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Further characterizations of Sobolev spaces

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Abstract. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions from $[0,+\infty)$ into $[0,+\infty)$. Under some suitable hypotheses on $(F_n)_{n\in\mathbb{N}}$, we prove that if $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, satisfies

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$ and moreover

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx,$$

where $K_{N,p}$ is a positive constant depending only on N and p. This extends some results in J. Bourgain and H.-M. Nguyen [A new characterization of Sobolev spaces, C. R. Math. Acad. Sci. Paris 343, 75–80 (2006)] and H.-M. Nguyen [Some new characterizations of Sobolev spaces, J. Funct. Anal. 237, 689–720 (2006)]. We also present some partial results concerning the case $p = 1$ and various open problems.

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1. Introduction

In [7], we established the following characterizations of Sobolev spaces:

Proposition 1 ([7, Theorem 2]). *Let* $1 < p < +\infty$. *Then*

(a) *There exists a constant* $C_{N,p}$*, depending only on* N *and* p*, such that*

$$\underset{|g(x)-g(y)|>\delta}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{\delta^p}{|x-y|^{N+p}}\,dx\,dy \le C_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx, \quad \forall \delta > 0,\ \forall g \in W^{1,p}(\mathbb{R}^N).$$

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(b) *If* $g \in L^p(\mathbb{R}^N)$ *satisfies*

$$\sup_{0<\delta<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

(c) *Moreover, for any* $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\delta \to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|>\delta} \frac{\delta^p}{|x-y|^{N+p}}\, dx\, dy = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx,$$

where $K_{N,p}$ *is defined by*

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e \cdot \sigma|^p\, d\sigma, \tag{1.1}$$

for any $e \in \mathbb{S}^{N-1}$.

Proposition 2 ([7, Theorem 3]). *Let* $1 < p < +\infty$. *Then*

(a) *For every* $g \in W^{1,p}(\mathbb{R}^N)$,

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|\le 1} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|>1} \frac{1}{|x-y|^{N+p}}\, dx\, dy \le C_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx,$$

where $C_{N,p}$ *is a positive constant depending only on* N *and* p.

(b) *If* $g \in L^p(\mathbb{R}^N)$ *satisfies*

$$\sup_{0<\varepsilon<1} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|\le 1} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|>1} \frac{1}{|x-y|^{N+p}}\, dx\, dy < +\infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

(c) *Moreover, for any* $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|\le 1} \frac{\varepsilon|g(x)-g(y)|^{p+\varepsilon}}{|x-y|^{N+p}}\, dx\, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx,$$

where $K_{N,p}$ *is defined by* (1.1).

A sharper version of assertion (b) of Proposition 1 was established by J. Bourgain and H.-M. Nguyen in [2]:

Proposition 3 ([2, Theorem 1]). *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$, be such that*

$$\sup_{n \in \mathbb{N}} \underset{|g(x)-g(y)|>\delta_n}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy < +\infty \tag{1.2}$$

for some sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers with $\lim_{n\to\infty} \delta_n = 0$. Then $g \in W^{1,p}(\mathbb{R}^N)$.

When $p = 1$, we have

Proposition 4. *Let $g \in L^1(\mathbb{R}^N)$ be such that*

$$\sup_{n \in \mathbb{N}} \underset{|g(x)-g(y)|>\delta_n}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\delta_n}{|x-y|^{N+1}}\, dx\, dy < +\infty \tag{1.3}$$

for some sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers with $\lim_{n\to\infty} \delta_n = 0$. Then $g \in BV(\mathbb{R}^N)$; moreover, there exists a constant c_N, depending only on N, such that

$$\liminf_{n\to\infty} \underset{|g(x)-g(y)|>\delta_n}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}} \frac{\delta_n}{|x-y|^{N+1}}\, dx\, dy \geq c_N \int_{\mathbb{R}^N} |\nabla g|\, dx. \tag{1.4}$$

Remark 1. Proposition 4 is not stated explicitly in [2], but its proof is implicit there (see the proof of [2, Theorem 1]).

The proof of Proposition 3 is much more involved than the one of Propositions 1 and 2 (see [2]).

In this paper, we generalize Propositions 1–3 as follows:

Theorem 1. *Let $1 < p < +\infty$ and $(F_n)_{n\in\mathbb{N}}$ be a sequence of functions from $[0, +\infty)$ into $[0, +\infty)$ such that*

(i) *$F_n(t)$ is a non-decreasing function with respect to t on $[0, +\infty)$, for all $n \in \mathbb{N}$.*
(ii) *$\int_0^1 F_n(t) t^{-(p+1)}\, dt = 1$ for all $n \in \mathbb{N}$.*
(iii) *$F_n(t)$ converges uniformly to 0 on every compact subset of $(0, +\infty)$ as n goes to infinity.*

Then

(a) *If $g \in W^{1,p}(\mathbb{R}^N)$, then for every $n \in \mathbb{N}$,*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\, dx\, dy \leq C_{N,p} \int_0^\infty F_n(t) t^{-(p+1)}\, dt \int_{\mathbb{R}^N} |\nabla g(x)|^p\, dx, \tag{1.5}$$

where $C_{N,p}$ is a positive constant depending only on N and p.

(b) *If $g \in L^p(\mathbb{R}^N)$ satisfies*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy < +\infty, \tag{1.6}$$

then $g \in W^{1,p}(\mathbb{R}^N)$ and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy \geq K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx. \tag{1.7}$$

(c) *Moreover, if*

$$\limsup_{n\to\infty} \int_0^\infty F_n(t)t^{-(p+1)}\,dt < +\infty, \tag{1.8}$$

then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy = K_{N,p}\int_{\mathbb{R}^N} |\nabla g(x)|^p\,dx, \qquad \forall g \in W^{1,p}(\mathbb{R}^N). \tag{1.9}$$

Here $K_{N,p}$ is defined by (1.1).

Remark 2. Many ideas used in the proof of Theorems 1 and 2 are borrowed from the method of J. Bourgain and H.-M. Nguyen in [2].

Remark 3. Propositions 1 and 3 follow from Theorem 1 by choosing

$$F_n(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \delta_n, \\ \dfrac{p\delta_n^p}{1-\delta_n^p} & \text{otherwise.} \end{cases} \tag{1.10}$$

To deduce Proposition 2 we choose

$$F_n(t) = \begin{cases} \varepsilon_n t^{p+\varepsilon_n} & \text{if } 0 \leq t \leq 1, \\ \varepsilon_n & \text{otherwise.} \end{cases}$$

Remark 4. We now make some comments about hypotheses (i)–(iii) on the sequence (F_n). The conclusion of Theorem 1 may fail if we do not assume (i). For example, let

$$F_n(t) = \begin{cases} nt^{p+1} & \text{if } 0 \leq t < 1/n, \\ 0 & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy = 0, \qquad \forall n \geq 1,\ \forall p > 1,$$

but $g \notin W^{1,p}(\mathbb{R}^N)$ for all $p > 1$.

Condition (ii) is a normalization condition. Indeed, if we assume

$$\lim_{n\to\infty}\int_0^1 F_n(t)t^{-(p+1)}\,dt = +\infty,$$

then g is a constant function (see Corollary 1).

Condition (iii) is also important. Indeed, the sequence $F_n(t) = t^{p+1}$, for all $n \geq 1$, satisfies conditions (i) and (ii). However, condition (1.6) is equivalent to $g \in W^{p/(p+1),p+1}(\mathbb{R}^N)$.

The analogue of assertion (b) in Theorem 1 for $p = 1$ is the following

Theorem 2. *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of functions from $[0, +\infty)$ into $[0, +\infty)$ satisfying* (i), (ii) *with $p = 1$ and* (iii). *Assume that $g \in L^1(\mathbb{R}^N)$ and g satisfies* (1.6) *with $p = 1$. Then $g \in BV(\mathbb{R}^N)$. Moreover, there exists a constant c_N, depending only on N, such that*

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+1}}\,dx\,dy \geq c_N\int_{\mathbb{R}^N}|\nabla g|\,dx. \tag{1.11}$$

Comparing with (1.7), we have

Question 1. *Can one replace c_N by $K_{N,1}$ in* (1.11)*?*

The reader can find further questions in Section 4.

Remark 5. Proposition 4 follows from Theorem 2 by choosing F_n as in (1.10) with $p = 1$.

For what concerns the analogues of assertions (a) and (c), A. Ponce has constructed a function $g \in W^{1,1}(\mathbb{R})$ such that

$$\lim_{\delta\to 0}\underset{|g(x)-g(y)|>\delta}{\int_{\mathbb{R}}\int_{\mathbb{R}}}\frac{\delta}{|x-y|^2}\,dx\,dy = +\infty.$$

Hence the analogues of these assertions for $p = 1$ do not hold (see [7]).

The proof of assertion (a) in Theorem 1 is similar to one in [7]; it is based on maximal functions. We present two methods of proof of assertion (b) in Theorem 1. The first one, is based on Proposition 3. The second one which relies heavily on Lemma 2 below, is more complicated but is interesting in its own right. For what concerns Theorem 2, we are able to apply the first method, but not the second due to lack of an analogue of Lemma 2 for $p = 1$. Lemma 2 is closely related to Proposition 3; its proof uses many ideas of J. Bourgain and H.-M. Nguyen from [2]. The proof of assertion (c) in Theorem 1 is also much more delicate than the one of assertion (c) in Propositions 1 and 2.

The paper is organized as follows. In Section 2 we will prove Theorems 1 and 2. In Section 3 we present another proof of assertion (b) in Theorem 1. Finally, in Section 4 we will discuss problems related to Γ-convergence.

2. Proofs of Theorems 1 and 2

2.1. Proof of assertion (a) in Theorem 1

Using the change of variables formula and Fubini's theorem, one gets

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+p}}\,dx\,dy = \int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_0^\infty \frac{F_n(|g(x+h\sigma)-g(x)|)}{h^{p+1}}\,dh\,dx\,d\sigma.$$

Consequently, to prove (1.5), it suffices to show that

$$\int_{\mathbb{R}^N}\int_0^\infty \frac{F_n(|g(x+h\sigma)-g(x)|)}{h^{p+1}}\,dh\,dx \le C_p\int_0^\infty F_n(t)t^{-(p+1)}\,dt\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx \tag{2.1}$$

for all $n\in\mathbb{N}$, where C_p is a positive constant depending only on p.

Without loss of generality we may assume that $\sigma = e_N$. Since $g\in W^{1,p}(\mathbb{R}^N)$, $g(x',\cdot)\in W^{1,p}(\mathbb{R})$ for almost every $x'=(x_1,\dots,x_{N-1})\in\mathbb{R}^{N-1}$.

Fix $x'\in\mathbb{R}^{N-1}$ such that $g(x',\cdot)\in W^{1,p}(\mathbb{R})$. Then

$$|g(x+he_N)-g(x)| \le h\fint_{x_N}^{x_N+h}\left|\frac{\partial g}{\partial x_N}(x',s)\right|ds \le hM_N\left(\frac{\partial g}{\partial x_N}\right)(x)$$

for almost every $(x_N,h)\in\mathbb{R}\times(0,+\infty)$, where $M_N(f)$ denotes the maximal function of f with respect to the variable x_N in the positive direction, i.e.,

$$M_N(f)(x',x_N) = \sup_{h>0}\fint_{x_N}^{x_N+h}|f(x',s)|\,ds. \tag{2.2}$$

Hence, since $F_n(t)$ is a non-decreasing function with respect to t,

$$F_n(|g(x+he_N)-g(x)|) \le F_n\left(hM_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right) \quad \text{for a.e. } (x_N,h)\in\mathbb{R}\times(0,+\infty),$$

which shows that

$$\int_{\mathbb{R}^N}\int_0^\infty \frac{F_n(|g(x+he_N)-g(x)|)}{h^{p+1}}\,dh\,dx \le \int_{\mathbb{R}^N}\int_0^\infty \frac{F_n(hM_N(\partial g/\partial x_N)(x))}{h^{p+1}}\,dh\,dx.$$

A direct computation yields

$$\int_{\mathbb{R}^N}\int_0^\infty \frac{F_n(|g(x+he_N)-g(x)|)}{h^{p+1}}\,dh\,dx \le \int_{\mathbb{R}^N}\left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx\int_0^\infty F_n(h)h^{-(p+1)}\,dh.$$

On the other hand, using the theory of maximal functions (see, e.g., [9, Chapter 1]), one finds

$$\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx_N\,dx' \le C_p\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\left|\frac{\partial g}{\partial x_N}(x)\right|^p dx_N\,dx'.$$

Consequently,

$$\int_{\mathbb{R}^N}\left|M_N\left(\frac{\partial g}{\partial x_N}\right)(x)\right|^p dx \le C_p\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx.$$

Therefore (2.1) is proved and (1.5) follows. □

2.2. Proof of assertion (c) in Theorem 1

The following lemma is useful in the proof of assertion (c) in Theorem 1.

