,1}) = 3$, the preceding remarks show that

$$\sum_{i=1}^{4}(\rho(\Theta_{[u_1^*,z_i]^\sharp})\rho(\Theta_{[u_1,z_i^*])^\sharp}) + \rho(\Theta_{[u_1,z_i^*]^\sharp})\rho(\Theta_{[u_1^*,z_i]^\sharp})) = \tfrac{1}{3}(EF + FE) + \tfrac{1}{2}H^2 = \tfrac{5}{6}I_2.$$

As $(f, [u_1^*, u_1]) = -1$, we deduce from (6.3.2) that $c = -\frac{5}{6}$, as wanted. As Theorem 6.1 gives a presentation of H by generators and relations, the result follows. □

Remark 6.5. It is immediate from the proof of Lemma 6.1 that for $\mathfrak{g}$ of type G_2 the following relation holds in H:

$$[\Theta_{u_1^*}, \Theta_{u_1}] = -\tfrac{1}{2}C + \tfrac{1}{3}(\Theta_{e_{2,1}}\Theta_{f_{2,1}} + \Theta_{f_{2,1}}\Theta_{e_{2,1}} + \tfrac{1}{2}\Theta_{h_{2,1}}^2) + \tfrac{1}{6}\Theta_{h_{2,1}}^2 - \tfrac{14}{9}. \qquad (6.3.3)$$

The expressions for all $[\Theta_u, \Theta_v]$ with $u, v \in \mathfrak{z}_\chi(1)$ can be derived from (6.3.3) by using the action of $\operatorname{ad}\Theta(\mathfrak{z}_\chi(0))$ on $\Theta(\mathfrak{z}_\chi(i))$, $i = 1, 2$. For example, it can be deduced easily that $[\Theta_{u_1^*}, \Theta_{u_2^*}]$ is a nonzero scalar multiple of $\Theta_{e_{2,1}}^2$. This implies that the span of all PBW monomials in C, $\Theta_{h_{2,1}}$, $\Theta_{e_{2,1}}$, $\Theta_{u_1^*}$, $\Theta_{u_2^*}$ is a subalgebra of H. It can be regarded as a Borel subalgebra of H.

6.4. As yet another application of Theorems 5.3 and 6.1 we are going to classify all irreducible finite-dimensional representations of H in the case where $\mathfrak{g}$ is of type C_n or G_2. Dimension formulae for these representations will be given. We shall rely on Joseph's theory of Goldie-rank polynomials. The reader will notice that our method is quite general and can be applied to any simple Lie algebra $\mathfrak{g}$. However, various problems remain in the general case, especially for Lie algebras of type E_7 and E_8. We hope to return to this interesting subject in the future.

Given $\nu \in \mathfrak{h}^*$ we denote by $I(\nu)$ the annihilator of the irreducible highest weight module $L(\nu)$ in $U(\mathfrak{g})$. Recall from the proof of Theorem 6.1 that $\lambda_0 = -\frac{1}{2}\varpi_n$ for $\mathfrak{g}$ of type C_n and $\lambda_0 = -\frac{2}{3}\varpi_2$ for $\mathfrak{g}$ of type G_2. Let $\Phi_0 = \{\alpha \in \Phi \mid \langle\lambda_0, \alpha\rangle \in \mathbb{Z}\}$. It is easy to see that Φ_0 coincides with the set of all *short* roots in Φ. In particular, Φ_0 is a root system in $\mathfrak{h}^*$ but not a closed subsystem of Φ. The set $\Pi_0 = \{\alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1} + \alpha_n\}$ is the basis of simple roots in Φ_0 contained in Φ^+. This implies that Φ_0 is of type D_n and A_2 when $\mathfrak{g}$ is of type C_n and G_2, respectively (our convention here is that $D_2 \cong A_1 \times A_1$ and $D_3 \cong A_3$). Note that $\lambda_0 = -\frac{d-1}{d}\varpi_n$ where

$$d = \frac{(\alpha_n, \alpha_n)}{(\alpha_{n-1}, \alpha_{n-1})} = \begin{cases} 2 & \text{when } \mathfrak{g} \text{ is of type } C_n, \\ 3 & \text{when } \mathfrak{g} \text{ is of type } G_2. \end{cases}$$

It is well-known that the subgroup $W_0 := \{w \in W \mid w(\lambda_0) - \lambda_0 \in \mathbb{Z}\Phi\}$ of W is generated by the reflections s_α with $\alpha \in \Phi_0$, hence identifies with the Weyl group of Φ_0. We note for further references that

$$\begin{aligned} \langle\lambda_0 + \mu + \rho, \alpha_{n-1} + \alpha_n\rangle &= \langle\lambda_0 + \mu + \rho, \alpha_{n-1}\rangle + d\langle\lambda_0 + \mu + \rho, \alpha_n\rangle \\ &= \langle\mu, \alpha_{n-1}\rangle + 1 + d\left(\langle\mu, \alpha_n\rangle - \frac{d-1}{d} + 1\right) \\ &= \langle\mu, \alpha_{n-1}\rangle + d\langle\mu, \alpha_n\rangle + 2. \end{aligned}$$

Theorem 6.2. *Let Φ_0^+ denote the set of all short roots in Φ^+.*

1. *If $\mathfrak{g}$ is of type C_n, $n \geq 2$, then to every $\mu \in P^+$ there corresponds a finite-dimensional simple H-module $V_H(\mu)$ such that*

$$\dim V_H(\mu) = \prod_{\alpha \in \Phi_0^+} \frac{\langle 2\mu + 2\rho - \varpi_n, \alpha \rangle}{\langle 2\rho - \varpi_n, \alpha \rangle}.$$

   Any finite-dimensional simple H-module is isomorphic to one of the modules $V_H(\mu)$, $\mu \in P^+$. The central characters of these modules are pairwise distinct.

2. *If $\mathfrak{g}$ is of type G_2, then to every $\mu = a\varpi_1 + b\varpi_2 \in P^+$ there correspond two finite-dimensional simple H-modules $V_H^{\pm}(\mu)$ such that*

$$\dim V_H^+(\mu) = \frac{(a+1)(a+3b+2)(2a+3b+3)}{6},$$
$$\dim V_H^-(\mu) = \frac{(a+1)(a+3b+3)(2a+3b+4)}{6}.$$

   Any finite-dimensional simple H-module is isomorphic to one of the modules $V_H^{\pm}(\mu)$, $\mu \in P^+$. The central characters of these modules are pairwise distinct.

