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Homological category weights and estimates for $\mathrm{cat}^1(X, \xi)$

Abstract. In this paper we study a new notion of category weight of homology classes developing further the ideas of E. Fadell and S. Husseini [3]. In the case of closed smooth manifolds the homological category weight is equivalent to the cohomological category weight of E. Fadell and S. Husseini but these two notions are distinct already for Poincaré complexes. An important advantage of the homological category weight is its homotopy invariance. We use the notion of homological category weight to study various generalizations of the Lusternik–Schnirelmann category which appeared in the theory of closed 1-forms and have applications in dynamics. Our primary goal is to compare two such invariants $\mathrm{cat}(X, \xi)$ and $\mathrm{cat}^1(X, \xi)$ which are defined similarly with reversion of the order of quantifiers. We compute these invariants explicitly for products of surfaces and show that they may differ by an arbitrarily large quantity. The proof of one of our main results, Theorem 8, uses an algebraic characterization of homology classes $z \in H_i(\tilde{X}; \mathbb{Z})$ (where $\tilde{X} \to X$ is a free abelian covering) which are movable to infinity of $\tilde{X}$ with respect to a prescribed cohomology class $\xi \in H^1(X; \mathbb{R})$. This result is established in Part II which can be read independently of the rest of the paper.

1. Introduction

In this paper we study various generalizations of the classical Lusternik–Schnirelmann category $\mathrm{cat}(X)$ which arise in topology of closed 1-forms. They are homotopy invariants of pairs (X, ξ) where X is a finite polyhedron and $\xi \in H^1(X; \mathbb{R})$ is a real cohomology class. Several potentially different notions

$$\mathrm{cat}(X, \xi) \le \mathrm{cat}^1(X, \xi) \le \mathrm{Cat}(X, \xi) \tag{1}$$

play different roles in application of the theory of closed one-forms to dynamics (see [4], [7], [6]); each of these invariants turns into the classical $\mathrm{cat}(X)$ when $\xi = 0$. One

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of the objectives of the present paper is to show that $\mathrm{cat}^1(X, \xi)$ can be distinct from $\mathrm{cat}(X, \xi)$ and moreover their difference can be arbitrarily large. At the moment we have no examples where $\mathrm{Cat}(X, \xi)$ is distinct from $\mathrm{cat}^1(X, \xi)$.

It is well-known that a most effective lower bound for the classical Lusternik–Schnirelmann category $\mathrm{cat}(X)$ is the cohomological cup-length, i.e. the largest number of cohomology classes of positive degree such that their cup-product is nontrivial. In our recent preprint [9] we established cohomological cup-length type lower bounds for $\mathrm{cat}(X, \xi)$ which use local systems of a special kind. In view of (1) all lower bounds for $\mathrm{cat}(X, \xi)$ hold for $\mathrm{cat}^1(X, \xi)$ as well. In order to distinguish between these two invariants one needs to have lower bounds for $\mathrm{cat}^1(X, \xi)$ which in general are not true for $\mathrm{cat}(X, \xi)$. Such lower bounds are found in the present paper.

Our main results are based on the idea of category weight which was initially introduced by E. Fadell and S. Husseini who proposed in [3] to attach "weights" to cohomology classes so that classes of higher weight contribute more into the cup-length estimate; see §2 for more detail. We would like to mention also papers of Y. Rudyak [13] and J. Strom [15] who suggested a useful modification of this notion. In this paper we propose yet another variation of this idea: we attach weights to homology classes (and not to cohomology classes as did the previous authors) and measure the "level of nonvanishing" of a cup-product $u_1 \cup \cdots \cup u_r$ by evaluating it $\langle u_1 \cup \cdots \cup u_r, z \rangle$ on homology classes of different weight. We show that the notion of category weight of homology classes has an important advantage of being homotopy invariant (unlike the weights of Fadell and Husseini). We prove that for closed manifolds the category weight of a homology class equals the category weight of Fadell and Husseini of the dual cohomology class. We also show that this statement is false for Poincaré complexes. The results about category weights of homology classes occupy Part I which can be read independently of the rest of the paper.

Part II also covers a story which may be read independently of Parts I and III. Here we study free abelian covers $p : \tilde{X} \to X$ and homology classes $z \in H_i(\tilde{X}; \mathbb{Z})$ which can be realized by singular cycles lying arbitrarily far in a specified direction. Such "directions" are parametrized by cohomology classes $\xi \in H^1(X; \mathbb{R})$ with $p^*(\xi) = 0$. Our result states that this property of z is equivalent to the existence of an infinite chain c' such that $\partial c' = c$ and c' is "automatically produced out of finite data" (see the discussion after Theorem 5). The main result of Part II generalizes Theorem 5.3 of [5] which treats the case of rank one cohomology classes. It also generalizes our previous result [8] covering the case of homology classes with coefficients in a field; in [8] our arguments use a different algebraic mechanism which fails to work over the integers.

In Part III we use the results of Parts I and II to obtain new cohomological lower bounds for $\mathrm{cat}^1(X, \xi)$. Our Theorem 8 gives in many cases stronger estimates than Theorem 5.6 of [5]; note that the latter theorem applies only in the special case of rank one cohomology classes although the results of the present paper are valid in full generality and do not impose this restriction. In Part III we also introduce a controlled version of $\mathrm{cat}^1(X, \xi)$ which behaves better under cartesian products. Finally, we compute $\mathrm{cat}^1(X, \xi)$ for products of surfaces as function of the cohomology class $\xi \in H^1(X; \mathbb{R})$. We compare our results with the computations of the invariant $\mathrm{cat}(X, \xi)$ completed in [9]. We conclude that $\mathrm{cat}^1(X, \xi)$ may exceed $\mathrm{cat}(X, \xi)$ by an arbitrarily large amount.

The following diagram illustrates dependence of parts of this paper:

Part I   Part II

      ↘   ↙
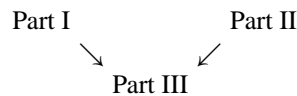
Part III

Parts I and II can be read independently, the results of Parts I & II are used in Part III.

Part I: Category weights of homology classes

Here we introduce and study the notion of category weight of homology classes which is somewhat dual to the cohomological notion introduced by E. Fadell and S. Husseini [3]; the homological category weight has the advantage of being homotopy invariant. In Part III we use this notion to obtain improved cohomological lower bounds for $\mathrm{cat}^1(X, \xi)$.

2. Basic definitions

The classical cohomological lower bound for the Lusternik–Schnirelmann category $\mathrm{cat}(X)$ states that $\mathrm{cat}(X) > n$ if there exist n cohomology classes of positive degree $u_j \in H^*(X; R_j)$, where $j = 1, \ldots, n$, such that their cup-product $u_1 \ldots u_n \in H^*(X; R)$ is nontrivial. Here R_j denotes a local coefficient system on X and R is the tensor product $R_1 \otimes \cdots \otimes R_n$.

E. Fadell and S. Husseini [3] improved this estimate by introducing the notion of a *category weight* $\mathrm{cwgt}(u)$ of a cohomology class $u \in H^q(X; R)$. Here is their definition:

Definition 1. *Let $u \in H^q(X; R)$ be a nonzero cohomology class where R is a local coefficient system on X. One says that $\mathrm{cwgt}(u) \geq k$ (where $k \geq 0$ is an integer) if for any closed subset $A \subset X$ with $\mathrm{cat}_X A \leq k$ one has $u|_A = 0 \in H^q(A; R)$.*

Recall that the inequality $\mathrm{cat}_X A \leq k$ means that A can be covered by k open subsets $U_i \subset X$ such that each inclusion $U_i \subset X$ is null-homotopic, $i = 1, \ldots, k$.

According to Definition 1 one has $\mathrm{cwgt}(u) \geq 0$ in general and $\mathrm{cwgt}(u) \geq 1$ for any nonzero cohomology class of positive degree. As Fadell and Husseini [3] showed, $\mathrm{cwgt}(u) > 1$ in some special situations, which allows improving the lower estimate for $\mathrm{cat}(X)$. Indeed, one has

$$\mathrm{cat}(X) \geq 1 + \sum_{i=1}^{n} \mathrm{cwgt}(u_i)$$

assuming that the cup-product $u_1 \ldots u_n$ is nonzero.

Y. Rudyak [13] and J. Strom [15] studied a modification of $\mathrm{cwgt}(u)$, called the strict category weight $\mathrm{swgt}(u)$. The latter has the advantage of being homotopy invariant. However, in some examples the strict category weight is considerably smaller than the original category weight of Fadell and Husseini.

In this paper we introduce and exploit a "dual" notion of category weight of homology classes. It has the geometric simplicity and clarity of category weight as defined by Fadell and Husseini but has a surprising advantage of being homotopy invariant.

Definition 2. *Let $z \in H_q(X; R)$ be a singular homology class with coefficients in a local system R and let $k \geq 0$ be a nonnegative integer. We say that* $\mathrm{cwgt}(z) \geq k$ *if for any closed subset $A \subset X$ with $\mathrm{cat}_X A \leq k$ there exists a singular cycle c in $X - A$ representing z. We say that* $\mathrm{cwgt}(z) = k$ *iff* $\mathrm{cwgt}(z) \geq k$ *and* $\mathrm{cwgt}(z) \ngeq k + 1$

In other words, $\mathrm{cwgt}(z) \geq k$ is equivalent to the fact that z can be realized by a singular cycle avoiding any prescribed closed subset $A \subset X$ with $\mathrm{cat}_X A \leq k$.

For example, $\mathrm{cwgt}(z) \geq 1$ iff z can be realized by a singular cycle avoiding any closed subset $A \subset X$ such that the inclusion $A \to X$ is homotopic to a constant map.

It will be convenient to define the category weight of the zero homology class as $+\infty$.

