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Sum-product theorems and incidence geometry

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Abstract. We prove the following theorems in incidence geometry.

1. There is $\delta > 0$ such that for any $P_1, \dots, P_4 \in \mathbb{C}^2$, and $Q_1, \dots, Q_n \in \mathbb{C}^2$, if there are $\le n^{(1+\delta)/2}$ distinct lines between P_i and Q_j for all i, j, then $P_1, \dots, P_4$ are collinear. If the number of the distinct lines is $< cn^{1/2}$, then the cross ratio of the four points is algebraic.
2. Given $c > 0$, there is $\delta > 0$ such that for any $P_1, P_2, P_3 \in \mathbb{C}^2$ noncollinear, and $Q_1, \dots, Q_n \in \mathbb{C}^2$, if there are $\le cn^{1/2}$ distinct lines between P_i and Q_j for all i, j, then for any $P \in \mathbb{C}^2 \setminus \{P_1, P_2, P_3\}$, we have δn distinct lines between P and Q_j.
3. Given $c > 0$, there is $\epsilon > 0$ such that for any $P_1, P_2, P_3 \in \mathbb{C}^2$ (respectively, $\mathbb{R}^2$) collinear, and $Q_1, \dots, Q_n \in \mathbb{C}^2$ (respectively, $\mathbb{R}^2$), if there are $\le cn^{1/2}$ distinct lines between P_i and Q_j for all i, j, then for any P not lying on the line $L(P_1, P_2)$, we have at least $n^{1-\epsilon}$ (resp. $n/\log n$) distinct lines between P and Q_j.

The main ingredients used are the subspace theorem, Balog–Szemerédi–Gowers theorem, and Szemerédi–Trotter theorem. We also generalize the theorems to higher dimensions, extend Theorem 1 to $\mathbb{F}_p^2$, and give the version of Theorem 2 over $\mathbb{Q}$.

0. Introduction

Notation.

- For $P \neq Q$, $L(P, Q)$ denotes the line through P, Q.
- Let A be a subset of a ring. Then $2A = \{a + a' : a, a' \in A\}$, $A^2 = \{aa' : a, a' \in A\}$.

We first prove the following two theorems.

Theorem 1. *There is $\delta > 0$ such that for any $P_1, \dots, P_4 \in \mathbb{C}^2$, and $Q_1, \dots, Q_n \in \mathbb{C}^2$, if*

$$|\{L(P_i, Q_j) : 1 \le i \le 4,\ 1 \le j \le n\}| \le n^{(1+\delta)/2}, \tag{0.1}$$

then $P_1, \dots, P_4$ are collinear. If

$$|\{L(P_i, Q_j) : 1 \le i \le 4,\ 1 \le j \le n\}| \le cn^{1/2}, \tag{0.2}$$

then the cross ratio of $P_1, \dots, P_4$ is algebraic.

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Theorem 2. *Given $c > 0$, there is $\delta > 0$ such that for any $P_1, P_2, P_3 \in \mathbb{C}^2$ noncollinear, and $Q_1, \dots, Q_n \in \mathbb{C}^2$, if*

$$|\{L(P_i, Q_j) : 1 \le i \le 3,\ 1 \le j \le n\}| \le cn^{1/2}, \tag{0.3}$$

then for any $P \in \mathbb{C}^2 \setminus \{P_1, P_2, P_3\}$, we have

$$|\{L(P, Q_j) : 1 \le j \le n\}| = \delta n. \tag{0.4}$$

Theorem 3. *Given $c > 0$, there is $\epsilon > 0$ such that for any $P_1, P_2, P_3 \in \mathbb{C}^2$ collinear, and $Q_1, \dots, Q_n \in \mathbb{C}^2$, if*

$$|\{L(P_i, Q_j) : 1 \le i \le 3,\ 1 \le j \le n\}| \le cn^{1/2}, \tag{0.5}$$

then for any $P \in \mathbb{C}^2 \setminus L(P_1, P_2)$, we have

$$|\{L(P, Q_j) : 1 \le j \le n\}| > n^{1-\epsilon}. \tag{0.6}$$

Remark 4. In Theorem 3, the bound $n^{1-\epsilon}$ in (0.6) is replaced by $n/\log n$ if the points are in $\mathbb{R}^2$ instead of $\mathbb{C}^2$.

Remark 5. In Remark 1.1 below, we see that assumption (0.3) does occur.

We will first interpret the geometric problems under consideration as sum-product problems. Roughly speaking, for Theorem 2, we want to show that given two sets $C, D \subset \mathbb{C}^2$ of about the same size, if $\{d_i/c_i : (c_i, d_i) \in C \times D,\ 1 \le i \le n\}$ is small, then $\{(d_i + b)/(c_i + a) : (c_i, d_i) \in C \times D,\ 1 \le i \le n\}$ is large, where a, b are fixed. So we want to have an upper bound on the number of solutions (c_i, d_i, c_j, d_j) of the equation

$$\frac{d_i + b}{c_i + a} = \frac{d_j + b}{c_j + a}.$$

This interpretation is introduced in Section 1. In Section 2, we use the subspace theorem to prove Theorem 2, for the case when the point P is not on any line connecting the P_i's. In Section 3, we use the Szemerédi–Trotter theorem to prove the corresponding case of Theorem 1. We also give a short proof using a theorem about convex functions by Elekes, Nathanson and Ruzsa [ENR]. The argument using the Szemerédi–Trotter theorem [S], besides applying over $\mathbb{C}$ (rather than $\mathbb{R}$), has the advantage that the set-up (reducing the problem to bounding the number of solutions of equations) was already used for the subspace theorem approach. Also, it generalizes easily to the prime field $\mathbb{F}_p$ setting. In Section 4, we use the sum-product theorem to take care of all the cases when more than two of the P_i's are at infinity. In Section 5, we generalize the theorems to high dimensions. In Section 6, we prove a stronger theorem over $\mathbb{Q}$ by using the λ_q constant (see [BC]).

