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Some examples of nil Lie algebras

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Abstract. Generalizing Petrogradsky's construction, we give examples of infinite-dimensional nil Lie algebras of finite Gelfand–Kirillov dimension over any field of positive characteristic.

The first examples of infinite-dimensional affine nil algebras were constructed by E. S. Golod and I. R. Shafarevich [GS]. These algebras have a strong exponential growth. Later L. Bartholdi and R. I. Grigorchuk [BG] showed that the Lie algebra associated to the "self-similar" Grigorchuk group is graded-nil and has Gelfand–Kirillov dimension 1. Using this result L. Bartholdi [B] was able to construct an infinite-dimensional affine graded-nil associative algebra of Gelfand–Kirillov dimension 2 over a finite field of characteristic 2. Recently T. Lenagan and A. Smoktunowicz [LS] constructed a family of infinite-dimensional affine nil algebras of finite Gelfand–Kirillov dimension over an arbitrary countable field.

In [P] V. Petrogradsky found an infinite-dimensional two-generated "self-similar" Lie algebra L over an arbitrary field of characteristic 2 such that (i) L is nil, (ii) $1 < \operatorname{GKdim} L < 2$.

In this paper we generalize Petrogradsky's construction and extend it to algebras over fields of arbitrary positive characteristic.

Let p be a prime number; F a field of characteristc p; $\widehat{T} = F[t_0, t_1, \dots]$ the algebra of truncated polynomials in countably many variables $t_0, t_1, \dots;\ t_i^p = 0$, $i \geq 0$. Let T be the subalgebra of $\widehat{T}$ consisting of polynomials with zero constant term. Write $\widehat{T}(k) = F[t_0, t_1, \dots, t_k]$ (for $k < 0$ we let $\widehat{T}(k) = F \cdot 1$) and consider the following Lie algebra of derivations of $\widehat{T}$:

$$\mathcal{D} = \Big\{ \sum_{i=1}^{\infty} a_i \partial_i \;\Big|\; a_i \in \widehat{T}(i-2) \Big\},$$

where $\partial_i = d/dt_i$. For $k \geq 1$, let

$$M_k = \mathcal{D} \cap \Big\{ \sum_{i \leq k-1} \widehat{T} \partial_i + \sum_{j=0}^{\infty} T^{1+j(p-1)} \partial_{k+j} \Big\}.$$

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In other words, a derivation $\sum_{i=1}^{\infty} a_i \partial_i$ lies in M_k if $a_i \in \widehat{T}(i-2) \cap T^{1+(i-k)(p-1)}$ for $i \geq k$. Clearly, $M_1 \subset M_2 \subset \cdots$.

Lemma 1. (a) $[M_k, M_q] \subseteq M_{\max(k,q)+1}$, (b) $a^p \in M_{k+1}$ *for all* $a \in M_k$.

Proof. (a) is obvious. Let us check (b). Let $d = \sum_{i=1}^{\infty} a_i \partial_i \in M_k$. Since $a_i \in \widehat{T}(i-2)$ it follows that $(a_i \partial_i)^p = a_i^p \partial_i^p = 0$. Now consider a commutator $[a_{i_1}\partial_{i_1}, \dots, a_{i_p}\partial_{i_p}] = b\partial_{i_s}$, where $b \in \widehat{T}$ and $i_s = \max\{i_1, \dots, i_p\}$.

The coefficient b is an expression in (repeated) $(\partial_{i_1}, \dots, \partial_{i_{s-1}}, \partial_{i_{s+1}}, \dots, \partial_{i_p})$-derivations of $a_{i_1}, \dots, a_{i_p}$. Let $i_s \geq k+1$. Then $a_{i_s} \in T^{1+(i_s-k)(p-1)}$, no more than $p-1$ derivations could have been applied to a_{i_s}, hence the derivative lies in $T^{1+(i_s-k)(p-1)-(p-1)} = T^{1+(i_s-k-1)(p-1)}$. □

Let $M = \bigcup_{k \geq 1} M_k$. From Lemma 1 it follows that M is a Lie subalgebra of $\mathcal{D}$, which is closed with respect to p-th powers. Let

$$\mathcal{D}_2 = \left\{ \sum_{i=1}^{\infty} a_i \partial_i \;\middle|\; a_i \in T^2 \cap \widehat{T}(i-2),\ i \geq 1 \right\}.$$

Lemma 2. *For each* $d \in M$ *there exists* $s \geq 1$ *such that* $d^{p^s} \in \mathcal{D}_2$.

