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Comparison results for Hessian equations via symmetrization

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1. Introduction

For a twice differentiable function u defined in a domain $\Omega \subset \mathbb{R}^n$ the k -Hessian operator ($k = 1, \dots, n$) is defined by

$$S_k(D^2u) = S_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.1}$$

where the λ 's are the eigenvalues of the Hessian matrix D^2u of u and S_k is the k^{th} elementary symmetric function. For example, for $k = 1$, $S_1(D^2u) = \Delta u$, while, for $k = n$, $S_n(D^2u) = \det D^2u$. Equations involving these operators, and some more general equations of the form

$$F(\lambda_1, \dots, \lambda_n) = f \quad \text{in } \Omega, \tag{1.2}$$

have been widely studied by many authors, who restrict their considerations to convenient cones of solutions with respect to which the operator in (1.2) is elliptic. Following [25] we define the cone Γ_k of ellipticity for (1.1) to be the connected component containing the positive cone $\Gamma^+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \forall i = 1, \dots, n\}$ of the set where S_k is positive. Thus Γ_k is an open, convex, symmetric cone with vertex at the origin and

$$\frac{\partial S_k}{\partial \lambda_i}(\lambda) > 0 \quad \text{on } \Gamma_k, \forall i = 1, \dots, n.$$

This implies that (1.1) is elliptic on functions $u \in C^2(\Omega)$ such that at each point of Ω the vector of eigenvalues of D^2u belongs to Γ_k .

Following [7] we call such functions *admissible* (with respect to S_k , $k \geq 2$) or *k-convex*. We will denote by $\Phi_0^k(\Omega)$ the set of admissible functions that vanish on the boundary of Ω .

In [12] Korevaar has shown that Γ_k can also be characterized as

$$\Gamma_k = \left\{ \lambda \in \mathbb{R}^n : S_k(\lambda) > 0, \frac{\partial S_k}{\partial \lambda_{i_1}}(\lambda) > 0, \dots, \frac{\partial^{k-1} S_k}{\partial \lambda_{i_1} \dots \partial \lambda_{i_{k-1}}}(\lambda) > 0, \right. \\ \left. \forall 1 \leq i_1 < \dots < i_{k-1} \leq n \right\};$$

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while in [11] Ivochkina proved that

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \dots, S_k(\lambda) > 0\}.$$

In this paper we deal with the following Dirichlet problem:

$$\begin{cases} S_k(D^2u) = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where g is a positive function satisfying the growth condition

$$g(x, u) \leq f(x) + \sigma|u|^p, \quad (1.4)$$

where $f > 0$ is a “smooth” function, $\sigma > 0$, $0 < p \leq k$. We also consider the symmetrized problem

$$\begin{cases} S_k(D^2v) = f^\# + \sigma|v|^p & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R, \end{cases} \quad (1.5)$$

where $f^\#$ is the spherically decreasing rearrangement of f and B_R is the ball centered at the origin, having the same k^{th} mean radius as Ω (see Section 2 for definitions). Our purpose is to compare solutions to problems (1.3) and (1.5).

Existence of solutions to (1.3) in $\Phi_0^k(\Omega)$ has been investigated by Caffarelli, Nirenberg and Spruck in [7], where they treat the case $f = f(x)$ and establish the classical solvability for the Dirichlet problem

$$\begin{cases} S_k(D^2u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if and only if $\partial\Omega$ satisfies

$$S_{k-1}(k_1, \dots, k_{n-1}) > c_0 > 0$$

where k_j denotes the principle curvatures of $\partial\Omega$, oriented so that convex domains have nonnegative curvatures. Many other authors have been interested in the problem of existence and regularity of classical or viscosity solutions to (1.3) under suitable hypotheses (see e.g. [20], [25], [26]). In any case existence and uniqueness of classical solutions to (1.3) are strictly related to the definition of the first eigenvalue of the Hessian operator $S_k(D^2u)$. In [13] and [24] for the case $k = n$ and in [26] for the general case $1 \leq k \leq n$, it is shown that there exists a positive constant λ_1 , which depends only on k , n and Ω , such that: (i) the problem

$$\begin{cases} S_k(D^2u) = |\lambda u|^k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

has a negative solution $\psi_1 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ for $\lambda = \lambda_1$; (ii) if $(\lambda^*, \psi^*) \in [0, \infty[\times (C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega}))$ is another solution to (1.6), then $\lambda^* = \lambda_1$, $\psi^* = \alpha\psi_1$ for some positive constant α . All these features suggest the well-known properties of the first eigenvalue

of linear second order elliptic operators. For this reason we refer to λ_1 as the *first eigenvalue* of the Hessian operator. Moreover, the following variational characterization holds true (see [26]):

$$(\lambda_1(\Omega))^k = \inf \left\{ - \int_{\Omega} u S_k(D^2u) dx : u \in \Phi_0^k(\Omega), \|u\|_{L^{k+1}(\Omega)} = 1 \right\}.$$

Concerning uniqueness results for solutions to problem (1.3) we refer to [13], [24], [25], [26]. No attempt has been made here to investigate existence or regularity questions for problem (1.3). Our aim is to prove comparison results by means of symmetrization techniques in the framework of papers [17], [4] in the case $k = n = 2$, and [22], [23] in the general case $1 \leq k \leq n$. In [17], [22], [23] the case $\sigma = 0$ is considered, while in [4] the case $\sigma > 0$ has been investigated when $k = n = 2$. Here we assume $\sigma > 0$ and $1 \leq k \leq n$. The paper is organized as follows. After introducing notation in Section 2, in Section 3 we state our main result and we prove a comparison result between a suitable rearrangement of the solution to (1.3) with the solution to the symmetrized problem (1.5). The rearrangements involved are those which preserve the so called “quermassintegrals” of level sets of solutions (see [17], [22], [23]) and the main tools are the Alexandrov–Fenchel inequalities (see Section 2). In the last section we prove some kind of Moser inequality related to Hessian integrals in the framework of papers by Moser [14] and Trudinger [19], [22].

