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Blow ups of complex solutions of the 3D Navier–Stokes system and renormalization group method

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Abstract. We consider complex-valued solutions of the three-dimensional Navier–Stokes system without external forcing on $\mathbb{R}^3$. We show that there exists an open set in the space of 10-parameter families of initial conditions such that for each family from this set there are values of parameters for which the solution develops blow up in finite time.

Keywords. Navier–Stokes system, renormalization group theory, fixed point, linearization near a fixed point, spectrum of the linearized group, Hermite polynomials.

Contents

1. Introduction . . . 267
2. Power series for solutions of the 3D Navier–Stokes systems and preliminary changes of variables . . . 269
3. The renormalization group equation . . . 270
4. The analysis of the equation (21) . . . 274
5. The linearization near fixed point . . . 281
6. The set of eigenfunctions of the group of linearized maps . . . 286
7. The choice of initial conditions and the initial part of the inductive procedure . . . 294
8. The list of remainders and their estimates . . . 302
9. Final steps in the proof of the main result. Formulation of the main theorem . . . 308
10. Critical value of parameters and behavior of solutions near the singularity point . . . 312
Appendix . . . 312

1. Introduction

There are many phenomena in nature which can be considered as some manifestation of blow ups, like hurricanes, tornadoes, sandstorms, etc. If we believe that the Navier–Stokes system describes well enough the motions of real gases and fluids under normal conditions, then it gives some reasons to expect that blow ups in solutions of this system also exist.

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In this paper we consider the 3D Navier–Stokes system for incompressible fluids moving without external forcing on $\mathbb{R}^3$ with viscosity equal to 1. We write the Fourier transform of an unknown function $u(x,t)$ as $-iv(k,t)$ and then for $v(k,t)$ we have the equation

$$v(k,t) = \exp\{-t|k|^2\}v(k,0) + \int_0^t \exp\{-(t-s)|k|^2\}\, ds \cdot \int_{\mathbb{R}^3} \langle v(k-k',s),k\rangle \cdot P_k v(k',s)\, dk'. \quad (1)$$

In this expression $v(k,0)$ is given by the initial condition and P_k is the orthogonal projection to the subspace orthogonal to k, i.e. $P_k v = v - \langle v,k\rangle \cdot k/\langle k,k\rangle$. The incompressibility condition takes the form $\langle v(k,t),k\rangle = 0$ for all $t > 0$ and $k \neq 0$. The formula (1) shows that the Navier–Stokes system is a genuinely infinite-dimensional dynamical system: the value $v(k,t)$ is determined by the integration over all "degrees of freedom" and previous moments of time.

The problem of blow ups in solutions of the Navier–Stokes system (NSS) appeared after classical works of J. Leray (see [Le]) where he proved the existence of weak solutions of NSS. O. Ladyzhenskaya proved the existence and uniqueness of strong solutions of 2-dimensional NSS in bounded domains (see [La]). Many important contributions to the modern understanding of fluid dynamics were given by E. Hopf [H], T. Kato [K], C. Foiaş and R. Temam [FT], V. Yudovich [Y], Giga and Miyakawa [GM] and others. However, the situation with the 3-dimensional NSS remained unclear.

In this paper we consider (1) in the space of real-valued functions $v(k,t)$. Certainly this does not mean that $iv(k,t)$ is the Fourier transform of a real-valued vector field. For such solutions the energy inequality does not hold. Detailed assumptions concerning the initial condition $v(k,0)$ will be discussed later (see §7). In all cases $v(k,0)$ will be bounded functions whose support is a neighborhood of some point $(0,0,k^{(0)})$. It follows from the incompressibility condition that the components $v_1(k,0)$, $v_2(k,0)$ of $v(k,0)$ are arbitrary functions of k while $v_3(k,0)$ can be found from the incompressibility condition $\langle v,k\rangle = 0$.

Various methods (see, for example, [K], [C], [S1]) allow one to prove in such cases the existence and uniqueness of classical solutions of (1) on finite time intervals. For these solutions (see, for example, [S2])

$$|v(k,t)| \le \mathrm{const} \exp\{-\mathrm{const}\sqrt{t} \cdot |k|\}, \qquad 0 \le t \le t_0. \quad (2)$$

Presumably, $v(k,t)$ has an asymptotics of this type but this requires more work. According to conventional wisdom, possible blow ups are connected with the violation of (2).

In this paper we fix t and consider one-parameter families of initial conditions $v_A(k,0) = Av(k,0)$, where A is a real parameter. We show that for some special $v(k,0)$ one can find critical values $A_{\mathrm{cr}} = A_{\mathrm{cr}}(t)$ such that the solution $v_{A_{\mathrm{cr}}}(k,s)$ blows up at t so that for $t' < t$ both the energy and the enstrophy are finite while at $t' = t$ they both become infinite. Even more, for $t' < t$ the solution decays exponentially outside some region depending on t. As $t' \uparrow t$ this region expands to an unbounded domain in $\mathbb{R}^3$.

Our basic approach is based on the renormalization group method which was so useful in probability theory, statistical physics and the theory of dynamical systems. It is rather difficult to give the exact formulation of our result in the introduction because it uses some notions, parameters, etc., which will appear in the later sections. Loosely speaking, we show that in l-parameter families of initial conditions, for $l = 10$, there exists an open set of such families such that for each family from this set, one can find values of parameters for which the solutions develop blow ups of the type we already described. The meaning of l is explained in §§4–6. We believe that our methods can be used for proving blow ups of complex solutions of equations with dissipation like Boussinesq equation, Kuramoto–Sivashinsky equation, quasi-geostrophic equation with viscosity and others. Recently we constructed blow ups of complex solutions of the Burgers system in $\mathbb{R}^n$ for any $n \geq 2$. For real-valued solutions one can write an equation describing fixed points of the corresponding renormalization group. However, the existence of its solution remains unclear.

2. Power series for solutions of the 3D Navier–Stokes systems and preliminary changes of variables

Our general approach is based upon the method of power series introduced in [S1], [S2]. We let $v_A(k, 0) = Av(k, 0)$, where $v(k, 0)$ is a real-valued function, and A is a real parameter. We write down the solution of (1) in the form

$$v_A(k,t) = \exp\{-t|k|^2\}Av(k,0) + \int_0^t \exp\{-(t-s)|k|^2\} \sum_{p>1} A^p g_p(k,s)\,ds. \quad (3)$$

The substitution of (3) into (1) gives the system of recurrent equations connecting the functions g_p:

$$g_1(k,s) = \exp\{-s|k|^2\}v(k,0), \quad (4)$$

$$g_2(k,s) = \int_{\mathbb{R}^3} \langle v(k-k',0), k\rangle P_k v(k',0) \cdot \exp\{-s|k-k'|^2 - s|k'|^2\}\,d^3k', \quad (5)$$

$$\begin{aligned} g_p(k,s) = {} & \int_0^s ds_2 \int_{\mathbb{R}^3} \langle v(k-k',0), k\rangle P_k g_{p-1}(k',s_2) \\ & \cdot \exp\{-s|k-k'|^2 - (s-s_2)|k'|^2\}\,d^3k' \\ & + \sum_{\substack{p_1+p_2=p \\ p_1,p_2>1}} \int_0^s ds_1 \int_0^s ds_2 \int_{\mathbb{R}^3} \langle g_{p_1}(k-k',s_1), k\rangle \\ & \cdot P_k g_{p_2}(k',s_2) \cdot \exp\{-(s-s_1)|k-k'|^2 - (s-s_2)|k'|^2\}\,d^3k' \\ + \int_0^s ds_1 \int_{\mathbb{R}^3} & \langle g_{p-1}(k-k',s_1), k\rangle P_k v(k',0) \cdot \exp\{-(s-s_1)|k-k'|^2 - s|k'|^2\}\,d^3k'. \end{aligned} \quad (6)$$

Clearly, $g_p(k,s) \perp k$ for every $p \geq 1$, $k \in \mathbb{R}^3$.

It follows from the results of [S2] that the series (3) converges for sufficiently small s and gives a classical solution of (1).

The formulas (4)–(6) resemble convolutions in probability theory. For example, if $C = \operatorname{supp} v(k,0)$ then $\operatorname{supp} g_p = \underbrace{C + \cdots + C}_{p \text{ times}}$. Therefore it is natural to expect that g_p satisfy some form of the limit theorem of probability theory. This question will be discussed in more detail in the next sections.

To simplify (4)–(6), we shall make some change of variables. Assume that we have some p. The terms in (6) with $p_1 \le p^{1/2}$ and $p_2 \le p^{1/2}$ will be called *boundary terms*. They will be treated as remainders and will be estimated later. Suppose that we have some number $k^{(0)}$ which will be assumed to be sufficiently large. Introduce the vector $\mathcal{K}^{(r)} = (0, 0, rk^{(0)})$. These will be the points near which each g_r will be concentrated, $p^{1/2} \le r \le p - p^{1/2}$. We write $k = \mathcal{K}^{(r)} + \sqrt{rk^{(0)}}\, Y$, $Y \in \mathbb{R}^3$. Thus instead of k we have the new variable $Y = (Y_1, Y_2, Y_3)$ which typically will take values $O(1)$. Put $\kappa^{(0,0)} = (0, 0, 1)$ and $\kappa^{(0)} = (0, 0, k^{(0)})$.

In all integrals over s_1, s_2 in (6) make another change of variables $s_j = s(1 - \theta_j/p_j^2)$, $j = 1, 2$. Instead of the variable of integration k' introduce Y' where $k' = \mathcal{K}^{(p_2)} + \sqrt{pk^{(0)}}Y'$. We write $\tilde{g}_r(Y, s) = g_r(\mathcal{K}^{(r)} + \sqrt{rk^{(0)}}Y, s)$, $\gamma = p_1/p$, $p_2/p = 1 - \gamma$. Then from (6),

$$
\begin{aligned}
\tilde{g}_p(Y,s) &= g_p(\mathcal{K}^{(p)} + \sqrt{pk^{(0)}}Y, s)\\
&= (pk^{(0)})^{5/2}\Bigg[\sum_{\substack{p_1,p_2>\sqrt{p}\\ p_1+p_2=p}} \int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2\, \frac{1}{p_1^2\cdot p_2^2}\\
&\quad\cdot \int_{\mathbb{R}^3} \left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, \left(1-\frac{\theta_1}{p_1^2}\right)s\right), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}}\right\rangle\\
&\quad\cdot P_{\kappa^{(0,0)}+Y/\sqrt{pk^{(0)}}}\tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, \left(1-\frac{\theta_2}{p_2^2}\right)s\right)\\
&\quad\cdot \exp\left\{-\theta_1\left|\kappa^{(0)} + \sqrt{k^{(0)}}\frac{Y-Y'}{\sqrt{p}\cdot\gamma}\right|^2 - \theta_2\left|\kappa^{(0)} + \frac{\sqrt{k^{(0)}}Y'}{\sqrt{p}(1-\gamma)}\right|^2\right\} d^3Y'\Bigg]. \qquad (7)
\end{aligned}
$$

This is the main recurrent relation which we shall study in the next sections. It is of some importance that in front of (7) we have the factor $p^{5/2}$ and inside the sum the factor $1/p_1^2 p_2^2$. Both are connected with the new scaling inherent to the Navier–Stokes system.

3. The renormalization group equation

As $p \to \infty$ the recurrent equation (7) takes some limiting form which will be derived in this section. All remainders which appear in this way are listed and estimated in §8.

The main contribution to (7) comes from p_1, p_2 of order p. If $Y, Y' = O(1)$ then $(Y - Y')/\sqrt{p}$, $Y'/\sqrt{p}$ are small compared to $\kappa^{(0)} = (0, 0, k^{(0)})$. Therefore the Gaussian term in (7) can be replaced by $\exp\{-(\theta_1+\theta_2)|k^{(0)}|^2\}$, s_1 and s_2 can be replaced by s, and the integrations over θ_1, θ_2 and Y' can be done separately. Thus instead of (7) we get a simpler recurrent relation:

$$\tilde{g}_p(Y,s) = \frac{1}{|k^{(0)}|^{3/2}} p^{5/2} \sum_{\substack{p_1,p_2>p^{1/2}\\ p_1+p_2=p}} \frac{1}{p_1^2 p_2^2} \int_{\mathbb{R}^3} \left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}} \right\rangle \\ \cdot P_{\kappa^{(0,0)}+Y/\sqrt{pk^{(0)}}} \tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\right) d^3Y'. \tag{8}$$

In view of incompressibility

$$\begin{aligned} &\sqrt{k^{(0)}} \left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}} \right\rangle \\ &\quad = \frac{1}{p_1} \left\langle g_{p_1}\left(\kappa^{(0)} p_1 + \frac{Y-Y'}{\sqrt{\gamma}} \sqrt{p_1}, s\right), \kappa^{(0)} p_1 + Y\gamma\sqrt{p} \right\rangle \\ &\quad = \frac{1}{p_1} \left\langle g_{p_1}\left(\kappa^{(0)} p_1 + \frac{Y-Y'}{\sqrt{\gamma}} \sqrt{p_1}, s\right), \kappa^{(0)} p_1 + \frac{Y-Y'}{\sqrt{\gamma}} \sqrt{p_1} \right\rangle \\ &\qquad + \frac{1}{p_1} \left\langle g_{p_1}\left(\kappa^{(0)} p_1 + \frac{Y-Y'}{\sqrt{\gamma}} \sqrt{p_1}, s\right), Y\gamma\sqrt{p} - (Y-Y')\sqrt{p} \right\rangle \\ &\quad = \frac{1}{\sqrt{p_1}} \left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), \frac{Y-Y'}{\sqrt{\gamma}} \right\rangle (\gamma - 1) \\ &\qquad + \frac{1}{\sqrt{p_2}} \left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), Y'\sqrt{1-\gamma} \right\rangle. \end{aligned} \tag{9}$$

Write $\tilde{g}_p$ in the form

$$\tilde{g}_p(Y,s) = \left(G_1^{(p)}(Y,s), G_2^{(p)}(Y,s), \frac{1}{\sqrt{p}} F^{(p)}(Y,s) \right). \tag{10}$$

Since $k = \tilde{\kappa}^{(0)} p + Y\sqrt{p}$, the incompressibility implies

$$\langle g_r(k,s), k \rangle = \langle g_r(k,s), k/r \rangle = 0 \tag{11}$$

and for $Y = O(1)$,

$$\frac{Y_1}{\sqrt{r}} G_1^{(r)}(Y,s) + \frac{Y_2}{\sqrt{r}} G_2^{(r)}(Y,s) + \frac{k^{(r)}}{\sqrt{r}} F^{(r)}(Y,s) = O\left(\frac{1}{r}\right). \tag{12}$$

In our approximation we replace (12) by

$$Y_1 G_1^{(r)}(Y,s) + Y_2 G_2^{(r)}(Y,s) + F^{(r)}(Y,s) = 0. \tag{13}$$

Thus for given Y_1, Y_2, Y_3 the component F_r can be expressed through $G_1^{(r)}, G_2^{(r)}$. This remains to be true even if we do not neglect the RHS of (12). Return back to (9). From (13),

$$\begin{aligned}
&\left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), \kappa^{(0)} + \frac{Y}{\sqrt{p}}\right\rangle \\
&\quad = \frac{1}{\sqrt{p}}\left[\frac{\gamma-1}{\sqrt{\gamma}}\left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), \frac{Y-Y'}{\sqrt{\gamma}}\right\rangle + \sqrt{1-\gamma}\left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), \frac{Y'}{\sqrt{1-\gamma}}\right\rangle\right] \\
&\quad = \frac{1}{\sqrt{p}}\left[\frac{\gamma-1}{\sqrt{\gamma}}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}} G_1^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right) + \frac{Y_2-Y_2'}{\sqrt{\gamma}} G_2^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right.\right. \\
&\qquad + \left.\frac{Y_3-Y_3'}{\sqrt{\gamma}} \frac{1}{\sqrt{p_1}} F^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right) + \sqrt{1-\gamma}\left(\frac{Y_1'}{\sqrt{1-\gamma}} G_1^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right. \\
&\qquad + \left.\left.\frac{Y_2'}{\sqrt{1-\gamma}} G_2^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right) + \frac{1}{\sqrt{p_2}} \frac{Y_3'}{\sqrt{1-\gamma}} F^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right)\right]. \qquad (14)
\end{aligned}$$

In our approximation the inner product in (14) can be replaced by

$$\begin{aligned}
&\frac{1}{\sqrt{p}}\left[\frac{\gamma-1}{\sqrt{\gamma}}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}} G_1^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right) + \frac{Y_2-Y_2'}{\sqrt{\gamma}} G_2^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right)\right. \\
&\qquad + \left.\sqrt{1-\gamma}\left(\frac{Y_1'}{\sqrt{1-\gamma}} G_1^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right) + \frac{Y_2}{\sqrt{1-\gamma}} G_2^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}} s\right)\right)\right]. \qquad (15)
\end{aligned}$$

According to the definition of the projector

$$\begin{aligned}
P_{\tilde{\kappa}^{(0)}+Y/\sqrt{p}}\tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\right) &= \tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\right) \\
&\quad - \frac{\langle \tilde{g}_{p_2}(Y'/\sqrt{1-\gamma}, s), \tilde{\kappa}^{(0)} + Y/\sqrt{p}\rangle(\tilde{\kappa}^{(0)} + Y/\sqrt{p})}{\langle \tilde{\kappa}^{(0)} + Y/\sqrt{p}, \tilde{\kappa}^{(0)} + Y/\sqrt{p}\rangle} \\
&= \tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\right) + O\left(\frac{1}{\sqrt{p_2}}\right). \qquad (16)
\end{aligned}$$

This shows that in the main order of magnitude the projector is the identity operator and we come to a simpler recurrent relation instead of (8):

$$\begin{aligned}
\tilde{g}_p(Y,s) &= \frac{1}{(k^{(0)})^2} \sum_{\substack{p_1, p_2 \geq p^{1/2} \\ p_1+p_2=p}} \frac{p^2}{p_1^2 p_2^2} \int_{\mathbb{R}^3} \left[\frac{\gamma-1}{\sqrt{\gamma}}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}} G_1^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right.\right. \\
&\quad + \left.\frac{Y_2-Y_2'}{\sqrt{\gamma}} G_2^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right) \\
&\quad + \left.\sqrt{1-\gamma}\left(\frac{Y_1'}{\sqrt{1-\gamma}} G_1^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right) + \frac{Y_2'}{\sqrt{1-\gamma}} G_2^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right)\right] \\
&\quad \cdot \tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\right) d^3Y'. \qquad (17)
\end{aligned}$$

The main assumption which we shall check below in the next sections concerns the asymptotic form of $\tilde{g}_p(Y,s)$ as $p \to \infty$: for some interval $S^{(p)} = [S_-^{(p)}, S_+^{(p)}]$ on the time axis and some $Z(s)$, $\Lambda(s)$, positive $\sigma^{(1)}$, $\sigma^{(2)}$ and for all $r < p$,

$$\begin{aligned}\tilde{g}_r(Y,s) &= Z(s)\Lambda(s)^r r \frac{\sigma^{(1)}}{2\pi} \exp\left\{-\frac{\sigma^{(1)}}{2}(|Y_1|^2 + |Y_2^2|)\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi}} \exp\left\{-\frac{\sigma^{(2)}}{2}|Y_3|^2\right\} \\ &\quad \cdot (H_1(Y_1,Y_2,Y_3) + \delta_1^{(r)}(Y,s), H_2(Y_1,Y_2,Y_3) + \delta_2^{(r)}(Y,s), \delta_3^{(r)}(Y,s)) \qquad (18)\end{aligned}$$

where

$$\delta_j^{(r)}(Y,s) \to 0 \quad \text{as } r \to \infty,\ j = 1,2,3.$$

Later we shall explain in more detail in what sense the convergence to zero takes place. The substitution of (18) into (17) gives

$$\begin{aligned}\tilde{g}_p(Y,s) &= \frac{1}{|k^{(0)}|^2} Z(s)^2 p \Lambda(s)^p \\ &\quad \cdot \sum_{\gamma = p_1/p} \frac{1}{p}\gamma^{1/2}(1-\gamma)^{1/2} \cdot \int_{\mathbb{R}^3} \left[\frac{\gamma - 1}{\sqrt{\gamma}}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} H_1\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right.\right. \\ &\quad \left. + \frac{Y_2 - Y_2'}{\sqrt{\gamma}} H_2\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right) + \sqrt{1-\gamma}\left(\frac{Y_1'}{\sqrt{1-\gamma}} H_1\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right. \\ &\quad \left.\left. + \frac{Y_2'}{\sqrt{1-\gamma}} H_2\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right)\right] H\left(\frac{Y'}{\sqrt{1-\gamma}}\right) \\ &\quad \cdot \frac{\sigma^{(1)}}{2\pi\gamma} \exp\left\{-\frac{\sigma^{(1)}}{2}\left(\frac{|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2}{\gamma}\right)\right\} \\ &\quad \cdot \frac{\sigma^{(1)}}{2\pi(1-\gamma)} \exp\left\{-\frac{\sigma^{(1)}}{2}\frac{|Y_1'|^2 + |Y_2'|^2}{1-\gamma}\right\} \\ &\quad \cdot \sqrt{\frac{\sigma^{(2)}}{2\pi\gamma}} \exp\left\{-\frac{\sigma^{(2)}}{2}\frac{|Y_3 - Y_3'|^2}{\gamma}\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}} \exp\left\{-\frac{\sigma^{(2)}}{2}\frac{|Y_3'|^2}{1-\gamma}\right\} d^3Y'. \qquad (19)\end{aligned}$$

Here

$$\begin{aligned}&H\left(\frac{Y'}{\sqrt{1-\gamma}}\right) \\ &\quad = \left(H_1\left(\frac{Y_1'}{\sqrt{1-\gamma}}, \frac{Y_2'}{\sqrt{1-\gamma}}, \frac{Y_3'}{\sqrt{1-\gamma}}\right), H_2\left(\frac{Y_1'}{\sqrt{1-\gamma}}, \frac{Y_2'}{\sqrt{1-\gamma}}, \frac{Y_3'}{\sqrt{1-\gamma}}\right), 0\right).\end{aligned}$$

We do not mention explicitly the dependence of H on s.