Proof. We follow the idea from [P]. Let $\lambda = (1 + \sqrt{1 + 4(p-1)})/2$, the positive root of $\lambda^2 - \lambda - (p-1) = 0$. We will define a $\mathbb{Z} + \mathbb{Z}\lambda$-grading on the algebra $\mathcal{D}$ by setting $\deg(t_i) = -\lambda^i$ and $\deg(\partial_i) = \lambda^i$. The degree of an arbitrary element from $\widehat{T}(i-2)\partial_i$ is greater than or equal to

$$\lambda^i - (p-1)(1 + \cdots + \lambda^{i-2}) = \lambda^i - (p-1)\frac{\lambda^{i-1} - 1}{\lambda - 1}$$
$$= \frac{\lambda^{i-1}(\lambda^2 - \lambda - (p-1)) + p - 1}{\lambda - 1} = \frac{p-1}{\lambda - 1} = \epsilon > 0.$$

Hence every homogeneous component of an element d from $\mathcal{D}$ has degree $\geq \epsilon$ and every homogeneous component of d^{p^s} has degree $\geq p^s\epsilon$.

Now let $d \in M_k$. Then by Lemma 1(b), $d^{p^s} \in M_{k+s}$, hence

$$d^{p^s} \in \sum_{i=1}^{k+s} \widehat{T}\partial_i + T^p \partial_{k+s+1} + T^{2p-1}\partial_{k+s+2} + \cdots.$$

The degree of an arbitrary nonzero element from $\sum_{i=1}^{k+s} \widehat{T}\partial_i$ is $\leq \lambda^{k+s}$. Hence, if the $\sum_{i=1}^{k+s} \widehat{T}\partial_i$-part of d^{p^s} is not equal to 0, then d^{p^s} has a nonzero homogeneous component of degree $\leq \lambda^{k+s}$. Hence $p^s\epsilon \leq \lambda^{k+s}$. However, since $\lambda < p$ it follows that $p^s\epsilon > \lambda^{k+s}$ for sufficiently large s, a contradiction. Hence for sufficiently large s we have $d^{p^s} \in T^p\partial_{k+s+1} + T^{2p-1}\partial_{k+s+2} + \cdots \subseteq \mathcal{D}_2$. □

Define

$$T^s\mathcal{D} = \Big\{\sum_{i=1}^{r} a_i d_i \;\Big|\; a_i \in T^s,\ d_i \in \mathcal{D},\ r \geq 1\Big\}.$$

Since $\bigcap_{s\geq 1} T^s\mathcal{D} = (0)$ the subspaces $T^s\mathcal{D} \cap \mathcal{D}$ define a topology on $\mathcal{D}$. Let

$$\mathcal{D}_{\text{fin}} = \Big\{\sum_{i=1}^{r} a_i \partial_i \;\Big|\; a_i \in \widehat{T}(i-2),\ r \geq 1\Big\}$$

and consider the closure of $\mathcal{D}_{\text{fin}}$ in the above topology: $\overline{\mathcal{D}}_{\text{fin}} = \bigcap_{s\geq 1}(\mathcal{D}_{\text{fin}} + (T^s\mathcal{D} \cap \mathcal{D}))$.

Lemma 3. *$\overline{\mathcal{D}}_{\text{fin}}$ is a subalgebra of $\mathcal{D}$, closed with respect to p-th powers.*

Proof. We have $[\mathcal{D}, T^s\mathcal{D}] \subseteq T^{s-1}\mathcal{D}$. This implies

$$[\mathcal{D}_{\text{fin}} + (T^k\mathcal{D} \cap \mathcal{D}), \mathcal{D}_{\text{fin}} + (T^s\mathcal{D} \cap \mathcal{D})] \subseteq \mathcal{D}_{\text{fin}} + (T^{\min(k,s)-1}\mathcal{D} \cap \mathcal{D}).$$

Hence, $[\overline{\mathcal{D}}_{\text{fin}}, \overline{\mathcal{D}}_{\text{fin}}] \subseteq \overline{\mathcal{D}}_{\text{fin}}$.