Lemma 1. *Assume that* $g \in W^{1,p}(\mathbb{R})$ *and* $(F_n)_{n\in\mathbb{N}}$ *satisfies hypotheses* (i)–(iii) *of Theorem* 1 *and* (1.8). *Then*

$$\lim_{n\to\mathbb{N}}\int_{\mathbb{R}}\int_0^\infty \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt = \int_{\mathbb{R}}|g'(t)|^p\,dt. \tag{2.3}$$

Proof. Since $g \in W^{1,p}(\mathbb{R})$,

$$|g(t+h)-g(t)| \le h \fint_t^{t+h}|g'(s)|\,ds \le h|M_+(g')(t)| \quad \text{for a.e. } (t,h)\in\mathbb{R}\times(0,+\infty),$$

where $M_+(g')$ denotes the maximal function of g' in the positive direction, i.e.,

$$M_+(g')(t) = \sup_{h>0}\fint_t^{t+h}|g'(s)|\,ds.$$

Hence, since F_n is a non-decreasing function, it follows that, for all measurable sets $A \subset \mathbb{R}$,

$$\begin{aligned}\int_A\int_0^\infty &\frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt\\ &\le \sup_{n\in\mathbb{N}}\int_0^\infty F_n(h)h^{-(p+1)}\,dh\int_A|M_+(g')(t)|^p\,dt. \end{aligned}\tag{2.4}$$

On the other hand, since $g \in W^{1,p}(\mathbb{R})$ and $1 < p < +\infty$, applying the theory of maximal functions (see [9, Chapter 1]), one gets $M_+(g') \in L^p(\mathbb{R})$ and

$$\int_{\mathbb{R}}|M_+(g')(x)|^p\,dx \le C\int_{\mathbb{R}}|g'(x)|^p\,dx. \tag{2.5}$$

Hereafter in this proof C will denote a positive constant depending only on p. Thus it follows from (2.4) that for each $\varepsilon \in (0,1)$, there exists a positive constant $k = k(\varepsilon) \geq 1$ such that

$$\int_{\mathbb{R}\setminus B} \int_0^\infty \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt + \int_{\mathbb{R}\setminus B} |g'(t)|^p\,dt \leq \varepsilon/2, \tag{2.6}$$

where

$$B := \{t \in [-k,k];\ |g'(t)| \leq k\}.$$

Set, for each $\tau > 0$,

$$A_\tau = \left\{t \in B;\ q(t) \leq \frac{|g(t+h)-g(t)|}{h} \leq |g'(t)| + \gamma \text{ for a.e. } h \in [0,\tau]\right\},$$

where $\gamma = \varepsilon/k^p$ and $q(t)$ is defined as follows:

$$q(t) = \begin{cases} |g'(t)| - \gamma & \text{if } |g'(t)| \geq \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Since $g \in W^{1,p}(\mathbb{R})$, it follows from (2.4) and (2.5) that one can choose τ sufficiently small such that

$$\int_{B\setminus A_\tau} \int_0^\infty \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt + \int_{B\setminus A_\tau} |g'(t)|^p\,dt \leq \varepsilon/2. \tag{2.8}$$

On the other hand, since F_n is a non-decreasing function,

$$\int_{A_\tau} \int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt \leq \int_{A_\tau} \int_0^\tau \frac{F_n((|g'(t)|+\gamma)h)}{h^{p+1}}\,dh\,dt.$$

A direct computation yields

$$\int_{A_\tau} \int_0^\tau \frac{F_n((|g'(t)|+\gamma)h)}{h^{p+1}}\,dh\,dt = \int_{A_\tau} (|g'(t)|+\gamma)^p \int_0^{(|g'(t)|+\gamma)\tau} F_n(s)s^{-(p+1)}\,ds\,dt.$$

Moreover, since $\int_0^1 F_n(t)t^{-(p+1)}\,dt = 1$ and $F_n(t)$ converges uniformly to 0 on every compact subset of $(0,+\infty)$ as n goes to infinity,

$$\lim_{n\to\infty} \int_{A_\tau} (|g'(t)|+\gamma)^p \int_0^{(|g'(t)|+\gamma)\tau} F_n(s)s^{-(p+1)}\,ds\,dt = \int_{A_\tau} (|g'(t)|+\gamma)^p\,dt.$$

Therefore,

$$\limsup_{n\to\infty} \int_{A_\tau} \int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt \leq \int_{A_\tau} (|g'(t)|+\gamma)^p\,dt. \tag{2.9}$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}}\left(\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh-|g'(t)|^p\right)dt \\ = \int_{\mathbb{R}\setminus B}\left(\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh-|g'(t)|^p\right)dt \\ + \int_{B\setminus A_\tau}\left(\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh-|g'(t)|^p\right)dt \\ + \int_{A_\tau}\left(\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh-|g'(t)|^p\right)dt.\end{aligned}$$

Thus combining (2.6), (2.8), and (2.9) yields

$$\begin{aligned}\int_{\mathbb{R}}\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt-\int_{\mathbb{R}}|g'(t)|^p\,dt \\ \le 3\varepsilon+\int_{A_\tau}((|g'(t)+\gamma|)^p-|g'(t)|^p)\,dt,\end{aligned} \tag{2.10}$$

when $n \ge n_\varepsilon$.

Since $(a+\gamma)^p \le a^p(1+C\gamma/a)$ if $\gamma \le a$ and $(a+\gamma)^p \le C\gamma$ if $a \le \gamma \le 1$, one has

$$\int_{A_\tau}((|g'(t)+\gamma|)^p-|g'(t)|^p)\,dt \le C\gamma k^p = C\varepsilon.$$

Here we use the fact that $A_\tau \subset B$ and the choice of $\gamma = \varepsilon/k^p$. Hence it follows from (2.10) that

$$\int_{\mathbb{R}}\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt-\int_{\mathbb{R}}|g'(t)|^p\,dt \le C\varepsilon, \qquad \forall n \ge n_\varepsilon. \tag{2.11}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{R}}\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt-\int_{\mathbb{R}}|g'(t)|^p\,dt \\ \ge -3\varepsilon+\int_{A_\tau}(q^p(t)-|g'(t)|^p)\,dt\end{aligned} \tag{2.12}$$

for all $n \ge n_\varepsilon$. Recall here that the function q is defined by (2.7). Since $(a-\gamma)^p + C\gamma a^{p-1} \ge a^p$ for all $a \ge \gamma > 0$ and $A_\tau \subset B$, one deduces that

$$\int_{A_\tau}(q^p(t)-|g'(t)|^p)\,dt \ge -C\gamma k^p = -C\varepsilon.$$

Thus, from (2.12),

$$\int_{\mathbb{R}}\int_0^\tau \frac{F_n(|g(t+h)-g(t)|)}{h^{p+1}}\,dh\,dt-\int_{\mathbb{R}}|g'(t)|^p\,dt \ge -C\varepsilon, \qquad \forall n \ge n_\varepsilon. \tag{2.13}$$

Combining (2.11) and (2.13) yields

$$\left| \int_{\mathbb{R}} \int_0^{\tau} \frac{F_n(|g(t+h) - g(t)|)}{h^{p+1}} \, dh \, dt - \int_{\mathbb{R}} |g'(t)|^p \, dt \right| \le C\varepsilon, \quad \forall n \ge n_\varepsilon. \tag{2.14}$$

On the other hand, since $g \in W^{1,p}(\mathbb{R})$, it follows that $g \in L^{\infty}_{\text{loc}}(\mathbb{R})$. Thus since $F_n(t)$ converges uniformly to 0 on every compact subset of $(0, +\infty)$, applying Lebesgue's dominated convergence theorem, one obtains

$$\lim_{n\to\infty} \int_{|t|\le k} \int_{\tau}^{m} \frac{F_n(|g(t+h) - g(t)|)}{h^{p+1}} \, dh \, dt = 0, \quad \forall m > 0.$$

Moreover, since F_n is a non-decreasing function, it follows that

$$\begin{aligned} \int_{|t|\le k} \int_{m}^{\infty} \frac{F_n(|g(t+h) - g(t)|)}{h^{p+1}} \, dh \, dt &\le 2k \int_m^{\infty} \frac{F_n(\|g'\|_{L^p} h^{(p-1)/p})}{h^{p+1}} \, dh \\ &\le \frac{2k}{m} \|g'\|^p_{L^p} \int_0^{\infty} F_n(t) t^{-(p+1)} \, dt. \end{aligned}$$

Hence using (1.8), one gets

$$\lim_{n\to\infty} \int_{|t|\le k} \int_{\tau}^{\infty} \frac{F_n(|g(t+h) - g(t)|)}{h^{p+1}} \, dh \, dt = 0. \tag{2.15}$$

Therefore the conclusion of Lemma 1 follows from (2.6), (2.14), and (2.15). □

Proof of assertion (c). We claim that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_0^{\infty} \frac{F_n(|g(x+h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx = \int_{\mathbb{R}^N} |\nabla g(x) \cdot \sigma|^p \, dx. \tag{2.16}$$

Without loss of generality, one may assume that $\sigma = e_N$. Take $x' \in \mathbb{R}^{N-1}$ such that $g(x', \cdot) \in W^{1,p}(\mathbb{R})$. As in (2.4), one gets

$$\begin{aligned} &\int_{\mathbb{R}} \int_0^{\infty} \frac{F_n(|g(x', x_N + h) - g(x', x_N)|)}{h^{p+1}} \, dh \, dx_N \\ &\qquad \le \int_0^{\infty} F_n(t) t^{-(p+1)} \, dt \int_{\mathbb{R}} |M_N(\partial g/\partial x_N)(x', x_N)|^p \, dx_N. \end{aligned}$$

Here M_N is defined by (2.2). On the other hand, by Lemma 1,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \int_0^{\infty} \frac{F_n(|g(x', x_N + h) - g(x', x_N)|)}{h^{p+1}} \, dh \, dx_N = \int_{\mathbb{R}} \left| \frac{\partial g}{\partial x_N}(x', x_N) \right|^p dx_N. \tag{2.17}$$

Thus, applying Lebesgue's dominated convergence theorem, one obtains

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_0^{\infty} \frac{F_n(|g(x+he_N) - g(x)|)}{h^{p+1}} \, dh \, dx = \int_{\mathbb{R}^N} |\nabla g(x) \cdot e_N|^p \, dx.$$

Therefore the conclusion of assertion (c) in Theorem 1 follows from (2.1), (2.16) and Lebesgue's dominated convergence theorem. □

2.3. Proof of assertion (b) in Theorem 1

Without loss of generality, we may assume that

$$M := 1 + \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{p+N}}\,dx\,dy < +\infty.$$

Thus since $F_n(t)$ is a non-decreasing function with respect to t,

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} F_n(2^{-(k+1)}) \underset{2^{-(k+1)}<|g(x)-g(y)|\le 2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{p+N}}\,dx\,dy < M. \tag{2.18}$$

On the other hand, by (i)–(iii) it follows that for each $s > 0$ there exists n such that $F_n(s) > 0$. Thus since F_n is a non-decreasing function,

$$\underset{|g(x)-g(y)|>s}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{p+N}}\,dx\,dy < +\infty, \qquad \forall s > 0.$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{\infty} F_n(2^{-(k+1)}) \underset{2^{-(k+1)}<|g(x)-g(y)|\le 2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{p+N}}\,dx\,dy \\
&\quad = \sum_{k=1}^{\infty} F_n(2^{-(k+1)}) \\
&\qquad \times \Bigg(\underset{|g(x)-g(y)|>2^{-(k+1)}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{p+N}}\,dx\,dy - \underset{|g(x)-g(y)|>2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{p+N}}\,dx\,dy \Bigg).
\end{aligned}$$