The last sum looks like a Riemann integral sum and the limit of (19) as $p \to \infty$ takes the form

$$\begin{aligned}
&\exp\left\{-\frac{\sigma^{(1)}}{2}(|Y_1|^2+|Y_2|^2)\right\}\frac{\sigma^{(1)}}{2\pi}\exp\left\{-\frac{\sigma^{(2)}|Y_3|^2}{2}\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi}}H(Y) \\
&= \frac{1}{|k^{(0)}|^2}Z(s)\int_0^1 \gamma^{1/2}(1-\gamma)^{1/2}\,d\gamma\int_{\mathbb{R}^3}\frac{\sigma^{(1)}}{2\pi\gamma}\exp\left\{-\frac{\sigma^{(1)}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)}{2\gamma}\right\} \\
&\quad\cdot\frac{\sigma^{(1)}}{2\pi(1-\gamma)}\exp\left\{-\frac{\sigma^{(1)}(|Y_1'|^2+|Y_2'|^2)}{2(1-\gamma)}\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi\gamma}}\exp\left\{-\frac{\sigma^{(2)}|Y_3-Y_3'|^2}{2\gamma}\right\} \\
&\quad\cdot\sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}}\exp\left\{-\frac{\sigma^{(2)}|Y_3'|^2}{2(1-\gamma)}\right\} \\
&\quad\cdot\left[-\frac{\gamma-1}{\sqrt{\gamma}}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2-Y_2'}{\sqrt{\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right. \\
&\quad\left.+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{1-\gamma}}\right)\right)\right] \\
&\quad\cdot H\left(\frac{Y'}{\sqrt{1-\gamma}}\right)d^3Y'. \qquad (20)
\end{aligned}$$

We take $Z(s) = (k^{(0)})^2$. Then the equation does not contain $k^{(0)}$. The integral over Y_3 is the usual convolution. Therefore we can look for functions H_1, H_2 depending only on Y_1, Y_2, i.e. $H_1(Y) = H_1(Y_1, Y_2)$, $H_2(Y) = H_2(Y_1, Y_2)$. Write down the equation for H_1, H_2 which does not contain Y_3:

$$\begin{aligned}
&\exp\left\{-\frac{\sigma^{(1)}}{2}|Y|^2\right\}\frac{\sigma^{(1)}}{2\pi}H(Y) = \int_0^1 d\gamma\int_{\mathbb{R}^2}\frac{\sigma^{(1)}}{2\pi\gamma}\exp\left\{-\frac{\sigma^{(1)}|Y-Y'|^2}{2\gamma}\right\}\frac{\sigma^{(1)}}{2\pi(1-\gamma)} \\
&\cdot\exp\left\{-\frac{\sigma^{(1)}}{2(1-\gamma)}|Y'|^2\right\}\left[-(1-\gamma)^{3/2}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2-Y_2'}{\sqrt{\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right. \\
&\left.+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right]H\left(\frac{Y'}{\sqrt{1-\gamma}}\right)d^2Y'. \qquad (21)
\end{aligned}$$

Here $Y = (Y_1, Y_2)$, $Y' = (Y_1', Y_2')$, $H(Y) = (H_1(Y_1, Y_2), H_2(Y_1, Y_2))$. This is our main equation for the fixed point of the renormalization group which we shall analyze in the next section (see also §7).

4. The analysis of the equation (21)

The solutions to the equation (21) have a natural scaling with respect to the parameter $\sigma = \sigma^{(1)}$. Namely, if we solve the equation (21) for $\sigma = 1$ and let the corresponding solution be $H(Y)$, then the general solution for arbitrary σ is given by the formula

$$H_\sigma(Y) = \sqrt{\sigma} H(\sqrt{\sigma} Y). \tag{22}$$

This is analogous to the usual scaling of the Gaussian fixed point in probability theory. Thus, it is enough to consider the equation (21) for $\sigma = 1$. We shall show that there exists a three-parameter family of solutions to the equation (21) for $\sigma = 1$. The equation (21) takes a simpler form if we use expansions over Hermite polynomials. All necessary facts about Hermite polynomials are collected in the Appendix. For $H(Y_1, Y_2) = (H_1(Y_1, Y_2), H_2(Y_1, Y_2))$, we write

$$H_j(Y_1, Y_2) = \sum_{m_1, m_2 \geq 0} h^{(j)}_{m_1,m_2} \operatorname{He}_{m_1}(Y_1) \operatorname{He}_{m_2}(Y_2), \qquad j = 1, 2, \tag{23}$$

where $\operatorname{He}_m(z)$ are the Hermite polynomials of degree m with respect to the Gaussian density $\frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}z^2\}$. We have (see (47))

$$z \operatorname{He}_m(z) = \operatorname{He}_{m+1}(z) + m \operatorname{He}_{m-1}(z), \qquad m > 0, \tag{24}$$

and

$$\operatorname{He}_0(z) = 1, \qquad z \operatorname{He}_0(z) = z = \operatorname{He}_1(z).$$

Also we use the formula (see (48))

$$\begin{aligned} \int_{\mathbb{R}^1} \operatorname{He}_{m_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}\right) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{|Y-Y'|^2}{2\gamma}\right\} \operatorname{He}_{m_2}\left(\frac{Y'}{\sqrt{1-\gamma}}\right) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{|Y'|^2}{2(1-\gamma)}\right\} dY' \\ = \gamma^{(m_1+1)/2}(1-\gamma)^{(m_2+1)/2} \operatorname{He}_{m_1+m_2}(Y) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{|Y|^2}{2}\right\}. \end{aligned} \tag{25}$$

Substituting (23) into (21) and using (24), (25), we come to the system of equations for the coefficients $h^{(j)}_{m_1,m_2}$ which is equivalent to (21):

$$\begin{aligned} h^{(j)}_{m_1,m_2} = \sum_{\substack{m_1'+m_1''=m_1 \\ m_2'+m_2''=m_2}} & [J^{(1)}_{m'm''}\{(B_1 h^{(1)})_{m_1',m_2'} + (B_2 h^{(2)})_{m_1',m_2'}\} h^{(j)}_{m_1'',m_2''} \\ & + J^{(2)}_{m'm''}\{h^{(1)}_{m_1',m_2'}(B_1 h^{(j)})_{m_1'',m_2''} + h^{(2)}_{m_1',m_2'}(B_2 h^{(j)})_{m_1'',m_2''}\}] \end{aligned} \tag{26}$$

where $m' = m_1' + m_2'$, $m'' = m_1'' + m_2''$ and

$$\begin{cases} J^{(1)}_{m'm''} = -\displaystyle\int_0^1 \gamma^{m'/2}(1-\gamma)^{(m''+3)/2}\, d\gamma, \\ J^{(2)}_{m'm''} = \displaystyle\int_0^1 \gamma^{(m'+1)/2}(1-\gamma)^{(m''+2)/2}\, d\gamma, \end{cases} \tag{27}$$

$$(B_1 h^{(j)})_{m_1',m_2'} = h^{(j)}_{m_1'-1,m_2'} + (m_1'+1) h^{(j)}_{m_1'+1,m_2'},$$

$$(B_2 h^{(j)})_{m_1',m_2'} = h^{(j)}_{m_1',m_2'-1} + (m_2'+1) h^{(j)}_{m_1',m_2'+1}.$$

To simplify the system (26), we shall look for solutions with $h^{(j)}(0,0)=0$, $j=1,2$. For $m_1+m_2=1$, we have

$$\begin{cases} h^{(1)}_{10} = J^{(1)}_{01}(h^{(1)}_{10}+h^{(2)}_{01})h^{(1)}_{10} + J^{(2)}_{10}(h^{(1)}_{10}h^{(1)}_{10}+h^{(2)}_{10}h^{(1)}_{01}),\\ h^{(1)}_{01} = J^{(1)}_{01}(h^{(1)}_{10}+h^{(2)}_{01})h^{(1)}_{01} + J^{(2)}_{10}(h^{(1)}_{01}h^{(1)}_{10}+h^{(2)}_{01}h^{(1)}_{01}),\\ h^{(2)}_{10} = J^{(1)}_{01}(h^{(1)}_{10}+h^{(2)}_{01})h^{(2)}_{10} + J^{(2)}_{10}(h^{(1)}_{10}h^{(2)}_{10}+h^{(2)}_{10}h^{(2)}_{01}),\\ h^{(2)}_{01} = J^{(1)}_{01}(h^{(1)}_{10}+h^{(2)}_{01})h^{(2)}_{01} + J^{(2)}_{10}(h^{(1)}_{01}h^{(2)}_{10}+h^{(2)}_{01}h^{(2)}_{01}), \end{cases}$$

where $J^{(1)}_{01}=-1/3$ and $J^{(2)}_{10}=1/6$. There are two cases:

Case 1: $h^{(1)}_{10}+h^{(2)}_{01}=-6$. In this case $(h^{(1)}_{10},h^{(1)}_{01},h^{(2)}_{10},h^{(2)}_{01})$ only needs to satisfy

$$(h^{(1)}_{10}+3)^2 = 9 - h^{(1)}_{01}h^{(2)}_{10}.$$

This is a two-parameter family of solutions.

Case 2: $h^{(1)}_{10}+h^{(2)}_{01}\neq -6$. In this case $(h^{(1)}_{10},h^{(1)}_{01},h^{(2)}_{10},h^{(2)}_{01})$ can be uniquely determined and we have $h^{(1)}_{10}=h^{(2)}_{01}=-2$, $h^{(1)}_{01}=h^{(2)}_{10}=0$.

In the remaining part of this paper we shall consider only Case 2 for which $h^{(1)}_{10}=h^{(2)}_{01}=-2$, $h^{(1)}_{01}=h^{(2)}_{10}=0$. Let us write down the recurrent relations for $m_1+m_2=2$, $j=1,2$:

$$\begin{cases} h^{(j)}_{20} = -(2J^{(2)}_{20}+4J^{(1)}_{02}+4J^{(2)}_{11})h^{(j)}_{20}+2J^{(1)}_{11}h^{(j)}_{10}h^{(1)}_{20}+h^{(j)}_{10}J^{(1)}_{11}h^{(2)}_{11},\\ h^{(j)}_{11} = -(2J^{(2)}_{20}+4J^{(1)}_{02}+4J^{(2)}_{11})h^{(j)}_{11}+J^{(1)}_{11}h^{(j)}_{01}(2h^{(1)}_{20}+h^{(2)}_{11})+J^{(1)}_{11}h^{(j)}_{10}(h^{(1)}_{11}+2h^{(2)}_{02}),\\ h^{(j)}_{02} = -(2J^{(2)}_{20}+4J^{(1)}_{02}+4J^{(2)}_{11})h^{(j)}_{02}+2J^{(1)}_{11}h^{(j)}_{01}h^{(2)}_{02}+h^{(j)}_{01}J^{(1)}_{11}h^{(1)}_{11}. \end{cases}$$

It is not difficult to check that the only solution to the above system is $h^{(j)}_{20}=h^{(j)}_{02}=h^{(j)}_{11}=0$. Solving the recurrent relations for $m_1+m_2=3$ gives us

$$\begin{cases} h^{(1)}_{03}=h^{(2)}_{30}=0,\\ h^{(1)}_{12}=h^{(2)}_{03},\\ h^{(1)}_{21}=h^{(2)}_{12},\\ h^{(1)}_{30}=h^{(2)}_{21}. \end{cases}$$

This shows that $(h^{(1)}_{12},h^{(1)}_{21},h^{(1)}_{30})$ can be considered as free parameters. For any $p\geq 4$, the recurrent relations for $m_1+m_2=p$ form a linear system of equations for the variables $\{h^{(j)}_{m_1,p-m_1}\}^{p}_{m_1=0}$ with coefficients depending on $h^{(j)}_{01}$ and $h^{(j)}_{10}$ only. In principle, they can be solved and an explicit expression for the solutions can be found. We emphasize here that if the free parameters take real values then the whole solution is also real.

It is not difficult to check that for any values of $(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})$, one can find all $h_{m_1,m_2}^{(j)}$ $(m_1 + m_2 \geq 4)$ by using (26). The solution we obtain is formal in the sense that it satisfies (26) but h_{m_1,m_2} with $m_1 + m_2 = p$ may not decay as $p \to \infty$. We are now ready to formulate the theorem concerning the existence of formal solutions to (26).

Theorem 4.1. *For any values of* $(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})$*, there exists a unique formal solution to the recurrent equation* (26).

Thus, Theorem 4.1 claims the existence of a three-parameter family of solutions of (21) parameterized by $h_{12}^{(1)}$, $h_{21}^{(1)}$ and $h_{30}^{(1)}$. It turns out that if $h_{12}^{(1)}$, $h_{21}^{(1)}$ and $h_{30}^{(1)}$ are sufficiently small, then $h_{m_1,m_2}^{(j)}$ decay as $m_1 + m_2 = d$ tends to infinity. Let us say that $h_{m_1,m_2}^{(j)}$ has *degree d* if $m_1 + m_2 = d$. For each $d \geq 4$, introduce the vector $h^{(d)} = (h_{0,d}^{(1)}, h_{1,d-1}^{(1)}, \ldots, h_{d,0}^{(1)}, h_{0,d}^{(2)}, \ldots, h_{d,0}^{(2)})^T$. The vector $h^{(d)}$ contains all terms of degree d. By the recurrent relation (26),

$$C^{(d)} h^{(d)} = b^{(d)} \tag{28}$$

where the vector $b^{(d)}$ contains terms of degree $\leq d-1$. Also $C^{(d)} \in \mathbb{R}^{(2d+2)\times(2d+2)}$ is a matrix with

$$C_{kl}^{(d)} = \begin{cases} 1 - \dfrac{16d - 16 + 32k}{(d+1)(d+3)(d+5)} & \text{if } 1 \leq k = l \leq d+1, \\ 1 - \dfrac{80d + 80 - 32k}{(d+1)(d+3)(d+5)} & \text{if } d+2 \leq k = l \leq 2d+2, \\ -\dfrac{32(d-k+2)}{(d+1)(d+3)(d+5)} & \text{if } 2 \leq k \leq d+1,\ l = d+k, \\ -\dfrac{32(k-d-1)}{(d+1)(d+3)(d+5)} & \text{if } d+2 \leq k \leq 2d+1,\ l = k-d, \\ 0 & \text{in all other cases.} \end{cases}$$

It is easy to check that if $d \geq 4$, then $C^{(d)}$ is nonsingular and as $d \to \infty$, $C^{(d)}$ converges to the identity matrix. This observation immediately implies the following lemma:

Lemma 4.2. *Let* $(C^{(d)})^{-1}$ *be the inverse matrix of* $C^{(d)}$ *for* $d \geq 4$. *There exists an absolute constant* $C_1 > 0$ *such that for all* $d \geq 4$,

$$\|(C^{(d)})^{-1}\| \leq C_1.$$

We are now ready to derive an estimate which gives the decay of solutions of the recurrent relation (26).

Theorem 4.3. *If* $|h_{12}^{(1)}|, |h_{21}^{(1)}|, |h_{30}^{(1)}| \leq \delta$ *and* δ *is sufficiently small, then for some* $C_2 > 0$ *and* $0 < \rho < 1/4$, *we have*

$$|h_{m_1,m_2}^{(j)}| \leq C_2 \frac{\rho^{m_1+m_2}}{\Gamma\left(\frac{m_1+m_2+7}{2}\right)} \qquad \forall m_1, m_2 \geq 0,\ j = 1, 2.$$

Proof. We begin by noting that $h^{(j)}_{m_1 m_2} = 0$ if $m_1 + m_2$ is even. This can be easily proven by using the recurrent relation (26) and the fact that $h^{(j)}_{00} = 0$ and $h^{(j)}_{m_1,m_2} = 0$ for $m_1 + m_2 = 2$. Let $0 < \rho_1 < 1$; ρ_1 will be chosen sufficiently small. We shall use induction on $m_1 + m_2$ where $m_1 + m_2$ is odd. According to the induction hypothesis

$$|h^{(j)}_{m_1,m_2}| \le \frac{\rho_1^{m_1+m_2+2}}{\Gamma\left(\frac{m_1+m_2+7}{2}\right)} g(m_1 + m_2) \tag{29}$$

for every $3 \le m_1 + m_2 \le d - 2$ where $d \ge L$ is an odd number and L will be chosen later to be sufficiently large. Also g is a function to be specified later. We shall comment on the choice of L and verify the induction hypothesis for $3 \le m_1 + m_2 \le L$ later. Let us show that the same inequality holds for $m_1 + m_2 = d$. Without any loss of generality, let us consider $j = 1$. The case $j = 2$ is similar. Fix m_1 and let $b^{(d)}_{m_1}$ be the $(m_1 + 1)$-th component of the vector $b^{(d)}$ in the equation (28). We now estimate $b^{(d)}_{m_1}$ using the induction hypothesis (29) and the equation (26):

$$\begin{aligned}
|b^{(d)}_{m_1}| \le & \sum_{m'=2}^{d-3} |J^{(1)}_{m',m''}| \cdot 2 \frac{\rho_1^{m'+3}}{\Gamma\left(\frac{m'+8}{2}\right)} \frac{\rho_1^{m''+2}}{\Gamma\left(\frac{m''+7}{2}\right)} (m'+1) g(m'+1) g(m'') \\
& + \sum_{m'=4}^{d-3} |J^{(1)}_{m',m''}| \cdot 2 \frac{\rho_1^{m'+1}}{\Gamma\left(\frac{m'+6}{2}\right)} \frac{\rho_1^{m''+2}}{\Gamma\left(\frac{m''+7}{2}\right)} (m'+1) g(m'-1) g(m'') \\
& + \sum_{m'=3}^{d-2} |J^{(2)}_{m',m''}| \cdot 2 \frac{\rho_1^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} \frac{\rho_1^{m''+3}}{\Gamma\left(\frac{m''+8}{2}\right)} (m'+1)(m''+1) g(m') g(m''+1) \\
& + \sum_{m'=3}^{d-4} |J^{(2)}_{m',m''}| \cdot 2 \frac{\rho_1^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} \frac{\rho_1^{m''+1}}{\Gamma\left(\frac{m''+6}{2}\right)} (m'+1) g(m') g(m''-1) \\
& + 12(|J^{(1)}_{2,d-2}| + |J^{(1)}_{d-1,1}| + |J^{(2)}_{d-2,2}| + |J^{(2)}_{1,d-1}|) \frac{\rho_1^{d}}{\Gamma\left(\frac{d+5}{2}\right)} g(d-2).
\end{aligned}$$

The last term on the RHS of the above inequality comes from the case where $h_{m'_1 m'_2}$ or $h_{m''_1 m''_2}$ is of degree one since the induction hypothesis holds only for $3 \le m_1 + m_2 \le d-2$. Also in the estimation of the first four terms we use the fact that for fixed (m', m_1), there are at most $\min\{m'+1, m''+1\}$ tuples of (m', m''_1, m'_2, m''_2) such that $m'_1 + m''_1 = m_1$, $m'_2 + m''_2 = m_2$, $m'_1 + m'_2 = m'$ and $m''_1 + m''_2 = m''$. By (27), we have

$$|J^{(1)}_{m',m''}| = \frac{\Gamma\left(\frac{m'+2}{2}\right)\Gamma\left(\frac{m''+5}{2}\right)}{\Gamma\left(\frac{m'+m''+7}{2}\right)}, \qquad |J^{(2)}_{m',m''}| = \frac{\Gamma\left(\frac{m'+3}{2}\right)\Gamma\left(\frac{m''+4}{2}\right)}{\Gamma\left(\frac{m'+m''+7}{2}\right)}$$

and for some constant $C_3 > 0$,

$$|J^{(1)}_{2,d-2}| + |J^{(1)}_{d-1,1}| + |J^{(2)}_{d-2,2}| + |J^{(2)}_{1,d-1}| \le \frac{C_3}{d^2}.$$

Therefore

$$
\begin{aligned}
|b_{m_1}^{(d)}| \le{}& \frac{2\rho_1^{d+5}}{\Gamma\left(\frac{d+7}{2}\right)} \sum_{m'=2}^{d-3} \frac{\Gamma\left(\frac{m'+2}{2}\right)(m'+1)}{\Gamma\left(\frac{m'+8}{2}\right)} \frac{\Gamma\left(\frac{m''+5}{2}\right)}{\Gamma\left(\frac{m''+7}{2}\right)} g(m'+1)g(m'') \\
&+ \frac{2\rho_1^{d+3}}{\Gamma\left(\frac{d+7}{2}\right)} \sum_{m'=4}^{d-3} \frac{\Gamma\left(\frac{m'+2}{2}\right)(m'+1)}{\Gamma\left(\frac{m'+6}{2}\right)} \frac{\Gamma\left(\frac{m''+5}{2}\right)}{\Gamma\left(\frac{m''+7}{2}\right)} g(m'-1)g(m'') \\
&+ \frac{2\rho_1^{d+5}}{\Gamma\left(\frac{d+7}{2}\right)} \sum_{m'=3}^{d-2} \frac{\Gamma\left(\frac{m'+3}{2}\right)(m'+1)}{\Gamma\left(\frac{m'+7}{2}\right)} \frac{\Gamma\left(\frac{m''+4}{2}\right)(m''+1)}{\Gamma\left(\frac{m''+8}{2}\right)} g(m')g(m''+1) \\
&+ \frac{2\rho_1^{d+3}}{\Gamma\left(\frac{d+7}{2}\right)} \sum_{m'=3}^{d-4} \frac{\Gamma\left(\frac{m'+3}{2}\right)(m'+1)}{\Gamma\left(\frac{m'+7}{2}\right)} \frac{\Gamma\left(\frac{m''+4}{2}\right)}{\Gamma\left(\frac{m''+6}{2}\right)} g(m')g(m''-1) \\
&+ \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \frac{C_3}{d^2} \frac{\Gamma\left(\frac{d+7}{2}\right)}{\Gamma\left(\frac{d+5}{2}\right)} \frac{12}{\rho_1^2} g(d-2) \\
\le{}& \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \rho_1 C_4 \Big(\sum_{m'=2}^{d-3} g(m'+1)g(m'') + \sum_{m'=4}^{d-3} g(m'-1)g(m'') \\
&+ \sum_{m'=3}^{d-2} g(m')g(m''+1) + \sum_{m'=3}^{d-4} g(m')g(m''-1) \Big) + \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} C_5 \frac{g(d-2)}{d\cdot\rho_1}
\end{aligned}
$$

where C_4, C_5 are some constants. Now we specify the choice of the function g. Let $g(m)$ be such that $g_1 = \alpha$ and

$$
g(m) = \sum_{p=1}^{m-1} g(p)g(m-p) \qquad \text{for } m > 1.
$$

By the method of formal power series it is not difficult to show that

$$
g(m) = \frac{1}{2}\frac{(2m-1)!!}{m!}(2\alpha)^m.
$$

Clearly, we have const $\le g(m+1)/g(m) \le$ const, and this immediately gives us

$$
\begin{aligned}
|b_{m_1}^{(d)}| &\le \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} C_6 \sum_{m'=1}^{d} g(m')g(d-m') + \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \frac{C_6}{d\rho_1} g(d) \\
&\le \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} g(d) \left(C_6\rho_1 + \frac{C_6}{d\cdot\rho_1} \right)
\end{aligned}
$$

where $C_6 > 0$ is some constant. Now by Lemma 4.2, we obtain

$$
|h_{m_1m_2}| \le \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} g(d) C_1 \left(C_6\rho_1 + \frac{C_6}{d\rho_1} \right).
$$

Choose ρ_1 so small that $C_1C_6\rho_1 < 1/2$ and $\rho_1 \cdot 4\alpha < 1/4$. Then take L so large that $\frac{C_1C_6}{\rho_1 L} < \frac{1}{2}$. This clearly implies

$$|h_{m_1,m_2}| \leq \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} g(d).$$

We now justify the induction hypothesis (29). Recall that our free parameters are $h_{12}^{(1)}$, $h_{21}^{(1)}$ and $h_{30}^{(1)}$. It is easy to check that if we set $h_{12}^{(1)} = h_{21}^{(1)} = h_{30}^{(1)} = 0$, then $h_{m_1m_2} = 0$ for any $m_1 + m_2 \geq 2$. Since L is fixed, and $0 < |h_{12}^{(1)}|, |h_{21}^{(1)}|, |h_{30}^{(1)}| < \delta$ with sufficiently small δ, then the induction hypothesis is satisfied. A simple estimate on g gives that

$$g(m) \leq (4\alpha)^m.$$

Thus the theorem is proven if one takes $\rho = 4\alpha\rho_1$.