This work is one more illustration of the relations between arithmetic combinatorics and point-line incidence geometry. Let us recall that presently the strongest results on the sum-product problem were obtained using the Szemerédi–Trotter theorem (due to Elekes and the second author). The results in this paper are another demonstration of the interplay between these two fields.

1. The set-up

Our strategy of proving Theorem 1 is to assume that P_1, P_2, P_3 are not collinear and get a large family of lines $L(P_4, Q_j)$ violating assumption (0.1). Therefore, the settings for Theorem 1 and Theorem 2 are the same. For simplicity, we describe the situation for Theorem 2 here and indicate the (small) difference when we prove Theorem 1.

We will work in the projective space $\mathbb{CP}^2 \cong (\mathbb{C}^3 \setminus \{0\})/\sim$, where $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for any $\lambda \neq 0$. We identify $\mathbb{C}^2$ with the affine space in $\mathbb{CP}^2$ defined by $z \neq 0$ via $(x, y) \mapsto (x, y, 1)$.

Let L_∞ be the line at infinity defined by $z = 0$. We may assume

(i) P_1, P_2, P_3 are $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. (Clearly, P_1 and P_2 lie on L_∞.)
(ii) No Q_i lies on L_∞.

In fact, let A be the 3×3 matrix with the vector P_i as the ith column. Since the P_i's are not collinear, the matrix A is invertible. Hence the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $P \mapsto A^{-1}P^T$ sends P_1, P_2, P_3 to $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. To see (ii), we notice that for any $Q = (1, d, 0) \in L_\infty$, the line $L(Q, P_3)$ is defined by $y = dx$. Assumption (0.3) implies that $|\{Q_i : Q_i \in L_\infty\}| \le cn^{1/2} \ll n$.

Let

$$Q_i = (c_i, d_i, 1),$$

$$C = \{c_i : 1 \le i \le n\}, \quad D = \{d_i : 1 \le i \le n\} \tag{1.1}$$

$$\mathcal{G} = \{(c_i, d_i) : 1 \le i \le n\}, \quad C^{-1} \underset{\mathcal{G}}{\times} D = \{d_i/c_i : 1 \le i \le n\}. \tag{1.2}$$

Then

$$|\mathcal{G}| = n \tag{1.3}$$

and assumption (0.3) implies

$$|C^{-1} \underset{\mathcal{G}}{\times} D| \le cn^{1/2}, \quad |C| = |D| = c'n^{1/2}, \tag{1.4}$$

since the lines $L(P_1, Q_i), L(P_2, Q_i), L(P_3, Q_i)$ are defined by $y = d_i z$, $x = c_i z$, $y = (d_i/c_i)x$, and $|C|\,|D| \ge n$.

Remark 1.1. Assumption (0.3) does occur. For example, if we let $Q_{i,j} = (2^i, 2^j, 1)$, $1 \le i, j \le N$, then

$$|\{L(P_1, Q_{i,j})\}_{i,j}| = |\{L(P_2, Q_{i,j})\}_{i,j}| = N, \quad |\{L(P_3, Q_{i,j})\}_{i,j}| = 2N - 1.$$

To be able to apply the tools from sum-product theory, we need the Laczkovich–Ruzsa version [LR] of the Balog–Szemerédi–Gowers theorem.

Theorem BSG-LR. *Let A, B be subsets of an abelian group with $|A| = |B| = N$, and let $G \subset A \times B$ with $|G| > K^{-1}N^2$. Define*

$$A \overset{G}{+} B = \{a + b : (a, b) \in G\}. \tag{1.5}$$

If $|A \overset{G}{+} B| < KN$, *then there are subsets* $A' \subset A$ *and* $B' \subset B$ *such that*

$$|A' + B'| < K^c N$$

and

$$|A'|, |B'| > K^{-c} N. \tag{1.6}$$

Remark 1.2. The absolute constant c in the above theorem is at most 8 (see [SSV]).

2. The proof of Theorem 2 for finite points

Let $N = n^{1/2}$. Take a point $P = (-a, -b, 1) \in \mathbb{C}^2$. The line $L(P, Q_i)$ has slope $(d_i + b)/(c_i + a)$. With the help of Theorem BSG-LR, Theorem 2 is reduced to the following

Theorem 2.1. *Let* $X = \{x_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ *and* $Y = \{y_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ *with* $|Y/X| \le cN$ *and* $|X| = |Y| = c'N$. *Fix* $a, b \in \mathbb{C}$. *Define*

$$Z = \left\{ \frac{y_i + b}{x_i + a} : 1 \le i \le N^2 \right\}.$$

Then $|Z| > \delta N^2$ *for some* $\delta > 0$.