However, the right hand side above equals

$$\begin{aligned}
&\sum_{k=1}^{\infty} 2^{p(k+1)} F_n(2^{-(k+1)}) \underset{|g(x)-g(y)|>2^{-(k+1)}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{2^{-(k+1)p}}{|x-y|^{p+N}}\,dx\,dy \\
&\qquad - \sum_{k=1}^{\infty} 2^{pk} F_n(2^{-(k+1)}) \underset{|g(x)-g(y)|>2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{2^{-kp}}{|x-y|^{p+N}}\,dx\,dy \\
&\quad = \sum_{k=2}^{\infty} 2^{pk}(F_n(2^{-k}) - F_n(2^{-(k+1)})) \underset{|g(x)-g(y)|>2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{2^{-kp}}{|x-y|^{p+N}}\,dx\,dy \\
&\qquad - F_n(1/4) \underset{|g(x)-g(y)|>1/2}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{p+N}}\,dx\,dy,
\end{aligned}$$

and, from hypothesis (iii) on F_n,

$$\lim_{n\to\infty} F_n(1/4) = 0.$$

Thus it follows from (2.18) that

$$\limsup_{n\to\infty} \sum_{k=2}^{\infty} 2^{pk}(F_n(2^{-k}) - F_n(2^{-(k+1)})) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g(x)-g(y)|>2^{-k}} \frac{2^{-kp}}{|x-y|^{p+N}}\,dx\,dy \le M. \tag{2.19}$$

We claim that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g(x)-g(y)|>2^{-k}} \frac{2^{-kp}}{|x-y|^{p+N}}\,dx\,dy < +\infty. \tag{2.20}$$

We prove (2.20) by contradiction. Suppose it does not hold. Then there exists $k_M \in \mathbb{N}$ such that for all $k \ge k_M$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N \atop |g(x)-g(y)|>2^{-k}} \frac{2^{-kp}}{|x-y|^{p+N}}\,dx\,dy > 2^{p+2}M.$$

Hence it follows from (2.19) that

$$\limsup_{n\to\infty} 2^{p+2}M \sum_{k=k_M}^{\infty} 2^{pk}(F_n(2^{-k}) - F_n(2^{-(k+1)})) \le M,$$

which shows that

$$\limsup_{n\to\infty} \sum_{k=k_M}^{\infty} 2^{pk}(F_n(2^{-k}) - F_n(2^{-(k+1)})) \le 2^{-(p+2)}. \tag{2.21}$$

On the other hand, since $p \ge 1$,

$$\begin{aligned}\sum_{k=k_M}^{\infty} 2^{pk}(F_n(2^{-k}) - F_n(2^{-(k+1)})) &= \sum_{k=k_M}^{\infty} 2^{pk}F_n(2^{-k}) - \sum_{k=1+k_M}^{\infty} 2^{p(k-1)}F_n(2^{-k})\\ &\ge \frac{1}{2}\sum_{k=1+k_M}^{\infty} 2^{pk}F_n(2^{-k}),\end{aligned}$$

and, since F_n is a non-decreasing function,

$$\begin{aligned}\int_0^{2^{-(1+k_M)}} F_n(t)t^{-(p+1)}\,dt &= \sum_{k=1+k_M}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} F_n(t)t^{-(p+1)}\,dt\\ &\le \sum_{k=1+k_M}^{\infty} F_n(2^{-k}) \int_{2^{-(k+1)}}^{2^{-k}} t^{-(p+1)}\,dt\\ &\le \frac{2^p}{p}\sum_{k=1+k_M}^{\infty} 2^{pk}F_n(2^{-k}).\end{aligned}$$

Thus it follows from (2.21) that

$$\limsup_{n\to\infty} 2^{-(p+1)} p \int_0^{2^{-(1+k_M)}} F_n(t) t^{-(p+1)}\, dt \le 2^{-(p+2)}.$$

This implies

$$\limsup_{n\to\infty} \int_0^{2^{-(1+k_M)}} F_n(t) t^{-(p+1)}\, dt \le \frac{1}{2}. \tag{2.22}$$

However, by (ii) and (iii), one gets

$$\lim_{n\to\infty} \int_0^{2^{-(1+k_M)}} F_n(t) t^{-(p+1)}\, dt = 1.$$

This contradicts (2.22), and proves (2.10). Thus by Proposition 3, it follows that $g \in W^{1,p}(\mathbb{R}^N)$.

In order to prove (1.7), we consider the sequence of functions G_n defined by

$$G_n(t) = \begin{cases} F_n(t) & \text{if } 0 \le t \le 1, \\ F_n(1) & \text{otherwise.} \end{cases}$$

This sequence satisfies hypotheses (i)–(iii) of Theorem 1 and (1.8). By assertion (c) of Theorem 1, (1.7) follows. □

2.4. *Proof of Theorem 2*

Applying the same method as in Section 2.3, one can prove that

$$\liminf_{k\to\infty} \underset{|g(x)-g(y)|>2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{2^{-k}}{|x-y|^{N+1}}\, dx\, dy < +\infty.$$

Thus by Proposition 4, one has $g \in BV(\mathbb{R}^N)$ and

$$\liminf_{k\to\infty} \underset{|g(x)-g(y)|>2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{2^{-k}}{|x-y|^{N+1}}\, dx\, dy \ge c \int_{\mathbb{R}^N} |\nabla g|\, dx.$$

Hereafter in this proof c denotes a constant depending only on N. Thus there exists a constant $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$,

$$\underset{|g(x)-g(y)|>2^{-k}}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{2^{-k}}{|x-y|^{N+1}}\, dx\, dy \ge c \int_{\mathbb{R}^N} |\nabla g|\, dx.$$

Applying the method of Section 2.3, one gets

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+1}}\,dx\,dy \ge c\int_0^{2^{-(1+k_0)}} F_n(t)t^{-2}\,dt \int_{\mathbb{R}^N} |\nabla g|\,dx$$
$$+ F_n(1/4) \underset{|g(x)-g(y)|>1/2}{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}} \frac{1}{|x-y|^{N+1}}\,dx\,dy.$$

On the other hand, since F_n converges uniformly to 0 on every compact subset of $(0,+\infty)$ and

$$\int_0^1 F_n(t)t^{-2}\,dt = 1,$$

it follows that

$$\lim_{n\to\infty}\int_0^{2^{-(1+k_0)}} F_n(t)t^{-2}\,dt = 1.$$

Therefore

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{N+1}}\,dx\,dy \ge c\int_{\mathbb{R}^N} |\nabla g|\,dx. \qquad \square$$

Theorems 1 and 2 have the following interesting consequence. It is motivated by the work of J. Bourgain, H. Brezis and P. Mironescu in [1] and [5].

Corollary 1. *Let $p \ge 1$ and $(F_n)_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions from $[0,+\infty)$ into $[0,+\infty)$ such that $F_n(1)$ is bounded and*

$$\limsup_{n\to\infty}\int_0^1 F_n(t)t^{-(p+1)}\,dt = +\infty.$$

Assume that $g \in L^p(\mathbb{R}^N)$ and g satisfies (1.6). *Then g is a constant function.*

Proof. Without loss of generality, one may assume that

$$\lim_{n\to\infty}\int_0^1 F_n(t)t^{-(p+1)}\,dt = +\infty.$$

For each $n \in \mathbb{N}$, set

$$G_n(t) = \begin{cases} \dfrac{F_n(t)}{\int_0^1 F_n(t)t^{-(p+1)}\,dt} & \text{if } 0 \le t \le 1, \\[2ex] \dfrac{F_n(1)}{\int_0^1 F_n(t)t^{-(p+1)}\,dt} & \text{otherwise.} \end{cases}$$

Then $(G_n)_{n\in\mathbb{N}}$ satisfies hypotheses (i)–(iii).

By Theorems 1 and 2, there exists a constant $c_{N,p} > 0$ such that

$$c_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy.$$

However,

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy \\ &\qquad \le \liminf_{n\to\infty} \frac{1}{\int_0^1 F_n(t) t^{-(p+1)} \, dt} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}} \, dx \, dy = 0. \end{aligned}$$

Thus it follows that $\int_{\mathbb{R}^N} |\nabla g|^p \, dx = 0$. Therefore g is a constant function. □

Remark 6. The conditions of Corollary 1 are satisfied by $F_n(t) = t^p$ for all $n \in \mathbb{N}$ with $p \ge 1$. Hence any function $g \in L^p(\mathbb{R}^N)$ satisfying

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|g(x) - g(y)|^p}{|x-y|^{N+p}} \, dx \, dy < +\infty$$

must be a constant. This was already observed in [5].

3. Another proof of assertion (b) in Theorem 1

First in Section 3.1 we present a fundamental lemma. Then in Section 3.2 we discuss a new proof of assertion (b) in Theorem 1.

3.1. A fundamental lemma

The following lemma will play an important role in this section.

Lemma 2 (Fundamental lemma). *Let $g \in L^p(\mathbb{R}^N)$, $1 < p < +\infty$. Assume that*

$$\iint_{\substack{K \times K \\ |g(x)-g(y)|>\varepsilon}} \frac{1}{|x-y|^{N+1}} \, dx \, dy < +\infty, \qquad \forall K \subset\subset \mathbb{R}^N, \ \forall \varepsilon > 0 \tag{3.1}$$

and

$$\liminf_{\varepsilon\to 0_+} \iint_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ \varepsilon<|g(x)-g(y)|<10\varepsilon}} \frac{\varepsilon^p}{|x-y|^{N+p}} \, dx \, dy < +\infty. \tag{3.2}$$

Then $g \in W^{1,p}(\mathbb{R}^N)$.

Remark 7. Condition (3.2) alone is not sufficient to show that $g \in W^{1,p}(\mathbb{R}^N)$ (in contrast with condition (1.2)). For example

$$g(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Surprisingly, the mild additional assumption (3.1) together with (3.2) implies that $g \in W^{1,p}(\mathbb{R}^N)$.

In order to prove Lemma 2, we need some useful lemmas. The first lemma, which was used in [2], is a direct consequence of a result due to J. Bourgain, H. Brezis, and P. Mironescu (see [1]).

Lemma 3. *Let g be a measurable function on the interval $[a, b]$ $(-\infty < a < b < +\infty)$, $y \in \mathbb{R}$, and $\delta > 0$. Set*

$$B = \{x \in [a, b];\ g(x) < y\}.$$

Assume that

$$0 < \frac{|[a, b] \cap B|}{b - a} < 1 \tag{3.3}$$

and

$$\underset{|g(x)-g(y)|>\delta}{\int_a^b \int_a^b} \frac{1}{|x - y|^2}\, dx\, dy < +\infty. \tag{3.4}$$

Then

$$|[a, b] \cap A_\tau| > 0, \qquad \forall \tau > \delta,$$

where $A_\tau := \{x \in [a, b];\ y \le g(x) < y + \tau\}$.

Hereafter $|A|$ denotes the Lebesgue measure of A for any measurable set $A \subset \mathbb{R}$.

Proof. We prove Lemma 3 by contradiction. Suppose that $|[a, b] \cap A_\tau| = 0$ for some $\tau > \delta$. Then from (3.4),

$$\int_B \int_{[a,b]\setminus B} \frac{1}{|x - y|^2}\, dx\, dy < +\infty.$$

This implies (see [1])

$$|B| = 0 \quad \text{or} \quad |[a, b] \setminus B| = 0,$$

which contradicts (3.3). □

The following lemma will be useful to prove Lemma 5. Estimate (3.6) was mentioned and used in [2]. Estimate (3.7) was also hidden there. It will play a role in the proof of Lemma 5. For the convenience of the reader, we will reproduce the proof.

