J. Eur. Math. Soc. 10, 399–413

Y.-P. Lee

Invariance of tautological equations I: conjectures and applications

Received September 18, 2005 and in revised form November 3, 2006

Abstract. The main goal of this paper is to introduce a set of conjectures on the relations in the tautological rings. In particular, this framework gives an efficient algorithm to calculate all tautological equations using only finite-dimensional linear algebra. Other applications include the proofs of *Witten's conjecture* on the relations between higher spin curves and Gelfand–Dickey hierarchy and *Virasoro conjecture* for target manifolds with conformal semisimple quantum cohomology, both for genus up to two.

1. Introduction

1.1. The tautological rings of moduli spaces of curves

Two basic references for tautological rings are [16] and [26].

Let $\overline{M}_{g,n}$ be the moduli stacks of stable curves. $\overline{M}_{g,n}$ are proper, irreducible, smooth Deligne–Mumford stacks. The Chow rings $A^*(\overline{M}_{g,n})$ over $\mathbb{Q}$ are isomorphic to the Chow rings of their coarse moduli spaces. The tautological rings $R^*(\overline{M}_{g,n})$ are subrings of $A^*(\overline{M}_{g,n})$, or subrings of $H^{2*}(\overline{M}_{g,n})$ via cycle maps, generated by some "geometric classes" which will be described below.

Convention 1. All Chow/cohomology/tautological rings are over $\mathbb{Q}$.

The first type of geometric classes are the *boundary strata*. $\overline{M}_{g,n}$ have natural stratification by topological types. The strata can be conveniently presented by their (dual) graphs, which can be described as follows. To each stable curve C with marked points, one can associate a dual graph Γ. Vertices of Γ correspond to irreducible components. They are labeled by their geometric genus. Draw an edge joining two vertices each time the two components intersect. For each marked point, one draws a half-edge incident to the vertex, with the same label as the point. Now, the stratum corresponding to Γ is the closure of the subset of all stable curves in $\overline{M}_{g,n}$ which have the same topological type as C.

Research partially supported by NSF and an AMS Centennial Fellowship.

Y.-P. Lee: Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, U.S.A.; e-mail: yplee@math.utah.edu

The second type of geometric classes are the Chern classes of tautological vector bundles. These include cotangent classes ψ_i, Hodge classes λ_k and κ-classes κ_l. (See [16], [25].)

To give a precise definition of the tautological rings, some natural morphisms between moduli stacks of curves will be used. The *forgetful morphisms*

$$\mathrm{ft}_i : \overline{M}_{g,n+1} \to \overline{M}_{g,n} \tag{1}$$

forget one of the $n+1$ marked points. The *gluing morphisms*

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}, \quad \overline{M}_{g-1,n+2} \to \overline{M}_{g,n} \tag{2}$$

glue two marked points to form a curve with a new node. Note that the boundary strata are the images (of the repeated applications) of the gluing morphisms, up to factors in $\mathbb{Q}$ due to automorphisms.

Definition 1. *The system of* tautological rings $\{R^*(\overline{M}_{g,n})\}_{g,n}$ *is the smallest system of* $\mathbb{Q}$*-unital subalgebras (containing classes of type one and two, and) closed under the forgetful and gluing morphisms.*

Remarks. (i) The phrase "(containing classes of type one and two, and)" can be removed from the definition as the type one classes can be generated by the fundamental classes of the boundary strata (units), and the type two classes can be generated by the type one classes under the natural morphisms.

(ii) The set of Hodge classes can be reconstructed *linearly* from the set of ψ-classes, κ-classes and boundary strata. Therefore, the Hodge classes can be omitted in the discussion of tautological rings.

Since the tautological rings $\{R^*(\overline{M}_{g,n})\}_{g,n}$ are defined by generators, this leaves us the task of finding the *relations* in order to understand their structure. Mumford [24], Getzler [7] and Faber [5] are among the pioneers in this direction.

1.2. The conjectures

Each dual graph Γ can be decorated by assigning a monomial, or more generally a polynomial, ψ to each half-edge and κ-classes to each vertex. The tautological classes in $R^k(\overline{M}_{g,n})$ can be represented by $\mathbb{Q}$-linear combinations of *decorated graphs*, which will be called simply graphs by abusing the language.

Convention 2. The graphical notations used here are different from some authors', e.g. those in [8]. *The edges here are considered as "gluing" two half-edges*, which are all labeled. Therefore, the consideration of automorphisms will lead to a discrepancy in constant factors. For example, the cycle class represented *here* by a graph consisting of a closed loop attached to a vertex is *twice* the class represented by the same graph in [8]. The reason for this convention is that it simplifies the splitting principle considerably in terms of graphs.

Define the operations $\mathfrak{r}_l$ on the spaces of decorated graphs $\{\Gamma\}$.

