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Sergiu Klainerman · Igor Rodnianski

Sharp L^1 estimates for singular transport equations

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Abstract. We provide L^1 estimates for a transport equation which contains singular integral operators. The form of the equation was motivated by the study of Kirchhoff–Sobolev parametrices in a Lorentzian space-time satisfying the Einstein equations. While our main application is for a specific problem in General Relativity we believe that the phenomenon which our result illustrates is of a more general interest.

1. Introduction

The goal of this paper is to prove an L^1-type estimate for solutions of the following transport equation:

$$\partial_t u(t,x) - a(t,x) M u(t,x) = g(t,x), \qquad u(0,x) = 0. \tag{1}$$

Here $a = a(t,x)$ and $g = g(t,x)$ are assumed to be smooth, compactly supported functions defined[1] on $[0,1] \times \mathbb{R}^2$ and M is a classical, translation invariant, Calderón–Zygmund operator in $\mathbb{R}^2$, given by a smooth[2] multiplier. Though, for simplicity, we shall proceed as if the equation (1) is scalar, all our results extend easily to systems, i.e. u and g take values in $\mathbb{R}^N$ and aM is an $N \times N$ matrix-valued operator.

Ideally, the desired estimate would take the form

$$\sup_{t\in[0,1]} \|u(t)\|_{L^1(\mathbb{R}^2)} \le C(\|a\|_{L^\infty([0,1]\times\mathbb{R}^2)})\, \|g\|_{L^1([0,1]\times\mathbb{R}^2)}.$$

It is well known, however, that such L^1-type estimates cannot possibly hold due to the failure of L^1 boundedness of Calderón–Zygmund operators. To illustrate this consider first the case of a constant coefficient transport equation with $a \equiv 1$. In this case we may write

$$u(t,x) = \int_0^t e^{(t-s)M} g(s)\, ds, \tag{2}$$

S. Klainerman, I. Rodnianski: Department of Mathematics, Princeton University, Princeton NJ 08544; e-mail: seri@math.princeton.edu, irod@math.princeton.edu

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[1] Similar results can be easily extended to higher dimensions.

[2] The smoothness assumption is only imposed to eliminate logarithmic divergences at infinity in $\mathbb{R}^2$, and is irrelevant to our main concerns.

where

$$e^{tM} = I + tM + \frac{1}{2}(tM)^2 + \cdots + \frac{1}{n!}(tM)^n + \cdots .$$

The problem of L^1 estimates for (1) is then reduced to the corresponding question for the operators M^n. Each M^n is a Calderón–Zygmund operator and as such does not map L^1 to L^1. A well known way to resolve this problem is to consider instead mapping properties from the Hardy space[3] $\mathcal{H}_1$ to L^1. Since translation invariant Calderón–Zygmund operators M map $\mathcal{H}_1$ into $\mathcal{H}_1$ (see [Ste2]) we easily infer that a solution u of the transport equation

$$\partial_t u - Mu = g, \qquad u(0, x) = 0,$$

belongs to the space $L^\infty([0, 1]; \mathcal{H}_1)$. Indeed,

$$\begin{aligned}\|u(t)\|_{\mathcal{H}_1} &\le \sum_{n=0}^{\infty} \int_0^t \frac{(t-s)^n}{n!} \|M^n g(s)\|_{\mathcal{H}_1}\, ds \le \sum_{n=0}^{\infty} \int_0^t \frac{C^n (t-s)^n}{n!} \|g(s)\|_{\mathcal{H}_1}\, ds \\ &\le e^{Ct} \int_0^t \|g(s)\|_{\mathcal{H}_1}\, ds.\end{aligned}$$

While this may be considered a satisfactory solution of the problem for the transport equation (1) with constant coefficients, the situation changes drastically in the variable coefficient case. Consider the transport equation

$$\partial_t u - a(x) M u = g, \qquad u(0, x) = 0, \tag{3}$$

with a time-independent coefficient $a(x)$. As before we may write

$$u(t, x) = \int_0^t e^{(t-s)aM} g(s)\, ds, \tag{4}$$

where

$$e^{taM} = I + taM + \frac{1}{2}(taM)^2 + \cdots + \frac{1}{n!}(taM)^n + \cdots .$$

The multiplication operator a and Calderón–Zygmund operator M do not commute[4]. We need instead that the operator aM has the same mapping properties as M, i.e. it maps $\mathcal{H}_1$ to itself, in which case we would easily conclude that solutions of the transport equation (3) belong to the space $L^\infty([0, 1]; \mathcal{H}_1)$. To ensure this condition we are led to the requirement that multiplication by the function $a = a(x)$ maps Hardy space into itself. It

[3] The classical Hardy space $\mathcal{H}_1$, defined by the norm $\|f\|_{\mathcal{H}_1} = \|f\|_{L^1(\mathbb{R}^2)} + \sup_{j=1,2} \|R_j f\|_{L^1(\mathbb{R}^2)}$, can be viewed as a logarithmic improvement of L^1. Here $R_j = (-\Delta)^{1/2}\partial_j$ are the standard Riesz operators in $\mathbb{R}^2$.