Lemma 4. *Let g be a measurable function on the interval $[a, b]$ $(-\infty < a < b < +\infty)$, $y \in \mathbb{R}$, $r > 0$, $s > 0$, and $\tau > \delta > 0$. Set*

$$B = \{x \in \mathbb{R};\ g(x) < y\}, \qquad A = \{x \in \mathbb{R};\ y \leq g(x) < y + \tau\}.$$

Assume that

$$\frac{|[a, b] \cap B|}{b - a} = r, \qquad \frac{|[a, b] \cap A|}{b - a} \leq s, \qquad r + s < 1,$$

and

$$\int_a^b \int_a^b_{|g(x)-g(y)|>\delta} \frac{1}{|x - y|^2}\, dx\, dy < +\infty. \tag{3.5}$$

Then there exists a subinterval $[c, d] \subset [a, b]$ $(a \leq c < d \leq b)$ such that

$$\frac{|[c, d] \cap B|}{d - c} = r, \qquad s/4 \leq \frac{|[c, d] \cap A|}{d - c} \leq s, \tag{3.6}$$

and

$$\frac{d - c}{b - a} \leq 4 \frac{|[a, b] \cap A|}{s(b - a)}. \tag{3.7}$$

Proof. Set $[a_1, b_1] = [a, b]$. Suppose that there exists $[a_k, b_k] \subset [a, b]$, $k \geq 1$, such that

$$\frac{|[a_k, b_k] \cap B|}{b_k - a_k} = r \quad \text{and} \quad \frac{|[a_k, b_k] \cap A|}{b_k - a_k} \leq s.$$

If

$$\frac{|[a_k, b_k] \cap A|}{b_k - a_k} \geq s/4.$$

then take $[c, d] = [a_k, b_k]$. Otherwise, by Lemma 3, one has

$$0 < \frac{|[a_k, b_k] \cap A|}{b_k - a_k} < s/4.$$

Take $s_k > 0$ such that $s/s_k \in \mathbb{Z}_+$ and

$$s_k/2 < \frac{|[a_k, b_k] \cap A|}{b_k - a_k} \leq s_k. \tag{3.8}$$

Then

$$s_k < s/2. \tag{3.9}$$

Set

$$\lambda_k = \frac{(b_k - a_k)s_k}{2s}. \tag{3.10}$$

Consider the function $\psi_k(t)$ defined as follows:

$$\psi_k(t) = |[t - \lambda_k, t + \lambda_k] \cap B|, \qquad \forall t \in [a_k + \lambda_k, b_k - \lambda_k].$$

We claim that there exists $t_k \in [a_k + \lambda_k, b_k - \lambda_k]$ such that $\psi_k(t_k)/(2\lambda_k) = r$.

To see this, we argue by contradiction. Suppose that $\psi_k(t)/(2\lambda_k) \neq r$ for all $t \in [a_k + \lambda_k, b_k - \lambda_k]$. Since ψ_k is a continuous function on $[a_k + \lambda_k, b_k - \lambda_k]$, we assume as well that $\psi_k(t)/(2\lambda_k) < r$ for all $t \in [a_k + \lambda_k, b_k - \lambda_k]$. Since $(b_k - a_k)/(2\lambda_k) = s/s_k \in \mathbb{Z}_+$, it follows that

$$|[a_k, b_k] \cap B| < 2r\lambda_k \frac{b_k - a_k}{2\lambda_k} = r(b_k - a_k).$$

This contradicts the fact that $|[a_k, b_k] \cap B|/(b_k - a_k) = r$.

Set $[a_{k+1}, b_{k+1}] = [t_k - \lambda_k, t_k + \lambda_k] \subset [a_k, b_k]$. Then

$$\frac{|[a_{k+1}, b_{k+1}] \cap B|}{b_{k+1} - a_{k+1}} = r \quad \text{and} \quad \frac{|[a_{k+1}, b_{k+1}] \cap A|}{b_{k+1} - a_{k+1}} \leq \frac{|[a_k, b_k] \cap A|}{2\lambda_k}.$$

Thus it follows from (3.8) and (3.10) that

$$\frac{|[a_{k+1}, b_{k+1}] \cap A|}{b_{k+1} - a_{k+1}} \leq \frac{|[a_k, b_k] \cap A|}{2\lambda_k} = \frac{|[a_k, b_k] \cap A|}{b_k - a_k} \frac{b_k - a_k}{2\lambda_k} \leq s_k \frac{b_k - a_k}{2\lambda_k} = s.$$

Moreover,

$$\frac{b_{k+1} - a_{k+1}}{b_k - a_k} = \frac{2\lambda_k}{b_k - a_k} = \frac{s_k}{s}.$$

Thus from (3.9), this implies

$$\frac{b_{k+1} - a_{k+1}}{b_k - a_k} \leq \frac{1}{2}. \tag{3.11}$$

On the other hand,

$$\underset{|g(x)-g(y)|>\delta}{\int_{a_k}^{b_k} \int_{a_k}^{b_k}} \frac{1}{|x - y|^2}\, dx\, dy \geq \iint\limits_{\substack{x \in [a_k, b_k] \cap B \\ y \in [a_k, b_k] \setminus (A \cup B)}} \frac{1}{|x - y|^2}\, dx\, dy,$$

which shows

$$\underset{|g(x)-g(y)|>\delta}{\int_{a_k}^{b_k} \int_{a_k}^{b_k}} \frac{1}{|x - y|^2}\, dx\, dy \geq r(1 - r - s). \tag{3.12}$$

Combining (3.5), (3.11), and (3.12) shows that the above process will stop at some $k \in \mathbb{Z}_+$. Then $[c, d] \subset [a, b]$ ($[c, d] = [a_k, b_k]$), and

$$\frac{|[c, d] \cap B|}{d - c} = r, \qquad s/4 \leq \frac{|[c, d] \cap A|}{d - c} \leq s.$$

If $k \geq 2$, then it follows from (3.8) and (3.10) that

$$\frac{d - c}{b - a} \leq \frac{b_2 - a_2}{b_1 - a_1} = \frac{2\lambda_1}{b_1 - a_1} = \frac{s_1}{s} \leq 2 \frac{|[a, b] \cap A|}{s(b - a)}.$$

Otherwise ($k = 1$), the estimate (3.7) holds clearly.

The proof is complete. □

The following lemma plays an important role in the proof of Lemma 6.

Lemma 5. *Let g be a measurable function on the interval $[a,b]$ $(-\infty < a < b < +\infty)$, $y \in \mathbb{R}$, and $\tau > \delta > 0$. Set*

$$\begin{cases} B_j = \{x \in \mathbb{R};\ g(x) < y + j\tau\}, \\ A_j = \{x \in \mathbb{R};\ y + j\tau \le g(x) < y + (j+1)\tau\}, \end{cases} \qquad \forall j \in \mathbb{Z}.$$

Assume that

$$\frac{|[a,b] \cap B_0|}{b-a} = \frac{1}{2}, \qquad \frac{|[a,b] \cap A_0|}{b-a} \le \frac{1}{8},$$

and

$$\int_a^b \int_a^b_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^2}\, dx\, dy < +\infty. \tag{3.13}$$

Then for each $r > 8$, there exist $m \in \mathbb{Z}_+$, $l_m \in \mathbb{Z}$, and $[c,d] \subset [a,b]$ $(c < d)$ such that

$$\begin{cases} |l_m| \le 2m, \\ \dfrac{|[c,d] \cap A_{l_m}|}{d-c} \dfrac{|[c,d] \cap A_{l_m+2}|}{d-c} \ge \dfrac{1}{4 \cdot 8^{m+1} r^{m+1}}, \\ d - c \le 4^m (8/r)^{m(m-1)/2} (b-a). \end{cases}$$

Proof. Set $j_1 = 0$. By Lemma 4, there exists $[a_1, b_1] \subset [a,b]$ such that

$$\frac{|[a_1,b_1] \cap B_{j_1}|}{b_1 - a_1} = \frac{1}{2} \quad \text{and} \quad \frac{1}{4 \cdot 8} \le \frac{|[a_1,b_1] \cap A_{j_1}|}{b_1 - a_1} \le \frac{1}{8}.$$

Suppose that there exist $[a_k, b_k]$ and j_k $(k \ge 1)$ such that

$$\begin{cases} |j_k| \le 2(k-1), \\ \dfrac{1}{2} - 2\displaystyle\sum_{i=1}^{k-1} \dfrac{1}{8^i} \le \dfrac{|[a_k,b_k] \cap B_{j_k}|}{b_k - a_k} \le \dfrac{1}{2} + 2\displaystyle\sum_{i=1}^{k-1} \dfrac{1}{8^i}, \\ \dfrac{1}{4 \cdot 8^k} \le \dfrac{|[a_k,b_k] \cap A_{j_k}|}{b_k - a_k} \le \dfrac{1}{8^k}. \end{cases} \tag{3.14}$$

Then we have the following cases:

Case 1:

$$\frac{|[a_k,b_k] \cap A_{j_k+1}|}{b_k - a_k} \ge \frac{1}{8^{k+1}} \quad \text{and} \quad \frac{|[a_k,b_k] \cap A_{j_k-1}|}{b_k - a_k} \ge \frac{1}{8^{k+1}}.$$

Set

$$m = k, \qquad l_m = j_k - 1, \qquad [c,d] = [a_k, b_k].$$

Then

$$\frac{|[c,d] \cap A_{l_m}|}{d-c} \frac{|[c,d] \cap A_{l_m+2}|}{d-c} \ge \frac{1}{8^{m+1} 8^{m+1}} \ge \frac{1}{4 \cdot 8^{m+1} r^{m+1}}. \tag{3.15}$$

Case 2:

$$\frac{|[a_k,b_k] \cap A_{j_k+1}|}{b_k - a_k} < \frac{1}{8^{k+1}} \quad \text{or} \quad \frac{|[a_k,b_k] \cap A_{j_k-1}|}{b_k - a_k} < \frac{1}{8^{k+1}}.$$

Case 2.1:
$$\frac{|[a_k, b_k] \cap A_{j_k+1}|}{b_k - a_k} < \frac{1}{8^{k+1}}.$$

Case 2.1.1:
$$\frac{|[a_k, b_k] \cap A_{j_k+2}|}{b_k - a_k} \geq \frac{1}{r^{k+1}}.$$

Set
$$m = k, \quad l_m = j_k, \quad [c, d] = [a_k, b_k].$$

Then
$$\frac{|[c, d] \cap A_{l_m}|}{d - c} \frac{|[c, d] \cap A_{l_m+2}|}{d - c} \geq \frac{1}{4 \cdot 8^{m+1} r^{m+1}}. \tag{3.16}$$

Case 2.1.2:
$$\frac{|[a_k, b_k] \cap A_{j_k+2}|}{b_k - a_k} < \frac{1}{r^{k+1}}.$$

Set
$$j_{k+1} = j_k + 2. \tag{3.17}$$

Then, from the first inequality of (3.14),
$$|j_{k+1}| \leq 2k. \tag{3.18}$$

Applying Lemma 4 with $s = 1/8^{k+1}$ and $B = B_{j_{k+1}}$, one gets $[a_{k+1}, b_{k+1}] \subset [a_k, b_k]$ such that
$$\begin{cases} \dfrac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} = \dfrac{|[a_k, b_k] \cap B_{j_{k+1}}|}{b_k - a_k}, \\ \dfrac{1}{4 \cdot 8^{k+1}} \leq \dfrac{|[a_{k+1}, b_{k+1}] \cap A_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \leq \dfrac{1}{8^{k+1}}, \\ \dfrac{b_{k+1} - a_{k+1}}{b_k - a_k} \leq \dfrac{4 \cdot 8^{k+1}}{r^{k+1}}. \end{cases} \tag{3.19}$$

Thus from (3.14) and (3.19),
$$\frac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \geq \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} \geq \frac{1}{2} - 2 \sum_{i=1}^{k} \frac{1}{8^i}$$

and, since $j_{k+1} = j_k + 2$ (see (3.17)),
$$\begin{aligned} \frac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} &= \frac{|[a_k, b_k] \cap B_{j_{k+1}}|}{b_k - a_k} \\ &= \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} + \frac{|[a_k, b_k] \cap (A_{j_k} \cup A_{j_k+1})|}{b_k - a_k} \\ &\leq \frac{1}{2} + 2 \sum_{i=1}^{k-1} \frac{1}{8^i} + \frac{1}{8^k} + \frac{1}{8^{k+1}}. \end{aligned}$$

Hence

$$\frac{1}{2} - 2\sum_{i=1}^{k}\frac{1}{8^i} \le \frac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \frac{1}{2} + 2\sum_{i=1}^{k}\frac{1}{8^i}. \tag{3.20}$$

Therefore from (3.18), (3.20) and the last two estimates of (3.19), one gets

$$\begin{cases} |j_{k+1}| \le 2k, \\ \dfrac{1}{2} - 2\displaystyle\sum_{i=1}^{k}\frac{1}{8^i} \le \dfrac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \dfrac{1}{2} + 2\displaystyle\sum_{i=1}^{k}\frac{1}{8^i}, \\ \dfrac{1}{4 \cdot 8^{k+1}} \le \dfrac{|[a_{k+1}, b_{k+1}] \cap A_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \dfrac{1}{8^{k+1}}, \\ \dfrac{b_{k+1} - a_{k+1}}{b_k - a_k} \le \dfrac{4 \cdot 8^{k+1}}{r^{k+1}}. \end{cases} \tag{3.21}$$