- **Cutting edges.** Cut one edge and create two new half-edges. Label two new half-edges with $i, j \notin \{1, \ldots, n\}$ in two different ways. Produce a formal sum of four graphs by decorating with an extra ψ^l the i-labeled new half-edges with coefficient $1/2$ and by decorating with an extra ψ^l the j-labeled new half-edges with coefficient $(-1)^{l-1}/2$. (By "extra" decoration we mean that ψ^l is multiplied by whatever decorations already there.) Produce more graphs by repeating the above procedure on the other edges of the original graph. Retain only the stable graphs. Take the formal sum of these final graphs.
- **Genus reduction.** For each vertex, produce l graphs. Reduce the genus of the vertex by one, add two new half-edges. Label two new half-edges i, j and decorate them with ψ^{l-1-m}, ψ^m (respectively) where $0 \leq m \leq l-1$. Do this for all vertices, and retain only the stable graphs. Take the formal sum of these graphs with coefficient $\frac{1}{2}(-1)^{m+1}$.
- **Splitting vertices.** Split one vertex in two. Add one new half-edge to each of the two new vertices. Label them i, j and decorate them with ψ^{l-1-m}, ψ^m (respectively) where $0 \leq m \leq l-1$. Produce new graphs by splitting the genus g between the two new vertices (g_1, g_2 such that $g_1 + g_2 = g$), and distributing to the two new vertices the (old) half-edges which belong to the original vertex, in all possible ways. The κ-classes on the given vertex are split between the two new vertices in a way similar to the half-edges. That is, consider each monomial of the κ-classes $\kappa_{l_1}, \ldots, \kappa_{l_p}$ on the split vertex as labeled by p special half-edges. When the vertex splits, distribute the p special edges in all possible ways. Do this for all vertices, and retain only the stable graphs. Take the formal sum of these graphs with coefficient $\frac{1}{2}(-1)^{m+1}$.

Remarks. (i) It is not difficult to see that the two new half-edges are symmetric for l odd and anti-symmetric for l even.

(ii) The (output) graphs might be disconnected. The stable graph here mean that each connected component is stable (and of non-negative dimension).

(iii) In Section 3.1 these operations will be expressed in another (equivalent) terminology (of *gwi*'s).

(iv) The λ-classes do not enter the discussion as they can be reconstructed from other classes [24], [6]. To include them explicitly, one may apply the elementary splittings of the Hodge bundle

$$0 \to \mathbb{E}_{g-1} \to \iota^*\mathbb{E}_g \to \mathcal{O}_{\Delta_0} \to 0$$

to the genus-reduced vertices and

$$0 \to \mathbb{E}_{g'} \to \iota^*\mathbb{E}_g \to \mathbb{E}_{g-g'} \to 0$$

to the split vertices.

Definition 2. *Define operators $\mathfrak{r}_l(\Gamma)$ from the vector spaces of decorated graphs of codimension k on $\overline{M}_{g,n}$ to those of codimension $k+l-1$ on $\overline{M}^{\bullet}_{g-1,n+2}$ to be the final sum of the decorated graphs (with $\mathbb{Q}$-coefficients) produced from the above three operations.*

Here • stands for possibly disconnected curves. Note that the arithmetic genus for a disconnected curve is defined to be

$$g(C) := \sum_{i=1}^{d} g(C_i) - d + 1,$$

where the C_i are connected components of C, $C = \coprod_{i=1}^{d} C_i$.

Let

$$\sum_i c_i \Gamma_i = 0$$

be a tautological equation in codimension k classes in $\overline{M}_{g,n}$. The triple (g, n, k) will be used.

Conjecture 1.

$$\mathfrak{r}_l\Big(\sum_i c_i \Gamma_i\Big) = 0 \tag{3}$$

for all l, modulo the tautological equations for (g', n') for which

- $g' < g$, *or*
- $g' = g$ *and* $n' < n$.

The set of equations (3) will be called the *R-invariance equations*. The reason for this name comes from its relation to Gromov–Witten theory and is explained in [20]. See also [19] for a brief account.

Remark 1. Conjecture 1 is now proved in [20].

Conjecture 2. *Let $E = \sum_i c_i \Gamma_i$ be a given linear combination of codimension k tautological classes in $\overline{M}_{g,n}$ and $k < 3g - 3 + n$.*

If $\mathfrak{r}_l(E) = 0$ for all l, modulo tautological equations in $\overline{M}_{g',n'}$ for (g', n') satisfying the above inductive conditions, then $E = 0$ is a tautological equation.

Remark 2. (i) By a theorem of Graber and Vakil [14], when $k = 3g - 3 + n$, i.e. in top codimension,

$$R^{\mathrm{top}}(\overline{M}_{g,n}) = \mathbb{Q}.$$

Therefore $R^{\mathrm{top}}(\overline{M}_{g,n})$ are considered well understood and used as part of inductive data.

(ii) Conjecture 1 means that the linear operators at the level of decorated graphs actually descend to operators at the level of tautological classes. That is,

$$\mathfrak{r}_l : R^k(\overline{M}^{\bullet}_{g,n}) \to R^{k+l-1}(\overline{M}^{\bullet}_{g-1,n+2}) \tag{4}$$

is well-defined. Conjecture 2 further asserts that the combination of these operators is injective.

Conjecture 3. *Conjecture* 2 *produces* all *tautological equations inductively.*

1.3. Results in low genus

Main Theorem ([13, 1, 2]). *The Invariance Conjectures hold for genus zero, genus one, and* $(g, n, k) = (2, 1, 2), (2, 2, 2), (2, 3, 2), (3, 1, 3)$.

The genus zero case: $R^k(\overline{M}_{0,n}) = A^k(\overline{M}_{0,n})$ is essentially due to S. Keel [17]. There are two sets of tautological equations in genus zero, namely the genus zero topological recursion relations (TRR) and WDVV equations. WDVV equations are consequences of TRRs. A TRR is based upon the fact that $\psi_i = 0$ on $\overline{M}_{0,3}$. A WDVV equation is based upon the fact that $A^1(\overline{M}_{0,4}) = \mathbb{Q}$.