2. Notation and preliminaries

Given a measurable function $u : \Omega \rightarrow \mathbb{R}$, we recall the definition of decreasing rearrangement of u . After defining the *distribution function* of u by

$$\mu(t) = \mathcal{L}^n(\{x \in \Omega : |u(x)| > t\}), \quad t \geq 0,$$

where \mathcal{L}^n denotes the n -dimensional Lebesgue measure in \mathbb{R}^n , the *decreasing rearrangement* of u is the distribution function of μ , i.e.

$$u^*(s) = \sup\{t \geq 0 : \mu(t) \geq s\}, \quad s \in [0, |\Omega|].$$

The following properties will prove useful in Section 3:

- (i) $\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(0,|\Omega|)}$, $p \geq 1$;
- (ii) (Hardy–Littlewood theorem) if $u, v : \Omega \rightarrow \mathbb{R}$ are measurable functions, then

$$\int_{\Omega} |uv| dx \leq \int_0^{|\Omega|} u^*(s)v^*(s) ds.$$

By the *spherically decreasing rearrangement* of u we mean

$$u^\#(x) = u^*(\omega_n|x|^n), \quad x \in \Omega^\#,$$

where ω_n denotes the measure of the unit ball of \mathbb{R}^n and $\Omega^\#$ is the ball centered at the origin, having the same measure as Ω (for more details see [18] and the references given there).

We go on with an appropriate definition of quermassintegral for non-convex domains (see [22]). For the basic theory of convex sets, the reader is referred, for example, to [6], [16]. Let Ω be a bounded domain in \mathbb{R}^n with boundary $\partial\Omega \in C^2$, having principal curvatures $k' = (k_1, \dots, k_{n-1})$ (oriented so that convex domains have non-negative curvatures). For $k = 1, \dots, n - 1$ we define the k^{th} mean curvature of $\partial\Omega$ by

$$H_k(\partial\Omega) = S_k(k_1, \dots, k_{n-1}),$$

while

$$H_0 = S_0 \equiv 1.$$

For $k = 0, \dots, n - 1$ the quermassintegral $V_k(\Omega)$ is then defined by

$$V_k(\Omega) = \frac{1}{n \binom{n-1}{k}} \int_{\partial\Omega} H_{n-k-1}(\partial\Omega) d\mathcal{H}^{n-1}, \tag{2.1}$$

where \mathcal{H}^{n-1} denotes the $(n - 1)$ -dimensional Hausdorff measure in \mathbb{R}^n . For $k = n$ we take $V_n(\Omega) = \mathcal{L}^n(\Omega)$. When $k \leq n - 2$, we restrict attention to quermassintegrals $V_k(\Omega)$ of domains Ω which are $(n - k - 1)$ -convex, that is

$$H_j(\partial\Omega) \geq 0, \quad j = 1, \dots, n - k - 1, \tag{2.2}$$

thereby ensuring that the integrand in (2.1) is non-negative. When $\partial\Omega$ is connected, (2.2) is equivalent to $H_{n-k-1}(\partial\Omega) \geq 0$, while for $k = 0$ we have to assume that Ω is $(n - 1)$ -convex, that is, the components of Ω are convex in the usual sense. We explicitly mention that in the latter case the following equality holds:

$$V_0(\Omega) = \omega_n \times \text{number of components of } \Omega.$$

When $k = n$, we clearly have $V_n(\Omega_1) \leq V_n(\Omega_2)$ whenever $\Omega_1 \subset \Omega_2$. For convex domains we also have $V_k(\Omega_1) \leq V_k(\Omega_2)$ whenever $\Omega_1 \subset \Omega_2$. But this property is not true in general when Ω_1 and Ω_2 are simply $(n - k - 1)$ -convex. We get around this difficulty by only considering functions belonging to $\Phi_0^k(\Omega)$, whose sublevel sets have the monotonicity property. Indeed, first of all, it is easy to see that, if $k > 0$ and $u \in \Phi_0^k(\Omega)$, then u is subharmonic and thus $u \leq 0$ in Ω ; so let $\Omega_t = \{x \in \Omega : u(x) < t\}$ and $\Omega_s = \{x \in \Omega : u(x) < s\}$, with $\min_{\Omega} u < s \leq t < 0$. If $\Sigma_s = \{x \in \Omega : u(x) = s\}$ and $\Sigma_t = \{x \in \Omega : u(x) = t\}$ are non-degenerate level surfaces with $s \leq t$, then (see [22])

$$V_{n-k+1}(\Omega_s) \leq V_{n-k+1}(\Omega_t). \tag{2.3}$$

One can also show a derivation formula for quermassintegrals of sublevel sets of a function $u \in \Phi_0^k(\Omega)$ (see [15]):

$$\frac{d}{dt} V_{n-k+1}(\Omega_t) = \frac{1}{\binom{n}{k}} \int_{\Sigma_t} H_{k-1}(\Sigma_t) |Du|^{-1} d\mathcal{H}^{n-1}. \tag{2.4}$$

Remark 2.1. Actually, (2.3) and (2.4) are still valid in a set wider than $\Phi_0^k(\Omega)$ (see [22]).