Case 2.2:

$$\frac{|[a_k, b_k] \cap A_{j_k+1}|}{b_k - a_k} \ge \frac{1}{8^{k+1}} \quad \text{and} \quad \frac{|[a_k, b_k] \cap A_{j_k-1}|}{b_k - a_k} < \frac{1}{8^{k+1}}.$$

Case 2.2.1:

$$\frac{|[a_k, b_k] \cap A_{j_k-2}|}{b_k - a_k} \ge \frac{1}{r^{k+1}}.$$

Set

$$m = k, \quad l_m = j_k - 2, \quad [c, d] = [a_k, b_k].$$

Then

$$\frac{|[c, d] \cap A_{l_m}|}{d - c}\,\frac{|[c, d] \cap A_{l_m+2}|}{d - c} \ge \frac{1}{4 \cdot 8^{m+1} r^{m+1}}. \tag{3.22}$$

Case 2.2.2:

$$\frac{|[a_k, b_k] \cap A_{j_k-2}|}{b_k - a_k} < \frac{1}{r^{k+1}}.$$

Set

$$j_{k+1} = j_k - 2. \tag{3.23}$$

Then from the first inequality of (3.14),

$$|j_{k+1}| \le 2k. \tag{3.24}$$

Applying Lemma 4 with $s = 1/8^{k+1}$ and $B = B_{j_{k+1}}$, one gets $[a_{k+1}, b_{k+1}] \subset [a_k, b_k]$ such that

$$\begin{cases} \dfrac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} = \dfrac{|[a_k, b_k] \cap B_{j_{k+1}}|}{b_k - a_k}, \\ \dfrac{1}{4 \cdot 8^{k+1}} \le \dfrac{|[a_{k+1}, b_{k+1}] \cap A_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \dfrac{1}{8^{k+1}}, \\ \dfrac{b_{k+1} - a_{k+1}}{b_k - a_k} \le \dfrac{4 \cdot 8^{k+1}}{r^{k+1}}. \end{cases} \tag{3.25}$$

Thus from the second estimate of (3.14),

$$\frac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} \le \frac{1}{2} + 2\sum_{i=1}^{k} \frac{1}{8^i}.$$

and, since $j_{k+1} = j_k - 2$ (see (3.23)),

$$\begin{aligned}\frac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} &= \frac{|[a_k, b_k] \cap B_{j_{k+1}}|}{b_k - a_k} \\ &= \frac{|[a_k, b_k] \cap B_{j_k}|}{b_k - a_k} - \frac{|[a_k, b_k] \cap (A_{j_k-1} \cup A_{j_k-2})|}{b_k - a_k} \\ &\ge \frac{1}{2} - 2\sum_{i=1}^{k-1} \frac{1}{8^i} - \frac{1}{8^{k+1}} - \frac{1}{r^{k+1}}.\end{aligned}$$

Hence

$$\frac{1}{2} - 2\sum_{i=1}^{k} \frac{1}{8^i} \le \frac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \frac{1}{2} + 2\sum_{i=1}^{k} \frac{1}{8^i}. \tag{3.26}$$

Therefore from (3.24), (3.26) and the last two estimates of (3.25), one has

$$\begin{cases} |j_{k+1}| \le 2k, \\ \dfrac{1}{2} - 2\displaystyle\sum_{i=1}^{k} \frac{1}{8^i} \le \dfrac{|[a_{k+1}, b_{k+1}] \cap B_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \dfrac{1}{2} + 2\displaystyle\sum_{i=1}^{k} \frac{1}{8^i}, \\ \dfrac{1}{4 \cdot 8^{k+1}} \le \dfrac{|[a_{k+1}, b_{k+1}] \cap A_{j_{k+1}}|}{b_{k+1} - a_{k+1}} \le \dfrac{1}{8^{k+1}}, \\ \dfrac{b_{k+1} - a_{k+1}}{b_k - a_k} \le \dfrac{4 \cdot 8^{k+1}}{r^{k+1}}. \end{cases} \tag{3.27}$$

On the other hand, from (3.14),

$$\int_{a_k}^{b_k} \int_{a_k}^{b_k}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^2}\, dx\, dy \ge \iint_{\substack{x \in [a_k, b_k] \cap B_{j_k} \\ y \in [a_k, b_k] \setminus (B_{j_k} \cup A_{j_k})}} \frac{1}{|x-y|^2}\, dx\, dy \gtrsim 1. \tag{3.28}$$

Thus it follows from inequality (3.28), the last inequalities of (3.21) and (3.27), and (3.13) that this process will stop at some $k \in \mathbb{Z}_+$. Thus from (3.15), (3.16) and (3.22), it suffices to show that ($[c, d] = [a_m, b_m]$)

$$d - c \le 4^m (8/r)^{m(m-1)/2} (b - a). \tag{3.29}$$

In fact, from the last inequality of (3.21) and (3.27),

$$b_i - a_i \le \frac{4 \cdot 8^i}{r^i}(b_{i-1} - a_{i-1}), \quad \forall 2 \le i \le m,$$

which shows, since $d - c = b_m - a_m$,

$$d - c \le 4^m (8/r)^{m(m+1)/2-1}(b - a) \le 4^m (8/r)^{m(m-1)/2}(b - a)$$

(the above inequality is evident when $m = 1$).

The proof is complete. □

Lemma 5 has the following consequence which is one of the main ingredients to establish Lemma 6.

Corollary 2. *Let* $1 < p < +\infty$*. Under the hypotheses of Lemma* 5*, there exist* $m \in \mathbb{Z}_+$ *and* $l_m \in \mathbb{Z}$ *such that*

$$|l_m| \le 2m$$

and

$$\iint\limits_{\substack{x\in[a,b]\cap A_{l_m}\\ y\in[a,b]\cap A_{l_m+2}}} \frac{1}{|x-y|^{p+1}}\,dx\,dy \ge c_p m(b-a)^{1-p}$$

for some positive constant c_p *depending only on* p*.*

Proof. Take $r = 16$. By Lemma 5, there exist $m \in \mathbb{Z}_+$, $l_m \in \mathbb{Z}$, and $[c, d] \subset [a, b]$ such that

$$\begin{cases} |l_m| \le 2m, \\ \dfrac{|[c,d]\cap A_{l_m}|}{d-c}\,\dfrac{|[c,d]\cap A_{l_m+2}|}{d-c} \ge \dfrac{1}{4\cdot 8^{m+1} r^{m+1}}, \\ d - c \le 4^m (8/r)^{m(m-1)/2}(b-a). \end{cases}$$

Hence

$$\begin{aligned} \frac{|[c,d]\cap A_{l_m}|\,|[c,d]\cap A_{l_m+2}|}{(d-c)^{p+1}} &\ge \frac{1}{4\cdot 8^{m+1} r^{m+1}}(d-c)^{1-p} \\ &\ge \frac{1}{4\cdot 8^{m+1} r^{m+1}} 4^{m(1-p)} (r/8)^{m(m-1)(p-1)/2}(b-a)^{1-p}, \end{aligned}$$

which shows that (since $p > 1$)

$$\frac{|[c,d]\cap A_{l_m}|\,|[c,d]\cap A_{l_m+2}|}{(d-c)^{p+1}} \ge c_p m(b-a)^{1-p}$$

for some positive constant c_p depending only on p.

On the other hand,

$$\iint\limits_{\substack{x\in[c,d]\cap A_{l_m}\\ y\in[c,d]\cap A_{l_m+2}}} \frac{1}{|x-y|^{p+1}}\,dx\,dy \ge \frac{|[c,d]\cap A_{l_m}|\,|[c,d]\cap A_{l_m+2}|}{(d-c)^{p+1}}.$$

Therefore,

$$\iint_{\substack{x\in[a,b]\cap A_{l_m}\\ y\in[a,b]\cap A_{l_m+2}}} \frac{1}{|x-y|^{p+1}}\,dx\,dy \ge c_p m(b-a)^{1-p}. \qquad \square$$

The following lemma plays a crucial role in this section.

Lemma 6. *Let g be a measurable function on a bounded interval $\mathcal{I}$ and $1 < p < +\infty$. Assume that*

$$\iint_{\substack{\mathcal{I}\times\mathcal{I}\\ |g(x)-g(y)|>\varepsilon}} \frac{1}{|x-y|^2}\,dx\,dy < +\infty, \qquad \forall \varepsilon > 0. \tag{3.30}$$

Then

$$\liminf_{\varepsilon\to 0_+} \iint_{\substack{\mathcal{I}\times\mathcal{I}\\ \varepsilon<|g(x)-g(y)|<10\varepsilon}} \frac{\varepsilon^p}{|x-y|^{p+1}}\,dx\,dy \ge c_p \frac{1}{|\mathcal{I}|^{p-1}}(\operatorname*{ess\,sup}_{\mathcal{I}} g - \operatorname*{ess\,inf}_{\mathcal{I}} g)^p \tag{3.31}$$

for some positive constant c_p depending only on p.

Proof. Case 1: f is bounded. We first follow the idea and the notations used in the proof of [2, Lemma 1].

By rescaling, we may assume $\mathcal{I} = [0,1]$. Set $s_+ = \operatorname{ess\,sup}_{\mathcal{I}} g$, $s_- = \operatorname{ess\,inf}_{\mathcal{I}} g$. Rescaling and translating g, one may also assume

$$s_+ = 1 \quad \text{and} \quad s_- = 0 \tag{3.32}$$

(unless g is constant on $\mathcal{I}$ in which case there is nothing to prove).

Take $0 < \delta \ll 1$ small enough to ensure that there are (density) points $t_+, t_- \in [40\delta, 1-40\delta] \subset [0,1]$ with

$$\begin{cases} \left|[t_+ - \tau, t_+ + \tau] \cap \left[g > \dfrac{3}{4}\right]\right| > \dfrac{9}{5}\tau, \\ \left|[t_- - \tau, t_- + \tau] \cap \left[g < \dfrac{1}{4}\right]\right| > \dfrac{9}{5}\tau, \end{cases} \qquad \forall 0 < \tau < 40\delta. \tag{3.33}$$

Take $K \in \mathbb{Z}_+$ such that

$$\delta < 2^{-K} \le 2\delta \tag{3.34}$$

and define

$$J = \{j \in \mathbb{Z}_+;\ 1/4 < j2^{-K} < 3/4\}.$$

Then

$$|J| \ge 2^{K-1} - 2 \approx 1/\delta. \tag{3.35}$$

For each j, set

$$A_j = \{x \in [0,1];\ (j-1)2^{-K} \le g(x) < j2^{-K}\} \quad \text{and} \quad B_j = \bigcup_{j'<j} A_{j'}.$$

Since $A_j \cap A_k = \emptyset$ for $j \neq k$, it follows from (3.35) that

$$\operatorname{card}(G) \ge 2^{K-2} - 3 \approx 1/\delta, \quad \text{where} \quad G = \{j \in J;\ |A_j| < 2^{-K+2}\}. \tag{3.36}$$

For each $j \in G$, set $\lambda_j = |A_j|$ and consider the function $\psi_j(t)$ defined as follows:

$$\psi_j(t) = |[t - 4\lambda_j, t + 4\lambda_j] \cap B_j|, \quad \forall t \in [40\delta, 1 - 40\delta].$$

Then, from (3.33),

$$\psi_j(t_+) < 4\lambda_j \quad \text{and} \quad \psi_j(t_-) > 4\lambda_j.$$

Thus, since ψ_j is a continuous function on the interval $[40\delta, 1-40\delta]$ containing two points t_+ and t_-, there exists $t_j \in [40\delta, 1-40\delta]$ such that

$$\psi_j(t_j) = 4\lambda_j.$$

In the rest of the proof we introduce a new way to estimate the left side of (3.31). Since $\lambda_j \lesssim \delta$, it follows from Corollary 2 that there exist $m_j \in \mathbb{Z}_+$ and $l_j \in \mathbb{Z}$ such that

$$|l_j - j| \le 2m_j$$

and

$$\iint\limits_{\substack{x \in \mathcal{I} \cap A_{l_j} \\ y \in \mathcal{I} \cap A_{l_j+2}}} \frac{1}{|x-y|^{p+1}}\, dx\, dy \ge c_p m_j \delta^{1-p}, \tag{3.37}$$

for some positive constant c_p depending only on p.