Proof. Let $I_z = \{i : (y_i + b)/(x_i + a) = z\}$. Then $\sum_{z \in Z} |I_z| = n = N^2$ and Cauchy–Schwarz gives

$$N^4 \le |Z| \sum |I_z|^2.$$

Now

$$\begin{aligned}\sum |I_z|^2 &= \left| \left\{ (i, j) : \frac{y_i + b}{x_i + a} = \frac{y_j + b}{x_j + a},\ 1 \le i, j \le n \right\} \right| \\ &\le \left| \left\{ (x, x', y, y') \in X \times X \times Y \times Y : \frac{y + b}{x + a} = \frac{y' + b}{x' + a} \right\} \right| \\ &= |\{(x, x', y, y') \in X \times X \times Y \times Y : x'y + bx' + ay = xy' + bx + ay'\}|. \end{aligned} \tag{2.1}$$

To bound (2.1), we invoke the subspace theorem [ESS], which gives an upper bound on the number of solutions of a linear equation in a multiplicative group.

A solution $(x_1, \ldots, x_m)$ of the equation

$$\sum_{i=1}^{m} c_i x_i = 1, \quad c_i \in \mathbb{C}, \tag{2.2}$$

is called *nondegenerate* if $\sum_{j=1}^{k} c_{i_j} x_{i_j} \ne 0$ for all k. The bound given below is due to Evertse, Schlickewei and Schmidt [ESS].

Subspace Theorem. *Let $\Gamma < \langle \mathbb{C}^*, \cdot \rangle$ be a subgroup of the multiplicative group of $\mathbb{C}$, and let the rank of Γ be r. Then*

$$\left|\left\{\text{nondegenerate solutions of } \sum_{i=1}^{m} c_i x_i = 1 \text{ in } \Gamma\right\}\right| < e^{(r+1)(6m)^{3m}}.$$

The formulation of the subspace theorem we need is the following (see [C2])

Corollary 2.2 ([C2]). *Let $\Gamma < \langle \mathbb{C}^*, \cdot \rangle$ be a subgroup of rank r and $A \subset \Gamma$ with $|A| = N$. Then the number of solutions in A of*

$$x_1 + \cdots + x_{2h} = 0 \tag{2.3}$$

is bounded by $N^{h-1}e^{rc} + N^h$, up to a constant depending on h. Here $c = c(h)$.

In order to apply the subspace theorem, we need the following (see [Fr], [R1], [Bi]).

Freiman's Lemma. *Let $\langle G, \cdot \rangle$ be a torsion-free abelian group and $A \subset G$ with $|A^2| < K|A|$. Then*

$$A \subset \{g_1^{j_1} \cdots g_d^{j_d} \,:\, j_i = 1, \ldots, \ell_i, \text{ and } g_i \in G\}, \tag{2.4}$$

where $d \le K$ and $\prod \ell_i < c(K)|A|$.

We let $\Gamma < \langle \mathbb{C}^*, \cdot \rangle$ be the subgroup generated by $g_1, \ldots, g_d$. Then the rank of Γ is bounded by $d \le K$ and the number of nondegenerate solutions of (2.2) in Γ is bounded by $e^{c_m K}$. We now obtain the subspace theorem under the product set assumption.

Notation. $d <_h f$ means $d \le c(h) f$, where $c(h)$ is a function of h.

Theorem 2.3 ([C2]). *Let $A \subset \mathbb{C}$ with $|A| = N$, and*

$$|A^2| < K|A|. \tag{2.5}$$

Then

$$|\{\text{solutions of } x_1 + \cdots + x_{2h} = 0 \text{ in } A\}| <_h N^{h-1}e^{cK} + N^h.$$

Theorem 2.3 gives N^3 as a bound on the number of solutions in A with $|A| = N$ to the equation

$$\xi_1 + \xi_2 + \xi_3 = \xi_4 + \xi_5 + \xi_6. \tag{2.6}$$

On the other hand, we expect (2.1) to be bounded by N^2. So we introduce a new variable z in (2.1), and let

$$x' = u'/z, \qquad x = u/z,$$

where $u, u' \in X^2$. Then the equation in (2.1) becomes

$$u'y + bu' + ayz = uy' + bu + ay'z. \tag{2.7}$$

A solution $(\xi_1, \ldots, \xi_6) \in X^2Y \times bX^2 \times aXY \times X^2Y \times bX^2 \times aXY$ of (2.6) is in one-to-one correspondence to a solution $(u', u, y', y, z) \in X^2 \times X^2 \times Y \times Y \times X$ of (2.7) by the following relations:

$$\xi_1 = u'y, \quad \xi_2 = bu', \quad \xi_3 = ayz, \quad \xi_4 = uy', \quad \xi_5 = bu, \quad \xi_6 = ay'z,$$

or

$$u' = \frac{\xi_2}{b}, \quad u = \frac{\xi_5}{b}, \quad y' = \frac{b\xi_4}{\xi_5}, \quad y = \frac{b\xi_1}{\xi_2}, \quad z = \frac{\xi_2\xi_3}{ab\xi_1}.$$

In order to apply Theorem 2.3, we take

$$A = X^2Y \cup bX^2 \cup aXY.$$

Then we have $|A^2| < K|A|$ by the following Proposition 2.26 in [TV].