Proof. By Theorem 5.3 the isoclasses of finite-dimensional simple H-modules are in 1-1 correspondence with the primitive ideals $\mathcal{I}$ of $U(\mathfrak{g})$ such that $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$. By Duflo's theorem, $\mathcal{I} = I(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. Let $\Phi_\lambda = \{\alpha \in \Phi \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$ and let Π_λ be the basis of simple roots of Φ_λ contained in $\Phi_\lambda \cap \Phi^+$. As explained in [19, p. 41], the equality $\mathcal{VA}(I(\lambda)) = \overline{\mathcal{O}}_{\min}$ holds if and only if $\dim \mathcal{O}_{\min} = |\Phi| - |\Phi_\lambda|$ and $\langle \lambda + \rho, \alpha \rangle > 0$ for all $\alpha \in \Pi_\lambda$ (the argument in [19] relies on the fact that $\mathcal{O}_{\min}$ is not a special orbit in the sense of Lusztig when $\mathfrak{g}$ is not simply laced). Since $\dim \mathcal{O}_{\min} = |\Phi| - |\Phi_0|$, we have $|\Phi_0| = |\Phi_\lambda|$. Now $\Phi_\lambda^\vee$ is a closed symmetric subsystem of the dual root system $\Phi^\vee \subset \mathfrak{h}$. The Borel–de Siebenthal algorithm implies that there is only one such subsystem in $\Phi^\vee$ of size $|\Phi_0|$, namely $\Phi_0^\vee$; see [4, Ch. VI, Sect. 4, Exerc. 4]. This shows $\Phi_\lambda = \Phi_0$ and $\Pi_\lambda = \Pi_0$.

Write $\lambda = \lambda_0 + \sum_{i=1}^n l_i \varpi_i$ with $l_i \in \Bbbk$. Since $\alpha_1, \dots, \alpha_{n-1} \in \Pi_0$, it must be that $l_i \in \mathbb{Z}_+$ for $1 \leq i \leq n-1$, while our earlier remarks show that $\langle \lambda + \rho, \alpha_{n-1} + \alpha_n \rangle = l_{n-1} + dl_n + 2$ is a positive integer. Hence $l_n \in \frac{1}{d}\mathbb{Z}$. Since $\Phi_\lambda \neq \Phi$ we have $\lambda \notin P$, giving $l_n \notin -\frac{1}{d} + \mathbb{Z}$. For $\mathfrak{g}$ of type C_n this says $l_n \in \mathbb{Z}$, while for $\mathfrak{g}$ of type G_2 we infer that either $l_n \in \mathbb{Z}$ or $l_n \in \frac{1}{3} + \mathbb{Z}$.

It is easy to see that s_{α_n} permutes the positive short roots in Φ. Therefore, $I(\lambda) = I(s_{\alpha_n} \cdot \lambda)$; see [13, (5.16)]. As $\alpha_n = -d\varpi_{n-1} + 2\varpi_n$ we have

$$\begin{aligned} s_{\alpha_n} \cdot \lambda &= s_{\alpha_n}\Big(\sum_{i=1}^{n-1}(l_i+1)\varpi_i + \Big(l_n + \frac{1}{d}\Big)\varpi_n\Big) - \rho = \lambda - \frac{dl_n+1}{d}\alpha_n \\ &= \lambda_0 + \sum_{i=1}^{n-2} l_i\varpi_i + (l_{n-1} + dl_n + 1)\varpi_{n-1} + \Big(l_n - \frac{2}{d}(dl_n+1)\Big)\varpi_n \\ &= \lambda_0 + \sum_{i=1}^{n-2} l_i\varpi_i + (l_{n-1} + dl_n + 1)\varpi_{n-1} - \Big(l_n + \frac{2}{d}\Big)\varpi_n. \end{aligned}$$

Thus replacing λ by $s_{\alpha_n} \cdot \lambda$ if necessary we may assume further that $l_n \geq 0$.

Suppose $l_n \in \mathbb{Z}$. Then the above discussion shows that $\lambda - \lambda_0 \in P^+$. It follows from Joseph's theory of Goldie-rank polynomials that

$$\mathrm{rk}(U(\mathfrak{g})/I(\lambda_0+\mu)) = c \prod_{\alpha\in\Phi_0^+} \langle \lambda_0+\mu+\rho, \alpha\rangle$$

for all $\mu \in P^+$, where c is a constant independent of μ; see [17, p. 303]. Recall that Theorem 5.3 associates to each $\mathcal{I} \in \mathfrak{X}$ with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}_{\min}$ an irreducible finite-dimensional H-module (up to isomorphism). Abusing notation we denote this module by $\varkappa^{-1}(\mathcal{I})$. We have already mentioned that $\mathcal{VA}(I(\lambda_0+\mu)) = \overline{\mathcal{O}}_{\min}$. Therefore, to each $\mu \in P^+$ there corresponds an irreducible finite-dimensional H-module $V_H(\mu) := \varkappa^{-1}(I(\lambda_0+\mu))$. By Theorem 5.3(2),

$$\dim V_H(\mu) = \mathrm{rk}(U(\mathfrak{g})/I(\lambda_0+\mu)) \quad (\forall \mu \in P^+).$$

Since $I(\lambda_0)$ is the Joseph ideal, Theorem 5.3 together with Proposition 6.1 gives $\dim V_H(0) = \dim H/H^+ = 1$. Therefore, $c^{-1} = \prod_{\alpha\in\Phi_0^+}\langle\lambda_0+\rho,\alpha\rangle$ and

$$\dim V_H(\mu) = \prod_{\alpha\in\Phi_0^+} \frac{\langle\lambda_0+\mu+\rho,\alpha\rangle}{\langle\lambda_0+\rho,\alpha\rangle}. \tag{6.4.1}$$

Since $\lambda_0+\rho+P^+$ is contained in the interior of the dominant Weyl chamber, the modules in the set $\{V_H(\mu) \mid \mu \in P_+\}$ have pairwise distinct central characters. This settles the case where $\mathfrak{g}$ is of type C_n.

Suppose $\mathfrak{g}$ of type G_2. For $\mu = a\varpi_1 + b\varpi_2 \in P^+$ we put $V_H^+(\mu) := V_H(\mu)$. Since $\Phi_0^+ = \{\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$ and $\lambda_0 = -\frac{2}{3}\varpi_2$, the dimension formula (6.4.1) reads

$$\dim V_H^+(\mu) = \frac{(a+1)(a+3b+2)(2a+3b+3)}{6}.$$

Now suppose $l_n \notin \mathbb{Z}$. Our earlier remarks show that $\mathfrak{g}$ is of type G_2 and $l_n \in \frac{1}{3}+\mathbb{Z}_+$. As a consequence, $\lambda \in \frac{1}{2}\lambda_0 + P^+$. For any $\mu \in P^+$ we have $\Phi_{\frac{1}{2}\lambda_0+\mu} = \Phi_0$. As $\frac{1}{2}\lambda_0 + \mu + \rho$ lies in the interior of the dominant Weyl chamber, the above argument applies, yielding $\mathcal{VA}(I(\frac{1}{2}\lambda_0+\mu)) = \overline{\mathcal{O}}_{\min}$. Theorem 5.3 shows that $V_H^-(\mu) := \varkappa^{-1}(I(\frac{1}{2}\lambda_0+\mu))$ is an irreducible finite-dimensional H-module with

$$\dim V_H^-(\mu) = \mathrm{rk}(U(\mathfrak{g})/I(\tfrac{1}{2}\lambda_0+\mu)).$$

In conjunction with the discussion in [17, p. 303] this entails that

$$\dim V_H^-(\mu) = \mathrm{rk}(U(\mathfrak{g})/I(\tfrac{1}{2}\lambda_0+\mu)) = c' \prod_{\alpha\in\Phi_0^+} \langle \tfrac{1}{2}\lambda_0+\mu+\rho,\alpha\rangle \tag{6.4.2}$$

for all $\mu \in P^+$ where c' is a constant independent of μ.