Set $i_0 = -1$ and

$$C_i = \{j \in G;\ l_j = i\}, \quad \forall i \in \mathbb{Z}.$$

For each $n \ge 1$, if

$$\{i \in \mathbb{Z};\ i \ge i_{n-1} + 1 \text{ and } C_i \neq \emptyset\} \neq \emptyset,$$

then set

$$\begin{aligned} i_n &= \inf\{i \in \mathbb{Z};\ i \ge i_{n-1} + 1 \text{ and } C_i \neq \emptyset\}, \\ k_n &= \max\{m_j;\ j \in G \text{ and } l_j = i_n\}. \end{aligned}$$

Then

$$k_n \gtrsim \operatorname{card}\{j \in G;\ l_j = i_n\}.$$

Hence it follows from (3.36) that

$$\sum_{n \ge 1,\, k_n \text{ exists}} k_n \gtrsim \operatorname{card}(G) \approx \frac{1}{\delta}. \tag{3.38}$$

On the other hand, from (3.37),

$$\int_{\mathcal{I}}\int_{\mathcal{I}}_{2^{-K}\le|g(x)-g(y)|\le 3\cdot 2^{-K}} \frac{\delta^p}{|x-y|^{p-1}}\,dx\,dy \ge \sum_{n\ge 1,\,k_n\text{ exists}} \iint_{\substack{x\in\mathcal{I}\cap A_{i_n}\\ y\in\mathcal{I}\cap A_{i_n+2}}} \frac{\delta^p}{|x-y|^{p+1}}\,dx\,dy$$

$$\ge c_p \sum_{n\ge 1,\,k_n\text{ exists}} k_n\delta. \tag{3.39}$$

Therefore the conclusion of Lemma 6 follows from (3.38), (3.39), and (3.34).

Case 2: f is unbounded. By the method used in Case 1, one has

$$\liminf_{\varepsilon\to 0_+} \iint_{\substack{\mathcal{I}\times\mathcal{I}\\ \varepsilon<|g(x)-g(y)|<10\varepsilon}} \frac{\varepsilon^p}{|x-y|^{p+1}}\,dx\,dy = +\infty. \qquad \square$$

Remark 8. It is interesting to compare Lemma 6 with Lemma 2 in [2] which asserts that, for each $p \ge 1$, there exists a positive constant c_p depending only on p such that, for any bounded interval $\mathcal{I}$ and for any measurable function g defined on $\mathcal{I}$,

$$\liminf_{\varepsilon\to 0_+} \iint_{\substack{\mathcal{I}\times\mathcal{I}\\ |g(x)-g(y)|>\varepsilon}} \frac{\varepsilon^p}{|x-y|^{p+1}}\,dx\,dy \ge c_p \frac{1}{|\mathcal{I}|^{p-1}}(\operatorname*{ess\,sup}_{\mathcal{I}} g - \operatorname*{ess\,inf}_{\mathcal{I}} g)^p.$$

Obviously, Lemma 6 implies this assertion for the case $p > 1$.

We are now ready to prove Lemma 2.

Proof of Lemma 2. Without loss of generality, we assume that

$$\sup_{n\in\mathbb{N}} \iint_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ \varepsilon_n<|g(x)-g(y)|<10\varepsilon_n}} \frac{\varepsilon_n^p}{|x-y|^{p+N}}\,dx\,dy < +\infty \tag{3.40}$$

for some sequence of positive numbers ε_n such that $\lim_{n\to\infty}\varepsilon_n = 0$.

Step 1: Proof of Lemma 2 when $N = 1$. This proof is similar to the one of [2, Theorem 1] for the case $N = 1$. We reproduce it here for the convenience of the reader.

Set

$$\tau_h(g)(x) = \frac{g(x+h)-g(x)}{h}, \qquad \forall x \in \mathbb{R},\ \forall 0 < h < 1.$$

For each $m \ge 2$, take $K \in \mathbb{R}_+$ such that $Kh > m$; then

$$\int_{-m}^{m} |\tau_h(g)(x)|^p\,dx \le \sum_{k=-K}^{K} \int_{kh}^{(k+1)h} |\tau_h(g)(x)|^p\,dx.$$

Thus, since

$$\int_a^{a+h} |\tau_h(g)(x)|^p \, dx \le \int_a^{a+h} \frac{1}{h^p} | \operatorname*{ess\,sup}_{x \in (a,a+2h)} g - \operatorname*{ess\,inf}_{x\in(a,a+2h)} g|^p \, dx,$$

it follows from Lemma 6 that, for some constant $c_p > 0$,

$$\int_{-m}^{m} |\tau_h(g)(x)|^p \, dx \le c_p \sup_{n\in\mathbb{N}} \iint\limits_{\substack{\mathbb{R}\times\mathbb{R} \\ \varepsilon_n < |g(x)-g(y)| < 10\varepsilon_n}} \frac{\varepsilon_n^p}{|x-y|^{p+1}} \, dx \, dy. \tag{3.41}$$

Since $m \ge 2$ is arbitrary, we deduce from (3.41) that

$$\int_{\mathbb{R}} |\tau_h(g)(x)|^p \, dx \le c_p \sup_{n\in\mathbb{N}} \iint\limits_{\substack{\mathbb{R}\times\mathbb{R} \\ \varepsilon_n < |g(x)-g(y)| < 10\varepsilon_n}} \frac{\varepsilon_n^p}{|x-y|^{p+1}} \, dx \, dy. \tag{3.42}$$

Therefore since (3.42) holds for all $0 < h < 1$, it follows that $g \in W^{1,p}(\mathbb{R})$ (see [4, Chapter 8]).

Step 2: Proof of Theorem 1 for $N \ge 2$. Using the change of variables formula and Fubini's theorem, one gets

$$\int\limits_{\substack{\mathbb{R}^N \\ \varepsilon < |g(x)-g(y)| < 10\varepsilon}} \int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{p+N}} \, dx \, dy = \int\limits_{\substack{\mathbb{S}^{N-1} \\ \varepsilon < |g(x+\sigma h)-g(x)| < 10\varepsilon}} \int\limits_{\mathbb{R}^N} \int_0^\infty \frac{1}{h^{p+1}} \, dh \, dx \, d\sigma.$$

Hence, it follows from (3.2) that

$$\liminf_{\varepsilon\to 0_+} \int\limits_{\substack{\mathbb{S}^{N-1} \\ \varepsilon < |g(x+\sigma h)-g(x)| < 10\varepsilon}} \int\limits_{\mathbb{R}^N} \int_0^\infty \frac{\varepsilon^p}{h^{p+1}} \, dh \, dx \, d\sigma < +\infty.$$

Applying Fatou's lemma, one has

$$\int_{\mathbb{S}^{N-1}} \liminf_{\varepsilon\to 0_+} \int\limits_{\substack{\mathbb{R}^N \\ \varepsilon < |g(x+\sigma h)-g(x)| < 10\varepsilon}} \int_0^\infty \frac{\varepsilon^p}{h^{p+1}} \, dh \, dx \, d\sigma < +\infty.$$

Thus for almost every $\sigma \in \mathbb{S}^{N-1}$,

$$\liminf_{\varepsilon\to 0_+} \int\limits_{\substack{\mathbb{R}^N \\ \varepsilon < |g(x+\sigma h)-g(x)| < 10\varepsilon}} \int_0^\infty \frac{\varepsilon^p}{h^{p+1}} \, dh \, dx < +\infty. \tag{3.43}$$

On the other hand, from (3.1),

$$\int\limits_{\substack{\mathbb{S}^{N-1} \\ |g(x+h\sigma)-g(x)| > \varepsilon}} \int\limits_{|x|<r} \int_0^r \frac{1}{h^{p+1}} \, dh \, dx \, d\sigma < +\infty, \qquad \forall r > 0, \ \forall \varepsilon > 0.$$

Hence

$$\int_{\substack{|x|<r}} \int_{\substack{0 \\ |g(x+h\sigma)-g(x)|>\varepsilon}}^{r} \frac{1}{h^{p+1}}\, dh\, dx < +\infty, \qquad \forall r > 0,\ \forall \varepsilon > 0, \tag{3.44}$$

for almost every $\sigma \in \mathbb{S}^{N-1}$.

Fix $\sigma \in \mathbb{S}^{N-1}$ such that conditions (3.43) and (3.44) are satisfied. We claim that

$$\frac{\partial g}{\partial \sigma} \in L^p(\mathbb{R}^N).$$

In fact, without loss of generality, suppose that $\sigma = e_N := (0, \dots, 0, 1)$. Then from (3.43), we have

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\substack{0 \\ \varepsilon<|g(x',x_N+h)-g(x',x_N)|<10\varepsilon}}^{\infty} \frac{\varepsilon^p}{h^{p+1}}\, dh\, dx_N\, dx' < +\infty.$$

Hence applying Fatou's lemma, one gets

$$\int_{\mathbb{R}^{N-1}} \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} \int_{\substack{\mathbb{R} \\ \varepsilon<|g(x',x_N)-g(x',y_N)|<10\varepsilon}} \frac{\varepsilon^p}{|x_N - y_N|^{p+1}}\, dx_N\, dy_N\, dx' < +\infty.$$

On the other hand, (3.44) gives

$$\iint_{\substack{K \times K \\ |g(x',x_N)-g(x',y_N)|>\varepsilon}} \frac{\varepsilon^p}{|x_N - y_N|^{p+1}}\, dx_N\, dy_N < +\infty, \qquad \forall K \subset\subset \mathbb{R}, \forall \varepsilon > 0,$$

for almost every $x' \in \mathbb{R}^{N-1}$. Therefore applying Lemma 2 for the case $N = 1$, one has $g(x', \cdot) \in W^{1,p}(\mathbb{R})$ for almost every $x' \in \mathbb{R}^{N-1}$ and moreover (see [7]),

$$\int_{\mathbb{R}^N} \left| \frac{\partial g}{\partial x_N}(x) \right|^p dx = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \left| \frac{\partial g}{\partial x_N}(x', x_N) \right|^p dx_N\, dx' < +\infty.$$

Since $\partial g / \partial \sigma \in L^p(\mathbb{R}^N)$ for almost every $\sigma \in \mathbb{S}^{N-1}$, we conclude that $g \in W^{1,p}(\mathbb{R}^N)$.

This completes the proof of the fundamental lemma 2. □

Remark 9. The constant 10 which appears in the condition "$\varepsilon < |g(x) - g(y)| < 10\varepsilon$" is a technical constant. We believe that 10 can be replaced by any positive constant strictly greater than 1, but we have not been able to prove this.

Remark 10. Lemma 2 is only proved in the case $1 < p < +\infty$. Lemma 2 clearly implies Proposition 3.

When $p = 1$ we have the following

Question 2. *Assume that $g \in L^1(\mathbb{R}^N)$ satisfies*

$$\iint\limits_{\substack{K\times K\\ |g(x)-g(y)|>\varepsilon}} \frac{1}{|x-y|^{N+1}}\,dx\,dy < +\infty, \qquad \forall K \subset\subset \mathbb{R}^N,\ \forall \varepsilon > 0$$

and

$$\liminf_{\varepsilon\to 0_+} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ \varepsilon<|g(x)-g(y)|<10\varepsilon}} \frac{\varepsilon}{|x-y|^{N+1}}\,dx\,dy < +\infty.$$

Does g belong to $BV(\mathbb{R}^N)$?

3.2. Proof of assertion (b) in Theorem 1

Without loss of generality, we may assume that

$$M := 1 + \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_n(|g(x)-g(y)|)}{|x-y|^{p+N}}\,dx\,dy < +\infty.$$

Thus since $F_n(t)$ is a non-decreasing function with respect to t,

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} 10^{p(k+1)} F_n(10^{-(k+1)}) \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{10^{-(k+1)}<|g(x)-g(y)|<10^{-k}} \frac{10^{-p(k+1)}}{|x-y|^{p+N}}\,dx\,dy \le M. \tag{3.45}$$

We claim that

$$\liminf_{k\to\infty} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{10^{-(k+1)}<|g(x)-g(y)|<10^{-k}} \frac{10^{-p(k+1)}}{|x-y|^{p+N}}\,dx\,dy < +\infty. \tag{3.46}$$

We prove (3.46) by contradiction. Suppose it does not hold. Then there exists $k_M \in \mathbb{N}$ such that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\limits_{10^{-(k+1)}<|g(x)-g(y)|<10^{-k}} \frac{10^{-p(k+1)}}{|x-y|^{p+N}}\,dx\,dy > 2\cdot 10^{p+1}M, \qquad \forall k \ge k_M.$$

Hence (3.45) implies

$$\sup_{n\in\mathbb{N}} 2\cdot 10^{p+1} M \sum_{k=k_M}^{\infty} 10^{p(k+1)} F_n(10^{-(k+1)}) \le M,$$

which shows that

$$\sup_{n\in\mathbb{N}} \sum_{k=k_M}^{\infty} 10^{-(k+1)} 10^{(k+2)(p+1)} F_n(10^{-(k+1)}) \le \frac{1}{2}. \tag{3.47}$$

On the other hand, since F_n converges uniformly to 0 on any compact subset of $(0, +\infty)$ and

$$\int_0^1 F_n(t)t^{-(p+1)}\,dt = 1$$

(see hypotheses (ii) and (iii) on F_n), one gets

$$\begin{aligned} 1 = \lim_{n\to\infty} \int_0^{10^{-(k_M+1)}} F_n(t)t^{-(p+1)}\,dt &= \lim_{n\to\infty} \sum_{k=k_M}^{\infty} \int_{10^{-(k+2)}}^{10^{-(k+1)}} F_n(t)t^{-(p+1)}\,dt \\ &\le \limsup_{n\to\infty} \sum_{k=k_M}^{\infty} 10^{-(k+1)}10^{(k+2)(p+1)} F_n(10^{-(k+1)}) \end{aligned}$$

(since $F_n(t)$ is a non-decreasing function with respect to t). This contradicts (3.47), and proves (3.46).