Now we introduce another useful notion. Let Ω be an $(n - k - 1)$ -convex domain with $k \leq n - 2$. We define the k^{th} mean radius of Ω by

$$\zeta_k(\Omega) = (V_k(\Omega)/\omega_n)^{1/k}, \quad k = 1, \dots, n, \quad \zeta_k(\emptyset) = 0.$$

The isoperimetric inequalities, or Alexandrov–Fenchel inequalities, for quermassintegrals contained in [21] provide

$$\zeta_l(\Omega) \leq \zeta_k(\Omega), \quad 1 \leq k \leq l \leq n. \tag{2.5}$$

We note that (2.5) includes the classical isoperimetric inequality when $l = n$ and $k = n - 1$.

Now we are able to symmetrize a function. Let $u \in \Phi_0^k(\Omega)$, $k = 1, \dots, n$; the k -symmetrand u_{k-1}^* of u is defined by

$$u_{k-1}^*(x) = \sup\{t \leq 0 : \zeta_{n-k+1}(\Omega_t) \leq |x|, Du \neq 0 \text{ on } \Sigma_t\} \tag{2.6}$$

for $|x| \leq R = \zeta_{n-k+1}(\Omega)$.

The following statements hold (see [22], [23]):

- (1) set $\tilde{u}_{k-1}(r) = u_{k-1}^*(|x|)$ for $r = |x|$; then $\tilde{u}_{k-1}(0) = \min_{\Omega} u$ and $\tilde{u}_{k-1}(R) = 0$;
- (2) \tilde{u}_{k-1} is a non-decreasing function on $[0, R]$;
- (3) $\tilde{u}_{k-1} \in C^{0,1}([0, R])$ and moreover

$$0 \leq \tilde{u}'_{k-1} \leq \sup_{\Omega} |Du| \quad \text{a.e.}$$

Remark 2.2. For $k = 1$ we find $u_0^*(x) = u^\#(x)$, the Schwarz symmetrand of u . For $k = n = 2$ we find $u_1^*(x) = u^*(2\pi|x|)$, where u^* is the rearrangement of u with respect to the perimeter of its level sets (see [17]).

By definition, through (2.5), it follows that $\zeta_k(\Omega_t) \leq \zeta_k(\{u_m^* < t\})$ for $k \geq n - m$, and equality holds when $n - m = k$; this implies

$$\|u\|_{L^p(\Omega)} \leq \|u_k^*\|_{L^p(B_R)}, \quad p \geq 1. \tag{2.7}$$

Let $u \in \Phi_0^k(\Omega)$ and let $\min_{\Omega} < t < 0$. We recall the following *Reilly equality* (see [15]):

$$\int_{\Omega_t} S_k(D^2u) dx = \frac{1}{k} \int_{\Sigma_t} |Du|^k H_{k-1}(\Sigma_t) d\mathcal{H}^{n-1}. \tag{2.8}$$

Finally, we consider the following functional, known as the *Hessian integral*:

$$I_{k,p}[u, \Omega] = \int_m^0 dt \int_{\Sigma_t} H_{k-1}(\Sigma_t) |Du|^{p-1} d\mathcal{H}^{n-1};$$

the following *Pólya–Szegő principle* holds (see [22]):

$$I_{k,p}[u, \Omega] \geq I_{k,p}[u_{k-1}^*, B_R], \quad p \geq 1. \tag{2.9}$$

In the radial case the Hessian integral can be written as follows:

$$I_{k,p}(u_{k-1}^*, B_R) = n \binom{n-1}{k-1} \omega_n \int_0^R f^p(\omega_n \rho^{n-k+1}) \rho^{n-k} d\rho \tag{2.10}$$

where $f(\omega_n|x|^{n-k+1}) = |\nabla u_{k-1}^*(x)|$.

3. Comparison results

In this section we will prove that the rearranged solution to problem (1.3) can be estimated by the solution to the conveniently symmetrized problem (1.5). In order to prove this comparison result we will adapt an argument which appears in [9] in the case of the p -laplacian operator and in [4] in the case $k = n = 2$.

Remark 3.1. Under the assumption that Ω is strictly $(k - 1)$ -convex, in [8] it is shown that, if $\Omega \in C^{3,1}$, f is a smooth positive function in Ω and $\sigma < (\lambda_1(\Omega))^k$ when $p = k$, then problem (1.3) has a solution in $C^{3,\alpha}(\Omega) \cap C(\overline{\Omega})$ for some $\alpha \in (0, 1)$. In fact, other results concerning the existence of classical or non-classical solution to (1.3) may be found, for instance, in [11], [10], [24], [25], [26]. For uniqueness of solutions to (1.3) we refer to [26] for the case $p < 1$ and to [25] for the case $1 \leq p \leq k$, provided Ω is sufficiently narrow or σ sufficiently small.

Remark 3.2. We now turn our attention to the radial case. Let $u(x) = \phi(r)$, $|x| = r$, be a radially symmetric function defined in the ball B_R . We have

$$\begin{aligned} u_{x_i} &= \phi' \frac{x_i}{r}, \\ u_{x_i x_i} &= \phi'' \frac{x_i^2}{r^2} + \phi' \frac{r^2 - x_i^2}{r^3}, \quad u_{x_i x_j} = \phi'' \frac{x_i x_j}{r^2} - \phi' \frac{x_i x_j}{r^3}, \quad i \neq j, \end{aligned}$$

for $i, j = 1, \dots, n$. At the point $x = (r, 0, \dots, 0)$, D^2u is diagonal, so $u_{x_1 x_1} = \phi''$ and $u_{x_j x_j} = \phi'/r$, $j \geq 2$. Since S_k is invariant under rotations,

$$\begin{aligned} S_k(D^2u) &= \frac{(n-1)!}{(n-k)!(k-1)!} \phi'' \left(\frac{\phi'}{r}\right)^{k-1} + \frac{(n-1)!}{(n-k-1)!k!} \left(\frac{\phi'}{r}\right)^k \\ &= \frac{(n-1)!}{(n-k)!(k-1)!} r^{-n+1} \left(\frac{r^{n-k}}{k} (\phi')^k\right)'. \end{aligned}$$