It also follows from Theorem 5.3 that C acts on $V_H^{\pm}(\mu)$ as $f^{\pm}(\mu)\,\mathrm{id}$ where $f^+(\mu) = (\lambda_0+\mu, \lambda_0+\mu+2\rho)$ and $f^-(\mu) = (\frac{1}{2}\lambda_0+\mu, \frac{1}{2}\lambda_0+\mu+2\rho)$. We have noted in the proof of Theorem 6.1 that $(\cdot,\cdot) = \frac{1}{3}(\cdot\,|\,\cdot)$. By [4, Table IX], $(\varpi_1\,|\,\varpi_1) = 2$, $(\varpi_1\,|\,\varpi_2) = 3$, and $(\varpi_2\,|\,\varpi_2) = 6$. Using this fact it is straightforward to see that $f^-(\mu) = f^-(a\varpi_1 + b\varpi_2)$ is a quadratic polynomial in a, b with all coefficients positive except for the constant term $f^-(0) = (-\frac{1}{3}\varpi_2, -\frac{1}{3}\varpi_2 + 2\rho) = -\frac{16}{9}$. Furthermore,

$$f^+(a\varpi_1 + b\varpi_2) = \tfrac{2}{3}a^2 + 2b^2 + 2ab + 2a + \tfrac{10}{3}b - \tfrac{28}{9}.$$

As a consequence, $f^+(\mu) > -\frac{16}{9}$ for all nonzero $\mu \in P^+$.

Let M be an H-module affording the representation ρ from Lemma 6.1. The above discussion shows that $M \cong V^{\pm}(\nu)$ for some $\nu \in P^+$. Since C acts on M as $-\frac{16}{9}\,\mathrm{id}$, the preceding remark yields $M \cong V_H^-(0)$. As $\dim M = 2$, this allows us to determine the scale factor c'. In view of (6.4.2) we then get

$$\dim V_H^-(\mu) = 2\prod_{\alpha\in\Phi_0^+} \frac{\langle -\frac{1}{3}\varpi_2 + \mu + \rho, \alpha\rangle}{\langle -\frac{1}{3}\varpi_2 + \rho, \alpha\rangle} = \frac{(a+1)(a+3b+3)(2a+3b+4)}{6}$$

for all $\mu \in P^+$. Since $(-\frac{2}{3}\varpi_2 + \rho + P^+) \cap (-\frac{1}{3}\varpi_2 + \rho + P^+) = \emptyset$ and the union $(-\frac{2}{3}\varpi_2 + \rho + P^+) \cup (-\frac{1}{3}\varpi_2 + \rho + P^+)$ is contained in the interior of the dominant Weyl chamber, the modules in the set $\{V_H^{\pm}(\mu) \mid \mu \in P^+\}$ have pairwise distinct central characters. This completes the proof. □

Remark 6.6. When $\mathfrak{g}$ is of type C_2 we have $\Phi_0^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$, $\varpi_1 = \varepsilon_1$, $\varpi_2 = \varepsilon_1 + \varepsilon_2$, and $\lambda_0 = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2)$; see [4, Table III]. In this case our dimension formula reads

$$\dim V_H(\mu) = \frac{(a+1)(a+2b+2)}{2} = \frac{(r-s+1)(r+s+2)}{2},$$

where $\mu = a\varpi_1 + b\varpi_2 = (a+b)\varepsilon_1 + b\varepsilon_2 = r\varepsilon_1 + s\varepsilon_2$ and $r, s \in \mathbb{Z}_+$, $r \geq s$. The same dimension formula can be found in [23] where it was obtained by a completely different method in the context of deformed symplectic oscillator algebras of rank one; see Remark 6.3.

7. Highest weight modules for H

7.1. Let Φ_e denote the set of all $\alpha \in \Phi$ with $\alpha(h) \in \{0, 1\}$, and put $\Phi_e^{\pm} = \Phi_e \cap \Phi^{\pm}$, $\Phi_{e,i}^{\pm} = \{\alpha \in \Phi_e^{\pm} \mid \alpha(h) = i\}$. Recall that $\mathfrak{z}_\chi$ is spanned by $\mathfrak{h}_e$, by all e_α with $\alpha \in \Phi_e$, and by e. Let $h_1, \dots, h_{l-1}$ be a basis of $\mathfrak{h}_e$, and let $\mathfrak{n}^{\pm}(i)$ be the span of all e_α with $\alpha \in \Phi_{e,i}^{\pm}$. Clearly, $\mathfrak{n}^+(0)$ and $\mathfrak{n}^-(0)$ are maximal nilpotent subalgebras of $\mathfrak{g}(0)^\natural$. Let $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$ be bases of $\mathfrak{n}^+(0)$ and $\mathfrak{n}^-(0)$ consisting of root vectors e_α with $\alpha \in \Phi$. Recall that the z_i's with $1 \leq i \leq 2s$ are root vectors for $\mathfrak{h}$. For $1 \leq i \leq s$, set $u_i = [e, z_i]$ and $u_i^* = [e, z_i^*]$. It follows from our discussion in 4.1 that u_i (resp. u_i^*) is a root vector for $\mathfrak{h}$ corresponding to the root $\beta + \gamma_i \in \Phi_{e,1}^-$ (resp. $\beta + \gamma_i^* \in \Phi_{e,1}^+$). Furthermore, $\{u_1, \dots, u_s, u_1^*, \dots, u_s^*\}$ is a $\Bbbk$-basis of $\mathfrak{z}_\chi(1)$.

Given a linear function λ on $\mathfrak{h}_e$ and $c \in \Bbbk$ we denote by $J_{\lambda,c}$ the linear span in H of all PBW monomials of the form

$$\prod_{i=1}^{t} \Theta_{y_i}^{l_i} \cdot \prod_{i=1}^{s} \Theta_{u_i}^{m_i} \cdot \prod_{i=1}^{\ell-1} (\Theta_{h_i} - \lambda(h_i))^{n_i} \cdot (C-c)^{n_\ell} \cdot \prod_{i=1}^{s} \Theta_{u_i^*}^{r_i} \cdot \prod_{i=1}^{t} \Theta_{x_i}^{q_i},$$

where $\sum_{i=1}^{\ell} n_i + \sum_{i=1}^{t} r_i + \sum_{i=1}^{s} q_i > 0$.