Formally $\mathrm{cwgt}(z) \geq k$ if z lies in the intersection

$$\bigcap_A \mathrm{Im}[H_q(X - A; R) \to H_q(X; R)]$$

where $A \subset X$ runs over all closed subsets with $\mathrm{cat}_X A \leq k$.

The relation $\mathrm{cwgt}(z) \leq k$ means that there exists a closed subset $A \subset X$ with $\mathrm{cat}_X A \leq k + 1$ such that any geometric realization of z intersects A. In particular, we obtain the following inequality:

$$\mathrm{cat}(X) \geq \mathrm{cwgt}(z) + 1 \tag{2}$$

for any nonzero homology class $z \in H_q(X; R)$. The last inequality can also be rewritten as

$$0 \leq \mathrm{cwgt}(z) \leq \mathrm{cat}(X) - 1 \leq \dim X \tag{3}$$

for any homology class.

Note that if X is path-connected and z is zero-dimensional, i.e. $z \in H_0(X)$, then $\mathrm{cwgt}(z) = \mathrm{cat}(X) - 1$.

Lemma 1. *Let $f : R \to R'$ be a morphism of local coefficient systems over X and let $f_* : H_q(X; R) \to H_q(X; R')$ be the induced map on homology. Then for any $z \in H_q(X; R)$ one has*

$$\mathrm{cwgt}(f_*(z)) \geq \mathrm{cwgt}(z). \tag{4}$$

Proof. The result follows directly by applying the definition. □

Lemma 2. *Assume that X is a simplicial polyhedron. Then $\mathrm{cwgt}(z) \geq k$ iff z can be realized in $X - A$ for any sub-polyhedron $A \subset X$ with $\mathrm{cat}_X A \leq k$.*

Proof. We only need to show the 'if' direction. Let $A \subset X$ be closed with $\mathrm{cat}_X A \leq k$. We need to show that z can be realized by a cycle in $X - A$. We have $A \subset U_1 \cup \cdots \cup U_k$ with each U_i open and null-homotopic in X. Passing to a fine subdivision of X, we can find a sub-polyhedron $B \subset X$ with $A \subset B \subset U_1 \cup \cdots \cup U_k$. Then $\mathrm{cat}_X B \leq k$ and z can be realized by a cycle lying in $X - B \subset X - A$. $\qquad\qquad\square$

Example 1. Assume that X is a closed 2-dimensional manifold, i.e. a compact surface. Let us show that any nonzero homology class $z \in H_1(X)$ has $\mathrm{cwgt}(z) \geq 1$. Indeed, it is easy to see that any closed subset $A \subset X$ which is null-homotopic in X lies in the interior of a disk $D^2 \subset X$; but $H_1(X - \mathrm{Int}\, D^2) \to H_1(X)$ is an isomorphism.

3. Homotopy invariance of $\mathrm{cwgt}(z)$

Lemma 3. *Let $f : X \to Y$ and $g : Y \to X$ be two continuous maps with $g \circ f \simeq 1_X$. Let R' be a local coefficient system over Y and $R = f^* R'$ be the induced local system over X. Given a homology class $z \in H_q(X; R)$, define $z' \in H_q(Y; R')$ by $z' = f_*(z)$. Then their category weights satisfy*

$$\mathrm{cwgt}(z') \geq \mathrm{cwgt}(z). \tag{5}$$

Proof. We start with the following well-known general remark. Let $B' \subset Y$ be a subset which is null-homotopic in Y. Then the set $B = f^{-1}(B') \subset X$ is null-homotopic in X. Indeed, since $1_X \simeq g \circ f$, the inclusion $B \to X$ is homotopic to the composition $B \xrightarrow{f} B' \xrightarrow{i} Y \xrightarrow{g} X$ where the inclusion $i : B' \to Y$ is null-homotopic by assumption.

Define $k = \mathrm{cwgt}(z)$. Assume that $A' \subset Y$ is a closed subset with $\mathrm{cat}_Y A' \leq k$. Consider $A = f^{-1}(A') \subset X$. Since $\mathrm{cat}_Y A' \leq k$ there exist open sets $U_1', \ldots, U_k' \subset Y$ covering A' with each $U_i' \to Y$ null-homotopic. Then the sets $U_i = f^{-1}(U_i') \subset X$ are open, cover A and are null-homotopic in X (by the above remark). This shows that $\mathrm{cat}_X A \leq k$.

Since $\mathrm{cwgt}(z) \geq k$, the class z can be realized by a singular cycle in $X - A$. Then the cycle $c' = f_*(c)$ in Y represents the class z' and is disjoint from A' as f maps $X - A$ into $Y - A'$. $\qquad\qquad\square$

As a corollary of the previous result we obtain homotopy invariance of the category weight:

Theorem 1. *If $f : X \to Y$ is a homotopy equivalence then for any homology class $z \in H_q(X; R)$ one has*

$$\mathrm{cwgt}(z) = \mathrm{cwgt}(f_*(z)). \tag{6}$$

Here $f_*(z) \in H_q(X; R')$ where $R' = g^* R$ is the local coefficient system over Y induced by the homotopy inverse $g : Y \to X$ of f.

4. Further properties of the category weight

Theorem 2. *Suppose that X is a metric space. Assume $u \in H^r(X; R)$, $z \in H_q(X; R')$ where R and R' are local systems over X. Then for the homology class $u \cap z \in H_{q-r}(X; R \otimes R')$ one has*

$$\mathrm{cwgt}(u \cap z) \geq \mathrm{cwgt}(u) + \mathrm{cwgt}(z). \tag{7}$$

Here $\mathrm{cwgt}(z)$ is the category weight of the homology class z as defined above in this paper and $\mathrm{cwgt}(u)$ is the category weight of u as defined by Fadell and Husseini [3].

Proof. Write $k = \mathrm{cwgt}(z)$, $l = \mathrm{cwgt}(u)$ and assume that $A \subset X$ is a closed subset with $\mathrm{cat}_X A \leq k + l$. We want to show that $u \cap z$ can be realized in the complement $X - A$. There exists an open cover $A \subset U_1 \cup \cdots \cup U_{k+l} \subset X$ with each $U_i \to X$ null-homotopic. Find open subsets $V_i \subset U_i$ such that $\bar{V}_i \subset U_i$ and $A \subset V_1 \cup \cdots \cup V_{k+l}$.

Define $B = V_1 \cup \cdots \cup V_l$ and let $C = A - B$. Clearly C is closed and satisfies $\mathrm{cat}_X C \leq k$. Hence z can be realized by a cycle avoiding C. In other words, $z = i_*(w)$ where $w \in H_q(X - C; R')$.

Since $\mathrm{cwgt}(u) \geq l$ we have $u|_{\bar{B}} = 0$ and thus $u = j_1^*(v)$ for some $v \in H^r(X, B; R)$. By statement 16 in [14, Chapter 5, §6], one has

$$j_*(u \cap z) = j_*(j_1^* v \cap z) = v \cap \bar{j}_*(i_* w) = 0$$

where $j : X \to (X, X - A)$, $\bar{j} : X \to (X, X - C)$ and $j_1 : X \to (X, B)$ are inclusions. By exactness, $j_*(u \cap z) = 0$ implies that $u \cap z$ lies in the image of $H_{q-r}(X - A; R \otimes R') \to H_{q-r}(X; R \otimes R')$. $\square$

As a corollary we obtain:

Corollary 4. *Suppose that X is a metric space and for some classes $z \in H_q(X; R)$ and $u \in H^q(X; R')$ the evaluation $\langle u, z \rangle \in R' \otimes R$ is nonzero. Then*

$$\mathrm{cat}(X) \geq \mathrm{cwgt}(z) + \mathrm{cwgt}(u) + 1. \tag{8}$$

Here $\mathrm{cwgt}(z)$ is the category weight of the homology class z as defined above in this paper and $\mathrm{cwgt}(u)$ is the category weight of u as defined by Fadell and Husseini [3].

Proof. This follows from inequality (2) combined with Theorem 2. $\square$

Inequality (8) allows us to improve the classical cohomological lower bound for the category $\mathrm{cat}(X)$ by taking into account the quality of the homology class z.

5. Manifolds and Poincaré complexes

In this section we prove that in the case of closed manifolds our notion of category weight coincides with the cohomological notion of Fadell and Husseini [3]. However, for Poincaré complexes these notions are distinct as we show by an example.

Theorem 3. *Suppose that X is a closed n-dimensional manifold, and $z \in H_q(X; R)$ where R is a local coefficient system. Let $u \in H^{n-q}(X; R \otimes \tilde{\mathbb{Z}})$ be the Poincaré dual cohomology class, i.e. $z = u \cap [X]$ (see below). Then*

$$\text{cwgt}(z) = \text{cwgt}(u). \tag{9}$$

Here $\tilde{\mathbb{Z}}$ denotes the orientation local system on X, i.e. for a point $x \in X$ the stalk of $\tilde{\mathbb{Z}}$ at x is $\tilde{\mathbb{Z}}_x = H_n(X, X - x; \mathbb{Z})$ (see [14]).

Proof. By the Poincaré duality theorem any homology class $z \in H_q(X; R)$ can be uniquely written as $z = u \cap [X]$ where $u \in H^{n-q}(X; R \otimes \tilde{\mathbb{Z}})$ and $[X] \in H_n(X; \tilde{\mathbb{Z}})$ is the fundamental class. Applying the inequality of Theorem 2 we find

$$\text{cwgt}(z) \geq \text{cwgt}(u) + \text{cwgt}([X]) = \text{cwgt}(u). \tag{10}$$

To obtain the inverse inequality one observes that if $A \subset X$ is a closed subset with $\text{cat}_X A \leq \text{cwgt}(z)$ then z can be realized by a singular cycle in the complement $X - A$ and the usual intersection theory for chains in manifolds shows that the cocycle Poincaré dual to z vanishes on A; hence $\text{cwgt}(u) \geq \text{cwgt}(z)$. □

Example 2. Let $X = \mathbb{RP}^n$ be the real projective space. For the unique nonzero homology class $z \in H_q(X; \mathbb{Z}_2)$ one has $\text{cwgt}(z) = n - q$. Indeed, the dual cohomology class is $\alpha^{n-q} \in H^{n-q}(X; \mathbb{Z}_2)$ where $\alpha \in H^1(X; \mathbb{Z}_2)$ is the generator. Clearly, $\text{cwgt}(\alpha^{n-q}) = n - q$.