If $f(x) = \psi(r)$ is also radially symmetric, then (1.5) becomes an ordinary differential equation on $[0, R]$, with limit conditions $\phi'(0) = \phi(R) = 0$, i.e.

$$\begin{cases} \frac{(n-1)!}{(n-k)!(k-1)!} r^{-n+1} \left(\frac{r^{n-k}}{k} (\phi')^k\right)' = f^\# + \sigma |\phi|^p & \text{in } (0, R), \\ \phi'(0) = \phi(R) = 0, \\ (\phi'', \phi'/r, \dots, \phi'/r) \in \Gamma_k. \end{cases}$$

The existence (and uniqueness) of a solution to such a problem can be proved following the lines of [5], [3]. Furthermore, if $(\phi'', \phi'/r, \dots, \phi'/r) \in \Gamma_k$, then $S_1(D^2\phi) = r^{-n+1}(r^{n-1}\phi')' > 0$. Thus $G(r) = r^{n-1}\phi'$ is an increasing function in $(0, R)$; since $G(0) = 0$, we deduce that $G \geq 0$ and hence ϕ is increasing in $(0, R)$.

Theorem 3.1. *Suppose that f, g are smooth functions defined on a $(k - 1)$ -convex domain Ω in \mathbb{R}^n such that (1.4) is satisfied, and let u, v be classical solutions to problems (1.3) and (1.5) respectively, with $\sigma < \min\{(\lambda_1(\Omega))^k, (\lambda_1(B_R))^k\}$ when $p = k$, and σ sufficiently small when $0 < p < k$. Then*

$$0 \geq u_{k-1}^*(x) \geq v(x), \quad x \in B_R. \tag{3.1}$$

Proof. Let $\min_{\Omega} u < t < 0$ and let us integrate the equation in (1.3) on the sublevel set Ω_t ; then

$$\int_{\Omega_t} S_k(D^2u) \, dx \leq \int_{\Omega_t} f \, dx + \sigma \int_{\Omega_t} (-u)^p \, dx. \tag{3.2}$$

By (2.8), the Hölder inequality and (2.4) we get

$$\begin{aligned} \int_{\Omega_t} S_k(D^2u) \, dx &= \frac{1}{k} \int_{\Sigma_t} |Du|^k H_{k-1}(\Sigma_t) \, d\mathcal{H}^{n-1} \geq \frac{1}{k} \frac{(\int_{\Sigma_t} H_{k-1}(\Sigma_t) \, d\mathcal{H}^{n-1})^{k+1}}{(\int_{\Sigma_t} H_{k-1}(\Sigma_t) |Du|^{-1} \, d\mathcal{H}^{n-1})^k} \\ &= \frac{1}{k} \frac{(\int_{\Sigma_t} H_{k-1}(\Sigma_t) \, d\mathcal{H}^{n-1})^{k+1}}{\binom{n}{k}^k (\frac{d}{dt} V_{n-k+1}(\Omega_t))^k}. \end{aligned}$$

We recall that, by definition,

$$\int_{\Sigma_t} H_{k-1}(\Sigma_t) \, d\mathcal{H}^{n-1} = n \binom{n-1}{n-k} V_{n-k}(\Omega_t) = n \binom{n-1}{n-k} \omega_n (\zeta_{n-k}(\Omega_t))^{(n-k)(k+1)},$$

so we can write

$$\int_{\Omega_t} S_k(D^2u) \, dx \geq \frac{n^{k+1} \binom{n-1}{n-k}^{k+1} \omega_n^{k+1} (\zeta_{n-k}(\Omega_t))^{(n-k)(k+1)}}{k \binom{n}{k}^k (\frac{d}{dt} V_{n-k+1}(\Omega_t))^k}.$$

On the other hand, by the definition of the k^{th} mean radius we have

$$\frac{d}{dt} V_{n-k+1}(\Omega_t) = \frac{\omega_n}{n-k+1} \zeta_{n-k+1}^{n-k}(\Omega_t) \frac{d}{dt} \zeta_{n-k+1}(\Omega_t);$$

thus, recalling the isoperimetric inequalities (2.5), we obtain

$$\int_{\Omega_t} S_k(D^2u) \, dx \geq \frac{n^{k+1}}{k \omega_n^{k-1}} \frac{\binom{n-1}{n-k}^{k+1}}{\binom{n}{k}^k} (n-k+1)^k \frac{\zeta_{n-k+1}^{n-k}(\Omega_t)}{(\frac{d}{dt} \zeta_{n-k+1}(\Omega_t))^k}. \tag{3.3}$$

Let us come back to (3.2). For the first integral on the right hand side of (3.2), by the Hardy–Littlewood inequality and (2.5), we obtain

$$\int_{\Omega_t} f \, dx \leq \int_0^{\mu(t)} f^*(s) \, ds \leq \int_0^{\omega_n \zeta_{n-k+1}^n(\Omega_t)} f^*(s) \, ds, \quad (3.4)$$

while for the second one we get

$$\begin{aligned} \int_{\Omega_t} (-u)^p \, dx &= \int_{-\infty}^t (-s)^p \mu'(s) \, ds = (-t)^p \mu(t) + p \int_{-\infty}^t (-s)^{p-1} \mu(s) \, ds \leq \\ &\leq \omega_n (-t)^p \zeta_{n-k+1}^n(\Omega_t) + \omega_n p \int_{-\infty}^t (-s)^{p-1} \zeta_{n-k+1}^n(\Omega_s) \, ds, \end{aligned}$$