Lemma 7.1. *The subspace $J_{\lambda,c}$ is a left ideal of the algebra H.*

Proof. For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^t$, $\mathbf{c}, \mathbf{d} \in \mathbb{Z}_+^s$, $\mathbf{m} \in \mathbb{Z}_+^\ell$, set

$$\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}) := \Big(\prod_{i=1}^{t} \Theta_{y_i}^{a_i}\Big)\Big(\prod_{i=1}^{s} \Theta_{u_i}^{c_i}\Big)\Big(\prod_{i=1}^{\ell-1} \Theta_{h_i}^{m_i}\Big) C^{m_l} \Big(\prod_{i=1}^{s} \Theta_{u_i^*}^{d_i}\Big)\Big(\prod_{i=1}^{t} \Theta_{x_i}^{b_i}\Big).$$

By [31, Theorem 4.6(ii)], the PBW monomials $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m})$ form a $\Bbbk$-basis of H. Note that $\deg_e(\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m})) = 4m_\ell + 3(|\mathbf{c}| + |\mathbf{d}|) + 2(|\mathbf{a}| + |\mathbf{b}|) + 2\sum_{i=1}^{\ell-1} m_i$.

Since $C - c$ is central in H we have $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m})(C - c) \in J_{\lambda,c}$. Relations (i) and (ii) of Theorem 6.1 imply that $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m})(\Theta_{h_i} - \lambda(h_i)) \in J_{\lambda,c}$ for $1 \le i \le \ell - 1$. Since $\Theta(\mathfrak{n}^+(0))$ is a Lie subalgebra of $\Theta(\mathfrak{z}_\chi(0))$, by Theorem 6.1, we also have $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}) \cdot \Theta_{e_\alpha} \in J_{\lambda,c}$ for all $\alpha \in \Phi_{e,0}^+$.

It remains to show that $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}) \cdot \Theta_{u_i^*} \in J_{\lambda,c}$ for all $i \le s$. We shall use induction on $\deg_e(\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}))$, so assume from now on that $\deg_e(\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m})) = N$ and $H^k \cdot \Theta_{u_i^*} \in J_{\lambda,c}$ for all $i \le s$ and all $k < N$. First note that the span of $u_1^*, \dots, u_s^*$ equals $\mathfrak{n}^+(1)$, hence is stable under the adjoint action of $\mathfrak{n}^+(0)$. Since we have already established that $H \cdot \Theta_{e_\alpha} \in J_{\lambda,c}$ for all $\alpha \in \Phi_{e,0}$, relation (ii) of Theorem 6.1 yields

$$\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{m}) \cdot \Theta_{u_i^*} \in \Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{0}, \mathbf{m}) \cdot \Theta(\mathfrak{n}^+(1)) + J_{\lambda,c}.$$

Thus we may assume that $\mathbf{d} = \mathbf{0}$. If $b_j = 0$ for all $j > i$, then $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{0}, \mathbf{m}) \cdot \Theta_{u_i^*} = \Theta(\mathbf{a}, \mathbf{b} + \mathbf{e}_i, \mathbf{c}, \mathbf{0}, \mathbf{m}) \in J_{\lambda,c}$. So suppose $\mathbf{b} = (b_1, \dots, b_k, 0, \dots, 0)$ where $b_k > 0$ and $k > i$. Then in view of [31, Theorem 4.6(iv)] and our induction assumption we have

$$\begin{aligned}\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{0}, \mathbf{m}) \cdot \Theta_{u_i^*} \in{}& \Theta(\mathbf{a}, \mathbf{b} + \mathbf{e}_i, \mathbf{c}, \mathbf{0}, \mathbf{m}) + \Theta(\mathbf{a}, \mathbf{b} - \mathbf{e}_k, \mathbf{c}, \mathbf{0}, \mathbf{m})[\Theta_{u_k^*}, \Theta_{u_i^*}] \\ &+ H^{N-2} \cdot \Theta_{u_k^*} \subseteq \Theta(\mathbf{a}, \mathbf{b} - \mathbf{e}_k, \mathbf{c}, \mathbf{0}, \mathbf{m})[\Theta_{u_k^*}, \Theta_{u_i^*}] + J_{\lambda,c}.\end{aligned}$$

Since $(f, [u_k^*, u_i^*]) = 0$, Theorem 6.1 shows that

$$[\Theta_{u_k^*}, \Theta_{u_i^*}] = \frac{1}{2} \sum_{j=1}^{2s} (\Theta_{[u_k^*, z_j]^\sharp} \Theta_{[u_i^*, z_j^*]^\sharp} + \Theta_{[u_i^*, z_j^*]^\sharp} \Theta_{[u_k^*, z_j]^\sharp}) \in \sum_{\alpha \in \Phi_{e,0}^+} H \cdot \Theta_{e_\alpha}$$

(one should take into account that $[u_i^*, z_j^*], [[u_i^*, z_j^*]^\sharp, [u_k^*, z_j]^\sharp] \in \bigcup_{\alpha \in \Phi_{e,0}^+} \Bbbk e_\alpha$ for all $j \le s$). So $\Theta(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{0}, \mathbf{m}) \cdot \Theta_{u_i^*} \in J_{\lambda,c}$, and the result follows by induction on N. □

7.2. Put $Z_H(\lambda, c) := H/J_{\lambda,c}$ and let v_0 denote the image of 1 in $Z_H(\lambda, c)$. Clearly, $Z_H(\lambda, c)$ is a cyclic H-module generated by v_0. We call $Z_H(\lambda, c)$ the *Verma module of level c corresponding to* λ. By Lemma 7.1, the vectors

$$\{\Theta_{y_1}^{l_1} \cdots \Theta_{y_t}^{l_t} \Theta_{u_1}^{m_1} \cdots \Theta_{u_s}^{m_s}(v_0) \mid l_1, \ldots, l_t, m_1, \ldots, m_s \in \mathbb{Z}_+\}$$

form a $\Bbbk$-basis of the Verma module $Z_H(\lambda, c)$. Let $Z_H^+(\lambda, c)$ denote the $\Bbbk$-span of all $\{\Theta_{y_1}^{l_1} \cdots \Theta_{y_t}^{l_t} \Theta_{u_1}^{m_1} \cdots \Theta_{u_s}^{m_s}(v_0)$ with $\sum_i l_i + \sum_i m_i > 0$. Let $Z_H^{\max}(\lambda, c)$ denote the sum of all H-submodules of $Z_H(\lambda, c)$ contained in $Z_H^+(\lambda, c)$, and put

$$L_H(\lambda, c) := Z_H(\lambda, c)/Z_H^{\max}(\lambda, c).$$

Proposition 7.1. *The following are true:*

(i) *$Z_H^{\max}(\lambda, c)$ is a unique maximal submodule of the Verma module $Z_H(\lambda, c)$ and hence $L_H(\lambda, c)$ is a simple H-module.*

(ii) *The simple H-modules $L_H(\lambda, c)$ and $L_H(\lambda', c')$ are isomorphic if and only if $\lambda = \lambda'$ and $c = c'$.*

(iii) *Any finite-dimensional simple H-module is isomorphic to one of the modules $L_H(\lambda, c)$ with $\lambda \in \mathfrak{h}_e^*$ satisfying $\lambda(h_\alpha) \in \mathbb{Z}_+$ for all $\alpha \in \Phi_{e,0}^+$. Furthermore, if $\mathfrak{g}$ is not of type* A *then c is a rational number.*