As stated, our solutions of (20) are determined by five parameters $\sigma^{(1)}, h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}, \sigma^{(2)}$. However, it turns out that these parameters are not independent and $\sigma^{(1)}$ can be expressed through $(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})$. Namely, let $G^{\sigma^{(1)},h_{12}^{(1)},h_{21}^{(1)},h_{30}^{(1)},\sigma^{(2)}}(Y)$ be the solution of (20). Then

$$G^{(\sigma^{(1)},h_{12}^{(1)},h_{21}^{(1)},h_{30}^{(1)},\sigma^{(2)})}(Y) = G^{(1,\sigma^{(1)}(h_{12}^{(1)}-1)+1,\sigma^{(1)}h_{21}^{(1)},\sigma^{(1)}(h_{30}^{(1)}-1)+1,\sigma^{(2)})}(Y).$$

This equality will be proven at the end of §6. We now formulate the final result concerning the existence of solutions of (21).

Theorem 4.2. *Let $\sigma^{(1)}, \sigma^{(2)} > 0$ and $h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}$ be sufficiently small. Then there exists a solution of* (20) *which has the following form:*

$$G^{(\sigma^{(1)},h_{12}^{(1)},h_{21}^{(1)},h_{30}^{(1)},\sigma^{(2)})}(Y_1, Y_2, Y_3) = \exp\left\{-\frac{\sigma^{(1)}}{2}(|Y_1|^2 + |Y_2|^2)\right\}$$
$$\cdot \frac{\sigma^{(1)}}{2\pi} \exp\left\{-\frac{\sigma^{(2)}}{2}|Y_3|^2\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi}}\sqrt{\sigma^{(1)}}H^{(h_{12}^{(1)},h_{21}^{(1)},h_{30}^{(1)})}(\sqrt{\sigma^{(1)}}Y_1, \sqrt{\sigma^{(1)}}Y_2).$$

Here $H^{(h_{12}^{(1)},h_{21}^{(1)},h_{30}^{(1)})}$ is the solution of (21) *with the given $h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}$.*

As already mentioned, the parameters $\sigma^{(1)}, h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}, \sigma^{(2)}$ are not independent and actually the set of solutions depends on four independent parameters (see Lemma 6.2).

From the estimate in Theorem 4.3 and from known asymptotic formulas for the Hermite polynomials it follows that the series giving $H^{(h_{12}^{(1)},h_{21}^{(1)},h_{30}^{(1)})}$ converges for every $Y = (Y_1, Y_2)$. Better estimates are also easily available.

5. The linearization near fixed point

Write $h_{12}^{(1)} = x^{(1)}$, $h_{21}^{(1)} = x^{(2)}$, $h_{30}^{(1)} = x^{(3)}$. Our fixed points have the following form:

$$G^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)},\sigma^{(2)})} = \frac{\sigma^{(1)}}{2\pi}\exp\left\{-\frac{\sigma^{(1)}(Y_1^2+Y_2^2)}{2}\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi}}\exp\left\{-\frac{\sigma^{(2)}Y_3^2}{2}\right\} \cdot (H_1^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}(Y_1,Y_2), H_2^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}(Y_1,Y_2), 0). \qquad (30)$$

Recall that

$$H^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})} = \sqrt{\sigma^{(1)}}H^{(1,x^{(1)},x^{(2)},x^{(3)})}(\sqrt{\sigma^{(1)}}Y_1, \sqrt{\sigma^{(1)}}Y_2)$$

and $H^{(1,x^{(1)},x^{(2)},x^{(3)})}$ are described in §4.

As already mentioned, the strategy of the proof of the main result is based on the method of renormalization group. At the p-th step of our procedure, we consider an interval on the time axis $S^{(p)} = [S_-^{(p)}, S_+^{(p)}]$ such that $S^{(p+1)} \subseteq S^{(p)}$. From our estimates it will follow that $\bigcap_p S^{(p)} = [S_-, S_+]$ is an interval of positive length. We want to find conditions under which $\tilde{g}_r(Y,s)$, $s \in S^{(p)}$, have a representation

$$\tilde{g}_r(Y,s) = Z(s)\Lambda(s)^r r\frac{\sigma^{(1)}}{2\pi}\exp\left\{-\frac{\sigma^{(1)}(Y_1^2+Y_2^2)}{2}\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi}}\exp\left\{-\frac{\sigma^{(2)}Y_3^2}{2}\right\} \cdot (H_1^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}(Y) + \delta_1^{(r)}(Y,s), H_2^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}(Y) + \delta_2^{(r)}(Y,s), \delta_3^{(r)}(Y,s))$$

where $\delta_1^{(r)}, \delta_2^{(r)}, \delta_3^{(r)}$ tend to zero as $r \to \infty$. The renormalization is based on the crucial observation (see above) that for large p, the sum over p_1 is a Riemann integral sum for an integral over γ varying from 0 to 1. Let us write

$$\tilde{g}_r(Y,s)\Lambda(s)^{-r}Z(s)^{-1}r^{-1}\exp\left\{\frac{\sigma^{(1)}(Y_1^2+Y_2^2)}{2}+\frac{\sigma^{(2)}Y_3^2}{2}\right\}\left(\frac{2\pi}{\sigma^{(1)}}\right)\left(\frac{2\pi}{\sigma^{(2)}}\right)^{1/2} = H^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}(Y_1,Y_2) + \delta^{(r)}(\gamma,Y,s) \qquad (31)$$

where $\delta^{(r)}(\gamma,Y,s) = \{\delta_j^{(r)}(\gamma,Y,s), 1 \le j \le 3\} = \tilde{\delta}^{(p)}(\gamma,Y,s)$, $\gamma = r/p$ and $\gamma \le 1$. It is natural to consider the set of functions $\{\tilde{\delta}^{(p)}(\gamma,Y,s)\}$ as a small perturbation of our fixed point (30). Recall that the third component of $H^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}$ is zero because of incompressibility and $\tilde{\delta}_3^{(p)}$ can be found from the incompressibility condition. When we go from p to $p+1$, then

$$\tilde{\delta}^{(p+1)}(\gamma,Y,s) = \tilde{\delta}^{(p)}\left(\frac{p+1}{p}\gamma, Y, s\right), \quad \gamma \le \frac{p}{p+1},$$

and the formula for $\tilde{\delta}^{(p+1)}(1, Y, s)$ follows from (21):

$$\exp\left\{-\frac{\sigma^{(1)}}{2}(|Y_1|^2+|Y_2|^2)-\frac{\sigma^{(2)}}{2}|Y_3|^2\right\}\frac{\sigma^{(1)}}{2\pi}\sqrt{\frac{\sigma^{(2)}}{2\pi}}\tilde{\delta}_j^{(p+1)}(1,Y,s)$$
$$=\int_0^1 d\gamma\int_{\mathbb{R}^3}\frac{\sigma^{(1)}}{2\pi\gamma}\sqrt{\frac{\sigma^{(1)}}{2\pi(1-\gamma)}}\frac{\sigma^{(2)}}{2\pi\gamma}\sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}}$$
$$\cdot\exp\left\{-\frac{\sigma^{(1)}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)}{2\pi\gamma}-\frac{\sigma^{(2)}|Y_3-Y_3'|^2}{2\pi\gamma}-\frac{\sigma^{(1)}(|Y_1'|^2+|Y_2'|^2)}{2\pi(1-\gamma)}\right.$$
$$\left.-\frac{\sigma^{(2)}|Y_3'|^2}{2\pi(1-\gamma)}\right\}\left\{\left[-(1-\gamma)^{3/2}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2-Y_2'}{\sqrt{\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.\right.$$
$$\left.+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right]$$
$$\cdot\tilde{\delta}_j^{(p+1)}\left(1-\gamma,\frac{Y'}{\sqrt{1-\gamma}},s\right)+\left[-(1-\gamma)^{3/2}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}\tilde{\delta}_1^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right.\right.$$
$$\left.+\frac{Y_2-Y_2'}{\sqrt{\gamma}}\tilde{\delta}_2^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right)+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}\tilde{\delta}_1^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right.$$
$$\left.\left.\left.+\frac{Y_2'}{\sqrt{1-\gamma}}\tilde{\delta}_2^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right)\right]H_j\left(\frac{Y'}{\sqrt{1-\gamma}},s\right)\right\}d^3Y',\quad j=1,2. \tag{32}$$

We did not include in the last expression terms which are quadratic in $\tilde{\delta}^{(p+1)}$ because in this section we consider only the linearized map.

Another way to introduce the semigroup of linearized maps is the following. Take $\theta > 0$ which later will tend to zero. Set $\gamma_j = (1+\theta)^{-j}$, $j = 0, 1, 2, \ldots$.

Define the linearized map L_θ corresponding to θ as follows:

1. for $\gamma_{j+1} \le \gamma \le \gamma_j$, $j = 1, 2, \ldots,$
$$L_\theta(\tilde{\delta}(\gamma, Y)) = \tilde{\delta}(\gamma(1+\theta), Y);$$
2. for $1/(1+\theta) \le \gamma \le 1$ the function $L_\theta(\tilde{\delta}(\gamma, Y))$ is given by the formula
$$L_\theta(\tilde{\delta}(\gamma, Y)) = \tilde{\delta}^{(p_1)}(1, Y, s)$$
where p_1 is found from the relation $p_1/p = \gamma$.

We remark that the value of $\tilde{\delta}^{(p_1)}(1, Y, s)$ is also found with the help of (32). In other words, at $\gamma = 1$ we use (32) to find the new $\tilde{\delta}^{(p+1)}(1, Y, s)$. Then we apply 1.

It is easy to see that the limits $\lim_{\theta\to 0, n\theta\to t} L_\theta^n = A^t$ exist and the operators A^t constitute a semigroup. The space where these operators act will be discussed later.

For $\gamma < 1$ and $t > 0$ such that $\gamma e^t < 1$ we can write
$$A^t\delta(\gamma, Y) = \delta(\gamma e^t, Y).$$

In our situation we introduce the following

Definition 5.1. *A function $\Phi_\alpha(Y)$, $Y=(Y_1,Y_2,Y_3)$, with values in $\mathbb{R}^3$ is called an* eigenfunction *if for the function $\Phi_\alpha(\gamma,Y)=\gamma^\alpha\Phi_\alpha(Y)$, we have*

$$\exp\left\{-\frac{\sigma^{(1)}}{2}(|Y_1|^2+|Y_2|^2)-\frac{\sigma^{(2)}}{2}|Y_3|^2\right\}\frac{\sigma^{(1)}}{2\pi}\sqrt{\frac{\sigma^{(2)}}{2\pi}}\Phi_{\alpha,j}(Y)$$
$$=\int_0^1 d\gamma\int_{\mathbb{R}^3}\frac{\sigma^{(1)}}{2\pi\gamma}\sqrt{\frac{\sigma^{(1)}}{2\pi(1-\gamma)}}\frac{\sigma^{(2)}}{2\pi\gamma}\sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}}$$
$$\cdot\exp\left\{-\frac{\sigma^{(1)}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)}{2\pi\gamma}-\frac{\sigma^{(2)}|Y_3-Y_3'|^2}{2\pi\gamma}-\frac{\sigma^{(1)}(|Y_1'|^2+|Y_2'|^2)}{2\pi(1-\gamma)}\right.$$
$$\left.-\frac{\sigma^{(2)}|Y_3'|^2}{2\pi(1-\gamma)}\right\}\left\{\left[-(1-\gamma)^{3/2}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2-Y_2'}{\sqrt{\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.\right.$$
$$\left.+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right]$$
$$\cdot\Phi_{\alpha,j}\left(1-\gamma,\frac{Y'}{\sqrt{1-\gamma}},s\right)+\left[-(1-\gamma)^{3/2}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}\Phi_{\alpha,1}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right.\right.$$
$$\left.+\frac{Y_2-Y_2'}{\sqrt{\gamma}}\Phi_{\alpha,2}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right)+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}\Phi_{\alpha,1}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right.$$
$$\left.\left.\left.+\frac{Y_2'}{\sqrt{1-\gamma}}\Phi_{\alpha,2}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right)\right]H_j\left(\frac{Y'}{\sqrt{1-\gamma}},s\right)\right\}d^3Y',\qquad j=1,2. \tag{33}$$

In the last expression, $\Phi_{\alpha,j}(\gamma,Y)$ is the j-th component of $\Phi_\alpha(Y)$.

The meaning of this definition is the following. Assume that our perturbation $\delta^{(r)}(Y)$ is proportional to $\delta^{(r)}(Y)=(r/p)^\alpha\Phi_\alpha(Y)$, for all $r<p$. If we apply (32) then in the main order of magnitude $\delta^{(p)}(Y)=\Phi_\alpha(Y)$. This will be important in the later constructions.

In §6 we shall study in more detail the set of eigenfunctions $\Phi_\alpha(Y)$. In particular, we shall show that they constitute a basis in the Hilbert space $L^2=L^2\left(\mathbb{R}^3\right)$ of square-integrable functions with respect to the weight $(\sigma^{(1)}/2\pi)^{3/2}\exp\{-\sigma^{(1)}Y^2/2\}$, $Y=(Y_1,Y_2,Y_3)$. If $\alpha>0$, $\alpha=0$, $\alpha<0$ then the corresponding eigenfunctions are called *unstable*, *neutral* and *stable* respectively.

The group A^t has several other important properties which will be used later. We consider the special fixed point $H^{(0)}$ for which $x_1=x_2=x_3=0$ and $H_1^{(0)}=-2y_1$, $H_2^{(0)}=-2y_2$. Its properties will be used in §7 and §9.

Consider the Hilbert space X of $\mathbb{R}^3$-valued functions $f(\gamma,Y)$ such that

$$\|f\|_X^2=\int_0^1 d\gamma\int_{\mathbb{R}^3}\frac{1}{(2\pi)^{3/2}}\exp\left\{-\frac{|Y|^2}{2}\right\}|f(\gamma,Y)|^2\,d^3Y.$$

For each $m_3\geq 0$, introduce the subspace X_{m_3} such that

$$X_{m_3}=\{f(\gamma,Y)=g(\gamma,Y_1,Y_2)\,\mathrm{He}_{m_3}(Y_3)\}.$$

It is clear that $X = \bigcup_{m_3 \geq 0} X_{m_3}$. Each subspace X_{m_3} is invariant under A^t. Indeed, if $\gamma e^t < 1$, then A^t acts only on the γ variable. Since the integral transformation (32) with respect to Y_3 is the usual convolution, the LHS of (32) can be written as the product of $\mathrm{He}_{m_3}(Y_3)$ and some functions of (Y_1, Y_2) (see Appendix 1). This implies the invariance of X_{m_3}. Using (32) we introduce the boundary operator T such that

$$
\begin{aligned}
&\exp\left\{-\frac{1}{2}(|Y_1|^2+|Y_2|^2)-\frac{1}{2}|Y_3|^2\right\}\frac{1}{2\pi}\sqrt{\frac{1}{2\pi}}(Tf)_j(Y)\\
&= \int_0^1 d\gamma \int_{\mathbb{R}^3} \frac{1}{2\pi\gamma}\sqrt{\frac{1}{2\pi(1-\gamma)}}\frac{1}{2\pi\gamma}\sqrt{\frac{1}{2\pi(1-\gamma)}}\\
&\cdot \exp\left\{-\frac{|Y_1-Y_1'|^2+|Y_2-Y_2'|^2}{2\pi\gamma}-\frac{|Y_3-Y_3'|^2}{2\pi\gamma}-\frac{|Y_1'|^2+|Y_2'|^2}{2\pi(1-\gamma)}\right.\\
&\left.-\frac{|Y_3'|^2}{2\pi(1-\gamma)}\right\}\left\{\left[-(1-\gamma)^{3/2}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2-Y_2'}{\sqrt{\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.\right.\\
&\left.+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right]\\
&\cdot f_j\left(1-\gamma,\frac{Y'}{\sqrt{1-\gamma}}\right)+\left[-(1-\gamma)^{3/2}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}f_1\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)\right.\right.\\
&\left.+\frac{Y_2-Y_2'}{\sqrt{\gamma}}f_2\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)+\gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}f_1\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)\right.\\
&\left.\left.\left.+\frac{Y_2'}{\sqrt{1-\gamma}}f_2\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right]H_j\left(\frac{Y'}{\sqrt{1-\gamma}}\right)\right\}d^3Y', \quad j=1,2. \qquad (34)
\end{aligned}
$$

In the above expression, $(Tf)_j$ and f_j denote the j-th components of Tf and f respectively. Note that while f is a function of γ and Y, Tf is a function of Y only. The operator T corresponds to the action of our linearized group at $\gamma = 1$ (see (32)). Our first lemma shows that when m_3 is large, the operator T is a contraction.

Lemma 1. *There exists a number $N_1 > 0$ and $0 < \eta_1 = \eta_1(N_1) < 1$ such that for all $m_3 \geq N_1$, and for all $f(\gamma, Y) \in X_{m_3}$, we have*

$$\|Tf\|_{L^2(G(Y))} \leq \eta_1 \|f\|_X$$

where $L^2(G(Y))$ is the Hilbert space of square-integrable functions of Y_1, Y_2 with respect to the Gaussian weight $G(Y) = (2\pi)^{-1}\exp\{-(Y_1^2+Y_2^2)/2\}$.