which, in terms of rearrangements, becomes

$$\begin{aligned} \int_{\Omega_t} (-u)^p \, dx &\leq \omega_n r^n (-\tilde{u}_{k-1}(r))^p + \omega_n p \int_0^r s^n (-\tilde{u}_{k-1}(s))^{p-1} \tilde{u}_{k-1}(s)' \, ds \\ &= n \omega_n \int_0^r s^{n-1} (-\tilde{u}_{k-1}(s))^p \, ds, \end{aligned} \quad (3.5)$$

with $r = \zeta_{n-k+1}(\Omega_t)$. Observing that

$$\frac{\zeta_{n-k+1}^{n-k}(\Omega_t)}{\left(\frac{d}{dt} \zeta_{n-k+1}(\Omega_t)\right)^k} = r^{n-k} (\tilde{u}_{k-1}(r)')^k,$$

by (3.3)–(3.5) we get

$$c_{n,k} r^{n-k} (\tilde{u}_{k-1}(r)')^k \leq \int_0^{\omega_n r^n} f^*(s) \, ds + n \omega_n \sigma \int_0^r s^{n-1} (-\tilde{u}_{k-1}(s))^p \, ds \quad (3.6)$$

where

$$c_{n,k} = \frac{n^{k+1}}{k \omega_n^{k-1}} \frac{\binom{n-1}{n-k}^{k+1}}{\binom{n}{k}^k} (n-k+1)^k.$$

Obviously the radial solution v to the symmetrized problem (1.5) satisfies

$$c_{n,k} r^{n-k} (\tilde{v}_{k-1}(r)')^k = \int_0^{\omega_n r^n} f^*(s) \, ds + n \omega_n \sigma \int_0^r s^{n-1} (-\tilde{v}_{k-1}(s))^p \, ds. \quad (3.7)$$

Let $U(r) = \int_0^r s^{n-1} \tilde{u}_{k-1}(s)^p \, ds$ and $V(r) = \int_0^r s^{n-1} \tilde{v}_{k-1}(s)^p \, ds$; let us prove that $U(r) \leq V(r)$ for all $r \in [0, R]$. Set

$$\varphi_1(s) = \frac{(U(s)^{(p+1)/p} - V(s)^{(p+1)/p})^+}{U(s)^{1/p}};$$

integrating by parts we obtain

$$\begin{aligned} \int_{U>V} (-\tilde{u}_{k-1}(r)')\varphi_1(r) dr &= \int_{U>V} \tilde{u}_{k-1}(r)\varphi_1'(r) dr \\ &= \int_{U>V} \tilde{u}_{k-1}(r) \left(r\tilde{u}_{k-1}(r)^p - \frac{p+1}{p} \left(\frac{V(r)}{U(r)} \right)^{1/p} r\tilde{v}_{k-1}(r)^p \right. \\ &\quad \left. + \frac{1}{p} \left(\frac{V(r)}{U(r)} \right)^{(p+1)/p} r\tilde{u}_{k-1}(r)^p \right) dr. \end{aligned} \tag{3.8}$$

Analogously, by considering

$$\varphi_2(s) = \frac{(U(s)^{(p+1)/p} - V(s)^{(p+1)/p})}{V(s)^{1/p}}$$

and integrating by parts we have

$$\begin{aligned} \int_{U>V} (-\tilde{v}_{k-1}(r)')\varphi_2(r) dr &= \int_{U>V} \tilde{v}_{k-1}(r)\varphi_2'(r) dr \\ &= \int_{U>V} \tilde{v}_{k-1}(r) \left(-r\tilde{v}_{k-1}(r)^p + \frac{p+1}{p} \left(\frac{U(r)}{V(r)} \right)^{1/p} r\tilde{v}_{k-1}(r)^p \right. \\ &\quad \left. - \frac{1}{p} \left(\frac{U(r)}{V(r)} \right)^{(p+1)/p} r\tilde{v}_{k-1}(r)^p \right) dr. \end{aligned} \tag{3.9}$$

Let $\gamma = (p + 1)/p$; subtracting (3.9) from (3.8) we get

$$\begin{aligned} \int_{U>V} [(-\tilde{u}_{k-1}(r)')\varphi_1(r) + \tilde{v}_{k-1}(r)'\varphi_2(r)] dr &= \\ \int_{U>V} r U^\gamma \left[\left(\frac{\tilde{u}_{k-1}^{1/(\gamma-1)}}{U} \right)^\gamma - \left(\frac{\tilde{v}_{k-1}^{1/(\gamma-1)}}{V} \right)^\gamma - \gamma \left(\frac{\tilde{v}_{k-1}^{1/(\gamma-1)}}{V} \right)^{\gamma-1} \left(\frac{\tilde{u}_{k-1}^{1/(\gamma-1)}}{U} - \frac{\tilde{v}_{k-1}^{1/(\gamma-1)}}{V} \right) \right] \\ + \int_{U>V} r V^\gamma \left[\left(\frac{\tilde{v}_{k-1}^{1/(\gamma-1)}}{V} \right)^\gamma - \left(\frac{\tilde{u}_{k-1}^{1/(\gamma-1)}}{U} \right)^\gamma - \gamma \left(\frac{\tilde{u}_{k-1}^{1/(\gamma-1)}}{U} \right)^{\gamma-1} \left(\frac{\tilde{v}_{k-1}^{1/(\gamma-1)}}{V} - \frac{\tilde{u}_{k-1}^{1/(\gamma-1)}}{U} \right) \right] \\ &= I_1 + I_2. \end{aligned}$$