Let $a \in \mathcal{D}_{\text{fin}} + (T^s\mathcal{D} \cap \mathcal{D})$. Thus there exist elements $c_1, \dots, c_r \in T^s$ and an integer $k \geq 1$ such that $a = \sum_{i=1}^{\infty} a_i \partial_i$ and all the coefficients $a_k, a_{k+1}, \dots$ lie in $\sum_{j=1}^{r} c_j \widehat{T}$. As above we notice that $(a_i\partial_i)^p = 0$ and $[a_{i_1}\partial_{i_1}, \dots, a_{i_p}\partial_{i_p}] = b\partial_{i_q}$, where the coefficient b is an expression in derivatives of $a_{i_1}, \dots, a_{i_p}$ and $i_q = \max\{i_1, \dots, i_p\}$. Suppose that $i_q \geq k$. Then $a_{i_q} \in \sum c_j \widehat{T}$ and b lies in the ideal of $\widehat{T}$ generated by the elements $\partial_{j_1} \cdots \partial_{j_l} c_j$, where $l \leq p-1$. This implies that $a^p \in \mathcal{D}_{\text{fin}} + (T^{s-p+1}\mathcal{D} \cap \mathcal{D})$. □

Now let

$$L = M \cap \overline{\mathcal{D}}_{\text{fin}}.$$

Lemma 4. *For each $a \in L$ there exists $s \geq 1$ such that $a^{p^s} \in T^2\mathcal{D}$.*

Proof. By Lemma 2 there exists $s \geq 1$ such that $a^{p^s} \in \mathcal{D}_2$. On the other hand, $a^{p^s} \in \mathcal{D}_{\text{fin}} + (T^2\mathcal{D} \cap \mathcal{D})$. We claim that $\mathcal{D}_2 \cap (\mathcal{D}_{\text{fin}} + T^2\mathcal{D}) \subseteq T^2\mathcal{D}$. Indeed, let $d \in \mathcal{D}_2$, $d = d' + d''$, $d' \in \mathcal{D}_{\text{fin}}$, $d'' \in T^2\mathcal{D}$. Then $d' = d - d'' \in \mathcal{D}_{\text{fin}} \cap \mathcal{D}_2 \subseteq T^2\mathcal{D}$. □

Lemma 5. *The associative subalgebra of* $\mathrm{End}_F(\widehat{T})$ *generated by $T^2\mathcal{D}$ is locally nilpotent.*

Proof. Consider a collection of elements $a_i' a_i'' d_i$, where $1 \leq i \leq r$, $a_i', a_i'' \in T$, and $d_i \in \mathcal{D}$. Let A be the subalgebra of T generated by a_i', a_i'', $1 \leq i \leq r$. Since $a_i'^p = a_i''^p = 0$ it follows that $A^{2r(p-1)+1} = (0)$. Now, $(a_{i_1}' a_{i_1}'' d_{i_1}) \cdots (a_{i_s}' a_{i_s}'' d_{i_s}) = \sum b_1 \cdots b_{2s} d_{j_1} \cdots d_{j_q}$, where $q \leq s$ and the b_i's are obtained from a_j', a_j'', $1 \leq j \leq r$, via (repeated) applications of the derivations d_k. Since there are only s derivations $d_{i_1}, \dots, d_{i_s}$ it follows that at least s of $b_1, \dots, b_{2s}$ lie in $\{a_{i_1}', a_{i_2}'', \dots, a_{i_s}', a_{i_s}''\}$. If $s = 2r(p-1)+1$, then $(a_{i_1}' a_{i_1}'' d_{i_1}) \cdots (a_{i_s}' a_{i_s}'' d_{i_s}) = 0$. □

Lemmas 4 and 5 imply

Proposition 1. *All elements of L are nilpotent.*

Simliarly to [P] we can find finitely generated nonnilpotent subalgebras in L and estimate their Gelfand–Kirillov dimensions.