Theorem 3 implies:

Corollary 5. *If X is a closed n-dimensional manifold then for any homology class $z \in H_q(X; R)$ with $q < n$ one has*

$$\text{cwgt}(z) \geq 1. \tag{11}$$

Indeed, if $q < n$ then the dual cohomology class u has positive degree and hence $\text{cwgt}(u) \geq 1$.

Consider now the case when X is an n-dimensional Poincaré complex. The first part of the proof of Theorem 3 is still applicable giving inequality (10) between category weights of the homology and cohomology classes. However, the second part of the proof fails. The following example shows that Theorem 3 is false for Poincaré complexes. It is a modification of an argument due to D. Puppe showing that the notion of category weight of cohomology classes is not homotopy invariant.

Example 3. Consider the lens space $L = S^{2n+1}/(\mathbb{Z}/p)$ where p is an odd prime and $\mathbb{Z}/p$ acts freely on S^{2n+1}. Denote by $r : S^{2n+1} \to L$ the quotient map. Let X be the mapping cylinder of r, i.e.

$$X = L \sqcup S^{2n+1} \times [0, 1]/\sim$$

where each point $(x, 0) \in S^{2n+1} \times [0, 1]$ is identified with $r(x) \in L$. Clearly X is homotopy equivalent to L and so it is a Poincaré complex. By a theorem of Krasnosel'skiĭ [10], the category of X equals $2n + 2$. Hence for $z = 1 \in H_0(X; \mathbb{Z}_2)$ one has

$$\mathrm{cwgt}(z) = \mathrm{cat}(X) - 1 = 2n + 1$$

(see above). The dual cohomology class u is the generator $u \in H^{2n+1}(X; \mathbb{Z}_2)$. Let us show that

$$\mathrm{cwgt}(u) = 1.$$

Indeed, consider the sphere $S = S^{2n+1} \times 1 \subset X$. The restriction $u|_S \in H^{2n+1}(S; \mathbb{Z}_2)$ coincides with the induced class $r^*(v)$ where $v \in H^{2n+1}(L; \mathbb{Z}_2)$ is the generator. Hence the cohomology class $u|_S$ is nonzero. However, the sphere S has category 2 and moreover $\mathrm{cat}_X S = 2$ (as the inclusion $S \to X$ is not null-homotopic).

The following simple construction gives non-manifolds for which the category weight can be explicitly calculated.

Lemma 6. *Let $X = X_1 \vee X_2$ be the wedge of two polyhedra X_1 and X_2 and let $z \in H_q(X; R)$ be the sum $z = z_1 + z_2$ where $z_i \in H_q(X_i; R_i)$ and $R_i = R|_{X_i}$. Then*

$$\mathrm{cwgt}(z) = \min\{\mathrm{cwgt}(z_1), \mathrm{cwgt}(z_2)\}. \tag{12}$$

Here $\mathrm{cwgt}(z_i)$ is the category weight of z_i viewed as a homology class of X_i.

Proof. The inequality $\mathrm{cwgt}(z) \leq \min\{\mathrm{cwgt}(z_1), \mathrm{cwgt}(z_2)\}$ is obvious. Let $A \subset X$ be a closed subset with $\mathrm{cat}_X A \leq k$ where $k = \min\{\mathrm{cwgt}(z_1), \mathrm{cwgt}(z_2)\}$. Then $A = A_1 \vee A_2$ where $A_i \subset X_i$ and $\mathrm{cat}_{X_i} A_i \leq k$, where $i = 1, 2$. One can realize z_i by a cycle avoiding A_i. The sum of these two cycles is a cycle representing z which avoids A. Thus we obtain the opposite inequality $\mathrm{cwgt}(z) \geq k$. $\square$

6. Strict category weight

The notion of strict category weight was introduced in [13]; it is a homotopy invariant variation of the category weight of Fadell and Husseini [3]. We use this notion in this paper and therefore recall the relevant definitions. We warn the reader that our terminology differs from [13] by 1 and is consistent with [3].

Definition 3. *Given a continuous map $\phi : A \to X$, we say that $\mathrm{cat}(\phi) \leq k$ if A can be covered by k open sets $A_1, \ldots, A_k$ such that each restriction $\phi|_{A_i}$ is null-homotopic. The* strict category weight *of a cohomology class $u \in H^q(X; R)$ (where R is a local coefficient system on X) is defined as the maximal integer k such that $\phi^*(u) = 0$ for any continuous map $\phi : A \to X$ with $\mathrm{cat}(\phi) \leq k$.*

The strict category weight is denoted by $\mathrm{swgt}(u)$. Clearly, one has

$$\mathrm{swgt}(u) \leq \mathrm{cwgt}(u)$$

and $\mathrm{swgt}(u) \geq 1$ for any cohomology class $u \in H^q(X; R)$ of positive degree $q > 0$.

Definition 4. *Let X be a closed smooth connected n-dimensional manifold. We define the strict category weight of a homology class $z \in H_q(X; R)$ (denoted $\mathrm{swgt}(z)$) as the strict category weight of the dual cohomology class $u \in H^{n-q}(X; R \otimes \tilde{\mathbb{Z}})$.*

A similar definition can be used in the case of Poincaré complexes, but we do not use it in such generality.

Proposition 7. *Let $z_i \in H_{q_i}(X_i; R_i)$ where X_i is a closed smooth orientable manifold of dimension n_i, $i = 1, 2$. Consider the cross-product*

$$z_1 \times z_2 \in H_q(X_1 \times X_2; R)$$

where $q = q_1 + q_2$ and R is the external tensor product $R = R_1 \boxtimes R_2$. Then

$$\mathrm{swgt}(z_1 \times z_2) \geq \mathrm{swgt}(z_1) + \mathrm{swgt}(z_2). \tag{13}$$

Proof. Let $u_i \in H^{n_i - q_i}(X_i; R_i)$ denote the dual of z_i, where $i = 1, 2$. Then the dual of $z_1 \times z_2$ is $u_1 \times u_2 \in H^{n-q}(X_1 \times X_2; R)$ where $n = n_1 + n_2$. Consider also the classes $u_1 \times 1 \in H^{n_1 - q_1}(X_1 \times X_2; R_1 \boxtimes \mathbb{Z})$ and $1 \times u_2 \in H^{n_2 - q_2}(X_1 \times X_2; \mathbb{Z} \boxtimes R_2)$.

Define $k_i = \mathrm{swgt}(z_i) = \mathrm{swgt}(u_i)$. Let $\phi : A \to X_1 \times X_2$ be a continuous map with $\mathrm{cat}(\phi) \leq k_1 + k_2$. Then A is the union of open subsets $A = A_1 \cup A_2$ such that $\mathrm{cat}(\phi)|_{A_i} \leq k_i$. We that $\phi^*(u_1 \times 1)|_{A_1} = 0$ and $\phi^*(1 \times u_2)|_{A_2} = 0$. This implies that the class $\phi^*(u_1 \times u_2) = \phi^*(u_1 \times 1) \cup \phi^*(1 \times u_2)$ vanishes. Hence $\mathrm{swgt}(z_1 \times z_2) \geq k_1 + k_2$. $\square$

Corollary 8. *Let X_i be closed orientable manifolds and $z_i \in H_{q_i}(X_i; R_i)$ where $q_i < \dim X_i$ for $i = 1, \ldots, k$. Consider $z = z_1 \times \cdots \times z_k \in H_q(X; R)$ where $X = X_1 \times \cdots \times X_k$, $q = q_1 + \cdots + q_k$ and $R = R_1 \boxtimes \cdots \boxtimes R_k$. Then*

$$\mathrm{cwgt}(z) \geq k. \tag{14}$$

This corollary is a source of examples of homology classes having high category weight.

Part II: Moving integral homology classes to infinity

In Part II we study conditions for an integral homology class $z \in H_i(\tilde{X}; \mathbb{Z})$ of a free abelian covering $\tilde{X} \to X$ to be movable to infinity with respect to a cohomology class $\xi \in H^1(X; \mathbb{R})$. The case of homology classes with coefficients in a field was studied in [8] using a different algebraic technique which is not applicable over $\mathbb{Z}$.

7. Abel–Jacobi maps and neighbourhoods of infinity

For the convenience of the reader we recall in this section the language introduced in [8]. Let X be a connected finite cell complex and $p : \tilde{X} \to X$ a regular covering having a free abelian group of covering transformations $H \simeq \mathbb{Z}^r$. Write $H_{\mathbb{R}} = H \otimes \mathbb{R}$; it is a vector space of dimension r containing H as a lattice.

Proposition 9. *There exists a canonical Abel–Jacobi map*

$$A : \tilde{X} \to H_{\mathbb{R}} \tag{15}$$

having the following properties:

(a) *A is H-equivariant; here H acts on $\tilde{X}$ by covering transformations and it acts on $H_{\mathbb{R}}$ by translations.*
(b) *A is proper (i.e. the preimage of a compact subset of $H_{\mathbb{R}}$ is compact).*
(c) *A is determined uniquely up to replacing it by a map $A' : \tilde{X} \to H_{\mathbb{R}}$ of the form $A' = A + F \circ p$ where $F : X \to H_{\mathbb{R}}$ is a continuous map.*

This fact is well-known; we refer to [8] for a detailed proof.