[4] If they did, we could write $(aM)^n$ as $a^n M^n$ and derive the estimate $\|u(t)\|_{L^1(\mathbb{R}^2)} \le C \int_0^t \frac{\|a\|^n_{L^\infty(\mathbb{R}^2)}(t-s)^n}{n!} \|M^n g(s)\|_{L^1(\mathbb{R}^2)}\, ds \le e^{Ct\|a\|_{L^\infty(\mathbb{R}^2)}} \int_0^t \|g(s)\|_{\mathcal{H}_1}\, ds$.

is well known however that multiplication by a bounded function does not preserve $\mathcal{H}_1$. Instead, such a function a should satisfy the Dini condition

$$\int_0^\infty \sup_{|x-y|\le\lambda} |a(x)-a(y)| \frac{d\lambda}{\lambda} < \infty$$

(see [Steg]). Functions satisfying the Dini condition cannot be sharply characterized in terms of the standard Lebesgue type spaces. Specifically, one can easily see that even if a is a single atom in the Besov space $B^0_{\infty,1}(\mathbb{R}^2)$ or even in $B^1_{2,1}(\mathbb{R}^2)$, both sharp Besov refinements of the $L^\infty(\mathbb{R}^2)$ space, this does not guarantee that the Dini condition is satisfied. Yet, in view of the specific applications we have in mind, we need to consider precisely the situation when a belongs to the space $B^1_{2,1}$, and allow even more general functions in the time-dependent case. As a consequence, to accomplish our goal we need to give up on the Hardy space $\mathcal{H}_1$ and consider in fact estimates[5] for solutions u of the transport equation (3) of the form

$$\sup_{t\in[0,1]} \|u(t)\|_{L^1(\mathbb{R}^2)} \le C(\|a\|_{B^1_{2,1}(\mathbb{R}^2)})N(g), \tag{5}$$

where the expression $N(g)$ reflects a logarithmic loss[6] relative to the L^1 norm of g. The proper definition of $N(g)$ is given below in (14). In the particular case of g with compact support, $N(g)$ becomes simply $\|g\|_{L^1(\mathbb{R}^2)} \log^+ \|g\|_{L^\infty(\mathbb{R}^2)} + 1$.

The key feature of estimate (5) is that only one logarithmic loss is present. This means that we are not able to attack the problem by merely considering the mapping properties of the operator aM. Indeed, the best we can prove is the estimate

$$\sup_{t\in[0,1]} \|aMg(t)\|_{L^1(\mathbb{R}^2)} \le C(\|a\|_{B^1_{2,1}(\mathbb{R}^2)})N(g),$$

which leads, by iteration, to a loss of $(\log^+ \|g\|_{L^\infty(\mathbb{R}^2)})^n$ for $(aM)^n$. Instead we analyze directly the mapping properties of the multilinear expressions

$$(a(x)M)^n = a(x)Ma(x)M \dots a(x)M \tag{6}$$

and their sums. Using commutator estimates and appropriate interpolations between the weak L^1 and L^2 mapping properties of the operators M we are able to show that in fact we lose only one logarithm for $\|(aM)^n g\|_{L^1}$, regardless of the exponent n. Note however that under our assumptions on $a(x)$ the commutator $[a(x), M]$ is not a bounded operator[7] on $L^1(\mathbb{R}^2)$ and thus the problem cannot be simply reduced to the weak-L^1 estimate for

[5] To prove such estimates we need the symbol $m(\xi)$ of M to be smooth at the origin, i.e., $|\partial^\alpha m(\xi)| \le c(1+|\xi|)^{-|\alpha|}$ for all $\xi \in \mathbb{R}^2$.

[6] Recall that according to the result of Stein [Ste1] the Hardy space $\mathcal{H}_1$ contains precisely such logarithmic loss, as the finiteness of the local $\mathcal{H}_1$ norm of g, i.e. the norm $\|g\|_{L^1} + \|R_j g\|_{L^1}$ computed over balls B, is equivalent to bounds on $\int_B |g(x)| \log^+ g(x)\, dx$.