The non-trivial known tautological equations in $g \leq 3$ are

(1) $(g, n, k) = (1, 4, 2)$, Getzler's genus one equation ([7]).
(2) $(g, n, k) = (2, 1, 2)$, Mumford–Getzler's equation.
(3) $(g, n, k) = (2, 2, 2)$, Getzler's genus two equation ([8]).
(4) $(g, n, k) = (2, 3, 2)$, Belorousski–Pandharipande's equation ([3]).
(5) $(g, n, k) = (3, 1, 3)$, a new relation ([2], [18]).

The genus one case will be discussed in Section 3.

Genus two and three. The checking of cases (2)–(4) is carried out in [1]. The calculation of (5) via Conjecture 2 is done in [2]. An equivalent form of the $(3, 1, 3)$ equation was independently discovered by Kimura–Liu [18] using a completely different method (equivalent in the sense that they use different vectors in $R^3(\overline{M}_{3,1})$.) By Getzler's Betty number and Hodge polynomial calculations in [8] and [9], these equations are the only tautological equations in the above cases.

Remark 3. (i) Here the phrase "the only tautological equations" should be taken with a grain of salt. The forgetful and gluing morphisms (1), (2) induce relations in $R^k(\overline{M}_{g,n})$ from lower (inductive) classes. For example, Getzler's equation in $(g = 1, n = 4, k = 2)$ induces an equation in $(2, 2, 3)$ by gluing two of the marked points; any equations in $\overline{M}_{g,n}$ will induce equations in $\overline{M}_{g,n+m}$. These induced equations will be taken into inductive data as well. The goal, of course, is to find new equations.

(ii) It is easy to see that if one equation is R-invariant, all induced equations are.

1.4. Relations to Gromov–Witten theory

This is the topic for a sequel to this paper [20], so the discussion will be brief and necessarily not precise.

1.4.1. Motivation of the conjectures. The motivation of the above conjectures comes from a study of Givental's axiomatic Gromov–Witten theory [10]–[12]. Givental has discovered some remarkable structures of the "moduli spaces" of semisimple axiomatic Gromov–Witten theories (or Frobenius manifolds). The "moduli space" of a given rank (i.e. the dimension of the Frobenius manifolds) is a "homogeneus space" of a "quantized loop group".

(i) The tautological equations hold for all geometric Gromov–Witten theories due to the fact that there is a natural stabilization morphism

$$\mathrm{st} : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}$$

from the moduli spaces of stable maps to moduli spaces of stable curves. It is natural to expect that the tautological equations hold for all axiomatic Gromov–Witten theories. Therefore, the tautological equations should be "invariant" under the action of the "quantized loop groups". This translates eventually to Conjecture 1.

(ii) Conversely, if all "universal equations" of (axiomatic) Gromov–Witten invariants are induced from the tautological equations on moduli spaces of curves, then one should be able to find the tautological equations from studying the universal equations in Gromov–Witten theory. The term "universal equations" means that the equations are valid and of the same form for all theories. Since the tautological equations in all Gromov–Witten theories are of the same form, they are invariant under the action of the "quantized loop groups". This gives Conjectures 2 and 3.

1.4.2. Applications in axiomatic Gromov–Witten theory. Combining the results of this paper with two separate papers [20], [1], the last one joint with Arcara, the following results are proved.

Theorem 1. (i) *All tautological equations discussed in Section* 1.3 *hold for Givental's axiomatic Gromov–Witten invariants.*
(ii) *The Virasoro Conjecture for semisimple conformal Frobenius manifolds holds up to genus two.*
(iii) *The Witten conjecture on the higher spin curves and Gelfand–Dickey hierarchies holds up to genus two.*

Remarks. (i) In joint work with Givental [13], it is shown that the conformality condition in Theorem 1(ii) can be removed for $g = 1$. The reason is that the uniqueness theorem of Dubrovin and Zhang [4] can be slightly modified to work in the non-conformal case. It was observed in [4, Section 6] that Getzler's equation [7] can be used to uniquely determine a genus one potential up to a linear combination of canonical coordinates u_i

$$\mathrm{const} + \sum_{i=1}^{N} c_i u_i ;$$

the Euler field then determines c_i in the conformal case. In the non-conformal case, we proved in [13] (see also Section 3) that Getzler's equation holds for the genus one potentials of any semisimple Frobenius manifold, without the conformality condition. It is not hard to see the c_i can be matched, and Virasoro constraints hold.

(ii) Theorem 1(ii) was independently proved by X. Liu [23].

(iii) The above theorems build upon many other authors' results, and will be discussed in [20].

2. Algorithm of finding tautological equations

2.1. Finiteness

Assuming the Invariance Conjectures, equation (4) says that

$$\mathfrak{r}_l : R^k(\overline{M}_{g,n}) \to R^{k+l-1}(\overline{M}^{\bullet}_{g-1,n+2})$$

is injective. In fact, the number of connected components in the image can go up by at most one. Therefore, if the resulting decorated graphs are connected, the genus must be reduced by one. If they are disconnected, then either the genus or the number of marked points (external half-edges) is reduced.

Lemma 1. $\mathfrak{r}_l$ *reduces the dimension of* Γ *by* l.

Proof. The proof is a straightforward case-by-case study. In the first step, the dimension remains the same after cutting an edge. Decorating with an extra ψ^l reduces the dimension by l. In the second step, reducing the genus and adding two half-edges changes the dimension by $-3+2=1$. Decorating with ψ^m and ψ^{l-1} reduces the dimension by $l-1$. In the third step, vertex splitting reduces the dimension by 3. Adding two half-edges increases the dimension by 2. Decoration reduces the dimension by $l-1$.