Both I_1 and I_2 are non-negative since the function $g(t) = t^\gamma$ is convex for all $\gamma > 1$. On the other hand, by (3.6) and (3.7),

$$\begin{aligned} I_1 + I_2 &\leq \int_{U>V} \left(\frac{U(r)^{(p+1)/p} - V(r)^{(p+1)/p}}{r^{(n-k)/k}} \right) \\ &\quad \times \left[\left(\frac{c_1 F(r) + c_2 U(r)}{U(r)^{k/p}} \right)^{1/k} - \left(\frac{c_1 F(r) + c_2 V(r)}{V(r)^{k/p}} \right)^{1/k} \right] dr, \end{aligned} \tag{3.10}$$

where $F(s) = \int_0^{\omega_n s^n} f^*(r) dr$, $c_1 = 1/c_{n,k}$, $c_2 = n\omega_n/c_{n,k}$. The right hand side of (3.10) is always negative, since the function $h(t) = (1 + t)/t^{k/p}$ is decreasing whenever $p \leq k$.

It follows that $|U > V| = 0$, i.e. $U(s) \leq V(s)$ for all $s \in (0, R)$. This estimate, combined with (3.6) and (3.7), gives (3.1) and the claim follows. \square

Remark 3.3. We recall that $V_0(\Omega) = \omega_n \times$ number of components of Ω . Since a C^2 domain is $(n - 1)$ -convex if and only if it is convex (see [21]), if u is a classical solution to (1.3) for $k = n$ and $\lambda(D^2u) \in \Gamma_n$, its level sets Ω_t are convex for a.e. t and this implies $V_0(\Omega_t) = \omega_n$ for a.e. t .

Remark 3.4. It immediately follows from (3.1) and (2.7) that

$$\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(B_R)}, \quad p \geq 1.$$

Proposition 3.1. *Under the assumptions of Theorem 3.1 we have*

$$I_{k,k+1}(u, \Omega) \leq I_{k,k+1}(v, B_R).$$

Proof. The above estimate is an easy consequence of the inequality

$$\begin{aligned} \int_{\Sigma_t} |Du|^k H_{k-1}(\Sigma_t) d\mathcal{H}^{n-1} &\leq k \int_0^{\omega_n \zeta_{n-k+1}(\Omega_t)} f^*(r) dr \\ &\quad + n\omega_n k \int_0^{\zeta_{n-k+1}(\Omega_t)} s^{n-1} (-\tilde{u}_{k-1}(s))^p ds \end{aligned}$$

and the equality holding for v . \square

4. Moser type inequalities for Hessian integrals in the limit case $p = n - k + 1$

In [22] Trudinger has proved some kind of Sobolev inequalities for functions in $A_{k-1}(\Omega)$. Namely, if Ω is a $(k - 1)$ -convex subset of \mathbb{R}^n , $k = 1, \dots, n$, and $u \in \Phi_0^k(\Omega)$, then

$$\|u\|_{L^q(\Omega)}^p \leq C I_{k,p}(u, \Omega)$$

for any $q \leq np/(n - k + 1 - p)$ if $p < n - k + 1$, and for $q \leq +\infty$ if $p > n - k + 1$, with a constant C depending on k, n, p and Ω . In the limit case $p = n - k + 1$, he has obtained the estimate

$$\|u\|_{L_\Psi(\Omega)}^p \leq C I_{k,p}(u, \Omega), \tag{4.1}$$

where $L_\Psi(\Omega)$ is the Orlicz space associated to the function

$$\Psi(t) = e^{|t|^{p'}} - 1.$$

This last result recovers the traditional one when $k = 1$ (see [19]); in this case it is well known that the estimate (4.1) has been made more precise by many authors in many directions (see, e.g., [14], [1], [2]). For example, in [14] it is proved that if $u \in C_0^\infty(\Omega)$ and $\|\nabla u\|_{L^n(\Omega)} \leq 1$, then

$$\int_{\Omega} e^{\beta|u(x)|^{n'}} dx \leq C(n)|\Omega|, \quad \forall \beta \leq \beta_n = (n\omega_n^{1/n})^{n'}. \tag{4.2}$$

As a matter of fact the integral in (4.2) is finite for any β , but it is uniformly bounded only for $\beta \leq \beta_n = (n\omega_n^{1/n})^{n'}$. Indeed, if $\beta_n = (n\omega_n^{1/n})^{n'}$, then a sequence $\{u_m\} \subset C_0^\infty(\Omega)$ can be constructed so that $\|\nabla u_m\|_{L^n(\Omega)} \leq 1$ and

$$\lim_{m \rightarrow +\infty} \int_{\Omega} e^{\beta |u_m(x)|^{n'}} dx = +\infty.$$

Similar questions arise concerning estimate (4.1). We will prove the following

Theorem 4.1. *Let Ω be a $(k - 1)$ -convex subset of \mathbb{R}^n , $k = 1, \dots, n$, and let $u \in \Phi_0^k(\Omega)$ be such that*

$$J_{k,p}(u, \Omega) = \frac{I_{k,p}(u, \Omega)}{\binom{n-1}{k-1}} \leq 1 \quad \text{for } p = n - k + 1.$$

Then

$$\int_{\Omega} e^{\beta |u(x)|^{(n-k+1)/(n-k)}} dx \leq c_{n,k} (V_{n-k+1}(\Omega))^{n/(n-k+1)} \tag{4.3}$$

for all $\beta \leq \beta_{n,k} = (n\omega_n^{1/(n-k+1)})^{(n-k+1)/(n-k)}$.

In order to prove the above theorem we need the following lemma due to Adams [1].