Proof. Since $f \in X_{m_3}$, we write by definition $f(\gamma, Y) = g(\gamma, Y_1, Y_2)\,\mathrm{He}_{m_3}(Y_3)$, where $g(\gamma, Y_1, Y_2)$ has an expansion in Hermite polynomials:

$$g(\gamma, Y_1, Y_2) = \sum_{m_1, m_2 \geq 0} \hat{g}(\gamma, m_1, m_2)\,\mathrm{He}_{m_1}(Y_1)\,\mathrm{He}_{m_2}(Y_2).$$

It is clear that

$$\|g\|_X = \left(\int_0^1 \sum_{m_1,m_2\geq 0} m_1!m_2!|\hat{g}(\gamma,m_1,m_2)|^2\,d\gamma\right)^{1/2}.$$

Since our special fixed point is the Hermite polynomial of degree one, we have

$$\begin{aligned}
(\widehat{Tg})_j(m_1,m_2) &= -\int_0^1 (1-\gamma)^{(m_1+m_2+m_3)/2}\cdot(-4)\cdot\hat{g}_j(1-\gamma,m_1,m_2)\,d\gamma\\
&+\int_0^1 \gamma(1-\gamma)^{(m_1+m_2+m_3+1)/2}\cdot(-2)\cdot(\hat{g}_j(1-\gamma,m_1-2,m_2)+m_1\hat{g}_j(1-\gamma,m_1,m_2))\,d\gamma\\
&+\int_0^1 \gamma(1-\gamma)^{(m_1+m_2+m_3+1)/2}\cdot(-2)\cdot(\hat{g}_j(1-\gamma,m_1,m_2-2)+m_2\hat{g}_j(1-\gamma,m_1,m_2))\,d\gamma\\
&-\delta_{j,1}\int_0^1 \gamma^{(m_1+m_2+m_3-1)/2}(1-\gamma)^2\cdot(-2)\cdot(\hat{g}_j(\gamma,m_1-2,m_2)+m_1\hat{g}_j(\gamma,m_1,m_2))\,d\gamma\\
&-\delta_{j,2}\int_0^1 \gamma^{(m_1+m_2+m_3-1)/2}(1-\gamma)^2\cdot(-2)\cdot(\hat{g}_j(\gamma,m_1,m_2-2)+m_2\hat{g}_j(\gamma,m_1,m_2))\,d\gamma\\
&+\int_0^1 \gamma^{(m_1+m_2+m_3+1)/2}(1-\gamma)\cdot(-2)\cdot\hat{g}_j(\gamma,m_1,m_2)\,d\gamma.
\end{aligned}$$

In the above expression $\delta_{j,1}$ and $\delta_{j,2}$ denote Kronecker delta functions. It follows easily from the Cauchy–Schwarz inequality that for some constant $C > 0$,

$$\begin{aligned}
|(\widehat{Tg})_j(m_1,m_2)|^2 &\leq \frac{C}{m_1+m_2+m_3+1}\int_0^1 |\hat{g}(\gamma,m_1,m_2)|^2\,d\gamma\\
&+\frac{C}{(m_1+m_2+m_3+1)^3}\\
&\cdot\left(\int_0^1 |\hat{g}(\gamma,m_1-2,m_2)|^2\,d\gamma+\int_0^1 |\hat{g}(\gamma,m_1,m_2-2)|^2\,d\gamma\right).
\end{aligned}$$

Now a simple application of the Minkowski inequality gives the result. □

Fix $m_3 \geq 0$ which may not be large. If $f(\gamma,Y) \in X_{m_3}$ has the property that $\hat{g}(\gamma,m_1,m_2) = 0$ for $m_1+m_2 \leq N_2$, then $T\hat{g}(m_1,m_2) = 0$ for $m_1+m_2 \leq N_2$. This allows us to introduce the subspace

$$X_{m_3,N_2} = \{f(\gamma,Y) \in X_{m_3} : \hat{g}(\gamma,m_1,m_2) = 0\ \forall m_1+m_2 \leq N_2\},$$

which is also invariant under T. The following lemma shows that for sufficiently large N_2, the operator T is also a contraction on the subspace X_{m_3,N_2}.

Lemma 2. *There exists a number* $N_2 > 0$ *and* $0 < \eta_2 = \eta_1(N_2) < 1$ *such that for all* $0 \leq m_3 \leq N_1$*, and* $f(\gamma,Y) \in X_{m_3,N_2}$*, we have*

$$\|Tf\|_{L^2(G(Y))} \leq \eta_2\|f\|_X.$$

Proof. This follows by the same arguments as in Lemma 1. We omit the details. □

From Lemmas 1 and 2, we conclude that our linearized group is contracting on the large subspace

$$W = \bigcup_{m_3 \geq N_1} X_{m_3} \cup \bigcup_{m_3 \geq 0,\, n \geq N_2} X_{m_3,n}.$$

The complement to this subspace is finite-dimensional in Y. It will be analyzed in the next section with the help of eigenfunctions of the group A^t.

6. The set of eigenfunctions of the group of linearized maps

In this section we show that all solutions of (21) studied in §4 have $l^{(u)} = 4$ unstable eigenvalues and $l^{(n)} = 6$ neutral eigenvalues. Therefore in the renormalization group approach we need 10-parameter families of initial conditions (see below).

As already mentioned, in the limit $p \to \infty$ the linearized maps generate a semigroup of operators acting in the space Δ of functions $f^{(j)}(\gamma, Y)$, $0 \leq \gamma \leq 1$, $Y \in \mathbb{R}^3$, $j = 1, 2$, which are continuous as functions of γ in the Hilbert space L^2. At $\gamma = 1$, the functions $f^{(j)}(\gamma, Y)$ satisfy the boundary condition which follows from (32):

$$\begin{aligned}
&\exp\left\{-\frac{\sigma^{(1)}}{2}(|Y_1|^2 + |Y_2|^2) - \frac{\sigma^{(2)}}{2}|Y_3|^2\right\}\frac{\sigma^{(1)}}{2\pi}\sqrt{\frac{\sigma^{(2)}}{2\pi}}f^{(j)}(1, Y)\\
&\quad = \int_0^1 d\gamma \int_{\mathbb{R}^3} \frac{\sigma^{(1)}}{2\pi\gamma}\sqrt{\frac{\sigma^{(2)}}{2\pi\gamma}}\frac{\sigma^{(1)}}{2\pi(1-\gamma)}\sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}}\\
&\qquad \cdot \exp\left\{-\frac{\sigma^{(1)}(|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2)}{2\gamma} - \frac{\sigma^{(2)}|Y_3 - Y_3'|^2}{2\gamma} - \frac{\sigma^{(1)}(|Y_1'|^2 + |Y_2'|^2)}{2(1-\gamma)}\right.\\
&\qquad \left. - \frac{\sigma^{(2)}|Y_3'|^2}{2(1-\gamma)}\right\}\left\{\left[-(1-\gamma)^{3/2}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}H_1\left(\frac{Y - Y'}{\sqrt{\gamma}}\right) + \frac{Y_2 - Y_2'}{\sqrt{\gamma}}H_2\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right)\right.\right.\\
&\qquad \left. + \gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}H_1\left(\frac{Y - Y'}{\sqrt{\gamma}}\right) + \frac{Y_2'}{\sqrt{1-\gamma}}H_2\left(\frac{Y - Y'}{\sqrt{\gamma}}\right)\right)\right]\\
&\qquad \cdot f^{(j)}\left(1-\gamma, \frac{Y'}{\sqrt{1-\gamma}}\right) + \left[-(1-\gamma)^{3/2}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}f^{(1)}\left(\gamma, \frac{Y - Y'}{\sqrt{\gamma}}\right)\right.\right.\\
&\qquad \left. + \frac{Y_2 - Y_2'}{\sqrt{\gamma}}f^{(2)}\left(\gamma, \frac{Y - Y'}{\sqrt{\gamma}}\right)\right) + \gamma^{1/2}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}f^{(1)}\left(\gamma, \frac{Y - Y'}{\sqrt{\gamma}}\right)\right.\\
&\qquad \left.\left.\left. + \frac{Y_2'}{\sqrt{1-\gamma}}f^{(2)}\left(\gamma, \frac{Y - Y'}{\sqrt{\gamma}}\right)\right)\right]H_j\left(\frac{Y'}{\sqrt{1-\gamma}}\right)\right\}d^3Y', \qquad j = 1, 2. \qquad (35)
\end{aligned}$$

Denote by $\mathcal{R}_p$ the linear operator which transforms $\{\delta^{(p)}(\gamma, Y, s)\}$ into $\{\delta^{(p+1)}(\gamma, Y, s)\}$. Here s is a parameter which plays no role in this section. As explained in §5, for each t the limit $\lim_{p\to\infty} \mathcal{R}_p^{tp} = A^t$ exists so that the operators A^t constitute a semigroup with infinitesimal generator $\mathcal{A} = \lim_{t\downarrow 0}(A^t - I)/t$. In our case $\mathcal{A}\delta(\gamma, Y, s) = \gamma \frac{\partial\delta(\gamma,Y,s)}{\partial\gamma}$ for $0 < \gamma < 1$, and for $\gamma = 1$ the function $\delta(1, Y, s)$ satisfies the boundary condition (33) in which $f(\gamma, Y) = \delta^{(p+1)}(1, Y, s)$.

If α is an eigenvalue of $\mathcal{A}$, then the corresponding eigenfunction has the form $\gamma^{\alpha}\Phi_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$ (see Lemma 5.1), where $\Phi_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$ satisfies the equation (33) with $f(\gamma, Y) = \gamma^{\alpha}\Phi_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$.

As before, for $\Phi^{(j)}_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$ the following scaling relation with respect to $\sigma^{(1)}, \sigma^{(2)}$ is valid:

$$\Phi^{(j)}_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y) \propto \Phi^{(j)}_{\alpha,1,1}(\sqrt{\sigma^{(1)}}Y_1, \sqrt{\sigma^{(1)}}Y_2, \sqrt{\sigma^{(2)}}Y_3).$$

Therefore it is enough to consider the above equation (33) for $\sigma^{(1)} = \sigma^{(2)} = 1$. We again use the expansion in Hermite polynomials:

$$\Phi^{(j)}_{\alpha,1,1}(Y) = \Phi^{(j)}_{\alpha}(Y) = \sum_{m_1,m_2,m_3} f^{(j)}_{\alpha}(m_1, m_2, m_3)\,\mathrm{He}_{m_1}(Y_1)\,\mathrm{He}_{m_2}(Y_2)\,\mathrm{He}_{m_3}(Y_3).$$

Here j takes values 1, 2, 3. Since in m_3 it is the usual convolution and H does not depend on Y_3, it is enough to look for solutions of (33) having the form $f_{m_1,m_2}\delta_{m_3}$. Put $\beta = \alpha + m_3/2$ and $f^{(j)}_{\beta}(m_1, m_2) = f^{(j)}_{\alpha}(m_1, m_2)\delta_{m_3}$. We arrive at the linear system of recurrent relations

$$\begin{aligned} f^{(j)}_{\beta}(m_1, m_2) = \sum_{\substack{m'_1+m''_1=m_1\\ m'_2+m''_2=m_2}} & J^{(1)}_{m',m''+2\beta}((B_1h^{(1)})_{m'_1,m'_2} + (B_2h^{(2)})_{m'_1,m'_2}) f^{(j)}_{\beta}(m''_1, m''_2)\\ & + J^{(2)}_{m',m''+2\beta} h^{(1)}_{m'_1,m'_2}(B_1 f^{(j)}_{\beta})(m''_1, m''_2)\\ & + J^{(2)}_{m',m''+2\beta} h^{(2)}_{m'_1,m'_2}(B_2 f^{(j)}_{\beta})(m''_1, m''_2)\\ & + J^{(1)}_{m'+2\beta,m''}((B_1 f^{(1)}_{\beta})(m'_1, m'_2) + (B_2 f^{(2)}_{\beta})(m'_1, m'_2)) h^{(j)}_{m''_1,m''_2}\\ & + J^{(2)}_{m'+2\beta,m''} f^{(1)}_{\beta}(m'_1, m'_2)(B_1h^{(j)})_{m''_1,m''_2}\\ & + J^{(2)}_{m'+2\beta,m''} f^{(2)}_{\beta}(m'_1, m'_2)(B_2h^{(j)})_{m''_1,m''_2}. \end{aligned} \tag{36}$$

Introduce the vector

$$\begin{aligned} f^{(d)}_{\beta} = (f^{(1)}_{\beta}(0, d), f^{(1)}_{\beta}(1, d-1), \ldots, f^{(1)}_{\beta}(d, 0),\\ f^{(2)}_{\beta}(0, d), f^{(2)}_{\beta}(1, d-1), \ldots, f^{(2)}_{\beta}(d, 0))^T. \end{aligned}$$

The vector $f^{(d)}_{\beta}$ contains all terms of degree d. The recurrent relation (34) can be written as

$$C^{(d)}_{\beta} f^{(d)}_{\beta} = b^{(d)}_{\beta}$$

where the vector $b_\beta^{(d)}$ contains terms of degree $\le d-1$. Also $C_\beta^{(d)} \in \mathbb{R}^{2(d+1)\times 2(d+1)}$ is a matrix whose (k,l) entry is

$$C_\beta^{(d)}(k,l)$$
$$= \begin{cases} 1 - \dfrac{16d+32\beta-16+32k}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)} & \text{if } 1 \le k = l \le d+1, \\ 1 - \dfrac{80d+160\beta+80-32k}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)} & \text{if } d+2 \le k = l \le 2d+2, \\ -\dfrac{32(d+2\beta-k+2)}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)} & \text{if } 2 \le k \le d+1,\ l = d+k, \\ -\dfrac{32(k-d-2\beta-1)}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)} & \text{if } d+2 \le k \le 2d+1,\ l = k-d, \\ 0 & \text{in all other cases.} \end{cases}$$

Note that $d + 2\Re(\beta) > -1$.

Lemma 6.1. *Assume $\Re(\beta) \ge 0$. There exists an integer $d_* > 0$, independent of β, such that for all $d \ge d_*$, the matrix $C_\beta^{(d)}$ is invertible.*

Proof. As d tends to infinity, $C_\beta^{(d)}$ tends to the identity matrix if $\Re(\beta) \ge 0$. A simple estimate on the diagonal and off-diagonal entries shows that for all β such that $\Re(\beta) \ge 0$ and sufficiently large d, the matrix $C_\beta^{(d)}$ becomes diagonally dominant. This implies the existence of the needed d_* and its independence of β.

A similar statement holds if we assume that $\Re(\beta) \ge -A$ for any given $A \le 0$. We formulate it as the following lemma.

Lemma 6.1′. *For any $A \ge 0$, there exists an integer $d_*(A) > 0$, which depends only on A, such that for all $d \ge d_*(A)$ and all β with $\Re(\beta) \ge -A$, the matrix $C_\beta^{(d)}$ is invertible.*

By Lemma 6.1, to find all eigenvalues of $\mathcal{A}$ amounts to solving the equation $\det(C_\beta^{(d)}) = 0$. The eigenvalue α is then found from the relation $\beta = \alpha + m_3/2$. Let

$$a_1 = \left(1 - \frac{16}{(d+2\beta+3)(d+2\beta+5)}\right) \Big/ \left(\frac{32}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)}\right).$$

Then a_1 is an eigenvalue of the matrix $\tilde{C}^{(d)} \in \mathbb{R}^{2(d+1)\times 2(d+1)}$ given by

$$\tilde{C}^{(d)}(k,l) = \begin{cases} k-1 & \text{if } 1 \le k = l \le d+1, \\ 2d+2-k & \text{if } d+2 \le k = l \le 2d+2, \\ d+2-k & \text{if } 2 \le k \le d+1,\ l = d+k, \\ k-d-1 & \text{if } d+2 \le k \le 2d+1,\ l = k-d, \\ 0 & \text{in all other cases.} \end{cases}$$

It is not difficult to find that the eigenvalues of $\tilde{C}^{(d)}$ are 0 and $d+1$ with algebraic multiplicity $d+2$ and d respectively. Solve the equations $a_1 = 0$ or $a_1 = d+1$ and use the condition $d + 2\Re(\beta) > -1$. The possible values of β are then given by

$$\beta = \frac{3-d}{2} \quad \text{or} \quad \frac{\sqrt{17}-4-d}{2}, \qquad d = 1, 2, 3, \ldots.$$

This fact immediately gives the following lemma.

Lemma 6.2. *Let* $(\tilde{C}_\beta^{(d)})^{-1}$ *be the inverse matrix of* $\tilde{C}_\beta^{(d)}$ *for* $d \geq d^*(\beta)$*, where* $d^*(\beta) = 3-2\beta$ *or* $\sqrt{17}-4-2\beta$ *is an integer. Then there exists an absolute constant* $C_2 > 0$ *such that for all* $d \geq d^*(\beta)$,

$$\|(\tilde{C}_\beta^{(d)})^{-1}\| \leq C_2.$$

We now state our theorem about the properties of the solutions to the recurrent relation (34).

Theorem 6.1. *The only possible values of* β *for which* (34) *has nonzero solutions* $f_\beta^{(j)}(m_1, m_2)$ *are given by*

$$\beta = \frac{3-m}{2} \quad \textit{or} \quad \frac{\sqrt{17}-4-m}{2}, \qquad m = 1, 2, \ldots.$$

The corresponding solutions $f_\beta^{(j)}(m_1, m_2)$ *have the following property:*

(a) $\beta = (\sqrt{17}-4-m)/2$. *In this case* $f_\beta^{(j)}(m_1, m_2) = 0$ *for any* $0 \leq m_1 + m_2 < m$. *For* $d = m$*, we have*

$$f_\beta^{(1)}(r, d-r) = -(d-r+1) f_\beta^{(2)}(r-1, d-r+1), \qquad r = 1, \ldots, d,$$

$f_\beta^{(1)}(0, d)$, $f_\beta^{(2)}(d, 0)$ *are free parameters.* $f_\beta^{(j)}(m_1, m_2)$ *for* $m_1 + m_2 \geq m+1$ *are uniquely determined if the values of the* $m+2$ *free parameters* $f_\beta^{(1)}(r, m-r)$, $r = 0, 1, \ldots, m$, *and* $f_\beta^{(2)}(m, 0)$ *are specified.*

(b) $\beta = (3-m)/2$. *In this case* $f_\beta^{(j)}(m_1, m_2) = 0$ *for any* $0 \leq m_1 + m_2 < m$. *For* $d = m$, *we have* $f_\beta^{(1)}(0, d) = f_\beta^{(2)}(d, 0) = 0$, *and*

$$f_\beta^{(1)}(r, d-r) = f_\beta^{(2)}(r-1, d-r+1), \qquad r = 1, \ldots, d,$$

are free parameters. $f_\beta^{(j)}(m_1, m_2)$ *for* $m_1 + m_2 \geq m+1$ *are uniquely determined if the values of the* m *free parameters* $f_\beta^{(1)}(r, m-r)$, $r = 1, \ldots, m$, *are specified.*

In both cases (a) *and* (b)*, the solutions* $f_\beta^{(j)}(m_1, m_2)$ *are zero for* $m_1 + m_2 = m + 1$, $m + 3, \ldots$. *Since* $f_\beta^{(j)}$ *depends linearly on the free parameters, for some* $C_3 > 0$ *and* $0 < \rho < 1/4000$ *we have*

$$|f_\beta^{(j)}(m_1, m_2)| \le C_3 \frac{\rho^{m_1+m_2+2\beta}}{\Gamma\left(\frac{m_1+m_2+2\beta+3}{2}\right)}, \quad \forall m_1, m_2 \ge 0,\ j = 1, 2.$$

Proof. (a) and (b) are obtained by straightforward computations. From the recurrent relation (34), by parity it is obvious that $f_\beta^{(j)}(m_1, m_2) = 0$ for $m_1 + m_2 = m + 1$. An easy induction shows that $f_\beta^{(j)}(m_1, m_2) = 0$ for $m_1 + m_2 = m + 3, m + 5, \ldots$. We now prove the decay estimate. The strategy of the proof is the same as in Theorem 4.3. From the proof of Theorem 4.3, it is clear that by choosing the parameters $(x^{(1)}, x^{(2)}, x^{(3)})$ sufficiently small, we have

$$|h_{m_1,m_2}^{(j)}| \le \frac{\rho^{m_1+m_2+2}}{\Gamma\left(\frac{m_1+m_2+7}{2}\right)} \quad \forall m_1, m_2 \ge 0,\ m_1 + m_2 \ge 3,\ j = 1, 2.$$