Remark 4. (i) By Lemma 1, $\mathfrak{r}_l\Gamma = 0$ when $k+l > \dim \overline{M}_{g,n}$. Therefore, one only has to check a finite number of l's.

(ii) There are only finitely many (g', n', k') involved in checking the validity of the invariance equation (3) for a given (g, n, k) and l. This observation can be easily verified from the definition of $\mathfrak{r}_l$. Therefore, it is a finite calculation to check (3) for any quadruple (g, n, k, l).

From the above discussions, one deduces

Corollary 1. *There is an algorithm to check Conjectures* 1 *and* 2 *for any given* (g, n, k).

2.2. The algorithm

The Invariance Conjectures have been used to "re-discover" all known tautological equations and discover a new one. Our basic strategy of finding the tautological equations for a given (g, n, k) is the following. Assume all tautological equations are known for (g', n', k') satisfying the inductive conditions in Conjecture 1.

(1) Find all tautological classes Γ_i in $R^k(\overline{M}_{g,n})$. Remove linear dependent classes from the induced equations as explained in Remark 3. Let $\{\Gamma_i\}_{i\in I}$ be the set of remaining vectors in the $\mathbb{Q}$-linear space $R^k(\overline{M}_{g,n})$. Let

$$E = \sum_{i\in I} c_i \Gamma_i$$

be a general element in $R^k(\overline{M}_{g,n})$ with *unknown* coefficients c_i.

(2) Apply the R-invariance condition (3) $\mathfrak{r}_l(E) = 0$ for $l = 1, \ldots, 3g - 3 + n - k$. For each l, $\mathfrak{r}_l(\Gamma)$ will be a $\mathbb{Q}$-linear combination of (disjoint unions of) Γ'_j as classes in $R^{k'}(\overline{M}_{g',n'})$, which are known by assumption. In particular, one knows all the relations between Γ'_j's.
(3) In each (g', n', k') pick a *basis* of $R^{k'}(\overline{M}_{g',n'})$. The part of the output of $\mathfrak{r}_l(E) = 0$ in (g', n', k') implies each component of the basis vanishes individually. Since all operations involved are *linear*, the vanishing gives linear equations in c_i.
(4) The above step produces enough linear equations on c_i's to determine them completely up to a few free variables, say c_1 and c_2. Write all other c_j's in terms of c_1 and c_2. Then $E = \sum_{i \in I} c_i \Gamma_i$ becomes

$$E = c_1 \Big(\sum_{j_1 \in J_1} d_{j_1} \Gamma_{j_1} \Big) + c_2 \Big(\sum_{j_2 \in J_2} d_{j_2} \Gamma_{j_2} \Big).$$

The output equations are

$$\sum_{j_1 \in J_1} d_{j_1} \Gamma_{j_1} = 0 \quad \text{and} \quad \sum_{j_2 \in J_2} d_{j_2} \Gamma_{j_2} = 0,$$

where the d_j are output constants. Note that J_1 and J_2 are not necessarily disjoint.
(5) If all c_i have to vanish after the above steps, then there are no (new) tautological equations for (g, n, k).

Remarks. (i) Step (4) will necessarily leave at least one free variable, as any equation $E = 0$ holds after multiplying by a constant.

(ii) In fact, the output of the R-invariance condition always highly *over-determines* the unknown coefficients c_i. To be able to solve c_i is usually a sign of correct calculations.

(iii) In all cases computed, $l = 1$ is enough. That is, $\mathfrak{r}_1(E) = 0$ already generates enough linear equations for c_i to determine them completely. It is not known whether this will hold in general. In particular, one might ask whether $\mathfrak{r}_1(E) = 0$ implies $\mathfrak{r}_l(E) = 0$ for $l \geq 2$. We do not have intuition leading to a guess.

As stated above, the above algorithm has given a *uniform* method of deriving all tautological equations, which were originally derived using many different methods. Furthermore, it is theoretically possible to program this algorithm in order to discover more tautological equations via robots. The only ingredient in this algorithm is linear algebra (and the efforts to obtain induced equations as explained in Remark 3). However, the dimension of $R^k(\overline{M}_{g,n})$ and the number of elements in the set $\{\Gamma_i\}_{i \in I}$ can grow. So the finite-dimensional linear algebra problem in question is not trivial.

Remark 5. An alternative way to the above (more satisfactory) algorithm goes through the following procedure.

- Calculate the rank of $R^k(\overline{M}_{g,n})$ to see if there is any new equation.
- If there is one, then apply the invariance condition equation (3) to obtain the coefficients of the equation.

This is actually the way which is employed to prove the derived tautological equations. However, the first step is usually not elementary.

3. Proof of Conjectures for $g = 1$

3.1. Notations

In this subsection, some notation will be introduced for future reference.

The following notions will be used to describe the same object.

(1) Tautological classes.
(2) Generic curves and Chern classes.
(3) Decorated graphs.
(4) Gwi's (defined below).

(1)⇔(2). For each tautological class, one may draw a generic curve of the given topological type, label marked points, and decorate the marked points and both sides of nodes with monomials of ψ classes and the components by κ classes.

(1)⇔(3). This has been explained in Sections 1.1–1.2.

(3)⇔(4). Given a decorated graph Γ.