Lemma 4.1. *Let $a(s, t)$ be a non-negative measurable function on $\mathbb{R} \times [0, +\infty[$ such that, for some $p \in]1, +\infty[$,*

$$a(s, t) \leq 1 \quad \text{for a.e. } 0 < s < t, \tag{4.4}$$

$$\sup_{t>0} \left(\int_{-\infty}^0 a(s, t)^{p'} ds + \int_0^{+\infty} a(s, t)^{p'} ds \right)^{1/p'} = \gamma < +\infty. \tag{4.5}$$

Suppose that a non-negative function $F(s)$ on \mathbb{R} satisfies

$$\int_{-\infty}^{+\infty} F(s)^p ds \leq 1.$$

Then there exists a constant $c_0 = c_0(p, \gamma)$ such that

$$\int_0^{+\infty} e^{-G(t)} dt \leq c_0, \quad \text{where } G(t) = t - \left(\int_{-\infty}^{+\infty} a(s, t) F(s) ds \right)^{p'}.$$

Proof of Theorem 4.1. Let us first observe that by the Pólya–Szegő principle (2.9) and by (2.7) we have

$$J_{k,p}(u_{k-1}^*, B_R) \leq J_{k,p}(u, \Omega),$$

$$\int_{\Omega} \exp(\beta |u|^{(n-k+1)/(n-k)}) dx \leq \int_{B_R} \exp(\beta |u_{k-1}^*|^{(n-k+1)/(n-k)}) dx;$$

then it is enough to prove the statement for $u(x) = u_{k-1}^*(x)$. Set

$$f(\omega_n |x|^{n-k+1}) = |\nabla u_{k-1}^*(x)|;$$

this implies that $f(s) = \omega_n(n-k+1)\tilde{u}'_{k-1}(s)(s/\omega_n)^{(n-k)/(n-k+1)}$. We have

$$-\tilde{u}_{k-1}(s) = \frac{1}{(n-k+1)\omega_n^{1/(n-k+1)}} \int_s^{V_{n-k+1}(\Omega)} f(r)r^{1/(n-k+1)} \frac{dr}{r}.$$

It follows that

$$\begin{aligned} & -\tilde{u}_{k-1}(e^{-(n-k+1)t/n}V_{n-k+1}(\Omega)) \\ &= \frac{1}{n\omega_n^{1/(n-k+1)}} \int_0^t f(e^{-(n-k+1)s/n}V_{n-k+1}(\Omega))(e^{-(n-k+1)s/n}V_{n-k+1}(\Omega))^{1/(n-k+1)} ds \\ &= \frac{1}{n\omega_n^{1/(n-k+1)}} \int_{-\infty}^{+\infty} a(s,t)F(s) ds, \end{aligned}$$

where

$$a(s,t) = \begin{cases} 0 & \text{if } s < 0 \text{ or } s > t, \\ 1 & \text{if } 0 \leq s \leq t, \end{cases}$$

$$F(s) = \begin{cases} f(e^{-(n-k+1)s/n}V_{n-k+1}(\Omega))(e^{-(n-k+1)s/n}V_{n-k+1}(\Omega))^{1/p} & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

It is clear that (4.4) and (4.5) are satisfied. On the other hand, by (2.10),

$$\begin{aligned} J_{k,p}(u_{k-1}^*, B_R) &= \frac{I_{k,p}(u_{k-1}^*, B_R)}{\binom{n-1}{k-1}} = n\omega_n \int_0^R f^p(\omega_n\rho^{n-k+1})\rho^{n-k} d\rho \\ &= \frac{n}{n-k+1} \int_0^{V_{n-k+1}(\Omega)} f^p(r) dr \\ &= \int_0^{+\infty} f(e^{-(n-k+1)s/n}V_{n-k+1}(\Omega))^{n-k+1} e^{-(n-k+1)s/n}V_{n-k+1}(\Omega) ds \\ &= \int_0^{+\infty} F(s)^p ds \leq 1. \end{aligned}$$

Then Lemma 4.1 gives

$$\int_0^{+\infty} \exp(-t + (|\tilde{u}_{k-1}(e^{-(n-k+1)t/n}V_{n-k+1}(\Omega))|n\omega_n^{1/(n-k+1)})^{(n-k+1)/(n-k)}) dt \leq c_0,$$

that is,

$$\begin{aligned} & \int_0^{V_{n-k+1}(\Omega)} \exp(|\tilde{u}_{k-1}(s)|n\omega_n^{1/(n-k+1)})^{\frac{n-k+1}{n-k}} s^{\frac{k-1}{n-k+1}} ds \\ & \leq c_0 \left(\frac{n-k+1}{n} \right) (V_{n-k+1}(\Omega))^{n/(n-k+1)} \end{aligned}$$

and the claim follows. \square

Remark 4.1 (Case $k \leq n/2$). As a matter of fact, by definition of $\Phi_0^k(\Omega)$, the integral in (4.3) is finite for every β , but it is uniformly bounded only for $\beta \leq \beta_{n,k}$. Let us show that, if $\beta > \beta_{n,k}$, then there exists a sequence $\{u_m\}_{m \in \mathbb{N}} \subset \Phi_0^k(\Omega)$ such that $J_{k,p}(u_m, \Omega) \leq 1$ and

$$\lim_{m \rightarrow +\infty} \int_{\Omega} e^{\beta |u_m(x)|^{(n-k+1)/(n-k)}} dx = +\infty.$$