By (i)–(iii) it follows that for each $s > 0$ there exists n such that $F_n(s) > 0$. Hence since F_n is a non-decreasing function,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{|g(x)-g(y)|>s} \frac{1}{|x-y|^{p+N}}\,dx\,dy < +\infty, \qquad \forall s > 0. \tag{3.48}$$

Therefore by Lemma 2, it follows from (3.46) and (3.48) that $g \in W^{1,p}(\mathbb{R}^N)$. $\square$

4. Γ-convergence

In this section we investigate some questions relating to Γ-convergence. In [8], A. C. Ponce studied similar questions in the context of [1].

We first recall the concept of Γ-convergence (see [3, 6]). One says that a sequence of functionals (I_n), with values in $[0, +\infty]$, Γ-*converges* to a functional I on $L^p(\mathbb{R}^N)$ when the following two conditions are satisfied:

(A) For every $g \in L^p(\mathbb{R}^N)$, there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ converging to g in $L^p(\mathbb{R}^N)$ and

$$\limsup_{n\to\infty} I_n(g_n) \le I(g).$$

(B) For every $g \in L^p(\mathbb{R}^N)$ and for every sequence $(g_n)_{n\in\mathbb{N}} \subset L^p(\mathbb{R}^N)$ converging to g in $L^p(\mathbb{R}^N)$, we have

$$I(g) \le \liminf_{n\to\infty} I_n(g_n).$$

We now take

$$I_{p,n}(g) = \iint_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g(x)-g(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\,dx\,dy$$

for some sequence (δ_n) converging to 0, and

$$I_p(g) = \begin{cases} \dfrac{K_{N,p}}{p} \displaystyle\int_{\mathbb{R}^N} |\nabla g|^p \, dx \\ \qquad \text{if } g \in W^{1,p}(\mathbb{R}^N) \text{ for } p > 1, \text{ resp. } g \in BV(\mathbb{R}^N) \text{ for } p = 1, \\ +\infty \quad \text{otherwise.} \end{cases}$$

When $1 < p < \infty$, property (A) is satisfied with $g_n = g$ by Proposition 1. A very interesting open question is

Question 3. *When $1 < p < \infty$, does $(I_{p,n})$ Γ-converge to I_p on $L^p(\mathbb{R}^N)$?*

In order to give a positive answer to Question 3, it would suffice to prove property (B), i.e., that one can take $c_{N,p} = (1/p)K_{N,p}$ in (4.3) below. This problem is open even for $N = 1$. A partial answer is given in Theorem 3 below.

The same question can be asked for $p = 1$:

Question 4. *Does $(I_{1,n})$ Γ-converge to I_1 on $L^1(\mathbb{R}^N)$?*

Here the situation is more delicate. As pointed out in Remark 5, there exists a function $g \in W^{1,1}(\mathbb{R})$ such that $\lim_{n\to\infty} I_{1,n}(g) = +\infty$ (while $I_1(g) < +\infty$). Hence we cannot argue as above by taking $g_n = g$ to prove property (A). However, (A) is still true:

Proposition 5. *Given any $g \in BV(\mathbb{R}^N)$ there exists a sequence $g_n \in C_c^\infty(\mathbb{R}^N)$ converging to g in $L^1(\mathbb{R}^N)$ and such that*

$$\limsup_{n\to\infty} I_{1,n}(g_n) \le K_{N,1} \int_{\mathbb{R}^N} |\nabla g| \, dx.$$

Proof. Let $(h_k)_{k\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^N)$ converging to g in $L^1(\mathbb{R}^N)$ and such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |\nabla h_k| \, dx = \int_{\mathbb{R}^N} |\nabla g| \, dx. \tag{4.1}$$

Using the same method as in the proof of [7, Lemma 3], one gets

$$\lim_{\delta\to 0} I_1(h_k, \delta) = K_{N,1} \int_{\mathbb{R}^N} |\nabla h_k| \, dx. \tag{4.2}$$

Thus there exists an increasing sequence n_k such that for all $n \ge n_k$,

$$I_{1,n}(h_k) \le K_{N,1} \int_{\mathbb{R}^N} |\nabla h_k| \, dx + \frac{1}{k}.$$

Define the sequence g_n by $g_n = h_k$ if $n_k < n \le n_{k+1}$, where $n_0 = 0$. Then for all $n_k < n \le n_{k+1}$,

$$I_{1,n}(g_n) \le K_{N,1} \int_{\mathbb{R}^N} |\nabla h_k| \, dx + \frac{1}{k},$$

which shows that

$$\limsup_{n\to\infty} I_{1,n}(g_n) \le K_{N,1} \int_{\mathbb{R}^N} |\nabla g|\, dx. \qquad \square$$

We now prove the following result which is a partial answer to Questions 3 and 4. It was announced in [7] for the one-dimensional case.

Theorem 3. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\mathbb{R}^N)$, $1 \le p < +\infty$, converging in $L^p(\mathbb{R}^N)$ to some $g \in L^p(\mathbb{R}^N)$, and $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with* $\lim_{n\to\infty}\delta_n = 0$. *Suppose that*

$$\sup_{n\in\mathbb{N}} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy < +\infty.$$

Then $g \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ and $g \in BV(\mathbb{R}^N)$ if $p = 1$. Moreover,

$$\liminf_{n\to\infty} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ |g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{N+p}}\, dx\, dy \ge c_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p\, dx \tag{4.3}$$

for some positive constant $c_{N,p}$.

We first prove a technical lemma which plays the same role as [2, Lemma 2] in the proof of [2, Theorem 1]. Its proof is based on that of [2, Lemma 2] and Egorov's theorem (see e.g. [10]).

Lemma 7. *Let $\mathcal{I}$ be a bounded interval, $(g_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\mathcal{I})$, $1 \le p < +\infty$, converging in $L^p(\mathcal{I})$ to some $g \in L^p(\mathcal{I})$, and $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with* $\lim_{n\to\infty}\delta_n = 0$. *Then*

$$\liminf_{n\to\infty} \iint\limits_{\substack{\mathcal{I}\times\mathcal{I} \\ |g_n(x)-g_n(y)|>\delta_n}} \frac{\delta_n^p}{|x-y|^{p+1}}\, dx\, dy \ge c_p \frac{1}{|\mathcal{I}|^{p-1}} \Big|\operatorname*{ess\,sup}_{\mathcal{I}} g - \operatorname*{ess\,inf}_{\mathcal{I}} g\Big|^p,$$

for some positive constant c_p depending only on p.

Proof. Without loss of generality suppose that $\mathcal{I} = [0,1]$, $\operatorname{ess\,sup}_{\mathcal{I}} g = s_+$, $\operatorname{ess\,inf}_{\mathcal{I}} g = s_-$ $(-\infty < s_-, s_+ < +\infty)$, and $s_+ - s_- = 1$.

Take $0 < \delta \ll 1$ small enough to ensure that there are (density) points $t_+, t_- \in [40\delta, 1-40\delta] \subset [0,1]$ with

$$\begin{cases} \left|[t_+ - 40\delta, t_+ + 40\delta] \cap \left[g > \frac{4}{5}s_+ + \frac{4}{5}s_-\right]\right| > 70\delta, \\ \left|[t_- - 40\delta, t_- + 40\delta] \cap \left[g < \frac{4}{5}s_- + \frac{1}{5}s_+\right]\right| > 70\delta. \end{cases}$$

We will assume as well that g_n converges to g for almost every $x \in \mathcal{I}$. Thus, by Egorov's theorem (see [10]), there exists a constant n_0 such that for all $n \geq n_0$, we have $\delta_n \leq \delta$ and

$$\begin{cases} \left| [t_+ - 40\delta, t_+ + 40\delta] \cap \left[g_n > \frac{3}{4}s_+ + \frac{1}{4}s_- \right] \right| > 60\delta, \\ \left| [t_- - 40\delta, t_- + 40\delta] \cap \left[g_n < \frac{3}{4}s_- + \frac{1}{4}s_+ \right] \right| > 60\delta. \end{cases}$$

Fix $n \geq n_0$, take $K \in \mathbb{Z}_+$ such that $\delta_n < 2^{-K} \leq 2\delta_n$ and set

$$J = \left\{ j \in \mathbb{Z}_+;\ \frac{3}{4}s_- + \frac{1}{4}s_+ < j2^{-K} < \frac{3}{4}s_+ + \frac{1}{4}s_- \right\}. \tag{4.4}$$

Then

$$|J| \geq 2^{K-1} - 2 \approx 1/\delta_n.$$

Define (for notational ease)

$$A_j = \{x \in [0,1];\ (j-1)2^{-K} \leq g_n(x) < j2^{-K}\}, \qquad B_j = \bigcup_{j' < j} A_{j'}, \qquad \forall j \in \mathbb{Z}.$$

Since the sets A_j are disjoint, it follows from (4.4) that

$$\operatorname{card}(G) \geq 2^{K-2} - 3 \approx 1/\delta_n,$$

where (for notational ease)

$$G = \{j \in J;\ |A_j| < 2^{-K+2}\}.$$

For each $j \in G$, set $\lambda_{1,j} = |A_j|$ and consider the function $\psi_j(t)$ defined as follows:

$$\psi_j(t) = |[t - 40\delta, t + 40\delta] \cap B_j|, \qquad \forall t \in [40\delta, 1 - 40\delta].$$

Applying the same method as in [2, Lemma 2], we deduce that there exist $\lambda_j > 0$ and $t_j \in [t - 40\delta, t + 40\delta]$ such that

$$\begin{cases} |[t_j - 40\lambda_j, t_j + 40\lambda_j] \cap B_j| = 40\lambda_j, \\ \lambda_j/4 \leq |[t_j - 40\lambda_j, t_j + 40\lambda_j] \cap A_j| \leq \lambda_j. \end{cases}$$

It follows that $\lambda_j \lesssim \delta_n$. The rest of the proof is similar to the one of [2, Lemma 2]. □

Proof of Theorem 3. The proof is the same as that of [2, Theorem 1], with the use of Lemma 7 in place of [2, Lemma 1]. The details are left to the reader. □

Finally, we present a special case where the Γ-convergence of a sequence of functionals can be established.

Consider the functionals J_n and J on $L^p(\mathbb{R}^N)$ defined as follows:

$$J_n(g) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{N+p}}\, dx\, dy, \qquad \forall n \in \mathbb{N}, \tag{4.5}$$

and

$$J(g) = \begin{cases} K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx \\ \qquad \text{if } g \in W^{1,p}(\mathbb{R}^N) \text{ for } p > 1, \text{ resp. } g \in BV(\mathbb{R}^N) \text{ for } p = 1, \\ +\infty \quad \text{otherwise}, \end{cases} \tag{4.6}$$

for all $g \in L^p(\mathbb{R}^N)$.

We have the following

Theorem 4. *Let* $1 \le p < +\infty$. *Assume* $F_n : [0, +\infty) \to [0, +\infty)$ *is convex on the interval* $[0, 1]$ *for all* $n \in \mathbb{N}$, *and satisfies hypotheses* (i)–(iii) *of Theorem* 1. *Then* (J_n) Γ-*converges to* J *on* $L^p(\mathbb{R}^N)$.