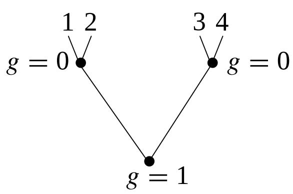
- To the vertices of Γ of genus $g_1, g_2, \ldots$, assign a product of "brackets" $\langle\rangle_{g_1}\langle\rangle_{g_2}\ldots$.
- Assign to each half-edge a symbol ∂^*. The external half-edges use numerical superscripts, $\partial^1, \ldots, \partial^n$, corresponding to their labeling. The two new half-edges use ∂^i, ∂^j. For each pair of half-edges coming from one and the same edge, the same superscript μ_i will be used. Otherwise, all half-edges should use different superscripts.
- For each given vertex $\langle\rangle_g$ with m half-edges, n external half-edges, and say two new half-edges, an insertion is placed at the vertex $\langle \partial^i \partial^j \prod_{a=1}^n \partial^a \prod_{b=1}^m \partial^{\mu_b}\rangle_g$.
- For each decoration of a half-edge with Chern classes c, assign a subscript to the corresponding half-edge, ∂_c^μ.

The output is called a *gwi*.

Remarks.

(i) Gwi is so named for its relations to Gromov–Witten invariants. Note, however, that gwi here stands for a vector in a $\mathbb{Q}$-algebra, rather than a rational number. The actual relation is recalled in [20]. Gwi notation is used in the proofs of this paper, mainly for typesetting convenience.
(ii) The convention on graphs adopted in Section 1.2 is meant to match the graphical and gwi notations. They often differ by a constant in some authors' conventions. (See, e.g., [8].)

Example. Let Γ be the following graph.



The corresponding gwi is

$$\langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^\nu \rangle \langle \partial^\mu \partial^\nu \rangle_1.$$

The *cutting edges* operation for $l = 1$ produces

$$\langle \partial^1 \partial^2 \partial^i \rangle \langle \partial^3 \partial^4 \partial^\nu \rangle \langle \partial_1^j \partial^\nu \rangle_1 + \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^i \rangle \langle \partial^\mu \partial_1^j \rangle_1.$$

Note that $\langle \partial^1 \partial^2 \partial_1^i \rangle$ has dimension -1. Therefore the corresponding graphs are removed from the output. Also, the i, j are symmetric. Hence, a factor of 2 is placed in front of a term instead of adding an additional term with i, j exchanged.

More generally, for any edge in an expression $\langle \ldots \partial^\mu \ldots \partial^\mu \ldots \rangle_g$, where the middle $\ldots$ might contain edges like $\ldots \rangle_h \langle \ldots$, the cutting edge operation for $\mathfrak{r}_l$ can be written as

$$\begin{aligned} \langle \ldots \partial^\mu \ldots \partial^\mu \ldots \rangle_g \mapsto & \frac{1}{2}(\langle \ldots \partial_l^i \ldots \partial^j \ldots \rangle_g + (-1)^{l-1} \langle \ldots \partial^i \ldots \partial_l^j \ldots \rangle_g) \\ & + \frac{1}{2}(\langle \ldots \partial^j \ldots \partial_l^i \ldots \rangle_g + (-1)^{l-1} \langle \ldots \partial_l^j \ldots \partial^i \ldots \rangle_g). \end{aligned} \quad (5)$$

The *genus reduction* for $\mathfrak{r}_l$ produces

$$\langle \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \ldots \rangle_g \mapsto \frac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \langle \partial_{l-1-m}^i \partial_m^j \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \ldots \rangle_{g-1}. \quad (6)$$

The *splitting vertices* operation for $l = 1$ produces

$$\langle \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \ldots \rangle_g \mapsto \frac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \sum_{g_1+g_2=g} \partial_{k_1}^{i_1} \partial_{k_2}^{i_2} \ldots (\langle \partial_{l-1-m}^i \rangle_{g_1} \langle \partial_m^j \rangle_{g_2}). \quad (7)$$

The symbol ∂_k^i considered as a "linear operator" on graphs is defined by the Leibniz rule:

$$\partial_k^i (\langle \partial_{l-1-m}^i \rangle_{g_1} \langle \partial_m^j \rangle_{g_2}) = \langle \partial_k^i \partial_{l-1-m}^i \rangle_{g_1} \langle \partial_m^j \rangle_{g_2} + \langle \partial_{l-1-m}^i \rangle_{g_1} \langle \partial_k^i \partial_m^j \rangle_{g_2}.$$

Convention 3. In the calculations when the κ-classes are not needed, as they are boundary classes in genus less than 3, a simplified notation is used:

$$\partial_k^\mu := \partial_{\psi^k}^\mu, \quad \partial^\mu := \partial_0^\mu.$$

To further simplify the notations, we write

$$\langle \ldots \rangle := \langle \ldots \rangle_{g=0}.$$

3.2. Reduction to Getzler's equation

By a result of E. Getzler (unpublished), the only (new) tautological equations in genus one are his equation in $(g = 1, n = 4, k = 2)$ [7] and TRR in $(g = 1, n = 1, k = 1)$. The $g = 1$ TRR expresses ψ-classes as boundary divisors

$$\langle \partial_1^i \rangle_1 = \frac{1}{24} \langle \partial^i \partial^\mu \partial^\mu \rangle. \tag{8}$$

Since this is an equation in top codimension, it is considered as part of the inductive data (and satisfies the invariance equation by Lemma 1). Therefore, one only has to check Conjectures 1 and 2 for Getzler's equation.