Proposition. *Let A, B be subsets of an abelian group with $|A| = |B| = N$. If $|A + B| < cN$, then*

$$|n_1A - n_2A + n_3B - n_4B| < c'N.$$

3. The proof of Theorem 1 for finite points

If we replace assumption (0.3) by assumption (0.1), then instead of (1.4) and Theorem 2.1, we have (3.1) and Theorem 3.1 below

$$n^{(1-\delta)/2} < |C| = |D| < n^{(1+\delta)/2}, \quad |C^{-1} \underset{\mathcal{G}}{\times} D| < n^{(1+\delta)/2}. \tag{3.1}$$

Theorem 3.1. *Let $X = \{x_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ and $Y = \{y_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ with*

$$N^{1-\delta} < |X| = |Y| < N^{1+\delta} \tag{3.2}$$

and

$$\left|\frac{Y}{X}\right| < N^{1+\delta}. \tag{3.3}$$

Fix $a, b \in \mathbb{C}$. Define

$$Z = \left\{\frac{y_i + b}{x_i + a} : 1 \le i \le N^2\right\}.$$

Then $|Z| > N^{1+\eta}$ for some $\eta = \eta(\delta) > \delta$.

Remark 3.2. Let δ' be the δ in (3.1). Then the δ in Theorem 3.1 is $(2c + 1)\delta'$ with an absolute constant c as in Theorem BSG-LR.

Similar to the argument from (2.1) to (2.7), we need to prove

$$E := |\{(u, u', y, y', z) \in X^2 \times X^2 \times Y \times Y \times X : u'y + bu' + ayz = uy' + bu + ay'z\}| < N^{4-\eta} \tag{3.4}$$

for some $\eta > 0$.

Rewriting the equation in (3.4) as

$$(y + b)u' - (y' + b)u + a(y - y')z = 0, \tag{3.5}$$

we see that (u', u) lies on the line $\ell_{y,y',z}$ defined by

$$S - \frac{y'+b}{y+b}T + \frac{a(y-y')z}{y+b} = 0. \tag{3.6}$$

Assume

$$E > N^{4-\eta}. \tag{3.7}$$

We will get a contradiction for η small. (See (3.14).)

We define

$$K = \{(y, y', z) \in Y \times Y \times X : |\ell_{y,y',z} \cap (X^2 \times X^2)| > N^{1-2\eta}\}. \tag{3.8}$$

Claim 1. *If* $3\delta < \eta$, *then*

$$|K| > \frac{E}{|X^2|}\ . \tag{3.9}$$

Proof. By (3.4)–(3.6) and (3.8),

$$E \le \sum_{y',y,z} |\ell_{y,y',z} \cap (X^2 \times X^2)| < |X^2|\,|K| + N^{1-2\eta}|X|\,|Y|^2,$$

and by (3.2), $N^{1-2\eta}|X|\,|Y|^2 < N^{1-2\eta+3(1+\delta)} < N^{4-\eta}$. The claim follows from (3.7).

Ruzsa's Inequality ([R2])**.** *Let* M *and* N *be finite subsets of an abelian group such that*

$$|M+N| \le \rho|M|.$$

Let $h \ge 1$ *and* $\ell \ge 1$. *Then*

$$|hN - \ell N| \le \rho^{h+\ell}|M|.$$

It follows from Ruzsa's inequality, (3.2) and (3.3) that

$$|X^2| < \left(\frac{N^{1+\delta}}{|X|}\right)^3 |X| < \frac{N^{3+3\delta}}{N^{2-2\delta}} = N^{1+5\delta}. \tag{3.10}$$

By (3.9), (3.7) and (3.10), we have

$$|K| > \frac{N^{4-\eta}}{N^{1+5\delta}} = N^{3-\eta-5\delta}. \tag{3.11}$$

Let

$$\mathcal{L} = \{\ell_{y,y',z} : (y, y', z) \in K\}. \tag{3.12}$$

Since for any (ξ, ς), there are at most $|Y| < N^{1+\delta}$ triples (y, y', z) such that

$$\xi = \frac{y'+b}{y+b}, \qquad \varsigma = \frac{a(y-y')z}{y+b},$$

for each line in $\mathcal{L}$ there are at most $N^{1+\delta}$ triples in K corresponding to it. Therefore,

$$|\mathcal{L}| > N^{2-\eta-6\delta}. \tag{3.13}$$

The following version of the Szemerédi–Trotter theorem over $\mathbb{C}$ is exactly what we need.

Szemerédi–Trotter Theorem ([S]). *Let $\mathcal{P} = C \times D \subset \mathbb{C}^2$ be a set of points and $\mathcal{L}$ be a set of lines such that $|\ell \cap \mathcal{P}| \geq k$ for any $\ell \in \mathcal{L}$. Then*

$$|\mathcal{P}|^2 > ck^3|\mathcal{L}|.$$

In the above theorem we take $\mathcal{P} = X^2 \times X^2$, $\mathcal{L}$ as in (3.12) and $k = N^{1-2\eta}$. Together with (3.10) and (3.13), we have

$$N^{4(1+5\delta)} > |X^2|^4 > c(N^{1-2\eta})^3|\mathcal{L}| > N^{5-7\eta-6\delta}.$$

This cannot happen if

$$\eta < \frac{1-26\delta}{7}. \tag{3.14}$$

Remark 3.3. The conditions that $\eta > 3\delta$ (cf. Claim 1) and (3.14) imply $\delta < 1/47$.

Remark 3.4. The case of $P_i, Q_j \in \mathbb{F}_p \times \mathbb{F}_p$ can be taken care of by the following theorem (see [B, Theorem 2.2]).