Proof. (a) For a root $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$ in Φ we put

$$\mathrm{ht}_\beta(\alpha) := \sum_{\alpha_i \neq \beta} n_i.$$

Clearly, $\mathrm{ht}_\beta(\alpha) = 0$ if and only if $\alpha = \pm\beta$. As all derivations of $\mathfrak{g}$ are inner, there is a unique $h_0 \in \mathfrak{h}$ such that $[h_0, e_\alpha] = \mathrm{ht}_\beta(\alpha) e_\alpha$ for all $\alpha \in \Phi$. As $[h_0, e_{\pm\beta}] = 0$ we see that $h_0 \in \mathfrak{h}_e$. Clearly, $\Theta_{h_0}(v_0) = \lambda(h_0)v_0$ and $Z_H(\lambda, c) = \Bbbk v_0 \oplus Z_H^+(\lambda, c)$. Since all y_i and z_i are root vectors for $\mathfrak{h}$ corresponding to negative roots different from $-\beta$, it follows from Theorem 6.1 that the subspace $Z_H^+(\lambda, c)$ decomposes into eigenspaces for Θ_{h_0} and the eigenvalues of Θ_{h_0} on $Z_H^+(\lambda, c)$ are of the form $\lambda(h_0) - k$ where k is a positive integer.

Let V be a nonzero H-submodule of $Z_H(\lambda, c)$. If $V \not\subseteq Z_H^+(\lambda, c)$, the above discussion shows that $v_0 \in V$. But then $V = Z_H(\lambda, c)$. Thus any proper submodule of $Z_H(\lambda, c)$ is contained in $Z_H^+(\lambda, c)$. As a consequence, $Z_H^{\max}(\lambda, c)$ is a unique maximal submodule of $Z_H(\lambda, c)$, proving (i).

(b) It follows from part (a) that each H-module $L_H(\lambda, c)$ decomposes into eigenspaces for Θ_{h_0}, the eigenvalues of Θ_{h_0} on $L_H(\lambda, c)$ lie in the set $\lambda(h_0) - \mathbb{Z}_+$, and the eigenspace $L_H(\lambda, c)_{\lambda(h_0)}$ is spanned by v_0. If $L_H(\lambda, c) \cong L_H(\lambda', c')$ as H-modules then it must be that $\lambda(h_0) \in \lambda'(h_0) - \mathbb{Z}_+$ and $\lambda'(h_0) \in \lambda(h_0) - \mathbb{Z}_+$. This implies that $\lambda(h_0) = \lambda'(h_0)$ and $L_H(\lambda, c)_{\lambda(h_0)} \cong L_H(\lambda', c')_{\lambda'(h_0)}$ as modules over the commutative subalgebra $\Theta(\mathfrak{h}_e) \oplus \Bbbk C$ of H. But then $\lambda = \lambda'$ and $c = c'$, hence (ii).

(c) Let M be a finite-dimensional simple H-module. Then $C \in Z(H)$ acts on M as c id for some $c \in \Bbbk$. Since $\Theta(\mathfrak{h}_e)$ is abelian, M contains at least one weight subspace for $\Theta(\mathfrak{h}_e)$.

From Theorem 6.1 it follows that the direct sum $\bigoplus_{\mu\in\mathfrak{h}_e^*} M_\mu$ of all weight subspaces of M is an H-submodule of M. Hence M decomposes into weight spaces relative to $\Theta(\mathfrak{h}_e)$. Note that any linear function on $\mathfrak{h}$ vanishing on $\mathfrak{h}_e$ is a scalar multiple of β. Since β is a simple root, any sum of roots from Φ_e^+ restricts to a nonzero function on $\mathfrak{h}_e$. But then the relation

$$\phi \geq \psi \;\Leftrightarrow\; \phi = \psi + \Big(\sum_{\gamma\in\Phi_e^+} r_\gamma \gamma\Big)_{|\mathfrak{h}_e}, \quad r_\gamma \in \mathbb{Z}_+, \qquad (\forall \phi, \psi \in \mathfrak{h}_e^*)$$

is a partial ordering on $\mathfrak{h}_e^*$. Since the set of $\Theta(\mathfrak{h}_e)$-weights of M is finite, it contains at least one maximal element with respect to this ordering, λ say. Let m be a nonzero vector in M_λ. Then $\Theta_{x_i}(m) = \Theta_{u_i^*}(m) = 0$ for all admissible i. As a consequence, there exists a homomorphism of H-modules $\xi\colon Z_H(\lambda, c) \to M$ such that $\xi(v_0) = m$. The simplicity of M implies that ξ is surjective, while part (a) yields $\operatorname{Ker}\xi = Z_H^{\max}(\lambda, c)$. Restricting M to the $\mathfrak{sl}_2$-triple $(\Theta_{e_\alpha}, \Theta_{h_\alpha}, \Theta_{e_{-\alpha}}) \subset H$ with $\alpha \in \Phi_{e,0}$ it is easy to observe that $\lambda(h_\alpha) \in \mathbb{Z}_+$ for all $\alpha \in \Phi_{e,0}^+$.

Finally, suppose $\mathfrak{g}$ is not of type A. Then $\mathfrak{z}_\chi(0)$ is a semisimple Lie algebra. By Weyl's theorem, M is a completely reducible $\Theta(\mathfrak{z}_\chi(0))$-module. Let $\mathfrak{g}_\mathbb{Q}$ be the $\mathbb{Q}$-form in $\mathfrak{g}$ spanned by the Chevalley system from 4.1, and $\mathfrak{z}_{\chi,\mathbb{Q}}(i) = \mathfrak{g}_\mathbb{Q} \cap \mathfrak{z}_\chi(i)$ where $i = 0, 1$. Choose $u, v \in \mathfrak{z}_{\chi,\mathbb{Q}}(1)$ with $(f, [u, v]) = 2$. Then $[u, z_i]^\sharp, [v, z_i^*]^\sharp \in \mathfrak{z}_{\chi,\mathbb{Q}}(0)$ for all i. The highest weight theory implies that there is a $\mathbb{Q}$-form in M stable under the action of $\Theta(\mathfrak{z}_{\chi,\mathbb{Q}}(0))$. It follows that $\operatorname{tr}_M(\Theta_{[u,z_i]^\sharp}\Theta_{[v,z_i^*]^\sharp}) \in \mathbb{Q}$ for $1 \leq i \leq 2s$. Since $\operatorname{tr}_M[\Theta_u, \Theta_v] = 0$, Theorem 6.1 entails that $(c - c_0)\dim M \in \mathbb{Q}$. Since $c_0 \in \mathbb{Q}$ by Theorem 6.1, we obtain $c \in \mathbb{Q}$. □

7.3. To determine the composition factors of the Verma modules $Z_H(\lambda, c)$ with their multiplicities we are going to establish a link between these H-modules and the $\mathfrak{g}$-modules obtained by parabolic induction from Whittaker modules for $\mathfrak{sl}_2(\Bbbk)$. The latter modules have been studied in much detail in [26, 28, 1], and it is known that their composition multiplicities can be calculated by using the Kazhdan–Lusztig algorithm. We are going to rely on Skryabin's equivalence (3.1); the Kazhdan filtration of H will play an important rôle too.