Let B_R be the ball centered at the origin such that $V_{n-k+1}(\Omega) = \omega_n R^{n-k+1} = 1$ and define

$$u_m(x) = \begin{cases} \frac{m^{-1/(n-k+1)}}{n\omega_n^{1/(n-k+1)}} \left(\frac{n}{n-k+1}\right)^{(n-k)/(n-k+1)} \log(\omega_n |x|^{n-k+1}) & \text{if } \omega_n |x|^{n-k+1} > e^{-m}, \\ a_m |x|^3 + b_m |x|^2 + d_m & \text{if } \omega_n |x|^{n-k+1} \leq e^{-m}, \end{cases}$$

where

$$\begin{aligned} a_m &= -\frac{2}{3} m^{-1/(n-k+1)} \omega_n^{2/(n-k+1)} \left(\frac{n}{n-k+1}\right)^{-1/(n-k+1)} e^{3m/(n-k+1)}, \\ b_m &= \frac{3}{2} m^{-1/(n-k+1)} \omega_n^{1/(n-k+1)} \left(\frac{n}{n-k+1}\right)^{-1/(n-k+1)} e^{2m/(n-k+1)}, \\ d_m &= -\frac{m^{(n-k)/(n-k+1)}}{n\omega_n^{1/(n-k+1)}} \left(\frac{n}{n-k+1}\right)^{(n-k)/(n-k+1)} \\ &\quad - \frac{5}{6} m^{-1/(n-k+1)} \frac{n-k+1}{n\omega_n^{1/(n-k+1)}} \left(\frac{n}{n-k+1}\right)^{(n-k)/(n-k+1)}. \end{aligned}$$

We explicitly mention that $u_m \in C^2(B_R)$ and $S_j(D^2u) > 0$ for any $1 \leq j \leq k$ so that $u \in \Phi_0^k(\Omega)$. First of all we compute $J_{k,p}(u_m, B_R)$. We have

$$f(\omega_n |x|^{n-k+1}) = |\nabla u_m(x)| = \begin{cases} \frac{m^{-1/(n-k+1)}}{n\omega_n^{1/(n-k+1)}} (n-k+1) \left(\frac{n}{n-k+1}\right)^{(n-k)/(n-k+1)} \frac{1}{|x|} & \text{if } \omega_n |x|^{n-k+1} > e^{-m}, \\ 3a_m |x|^2 + 2b_m |x| & \text{if } \omega_n |x|^{n-k+1} \leq e^{-m}; \end{cases}$$

then, setting $r_m = (e^{-m}/\omega_n)^{1/(n-k+1)}$, we have

$$\begin{aligned} I_{k,p}(u_m, B_R) &= n \binom{n-1}{k-1} \omega_n \int_0^R f^p(\omega_n \rho^{n-k+1}) \rho^{n-k} d\rho \\ &= n \binom{n-1}{k-1} \omega_n \left[\int_0^{r_m} f^p(\omega_n \rho^{n-k+1}) \rho^{n-k} d\rho + \int_{r_m}^R f^p(\omega_n \rho^{n-k+1}) \rho^{n-k} d\rho \right] \\ &= \frac{c(n,k)}{m} + \binom{n-1}{k-1}, \end{aligned}$$

where $c(n, k)$ is a positive constant depending only on n and k . This implies

$$\lim_{m \rightarrow +\infty} I_{k,p}(u_m, B_R) = \binom{n-1}{k-1}$$

and then

$$\lim_{m \rightarrow +\infty} J_{k,p}(u_m, B_R) = 1.$$

Furthermore, we have

$$\begin{aligned} \int_{B_R} e^{\beta|u_m(x)|^{(n-k+1)/(n-k)}} dx &\geq \int_0^{e^{-m}} \exp\left(\frac{m\beta}{\beta_{n,k}} \frac{n}{n-k+1}\right) s^{(k-1)/(n-k+1)} ds \\ &= \frac{n-k+1}{n} \exp\left[\frac{n}{n-k+1} \left(\frac{\beta}{\beta_{n,k}} - 1\right) m\right], \end{aligned}$$

which tends to $+\infty$ as m goes to $+\infty$ when $\beta > \beta_{n,k}$.

Remark 4.2. When $u \in C^2(\Omega)$ is a convex function, vanishing on the boundary $\partial\Omega$, estimate (4.3) can be improved in the sense that, if $J_{k,p}(u, \Omega) \leq 1$, then there exists a positive constant $C = C(n, k)$ such that $\|u\|_{L^\infty(\Omega)} \leq C$. In fact, in this case the k -symmetrand of u , u_{k-1}^* , is a radially symmetric, convex function in the ball B_R , vanishing on ∂B_R and belonging to $C^2(B_R) \cap C(\overline{B_R})$. By the Pólya–Szegő principle and (2.7) we know that

$$\|u\|_{L^\infty(\Omega)} \leq \|u_{k-1}^*\|_{L^\infty(B_R)}, \quad I_{k,p}(u, \Omega) \geq I_{k,p}(u_{k-1}^*, B_R),$$

so, once again, it is enough to consider u_{k-1}^* . Without loss of generality we suppose that $V_{n-k+1}(\Omega) = V_{n-k+1}(B_R) = 1$; we obtain, recalling that $J_{k,p}(u, \Omega) \leq 1$,

$$\begin{aligned} \|u_{k-1}^*\|_{L^\infty(B_R)} &= |u_{k-1}^*(0)| = \int_0^1 (\tilde{u}_{k-1}(s))' ds \\ &\leq \int_0^1 \frac{1}{1-s} \left(\int_s^1 (\tilde{u}_{k-1}(r))' dr \right) ds = \int_0^1 (\tilde{u}_{k-1}(s))' |\log(1-s)| ds \\ &\leq \left(\int_0^1 (\tilde{u}_{k-1}(s))'^{n-k+1} s^{n-k} ds \right)^{\frac{1}{n-k+1}} \left(\int_0^1 \frac{|\log(1-s)|^{(n-k+1)/(n-k)}}{s} ds \right)^{\frac{n-k}{n-k+1}} \\ &\leq C(n, k). \end{aligned}$$

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