Szemerédi–Trotter Theorem for $\mathbb{F}_p$. *Let $\mathcal{P} \subset \mathbb{F}_p$ be a set of points, and $\mathcal{L}$ be a set of lines such that*

$$|\mathcal{P}|, |\mathcal{L}| \leq M < p^\alpha \quad \textit{for some } 0 < \alpha < 2. \tag{3.15}$$

Let $\mathcal{I} = \{(p, \ell) \in \mathcal{P} \times \mathcal{L} : p \in \ell\}$ be the incidence relation. Then

$$|\mathcal{I}| < cM^{3/2-\gamma} \quad \textit{for some } \gamma = \gamma(\alpha) > 0. \tag{3.16}$$

In (3.15), take $\mathcal{P} = X^2 \times X^2$, $\mathcal{L}$ as in (3.12), and $M = N^{2+10\delta}$ (cf. (3.10)). By (3.13) (which follows from the assumption that $E > N^{4-\eta}$), we may assume $|\mathcal{L}| = N^{2-\eta-6\delta}$. Since each line in $\mathcal{L}$ contains at least $N^{1-2\eta}$ points, we have

$$|\mathcal{I}| \geq |\mathcal{L}|N^{1-2\eta}. \tag{3.17}$$

Hence

$$cN^{(2+10\delta)(3/2-\gamma)} > N^{2-\eta-6\delta}N^{1-2\eta}.$$

This is a contradiction if δ and η are small. Therefore (3.4) holds, and Theorem 3.1 is true over $\mathbb{F}_p$.

Remark 3.5. The finite points case of Theorem 1 over $\mathbb{R}$ also follows from the following theorem by Elekes, Nathanson and Ruzsa [ENR].

Theorem ENR. *Let $S \subset \mathbb{R}$ be finite and let f be a piecewise convex function (i.e. $f' > 0$). Then*

$$|2S| + |2f(S)| \geq c|S|^{5/4}.$$

Proof of Remark 3.5. Similar to the way we derive the assumption of Theorem 3.1, we will start with (3.1) and use Theorem BSG-LR (twice, this time). Let

$$\mathcal{G} = \{(c_i, d_i) \in C \times D : 1 \le i \le N^2\}. \tag{3.18}$$

Assume

$$N^{1-\delta} < |C| = |D| < N^{1+\delta}, \quad |\mathcal{G}| \sim N^2, \tag{3.19}$$

$$\left|\left\{\frac{d_i}{c_i} : (c_i, d_i) \in \mathcal{G}\right\}\right| < N^{1+\delta}, \tag{3.20}$$

$$\left|\left\{\frac{d_i + b}{c_i + a} : (c_i, d_i) \in \mathcal{G}\right\}\right| < N^{1+\eta}. \tag{3.21}$$

First, from (3.20), we obtain $C' \subset C$ and $D' \subset D$ such that

$$|C'| \sim |C|, \quad |D'| \sim |D|, \quad |\mathcal{G} \cap (C' \times D')| \sim N^2$$

and

$$\left|\frac{D'}{C'}\right| \lesssim N^{1+\delta}. \tag{3.22}$$

Let

$$\mathcal{G}' = \mathcal{G} \cap (C' \times D').$$

Applying Theorem BSG-LR again, we obtain $X \subset C' \subset C$ and $Y \subset D' \subset D$ such that

$$|X| \sim |C'| \sim |C|, \quad |Y| \sim |D'| \sim |D|, \quad |\mathcal{G}' \cap (X \times Y)| \sim N^2,$$

$$\left|\frac{Y}{X}\right| \le \left|\frac{D'}{C'}\right| \lesssim N^{1+\delta}, \tag{3.23}$$

$$\left|\frac{Y + b}{X + a}\right| \lesssim N^{1+\eta}. \tag{3.24}$$

The bound (3.23) implies that

$$|\log Y - \log X| \lesssim N^{1+\delta}. \tag{3.25}$$

Ruzsa's inequality and (3.25) give

$$|2 \log X| \lesssim N^{1+5\delta}. \tag{3.26}$$

Assume $\delta < 1/20$. In Theorem ENR, we take $S = \log X$, and let f be the convex function $f(s) = \log(e^s + a)$. Then

$$|2 \log(X + a)| > N^{5/4}. \tag{3.27}$$

On the other hand, (3.24) implies

$$|\log(Y+b) - \log(X+a)| \lesssim N^{1+\eta}. \tag{3.28}$$

Again, applying Ruzsa's inequality to (3.28) gives

$$|2\log(X+a)| \lesssim N^{1+5\eta},$$

which contradicts (3.27) if $\eta < 1/20$.

4. The cases of points at infinity

In this section we handle all the cases when more than two of the P_i's are at infinity.

Let $P = (1, -1/d, 0) \in L_\infty$. Then the lines $L(P, Q_i)$ are defined by

$$x + dy - (c_i + dd_i)z = 0.$$

To prove Theorems 1 and 2, we need the following two theorems.

Theorem 4.1. *Let $X = \{x_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ and $Y = \{y_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ with*

$$N^{1-\delta} < |X| = |Y| < N^{1+\delta} \tag{4.1}$$

and

$$\left|\frac{Y}{X}\right| < N^{1+\delta}. \tag{4.2}$$

Fix $d \in \mathbb{C}$. Define

$$Z = \{x_i + dy_i : 1 \le i \le N^2\}. \tag{4.3}$$

Then

$$|Z| > N^{1+\eta} \quad \textit{for some } \eta = \eta(\delta) \ge \delta. \tag{4.4}$$

Theorem 4.2. *Let $X = \{x_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ and $Y = \{y_i \in \mathbb{C}^2 : 1 \le i \le N^2\}$ with*

$$|X| = |Y| = c'N \quad \textit{and} \quad \left|\frac{Y}{X}\right| < cN.$$

Fix $d \in \mathbb{C}$. Define $Z = \{x_i + dy_i : 1 \le i \le N^2\}$. Then $|Z| > \delta N^2$ for some $\delta > 0$.