Let $\xi \in H^1(X; \mathbb{R})$ be a cohomology class with the property

$$p^*(\xi) = 0 \in H^1(\tilde{X}; \mathbb{R}).$$

Such a class ξ can be viewed either as a homomorphism $\xi : H \to \mathbb{R}$ or as a linear functional $\xi_{\mathbb{R}} : H_{\mathbb{R}} \to \mathbb{R}$.

Definition 5. *A subset $N \subset \tilde{X}$ is called a* neighbourhood of infinity *in $\tilde{X}$ with respect to the cohomology class ξ if N contains the set*

$$\{x \in \tilde{X}; \ \xi_{\mathbb{R}}(A(x)) > c\} \subset N, \tag{16}$$

for some real $c \in \mathbb{R}$. Here $A : \tilde{X} \to H_{\mathbb{R}}$ is an Abel–Jacobi map for the covering $p : \tilde{X} \to X$.

See [8] for more details.

8. Homology classes movable to infinity

Let G be an abelian group (the coefficient system). We mainly have in mind the cases of $G = \mathbb{Z}$ or $G = \mathbf{k}$ a field.

Definition 6 (see [4, §5]). *A homology class $z \in H_i(\tilde{X}; G)$ is said to be* movable to infinity *of $\tilde{X}$ with respect to a nonzero cohomology class $\xi \in H^1(X; \mathbb{R})$, $p^*(\xi) = 0$, if in any neighbourhood N of infinity with respect to ξ there exists a (singular) cycle with coefficients in G representing z.*

Equivalently, a homology class $z \in H_i(\tilde{X}; G)$ is movable to infinity with respect to $\xi \in H^1(X; \mathbb{R})$ if z lies in the intersection

$$\bigcap_N \text{Im}[H_i(N; G) \to H_i(\tilde{X}; G)] \tag{17}$$

where N runs over all neighbourhoods of infinity in $\tilde{X}$ with respect to ξ. This can also be expressed by saying that z lies in the kernel of the natural homomorphism

$$H_i(\tilde{X}; G) \to \varprojlim H_i(\tilde{X}, N; G) \tag{18}$$

where in the inverse limit N runs over all neighbourhoods of infinity in $\tilde{X}$ with respect to ξ.

    The following theorem proven in [8] gives an explicit description of all movable homology classes in the case when $G = \mathbf{k}$ is a field. It generalizes the result of [4, §5] treating the simplest case of infinite cyclic covers $q : \tilde{X} \to X$.

Theorem 4. *Let X be a finite cell complex and $q : \tilde{X} \to X$ be a regular covering having a free abelian group of covering transformations $H \simeq \mathbb{Z}^r$. Let $\xi \in H^1(X; \mathbb{R})$ be a nonzero cohomology class of rank r satisfying $q^*(\xi) = 0$. The following properties of a nonzero homology class $z \in H_i(\tilde{X}; \mathbf{k})$ (where $\mathbf{k}$ is a field) are equivalent:*

(A) *z is movable to infinity with respect to ξ.*
(B) *Any singular cycle c in $\tilde{X}$ realizing the class z bounds an infinite singular chain c' in $\tilde{X}$ containing only finitely many simplices lying outside every neighbourhood of infinity $N \subset \tilde{X}$ with respect to ξ.*
(C) *There exists a nonzero element $x \in \mathbf{k}[H]$ such that $x \cdot z = 0$.*

Later in this paper (see §9) we will describe the set of homology classes with integral coefficients which are movable to infinity.

9. Integral homology classes movable to infinity

To get an analogue of Theorem 4 in the case of integral coefficients, we need another definition.

Definition 7. *Let H be a group and $\xi : H \to \mathbb{R}$ a homomorphism. A nonzero element $\Delta \in \mathbb{Z}[H]$ is said to have ξ-lowest coefficient 1 if $\Delta = (1 - y)h$ with $h \in H$ and $y = \sum a_j g_j$, where the $g_j \in H$ satisfy $\xi(g_j) > 0$ and $a_j \in \mathbb{Z}$.*

Theorem 5. *Let X be a finite cell complex and $p : \tilde{X} \to X$ be a regular covering having a free abelian group of covering transformations $H \simeq \mathbb{Z}^r$. Let $\xi \in H^1(X; \mathbb{R})$ be a nonzero cohomology class of rank r satisfying $p^*(\xi) = 0$. The following properties of a nonzero integral homology class $z \in H_i(\tilde{X}; \mathbb{Z})$ are equivalent:*

(A) *z is movable to infinity with respect to ξ.*

(B) *Any singular cycle c in $\tilde{X}$ realizing the class z bounds an infinite singular chain c' in $\tilde{X}$ with integral coefficients containing only finitely many simplices lying outside every neighbourhood of infinity $N \subset \tilde{X}$ with respect to ξ.*

(C) *There exists a nonzero element $\Delta \in \mathbb{Z}[H]$ with ξ-lowest coefficient 1 such that $\Delta \cdot z = 0$.*

This result improves Theorem 5.3 of [5] which treats the case of rank one cohomology classes, $r = 1$. Movability to infinity of homology classes with coefficients in a field was studied in [4] ($r = 1$ case) and in [8] ($r \geq 1$).

Note that the implications (C)⇒(B)⇒(A) of Theorem 5 are straightforward (see below); the only nontrivial statement is the implication (A)⇒(C). Let us explain why (C)⇒(B). Suppose that $\Delta \cdot z = 0 \in H_i(\tilde{X}; \mathbb{Z})$ where $\Delta \in \mathbb{Z}[H]$ has ξ-lowest coefficient 1. Without loss of generality we may assume that $\Delta = 1 - y$ where $y \in \mathbb{Z}[H]$ is ξ-positive, i.e. y is a finite sum of the form $\sum a_j g_j$ where $g_j \in H, \xi(g_j) > 0$, and $a_j \in \mathbb{Z}$. Let c be a chain representing the class z. Then the cycle $\Delta \cdot c$ bounds, i.e. $(1 - y) \cdot c = \partial c_1$ where c_1 is a finite chain in $\tilde{X}$. Set $c' = c_1 + yc_1 + y^2 c_1 + \cdots$. Then $\partial c' = c$ and c' has finitely many simplices lying outside every neighbourhood of infinity $N \subset \tilde{X}$ with respect to ξ.

The main part of the proof consists in establishing the vanishing of the $\lim^1$ term in the following exact sequence:

$$0 \to \varprojlim{}^1 H_{q+1}(\tilde{X}, N; \mathbb{Z}) \to H_q(X; \widehat{\mathbb{Z}[H]}_\xi) \to \varprojlim H_q(\tilde{X}, N; \mathbb{Z}) \to 0. \qquad (19)$$

This exact sequence was described in §6 of [8]. Formally, the proof of the exactness of (19) given in [8] assumes that the ring of coefficients is a field but it works equally well in the case $\mathbb{Z}$ with no modifications. In the exact sequence (19) lim and $\lim^1$ are taken relative to the system of neighbourhoods of infinity $N \subset \tilde{X}$ with respect to ξ. The symbol $\widehat{\mathbb{Z}[H]}_\xi$ in (19) denotes the Novikov completion of the group ring $\mathbb{Z}[H]$ (see [11], [12]). Recall that elements of the group ring $\mathbb{Z}[H]$ are finite sums of the form $\sum a_i g_i$ where $a_i \in \mathbb{Z}$ and $g_i \in H$; the ring $\widehat{\mathbb{Z}[H]}_\xi$ also includes all countable sums $\sum a_i g_i$ having the property $\lim_{i \to +\infty} \xi(g_i) = +\infty$.

Proposition 10. *Under the conditions of Theorem 5 one has*

$$\varprojlim{}^1 H_q(\tilde{X}, N; \mathbb{Z}) = 0, \qquad (20)$$

where N runs over all neighbourhoods of infinity in $\tilde{X}$ with respect to ξ partially ordered by reverse inclusion.

Proposition 10 gives the implication (A)⇒(B) of Theorem 5. Indeed, using Definition 6 combined with (20) we see that a homology class $z \in H_q(\tilde{X}; \mathbb{Z})$ is movable to infinity with respect to ξ if and only if a cycle $c \in C_q(\tilde{X})$ representing z bounds a chain $c' \in C_q(\tilde{X}) \otimes \widehat{\mathbb{Z}[H]}_\xi$, i.e. $\partial c' = c$. Here $C_*(\tilde{X})$ denote the cellular chain complex of $\tilde{X}$ with

integral coefficients. One can view c' as an infinite chain in $\tilde{X}$ having finitely many terms outside any given neighbourhood of infinity in $\tilde{X}$ with respect to ξ.

To see that (B)$\Rightarrow$(C), let $S_\xi \subset \mathbb{Z}[H]$ be the subset consisting of elements with ξ-lowest coefficient 1 and $\Lambda_\xi = S_\xi^{-1}\mathbb{Z}[H]$ the localization. By [6, Lemma 1.13] the inclusion $\Lambda_\xi \to \widehat{\mathbb{Z}[H]}_\xi$ is faithfully flat so that the change of coefficients $H_*(X; \Lambda_\xi) \to H_*(X; \widehat{\mathbb{Z}[H]}_\xi)$ is injective as well. The result follows.

10. Proof of Theorem 5

First we discuss some commutative algebra. Recall our notations. $H = \mathbb{Z}^r$ is a free abelian group and $\xi : H \to \mathbb{R}$ is an injective group homomorphism. We denote by A the Novikov ring $\widehat{\mathbb{Z}[H]}_\xi$ and by A_0 its subring $\widehat{\mathbb{Z}[H_0]}_\xi$ where $H_0 = \{g \in H; \xi(g) \geq 0\}$. Elements of A_0 are countable formal sums of the form $\sum_j a_j g_j$ where $a_j \in \mathbb{Z}$ and $\xi(g_j)$ tends to $+\infty$.