Let $\mathfrak{s}_\beta$ denote the subalgebra of $\mathfrak{g}$ spanned by $(e, h, f) = (e_\beta, h_\beta, f_\beta)$, and put

$$\mathfrak{p}_\beta := \mathfrak{s}_\beta + \mathfrak{h} + \sum_{\alpha\in\Phi^+} \Bbbk e_\alpha, \quad \mathfrak{n}_\beta := \sum_{\alpha\in\Phi^+\setminus\{\beta\}} \Bbbk e_\alpha, \quad \widetilde{\mathfrak{s}}_\beta := \mathfrak{h}_e \oplus \mathfrak{s}_\beta.$$

Clearly, $\mathfrak{p}_\beta = \widetilde{\mathfrak{s}}_\beta \oplus \mathfrak{n}_\beta$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{n}_\beta$ and $\widetilde{\mathfrak{s}}_\beta$ is a Levi subalgebra of $\mathfrak{p}_\beta$. Let $C_\beta = ef + fe + \frac{1}{2}h^2 = 2ef + \frac{1}{2}h^2 - h$ be the Casimir element of $U(\mathfrak{s}_\beta)$. Given $\lambda \in \mathfrak{h}_e^*$ and $c \in \Bbbk$ we denote by $I_\beta(\lambda, c)$ the left ideal of $U(\mathfrak{p}_\beta)$ generated by $f - 1$, $C_\beta - c$, all $h - \lambda(h)$ with $h \in \mathfrak{h}_e$, and all e_γ with $\gamma \in \Phi^+ \setminus \{\beta\}$.

Define $Y(\lambda, c) := U(\mathfrak{p}_\beta)/I_\beta(\lambda, c)$, a $\mathfrak{p}_\beta$-module with the trivial action of $\mathfrak{n}_\beta$, and let $1_{\lambda,c}$ denote the image of 1 in $Y(\lambda, c)$. Since $f.1_{\lambda,c} = 1_{\lambda,c}$, we have

$$e.1_{\lambda,c} = \tfrac{1}{2}(C_\beta - \tfrac{1}{2}h^2 + h).1_{\lambda,c} = (-\tfrac{1}{4}h^2 + \tfrac{1}{2}h + \tfrac{1}{2}c).1_{\lambda,c}.$$

Together with the PBW theorem this shows that the vectors $\{h^k \cdot 1_{\lambda,c} \mid k \in \mathbb{Z}_+\}$ form a $\Bbbk$-basis of $Y(\lambda, c)$ (the independence of these vectors follows from the fact that $Y(\lambda, c)$ is infinite-dimensional). We mention for completeness that $Y(\lambda, c)$ is isomorphic to a Whittaker module for $\mathfrak{s}_\beta \cong \mathfrak{sl}_2(\Bbbk)$.

The above discussion shows that the vectors

$$m(\mathbf{i}, \mathbf{j}, \mathbf{k}, l) := z_1^{i_1} \cdots z_s^{i_s} \cdot y_1^{j_1} \cdots y_t^{j_t} \cdot u_1^{k_1} \cdots u_s^{k_s} \cdot h^l(1_{\lambda,c})$$

with $\mathbf{i}, \mathbf{k} \in \mathbb{Z}_+^s$, $\mathbf{j} \in \mathbb{Z}_+^t$, and $l \in \mathbb{Z}_+$ form a $\Bbbk$-basis of the induced $\mathfrak{g}$-module

$$M(\lambda, c) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\beta)} Y(\lambda, c).$$

7.4. Recall from 4.1 that each z_i^* with $1 \le i \le s$ is a root vector for $\mathfrak{h}$ corresponding to $\gamma_i^* = -\beta - \gamma_i \in \Phi^+$. Put $\delta = \frac{1}{2}(\gamma_1^* + \cdots + \gamma_s^*)$ and $\rho_0 = \rho - 2\delta - (s+1)\beta = \sum_{\alpha \in \Phi_{e,0}^+} \alpha$. Since the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}_e$ is nondegenerate, for any $\eta \in \mathfrak{h}_e^*$ there exists a unique $t_\eta \in \mathfrak{h}_e$ such that $\varphi = (t_\eta, \cdot)$. Hence $(\cdot, \cdot)$ induces a nondegenerate bilinear form on $\mathfrak{h}_e^*$ via $(\mu, \nu) := (t_\mu, t_\nu)$ for all $\mu, \nu \in \mathfrak{h}_e^*$. Given a linear function φ on $\mathfrak{h}$ we denote by $\bar{\varphi}$ the restriction of φ to $\mathfrak{h}_e$.

Theorem 7.1. *Each $\mathfrak{g}$-module $M(\lambda, c)$ is an object of the category $\mathcal{C}$. Furthermore,* $\mathrm{Wh}(M(\lambda, c)) \cong Z_H(\lambda + \bar{\delta}, c + (\lambda + 2\bar{\rho}, \lambda))$ *as H-modules.*

Proof. Put $M := M(\lambda, c)$, and let M_0 (resp. M_1) denote the $\Bbbk$-span of all $m(\mathbf{i}, \mathbf{j}, \mathbf{k}, l)$ in M with $|\mathbf{i}| + l = 0$ (resp. $|\mathbf{i}| + l > 0$). Clearly, $M = M_0 \oplus M_1$ as vector spaces. Let $\mathrm{pr}\colon M = M_0 \oplus M_1 \twoheadrightarrow M_0$ denote the first projection.

If $|\mathbf{i}| + 2|\mathbf{j}| + 3|\mathbf{k}| + 2l = k$, we say that $m(\mathbf{i}, \mathbf{j}, \mathbf{k}, l)$ has *Kazhdan degree* k. Let M^k denote the $\Bbbk$-span in M of all $m(\mathbf{i}, \mathbf{j}, \mathbf{k}, l)$ of Kazhdan degree $\le k$. Then $\{M^k \mid k \in \mathbb{Z}_+\}$ is an increasing filtration in M and $M^0 = \Bbbk 1_{\lambda,c}$. Taking $U(\mathfrak{g})$ with its Kazhdan filtration (as defined in [10] for example) we can thus regard M as a filtered $U(\mathfrak{g})$-module.

Let $z = \lambda f + \sum_{i=1}^s \mu_i z_i^* \in \mathfrak{m}_\chi$ where $\lambda, \mu_i \in \Bbbk$. Since $z_i^* \in \mathfrak{n}_\beta$ for $1 \le i \le s$ and $f.1_{\lambda,c} = 1_{\lambda,c}$, we have $z.1_{\lambda,c} = \lambda \cdot 1_{\lambda,c} = \chi(z) \cdot 1_{\lambda,c}$. Since z acts locally nilpotently on $U(\mathfrak{g})$, we deduce that $z - \chi(z)$ acts locally nilpotently on M for all $z \in \mathfrak{m}_\chi$. As a consequence, M is an object of $\mathcal{C}_\chi$. By our discussion in 3.1, $\mathrm{Wh}(M) \ne 0$, the algebra H acts on M, and $M \cong Q_\chi \otimes_H \mathrm{Wh}(M)$ as $\mathfrak{g}$-modules.