To prove Theorem 4.1, we assume the contrary that

$$|Z| < N^{1+\eta} \tag{4.5}$$

for some $\eta = \eta(\delta) \ge \delta$. We will show that this cannot happen if η is small.

Let $A = X$, $B = dY$, where X, Y satisfy the assumptions of Theorem 4.1. Applying Theorem BSG-LR to A and B, we have

$$N^{1-\eta} < |A| = |B| < N^{1+\eta}, \tag{4.6}$$

$$\left|\frac{B}{A}\right| < N^{1+\eta}, \tag{4.7}$$

$$|A+B| < N^{1+\eta}. \tag{4.8}$$

By the same argument as that to obtain (3.10), (4.6)–(4.8) implies

$$|2A|, |A^2| < N^{1+5\eta}.$$

On the other hand, (4.6) and the sum-product theorem below imply

$$|2A| + |A^2| > N^{\frac{14}{11}(1-\eta)}.$$

This is a contradiction if $\eta < 1/23$.

Theorem (Solymosi [S])**.**

$$|2A| + |A^2| > |A|^{\frac{14}{11}-\epsilon}.$$

Remark 4.3. Let η' be the η in (4.5). Then the η in (4.6)–(4.8) is bounded by $c\eta'$, where $c \le 8$ is an absolute constant. (See Remark 1.2.) For example, if $\eta' = \delta$, we can take $\eta \le (2c+1)\delta$.

The proof of Theorem 4.2 by using the subspace theorem is rather straightforward, since as in the proof of Theorem 2.1, it suffices to show that

$$|\{(x, x', y, y') \in X \times X \times Y \times Y : x + dy = x' + dy'\}| < \frac{1}{\delta} N^2.$$

Proof of Theorem 3. Since P_1, P_2, P_3 are collinear, we may assume that $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (1,-1,0) \in L_\infty$. Assumption (0.5) means that $|C|, |D|, |C+D| \lesssim N$. For a point $P = (-a,-b,1) \notin L_\infty$, the family of lines $\{L(P, Q_j)\}_j$ corresponds to $\{\frac{d_i+b}{c_i+a} : (c_i, d_i) \in C \times D,\ 1 \le i \le N^2\}$. Applying the theorems below to the sets $C+a$, $D+b$, and by Ruzsa's inequality, we have $|(C+a)(D+b)| \sim N^{2-\epsilon}$ (respectively, $N^2/\log N$). This together with the Balog–Szemerédi–Gowers theorem implies that $|\{L(P, Q_j)\}_j| \gtrsim N^{2-\epsilon}$ (respectively, $N^2/\log N$).

Theorem ([C1])**.** *Let $A \subset \mathbb{C}$ be a finite set with $|2A| \sim |A|$. Then*

$$|A^2| > |A|^{2-\epsilon} \quad \text{for some } \epsilon > 0.$$

Theorem (Elekes–Ruzsa [ER])**.** *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$|A+A|^4 \cdot |A^2| \cdot \log|A| > |A|^6.$$

The special case of Theorem 1. Assume (0.2) holds. Then $P_1, \dots, P_4$ are collinear. After a Möbius transformation, we may assume that the four points are $P_1 = (1,0,0)$, $P_2 = (1,-1,0)$, $P_3 = (0,1,0)$, $P_4 = (1,-1/d,0) \in L_\infty$. The lines $\{L(P_i, Q_j)\}_j$ for $i = 1,\dots,4$ correspond to C, $C+D$, D and $\{c_i + dd_i : (c_i, d_i) \in C \times D, 1 \le i \le N^2\}$ respectively. Since $|C| \sim |D| \sim |C+D| \sim N$, we have $C' \subset C$ with $|C'| \sim N$ and $C' \subset a + D$ for some a. Hence $C' + dD \subset a + (D + dD)$ and our conclusion follows from the following theorem.

Theorem (Konyagin–Laba [KL]). *Let $t \in \mathbb{C}$ be transcendental. Then*

$$|A + tA| > \frac{|A| \log |A|}{\log \log |A|}.$$

5. Higher dimensional cases

The case of $\mathbb{C}^k$ with $k > 2$ follows easily from the case of $k = 2$.

Theorem 5.1. *There is $\delta > 0$ such that for any $P_1, \dots, P_{k+2}, Q_1, \dots, Q_n \in \mathbb{C}^k$, if*

$$|\{L(P_i, Q_j) : 1 \le i \le k+2,\ 1 \le j \le n\}| \le n^{(k-1+\delta)/k}, \tag{5.1}$$

then $P_1, \dots, P_{k+2}$ lie on a hyperplane.