Example 1. Consider the elements

$$v_n = \partial_n + \sum_{i=n+1}^{\infty} (t_{n-1} \cdots t_{i-2})^{p-1} \partial_i$$

from L, $n \geq 1$. We have

$$[v_i, v_{i+1}] = -t_i^{p-2} v_{i+2}; \quad [v_i, v_j] = -(t_{i-1} \cdots t_{j-3})^{p-1} t_{j-1}^{p-2} v_{j+1} \quad \text{for } j \geq i+2;$$

$$v_i(t_j) = \begin{cases} 0, & j < i, \\ 1, & j = i, \\ (t_{i-1} \cdots t_{j-2})^{p-1} & j > i. \end{cases}$$

It is easy to see that the Lie algebra $\mathcal{L} = \mathrm{Lie}\langle v_1, v_2 \rangle$ generated by v_1, v_2 is not nilpotent. Just as in [P], we will find its Gelfand–Kirillov dimension.

Proposition 2. $\mathrm{GKdim}\, \mathcal{L} = \log_\lambda p$.

Clearly, $1 < \log_\lambda p < 2$ and $\log_\lambda p \to 2$ as $p \to \infty$.

We first prove several lemmas.

Lemma 6. *We have*

$$\mathcal{L} \subseteq \mathrm{Span}(t_0^{i_0} \cdots t_{n-2}^{i_{n-2}} v_n \mid n \geq 1;\ 0 \leq i_0, \ldots, i_{n-4} \leq p-1;\ 0 \leq i_{n-3}, i_{n-2} \leq p-2).$$

Proof. Denote the right hand side of this inclusion by V. Since all v_i are in V, it suffices to prove that $[V, v_1], [V, v_2] \subseteq V$. Let $a = t_0^{i_0} t_1^{i_1} \cdots t_{n-2}^{i_{n-2}}$, $a v_n \in V$. Consider

$$\begin{aligned} [a v_n, v_1] &= a[v_n, v_1] - v_1(a) v_n \\ &= -a(t_0 \cdots t_{n-3})^{p-1} t_{n-1}^{p-2} v_{n+1} - \sum_{s=0}^{n-2} t_0^{i_0} \cdots t_{s-1}^{i_{s-1}} v_1(t_s^{i_s}) t_{s+1}^{i_{s+1}} \cdots t_{n-2}^{i_{n-2}} v_n. \end{aligned}$$

The first term clearly lies in V and so do all the summands in the second term for $s \leq n-3$. For $s = n-2$ we have

$$t_0^{i_0} \cdots t_{n-3}^{i_{n-3}} v_1(t_{n-2}^{i_{n-2}}) v_n = i_{n-2} t_0^{i_0} \cdots t_{n-3}^{i_{n-3}} (t_0 \cdots t_{n-4})^{p-1} t_{n-2}^{i_{n-2}-1} v_n \in V.$$

Similarly, $[a v_n, v_2] \in V$. □

Observe that the Lie algebra $\mathcal{L}$ is a graded subalgebra of the $\mathbb{Z}+\mathbb{Z}\lambda$-graded algebra $\mathcal{D}$ with $\deg(v_1)=\lambda$, $\deg(v_2)=\lambda^2$. Let $\mathcal{L}_d$ be the sum of the homogeneous subspaces of $\mathcal{L}$ of degree $\le d$.

Suppose that $v = t_0^{i_0}\cdots t_{n-4}^{i_{n-4}} t_{n-3}^{i_{n-3}} t_{n-2}^{i_{n-2}} v_n \in \mathcal{L}_d$ with $0 \le i_0,\dots,i_{n-4} \le p-1$ and $0 \le i_{n-3}, i_{n-2} \le p-2$. Then

$$\begin{aligned}\lambda^n - (p-1)(1+\lambda+\cdots+\lambda^{n-4}) - (p-2)(\lambda^{n-3}+\lambda^{n-2}) \\ &= \lambda^n-(p-1)(1+\lambda+\cdots+\lambda^{n-2})+\lambda^{n-3}+\lambda^{n-2}\\ &= \frac{p-1}{\lambda-1}+\lambda^{n-3}+\lambda^{n-2} \le \deg(v)\le d.\end{aligned}$$

This implies that $2\lambda^{n-3}\le d$ and therefore $n \le \log_\lambda(d/2)+3$.