As a consequence of Theorem 4, we have

Corollary 3. *Let* F_n *be defined as follows:*

$$F_n(t) = \begin{cases} \varepsilon_n t^{p+\varepsilon_n} & \text{if } 0 \le t \le 1, \\ \varepsilon_n & \text{otherwise}, \end{cases}$$

for some sequence (ε_n) *converging to* 0. *Then* (J_n) Γ-*converges to* J.

In order to prove Theorem 4, one needs the following two lemmas.

Lemma 8. *Let* $1 \le p < +\infty$ *and* $F_n : [0, +\infty) \to [0, +\infty)$ *satisfy hypotheses* (i)–(iii) *of Theorem* 1 *and* $(g_n)_{n \in \mathbb{N}} \subset L^p(\mathbb{R}) \cap C_b^2(\mathbb{R})$. *Assume that* g_n *converges to* g *in* $L^p(\mathbb{R})$ *and* $C_b^2(\mathbb{R})$. *Then*

$$\liminf_{n \to \infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x+h) - g_n(x)|)}{h^{p+1}} \, dh \, dx \ge \int_{\mathbb{R}} |g'|^p \, dx.$$

We recall that

$$C_b^2(\mathbb{R}^N) = \{g \in C^2(\mathbb{R}^N);\ \|g\|_{L^\infty} + \|Dg\|_{L^\infty} + \|D^2 g\|_{L^\infty} < +\infty\},$$

with the norm

$$\|g\|_{C_b^2} = \|g\|_{L^\infty} + \|Dg\|_{L^\infty} + \|D^2 g\|_{L^\infty}, \qquad \forall g \in C_b^2(\mathbb{R}^N).$$

Proof. Fix $\varepsilon > 0$ arbitrary. Take $\delta > 0$ such that

$$\int_{A_\delta} |g'|^p \, dx > (1 - \varepsilon) \int_{\mathbb{R}} |g'|^p \, dx, \tag{4.7}$$

where

$$A_\delta := \{x \in \mathbb{R};\ |g'(x)| > 2\delta\}.$$

Since g_n converges to g in $C^1(\mathbb{R})$, there exists some $n_\varepsilon \in \mathbb{N}$ such that

$$|g_n'(x)| > \delta, \qquad \forall x \in A_\delta.$$

Hence since (g_n) is bounded in $C_b^2(\mathbb{R})$, it follows that

$$|g_n(x+h) - g_n(x)| \geq (1-\varepsilon)h|g'(x)|, \quad \forall x \in A_\delta,\ \forall 0 < h < \tau,$$

for some $\tau > 0$ and for all $n \geq n_\varepsilon$. Thus since F_n is non-decreasing, one gets

$$\liminf_{n\to\infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x+h) - g_n(x)|)}{h^{p+1}}\, dh\, dx \geq \liminf_{n\to\infty} \int_{A_\delta} \int_0^\tau \frac{F_n((1-\varepsilon)h|g_n'(x)|)}{h^{p+1}}\, dh\, dx. \tag{4.8}$$

On the other hand, from (i),

$$\liminf_{n\to\infty} \int_{A_\delta} \int_0^\tau \frac{F_n((1-\varepsilon)h|g_n'(x)|)}{h^{p+1}}\, dh\, dx \geq \liminf_{n\to\infty} (1-\varepsilon)^p \int_{A_\delta} |g_n'|^p\, dx \int_0^{\tau\delta/2} F_n(t) t^{-(p+1)}\, dt.$$

Thus it follows from (ii) and (iii) that

$$\liminf_{n\to\infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x+h) - g_n(x)|)}{h^{p+1}}\, dh\, dx \geq (1-\varepsilon)^p \liminf_{n\to\infty} \int_{A_\delta} |g_n'|^p\, dx,$$

which implies, from (4.7),

$$\liminf_{n\to\infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x+h) - g_n(x)|)}{h^{p+1}}\, dh\, dx \geq (1-\varepsilon)^{p+1} \int_{\mathbb{R}} |g'|^p\, dx.$$

Therefore, since $\varepsilon > 0$ is arbitrary,

$$\liminf_{n\to\infty} \int_{\mathbb{R}} \int_0^\infty \frac{F_n(|g_n(x+h) - g_n(x)|)}{h^{p+1}}\, dh\, dx \geq \int_{\mathbb{R}} |g'|^p\, dx.$$

Lemma 9. *Let* $1 \leq p < +\infty$ *and* $F_n : [0, +\infty) \to [0, +\infty)$ *satisfy hypotheses* (i)–(iii) *of Theorem* 1 *and* $g \in C_c^2(\mathbb{R}^N)$*. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{p+1}}\, dx\, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p\, dx.$$

Proof. Without loss of generality, one may assume that

$$\operatorname{supp} g \subset \{x \in \mathbb{R}^N;\ |x| \leq 1\}. \tag{4.9}$$

Using the change of variables formula one has

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x-y|^{p+1}}\, dx\, dy = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^\infty \frac{F_n(|g(x+h\sigma) - g(x)|)}{h^{p+1}}\, dh\, dx\, d\sigma.$$

Thus by Lemma 8 and Fatou's lemma, one gets

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_n(|g(x) - g(y)|)}{|x - y|^{p+1}} \, dx \, dy \geq K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.$$

Hence it suffices to prove that

$$\limsup_{n\to\infty} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^\infty \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \leq K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.$$

Fix $\varepsilon > 0$ (arbitrary). Define

$$A_\varepsilon := \{(\sigma, x) \in \mathbb{S}^{N-1} \times \mathbb{R}^N;\ |\nabla g(x) \cdot \sigma| > 2\varepsilon\}.$$

Choose $0 < \tau < 1$ such that $\tau \|g\|_{C_b^2} < \varepsilon^2$. Then

$$|g(x + h\sigma) - g(x)| < (1 + \varepsilon) h |\nabla g(x) \cdot \sigma|, \qquad \forall 0 < h < \tau, \forall (\sigma, x) \in A_\varepsilon.$$

Thus it follows from (i) that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{A_\varepsilon} \int_0^\tau \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \\
&\qquad \leq \limsup_{n\to\infty} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^\tau \frac{F_n(|(1 + \varepsilon) \nabla g(x) \cdot \sigma|)}{h^{p+1}} \, dh \, dx \, d\sigma,
\end{aligned}$$

which implies

$$\limsup_{n\to\infty} \int_{A_\varepsilon} \int_0^\tau \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \leq (1 + \varepsilon)^p K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx.$$

On the other hand, from the choice of τ,

$$\int_{A_\varepsilon^c} \int_0^\tau \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \leq \int_{\mathbb{S}^{N-1}} \int_{|x|<2} \int_0^\tau \frac{F_n(3\varepsilon h)}{h^{p+1}} \, dh \, dx \leq C \varepsilon^p,$$

where A_ε^c denotes the complement of A_ε in $\mathbb{S}^{N-1} \times \mathbb{R}^N$. Hereafter in this proof, C denotes a constant independent of n. Thus

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_0^\tau \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma \\
&\qquad \leq (1 + \varepsilon)^p K_{N,p} \int_{\mathbb{R}^N} |\nabla g|^p \, dx + C \varepsilon^p. \qquad (4.10)
\end{aligned}$$

On the other hand, it follows from (iii) that

$$\lim_{n\to\infty} \int_{\mathbb{S}^{N-1}} \int_{|x|\leq 2} \int_\tau^\infty \frac{F_n(|g(x + h\sigma) - g(x)|)}{h^{p+1}} \, dh \, dx \, d\sigma = 0. \qquad (4.11)$$

From (4.9), one gets

$$\int_{\mathbb{S}^{N-1}}\int_{\tau}^{\infty}\int_{|x|\geq 2}\frac{F_n(|g(x+h\sigma)-g(x)|)}{h^{p+1}}\,dh\,dx\,d\sigma = \int_{\mathbb{S}^{N-1}}\int_{\tau}^{\infty}\int_{|x|\geq 2}\frac{F_n(|g(x+h\sigma)|)}{h^{p+1}}\,dh\,dx\,d\sigma.$$

Also, from (4.9),

$$\int_{\mathbb{S}^{N-1}}\int_{\tau}^{\infty}\int_{|x|\geq 2}\frac{F_n(|g(x+h\sigma)|)}{h^{p+1}}\,dh\,dx\,d\sigma \leq \int_{\mathbb{S}^{N-1}}\int_{\tau}^{\infty}\int_{\mathbb{R}^N}\frac{F_n(|g(x)|)}{h^{p+1}}\,dh\,dx\,d\sigma.$$

Thus

$$\lim_{n\to\infty}\int_{\mathbb{S}^{N-1}}\int_{\tau}^{\infty}\int_{|x|\geq 2}\frac{F_n(|g(x+h\sigma)-g(x)|)}{h^{p+1}}\,dh\,dx\,d\sigma = 0. \tag{4.12}$$

Combining (4.11) and (4.12) yields

$$\lim_{n\to\infty}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_{\tau}^{\infty}\frac{F_n(|g(x+h\sigma)-g(x)|)}{h^{p+1}}\,dh\,dx\,d\sigma = 0. \tag{4.13}$$

Hence it follows from (4.10) and (4.13) that

$$\limsup_{n\to\infty}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_{0}^{\infty}\frac{F_n(|g(x+h\sigma)-g(x)|)}{h^{p+1}}\,dh\,dx\,d\sigma \leq (1+\varepsilon)^p K_{N,p}\int_{\mathbb{R}^N}|\nabla g|^p\,dx + C\varepsilon^p.$$

Therefore, since $\varepsilon > 0$ is arbitrary, one has

$$\limsup_{n\to\infty}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N}\int_{0}^{\infty}\frac{F_n(|g(x+h\sigma)-g(x)|)}{h^{p+1}}\,dh\,dx\,d\sigma \leq K_{N,p}\int_{\mathbb{R}^N}|\nabla g|^p\,dx. \quad \square$$

Proof of Theorem 4.

Step 1: Proof of property (A). By the same method as in the proof of Proposition 5, property (A) follows from Lemma 9.

Step 2: Proof of property (B). We use some ideas of A. Ponce in the proof of [8, Lemma 12.2]. Let (g_n) be a sequence converging to g in $L^p(\mathbb{R}^N)$. Let (ρ_δ) be a sequence of smooth mollifiers. Set

$$g_{n,k}(x) = \begin{cases} g_n(x) & \text{if } |g_n(x)| \leq k, \\ \dfrac{kg_n(x)}{|g_n(x)|} & \text{otherwise,} \end{cases}$$

for all $k \in \mathbb{N}$, and

$$g_{n,k,\delta} = g_{n,k} * \rho_\delta, \qquad \forall n \in \mathbb{N},\ \forall k \in \mathbb{N}.$$

Then, since F_n is convex on [0, 1], it follows from (i) and (iii) that

$$\liminf_{n\to\infty} J_n(g_{n,k,\delta}) \le \liminf_{n\to\infty} J_n(g_{n,k}). \tag{4.14}$$

On the other hand, from Fatou's lemma and Lemma 8, one gets

$$\liminf_{n\to\infty} J_n(g_{n,k,\delta}) \ge J(h_{k,\delta}), \tag{4.15}$$

where

$$h_k(x) = \begin{cases} g(x) & \text{if } |g(x)| \le k, \\ \dfrac{kg(x)}{|g(x)|} & \text{otherwise,} \end{cases}$$

and

$$h_{k,\delta} = h_k * \rho_\delta.$$

Thus, since

$$\liminf_{\delta\to 0} J(h_{k,\delta}) \ge J(h_k),$$

it follows from (4.14) and (4.15) that

$$\liminf_{n\to\infty} J_n(g_{n,k}) \ge J(h_k).$$

On the other hand, from (i),

$$J_n(g_{n,k}) \le J(g_n).$$

Therefore,

$$\liminf_{n\to\infty} J_n(g_n) \ge J(h_k), \qquad \forall k \in \mathbb{N}.$$

This implies

$$\liminf_{n\to\infty} J_n(g_n) \ge J(g). \qquad \square$$

However, without the assumption on the convexity of (F_n) on [0, 1] in Theorem 4, one has

Question 5. *Assume that* $F_n : [0, +\infty) \to [0, +\infty)$ *satisfies hypotheses* (i)–(iii) *of Theorem* 1*, and* J_n *and* J *are defined by* (4.5) *and* (4.6)*. Does* (J_n) Γ*-converge to* J *on* $L^p(\mathbb{R}^N)$ *for all* $p \ge 1$*?*

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