Now observe that

$$z_k^*.m(\mathbf{i}, \mathbf{j}, \mathbf{k}, l) \in i_k \cdot m(\mathbf{i} - \mathbf{e}_k, \mathbf{j}, \mathbf{k}, l) + \mathrm{span}\,\{m(\mathbf{i}', \mathbf{j}', \mathbf{k}', l') \mid |\mathbf{i}'| \ge |\mathbf{i}|\}$$

for all $k \le s$, and

$$(f - 1).m(\mathbf{i}, \mathbf{j}, \mathbf{k}, l) \in 2^l \cdot m(\mathbf{i}, \mathbf{j}, \mathbf{k}, 0) + \mathrm{span}\,\{m(\mathbf{i}', \mathbf{j}', \mathbf{k}', l') \mid l' > 0\} \quad \text{when } l > 0.$$

From this it is immediate that the map $\mathrm{pr}\colon \mathrm{Wh}(M) \to M_0$ is injective.

Note that $1_{\lambda,c} \in \mathrm{Wh}(M)$, and for all $h \in \mathfrak{h}_e$ we have

$$\Theta_h(1_{\lambda,c}) = \Big(h + \frac{1}{2}\sum_{i=1}^{2s}[h, z_i^*]z_i\Big)(1_{\lambda,c}) = \Big(h + \frac{1}{2}\sum_{i=1}^{s}[[h, z_i^*], z_i]\Big)(1_{\lambda,c})$$
$$= \Big(\lambda(h) + \frac{1}{2}\sum_{i=1}^{s}\gamma_i^*(h)f\Big)\cdot 1_{\chi,c} = (\lambda+\delta)(h)\cdot 1_{\chi,c}.$$

Suppose $x \in \mathfrak{z}_\chi(0)$ is a root vector for $\mathfrak{h}$ corresponding to root $\gamma \in \Phi_{e,0}^+$. Then $x \in \mathfrak{n}_\beta$ and $[[x, z_i^*], z_i] \in \mathfrak{n}_\beta$ for all $i \le s$. Therefore,

$$\Theta_x(1_{\lambda,c}) = \Big(x + \frac{1}{2}\sum_{i=1}^{2s}[x, z_i^*]z_i\Big)(1_{\lambda,c}) = \Big(h + \frac{1}{2}\sum_{i=1}^{s}[[x, z_i^*], z_i]\Big)(1_{\lambda,c}) = 0.$$

Recall from (2.5) that for any positive root vector $u \in \mathfrak{z}_\chi(1)$ we have

$$z_u = -\frac{1}{3}\sum_{i=1}^{2s}\langle z_i^*, [u, [z_i, z_i^*]]\rangle z_i = \frac{1}{3}\sum_{i=1}^{2s}\langle z_i^*, [u, f]\rangle z_i \in \mathfrak{n}_\beta.$$

This implies that

$$\Theta_u(1_{\lambda,c}) = \Big(u + \sum_{i=1}^{2s}[u, z_i^*]z_i + \frac{1}{3}\sum_{i,j=1}^{2s}[uz_i^*z_j^*]z_jz_i + z_u\Big)(1_{\lambda,c})$$
$$= \Big(\sum_{i=1}^{s}[[u, z_i^*], z_i] + \frac{1}{3}\sum_{i,j=1}^{s}[uz_i^*z_j^*]z_jz_i + \frac{1}{3}\sum_{i=1}^{s}\sum_{j=s+1}^{2s}[uz_i^*z_j^*]z_jz_i\Big)(1_{\lambda,c})$$
$$= \frac{1}{3}\Big(\sum_{i,j=1}^{s}[[uz_i^*z_j^*], z_j]z_i + \sum_{i,j=1}^{s}z_j[[uz_i^*z_j^*], z_i] - \sum_{i=1}^{s}[uz_i^*z_i][z_i^*, z_i]\Big)(1_{\lambda,c})$$
$$= \frac{1}{3}\Big(\sum_{i,j=1}^{s}[[[uz_i^*z_j^*], z_j], z_i] - \sum_{i=1}^{s}[uz_i^*z_i]f\Big)(1_{\lambda,c}) \in \mathfrak{n}_\beta.1_{\lambda,c} = 0.$$

Therefore, $\Theta_{u_i^*}(1_{\lambda,c}) = 0$ for all $i \le s$. Our discussion in 4.4 shows that

$$C(1_{\lambda,c}) = \Big(2e + \frac{h^2}{2} - (s+1)h + C_0 + 2\sum_{i=1}^{2s}[e, z_i^*]z_i\Big)(1_{\lambda,c})$$
$$= \Big(C_\beta - sh + C_0 + 2\sum_{i=1}^{s}[[e, z_i^*], z_i]\Big)(1_{\lambda,c}). \tag{7.4.1}$$

As $[[[e, z_i^*], z_i], f] = [[[e, f], z_i^*], z_i] = [[h, z_i^*], z_i] = -f$ we have $[[e, z_i^*], z_i] - \frac{1}{2}h \in \mathfrak{h}_e$ for all $i \le s$. Let x be an arbitrary element in $\mathfrak{h}_e$. Then $(x, h) = 0$, $\beta(x) = 0$, and

$$(x, [[e, z_i^*], z_i] - \tfrac{1}{2}h) = ([x, [e, z_i^*]], z_i]) = \gamma_i^*(x)([e, z_i^*], z_i) = \gamma_i^*(x),$$

that is, $[[e, z_i^*], z_i] - \frac{1}{2}h = t_{\bar{\gamma}_i^*}$ for all $i \leq s$; see our discussion at the beginning of this subsection. But then

$$\Big(2\sum_{i=1}^{s}[[e, z_i^*], z_i] - sh\Big)(1_{\lambda,c}) = \Big(2\sum_{i=1}^{s} t_{\bar{\gamma}_i^*}\Big)(1_{\lambda,c}) = 4(\lambda, \bar{\delta}) \cdot 1_{\lambda,c}. \qquad (7.4.2)$$

Since $C_0 = \sum a_i b_i$ is a Casimir element of $U(\mathfrak{z}_\chi(0))$ and all positive root vectors in $\mathfrak{z}_\chi(0)$ annihilate $1_{\lambda,c}$, it is straightforward to see that $C_0(1_{\lambda,c}) = (\lambda, \lambda + 2\bar{\rho}_0) \cdot 1_{\lambda,c}$. In conjunction with (7.4.1) and (7.4.2) this yields

$$C(1_{\lambda,c}) = (c + (\lambda, \lambda + 2\bar{\rho}_0) + 4(\lambda, \bar{\delta})) \cdot 1_{\lambda,c} = (c + (\lambda, \lambda + 2\bar{\rho})) \cdot 1_{\lambda,c}.$$

Put $\lambda' := \lambda + \bar{\delta}$ and $c' := c + (\lambda, \lambda + 2\bar{\rho})$. Let V_0 denote the H-submodule of M generated by $1_{\lambda,c}$. The above discussion shows that the left ideal $J_{\lambda',c'}$ of H annihilates $1_{\lambda,c}$. Therefore, V_0 is a homomorphic image of the Verma module $Z_H(\lambda', c')$.