Now,

$$\dim \mathcal{L}_d \le \sum_{n\le \log_\lambda(d/2)+3} p^{n-3} \le p^{\log_\lambda(d/2)+3} \le c d^{\log_\lambda p},$$

where c is a constant which depends on p.

Now let us estimate the dimension of $\mathcal{L}_d$ from below. In what follows we will assume that $p\ge 3$. For $p=2$ the assertion was proved in [PS].

Lemma 7. *For each $n\ge 1$ there exists $a_{2n-3}\in \widehat{T}(2n-3)$ such that*

$$v_{2n}\in\mathcal{L},\quad t_{2n-1}^{p-2}v_{2n+1}\in\mathcal{L},\quad v'_{2n-1}=v_{2n-1}+a_{2n-3}t_{2n-2}^{p-2}v_{2n}\in\mathcal{L}.$$

Proof. For $n=1$ the assertion is obvious. Let us assume it for n and prove for $n+1$. We have

$$t_{2n-1}^{p-2}t_{2n}^{p-2}v_{2n+2}=[t_{2n-1}^{p-2}v_{2n+1}, v_{2n}]\in\mathcal{L}.$$

Since $[v_{2n+2},v_{2n}]=t_{2n-1}^{p-1}t_{2n+1}^{p-2}v_{2n+3}$ and since $p\ge 3$, it follows that

$$[t_{2n-1}^{p-2}t_{2n}^{p-2}v_{2n+2}, v_{2n}]=(p-2)t_{2n-1}^{p-2}t_{2n}^{p-3}v_{2n+2}.$$

Repeating this $p-2$ times we get

$$(p-2)!t_{2n-1}^{p-2}v_{2n+2}\in\mathcal{L}.$$

For $i>0$ we have $[t_{2n-1}^{i}v_{2n+2}, v_{2n}]=0$. Hence

$$[t_{2n-1}^{i}v_{2n+2}, v'_{2n-1}]=[t_{2n-1}^{i}v_{2n+2}, v_{2n-1}].$$

Commuting $p-2$ times, we get

$$\begin{aligned}[\dots[t_{2n-1}^{p-2}v_{2n+2}, \underbrace{v'_{2n-1}],\dots,v'_{2n-1}}_{p-2}] &= [\dots[t_{2n-1}^{p-2}v_{2n+2}, v_{2n-1}],\dots,\dots,v_{2n-1}]\\ &=(p-2)!v_{2n+2}\in\mathcal{L},\end{aligned}$$

which proves the first inclusion of the lemma.

Now,

$$[v_{2n+2}, t_{2n-1}^{p-2} v_{2n+1}] = t_{2n-1}^{p-2} t_{2n+1}^{p-2} v_{2n+3} \in \mathcal{L}.$$

As above, for any $i > 0$ we have

$$[t_{2n-1}^{i} t_{2n+1}^{p-2} v_{2n+3}, v'_{2n-1}] = [t_{2n-1}^{i} t_{2n+1}^{p-2} v_{2n+3}, v_{2n-1}].$$

Hence,

$$[\dots[t_{2n-1}^{p-2} t_{2n+1}^{p-2} v_{2n+3}, \underbrace{v'_{2n-1}], \dots, v'_{2n-1}]}_{p-2} = [\dots[t_{2n-1}^{p-2} t_{2n+1}^{p-2} v_{2n+3}, \underbrace{v_{2n-1}], \dots, v_{2n-1}]}_{p-2}$$
$$= (p-2)! t_{2n+1}^{p-2} v_{2n+3},$$

which proves the second inclusion of the lemma.