It is well-known that A is a principal ideal domain but A_0 is not. Our goal is to obtain some partial results about properties of modules over the ring A_0 resembling those of modules over principal ideal domains.

Definition 8. *Let M be an A_0-module. A sequence of elements $m_1, \ldots, m_k \in M$ is a quasi-basis for M if (1) for any $m \in M$ there exists $g \in H_0$ such that gm can be represented in the form $gm = \sum a_j m_j$ where $a_j \in A_0$, and (2) there are no nontrivial relations $\sum a_j m_j = 0$.*

Lemma 11. *Let $f : A_0^n \to A_0^m$ be a homomorphism of finitely generated free A_0-modules. Then there exist quasi-bases $d_1, \ldots, d_n \in A_0^n$ and $e_1, \ldots, e_m \in A_0^m$ and an integer $\mu \leq \min\{n, m\}$ such that for any $j \leq \mu$ one has*

$$f(d_j) = a_j e_j, \quad where \quad a_j \in A_0, \ a_j \neq 0, \tag{21}$$

and $f(d_j) = 0$ for $j > \mu$.

Proof. Localizations of A_0^n and A_0^m with respect to the multiplicative set H lead to free modules over the principal ideal domain A. Hence, applying the standard theory, we find free bases $d'_1, \ldots, d'_n \in A^n$ and $e'_1, \ldots, e'_m \in A^m$ and an integer $\mu \leq \max\{n, m\}$ such that $f(d'_j) = a'_j e'_j$ for $j \leq \mu$ (where $a'_j \in A$, $a'_j \neq 0$) and $f(d'_j) = 0$ for $j > \mu$. Choose $g \in H_0$ such that $gd'_j \in A_0^n$ and $ge'_j \in A_0^m$ for all j. Choose $g' \in H_0$ such that $g'a'_j \in A_0$ for all j. Now set $d_j = gg'd'_j$, $e_j = ge'_j$, and $a_j = g'a'_j$. We obtain quasi-bases d_j and e_j and clearly the relations (21) hold. $\square$

Lemma 12. *Let C_* be a free finitely generated[1] chain complex over A_0. Then there exist quasi-bases $e_1^q, e_2^q, \ldots, e_{r_q}^q \in C_q$ (where $q \in \mathbb{Z}$ and r_q denotes the rank of C_q) and*

---

[1] By this we mean that each A_0-module C_q is finitely generated and only finitely many modules C_q are nonzero.

integers $\mu_q \leq \min\{r_q, r_{q-1}\}$ such that the differential $d : C_q \to C_{q-1}$ is given by

$$d(e_j^q) = \begin{cases} a_j^q e_j^{q-1} & \text{for } j \leq \mu_q, \\ 0 & \text{for } j > \mu_q, \end{cases} \tag{22}$$

and the elements $a_j^q \in A_0$ are nonzero.

Proof. The proof essentially repeats the arguments of Lemma 11. In the first step we construct a basis f_j^q of the localized chain complex $C_q' = A \otimes_{A_0} C_q$ over the principal ideal domain A such that all differentials $d : C_q' \to C_{q-1}'$ have the diagonal form $d(f_j^q) = \alpha_j^q f_j^{q-1}$ with $\alpha_j^q \in A$. In the second step one multiplies the basis f_j^q by a suitable group element $g^q \in H_0$ so that (1) the elements $e_j^q = g^q f_j^q$ lie in the original complex C_q and (2) the elements $a_j^q = g^q (g^{q-1})^{-1} \alpha_j^q$ lie in A_0.   □

Lemma 13. *Let C_* be a free finitely generated chain complex over A_0. Then there exists a finitely generated free chain subcomplex $D_* \subset C_*$ such that $g C_* \subset D_*$ for some group element $g \in H_0$ and $H_j(D_*)$ is isomorphic to a finite direct sum of cyclic[2] A_0-modules.*

Proof. Apply Lemma 12 and take for $D_q \subset C_q$ the A_0-submodule generated by the elements $e_1^q, \ldots, e_{r_q}^q$.   □

Next we apply the above results to obtain the following corollary.

Corollary 14. *Let C_* be a free finitely generated chain complex over A_0. Let $\bar{C}_* = A \otimes_{A_0} C_*$ be the localized chain complex and $i : C_* \to \bar{C}_*$ the inclusion. Then for any q there exists a group element $g = g^q \in H_0$ such that the kernel of the induced map $i_* : H_q(C) \to H_q(\bar{C})$ coincides with the kernel of multiplication by g on $H_q(C)$.*

Proof. As a preparation, consider a nonzero cyclic A_0-module $M_0 = A_0/(aA_0)$ and the associated A-module $M = A/(aA)$. Here $a \in A_0$ is a noninvertible element. Write a in the form $a = g(\alpha + h\beta)$ where $g \in H_0$, $\alpha \in \mathbb{Z}$, $\alpha \neq 0$, $\beta \in A_0$ and $h \in H$ is such that $\xi(h) > 0$. Note that M is trivial if and only if a is invertible in A, i.e. when $\alpha = \pm 1$. Similarly, M_0 is trivial iff a is invertible in A_0, i.e. when $g = 0$ and $\alpha = \pm 1$. We will say that M_0 is a *cyclic module of the first (second) kind* if $\alpha = \pm 1$ (or $|\alpha| > 1$, respectively). We see that for a cyclic module M_0 of the first kind there exists $g \in H_0$ such that $g M_0 = 0$ and the corresponding module M is trivial. For a cyclic module M_0 of the second kind, there is a $g \in H$ such that $g M_0 \to M$ is injective. Indeed, with $a = g(\alpha + h\beta)$ as above and $M_0 = A_0/(aA_0)$, we get $g M_0 = A_0/(\alpha + h\beta A_0)$, which injects into $A/(\alpha + h\beta A)$.

Apply Lemma 13 to obtain a subcomplex $D_* \subset C_*$ such that $g' C_* \subset D_*$ for some $g' \in H_0$ and $H_*(D)$ is a finite direct sum of cyclic A_0-modules. One finds that $A \otimes_{A_0} D_* = A \otimes_{A_0} C_* = \bar{C}_*$ and $H_q(\bar{C}) = A \otimes_{A_0} H_q(C) = A \otimes_{A_0} H_q(D)$ since A is flat over A_0. Also, the kernel of the map restricted to the summands of cyclic modules of the second

---

[2] That is, modules of the form $A_0/(aA_0)$ where $a \in A_0$.

kind can be annihilated by multiplication with a suitable element of H_0. Thus there exists an $h \in H_0$ such that the kernel of the map $H_q(D) \to H_q(\bar{C})$ coincides with the kernel of $h : H_q(D) \to H_q(D)$.

Now set $g = hg' \in H_0$. Let us show that the kernel of $i_* : H_q(C) \to H_q(\bar{C})$ coincides with the kernel of multiplication $g_* : H_q(C) \to H_q(C)$ by g. Consider the following diagram:

$$
\begin{array}{ccccccc}
H_q(C) & \xrightarrow{g'_*} & H_q(D) & \xrightarrow{h_*} & H_q(D) & \longrightarrow & H_q(C) \\
i_* \downarrow & & j_* \downarrow & & j_* \downarrow & & \downarrow i_* \\
H_q(\bar{C}) & \xrightarrow{g'_*} & H_q\bar{C} & \xrightarrow{h_*} & H_q(\bar{C}) & \xrightarrow{\mathrm{id}} & H_q(\bar{C})
\end{array}
$$

The composition of the upper horizontal row is multiplication by g, i.e. the map $g_* : H_q(C) \to H_q(C)$. Every map appearing in the lower horizontal row is an isomorphism. From the previous paragraph we know that $\mathrm{Ker}(j_*) = \mathrm{Ker}(h_*)$. Therefore, examining the diagram, we find that $\mathrm{Ker}(i_*) = \mathrm{Ker}(g_*)$ as claimed. $\square$

The vanishing of the lim^1 term of the exact sequence (19) (i.e. Proposition 10, see above) would follow once one has the Mittag-Lefler condition (see [16, Prop. 3.5.7]), which in our case states:

Proposition 15. *For any neighbourhood of infinity $N \subset \tilde{X}$ with respect to ξ there exists a neighbourhood of infinity $N' \subset N$ such that for any neighbourhood of infinity $N'' \subset N'$ one has*

$$\mathrm{Im}[H_q(\tilde{X}, N'') \to H_q(\tilde{X}, N)] = \mathrm{Im}[H_q(\tilde{X}, N') \to H_q(\tilde{X}, N)]. \tag{23}$$

The homology groups appearing in (23) are with coefficients in $\mathbb{Z}$ and all neighbourhoods of infinity are with respect to a fixed cohomology class ξ.

The equality (23) can be expressed by saying that any cycle in $\tilde{X}$ relative to N which can be refined to a cycle relative to N' can be refined to a cycle relative to an arbitrary neighbourhood of infinity $N'' \subset N' \subset \tilde{X}$.

Proof of Proposition 15. Let $C_*(\tilde{X})$ denote the cellular chain complex of $\tilde{X}$. It is a complex of finitely generated free $\mathbb{Z}[H]$-modules. Let $N \subset \tilde{X}$ be a cellular neighbourhood of infinity with respect to ξ as described in Lemma 3 of [8]. The cellular chain complex $C_*(N)$ is free and finitely generated over $\mathbb{Z}[H_0]$ where $H_0 = \{g \in H; \xi(g) \geq 0\}$. Consider the completed chain complexes $C'_*(N) = A_0 \otimes_{\mathbb{Z}[H_0]} C_*(N)$ and $C'_*(\tilde{X}) = A \otimes_{\mathbb{Z}[H]} C_*(\tilde{X})$. Recall that $A = \widehat{\mathbb{Z}[H]}_\xi$ is the Novikov ring and $A_0 = \widehat{\mathbb{Z}[H_0]}_\xi$. The canonical inclusions $C_*(N) \to C'_*(N)$ and $C_*(\tilde{X}) \to C'_*(\tilde{X})$ determine a chain homomorphism

$$F : C_*(\tilde{X})/C_*(N) \xrightarrow{\cong} C'_*(\tilde{X})/C'_*(N) \tag{24}$$

which is an isomorphism. Injectivity of F is equivalent to $C_*(\tilde{X}) \cap C'_*(N) = C_*(N)$ (which is obvious) and surjectivity of F is equivalent to $C_*(\tilde{X}) + C'_*(N) = C'_*(\tilde{X})$. The latter follows from the equality $\mathbb{Z}[H] + A_0 = A$ for subrings of A.