Our induction hypothesis for $f_\beta^{(j)}(m_1, m_2)$ says

$$|f_\beta^{(j)}(m_1, m_2)| \le \frac{\rho^{m_1+m_2+2\beta}}{\Gamma\left(\frac{m_1+m_2+2\beta+3}{2}\right)} \quad \forall m \le m_1 + m_2 < d,\ j = 1, 2,$$

where $d \ge L$ and $d - m$ is an even number (note that $f_\beta^{(j)}(m_1, m_2) = 0$ for $m_1 + m_2 = m + 1, m + 3, \ldots$). We assume that L is sufficiently large and will verify the induction assumption for $m \le d \le L$ later. Now for $m_1 + m_2 = d$, by Lemma 6.2, we have

$$\begin{aligned}
|f_\beta^{(j)}(m_1, m_2)| \le{}& C_2 \sum_{m'=2}^{d-m} (m' + 1)|J_{m',m''+2\beta}^{(1)}| \cdot 2(m' + 1) \frac{\rho^{m'+3}}{\Gamma\left(\frac{m'+8}{2}\right)} \frac{\rho^{m''+2\beta}}{\Gamma\left(\frac{m''+2\beta+3}{2}\right)} \\
&+ C_2 \sum_{m'=4}^{d-m} (m' + 1)|J_{m',m''+2\beta}^{(1)}| \cdot 2 \frac{\rho^{m'+1}}{\Gamma\left(\frac{m'+6}{2}\right)} \frac{\rho^{m''+2\beta}}{\Gamma\left(\frac{m''+2\beta+3}{2}\right)} \\
&+ C_2 |J_{2,d-2+2\beta}^{(1)}| \cdot 4 \frac{\rho^{d-2+2\beta}}{\Gamma\left(\frac{d+2\beta+1}{2}\right)} \\
&+ C_2 \sum_{m'=3}^{d-m+1} (m' + 1)|J_{m',m''+2\beta}^{(2)}| \cdot 2 \frac{\rho^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} (m'' + 1) \frac{\rho^{m''+2\beta+1}}{\Gamma\left(\frac{m''+2\beta+4}{2}\right)} \\
&+ C_2 \sum_{m'=3}^{d-m-1} (m' + 1)|J_{m',m''+2\beta}^{(2)}| \cdot 2 \frac{\rho^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} \frac{\rho^{m''+2\beta-1}}{\Gamma\left(\frac{m''+2\beta+2}{2}\right)} \\
&+ C_2 |J_{1,d-1+2\beta}^{(2)}| \cdot 4 \frac{\rho^{d-1+2\beta}}{\Gamma\left(\frac{d+2\beta+2}{2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&+ C_2 \sum_{m'=m-1}^{d-3} (m''+1)|J^{(1)}_{m'+2\beta,m''}| \cdot 2(m'+1) \frac{\rho^{m'+2\beta+1}}{\Gamma\big(\frac{m'+2\beta+4}{2}\big)} \frac{\rho^{m''+2}}{\Gamma\big(\frac{m''+7}{2}\big)} \\
&+ C_2 \sum_{m'=m+1}^{d-3} (m''+1)|J^{(1)}_{m'+2\beta,m''}| \cdot 2 \frac{\rho^{m'+2\beta-1}}{\Gamma\big(\frac{m'+2\beta+3}{2}\big)} \frac{\rho^{m''+2}}{\Gamma\big(\frac{m''+7}{2}\big)} \\
&+ C_2 |J^{(1)}_{d-1+2\beta,1}| \cdot 4 \frac{\rho^{d-2+2\beta}}{\Gamma\big(\frac{d+2\beta+1}{2}\big)} \\
&+ C_2 \sum_{m'=m}^{d-2} (m''+1)|J^{(2)}_{m'+2\beta,m''}| \cdot 2 \frac{\rho^{m'+2\beta}}{\Gamma\big(\frac{m'+2\beta+3}{2}\big)} \frac{\rho^{m''+3}}{\Gamma\big(\frac{m''+8}{2}\big)} (m''+1) \\
&+ C_2 \sum_{m'=m}^{d-4} (m''+1)|J^{(2)}_{m'+2\beta,m''}| \cdot 2 \frac{\rho^{m'+2\beta}}{\Gamma\big(\frac{m'+2\beta+3}{2}\big)} \frac{\rho^{m''+1}}{\Gamma\big(\frac{m''+6}{2}\big)} \\
&+ C_2 |J^{(2)}_{d-2+2\beta,2}| \cdot 4 \frac{\rho^{d-2+2\beta}}{\Gamma\big(\frac{d+2\beta+1}{2}\big)} \\
\le\; & C_2 \frac{\rho^{d+2\beta}}{\Gamma\big(\frac{d+2\beta+3}{2}\big)} \Bigg(\sum_{m'=2}^{d-m} \frac{8\rho^3}{d+2\beta+5} + \sum_{m'=4}^{d-m} \frac{8\rho}{d+2\beta+5} + \frac{8}{\rho^2(d+2\beta+5)} \\
&+ \sum_{m'=3}^{d-m+1} \frac{8\rho^3}{d+2\beta+5} \frac{2(m''+1)}{d+2\beta+3} + \sum_{m'=3}^{d-m-1} \frac{8\rho}{d+2\beta+5} + \frac{8}{\rho(d+2\beta+5)} \\
&+ \sum_{m'=m-1}^{d-3} \frac{8\rho^3}{d+2\beta+5} \frac{2(m'+1)}{d+2\beta+3} + \sum_{m'=m+1}^{d-3} \frac{8\rho}{d+2\beta+5} + \frac{16}{\rho^2(d+2\beta+5)} \\
&+ \sum_{m'=m}^{d-2} \frac{16\rho^3}{d+2\beta+5} + \sum_{m'=m}^{d-4} \frac{8\rho}{d+2\beta+5} + \frac{16}{\rho^2(d+2\beta+5)} \Bigg) \\
\le\; & C_2 \frac{\rho^{d+2\beta}}{\Gamma\big(\frac{d+2\beta+3}{2}\big)} \cdot 2000\rho \le \frac{\rho^{d+2\beta}}{\Gamma\big(\frac{d+2\beta+3}{2}\big)}
\end{aligned}
$$

where in the second last inequality above we have used the fact that $d \ge L$ and L is sufficiently large so that $d/(d+2\beta+3) \le 2$. It suffices to take $L = 2m$. To check the inductive assumption for $m \le d \le 2m$, we recall that $f^{(j)}_{\beta}(m_1, m_2)$ depends linearly on several free parameters. If we let them be sufficiently small, then it is clear that the inductive assumption is satisfied for $m \le d \le 2m$. Our theorem is proved.

We now formulate our main theorem about the spectrum of the linearized operator.

Theorem 6.2. *The spectrum of the operator $\mathcal{A}$ consists of the following eigenvalues:*

$$\operatorname{spec}(\mathcal{A}) = \{1, 1/2, 0, \lambda^{(1)}_m : \lambda^{(2)}_m, m \ge 1\}$$

where $\lambda^{(1)}_m = -m/2$, $\lambda^{(2)}_m = (\sqrt{17} - 4 - m)/2$, $m \ge 1$.

The first three eigenvalues have multiplicities $\nu_1 = 1$, $\nu_{1/2} = 3$, $\nu_0 = 6$. *The eigenvalues* $\lambda_m^{(1)}$, $\lambda_m^{(2)}$ *correspond to the stable part of the spectrum and also have finite multiplicities given by:* $\nu_{\lambda_m^{(1)}} = (m+3)(m+4)/2$, $\nu_{\lambda_m^{(2)}} = m(m+5)/2$.

Each $\alpha \in \operatorname{spec}(\mathcal{A})$ *corresponds in a one-to-one way to the Hermite polynomial* $\mathrm{He}_{m_1}(Y_1)\,\mathrm{He}_{m_2}(Y_2)\,\mathrm{He}_{m_3}(Y_3)$.

For each $\alpha \in \operatorname{spec}(\mathcal{A})$, *the eigenfunctions* $f_\alpha^{(j)}(m_1, m_2, m_3)$ *have the following property:*

(a) $f_\alpha^{(j)}(m_1, m_2, m_3)$ *is compactly supported in the* m_3 *variable, i.e., there exists an integer* $m_3^* = m_3^*(\alpha)$ *such that*

$$f_\alpha^{(j)}(m_1, m_2, m_3) = 0 \quad \text{if } m_3 > m_3^*.$$

(b) $f_\alpha^{(j)}(m_1, m_2, m_3)$ *decays faster than exponentially, more precisely, there exist constants* $C_3 = C_3(\alpha) > 0$ *and* $0 < \rho < 1/4000$ *such that*

$$|f_\alpha^{(j)}(m_1, m_2, m_3)| \le C_3 \frac{\rho^{m_1+m_2+m_3+2\alpha}}{\Gamma\left(\frac{m_1+m_2+m_3+2\alpha+3}{2}\right)}, \qquad \forall m_1, m_2, m_3 \ge 0.$$

The system of eigenfunctions is complete in the following sense. Let $\Gamma^{(s)}$ *be the stable linear subspace of* Δ *generated by all eigenfunctions with* $\Re(\lambda) < 0$, $\Gamma^{(u)}$ *be the unstable subspace generated by all eigenfunctions with eigenvalues* $\lambda > 0$, *and* $\Gamma^{(n)}$ *be the neutral subspace generated by all eigenfunctions with eigenvalue* $\lambda = 0$. *Then* $\dim \Gamma^{(u)} = 4$, $\dim \Gamma^{(n)} = 6$ *and*

$$\Delta = \Gamma^{(u)} + \Gamma^{(n)} + \Gamma^{(s)}.$$

Proof. By Lemma 6.1, we only need to examine β for which $\det(C_\beta^{(d)}) = 0$. From previous arguments, we know that for $d \ge 1$, $\beta = -(d-3)/2$ or $(\sqrt{17} - 4 - d)/2$. We discuss the spectrum separately in the following three cases.

Unstable spectrum: $\alpha = 1, 1/2$.

(a) $\alpha = 1$. Since $\beta = \alpha + m_3/2$, the only possibility is that $\beta = 1$, $d = 1$ and $m_3 = 0$. The eigenspace is one-dimensional with $f_{000}^{(1)} = f_{000}^{(2)} = f_{010}^{(1)} = f_{100}^{(2)} = 0$, $f_{100}^{(1)} = f_{010}^{(2)}$ is a free parameter and the remaining part of all higher degree terms ($f_{m_1,m_2,0}^{(j)}$ with $m_1 + m_2 \ge 2$) is uniquely determined once we specify $f_{100}^{(1)}$.

(b) $\alpha = 1/2$. Possible cases are $m_3 = 0$, $\beta = 1/2$, $d = 0, 2$ or $m_3 = 1$, $\beta = 1$, $d = 1$. In the first case we have $f_{m_1,m_2,0}^{(j)} = 0$ for $m_1 + m_2 \le 1$, $f_{110}^{(1)} = f_{020}^{(2)}$, $f_{200}^{(1)} = f_{110}^{(2)}$ are two free parameters, all other terms of higher degree ($f_{m_1,m_2,0}^{(j)}$ with $m_1 + m_2 \ge 3$) are uniquely determined once we specify the above four parameters. In the second case we have $f_{001}^{(1)} = f_{001}^{(2)} = f_{011}^{(1)} = f_{101}^{(2)} = 0$, $f_{101}^{(1)} = f_{011}^{(2)}$ is a free parameter and the remaining part of all higher degree terms ($f_{m_1,m_2,1}^{(j)}$ with $m_1 + m_2 \ge 2$) is uniquely determined once we specify $f_{101}^{(1)}$. Putting two cases together, we see that the dimension of the eigenspace is 3.

This gives $\dim \Gamma^{(u)} = 4$.

Neutral spectrum: Here we have $\alpha = 0$, and three cases.

(a) $m_3 = 2$. Then $\beta = 1$. The eigenspace is one-dimensional with $f^{(1)}_{002} = f^{(2)}_{002} = f^{(1)}_{012} = f^{(2)}_{102} = 0$, $f^{(1)}_{102} = f^{(2)}_{012}$ is a free parameter and the remaining part of all higher degree terms ($f^{(j)}_{m_1,m_2,2}$ with $m_1 + m_2 \geq 2$) is uniquely determined once we specify $f^{(1)}_{102}$. This eigenvector is connected with $\partial/\partial\sigma^{(2)}$ which corresponds to the variation of the parameter $\sigma^{(2)}$ of the fixed point.

(b) $m_3 = 1$. Then $\beta = 1/2$. We have $f^{(j)}_{m_1,m_2,1} = 0$ for $m_1 + m_2 \leq 1$, $f^{(1)}_{111} = f^{(2)}_{021}$, $f^{(1)}_{201} = f^{(2)}_{111}$ are two free parameters, all other terms of higher degree ($f^{(j)}_{m_1,m_2,1}$ with $m_1 + m_2 \geq 3$) are uniquely determined once we specify the above two parameters. Clearly the eigenspace is two-dimensional. This eigenspace does not correspond to any change of parameters of the fixed point.

(c) $m_3 = 0$. Then $\beta = 0$. We have $f^{(j)}_{m_1,m_2,0} = 0$ for $m_1 + m_2 \leq 2$, $f^{(1)}_{030} = f^{(2)}_{300} = 0$, $f^{(1)}_{120} = f^{(2)}_{030}$, $f^{(1)}_{210} = f^{(2)}_{120}$, $f^{(1)}_{300} = f^{(2)}_{210}$ are three free parameters. All other terms of higher degree ($f^{(j)}_{m_1,m_2,0}$ with $m_1 + m_2 \geq 4$) are uniquely determined once we specify the above three parameters. This eigenspace corresponds to ($\partial/\partial x^{(1)}$, $\partial/\partial x^{(2)}$, $\partial/\partial x^{(3)}$).

Putting all three cases together, we see that $\dim \Gamma^{(n)} = 6$.

Stable spectrum: $\Re(\alpha) < 0$.

There are two cases.

Case 1: $\alpha = -m/2, m \geq 1$. Recall that $\beta = \alpha + m_3/2$, and m_3 satisfies $0 \leq m_3 \leq m+2$. By Theorem 6.3, for each such β, the number of free parameters is $3 - 2\beta$. Then the total multiplicity ν_α is given by

$$\nu_\alpha = \sum_{m_3=0}^{m+2} [3 - (-m + m_3)] = \frac{(m+3)(m+4)}{2}.$$

Case 2: $\alpha = (\sqrt{17} - 4 - m)/2, m \geq 1$. Now $\beta = \alpha + m_3/2$ and $0 \leq m_3 \leq m - 1$. By Theorem 6.3, we have

$$\nu_\alpha = \sum_{m_3=0}^{m-1} (m - m_3 + 2) = \frac{m(m+5)}{2}.$$

It follows easily that the eigenfunction $f^{(j)}_\alpha(m_1, m_2, m_3)$ is compactly supported in the m_3 variable. By Theorem 6.3, the decay estimate on $f^{(j)}_\alpha(m_1, m_2, m_3)$ is obvious.

It turns out that the eigenvector corresponding to $\partial/\partial\sigma^{(1)}$ is in the eigenspace spanned by the eigenvectors ($\partial/\partial x^{(1)}$, $\partial/\partial x^{(2)}$, $\partial/\partial x^{(3)}$). More precisely, we have the following:

Lemma 6.3. *Let* $t_1 = x^{(1)} - 1$, $t_2 = x^{(2)}$, $t_3 = x^{(3)} - 1$. *Then*

$$\tilde{G}^{(\sigma^{(1)},t_1,t_2,t_3,\sigma^{(2)})}(Y) = G^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)},\sigma^{(2)})}(Y) \tag{37}$$

where $G^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)},\sigma^{(2)})}$ *is defined in* (30). *The function* $\tilde{G}$ *satisfies the following scaling relation:*

$$\tilde{G}^{(\sigma^{(1)},t_1,t_2,t_3,\sigma^{(2)})}(Y) = \tilde{G}^{(1,\sigma^{(1)}t_1,\sigma^{(1)}t_2,\sigma^{(1)}t_3,\sigma^{(2)})}(Y). \tag{38}$$

Proof. Let $f^{(j),0}_{m_1,m_2,0}$ correspond to the eigenvector $\partial/\partial\sigma^{(1)}$. Then a simple calculation shows that

$$f^{(j),0}_{m_1,m_2,0} = (m_1 + m_2 - 1)h^{(j)}_{m_1 m_2} + h^{(j)}_{m_1-2,m_2} + h^{(j)}_{m_1,m_2-2}.$$

If $f^{(j),1}_{m_1,m_2,0}$, $f^{(j),2}_{m_1,m_2,0}$ and $f^{(j),3}_{m_1,m_2,0}$ correspond to the eigenvectors $\partial/\partial x^{(1)}$, $\partial/\partial x^{(1)}$, and $\partial/\partial x^{(3)}$ respectively, then clearly we have

$$f^{(j),0}_{m_1,m_2,0} = (x^{(1)} - 1)f^{(j),1}_{m_1,m_2,0} + x^{(2)} f^{(j),2}_{m_1,m_2,0} + (x^{(3)} - 1)f^{(j),3}_{m_1,m_2,0}.$$

This immediately gives

$$\left[\sigma^{(1)}\frac{\partial}{\partial\sigma^{(1)}} - (x^{(1)}-1)\frac{\partial}{\partial x^{(1)}} - x^{(2)}\frac{\partial}{\partial x^{(2)}} - (x^{(3)}-1)\frac{\partial}{\partial x^{(3)}}\right]G^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)},\sigma^{(2)})}(Y) = 0.$$

Regarding this as a transport equation in the variables $(\sigma^{(1)}, t_1, t_2, t_3)$, we can easily find that $\tilde{G}$ satisfies the scaling (35). The lemma is proved.

This lemma actually shows in what sense the parameters $\sigma^{(1)}$, $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are dependent.

As was shown in §4, we have the five-parameter family of fixed points $G^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)},\sigma^{(2)})}$. We use the notation $\pi = (\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}, \sigma^{(2)})$ and write $G^{(\pi)}$ instead of $G^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)},\sigma^{(2)})}$. The spectrum of the linearization of the equation for the fixed point does not depend on π (see §5) and has $l^{(u)} = 4$ unstable eigenvectors $\Phi^{(u)}_j(Y_1, Y_2, Y_3)$, $1 \le j \le l^{(u)} = 4$, and $l^{(n)} = 6$ neutral eigenvectors $\Phi^{(n)}_{j'}(Y_1, Y_2, Y_3)$, $1 \le j' \le l^{(n)} = 6$.

7. The choice of initial conditions and the initial part of the inductive procedure

The equation (21) for the fixed point which was derived in §3 is nontypical from the point of view of the renormalization group theory because it contains integration over γ, $0 \le \gamma \le 1$. On the other hand, since we consider the Cauchy problem for (1) we are given only the initial condition $v(k, 0)$ which produces through the recurrent relations (4), (5), (6) the whole set of functions $g_r(k, s)$. For large p and $r \le p$ they can be considered as depending on $\gamma = r/p$ and our procedure is organized in such a way that for γ which are away from zero, $\tilde{g}_r$ are close to their limits.

We take $k^{(0)}$ which will be assumed to be sufficiently large, introduce the neighborhood

$$A_1 = \{k : |k - \kappa^{(0)}| \le D_1\sqrt{k^{(0)} \ln k^{(0)}}\}$$

where $\kappa^{(0)} = (0, 0, k^{(0)})$ and D_1 is also sufficiently large. Our initial conditions will be zero outside A_1. Inside A_1 they have the form

$$v(k, 0) = \frac{1}{2\pi} \exp\left\{-\frac{Y_1^2 + Y_2^2}{2}\right\}\Big(H^{(0)}(Y_1, Y_2) + \sum_{j=1}^{4} b_j^{(u)}\Phi_j^{(u)}(Y_1, Y_2, Y_3)$$
$$+ \sum_{j'=1}^{6} b_{j'}^{(n)}\Phi_{j'}^{(n)}(Y_1, Y_2, Y_3) + \Phi(Y_1, Y_2, Y_3; b^{(u)}, b^{(n)})\Big)\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{Y_3^2}{2}\right\}. \quad (39)$$

In this expression $k = k^{(0)} + \sqrt{k^{(0)}}Y$ and $H^{(0)}(Y_1, Y_2) = (H_1^{(0)}(Y_1, Y_2), H_2^{(0)}(Y_1, Y_2), 0)$ is the fixed point of our renormalization group (see §4) corresponding to the parameters $\sigma_1^{(1)} = \sigma_1^{(2)} = 1$, $x_1 = x_2 = x_3 = 0$. For this fixed point $H^{(0)}(Y_1, Y_2) = (H_1^{(0)}(Y_1, Y_2), H_2^{(0)}(Y_1, Y_2))$ and $H_1^{(0)}(Y_1, Y_2) = -2Y_1$, $H_2^{(0)}(Y_1, Y_2) = -2Y_2$, which are Hermite polynomials of degree one. This fixed point has some special properties which will be used below. Also $\Phi_j^{(u)}$, $\Phi_{j'}^{(n)}$ are unstable and neutral eigenfunctions of the linearized group corresponding to $H^{(0)}$ (see §§5, 6), $b_j^{(u)}$ and $b_{j'}^{(n)}$ are our main parameters. We assume that their values satisfy the inequalities

$$-\rho_1 \le b_j^{(u)}, b_{j'}^{(n)} \le \rho_1,$$

where ρ_1 is a positive constant. Our numerical studies show that it is enough to take $\rho_1 = 3/4$. Each function $\Phi(Y_1, Y_2, Y_3; b^{(u)}, b^{(n)})$, $b^{(u)} = \{b_j^{(u)}\}$, $b^{(n)} = \{b_{j'}^{(n)}\}$, is small in the sense that it satisfies the inequalities

$$\sup_{Y,b} |\Phi(Y_1, Y_2, Y_3; b^{(u)}, b^{(n)})| \le D_2,$$
$$\sup \|\Phi(Y_1, Y_2, Y_3; \bar{b}^{(u)}, \bar{b}^{(n)}) - \Phi(Y_1, Y_2, Y_3; \bar{\bar{b}}^{(u)}, \bar{\bar{b}}^{(n)})\|$$
$$\le D_2(|\bar{b}^{(u)} - \bar{\bar{b}}^{(u)}| + |\bar{b}^{(n)} - \bar{\bar{b}}^{(n)}|).$$

Due to the presence of $b^{(u)}, b^{(n)}$, we have $l = l^{(u)} + l^{(n)} = 10$-parameter families of initial conditions, due to the presence of Φ we have an open set in the space of such families.

Let

$$A_r = \{k : |k - r\kappa^{(0)}| \le D_1\sqrt{rk^{(0)} \ln(rk^{(0)})}\},$$

and let the variable Y be such that $k = r\kappa^{(0)} + \sqrt{rk^{(0)}}Y$. Assume that for some p and all $r < p$ and $|Y| \le D_1\sqrt{\ln(rk^{(0)})}$,

$$g_r(r\kappa^{(0)} + \sqrt{rk^{(0)}}Y, s) = Z_p(s)\Lambda_p(s)r\tilde{g}_r(Y, s)$$

and

$$\tilde{g}_r(Y,s) = \frac{1}{2\pi}\exp\left\{-\frac{Y_1^2+Y_2^2}{2}\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{Y_3^2}{2}\right\}$$
$$\cdot\Bigg(H_1^{(0)}(Y_1,Y_2)+\delta_1^{(r)}(Y_1,Y_2,Y_3),\ H_2^{(0)}(Y_1,Y_2)+\delta_2^{(r)}(Y_1,Y_2,Y_3),$$
$$\frac{1}{\sqrt{rk^{(0)}}}(F^{(r)}(Y_1,Y_2)+\delta_3^{(r)}(Y_1,Y_2,Y_3))\Bigg)$$

where in view of incompressibility

$$H_1^{(0)}Y_1+H_2^{(0)}Y_2+F^{(r)}=0. \tag{40}$$

$Z_p(s)$, $\Lambda_p(s)$ are functions of s defined for $s\in[S_-^{(p)},S_+^{(p)}]$. Actually, as will be seen later, $Z_p(s)$ is a constant which does not depend on p and s. The expression for $\Lambda_p(s)$ depends on the form of our fixed point $H^{(0)}$ (see below).

Set $p_0=N$ where N is an integer. Actually we will take $N=50$. The initial part of our procedure goes for $p\le p_0$. It is discussed in this section. The part corresponding to $p>p_0$ is discussed in §9.

Returning to (6) take the term with some p_1, p_2, $p_1+p_2=p$, and introduce the new integration variable Y' where $k'=p_2\kappa^{(0)}+\sqrt{pk^{(0)}}Y'$. Introduce also the variables θ_1, θ_2, $0\le\theta_1\le(p_1k^{(0)})^2$, $0\le\theta_2\le(p_2k^{(0)})^2$ where $s_1=s(1-\theta_1/(p_1k^{(0)})^2)$, $s_2=s(1-\theta_2/(p_2k^{(0)})^2)$.