Theorem 5.2. *Given $c > 0$, there is $\delta > 0$ such that for any $P_1, \dots, P_{k+1} \in \mathbb{C}^k$ not contained in any hyperplane, and any $Q_1, \dots, Q_n \in \mathbb{C}^k$, if*

$$|\{L(P_i, Q_j) : 1 \le i \le k+1,\ 1 \le j \le n\}| \le cn^{(k-1)/k}, \tag{5.2}$$

then for any $P \in \mathbb{C}^k \setminus \{P_1, \dots, P_{k+1}\}$ we have

$$|\{L(P, Q_j) : 1 \le j \le n\}| = \delta n. \tag{5.3}$$

The set-up is similar to that of the $\mathbb{C}^2$ case. We work on $\mathbb{CP}^k$ instead of $\mathbb{C}^k$. Assuming $P_1, \dots, P_{k+1}$ are not contained in any hyperplane, after a linear transformation we may assume that $P_1 = (1, 0, \dots, 0)$, $P_2 = (0, 1, 0, \dots, , 0), \dots, P_{k+1} = (0, \dots, 0, 1)$. By the same reasoning as before, we may assume that the Q_j's all lie in the affine space. Hence we may set

$$Q_j = (c_1, \dots, c_k)^{(j)} := (c_1^{(j)}, \dots, c_k^{(j)}) \in \mathbb{R}^k \subset \mathbb{C}^k,$$

where $j = 1, \dots, n$.

Let $N = n^{1/k}$. Assumption (5.2) implies

$$|\{(c_2, \dots, c_k)^{(j)}\}_{j=1}^{N^k}|, |\{(c_1, c_3, \dots, c_k)^{(j)}\}_{j=1}^{N^k}|, \dots, |\{(c_1, \dots, c_{k-1})^{(j)}\}_{j=1}^{N^k}| < N^{k-1} \tag{5.4}$$

and

$$|\{(c_2/c_1, \dots, c_k/c_1)^{(j)}\}_{j=1}^{N^k}| < N^{k-1}. \tag{5.5}$$

For a finite point $P = (-a_1, \dots, -a_k, 1)$, the family of lines $\{L(P, Q_j) : 1 \le j \le N^k\}$ corresponds one-to-one to

$$Z = \left\{ \left(\frac{c_2 + a_2}{c_1 + a_1}, \dots, \frac{c_k + a_k}{c_1 + a_1} \right)^{(j)} : 1 \le j \le N^k \right\}.$$

Hence (5.3) is equivalent to

$$|Z| = \delta N^k \tag{5.6}$$

for some $\delta > 0$. Let $C_i = \{c_i^{(j)} : j = 1, \ldots, N^k\}$. We will show that

$$|C_i| = cN \quad \text{for } i = 1, \ldots, k. \tag{5.7}$$

For simpler notations and without losing generality, we give an argument for the case $k = 4$. Let

$$A = \{Q_1, \ldots, Q_{N^4}\},$$

and let $p_{j_1\cdots j_m}(x_1, \ldots, x_4) = (x_{j_1}, \ldots, x_{j_m})$ be the projection to the j_1-th, …, j_m-th coordinates.

First, we may assume

$$|p_{123}^{-1}(c_1, c_2, c_3) \cap A| \gtrsim N \quad \text{for all } (c_1, c_2, c_3) \in p_{123}(A). \tag{5.8}$$

In fact, let $A^c = \{(c_1, \ldots, c_4) \in A : |p_{123}^{-1}(c_1, c_2, c_3) \cap A| = o(N)\}$. Then

$$|A^c| \le o(N)N^3 = o(N^4), \tag{5.9}$$

and A^c can be ignored.

Next, we see that for the set A considered in (5.8), the bound $|p_{124}(A)| \lesssim N^3$ implies

$$|p_{12}(A)| \lesssim N^2. \tag{5.10}$$

Indeed,

$$N^3 \gtrsim |p_{124}(A)| > |p_{12}(A)| \cdot \min_{(c_1,c_2)\in p_{12}(A)} |p_{124}(p_{12}^{-1}(c_1, c_2) \cap A)| \gtrsim |p_{12}(A)|\, N. \tag{5.11}$$

The last inequality is because of (5.8). Similarly, we have $|p_{13}(A)|, |p_{23}(A)| \lesssim N^2$.

Using (5.10) instead of (5.4), by the same reasoning as for (5.8), shrinking the set A in (5.8) a bit, we may assume

$$|p_{12}^{-1}(c_1, c_2) \cap A| \gtrsim N^2 \quad \text{for all } (c_1, c_2) \in p_{12}(A). \tag{5.12}$$

Therefore, (5.4) and (5.12) imply

$$N^3 \gtrsim |p_{134}(A)| \gtrsim |p_1(A)| \cdot \min_{c_1\in p_1(A)} |p_{134}(p_1^{-1}(c_1) \cap A)| > |p_1(A)|\, N^2, \tag{5.13}$$

which implies

$$|C_1| = |p_1(A)| \lesssim N. \tag{5.14}$$

Similarly, we have $|C_2|, |C_3| \lesssim N$ for $|A| \sim N^4$.

Repeating this process on the set A obtained in (5.12) with different projections, we have $|C_4| = |p_4(A)| \lesssim N$. Now (5.7) follows from $N^4 \le |C_1|\,|C_2|\,|C_3|\,|C_4| \lesssim N^4$.