The short exact sequence of chain complexes over A_0,

$$0 \to C'_*(N) \to C'_*(\tilde{X}) \to C_*(\tilde{X})/C_*(N) \to 0,$$

gives the exact sequence

$$\cdots \to H'_q(N) \xrightarrow{i_*} H'_q(\tilde{X}) \to H_q(\tilde{X}, N) \xrightarrow{\partial} H'_{q-1}(N) \to \cdots$$

where $H'_*(N)$ denotes the homology of the complex $C'_*(N)$ and similarly for $H'_*(\tilde{X})$; the symbol $H_q(\tilde{X}, N)$ denotes $H_q(\tilde{X}, N; \mathbb{Z})$.

Applying Corollary 14 to the subcomplex $C'_*(N) \subset C'_*(\tilde{X})$ we find a group element $g \in H_0$ such that $\mathrm{Ker}[i_* : H'_{q-1}(N) \to H'_{q-1}(\tilde{X})]$ coincides with $\mathrm{Ker}[g_* : H'_{q-1}(N) \to H'_{q-1}(N)] = \mathrm{Ker}[j_* : H'_{q-1}(N) \to H'_{q-1}(g^{-1}N)]$. Here $j : N \to g^{-1}N$ is the inclusion. Setting $N' = gN \subset N$ we obtain

$$\mathrm{Ker}[i_* : H'_{q-1}(N') \to H'_{q-1}(\tilde{X})] = \mathrm{Ker}[i_* : H'_{q-1}(N') \to H'_{q-1}(N)]. \quad (25)$$

Now, consider the following commutative diagram:

$$
\begin{array}{ccccc}
H'_q(N) & \longrightarrow & H'_q(\tilde{X}) & \xrightarrow{\beta} & H_q(\tilde{X}, N) \\
\downarrow & & \downarrow & & \downarrow= \\
H_q(N, N') & \xrightarrow{\tau} & H_q(\tilde{X}, N') & \xrightarrow{\alpha} & H_q(\tilde{X}, N) \\
\downarrow & & \downarrow\partial & & \\
H'_{q-1}(N') & \underset{=}{\xrightarrow{\sigma}} & H'_{q-1}(N') & & \\
\downarrow\gamma & & \downarrow & & \\
H'_{q-1}(N) & \longrightarrow & H'_{q-1}(\tilde{X}) & &
\end{array}
$$

Clearly $\mathrm{Im}\,\beta \subset \mathrm{Im}\,\alpha$. The inverse inclusion $\mathrm{Im}\,\alpha \subset \mathrm{Im}\,\beta$ would follow once we know that for any $x \in H_q(\tilde{X}, N')$ there exists $y \in H_q(N, N')$ such that $\partial\tau(y) = \partial(x) \in H'_{q-1}(N')$. Now, equality (25) says that $\gamma \circ \sigma^{-1} \circ \partial = 0$ is trivial, which (using exactness properties of the diagram above) means that for any $x \in H_q(\tilde{X}, N')$ an element $y \in H_q(N, N')$ with the above mentioned property exists. This shows that $\mathrm{Im}\,\alpha = \mathrm{Im}\,\beta$, i.e.

$$\mathrm{Im}[H_q(\tilde{X}, N') \to H_q(\tilde{X}, N)] = \mathrm{Im}[H'_q(\tilde{X}) \to H_q(\tilde{X}, N)]. \quad (26)$$

For any neighbourhood of infinity $N'' \subset N'$ one has the following diagram

$$
\begin{array}{ccc}
H_q(\tilde{X}, N'') & \longrightarrow & H_q(\tilde{X}, N') \\
\uparrow & \overset{\gamma}{\searrow} & \downarrow\alpha \\
H'_q(\tilde{X}) & \underset{\beta}{\longrightarrow} & H_q(\tilde{X}, N)
\end{array}
$$

which gives $\mathrm{Im}\,\beta \subset \mathrm{Im}\,\gamma \subset \mathrm{Im}\,\alpha$; but since we already know that $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\beta$ coincide we obtain $\mathrm{Im}\,\gamma = \mathrm{Im}\,\alpha$, i.e. (23).

This completes the proof of Proposition 15 for the specially chosen neighbourhood N. If $N_1 \subset \tilde{X}$ is an arbitrary neighbourhood of infinity with respect to ξ then $g_1 N \subset N_1$ and we easily see that for any $N'' \subset g_1 N'$ one has $\mathrm{Im}[H_q(\tilde{X}, N'') \to H_q(\tilde{X}, N_1)] = \mathrm{Im}[H_q(\tilde{X}, g_1 N') \to H_q(\tilde{X}, N_1)]$, i.e. (23) is satisfied. $\qquad \square$

Part III: Cohomological estimates for $\mathrm{cat}^1(X, \xi)$

In Part III we combine the results of Parts I and II to obtain new cohomological lower bounds for $\mathrm{cat}^1(X, \xi)$. This allows us to compute explicitly $\mathrm{cat}^1(X, \xi)$ in some examples. Finally, we compare $\mathrm{cat}^1(X, \xi)$ with the values of a similar invariant $\mathrm{cat}(X, \xi)$ and conclude that their difference can be arbitrarily large.

11. Line bundles, algebraic integers and movability of homology classes

Let X be a finite cell complex and $\xi \in H^1(X; \mathbb{R})$ be a nonzero cohomology class. ξ determines the obvious homomorphism $H_1(X; \mathbb{Z}) \to \mathbb{R}$. Its kernel will be denoted $\mathrm{Ker}(\xi)$. The factor group $H = H_1(X; \mathbb{Z})/\mathrm{Ker}(\xi)$ is a finitely generated free abelian group which is naturally isomorphic to the group of periods of ξ. The rank of H is equals the rank of the class ξ; it is denoted by $r = \mathrm{rk}(\xi)$. Consider the covering $p : \tilde{X} \to X$ corresponding to $\mathrm{Ker}(\xi)$. This covering has H as the group of covering transformations.

Let $\mathcal{V}_\xi = (\mathbb{C}^*)^r = \mathrm{Hom}(H, \mathbb{C}^*)$ denote the variety of all complex flat line bundles L over X such that the induced flat line bundle $p^* L$ over $\tilde{X}$ is trivial. If $t_1, \ldots, t_r \in H$ is a basis, then the monodromy of $L \in \mathcal{V}_\xi$ along t_i is a nonzero complex number $x_i \in \mathbb{C}^*$ and the numbers $x_1, \ldots, x_r \in \mathbb{C}^*$ form a coordinate system on $\mathcal{V}_\xi$. Given a flat line bundle $L \in \mathcal{V}_\xi$ the monodromy representation of L is the ring homomorphism

$$\mathrm{Mon}_L : \mathbb{Z}[H] \to \mathbb{C} \tag{27}$$

sending each $t_i \in H$ to $x_i \in \mathbb{C}^*$.

The dual bundle $L^* \in \mathcal{V}_\xi$ is such that $L \otimes L^*$ is trivial; if $x_1, \ldots, x_r \in \mathbb{C}^*$ are coordinates of L then $x_1^{-1}, \ldots, x_r^{-1} \in \mathbb{C}^*$ are coordinates of L^*.

Any nonzero element $P \in \mathbb{Z}[H]$ can be written as $P = \sum_{i=1}^k a_i h_i$ where $a_i \in \mathbb{Z}$, $a_i \neq 0$, $h_i \in H$ and $\xi(h_1) < \cdots < \xi(h_k)$. The nonzero integer a_k is called the ξ-*top coefficient* of P.

The following notion was introduced in [6, Definition 1.53].

Definition 9. *A flat line bundle $L \in \mathcal{V}_\xi$ is called a ξ-algebraic integer if the kernel of the monodromy homomorphism $\mathrm{Mon}_L : \mathbb{Z}[H] \to \mathbb{C}$ contains a nonzero polynomial $P \in \mathbb{Z}[H]$ having ξ-top coefficient* 1.

Theorem 6. *Let $L \in \mathcal{V}_\xi$ be a complex flat line bundle over X which is not a ξ-algebraic integer. Suppose that for some $v \in H^q(X; L)$ and $z \in H_q(\tilde{X}; \mathbb{Z})$ one has $\langle v, p_*(z) \rangle \neq 0 \in \mathbb{C}$ where $p_* : H_q(\tilde{X}; \mathbb{Z}) \to H_q(X; L^*)$ is the obvious coefficient map. Then the class z is not movable to infinity of $\tilde{X}$ with respect to ξ.*

Proof. We will show that if a homology class $z \in H_q(\tilde{X}; \mathbb{Z})$ is movable to infinity with respect to ξ then $p_*(z) = 0 \in H_q(X; L^*)$ for any $L \in \mathcal{V}_\xi$ which is not a ξ-algebraic integer. This statement clearly implies the theorem.

Let $S_\xi \subset \Lambda = \mathbb{Z}[H]$ denote the set of all nonzero Laurent polynomials $P \in \Lambda$ having ξ-lowest coefficient 1. The monodromy homomorphism $\mathrm{Mon}_{L^*} : \Lambda \to \mathbb{C}$ is injective when restricted to S_ξ (because of our assumption that L is not a ξ-algebraic integer). Hence $\mathrm{Mon}_{L^*} : \Lambda \to \mathbb{C}$ extends to the localized ring $\Lambda_\xi = S_\xi^{-1}\Lambda$.