Then from (6),

$$g_p(p\kappa^{(0)}+\sqrt{pk^{(0)}}Y,s)=Z_p(s)\Lambda_p(s)p\tilde{g}_p(Y,s)$$
$$=(pk^{(0)})^{5/2}\int_0^{((p-1)k^{(0)})^2}d\theta_2\int_{\mathbb{R}^3}\exp\left\{-\theta_2\left|\kappa^{(0,0)}+\frac{Y'}{\sqrt{(p-1)k^{(0)}}}\right|^2\right\}$$
$$\cdot\ Z_p\left(s\left(1-\frac{\theta_2}{((p-1)k^{(0)})^2}\right)\right)\Lambda_{p-1}\left(s\left(1-\frac{\theta_2}{((p-1)k^{(0)})^2}\right)\right)(p-1)Z_1(s)\Lambda_1(s)$$
$$\cdot\left\langle\tilde{g}_1((Y-Y')\sqrt{pk^{(0)}},0),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\right\rangle$$
$$\cdot\ P_{\kappa^{(0,0)}+Y/\sqrt{pk^{(0)}}}\tilde{g}_{p-1}\left(Y'\sqrt{\frac{p}{p-1}},s\left(1-\frac{\theta_2}{((p-1)k^{(0)})^2}\right)\right)d^3Y'$$
$$+\,p\sum_{\substack{p_1+p_2=p\\p_1,p_2>1}}\frac{1}{p}\frac{(pk^{(0)})^{5/2}p_1p_2}{(p_1k^{(0)})^2(p_2k^{(0)})^2}\int_0^{(p_1k^{(0)})^2}d\theta_1\int_0^{(p_2k^{(0)})^2}d\theta_2$$
$$\cdot\int_{\mathbb{R}^3}\left(\frac{1}{2\pi}\right)^{3/2}\exp\left\{-\frac{(Y_1-Y_1')^2+(Y_2-Y_2')^2+(Y_3-Y_3')^2}{2\gamma}\right\}$$
$$\cdot\left\langle\tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}},s\left(1-\frac{\theta_1}{(p_1k^{(0)})^2}\right)\right),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\right\rangle$$
$$\cdot\ P_{\kappa^{(0,0)}+Y/\sqrt{pk^{(0)}}}\tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}},s\left(1-\frac{\theta_2}{(p_2k^{(0)})^2}\right)\right)$$

$$
\begin{aligned}
&\cdot\; Z_{p_1}\Big(s\Big(1-\frac{\theta_1}{(p_1k^{(0)})^2}\Big)\Big)\Lambda_{p_1}\Big(s\Big(1-\frac{\theta_1}{(p_1k^{(0)})^2}\Big)\Big)Z_{p_2}\Big(s\Big(1-\frac{\theta_2}{(p_2k^{(0)})^2}\Big)\Big)\\
&\cdot\; \Lambda_{p_2}\Big(s\Big(1-\frac{\theta_2}{(p_2k^{(0)})^2}\Big)\Big)\Big(\frac{1}{2\pi}\Big)^{3/2}\exp\Big\{-\frac{(Y_1')^2+(Y_2')^2+(Y_3')^2}{2(1-\gamma)}\Big\}\\
&\cdot\; \exp\Big\{-\theta_1\Big|\kappa^{(0,0)}+\frac{Y-Y'}{\gamma\sqrt{pk^{(0)}}}\Big|^2\Big\}\exp\Big\{-\theta_2\Big|\kappa^{(0,0)}+\frac{Y'}{(1-\gamma)\sqrt{pk^{(0)}}}\Big|^2\Big\}\\
&+\frac{(pk^{(0)})^{5/2}(p-1)}{((p-1)k^{(0)})^2}\int_0^{((p-1)k^{(0)})^2}d\theta_1\int_{\mathbb{R}^3}\exp\Big\{-\theta_1\Big|\kappa^{(0,0)}+\frac{Y-Y'}{\sqrt{(p-1)k^{(0)}}}\Big|^2\Big\}\\
&\cdot\; Z_{p-1}\Big(s\Big(1-\frac{\theta_1}{((p-1)k^{(0)})^2}\Big)\Lambda_{p-1}\Big(s\Big(1-\frac{\theta_1}{((p-1)k^{(0)})^2}\Big)\Big)Z_1(s)\Lambda_1(s)\\
&\cdot\; \Big\langle \tilde g_{p-1}\Big((Y-Y')\sqrt{\frac{p}{p-1}},s\Big(1-\frac{\theta_1}{((p-1)k^{(0)})^2}\Big)\Big),\kappa^{(0,0)}+\frac{Y-Y'}{\sqrt{pk^{(0)}}}\Big\rangle\\
&\cdot\; P_{\kappa^{(0,0)}+Y/\sqrt{pk^{(0)}}}\tilde g_1(Y'\sqrt{p},s)\,d^3Y'. \qquad (41)
\end{aligned}
$$

Here $\gamma = p_1/p$ and $\kappa^{(0,0)} = (0,0,1)$. Now we shall modify (41) for $p_1, p_2 > 1$ similarly to what we did in §3. Later we discuss the terms with $p_1 = 1$ and $p_2 = 1$ which will be included in the remainders. The modification consists of four steps.

Step 1. All terms $s(1-\theta_1/(p_1k^{(0)})^2)$, $s(1-\theta_2/(p_2k^{(0)})^2)$ are replaced by s.

Step 2. Write

$$\frac{(pk^{(0)})^{5/2}p_1p_2}{(p_1k^{(0)})^2(p_2k^{(0)})^2}=\frac{(pk^{(0)})^{1/2}}{(k^{(0)})^2\gamma(1-\gamma)}.$$

Step 3. Consider the inner product

$$(pk^{(0)})^{1/2}\Big\langle \tilde g_{p_1}\Big(\frac{Y-Y'}{\sqrt{\gamma}},s\Big),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\Big\rangle.$$

Up to remainders and from (40) it equals

$$
\begin{aligned}
&\Big(\frac{1}{2\pi}\Big)^{3/2}\exp\Big\{-\frac{(Y_1-Y_1')^2+(Y_2-Y_2')^2+(Y_3-Y_3')^2}{2\gamma}\Big\}\Big[H_1^{(0)}\Big(\frac{Y_1-Y_1'}{\sqrt{\gamma}},\frac{Y_2-Y_2'}{\sqrt{\gamma}}\Big)Y_1\\
&\quad+H_2^{(0)}\Big(\frac{Y_1-Y_1'}{\sqrt{\gamma}},\frac{Y_2-Y_2'}{\sqrt{\gamma}}\Big)Y_2+\frac{1}{\sqrt{\gamma}}F^{(p_1)}\Big(\frac{Y-Y'}{\sqrt{\gamma}},s\Big)\Big]\\
&=\Big(\frac{1}{2\pi}\Big)^{3/2}\exp\Big\{-\frac{(Y_1-Y_1')^2+(Y_2-Y_2')^2+(Y_3-Y_3')^2}{2\gamma}\Big\}\\
&\quad\cdot\Big[H_1^{(0)}\Big(\frac{Y_1-Y_1'}{\sqrt{\gamma}},\frac{Y_2-Y_2'}{\sqrt{\gamma}}\Big)Y_1+H_2^{(0)}\Big(\frac{Y_1-Y_1'}{\sqrt{\gamma}},\frac{Y_2-Y_2'}{\sqrt{\gamma}}\Big)Y_2\\
&\quad-H_1^{(0)}\Big(\frac{Y_1-Y_1'}{\sqrt{\gamma}},\frac{Y_2-Y_2'}{\sqrt{\gamma}}\Big)\frac{Y_1-Y_1'}{\gamma}-H_2^{(0)}\Big(\frac{Y_1-Y_1'}{\sqrt{\gamma}},\frac{Y_2-Y_2'}{\sqrt{\gamma}}\Big)\frac{Y_2-Y_2'}{\gamma}\Big]
\end{aligned}
$$

$$= \left(\frac{1}{2\pi}\right)^{3/2} \exp\left\{-\frac{(Y_1 - Y_1')^2 + (Y_2 - Y_2')^2 + (Y_3 - Y_3')^2}{2\gamma}\right\}$$
$$\cdot \left\{-\frac{\gamma - 1}{\sqrt{\gamma}}\left[H_1^{(0)}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}, \frac{Y_2 - Y_2'}{\sqrt{\gamma}}\right)\frac{Y_1 - Y_1'}{\sqrt{\gamma}}\right.\right.$$
$$\left. + H_2^{(0)}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}, \frac{Y_2 - Y_2'}{\sqrt{\gamma}}\right)\frac{Y_2 - Y_2'}{\sqrt{\gamma}}\right]$$
$$+ \sqrt{1-\gamma}\left[H_1^{(0)}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}, \frac{Y_2 - Y_2'}{\sqrt{\gamma}}\right)\frac{Y_1'}{\sqrt{1-\gamma}}\right.$$
$$\left.\left. + H_2^{(0)}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}}, \frac{Y_2 - Y_2'}{\sqrt{\gamma}}\right)\frac{Y_2'}{\sqrt{1-\gamma}}\right]\right\}.$$

Let us stress again that $H_j^{(0)}(Y, s)$ depends only on Y_1, Y_2 and s. With respect to Y_3 we have the usual convolution.

Step 4. Replace the projection operator by the identity operator. It is not the reduction to the Burgers system because the incompressibility condition is preserved.

Now we shall modify the first and the last terms in (41). For the first one we can write

$$\frac{(pk^{(0)})^{5/2}(p-1)}{((p-1)k^{(0)})^2}\int_0^{((p-1)k^{(0)})^2} d\theta_2 \int_{\mathbb{R}^3} \exp\left\{-\theta_2\left|\kappa^{(0,0)} + \frac{Y'}{\sqrt{(p-1)k^{(0)}}}\right|^2\right\}$$
$$\cdot \exp\{-s|\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}|^2\}\left\langle v(\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}, 0), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}}\right\rangle$$
$$\cdot P_{\kappa^{(0)}+Y/\sqrt{pk^{(0)}}}\tilde{g}_{p-1}\left(Y'\sqrt{\frac{p}{p-1}}, s\left(1 - \frac{\theta_2}{((p-1)k^{(0)})^2}\right)\right) d^3Y'. \tag{42}$$

The factor $p-1$ comes from the inductive assumption concerning g_{p-1}. As before, we replace $\exp\{-\theta_2|\kappa^{(0,0)} + Y'/\sqrt{(p-1)k^{(0)}}|^2\}$ by $\exp\{-\theta_2\}$, $P_{\kappa^{(0)}+Y/\sqrt{pk^{(0)}}}$ by the identity operator and $\tilde{g}_{p-1}(Y'\sqrt{p/(p-1)}, s(1-\theta_2/((p-1)k^{(0)})^2))$ by $\tilde{g}_{p-1}(Y'\sqrt{p/(p-1)}, s)$. All corrections are included in the remainder terms.

For the Gaussian term in $v(\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}, 0)$ we can write $(2\pi)^{-3/2} \cdot \exp\{|Y - Y'|^2 p/2\}$. This shows that typically $Y - Y' = O(1/\sqrt{p})$. For the third component $F^{(1)}$ of $v(\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}, 0)$, by using the incompressibility condition we can write

$$F^{(1)}(\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}, 0) = -\frac{1}{\sqrt{k^{(0)}}}$$
$$\cdot \left((Y_1 - Y_1')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p}) + (Y_2 - Y_2')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p}) + O\left(\frac{1}{\sqrt{k^{(0)}}}\right)\right)$$
$$\cdot \exp\left\{-\frac{p|Y - Y'|^2}{2}\right\}.$$

For the product in (42) we have

$$(\sqrt{2\pi})^3 \cdot \sqrt{pk^{(0)}}\left\langle v(\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}, 0), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}}\right\rangle$$
$$= \exp\left\{-\frac{p|Y - Y'|^2}{2}\right\}\Big[H_1^{(0)}((Y - Y')\sqrt{p})Y_1 + H_1^{(0)}((Y - Y')\sqrt{p})Y_2$$
$$-\sqrt{p}((Y_1 - Y_1')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p}) + (Y_2 - Y_2')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p})) + O\left(\frac{1}{\sqrt{k^{(0)}}}\right)\Big].$$

The expression in the square brackets grows as $\sqrt{p}$ and therefore

$$(\sqrt{2\pi})^3\sqrt{pk^{(0)}}\left\langle v(\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}, 0), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}}\right\rangle$$

can be replaced by

$$-\sqrt{p}\Big[(Y_1 - Y_1')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p}) + (Y_2 - Y_2')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p})$$
$$-\frac{1}{\sqrt{p}}(H_1^{(0)}((Y - Y')\sqrt{p})Y_1 + H_1^{(0)}((Y - Y')\sqrt{p})Y_2)\Big].$$

Further,

$$\exp\{-s|\kappa^{(0)} + (Y - Y')\sqrt{pk^{(0)}}|^2\}$$
$$= \exp\{-s|k^{(0)}|^2\}\exp\{-2sk^{(0)}\langle\kappa^{(0,0)}, (Y - Y')\sqrt{p}\sqrt{k^{(0)}}\rangle\}\exp\{-s|Y - Y'|^2pk^{(0)}\}.$$

The first factor takes values $O(1)$, the others can be written as $1 + O(1/\sqrt{k^{(0)}})$. The term of the main order of magnitude of (42) takes the form

$$p\exp\{-s(k^{(0)})^2\}\frac{p-1}{p}\frac{1}{p}$$
$$\cdot\Big[-\int_{\mathbb{R}^3}[(Y_1 - Y_1')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p}) + (Y_2 - Y_2')\sqrt{p}H_1^{(0)}((Y - Y')\sqrt{p})]$$
$$+\frac{1}{\sqrt{p}}[H_1^{(0)}((Y - Y')\sqrt{p})Y_1 + H_1^{(0)}((Y - Y')\sqrt{p})Y_2]\left(\frac{p}{2\pi}\right)^{3/2}\exp\left\{-\frac{|Y - Y'|^2p}{2}\right\}$$
$$\cdot\left(\frac{1}{2\pi}\right)^{3/2}\exp\left\{-\frac{|Y'|^2p}{2(p-1)}\right\}H^{(0)}\left(Y'\sqrt{\frac{p}{p-1}}\right)d^3Y'\Big].$$

A similar expression can be written for the last term in (41). We choose the initial interval on the time axis in the form $S^{(1)} = [1/4(k^{(0)})^2, 3/8(k^{(0)})^2]$. Due to our choice of the interval $S^{(1)}$ the product $s(k^{(0)})^2$ is $O(1)$. During the first part of our procedure $S^{(p)} = S^{(1)}$, $p \le p_0$.

Now we derive recurrent formulas for $\Lambda_p(s)$. It is clear that one neutral eigenvector corresponding to $\alpha = 0$ is associated with multiplication by $\Lambda(s)$. In the case of an arbitrary fixed point, at each step p, we must renormalize $\Lambda_p(s)$ in such a way that the projection to this neutral eigenvector is zero. The situation turns out to be simpler for our special fixed point $H^{(0)}$. In this case the integral in (41) containing $H^{(0)}$ gives us the product of $H^{(0)}$, the Gaussian term and a polynomial in γ. The function $H^{(0)}$ and the Gaussian term can be taken out of the summation over γ and this gives us the following recurrent system for $\Lambda_p(s)$:

$$\Lambda_p(s) = \sum_{p_1+p_2=p} \frac{1}{p}\frac{1}{(k^{(0)})^2}(6\gamma^2-10\gamma+4)\Lambda_{p_1}(s)\Lambda_{p_2}(s)(1-e^{-s(p_1k^{(0)})^2})(1-e^{-s(p_2k^{(0)})^2}) \tag{43}$$

where the factor $6\gamma^2 - 10\gamma + 4$ comes from the integral of $H^{(0)}$ with itself.

In two separate papers [Li], [S3], we prove that the asymptotics of $\Lambda_p(s)$ is given by

$$\Lambda_p(s) = (k^{(0)})^2\Lambda(s)^p\left(1 + O\left(\frac{1}{p^{3/2}}\right)\right), \tag{44}$$

where $\Lambda(s) > 0$ is a limiting constant independent of p. This result will be used in the proof of our main theorem.

Now we shall discuss the behavior of all remainders $\delta^{(r)}$, $r < p$. We make the following inductive assumption:

$$\begin{aligned}\delta^{(r)}(Y,s) = &\sum_{j=1}^{4}(b_{j,r}^{(u)} + \beta_{j,r}^{(u)})\gamma^{\alpha_j^{(u)}}\Phi_j^{(u)}(Y)\\ &+\sum_{j'=1}^{6}(b_{j',r}^{(n)} + \beta_{j',r}^{(n)})\Phi_{j'}^{(n)}(Y) + \Phi_r^{(st)}(Y,s), \qquad \gamma = \frac{r}{p-1}.\end{aligned}$$

Here $b_{j,r}^{(u)} = (p-1)^{\alpha_j}b_j^{(u)}\gamma^{\alpha_j^{(u)}}$, $b_{j',r}^{(n)} = b_{j'}^{(n)}$, and the corrections $\beta_{j,r}^{(u)}$, $\beta_{j',r}^{(n)}$ are small compared to $b_{j,r}^{(u)}$, $b_{j',r}^{(n)}$ respectively. Also $\{\Phi_r^{(st)}(Y,s)\}$, $1 \le r < p$, belongs to the stable subspace of our fixed point.

As we go from $p-1$ to p, the variable $\gamma = r/(p-1)$ changes to $\gamma' = r/p = \gamma(p-1)/p$. Therefore

$$\begin{aligned}(b_{j,r}^{(u)} + \beta_{j,r}^{(u)})\gamma^{\alpha_j^{(u)}}\Phi_j^{(u)} =&((p-1)^{\alpha_j^{(u)}}b_j^{(u)} + \beta_{j,r}^{(u)})\left(\frac{p}{p-1}\right)^{\alpha_j^{(u)}}(\gamma')^{\alpha_j^{(u)}}\Phi_j^{(u)}\\ =&\left(p^{\alpha_j^{(u)}}b_j^{(u)} + \left(\frac{p}{p-1}\right)^{\alpha_j^{(u)}}\beta_{j,r}^{(u)}\right)(\gamma')^{\alpha_j^{(u)}}\Phi_j^{(u)}.\end{aligned}$$

The formulas for the part involving the neutral eigenfunctions are similar and even simpler because $\alpha_{j'}^{(n)} = 0$. Thus the main terms in the expressions containing unstable and neutral eigenvalues preserve their form. The norm of $\{\Phi_r^{st}\}$ decreases.

In this section we consider our process till $p \le p_0$ where p_0 will be specified later. In the initial part of our procedure with $p \le p_0$ we use the discrete recurrent formulas and get small corrections $\beta_{j,r}^{(u)}$, $\beta_{j',r}^{(n)}$ and $\Phi^{(st)}$. We consider four types of terms.

(a$_1$) In the expression for $\delta^{(p)}(Y,s)$ there are terms which depend linearly on all $\delta^{(r)}(Y,s)$. Especially important are the terms $p^{\alpha_j^{(u)}} b_j^{(u)}(\gamma)^{\alpha_j^{(u)}}$, $\Phi_j^{(u)}$ and $b_{j'}^{(n)}\Phi_{j'}^{(n)}$. In the limiting regime $p\to\infty$ they produce the integral over γ which gives

$$p^{\alpha_j^{(u)}} b_j^{(u)}\left(1+\frac{1}{p}\right)^{\alpha_j^{(u)}}\Phi_j^{(u)} = (p+1)^{\alpha_j^{(u)}} b_j^{(u)}\Phi_j^{(u)},$$

in view of the definition of the eigenfunctions (see §5) and the condition $\gamma' = 1$. The same statement holds true for the neutral eigenfunctions.
However, for finite p, the sums over γ differ from the corresponding integrals. The difference produces some corrections which we expand according to our decomposition of the whole space onto unstable, neutral subspaces and the stable subspace. Corresponding terms are denoted as $\beta_{j,p,1}^{(u)}$, $\beta_{j',p,1}^{(n)}$ and $\Phi_{p,1}^{(st)}$. These corrections decay as $O(1/p)$.

(a$_2$) The term which contains all corrections arising during the four steps of our procedure (see above). All these corrections depend on $k^{(0)}$ and are smaller than $1/(k^{(0)})^{\mu_1}$ for some positive constant μ_1.

(a$_3$) The term $\tilde{\beta}_p$ which is a linear function of all $\beta_{j,r}^{(u)}, \beta_{j,r}^{(n)}$, $1\le r\le p-1$. We use the Hilbert space $X^{(p)}$ consisting of functions $f=\{f_r(Y), 1\le r<p\}$, and

$$\|f\|^2 = \frac{1}{p-1}\sum_{r=1}^{p-1}\|f_r(Y)\|_{L^2}^2$$

where $\|\cdot\|_{L^2}$ is the norm in the space of square-integrable functions of Y. It follows easily from §§5, 6 that for some constant C_1,

$$\|\tilde{\beta}_p\|_{L^2} \le C_1\Big(\sum_{j=1}^{4}\|\beta_j^{(u)}\| + \sum_{j'=1}^{6}\|\beta_{j'}^{(n)}\|\Big).$$

Therefore

$$\begin{aligned}
&\|\{\beta_{j,r}^{(u)},\beta_{j',r}^{(n)}\}, 1\le r\le p-1, \tilde{\beta}_p\|_{X(p+1)}^2\\
&\qquad = \frac{1}{p}\sum_{r=1}^{p}(\|\beta_{j,r}^{(u)}\|^2 + \|\beta_{j',r}^{(n)}\|^2)\\
&\qquad = \frac{p-1}{p}\|\{\beta_{j,r}^{(u)},\beta_{j',r}^{(n)}, 1\le r\le p-1\}\|_{X(p-1)}^2 + \frac{1}{p}\|\tilde{\beta}_p\|^2\\
&\qquad \le \|\{\beta_{j,r}^{(u)},\beta_{j',r}^{(n)}, 1\le r\le p-1\}\|^2\left(\frac{p-1}{p}+\frac{C_1}{p}\right)\\
&\qquad \le \|\{\beta_{j,r}^{(u)},\beta_{j',r}^{(n)}, 1\le r\le p-1\}\|^2\left(1+\frac{C_2}{p}\right)
\end{aligned}$$

for another constant C_2. Iterating this estimate we get

$$\|\{\beta_{j,r}^{(u)},\beta_{j',r}^{(n)}, 1\le r\le p\}\| \le \|\beta_{j,1}^{(u)},\beta_{j',1}^{(n)}\|\prod_{q=1}^{p}\left(1+\frac{C_2}{q}\right).$$

Here $\|\beta_{j,1}^{(u)}, \beta_{j',1}^{(n)}\| \le 1/(k^{(0)})^{\mu_2}$ for another constant μ_2 and $\prod_{q=1}^{p}(1+C_2/q) \le p_0^{C_3}$ for another constant C_3. This gives the estimate of the linear part.