3.3. Getzler's equation in $(g, n, k) = (1, 4, 2)$

The calculation here is reproduced from a joint work with A. Givental.

Theorem 2 ([13]). *Getzler's equation is the only (new) codimension* 2 *equation in* $\overline{M}_{1,4}$ *which satisfies the invariance equation* (3). *Furthermore, the invariance equation determines the coefficients of Getzler's equation up to a common scaling constant.*

The proof is divided into the following six steps.

3.3.1. Step 1: Enumerate all boundary strata. There are nine codimension two boundary strata in $\overline{M}_{1,4}$, when the ordering of the four external half-edges is ignored. Equivalently, one may symmetrize the four external half-edges by the S_4 permutations. Let $\partial^1, \partial^2, \partial^3, \partial^4$ denote the four external half-edges. A general element can be written as

$$\begin{aligned} E = \sum_{S_4 \text{ permutations}} & (c_1 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^\nu \rangle \langle \partial^\mu \partial^\nu \rangle_1 \\ & + c_2 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^\mu \partial^\nu \rangle \langle \partial^4 \partial^\nu \rangle_1 \\ & + c_3 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\ & + c_4 \langle \partial^1 \partial^2 \partial^3 \partial^\mu \rangle \langle \partial^4 \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\ & + c_5 \langle \partial^1 \partial^2 \partial^3 \partial^\mu \rangle \langle \partial^4 \partial^\mu \partial^\nu \partial^\nu \rangle \\ & + c_6 \langle \partial^1 \partial^2 \partial^3 \partial^4 \partial^\mu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle \\ & + c_7 \langle \partial^1 \partial^2 \partial^\mu \partial^\nu \rangle \langle \partial^3 \partial^4 \partial^\mu \partial^\nu \rangle \\ & + c_8 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^\mu \partial^\nu \partial^\nu \rangle \\ & + c_9 \langle \partial^1 \partial^\mu \partial^\nu \rangle \langle \partial^2 \partial^3 \partial^4 \partial^\mu \partial^\nu \rangle). \end{aligned}$$

3.3.2. Step 2: Apply the invariance equation. The invariance equation produces, after applying a genus zero TRR,

$$\begin{aligned}
0 = \mathfrak{r}_1 E = \sum_{S_4} \mathrm{S}_{ij} (&2c_1 \langle \partial^1 \partial^2 \partial^j \rangle \langle \partial^3 \partial^4 \partial^\mu \rangle \langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
&- c_1 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^\nu \rangle \langle \partial^i \partial^\nu \partial^\mu \rangle \langle \partial^j \rangle_1 \\
&+ c_2 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^j \partial^\mu \rangle \langle \partial^i \partial^4 \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
&- c_2 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^4 \partial^\nu \rangle \langle \partial^j \rangle_1 \\
&+ c_3 \langle \partial^1 \partial^2 \partial^j \rangle \langle \partial^3 \partial^4 \partial^i_1 \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
&+ c_3 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^i_1 \partial^\nu \rangle \langle \partial^j \rangle_1 \\
&- c_3 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^i \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
&- 2c_3 \langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^i \partial^\mu \rangle \langle \partial^4 \partial^j \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
&+ c_4 \langle \partial^1 \partial^2 \partial^3 \partial^i_1 \rangle \langle \partial^4 \partial^j \partial^\nu \rangle \langle \partial^\nu \rangle_1 \\
&- 3c_4 \langle \partial^1 \partial^2 \partial^i \rangle \langle \partial^3 \partial^j \partial^\mu \rangle \langle \partial^4 \partial^\mu \partial^\nu \rangle \langle \partial^\nu \rangle_1) \\
&+ \text{genus-zero-only terms.}
\end{aligned}$$

Here S_{ij} is the symmetrization operator of the indices i, j.

3.3.3. Step 3: Genus one terms. The basic strategy is to find a basis, express the vector in terms of the basis, and set the coefficients to 0.

It is easy to see that the terms containing $\langle \partial^j \rangle_1$ give the condition (after applying a genus zero TRR)

$$-c_1 - c_2 + c_3 = 0.$$

The terms containing $\langle \partial^* \partial^{**} \partial^j \rangle \langle \partial^\nu \rangle_1$ give the equation

$$2c_1 - 3c_4 = 0.$$

The terms containing $\langle \partial^* \partial^{**} \partial^\nu \rangle \langle \partial^\nu \rangle_1$ give the equation

$$c_2 - 2c_3 + c_4 = 0.$$

3.3.4. Step 4: Genus zero terms. For the terms involving geometric genus zero graphs only, the only relations are WDVV, after stripping off all descendents by a genus zero TRR.