The homomorphism $p_* : H_q(\tilde{X}; \mathbb{Z}) \to H_q(X; L^*)$ can be decomposed as

$$p_* : H_q(\tilde{X}; \mathbb{Z}) = H_q(X; \Lambda) \xrightarrow{\alpha} H_q(X; \Lambda_\xi) \to H_q(X; L^*)$$

and the module in the middle equals $H_q(X; \Lambda_\xi) = S_\xi^{-1} H_q(\tilde{X}; \mathbb{Z})$. If $z \in H_q(\tilde{X}; \mathbb{Z})$ is movable to infinity with respect to ξ then $\Delta \cdot z = 0$ for some $\Delta \in S_\xi$ and hence $\alpha(z) = 0$ and $p_*(z) = 0$. $\square$

12. Definition and properties of $\mathrm{cat}^1(X, \xi)$

Let X be a finite polyhedron and $\xi \in H^1(X; \mathbb{R})$ a cohomology class with real coefficients. Let ω be a closed 1-form on X representing ξ (see [4] for the formalism of closed 1-forms on topological spaces).

Definition 10. *Let N be a positive integer. A subset $A \subset X$ is said to be N-movable with respect to ω if there exists a continuous homotopy $h_t : A \to X$, $t \in [0, 1]$, such that $h_0 : A \to X$ is the inclusion and for any point $x \in A$ we have*

$$\int_{h_1(x)}^{x} \omega > N$$

where the integral is calculated along the path $t \mapsto h_{1-t}(x) \in X$, $t \in [0, 1]$.

Recall that for $A \subset X$, $\mathrm{cat}_X(A)$ denotes the Lusternik–Schnirelmann category of A in X, i.e. the minimal integer k such that A can be covered by k open sets in X each of which is null-homotopic in X.

The following notion has been introduced in [7].

Definition 11. *Let X be a finite polyhedron and $\xi \in H^1(X; \mathbb{R})$. Fix a closed 1-form ω in ξ. The number $\mathrm{cat}^1(X, \xi)$ is the minimal integer k such that there exists a closed subset $A \subset X$ with $\mathrm{cat}_X(X - A) \leq k$ and such that A is N-movable with respect to ω for any positive integer N.*

By reversing the order of quantifiers one obtains another notion originally introduced in [4].

Definition 12. *Let X be a finite polyhedron and $\xi \in H^1(X; \mathbb{R})$. Fix a closed 1-form ω in ξ. The number $\text{cat}(X, \xi)$ is the minimal integer k such that for any positive integer N there exists a closed subset $A \subset X$ which is N-movable with respect to ω and such that $\text{cat}_X(X - A) \leq k$.*

It is easy to see that neither $\text{cat}^1(X, \xi)$ nor $\text{cat}(X, \xi)$ depend on the choice of ω. Furthermore both notions are homotopy invariants of the pair (X, ξ) (see [4, 7]). Another observation is that for $\xi = 0$ we get the ordinary Lusternik–Schnirelmann category $\text{cat}(X, \xi) = \text{cat}^1(X, \xi) = \text{cat}(X)$.

It follows straightforwardly from the definitions that

$$\text{cat}(X, \xi) \leq \text{cat}^1(X, \xi) \leq \text{cat}(X).$$

We show later in this paper that for some pairs (X, ξ) one has

$$\text{cat}(X, \xi) < \text{cat}^1(X, \xi)$$

and that the difference between $\text{cat}^1(X, \xi)$ and $\text{cat}(X, \xi)$ can indeed be arbitrarily large.

13. The main estimate

Theorem 7. *Let X be a finite cell complex and $\xi \in H^1(X; \mathbb{R})$. Let $L \in \mathcal{V}_\xi$ be a complex flat line bundle over X which is not a ξ-algebraic integer. Assume that for some $u \in H^q(X; L)$ and $z \in H_q(X; \Lambda)$ the evaluation $\langle u, p_*(z) \rangle \in \mathbb{C}$ is nonzero where $p_* : H_q(X; \Lambda) \to H_q(X; L^*)$ is the coefficient homomorphism. Then[3]*

$$\text{cat}^1(X, \xi) \geq \text{cwgt}(z) + 1. \tag{28}$$

Proof. Set $k = \text{cwgt}(z)$ and assume the contrary, i.e. that $\text{cat}^1(X, \xi) \leq k$. Then there exists a closed subset $A \subset X$ with $\text{cat}_X A \leq k$ such that the complement $F = X - A$ is N-movable for any $N > 0$ with respect to a closed 1-form ω on X representing ξ. Applying the definition, we find that z can be realized by a singular cycle c in $X - A = F$ with coefficients in the local system Λ.

Consider the covering $p : \tilde{X} \to X$ corresponding to $\text{Ker}(\xi)$. Viewed differently, the cycle c is a usual singular cycle in $\tilde{X}$ lying in the set $\tilde{F} = p^{-1}(F)$. Since F is N-movable for any N we find that any cycle in $\tilde{F}$ is movable to infinity with respect to ξ. Thus we obtain a contradiction with Theorem 6. $\qquad\square$

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[3] The group $H_q(X; \Lambda)$ is naturally isomorphic to $H_q(\tilde{X}; \mathbb{Z})$. However, the category weights of z viewed as element of $H_q(X; \Lambda)$ or of $H_q(\tilde{X}; \mathbb{Z})$ are in general different. In inequality (28) the symbol $\text{cwgt}(z)$ denotes the category weight of z regarded as an element of $H_q(X; \Lambda)$.

Theorem 8. *Let X be a finite cell complex and $\xi \in H^1(X; \mathbb{R})$. Let $L \in \mathcal{V}_\xi$ be a complex flat line bundle over X which is not a ξ-algebraic integer. Suppose that for an integral homology class $z \in H_q(\tilde{X}; \mathbb{Z}) = H_q(X; \Lambda)$ and some cohomology classes $u \in H^d(X; L)$ and $u_i \in H^{d_i}(X; \mathbb{C})$, where $d_i > 0$ for $i = 1, \ldots, k$, the evaluation $\langle u \cup u_1 \cup \cdots \cup u_k, p_*(z) \rangle \in \mathbb{C}$ is nonzero. Here $p_*(z) \in H_d(X; L^*)$, $q = d + d_1 + \cdots + d_k$. Then*

$$\mathrm{cat}^1(X, \xi) \geq \mathrm{cwgt}(z) + k + 1. \tag{29}$$

Here $\mathrm{cwgt}(z)$ denotes the category weight of z viewed as a homology class of X with local coefficient system Λ.

Proof. First observe that we may assume that the classes $u_1, \ldots, u_k$ are integral, i.e. lie in $H^*(X; \mathbb{Z})$. Indeed, the product $\langle u \cup u_1 \cup \cdots \cup u_k, p_*(z) \rangle$ is a multilinear function of $u_1, \ldots, u_k$; since the integral cohomology classes generate $H^*(X; \mathbb{C})$, vanishing of this function on all integral combinations would imply vanishing in general.

Define $z' = p^*(u_1 \cup \cdots \cup u_k) \cap z \in H_d(\tilde{X}; \mathbb{Z}) = H_d(X; \Lambda)$. Then

$$\langle u, p_*(z') \rangle = \langle u \cup u_1 \cup \cdots \cup u_k, p_*(z) \rangle \neq 0 \in \mathbb{C}.$$

Applying the previous theorem we find $\mathrm{cat}^1(X, \xi) \geq \mathrm{cwgt}(z') + 1$. Now, Theorem 2 gives $\mathrm{cwgt}(z') \geq k + \mathrm{cwgt}(z)$. This completes the proof. $\square$

Remark. Consider the statement of Theorem 8 in the special case $\xi = 0$. Then the variety $\mathcal{V}_\xi$ contains the trivial line bundle $L = \mathbb{C}$ only and $L = \mathbb{C}$ is not a ξ-algebraic integer. Hence Theorem 8 gives the inequality

$$\mathrm{cat}(X) \geq \mathrm{cwgt}(z) + k + 1$$

under the assumption that

$$\langle u_1 \cup \cdots \cup u_k, z \rangle \neq 0$$

where $u_i \in H^{d_i}(X; \mathbb{C})$, $d_i > 0$ and $z \in H_d(X; \mathbb{C})$, $d = d_1 + \cdots + d_k$. This claim is a special case of (8).

Example 4. Let $X = \Sigma$ be a closed orientable surface of genus $g > 1$ and $\xi \neq 0 \in H^1(X; \mathbb{R})$. Fix a flat line bundle $L \in \mathcal{V}_\xi$ which is transcendental (see [9, §6]). Then $H^1(X; L)$ has dimension $2g - 2 > 0$. Pick a nonzero class $u \in H^1(X; L)$. By Proposition 6.5 from [9] there exists a homology class $z \in H_1(X; \Lambda)$ such that $\langle u, p_*(z) \rangle \neq 0$. Since $\mathrm{cwgt}(z) \geq \mathrm{swgt}(z) \geq 1$ we get $\mathrm{cat}^1(\Sigma, \xi) \geq 2$ by applying Theorem 7. Since $\mathrm{cat}^1(X, \xi) \leq \dim X$ in general for $\xi \neq 0$, we find

$$\mathrm{cat}^1(\Sigma, \xi) = 2 \tag{30}$$

for any nonzero $\xi \in H^1(\Sigma; \mathbb{R})$.