(a_4) All terms which are quadratic functions of all remainders. Since all previous terms were already estimated, the quadratic terms are much smaller than the previous ones.

The sum of all these terms gives $\beta_{j,p}^{(u)}, \beta_{j',p}^{(n)}, \Phi_p^{(st)}$.

We take $N = 50$. For all $p \le 50$ all remainders were found numerically by computer using the exact recurrent relations (7). At $N = 50$ we make the first rescaling. Put $b_j^{(u,1)} = p^{\alpha_j} b_j^{(u)} + \beta_{j,p}^{(u)}$ for $1 \le j \le 4$ and $b_{j'}^{(n,1)} = b_{j'}^{(n)} + \beta_{j',p}^{(n)}$ for $1 \le j' \le 6$ and $p = 50$. These are our new rescaled variables. All previous expressions for $\delta^{(r)}(Y, s)$, $r < N$, can be written as functions of these new variables:

$$b_j^{(u)} p^{\alpha_j} + \beta_{j,r}^{(u)} = b_j^{(u,1)} + \beta_{j,r}^{(u,1)}, \quad 1 \le j \le 4,$$

where $\beta_{j,r}^{(u,1)} = \beta_{j,r}^{(u)} - \beta_{j,p}^{(u)}$ and

$$b_{j'}^{n)} + \beta_{j',r}^{(n)} = \beta_{j',1}^{(n,1)} + \beta_{j',r}^{(n,1)}, \quad 1 \le j' \le 6,$$

where $\beta_{j',r}^{(n,1)} = \beta_{j',r}^{(u)} - \beta_{j',p}^{(n)}$. The change in the expression for $\Phi_r^{(st)}$ is just the change of the variables $b_j^{(u)}, b_{j'}^{(n)}$. Numerically it was shown that ρ_1 can be chosen in such a way that the set $b_j^{(u)}, b_{j'}^{(n)}$ for which $-\rho_1/2 \le b_j^{(u,1)}, b_{j'}^{(n,1)} \le \rho_1/2$ is contained in the original set $-\rho_1 \le b_j^{(u)}, b_{j'}^{(n)} \le \rho_1$. We use this procedure till $p = p_0 = N$. The procedure for $p > p_0$ will be discussed in §9.

8. The list of remainders and their estimates

At the beginning of §7 we described 10-parameter families of initial conditions which we consider in this paper. We mentioned above that for each p we have an interval $S^{(p)} = [S_-^{(p)}, S_+^{(p)}]$ on the time axis. Actually these intervals will be changed only when $p = p_n = (1+\epsilon)^n$ where $\epsilon > 0$ is a constant. Therefore we shall write $S^{(n)} = [S_-^{(p_n)}, S_+^{(p_n)}]$ and hope that there will be no confusion.

In this and the next section we consider $p > p_0$. Each function $\tilde{g}_r(Y, s)$, $3 \le r < p$, has the following representation: in the domain $|Y| \le D_1\sqrt{\ln(rk^{(0)})}$, $Y = (Y_1, Y_2, Y_3) \in \mathbb{R}^3$,

$$\tilde{g}_r(Y, s) = Z(s)\Lambda_r(s) r \frac{1}{2\pi} \exp\left\{\frac{1}{2}(|Y_1|^2 + |Y_2|^2)\right\} \cdot \sqrt{\frac{1}{2\pi}} \exp\left\{-\frac{1}{2}|Y_3|^2\right\}(H^{(0)}(Y_1, Y_2) + \delta^{(r)}(Y, s));$$

in the domain $|Y| > D_1\sqrt{\ln(rk^{(0)})}$,

$$\frac{1}{2\pi}\exp\left\{-\frac{1}{2}(|Y_1|^2+|Y_2|^2)\right\}\sqrt{\frac{1}{2\pi}}\exp\left\{-\frac{1}{2}|Y_3|^2\right\}|H^{(0)}(Y_1,Y_2)+\delta^{(r)}(Y,s)| \le \Lambda_r(s)r\frac{1}{r^{\lambda_1-1}}$$

for a constant $\lambda_1 > 0$. We use the formula (7) to get $\tilde{g}^{(p)}(Y,s)$. New remainders appear in one of the following ways.

Type I. The recurrent relation (7) does not coincide with the equation for the fixed point and actually is some perturbation of this equation. The difference produces some remainders which tend to zero as $p \to \infty$.

Type II. For the limiting equation all eigenvectors in the linear approximation are multiplied by some constant. In the equation (7) this is no longer true and the difference generates some remainders (see also §9).

Type III. The remainders which are quadratic functions of all previous remainders.

8.1. The remainders of Type 1

We define the *domain A* to be the set $\{|Y| \le D_1\sqrt{\ln(rk^{(0)})}\}$ and the *domain B* to be the set $\{|Y| > D_1\sqrt{\ln(rk^{(0)})}\}$. The estimates will be done separately in each domain. We include the first, the second and the last two terms in (7) in the remainders. We shall estimate only the first one, the others are estimated in the same way.

Domain A: We have

$$\begin{aligned}\beta_p^{(1)}(Y,s) &= (p+1)^{5/2}\\ &\cdot \frac{1}{sp^2}\int_0^{p^2} d\theta_2\int_{\mathbb{R}^3}\left\langle v((k^{(0)}+(Y-Y')\sqrt{p+1},0);b),k^{(0)}+\frac{Y}{\sqrt{p+1}}\right\rangle\\ &\cdot P_{k^{(0)}+Y/\sqrt{p+1}}\tilde{g}_p\left(Y',\left(1-\frac{\theta_2}{p^2}\right)s\right)\\ &\cdot \exp\left\{-|k^{(0)}+(Y-Y')\sqrt{p+1}|^2-\frac{\theta_2}{p^2}|k^{(0)}p+Y'\sqrt{p+1}|^2\right\}d^3Y'.\end{aligned}$$

Here b means the collection of all parameters in the definition of $v(k;0)$. The main contribution to the integral comes from $Y-Y' = O(1/\sqrt{p+1})$. In this domain in the main order of magnitude

$$\langle v(k^{(0)}+(Y-Y')\sqrt{p+1},0;b),k^{(0)}\rangle = O(1).$$

Assuming that $v(k^{(0)}+(Y-Y')\sqrt{p+1},0;b)$ is differentiable with respect to the first three variables we see that the inner product

$$\left\langle v(k^{(0)}+(Y-Y')\sqrt{p+1},0;\alpha),k^{(0)}+\frac{Y}{\sqrt{p+1}}\right\rangle$$

is of order $O(1)$. For $\tilde{g}_p$ we can write using our inductive assumptions

$$\tilde{g}_p\left(Y',\left(1-\frac{\theta_2}{p^2}\right)s\right)=\Lambda_p(s)p\frac{1}{2\pi}\sqrt{\frac{1}{2\pi}}\exp\left\{-\frac{|Y_1|^2+Y_2|^2}{2}\right\}$$
$$\cdot\exp\left\{-\frac{|Y_3|^2}{2}\right\}\mathcal{H}^{(p)}\left(Y',\left(1-\frac{\theta_2}{p^2}\right)s\right)$$

where $\mathcal{H}^{(p)}(Y,s)=H^{(0)}(Y_1,Y_2)+\delta^{(p)}(Y,s)$. Also

$$\exp\left\{-\frac{\theta_2}{p^2}|k^{(0)}p+Y'\sqrt{p+1}|^2\right\}=\exp\left\{-\theta_2\left|k^{(0)}+\frac{Y'\sqrt{p+1}}{p}\right|^2\right\}$$

and in the main order of magnitude the integration over θ_2 does not depend on Y'. Thus we can write

$$|\beta_p^{(1)}(Y,s)|\le\Lambda_p(s)p\exp\left\{-\frac{1}{2}(|Y_1|^2+|Y_2|^2)\right\}\exp\left\{-\frac{1}{2}|Y_3|^2\right\}\frac{D_4}{p}. \tag{45}$$

Here and later various constants whose exact values play no role in the arguments will be denoted by the letter D with indices. Since in the expression for $\tilde{g}_{p+1}$ we have the factors $\Lambda_p(p+1)\exp\{-\frac{1}{2}(|Y_1|^2+|Y_2|^2)\}\frac{1}{2\pi}\sqrt{\frac{1}{2\pi}}\exp\{-\frac{1}{2}|Y_3|^2\}$, the estimate (45) shows that $|\beta_p^{(1)}(Y,s)|$ is much smaller than $\tilde{g}_{p+1}$ by a factor of $O(1/p)$. This is good enough for our purposes. We do not discuss the errors which follow from the fact that the expressions in the previous formulas depend on θ_2.

Domain B: The smallness of $\beta_p^{(1)}(Y,s)$ in this case follows easily from several inequalities and arguments.

1°: $|Y|\le D_4\sqrt{pk^{(0)}}$ because $|k|\le D_5pk^{(0)}$.
2°: $|Y-Y'|\le D_6\sqrt{k^{(0)}}$ because $v(k,0;b)$ has a compact support.
3°: If $|Y-Y'|\le 2s_+/\sqrt{p}$ (recall that $S=[S_-,S_+]$, we write here $s_+=S_+$), then

$$\exp\{-|k^{(0)}+(Y-Y')\sqrt{p+1}|^2\}\le 1.$$

If $|Y-Y'|\ge 2s_+/\sqrt{p}$ then

$$\exp\{-|k^{(0)}+(Y-Y')\sqrt{p+1}|^2\}\le\exp\left\{-\frac{s_+}{4}|Y-Y'|^2\right\}.$$

4°: If $|Y'|\ge D_7\sqrt{p}$ then

$$\exp\left\{-\frac{\theta_2}{p^2}|k^{(0)}p+Y'\sqrt{p+1}|^2\right\}\le\exp\{-D_8\theta_2\}.$$

5°: If $|Y'|\le D_7\sqrt{p}$ then

$$\exp\left\{-\frac{\theta_2}{p^2}|k^{(0)}p+Y'\sqrt{p+1}|^2\right\}\le 1.$$

6°: We have

$$\exp\left\{-\frac{1}{2}(|Y_1'|^2+|Y_2'|^2)-\frac{1}{2}|Y_3'|^2\right\}$$
$$=\exp\left\{-\frac{1}{2}(|Y_1-(Y_1-Y_1')|^2+|Y_2-(Y_2-Y_2')|^2)-\frac{1}{2}|Y_3-(Y_3-Y_3')|^2\right\}$$
$$=\exp\left\{-\frac{1}{2}(|Y_1|^2+|Y_2|^2)-\frac{1}{2}(|Y_3|^2)\right\}$$
$$\cdot\exp\left\{(Y_1(Y_1-Y_1')+Y_2(Y_2-Y_2'))+Y_3(Y_3-Y_3')\right.$$
$$\left.-\frac{1}{2}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)-\frac{1}{2}|Y_3-Y_3'|^2\right\}.$$

If $|Y-Y'|\le 2s_+/\sqrt{p}$ then

$$\exp\left\{(Y_1(Y_1-Y_1')+Y_2(Y_2-Y_2'))+Y_3(Y_3-Y_3')\right.$$
$$\left.-\frac{1}{2}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)-\frac{1}{2}|Y_3-Y_3'|^2\right\}\le D_8.$$

If $|Y-Y'|>2s_+/\sqrt{p}$ then we have an integral of a function which is the product of some Gaussian factor and $|\mathcal{H}^{(p)}(Y)|$. A direct estimate shows as before that in this case

$$|\beta_p^{(1)}(Y,s)|\le\Lambda_p(s)pe^{-\frac{1}{2}(|Y_1|^2+Y_2|^2)}e^{-\frac{1}{2}|Y_3|^2}\frac{D_8}{p^{3/2}},$$

which is also good for us.

In the same way one can estimate terms with relatively small p_1 and $p-p_1$ (i.e., $p_1\le\sqrt{p}$ or $p_1\ge p-\sqrt{p}$). The remainders will be of order $(1/\sqrt{p_1})(1/p)$. The next set of remainders comes from splitting the integration over θ and Y' (see (7) and beginning of §3). We may assume that $p_1>\sqrt{p}$ or $p_1<p-\sqrt{p}$ because other terms were estimated before. Put

$$\tilde{\tilde{g}}_{p+1}(Y,s)=(p+1)^{5/2}\sum_{\substack{p_1+p_2=p_{+1}\\ p_1,p_2>\sqrt{p}}}\int_0^{p_1^2}d\theta_1\int_0^{p_2^2}d\theta_2\,\frac{1}{p_1^2p_2^2}$$
$$\cdot\int_{\mathbb{R}^3}\left\langle\tilde{g}_{p_1}\left((Y-Y')\frac{(1-\theta_1/p_1^2)^{1/2}}{\sqrt{\gamma}},\left(1-\frac{\theta_1}{p_1^2}\right)s\right),k^{(0)}+\frac{Y}{\sqrt{p+1}}\right\rangle$$
$$\cdot P_{k^{(0)}+\frac{Y}{\sqrt{p+1}}}\tilde{g}_{p_2}\left(\frac{Y'(1-\theta_2/p_2^2)^{1/2}}{\sqrt{1-\gamma}},\left(1-\frac{\theta_2}{p_2^2}\right)s\right)$$
$$\cdot\exp\left\{-\theta_1\left|k^{(0)}+\frac{Y-Y'}{\sqrt{p+1}\,\gamma}\right|^2-\theta_2\left|k^{(0)}+\frac{Y-Y'}{\sqrt{p+1}\,\gamma}\right|^2\right\}.$$

Using the inductive assumption we can rewrite the last expression as follows:

$$\begin{aligned}\tilde{\tilde{g}}_{p+1}(Y,s) = (p+1) \sum_{\substack{p_1+p_2=p+1\\ p_1,p_2>\sqrt{p}}} \int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2 \\ \cdot \Lambda_{p_1}(s)\Lambda_{p_2}(s)\frac{1}{\gamma(1-\gamma)}\frac{1}{p+1}\exp\left\{-\frac{1}{2}\frac{|Y_1-Y_1'|^2+|Y_2-Y_2'|^2}{\gamma}\right. \\ \left.-\frac{1}{2}\frac{|Y_3-Y_3'|^2}{\gamma}-\frac{1}{2}\frac{|Y_1'|^2+|Y_2'|^2}{(1-\gamma)}-\frac{1}{2}\frac{|Y_3'|^2}{1-\gamma}\right\} \\ \cdot p^{1/2}\left\langle \mathcal{H}^{(p_1)}\left(Y-Y', s\left(1-\frac{\theta_1}{p_1^2}\right)\right), \sqrt{s}k^{(0)}+\frac{Y}{\sqrt{p}}\right\rangle \\ \cdot P_{\sqrt{s}k^{(0)}+Y/\sqrt{p}}\mathcal{H}^{(p_2)}\left(Y', s\left(1-\frac{\theta_2}{p_2^2}\right)\right) d^3Y'.\end{aligned}$$

As explained before, in the domain A due to incompressibility, the inner product

$$\left\langle \mathcal{H}^{(p_1)}\left(Y-Y'; s\left(1-\frac{\theta_1}{p_1^2}\right)\right), \sqrt{s}k^{(0)}+\frac{Y}{\sqrt{p}}\right\rangle$$

takes values $O(1/\sqrt{p})$ because the first two components of the vector $\sqrt{s}k^{(0)}+Y/\sqrt{p}$ are of order $O(1/\sqrt{p})$. Therefore the product

$$\sqrt{p}\left\langle \mathcal{H}^{(p_1)}\left(Y-Y', s\left(1-\frac{\theta_1}{p_1^2}\right)\right), \sqrt{s}k^{(0)}+\frac{Y}{\sqrt{p}}\right\rangle$$

takes values of order $O(1)$.

The remainder can be written in the following form:

$$\begin{aligned}\beta_p^{(2)}(Y,s) = \sum_{\substack{p_1+p_2=p+1\\ p_1,p_2>\sqrt{p}}} \frac{1}{\gamma(1-\gamma)}\frac{1}{p}\int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2 \\ \cdot \Lambda_{p_1}(s)\Lambda_{p_2}(s)\frac{1}{\Lambda_p(s)}\int_{\mathbb{R}^3}\exp\left\{-\frac{|Y_1-Y_1'|^2+|Y_2-Y_2'|^2}{2\gamma}\right. \\ \left.-\frac{1}{2\gamma}\frac{|Y_3-Y_3'|^2}{2\gamma}-\frac{|Y_1'|^2+Y_2'|^2}{2(1-\gamma)}-\frac{|Y_3'|^2}{2(1-\gamma)}\right\} \\ \cdot \left\langle \mathcal{H}^{(p_1)}\left(Y-Y', s\left(1-\frac{\theta_1}{p_1^2}\right)\right), \sqrt{s}k^{(0)}+\frac{Y}{\sqrt{p}}\right\rangle \\ \cdot P_{\sqrt{s}k^{(0)}+Y/\sqrt{p}}\mathcal{H}^{(p_2)}\left(Y', \left(1-\frac{\theta_2}{p_2^2}\right)s\right) \\ \cdot \exp\left\{-\theta_1\left|\sqrt{s}k^{(0)}+\frac{Y-Y'}{\sqrt{p}\gamma}\right|^2-\theta_2\left|\sqrt{s}k^{(0)}+\frac{Y'}{\sqrt{p}(1-\gamma)}\right|^2\right\}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{p_1+p_2=p+1\\ p_1,p_2>1}} \frac{1}{\gamma(1-\gamma)}\frac{1}{p}\int_0^{p_1^2} \exp\{-\theta_1 s\}\,d\theta_1 \int_0^{p_2^2} \exp\{-\theta_2 s\}\,d\theta_2 \\
&\cdot \int_{\mathbb{R}^3} \exp\biggl\{-\frac{|Y_1-Y_1'|^2+|Y_2-Y_2'|^2}{2\gamma} - \frac{|Y_3-Y_3'|^2}{2\gamma} \\
&- \frac{|Y_1'|^2+Y_2'|^2}{2(1-\gamma)} - \frac{|Y_3'|^2}{2(1-\gamma)}\biggr\} \\
&\cdot p^{1/2}\Bigl\langle \mathcal{H}^{(p_1)}(Y-Y'), \sqrt{s}k^{(0)} + \frac{Y}{\sqrt{p}}\Bigr\rangle P_{\sqrt{s}k^{(0)}+Y/\sqrt{p}}\mathcal{H}^{(p_2)}(Y',s)\,d^3Y'.
\end{aligned}
$$

We did not include the factor $\Lambda^{p-1}p$ because it is a part of the inductive assumption. This remainder is estimated in the following way.

First we consider

$$
R_1 = \left(\left|\sqrt{s}k^{(0)} + \frac{Y-Y'}{\sqrt{p}\gamma}\right|^2 - s\right) + \left(\left|\sqrt{s}k^{(0)} + \frac{Y'}{\sqrt{p}(1-\gamma)}\right|^2 - s\right).
$$

As before, consider the domain where

$$
|Y-Y'| \le D_9\sqrt{\ln(pk^{(0)})}, \qquad |Y'| \le D_{10}\sqrt{\ln(pk^{(0)})}.
$$

We write

$$
R_1 = \frac{|Y-Y'|^2}{p\gamma_1^2} + \frac{|Y'|^2}{p\gamma_2^2} + D_{11}\left(\frac{|Y-Y'|}{\sqrt{p}\gamma} + \frac{|Y'|}{\sqrt{p}(1-\gamma)}\right).
$$

In the domain A,

$$
|R_1| \le \frac{D_{12}\ln(pk^{(0)})}{pk^{(0)}}.
$$

Therefore

$$
\begin{aligned}
R_2 =\ & \exp\left\{-\theta_1\left|\sqrt{s}k^{(0)} + \frac{Y-Y'}{\sqrt{p}\gamma}\right|^2 - \theta_2\left|\sqrt{s}k^{(0)} + \frac{Y'}{\sqrt{p}\gamma^2}\right|^2\right\} \\
& - \exp\{-\theta_1 s\}\exp\{-\theta_2 s\} \\
=\ & \exp\{-(\theta_1+\theta_2)s)\}\left[\exp\left\{-\theta_1\left(\left|\sqrt{s}k^{(0)} + \frac{Y-Y'}{\sqrt{p}\gamma}\right|^2 - s\right)\right.\right. \\
& \left.\left.\cdot \exp\left\{-\theta_2\left(\left|\sqrt{s}k^{(0)} + \frac{Y'}{\sqrt{p}(1-\gamma)}\right|^2 - s\right)\right\} - 1\right]
\end{aligned}
$$

and in the domain A,

$$
|R_2| \le \exp\{-(\theta_1+\theta_2)s\}\left(\frac{\theta_1 D_{13}}{\sqrt{p}\,\gamma} + \frac{\theta_2 \ln p}{\sqrt{p}(1-\gamma)}\right).
$$

This shows that in the domain A we can replace the exponent

$$\exp\left\{-\theta_1\left|\sqrt{s}k^{(0)}+\frac{Y-Y'}{\sqrt{p}\gamma}\right|^2-\theta_2\left|\sqrt{s}k^{(0)}+\frac{Y'}{\sqrt{p}(1-\gamma)}\right|^2\right\}$$

by $\exp\{-(\theta_1+\theta_2)s(k^{(0)})^2\}$ and the remainder will be no more than $D_{14}\ln p/\sqrt{p}$. This is enough for our purposes. In the domain B the estimates are similar because again the main contribution to the integral comes from $|Y-Y'|\le D_9\sqrt{\ln p}$, $|Y'|\le D_{10}\sqrt{\ln p}$. In other words, in the domain B we can replace the product of the Gaussian factors and $\mathcal{H}^{(p)}$ by

$$\exp\left\{-\frac{1}{2}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)-\frac{1}{2}|Y_3-Y_3'|^2-\frac{1}{2}(|Y_1'|^2+|Y_2'|^2)-\frac{1}{2}|Y_3'|^2\right\}.$$

This is also enough for our purposes.