Finally,

$$\begin{aligned}\mathcal{L} \ni [&t_{2n-1}^{p-2} v_{2n+1}, v'_{2n-1}] \\ &= [t_{2n-1}^{p-2} v_{2n+1}, v_{2n-1}] + [t_{2n-1}^{p-2} v_{2n+1}, a_{2n-3} t_{2n-2}^{p-2} v_{2n}] \\ &= (p-2) t_{2n-1}^{p-3} v_{2n+1} + t_{2n-1}^{p-2} t_{2n}^{p-2} v_{2n+2} + a_{2n-3} t_{2n-2}^{p-2} t_{2n-1}^{p-2} t_{2n}^{p-2} v_{2n+2}.\end{aligned}$$

Write

$$A_i = t_{2n-1}^{i} v_{2n+1}, \quad B_i = t_{2n-1}^{i} t_{2n}^{p-2} v_{2n+2}, \quad C_i = a_{2n-3} t_{2n-2}^{p-2} t_{2n-1}^{i} t_{2n}^{p-2} v_{2n+2}.$$

For any $i > 0$ we have

$$\begin{aligned}&[A_i, v'_{2n-1}] = [A_i, v_{2n-1}] + [A_i, a_{2n-3} t_{2n-2}^{p-2} v_n] = i A_{i-1} + B_i + C_i, \\ &[B_i, v'_{2n-1}] = [B, v_{2n-1}] = i B_{i-1}, \\ &[C_i, v'_{2n-1}] = [C_i, v_{2n-1}] = i C_{i-1}.\end{aligned}$$

Therefore, for all $0 \le i \le p-2$ we have

$$\mathcal{L} \ni [\dots[A_{p-2}, \underbrace{v'_{2n-1}], \dots, v'_{2n-1}]}_{i}$$
$$= \frac{(p-2)!}{(p-i-1)!}((p-i-1)A_{p-i-2} + i B_{p-i-1} + i C_{p-i-1}).$$

In particular, for $i = p-2$ we get

$$\mathcal{L} \ni A_0 + (p-2)B_1 + (p-2)C_1 = v_{2n+1} + a_{2n-1} t_{2n}^{p-2} v_{2n+2},$$

where $a_{2n-1} = -2(t_{2n-2} + a_{2n-3}) t_{2n-2}^{p-2} t_{2n-1} \in T(2n-1)$. $\square$

Lemma 8. *The algebra $\mathcal{L}$ contains all elements of the type*

$$t_0^{p-1} t_1^{i_1} \cdots t_{2n-3}^{i_{2n-3}} t_{2n-1}^{p-2} v_{2n+1} \tag{$*$}$$

with $0 \le i_k \le p-1$ *for* $k = 1, \dots, 2n-3$ *and* $n \ge 2$.

Proof. Consider in $\mathcal{L}$ the element $l = [v_{2n}, v_1] = (t_0 \cdots t_{2n-3})^{p-1} t_{2n-1}^{p-2} v_{2n+1}$. Note that

$$[l, v'_{2n-3}] = [l, v_{2n-3}] = (p-1)(t_0 \cdots t_{2n-2})^{p-1} t_{2n-3}^{p-2} t_{2n-1}^{p-2} v_{2n+1} \in \mathcal{L}.$$

Continuing in this way, we find that $(t_0 \cdots t_{2n-2})^{p-1} t_{2n-3}^{i} t_{2n-1}^{p-2} v_{2n+1} \in \mathcal{L}$ for all $i = p-1, \dots, 0$.

Assume that, for some $1 \le k \le 2n-3$, $\mathcal{L}$ contains all elements of the type

$$l_k = (t_0 \cdots t_k)^{p-1} t_{k+1}^{i_{k+1}} \cdots t_{2n-3}^{i_{2n-3}} t_{2n-1}^{p-2} v_{2n+1}, \qquad 0 \le i_s \le p-1,\ s = k+1, \dots, 2n-3.$$

If k is even then $v_k \in \mathcal{L}$, and we obtain elements in $\mathcal{L}$ of type l_{k-1} by commuting l_k with v_k. If k is odd then $v'_k \in \mathcal{L}$, and it is easy to see that $[l_k, v'_k] = [l_k, v_k]$, which again gives all elements of type l_{k-1}. Downward induction on k proves that the elements of type l_0 lie in $\mathcal{L}$, which are exactly the elements $(*)$. □