Getting back to the case of any $k > 2$, we let $B = \{Q_1, \ldots, Q_{N^k}\}$. We will show that

$$|\{(c_i/c_1)^{(j)} : 1 \le j \le N^k\}| \sim N \quad \text{for all } i. \tag{5.15}$$

Let

$$C_{1i} = \{(c_1, c_i) \in C_1 \times C_i : |p_{1i}^{-1}(c_1, c_i) \cap B| \gtrsim N^{k-2}\}. \tag{5.16}$$

Since $|B| \sim N^k$, by the same reasoning as for (5.8) we have

$$|C_{1i}| \sim N^2. \tag{5.17}$$

Let π_i be the projection

$$\{(c_2/c_1, \dots, c_k/c_1)^{(j)} : (c_1, c_i)^{(j)} \in C_{1i}\} \to \{(c_i/c_1)^{(j)} : (c_1, c_i)^{(j)} \in C_{1i}\}.$$

The fiber of π_i at (c_1, c_2) corresponds one-to-one to $p_{1i}^{-1}(c_1, c_i) \cap B$. Hence the image of π_i has size $\lesssim N$ by (5.5). We replace B by $p_{1i}^{-1}(C_{1i}) \cap B$. (Note that (5.16) and (5.17) imply $|p_{1i}^{-1}(C_{1i}) \cap B| \sim N^k$.). We do this for each i (and shrink B a little if necessary.). Thus (5.15) is proved.

To prove (5.6), we want to show that under condition (5.15),

$$\left|\left\{(c_1, \dots, c_k, c_1', \dots, c_k') \in C_1 \times \cdots \times C_k \times C_1 \times \cdots \times C_k : \frac{c_i + a_i}{c_1 + a_1} = \frac{c_i' + a_i}{c_1' + a_1}, \forall i\right\}\right| \lesssim N^k. \tag{5.18}$$

It follows from the case of $\mathbb{C}^2$ that

$$\frac{c_2 + a_2}{c_1 + a_1} = \frac{c_2' + a_2}{c_1' + a_1} \tag{5.19}$$

has $\lesssim N^2$ solutions in c_1, c_2, c_1', c_2'. Fixing c_1, c_1', the equation

$$\frac{c_3 + a_3}{c_1 + a_1} = \frac{c_3' + a_3}{c_1' + a_1} \tag{5.20}$$

has at most N choices of c_3 (then c_3' is determined). Hence (5.19) and (5.20) together have $\lesssim N^3$ solutions in $c_1, c_2, c_3, c_1', c_2', c_3'$. Therefore, (5.18) follows by induction, and the finite point case of Theorem 5.2 is proved.

Only set theory is used in the argument above, hence Theorem 5.1, the other case of Theorem 5.2, and the case of $\mathbb{F}_p$ are proved in exactly the same way.

Remark 5.3. Theorems 5.1 and 5.2 are true if we replace $\mathbb{C}^k$ by $\mathbb{F}_p^k$.

6. Theorem 2 over $\mathbb{Q}$

We have a stronger result by using the λ_q constant, when the points are in $\mathbb{Q}^2$.

Theorem 6.1. *Given $\epsilon > 0$, there is $\delta > 0$ such that for any $P_1, P_2, P_3 \in \mathbb{Q}^2$ non-collinear, and $Q_1, \dots, Q_n \in \mathbb{Q}^2$, if*

$$|\{L(P_i, Q_j) : 1 \le i \le 3,\ 1 \le j \le n\}| \le n^{1/2+\epsilon}, \tag{6.1}$$

then for any $P \in \mathbb{Q}^2 \setminus \{P_1, P_2, P_3\}$, we have

$$|\{L(P, Q_j) : 1 \le j \le n\}| > n^{1-\delta}. \tag{6.2}$$

We use the same set-up as for the $\mathbb{C}$ case. Given a set $A \subset \mathbb{Q}$ with $N^{1-\epsilon} < |A| < N^{1+\epsilon}$ and $|A^2| < N^{1+5\epsilon}$, we want to bound the number of solutions $\xi_1, \dots, \xi_6 \in A$ in the following equation by $N^{3+\delta}$ for some $\delta(\epsilon) > 0$:

$$\xi_1 + \xi_2 + \xi_3 = \xi_4 + \xi_5 + \xi_6. \tag{6.3}$$

We use the λ_q constant of A for this. We recall

Definition. *Let $A \subset \mathbb{Z}$ be finite. The λ_q* constant *of A is*

$$\lambda_{q,A} = \frac{\| \sum_{a \in A} e(ax) \|_q}{\sqrt{|A|}}, \quad \textit{where} \quad e(\theta) = e^{2\pi i \theta}.$$

Proposition ([BC]). *Given $\varepsilon > 0$ and $q > 2$, there exists $\delta = \delta(q, \varepsilon)$ such that if $A \subset \mathbb{Z}$ with $|A^2| < |A|^{1+\varepsilon}$, then*

$$\lambda_q(A) < |A|^\delta,$$

where $\delta \to 0$ as $\varepsilon \to 0$. Therefore, $\| \sum_{a \in A} e(ax) \|_q < |A|^{1/2+\delta_6}$.

Define $r(\eta) = |\{(\xi_1, \xi_2, \xi_3) \in A \times A \times A : \eta = \xi_1 + \xi_2 + \xi_3\}|$. In the proposition above, we take $q = 6$. Then

$$\begin{aligned} |\{(\xi_1, \dots, \xi_6) : \xi_1 + \xi_2 + \xi_3 = \xi_4 + \xi_5 + \xi_6\}| &= \sum r(\eta)^2 \\ &= \Big\| \Big(\sum_{a \in A} e(ax) \Big)^3 \Big\|_2^2 = \Big\| \sum_{a \in A} e(ax) \Big\|_6^6 < (N^{(1+\epsilon)(1/2+\delta_6)})^6 = N^{3+\delta}. \end{aligned}$$

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