The next remainder of Type I comes from the difference between the sum over γ and the corresponding integral. The remainder $\beta_p^{(3)}(Y,s)$ is the difference between the sum

$$\sum_{\substack{p_1+p_2=p+1\\ p_1,p_2>\sqrt{p}}}\sqrt{\gamma}\sqrt{(1-\gamma)}\frac{1}{p}\int_{\mathbb{R}^3}\exp\left\{-\frac{(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)}{2\gamma}\right.$$
$$\left.-\frac{(|Y_3-Y_3'|)^2}{2\gamma}-\frac{|Y_1'|^2+|Y_2'|^2}{2(1-\gamma)}-\frac{|Y_3'|^2}{2(1-\gamma)}\right\}\left(\frac{1}{2\pi\gamma}\right)^{3/2}\left(\frac{1}{2\pi(1-\gamma)}\right)^{3/2}$$
$$\cdot\, p^{1/2}\left\langle\mathcal{H}^{(p_1)}((Y-Y')),\sqrt{s}k^{(0)}+\frac{Y}{\sqrt{p}}\right\rangle P_{\sqrt{s}k^{(0)}+Y/\sqrt{p}}\mathcal{H}^{(p_2)}(Y',s)\,d^3Y'$$

and the corresponding integral over γ from 0 to 1. It is easy to check that this difference is not more than $D_{15}/\sqrt{p}$.

8.2. The remainders of Type II and III

All remainders of Type II appear because we use sums (over p_1) instead of integrals. We use a linear interpolation to define $\delta(\gamma,Y,s)$ for all γ. From our inductive assumptions it follows that $|\delta_p(\gamma,Y,s)|\le D_{16}/\sqrt{p}$. Therefore, the remainders which follow from the difference between the sum and the integral also satisfy this estimate.

It remains to consider the quadratic expressions of $\delta_p(\gamma,Y,s)$. The Gaussian density is present in all these expressions. Therefore, all the remainders are not more than D_{17}/p.

9. Final steps in the proof of the main result. Formulation of the main theorem

In this section we consider our procedure for $p>p_0=N$. Introduce the sequence $p_m=(1+\epsilon)p_{m-1}=(1+\epsilon)^m p_0$, $m>0$, where $\epsilon>0$ is small (see below). These are the values of p when we make the change of parameters, i.e. rescaling. For $p\neq p_m$, no changes are done.

In §7 the choice of the fixed point $H^{(0)}$ was explained and the corresponding functions $\Phi_j^{(u)}$, $\Phi_{j'}^{(n)}$ were introduced. Also we have the stable subspace of our linearized map. Consider p with $p_m < p < p_{m+1}$, $m \geq 0$. By induction, assume that we have an interval $S^{(m)} = [S_-^{(m)}, S_+^{(m)}]$ on the time axis such that for all $s \in S^{(m)}$, $r < p$, we have the representation (see also (44) for the definition of $\Lambda(s)$)

$$\tilde{g}_r(Y,s) = Z(s)\Lambda(s)^r r(H^{(0)}(Y) + \delta^{(r)}(Y,s)) \cdot \frac{1}{2\pi}\exp\left\{-\frac{Y_1^2+Y_2^2}{2}\right\}\sqrt{\frac{1}{2\pi}}\exp\left\{-\frac{Y_3^2}{2}\right\}, \quad r < p.$$

If $\gamma = r/(p-1)$, then

$$\delta^{(r)}(Y,s) = \sum_{j=1}^{4}(b_{j,p-1}^{(u,m)} + \beta_{j,r}^{(u)})\gamma^{\alpha_j^{(u)}}\Phi_j^{(u)}(Y) + \sum_{j'=1}^{6}(b_{j',p-1}^{(n,m)} + \beta_{j',r}^{(n)})\Phi_j^{(n)}(Y) + \Phi^{(st)}(Y,\gamma).$$

Note that in this expression $b_{j,p-1}^{(u,m)}$, $b_{j',p-1}^{(n,m)}$ do not depend on r. The terms $\beta_{j,r}^{(u)}$, $\beta_{j',r}^{(n)}$ are small corrections to the main terms $b_{j,p-1}^{(u,m)}$, $b_{j',p}^{(n,m)}$ and are also functions of $b_j^{(u,m)}$, $b_{j'}^{(n,m)}$. Our next inductive assumption says that $b_{j,p_m}^{(u,m)}$, $b_{j',p_m}^{(n,m)}$ satisfy the inequalities

$$-B_1\rho_1^m \leq b_{j,p_m}^{(u,m)}, b_{j',p_m}^{(n,m)} \leq B_1\rho_1^m$$

where B_1 is a positive constant. We can take $B_1 = 2$. The inductive assumption concerning the corrections $\beta_{j,r}^{(u)}$, $\beta_{j',r}^{(n)}$ says that

$$|\beta_{j,r}^{(u)}|, |\beta_{j',r}^{(n)}| \leq B_2\rho_2^m \quad \text{for all } 1 \leq j_1 \leq 4,\ 1 \leq j_1' \leq 6,$$

$$\left|\frac{\partial\beta_{j,r}^{(u)}}{\partial b_{j_1,r}^{(u,m)}}\right|, \left|\frac{\partial\beta_{j,r}^{(u)}}{\partial b_{j_1',r}^{(n,m)}}\right|, \left|\frac{\partial\beta_{j',r}^{(n)}}{\partial b_{j_1,r}^{(u,m)}}\right|\left|\frac{\partial\beta_{j',r}^{(n)}}{\partial b_{j_1',r}^{(n,m)}}\right| \leq B_3\rho_2^m.$$

Here $0 < \rho_2 < \rho_1$ and B_2, B_3 are other constants.

The function $\Phi^{(st)}(Y,\gamma)$ belongs to the stable subspace of the linearized semigroup (see §6) and satisfies the inequality

$$\|\Phi^{(st)}(Y,\gamma)\|_X \leq B_4\rho_3^m,$$

where $B_4 > 0$ and $0 < \rho_3 < 1$ are constants. At one step of our procedure $p-1$ is replaced by p, γ is replaced by $\gamma' = \gamma(p-1)/p$ and $\gamma^{\alpha_j^{(u)}}$ is replaced by $\left(1+\frac{1}{p-1}\right)^{\alpha_j^{(u)}}(\gamma')^{\alpha_j^{(u)}}$, $b_j^{(u,m)} + \beta_{j,r}^{(u)}$ is replaced by $(b_j^{(u,m)} + \beta_{j,r}^{(u)})\left(1+\frac{1}{p-1}\right)^{\alpha_j^{(u)}}$. At the end of the interval $p_m < p < p_{m+1}$ the variable $b_{j,p_m}^{(u,m)}$ acquires the factor

$$\prod_{p_m<p<p_{m+1}}\left(1+\frac{1}{p-1}\right)^{\alpha_j^{(u)}} \approx (1+\epsilon)^{\alpha_j^{(u)}}.$$

For the neutral part of the spectrum the variable $b_{j',p_m}^{(n,m)}$ remains the same because $\alpha_{j'}^{(n)} = 0$. The stable part is contracting.

Now we discuss $\delta^{(p)}(Y, 1)$ using (7). As in §7, $\delta^{(p)}(Y, 1)$ consists of three parts.

Part I. In all $\delta^{(r)}$, $r < p$, the main term is the one which contains the basic parameters $b^{(u,m)}_{j,p}$, $b^{(n,m)}_{j',p}$. Consider the terms in (7) which are linear in $b^{(u)}_{j,p-1}$, $b^{(n)}_{j',p-1}$. As follows from the definition of the linearized group and its spectrum, for unstable eigenvectors we get the factor $\left(1+\frac{1}{p-1}\right)^{\alpha^{(u)}_j} b^{(u,m)}_{j,p-1}$. For the neutral part we get the factor 1. We put $b^{(u,m)}_{j,p} = b^{(u,m)}_{j,p-1}\left(1+\frac{1}{p}\right)^{\alpha^{(u)}_j}$, $b^{(n,m)}_{j',p} = b^{(n)}_{j',p-1}$. The vector corresponding to the stable subspace is transformed accordingly.

Part II. All remainders which arise because the formulas for finite p are different from the limiting formulas. These remainders were discussed in §6. The result is written as a linear combination of $\Phi^{(u)}_j$, $\Phi^{(n)}_{j'}$ and a vector from the stable subspace. The corresponding terms are included in $\beta^{(u)}_{j,p}$, $\beta^{(n)}_{j',p}$ and the function from the stable subspace.

Part III. The term which is the sum of all quadratic functions of all $\delta^{(r)}$. Again we expand it using the functions $\Phi^{(u)}_j$, $\Phi^{(n)}_{j'}$ and the stable subspace. The result is included in $\beta^{(u,m)}_{j,p}$, $\beta^{(n,m)}_{j',p}$ and $\Phi^{(st)}_p(Y)$ from the stable subspace.

Finally, we have

$$b^{(u,m)}_{j,p} = b^{(u,m)}_{j,p-1}\left(1+\frac{1}{p}\right)^{\alpha^{(u)}_j}, \quad b^{(n,m)}_{j,p} = b^{(n,m)}_{j,p-1}$$

and the formulas for $\beta^{(u,m)}_{j,p}$, $\beta^{(n,m)}_{j',p}$ and $\Phi^{(st)}_p(Y)$.

This procedure is used until $p < p_{m+1}$. When $p = p_{m+1}$, then in addition we make rescaling and introduce new variables

$$b^{(u,m+1)}_{j,p_{m+1}} = b^{(u,m)}_{j,p_{m+1}} + \beta^{(u,m)}_{j,p_{m+1}}, \qquad b^{(n,m+1)}_{j',p_{m+1}} = b^{(n,m)}_{j',p_{m+1}} + \beta^{(n,m)}_{j',p_{m+1}}.$$

Let $\Delta^{(m+1)}_{m+1} = [-B_1\rho_1^{m+1}, B_1\rho_1^{m+1}]$ and

$$\Delta^{(m+1)}_m = \{(b^{(u,m)}_{j,p_m}, b^{(n,m)}_{j',p_m}) : \; -B_1\rho_1^{m+1} \le b^{(u)}_{j,p_{m+1}}, b^{(n)}_{j',p_{m+1}} \le B_1\rho_1^{m+1}\}.$$

It follows easily from the estimates of $\beta^{(u,m+1)}_{j,p_{m+1}}$, $\beta^{(n,m+1)}_{j',p_{m+1}}$ that $\Delta^{(m+1)}_m \subseteq \Delta^{(m)}_m$. If

$$\Delta^{(m)}_0 = \{(b^{(u)}_j, b^{(n)}_{j'}) : (b^{(u,m)}_{j,m}, b^{(n,m)}_{j',m}) \in \Delta^{(m)}_m\},$$

then $\Delta^{(m)}_0$ is a decreasing sequence of closed intervals. The intersection $\bigcap_m \Delta^{(m)}_0$ gives us the values of parameters for which $\delta^{(p)} \to 0$ as $p \to \infty$.

We also make some shortening of the time interval $S^{(m)}$. In the formulas for $\delta^{(r)}$ there are several remainders which appear because we replace in all expressions s' and s'' by s. We estimate these remainders using the fact that our functions satisfy with respect to the time variable the Lipschitz condition with respect to the time variable. The maxima of these functions decay as some power of p. We choose the interval $S^{(m+1)} \subset S^{(m)}$ so that for all $s \in S^{(m+1)}$ the basic inclusion $\Delta^{(m+1)}_m \subset \Delta^{(m)}_m$ remains valid. The difference $S^{(m)} \setminus S^{(m+1)}$ consists of two intervals whose lengths decay exponentially with m. Therefore $\bigcap_m S^{(m)} = [S_-, S_+]$ is an interval of positive length.

The transformation $(b^{(u,m+1)}_{j,p_{m+1}}, b^{(n,m+1)}_{j',p_{m+1}}) \to (b^{(u,m)}_{j,p_m}, b^{(n,m)}_{j',p_m})$ is given by smooth functions and is close to the identity map. The last step in the renormalization procedure is the replacement in all $\delta^{(r)}$, $r < p_{m+1}$, of the variables $b^{(u,m)}_{j,p_m}, b^{(n,m)}_{j',p_m}$ by their expressions through $b^{(u,m+1)}_{j,p_{m+1}}, b^{(n,m+1)}_{j',p_m+1}$. The form of $\delta^{(r)}$ in the new variables remains essentially the same.

The choice of constants

The main constants which are used in the construction are the following:

1. $k^{(0)}$ determines the center of the domain where $v(k, 0)$ is concentrated;
2. D_1 is the constant which determines the size of the neighborhood where $v(k, 0)$ is concentrated;
3. ρ_1, B_1 determine the size of the intervals where $b^{(u)}_j, b^{(n)}_{j'}$ vary;
4. ρ_2, B_2 determine the upper bounds of the perturbations $\beta^{(u)}_{j,r}, \beta^{(n)}_{j',r}$;
5. λ_1 determines the power of decay of g_r in the domain B;
6. N is the number of steps where the procedure was done numerically;
7. ϵ determines the values of p where the renormalization is done.

The value of $k^{(0)}$ should be sufficiently large. The constant B_1 should be small but $\rho_1 < 1$ should not be too small in order to make the corrections coming from the quadratic part of our formulas smaller than the main term in the linear part. Moreover, it cannot be too small in order that we could choose the next interval $\Delta^{(m+1)}$. The parameter λ_1 is a function of D_1. The value of D_1 determines the estimates in the domain B which decay as $1/(k^{(0)})^{\lambda_1}$. The value of ϵ is chosen so small that we can write with a good precision the action of the linearized renormalization group.

Now we formulate the main result of this paper.

Theorem 1 (Main Theorem). *Take a* 10*-parameter family of initial conditions described in* §7 *with all constants satisfying the needed inequalities. Then one can find an interval* $S = [S_-, S_+]$*, functions* $Z(s)$*,* $\Lambda(s)$*, and values* $b^{(u)}_j = b^{(u)}_j(s)$*,* $b^{(n)}_{j'} = b^{(n)}_{j'}(s)$ *of parameters so that*

(a$_1$) *For* $Y = (Y_1, Y_2, Y_3)$*,* $|Y| \le D_1\sqrt{pk^{(0)}}$*,*

$$\begin{aligned}\tilde{g}_p(Y, s) &= g_p(k^{(p)} + \sqrt{pk^{(0)}}\,Y, s)\\ &= pZ(s)\Lambda(s)^p \exp\left(-\frac{Y_1^2 + Y_2^2 + Y_3^2}{2}\right)(H^{(0)}_1(Y_1, Y_2)\\ &\quad + \delta^{(p)}_1(Y, s), H^{(0)}_2(Y_1, Y_2) + \delta^{(p)}_2(Y, s), \delta^{(p)}_3(Y, s))\end{aligned}$$

and $\sup_{Y,j} |\delta^{(p)}_j(Y, s)| \to 0$ *as* $p \to \infty$*. Here* $H^{(0)}(Y_1, Y_2)$ *is the fixed point of our renormalization group for which* $x_1 = x_2 = x_3 = 0$*.*

(a$_2$) *For* $Y = (Y_1, Y_2, Y_3)$, $|Y| > D_1\sqrt{pk^{(0)}}$,

$$|\tilde{g}_p(Y, s)| \le \frac{1}{(pk^{(0)})^{\lambda_1}}.$$

The function $\Lambda(s)$ *is strictly increasing on* S. *Moreover, for* $s \in S$, *we have*

$$\Lambda'(s) \ge B > 0$$

where $B > 0$ *is another constant independent of* s.

10. Critical value of parameters and behavior of solutions near the singularity point

We return to the first formulas:

$$v_A(k, t) = \exp\{-t|k|^2\}A \cdot v(k, 0) + \int_0^t \exp\{-(t-s)|k|^2\} \sum_{p>1} A^p g_p(k, s)\, ds. \quad (46)$$

Take $t \in [S_-, S_+]$ and find the values of the parameters $b_j^{(u)}$, $b_{j'}^{(n)}$ for which the main theorem holds. Put $A_{\mathrm{cr}}(t) = \Lambda(t)^{-1}$. If so then $A^p g_p(k, t)$ is concentrated in the domain with center at $\kappa^{(0)}p/\sqrt{t}$ having the size $O(\sqrt{p})$ and there it takes values $O(p)$. This immediately implies that at t the energy is infinite.

Consider $t' < t$ and write $\Delta t = t - t'$. It follows from the properties of $\Lambda(s)$ (see the formulation of the main theorem) that $\Lambda(t')/\Lambda(t) = 1 - B\Delta t + O(\Delta t)$ for some constant $B > 0$. Since $A_{\mathrm{cr}}^p \Lambda(t')^p = A_{\mathrm{cr}}^p \Lambda(t)^p (\Lambda(t')/\Lambda(t))^p = (1 - B\Delta t + o(\Delta t))^p$, it is clear that the terms in (46) with $p \le O(1/\Delta t)$ are close to the limiting terms corresponding to t. For $p \gg O(1/\Delta t)$ the product $A_{\mathrm{cr}}^p \Lambda(t')^p$ tends exponentially to zero and dominates the other terms in the expression for g_p. Therefore for $t' < t$ both the energy and the enstrophy are finite.

In the domain $|k| \le O(1/\Delta t)$, the solutions grow as $|k|^{3/2}$. The extra factor $|k|^{1/2}$ appears because for any k the values of p for which the terms in (46) give the essential contribution to the solution belong to an interval of size $O(\sqrt{|k|}) = O(\sqrt{p})$. From this argument one can easily derive that $E(t') = O(1)/(\Delta t)^5$ and $\Omega(t') = O(1)/(\Delta t)^7$.

Appendix: Hermite polynomials and their basic properties

Take $\sigma > 0$ and write

$$\mathrm{He}_n^{(\sigma)}(x) = (-1)^n e^{\sigma x^2/2} \frac{d^n}{dx^n} e^{-\sigma x^2/2}, \quad n \ge 0.$$

It is clear that $\mathrm{He}_n^{(\sigma)}(x) = \sigma^n x^n + \cdots$, where dots mean terms of smaller degree. We shall call $\mathrm{He}_n^{(\sigma)}$ the n-th *Hermite polynomial*. It is clear that $\mathrm{He}_0^{(\sigma)}(x) = 1$, $\mathrm{He}_1^{(\sigma)}(x) = \sigma x$, $\mathrm{He}_2^{(\sigma)}(x) = \sigma^2 x^2 - \sigma$ and so on. In general, $\mathrm{He}_n^{(\sigma)}(x) = \sigma^{n/2}\,\mathrm{He}_n^{(1)}(\sqrt{\sigma} x)$. It is easy to check that

$$\sigma x\, \mathrm{He}_n^{(\sigma)}(x) = \mathrm{He}_{n+1}^{(\sigma)}(x) + \sigma n\, \mathrm{He}_{n-1}^{(\sigma)}(x). \quad (47)$$

The Fourier transform of $\mathrm{He}_m^{(\sigma)}(x)e^{-\sigma x^2/2}\sqrt{\sigma/2\pi}$ is $(i\lambda)^m e^{-\lambda^2/2\sigma}$. This implies the formula for the convolution:

$$\int_{\mathbb{R}^1} \mathrm{He}_{m_1}^{(\sigma)}(x-y)e^{-\sigma(x-y)^2/2}\sqrt{\frac{\sigma}{2\pi}}\,\mathrm{He}_{m_2}^{(\sigma)}(y)e^{-\sigma y^2/2}\sqrt{\frac{\sigma}{2\pi}}\,dy = \mathrm{He}_{m_1+m_2}^{(\sigma/2)}(x)e^{-\sigma x^2/2}\sqrt{\frac{\sigma}{4\pi}}. \tag{48}$$

Take positive γ_1, γ_2 with $\gamma_1 + \gamma_2 = 1$ and consider the convolution of $\mathrm{He}_{m_1}^{(\sigma)}(x/\sqrt{\gamma_1})e^{-\sigma x^2/2\gamma_1}\sqrt{\sigma/2\pi\gamma_1}$ and $\mathrm{He}_{m_2}^{(\sigma)}(x/\sqrt{\gamma_2})e^{-\sigma x^2/2\gamma_2}\sqrt{\sigma/2\pi\gamma_2}$. Their Fourier transforms are $(i\lambda\sqrt{\gamma_1})^{m_1}e^{-\lambda^2\gamma_1/2\sigma}$ and $(i\lambda\sqrt{\gamma_2})^{m_2}e^{-\lambda^2\gamma_2/2\sigma}$ respectively. The product of these two functions is $\gamma_1^{m_1/2}\gamma_2^{m_2/2}(i\lambda)^{m_1+m_2}e^{-\lambda^2/2\sigma}$. Therefore the convolution is $\gamma_1^{m_1/2}\gamma_2^{m_2/2}\,\mathrm{He}_{m_1+m_2}^{(\sigma)}(x)e^{-\sigma x^2/2}$.

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