End of proof of Proposition 2. Let $v = t_0^{p-1} t_1^{i_1} t_2^{i_2} \cdots t_{2n-3}^{i_{2n-3}} t_{2n-1}^{p-2} v_{2n+1}$ where $0 \le i_k \le p-1$ for $k = 1, \dots, 2n-3$ and $n \ge 2$. Then $\deg(v) < \deg(v_{2n+1}) = \lambda^{2n+1}$. If $n \le \frac{1}{2}(\log_\lambda d - 1)$, then $\lambda^{2n+1} \le d$ and therefore $v \in \mathcal{L}_d$. This implies

$$\dim \mathcal{L}_d \ge \sum_{2 \le n \le \frac{1}{2}(\log_\lambda d - 1)} p^{2n-3} \ge p^{\log_\lambda d - 4} = \frac{1}{p^4} d^{\log_\lambda p}.$$

Summarizing we get

$$\frac{1}{p^4} d^{\log_\lambda p} \le \dim \mathcal{L}_d \le c d^{\log_\lambda p}.$$

This implies

$$\lim_{d \to \infty} \frac{\ln \dim \mathcal{L}_d}{\ln d} = \log_\lambda p.$$

Let $\mathcal{L}_{(d)}$ denote the span of all commutators in v_1, v_2 of length $\le d$. It is easy to see that

$$\mathcal{L}_{\lambda d} \subseteq \mathcal{L}_{(d)} \subseteq \mathcal{L}_{\lambda^2 d}.$$

Hence

$$\operatorname{GKdim} \mathcal{L} = \lim_{d \to \infty} \frac{\ln \dim \mathcal{L}_{(d)}}{\ln d} = \lim_{d \to \infty} \frac{\ln \dim \mathcal{L}_d}{\ln d} = \log_\lambda p. \qquad \square$$

Let A be the associative subalgebra of $\operatorname{End}_F(\widehat{T})$ generated by $\mathcal{L}$.

Proposition 3. $\operatorname{GKdim} A \le 2 \log_\lambda p$.

Proof. The proof is similar to that in [PS]. That is why we will only sketch it. First, notice that A is contained in the span of operators of the type

$$a = t_0^{\alpha_0} \cdots t_{n-2}^{\alpha_{n-2}} v_1^{\beta_1} \cdots v_n^{\beta_n},$$

where $0 \le \alpha_0, \ldots, \alpha_{n-3} \le p-1,\ 0 \le \alpha_{n-2} \le p-2,\ 0 \le \beta_1, \ldots, \beta_n \le p-1, \beta_n \ge 1$. Let $a \in A_d$. Then

$$d \ge \deg(a) = \sum_{i=1}^{n} \beta_i \lambda^i - \sum_{j=0}^{n-2} \alpha_j \lambda^j \ge \lambda^n - (p-1) \sum_{j=0}^{n-3} \lambda^j - (p-2)\lambda^{n-2}$$

$$= \frac{p-1}{\lambda-1} + \lambda^{n-2} > \lambda^{n-2}.$$

Hence, $n < \log_\lambda d + 2 = r$. For each n the number of such monomials is less than p^{2n-1}. Hence,

$$\dim A_d \le \sum_{n<r} p^{2n-1} < p^{2r-1} = p^{2\log_\lambda d + 3}.$$

Now it remains to notice that

$$\lim_{d \to \infty} \frac{\ln p^{2\log_\lambda d+3}}{\ln d} = 2\log_\lambda p. \qquad \square$$

Example 2. The Lie algebra $L_m,\ m \ge 1$, generated by the derivations $\partial_1, \ldots, \partial_m$, $\partial_{m+1} + \sum_{i=2}^{\infty} (t_1 \cdots t_{i-1})^{p-1} \partial_{m+i}$ is not nilpotent. The associative subalgebra of $\mathrm{End}_F(\widehat{T})$ generated by L_m has finite Gelfand–Kirillov dimension.

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