Note that $\mathrm{cat}(\Sigma, \xi) = 1$ for any $\xi \neq 0$ (see Theorem 12 in [9]). This gives a first instance where

$$\mathrm{cat}(X, \xi) < \mathrm{cat}^1(X, \xi). \tag{31}$$

14. A controlled version of $\mathrm{cat}^1(X, \xi)$

We have seen in Example 4 that $\mathrm{cat}(X, \xi)$ and $\mathrm{cat}^1(X, \xi)$ can indeed be different. In order to show that the difference between them can be arbitrarily large, we have to introduce a controlled version of $\mathrm{cat}^1(X, \xi)$ which behaves better under cartesian products. The following discussion is very similar to [9, Section 9].

Let ω be a continuous closed 1-form on a finite cell complex X. Let $\xi = [\omega] \in H^1(X; \mathbb{R})$ be the cohomology class represented by ω.

Definition 13. *Let N and C be two positive integers. A subset $A \subset X$ is N-movable with respect to ω with control C if there exists a continuous homotopy $h_t : A \to X$, $t \in [0, 1]$, such that (1) $h_0 : A \to X$ is the inclusion; (2) for any point $x \in A$ one has*

$$\int_x^{h_1(x)} \omega < -N, \tag{32}$$

where the integral is calculated along the path $t \mapsto h_t(x) \in X$, $t \in [0, 1]$, and (3) for any point $x \in A$ and for any $t \in [0, 1]$ one has

$$\int_x^{h_t(x)} \omega \le C. \tag{33}$$

Definition 14. *Fix a closed 1-form ω representing ξ. The number $\mathrm{ccat}^1(X, \xi)$ is the minimal integer k with the property that there exists $C > 0$ and a closed subset $A \subset X$ with $\mathrm{cat}_X(X - A) \le k$ such that A is N-movable with control C with respect to ω for every positive integer N.*

Lemma 16. *The following properties hold for $\mathrm{ccat}^1(X, \xi)$.*

(1) *We have $\mathrm{cat}^1(X, \xi) \le \mathrm{ccat}^1(X, \xi)$.*
(2) *If X is connected and $\xi \ne 0$, then $\mathrm{ccat}^1(X, \xi) \le \mathrm{cat}(X) - 1$.*
(3) *If $\xi = 0$, then $\mathrm{ccat}^1(X, \xi) = \mathrm{cat}(X)$.*
(4) *If $\phi : Y \to X$ is a homotopy equivalence and $\xi \in H^1(X; \mathbb{R})$, then*

$$\mathrm{ccat}^1(X, \xi) = \mathrm{ccat}^1(Y, \phi^*\xi).$$

Proof. The first assertion is obvious, the remaining assertions are obtained by repeating the arguments given in [4] and [9]. □

Remark 1. It is worth pointing out that the applications of $\mathrm{cat}^1(X, \xi)$ to dynamics described in [7] also hold with the potentially larger quantity $\mathrm{ccat}^1(X, \xi)$ (cf. [9, Remark 9.9]).

The desired product inequality now reads as follows.

Theorem 9. *Let X and Y be finite cell complexes and let $\xi_X \in H^1(X; \mathbb{R})$ and $\xi_Y \in H^1(Y; \mathbb{R})$ be real cohomology classes. Assume that*

$$\text{ccat}^1(X, \xi_X) > 0 \quad or \quad \text{ccat}^1(Y, \xi_Y) > 0. \tag{34}$$

Then

$$\text{ccat}^1(X \times Y, \xi) \le \text{ccat}^1(X, \xi_X) + \text{ccat}^1(Y, \xi_Y) - 1, \tag{35}$$

where

$$\xi = \xi_X \times 1 + 1 \times \xi_Y. \tag{36}$$

We skip the proof since it is fully analogous to the proof of the similar statement for $\text{ccat}(X, \xi)$ given in [9, Theorem 9].

15. Calculation of $\text{cat}^1(X, \xi)$ for products of surfaces

Theorem 10. *Let M^{2k} denote the product $\Sigma_1 \times \cdots \times \Sigma_k$ where each Σ_i is a closed orientable surface of genus $g_i > 1$. Given a cohomology class $\xi \in H^1(M^{2k}; \mathbb{R})$, one has*

$$\text{cat}^1(M^{2k}, \xi) = \text{ccat}^1(M^{2k}, \xi) = 1 + k + r \tag{37}$$

where r denotes the number of indices $i \in \{1, \ldots, k\}$ such that the cohomology class $\xi|_{\Sigma_i} \in H^1(\Sigma_i; \mathbb{R})$ vanishes. In particular

$$\text{cat}^1(M^{2k}, \xi) = \text{ccat}^1(M^{2k}, \xi) = 1 + k \tag{38}$$

assuming that $\xi|_{\Sigma_i} \ne 0 \in H^1(\Sigma_i; \mathbb{R})$ for any $i = 1, \ldots, k$.

Proof. After rearranging the surfaces we may assume that $\xi_i = \xi|_{\Sigma_i}$ is nonzero for $i = 1, \ldots, k - r$ and $\xi_i = 0$ for $i > k - r$.

Note that $\text{ccat}^1(\Sigma_i, \xi_i) > 0$ for any $i = 1, \ldots, k$. Indeed, otherwise applying Theorem 10 of [9] we would get $\chi(\Sigma_i) = 0$, contradicting our assumption $g_i > 0$. Hence we may apply the inequality of Theorem 9 several times to obtain

$$\text{ccat}^1(M^{2k}, \xi) \le \sum_{i=1}^{k} \text{ccat}^1(\Sigma_i, \xi_i) - (k - 1).$$

By Example 4 and Lemma 16 we have

$$\text{ccat}^1(\Sigma_i, \xi_i) = \text{cat}^1(\Sigma_i, \xi_i) = \begin{cases} 2 & \text{if } i \le k - r, \\ 3 & \text{if } i > k - r, \end{cases}$$

and thus

$$\text{ccat}^1(M^{2k}, \xi) \le 2(k - r) + 3r - (k - 1) = k + r + 1. \tag{39}$$

Next we prove the opposite inequality to (39). Let $L \in \mathcal{V}_\xi$ be transcendental. Define $H = \pi_1(M)/\mathrm{Ker}(\xi)$ and $L_i = L|_{\Sigma_i}$ and $H_i = \pi_1(\Sigma_i)/\mathrm{Ker}(\xi_i)$. It follows that L_i is also transcendental. Choose $u_i' \in H^1(\Sigma_i; L_i)$ and $z_i \in H_1(\tilde{\Sigma}_i; \mathbb{Z})$ such that $\langle u_i', p_*(z_i) \rangle \neq 0$ as in Example 4. Here $p_i : \tilde{\Sigma}_i \to \Sigma_i$ is the covering space corresponding to $\mathrm{Ker}(\xi_i)$ and $p_{i*} : H_*(\tilde{\Sigma}_i; \mathbb{Z}) \to H_i(\Sigma_i; L_i^*)$. Note that for $i > k - r$ we simply have $\tilde{\Sigma}_i = \Sigma_i$ and $L_i = \mathbb{C}$. Now, $\mathrm{Ker}(\xi_1) \times \cdots \times \mathrm{Ker}(\xi_k) \subset \mathrm{Ker}(\xi)$ so there is a covering map

$$q : \tilde{\Sigma}_1 \times \cdots \times \tilde{\Sigma}_k \to \tilde{M}$$

where $\tilde{M}$ is the covering space of M corresponding to $\mathrm{Ker}(\xi)$. Let

$$z' = z_1 \times \cdots \times z_k \in H_k(M; \mathbb{Z}[H_1 \times \cdots \times H_k]) \cong H_k(\tilde{\Sigma}_1 \times \cdots \times \tilde{\Sigma}_k; \mathbb{Z}),$$

$$z = q_*(z') \in H_k(M; \mathbb{Z}[H]) \cong H_k(\tilde{M}; \mathbb{Z}).$$

It follows from Corollary 8 and Lemma 1 that $\mathrm{cwgt}(z) \geq k$ (where z is viewed as an element of $H_k(M; \mathbb{Z}[H])$).

Define

$$u = u_1' \times \cdots \times u_{k-r}' \times 1 \times \cdots \times 1 \in H^{k-r}(M; L),$$

$$u_j = p_{k-r+j}^* u_{k-r+j}' \in H^1(M; \mathbb{C}), \quad j = 1, \ldots, r,$$

where $p_{k-r+j} : M \to \Sigma_{k-r+j}$ is the projection. Notice that

$$\langle u \cup u_1 \cup \cdots \cup u_r, p_*(z) \rangle = \pm \prod_{i=1}^{k} \langle u_i', p_{i*}(z_i) \rangle \neq 0.$$

Theorem 8 and Corollary 8 apply and give

$$\mathrm{cat}^1(M, \xi) \geq \mathrm{cwgt}(z) + r + 1 \geq k + r + 1.$$

Combining this with (39) we obtain

$$\mathrm{cat}^1(M, \xi) = \mathrm{ccat}^1(M, \xi) = k + r + 1$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We now want to compare the values of $\mathrm{cat}^1(M, \xi)$ with the invariant $\mathrm{cat}(M, \xi)$ (see Definition 12) for products of surfaces $M = \Sigma_1 \times \cdots \times \Sigma_k$ where each Σ_i is a closed orientable surface of genus $g_i > 1$. It was shown in [9, Thm. 17] that

$$\mathrm{cat}(M, \xi) = 1 + 2r \tag{40}$$

where r denotes the number of indices $i \in \{1, \ldots, k\}$ such that $\xi|_{\Sigma_i} = 0$.

Corollary 17. *Under the assumptions of Theorem* 10 *the difference*

$$\mathrm{cat}^1(M, \xi) - \mathrm{cat}(M, \xi) \tag{41}$$

equals the number of indices $i \in \{1, \ldots, k\}$ *such that* $\xi|_{\Sigma_i} \neq 0 \in H^1(\Sigma_i; \mathbb{R})$.

Corollary 17 leads to the following statement which is one of the main results of this paper:

Corollary 18. *The difference* (41) *can be arbitrarily large.*

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