(a) Those terms containing a factor $\langle \partial^*, \partial^{**}, \partial^{***}, \partial^i \rangle$ give the equation

$$\sum_{S_4} \mathrm{S}_{ij} \langle \partial^1 \partial^2 \partial^3 \partial^i \rangle [c_5 \langle \partial^4 \partial^j_1 \partial^\nu \partial^\nu \rangle - 4c_6 \langle \partial^4 \partial^j \partial^\mu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle - c_9 \langle \partial^4 \partial^\mu \partial^\nu \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle] = 0,$$

which gives the condition

$$c_5 - 4c_6 - c_9 = 0.$$

(b) Those terms containing a factor $\langle\partial^*\partial^{**}\partial^i\rangle$ give the equation

$$
\begin{aligned}
\sum_{S_4} \mathrm{S}_{ij}\langle\partial^1\partial^2\partial^i\rangle\Big[&\frac{1}{12}c_1\langle\partial^3\partial^4\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\mu\partial^\nu\rangle\\
&-3c_5\langle\partial^3\partial^j\partial^\nu\rangle\langle\partial^4\partial^\mu\partial^\mu\partial^\nu\rangle\\
&-6c_6\langle\partial^3\partial^4\partial^j\partial^\nu\rangle\langle\partial^\mu\partial^\mu\partial^\nu\rangle\\
&-2c_7\langle\partial^3\partial^4\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\\
&+c_8\langle\partial^3\partial^4\partial^j_1\partial^\nu\partial^\nu\rangle\\
&-c_8\langle\partial^3\partial^4\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\partial^\nu\rangle\\
&-3c_9\langle\partial^3\partial^\mu\partial^\nu\rangle\langle\partial^4\partial^j\partial^\mu\partial^\nu\rangle\Big]=0.
\end{aligned}
\tag{9}
$$

All graphs are disconnected with two components. One common connected component is $\langle\partial^1\partial^2\partial^i\rangle$. The other one is a stratum isomorphic to $\overline{M}_{0,3}\times\overline{M}_{0,4}\subset\overline{M}_{0,5}$. Due to different labeling, there are five different strata for the second component, up to the obvious permutation symmetry in ∂^3, ∂^4:

$$
\begin{aligned}
\vec{v}_1&=\langle\partial^3\partial^4\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\mu\partial^\nu\rangle,\\
\vec{v}_2&=\langle\partial^3\partial^j\partial^\nu\rangle\langle\partial^4\partial^\mu\partial^\mu\partial^\nu\rangle,\\
\vec{v}_3&=\langle\partial^3\partial^\mu\partial^\nu\rangle\langle\partial^4\partial^j\partial^\mu\partial^\nu\rangle,\\
\vec{v}_4&=\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^3\partial^4\partial^\mu\partial^\nu\rangle,\\
\vec{v}_5&=\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^3\partial^4\partial^j\partial^\nu\rangle.
\end{aligned}
$$

The WDVV equations induce three linear relations in $\vec{v}_1,\dots,\vec{v}_5$, two of them being independent:

$$
\begin{aligned}
\vec{v}_1+\vec{v}_5&=2\vec{v}_3,\\
\vec{v}_2+\vec{v}_5&=\vec{v}_3+\vec{v}_4,\\
\vec{v}_1+\vec{v}_4&=\vec{v}_2+\vec{v}_3.
\end{aligned}
$$

Thus, one can write $\vec{v}_1$ and $\vec{v}_2$ in terms of $\vec{v}_3$, $\vec{v}_4$, $\vec{v}_5$. Equation (9) then gives

$$
\begin{aligned}
\frac{1}{6}c_1-3c_5+3c_9&=0,\\
-3c_5-2c_7+2c_8&=0,\\
-\frac{1}{12}c_1+3c_5-6c_6&=0.
\end{aligned}
$$

(c) The remaining terms, after applying a genus zero TRR, contain no descendents. Therefore the only relations are WDVV and their derivatives. However, WDVV and their derivatives do not change the sum of the coefficients, therefore the sum has to vanish. This gives another equation

$$
-\frac{1}{2}c_1-\frac{11}{24}c_2-\frac{11}{24}c_3-\frac{11}{24}c_4+3c_6-3c_8=0.
$$

3.3.5. Step 5: Final equation. Combining the above equations, one can express all coefficients in terms of c_3 and c_9:

$$c_1 = -3c_3, \quad c_2 = 4c_3, \quad c_4 = -2c_3, \quad c_5 = -\frac{1}{6}c_3 - c_9,$$
$$c_6 = -\frac{1}{24}c_3 - \frac{1}{2}c_9, \quad c_7 = \frac{1}{4}c_3 + c_9, \quad c_8 = -\frac{1}{2}c_9.$$

That is,

$$E = -c_3(\text{Getzler's coefficients}) + c_9(T) = 0,$$

where T is a sum of (geometric) genus zero graphs. It is easy to see that $T = 0$ by WDVV. Therefore, the $l = 1$ case is established.

3.3.6. Step 6: $l = 2$. By the same computation, one is led to

$$\begin{aligned}\mathfrak{r}_2 E = \sum_{S_4} \mathrm{A}_{ij} \Bigg[& 6\langle \partial^1 \partial^2 \partial^j \rangle \langle \partial^3 \partial^4 \partial^\nu \rangle \langle \partial^i_2 \partial^\nu \rangle_1 \\ & - 4\langle \partial^1 \partial^2 \partial_\mu \rangle \langle \partial^3 \partial^\mu \partial^j \rangle \langle \partial^4 \partial^i_2 \rangle_1 \\ & + \frac{1}{24} \langle \partial^1 \partial^2 \partial^3 \partial^4 \partial^i_2 \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle \\ & + 3\langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^4 \partial^\nu \rangle \langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^j_1 \rangle_1 \\ & - 4\langle \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^3 \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^4 \partial^\nu \rangle \langle \partial^j_1 \rangle_1 \\ & - \frac{4}{24} \langle \partial^i_1 \partial^1 \partial^2 \partial^3 \rangle \langle \partial^j \partial^4 \partial^\nu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle \\ & - \frac{6}{24} \langle \partial^i_1 \partial^1 \partial^2 \partial^\mu \rangle \langle \partial^j \partial^3 \partial^4 \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle \Bigg],\end{aligned}$$

where A_{ij} is the anti-symmetrizer. By easy application of genus one TRR and WDVV and the antisymmetric property of i, j, the first term cancels with the seventh term; the second term cancels with the sixth term; the third, fourth and fifth terms combine to vanish.