We claim that the restriction of $\mathrm{pr}\colon M \twoheadrightarrow M_0$ to V_0 is surjective. Recall that M_0 is spanned by all $m(\mathbf{0}, \mathbf{j}, \mathbf{k}, 0)$ with $\mathbf{j} \in \mathbb{Z}_+^t$ and $\mathbf{k} \in \mathbb{Z}_+^s$. Clearly, $m(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0) = 1_{\lambda,c} \in \mathrm{pr}(V_0)$. Assume that all vectors $m(\mathbf{0}, \mathbf{j}, \mathbf{k}, 0)$ of Kazhdan degree $2|\mathbf{j}| + 3|\mathbf{k}| < n$ are in $\mathrm{pr}(V_0)$. Now let $m(\mathbf{0}, \mathbf{a}, \mathbf{b}, 0) \in M_0$ be such that $2|\mathbf{a}| + 3|\mathbf{b}| = n$ and $|\mathbf{a}| + |\mathbf{b}| = k$, and denote by $M_{n,k}$ the span of all $m(\mathbf{i}, \mathbf{j}, \mathbf{k}, l)$ of Kazhdan degree n with $|\mathbf{i}|+|\mathbf{j}|+|\mathbf{k}|+l > k$. Assume that all vectors $m(\mathbf{0}, \mathbf{j}, \mathbf{k}, 0)$ of Kazhdan degree n with $|\mathbf{j}|+|\mathbf{k}| > k$ are in $\mathrm{pr}(V_0)$. Since M is a filtered $U(\mathfrak{g})$-module, it follows from Lemmas 2.3 and 2.5 that

$$\Theta_{y_1}^{a_1} \cdots \Theta_{y_t}^{a_t} \Theta_{u_1}^{b_1} \cdots \Theta_{u_s}^{b_s}(1_{\lambda,c}) \in m(\mathbf{0}, \mathbf{a}, \mathbf{b}, 0) + M_{n,k} + M^{n-1}. \qquad (7.4.3)$$

In view of our assumptions on n and k we get $m(\mathbf{0}, \mathbf{a}, \mathbf{b}, 0) \in \mathrm{pr}(V_0 + M^{n-1} + M_{n,k}) = \mathrm{pr}(V_0)$. Our claim now follows by double induction on n and k. Since we have already established that $\mathrm{pr}\colon \mathrm{Wh}(M) \to M_0$ is injective, this yields $\mathrm{Wh}(M) = V_0$.

Using (7.4.3) it is easy to observe that the vectors $\Theta_{y_1}^{a_1} \cdots \Theta_{y_t}^{a_t} \Theta_{u_1}^{b_1} \cdots \Theta_{u_s}^{b_s}(1_{\lambda,c})$ with $\mathbf{a} \in \mathbb{Z}_+^t$ and $\mathbf{b} \in \mathbb{Z}_+^s$ are linearly independent over $\Bbbk$. Hence it follows from Lemma 7.1 and our discussion at the beginning of 7.2 that $V_0 \cong Z_H(\lambda', c')$ as H-modules. □

Remark 7.1. Combined with Skryabin's equivalence and the main results of Miličić–Soergel [27] and Backelin [1], Theorem 7.1 implies that the composition multiplicities of the Verma modules $Z_H(\lambda, c)$ can be computed with the help of inverse parabolic Kazhdan–Lusztig polynomials associated with the pair $(W, \langle s_\beta \rangle)$. This confirms in the minimal nilpotent case the Kazhdan–Lusztig conjecture for finite $\mathcal{W}$-algebras as formulated by de Vos and van Driel in [8]. Recall that our construction of H_χ is a special instance of quantum Hamiltonian reduction where the constraints imposed are read off from the $\mathfrak{sl}_2$-triple $(e_\beta, h_\beta, e_{-\beta})$. In the physics literature the algebra H appears undercover under the name of a finite $\mathcal{W}$-algebra associated with the minimal embedding $\mathfrak{sl}_2(\Bbbk) \hookrightarrow \mathfrak{g}$.

Remark 7.2. It would be interesting to relate the (Kazhdan) filtered algebra H to the Becchi–Rouet–Stora–Tyutin (BRST) quantisation of the Poisson algebra $\mathrm{gr}\, H$. We recall that the Poisson structure on $\mathrm{gr}\, H$ is determined in [10]. It would be important for the characteristic p theory to determine all $(\lambda, c) \in \mathfrak{h}_e^* \times \Bbbk$ such that the simple H-module $L_H(\lambda, c)$ is finite-dimensional; see Proposition 7.1(iii).

Given $t \in \Bbbk$ we let H_t denote the factor algebra $H/(C-t)$ where $(C-t)$ is the two-sided ideal of H generated by the central element $C-t$. It is clear from the definition that each $L(\lambda, t)$ is an H_t-module.

Corollary 7.1. *If $\mathfrak{g}$ is of type C_n or G_2, then any finite-dimensional H_t-module is completely reducible.*

Proof. It suffices to show that $\mathrm{Ext}^1_{H_t}(M, N) = 0$ for any two finite-dimensional simple H_t-modules M and N. If the H_t-modules M and N are not isomorphic, then they have distinct central characters; see Theorem 6.2. Thus it remains to show that $\mathrm{Ext}^1_{H_t}(M, M) = 0$. By Proposition 7.1(iii), $M \cong L_H(\lambda, t)$ for some $\lambda \in \mathfrak{h}_e^*$. Let V be a finite-dimensional H_t-module containing M as a submodule and such that $V/M \cong M$ as H_t-modules. As $\Theta(\mathfrak{z}_\chi(0))$ is a semisimple Lie subalgebra of H, Weyl's theorem shows that V decomposes into weight spaces relative to $\Theta(\mathfrak{h}_e)$, say $V = \bigoplus_{\mu \in X(V)} V_\mu$. Moreover, the set of $\Theta(\mathfrak{h}_e)$-weights of V coincides with that of $L(\lambda, c)$, showing that $\mu \leq \lambda$ for all $\mu \in X(V)$. It follows from our discussion in 7.2 that $\dim M_\lambda = 1$ and $\dim V_\lambda = 2$. Let $v \in V_\lambda \setminus M_\lambda$ and let M' denote the H-submodule of V generated by v. By construction, the left ideal $J_{\lambda,t}$ of H annihilates v, showing that M' is a homomorphic image of the Verma module $Z_H(\lambda, t)$. But then the $\Theta(\mathfrak{h}_e)$-weight space M'_λ is 1-dimensional, implying $(M \cap M')_\lambda = M_\lambda \cap M'_\lambda = 0$. Consequently, $M \cap M' = 0$. The irreducibility of V/M now entails that $V = M \oplus M'$ and $M' \cong M$. Then $\mathrm{Ext}^1_{H_t}(M, M) = 0$, completing the proof. □

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