By Lemma 1, it is enough to check R-invariance for $l = 1$ and $l = 2$. The proof is complete.

Acknowledgments. Special thanks are due to my collaborators D. Arcara ([1, 2]) and A. Givental ([13]) for many useful discussions during our joint projects related to this paper; to R. Cavalieri and R. Vakil for their interest in and clarification of the conjectures; to the referee, and participants of WAGS Spring 2004 and Moduli Space of Curves and Gromov–Witten Theory Workshop, for suggestions on exposition.

References

[1] Arcara, D., Lee, Y.-P.: Tautological equations in genus two via invariance conjecture. Bull. Inst. Math. Acad. Sin. **2**, 1–27 (2007) Zbl pre05238071 MR 2294106

[2] Arcara, D., Lee, Y.-P.: Tautological equation in $\overline{M}_{3,1}$ via invariance conjecture. math.AG/0503184

[3] Belorousski, P., Pandharipande, R.: A descendent relation in genus 2. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29**, 171–191 (2000) Zbl 0981.81063 MR 1765541

[4] Dubrovin, B., Zhang, Y.: Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation. Comm. Math. Phys. **198**, 311–361 (1998) Zbl 0923.58060 MR 1672512

[5] Faber, C.: A conjectural description of the tautological ring of the moduli space of curves. In: Moduli of Curves and Abelian Varieties, Aspects Math., E33, Vieweg, Braunschweig, 109–129 (1999) Zbl 0978.14029 MR 1722541

[6] Faber, C., Pandharipande, R.: Hodge integrals and Gromov–Witten theory. Invent. Math. **139**, 173–199 (2000) Zbl 0960.14031 MR 1728879

[7] Getzler, E., Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov–Witten invariants. J. Amer. Math. Soc. **10**, 973–998 (1997) Zbl 0909.14002 MR 1451505

[8] Getzler, E., Topological recursion relations in genus 2. In: Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997), World Sci., River Edge, NJ, 73–106 (1998) Zbl 1021.81056 MR 1672112

[9] Getzler, E., Looijenga, E.: The Hodge polynomial of $\overline{M}_{3,1}$. math.AG/9910174

[10] Givental, A.: Semisimple Frobenius structures at higher genus. Int. Math. Res. Not. **2001**, no. 23, 1265–1286 Zbl 1074.14532 MR 1866444

[11] Givental, A.: Gromov–Witten invariants and quantization of quadratic Hamiltonians. Moscow Math. J. **1**, 551–568, 645 (2001) Zbl 1008.53072 MR 1901075

[12] Givental, A.: Symplectic geometry of Frobenius structures. math.AG/0305409

[13] Givental, A., Lee, Y.-P.: Invariance of tautological rings at genus one. Draft

[14] Graber, T., Vakil, R.: On the tautological ring of $\overline{\mathcal{M}}_{g,n}$. Turkish J. Math. **25**, 237–243 (2001) Zbl 1040.14007 MR 1829089

[15] Graber, T., Vakil, R.: Relative virtual localization and vanishing of tautological classes on moduli spaces of curves. Duke Math. J. **130**, 1–37 (2005) Zbl pre05004321 MR 2176546

[16] Harris, J., Morrison, I.: Moduli of Curves. Grad. Texts in Math. 187, Springer, New York (1998) Zbl 0913.14005 MR 1631825

[17] Keel, S.: Intersection theory of moduli space of stable n-pointed curves of genus zero. Trans. Amer. Math. Soc. **330**, 545–574 (1992) Zbl 0768.14002 MR 1034665

[18] Kimura, T., Liu, X.: A genus-3 topological recursion relation. Comm. Math. Phys. **262**, 645–661 (2006) Zbl 1105.14074 MR 2202306

[19] Lee, Y.-P.: Witten's conjecture and Virasoro conjecture for genus up to two. In: Gromov–Witten Theory of Spin Curves and Orbifolds, Contemp. Math. 403, Amer. Math. Soc., 31–42 (2006) Zbl 1114.14034 MR 2234883

[20] Lee, Y.-P. (and Y. Iwao in Appendix): Invariance of tautological equations II: Gromov–Witten theory. math.AG/0605708

[21] Lee, Y.-P.: Witten's conjecture, Virasoro conjecture, and invariance of tautological equations. math.AG/0311100

[22] Lee, Y.-P., Pandharipande, R.: Frobenius manifolds, Gromov–Witten theory, and Virasoro constraints. In preparation

[23] Liu, X.: Genus-2 Gromov–Witten invariants for manifolds with semisimple quantum cohomology. Amer. J. Math. **129**, 463–498 (2007) Zbl pre05152574 MR 2306043

[24] Mumford, D.: Towards an enumerative geometry of the moduli space of curves. In: Arithmetic and Geometry, M. Artin and J. Tate (eds.), Part II, Birkhäuser, 271–328 (1983) Zbl 0554.14008 MR 0717614

[25] Pandharipande, R.: Three questions in Gromov–Witten theory. In: Proc. Int. Congress Math., Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 503–512 (2002) Zbl 1047.14043 MR 1957060

[26] Vakil, R.: The moduli space of curves and Gromov–Witten theory. math.AG/0602347