[7] The classical result of Coifman–Rochberg–Weiss [CRW] requires only that $a \in$ BMO for the commutator to be bounded on L^p with $p \in (1,\infty)$. Extensions of this result from L^p to the Hardy space $\mathcal{H}_1$ however impose once again a Dini type condition on a.

the Calderón–Zygmund operator M^n. Instead, using the assumption that $a \in B^1_{2,1}$ we first reduce the problem to the case where in the multilinear expression (6) the function a is replaced by its atoms

$$Ma_{k_1}M \dots a_{k_{n-1}}M$$

with $a_k = P_k a$ and the Littlewood–Paley projection P_k associated with the dyadic band of frequencies of size 2^k. We then decompose

$$M = M_{\geq k_1} + M_{<k_1} = P_{<k_1}M + P_{\geq k_1}M$$

and observe that $[M_{\geq k_1}, a_{k_1}]$ is a bounded operator on L^1. It follows that

$$\begin{aligned} Ma_{k_1}M \dots a_{k_{n-1}}M &= a_{k_1}M_{\geq k_1}M \dots a_{k_{n-1}}M + [M_{\geq k_1}, a_{k_1}]M \dots a_{k_{n-1}}M \\ &\quad + M_{<k_1}a_{k_1}M \dots a_{k_{n-1}}M. \end{aligned}$$

We now proceed inductively. The first two terms can be reduced to the problem of L^1 estimates for the multilinear expressions $M^2a_{k_2} \dots a_{k_{n-1}}M$ and $M \dots a_{k_{n-1}}M$, each containing only $n-1$ Calderón–Zygmund operators and $n-2$ atoms a_{k_i}. The remaining term $M_{<k_1}a_{k_1}M \dots a_{k_{n-1}}M$ can be written in the form

$$M_{<k_1}a_{k_1}Ma_{k_2} \dots a_{k_{n-1}}M = \sum_{l_2,\dots,l_{n-1}} M_{<k_1}a_{k_1}M_{k_1}a_{k_2}M_{l_2} \dots a_{k_{n-1}}M_{l_{n-1}}.$$

The operator $M_{<k_1}$ is handled with the help of the weak-L^1 estimate, which comes on one hand with a logarithmic loss but on the other hand has a certain important redeeming property in the choice of the constants, which are, in particular, dependent on the multi-index $l_1, \dots, l_n$. The remaining argument consists in showing that the operator $M_{k_1}a_{k_2}M_{l_2} \dots a_{k_{n-1}}M_{l_{n-1}}$ is bounded on L^1 with the bound reflecting exponential gains in the differences of either of the adjacent frequencies $|l_m - l_{m-1}|$ or $|k_m - k_{m-1}|$.

The problem of L^1 estimates for the transport equation (1) with variable time-dependent coefficient $a(t,x)$ exemplifies even more the need for such multilinear estimates. In this case a solution u does not quite have an exponential map representation similar to (4). Instead it can be written in the form

$$u(t) = \int_0^t T\{e^{\int_s^t a(\tau)M\,d\tau}\}g(s)\,ds.$$

Here T is the Quantum Field Theory (QFT) notation for the time ordered product. Thus, we have

$$\begin{aligned} u(t) &= \int_0^t \sum_{n=0}^{\infty} \frac{1}{n!} T\left\{\int_s^t \int_s^t \dots \int_s^t a(t_1)Ma(t_2)M \dots a(t_n)M\,dt_1 \dots dt_n\right\} g(s)\,ds \\ &= \int_0^t \sum_{n=0}^{\infty} \int_0^t a(t_1)M\,dt_1 \int_0^{t_1} a(t_2)M\,dt_2 \dots \int_0^{t_{n-1}} a(t_n)M \int_0^{t_n} g(s)\,ds. \end{aligned} \tag{7}$$

The time ordering T arranges variables $t_1, \dots, t_n$ in the decreasing order $t_1 \geq \dots \geq t_n$. Our method for deriving L^1 estimates for solutions of the transport equation (1) involves analyzing each of the multilinear expressions in the above expansion. As in the case of the time-independent coefficient a we will be able to derive an L^1 estimate with a logarithmic loss under the assumption that a is a $B^1_{2,1}$-valued function with an appropriate (in fact L^1) time dependence. The infinite series representation (7) will also help us to uncover another phenomenon. In the case when the time-dependent coefficient a can be written as a time derivative of a function b, i.e., $a = \partial_t b$, the L^1 estimate for solutions of the transport equation (1) does not require Besov regularity of the coefficient a and instead needs $L^2([0,1]; H^1)$ regularity of a together with $L^2([0,1]; H^2)$ regularity of b. Our main result is the L^1 estimate for solutions of the transport equation (1) with the coefficient $a = \partial_t b + c$ with $c \in L^1([0,1]; B^1_{2,1})$ and b satisfying the above conditions.

To treat this general case we consider multilinear expressions appearing in (7) and decompose each of the $a(t_i)$ into its Littlewood–Paley components to form a term

$$J_{n,\mathbf{k}}(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_n} a_{k_1}(t_1) M a_{k_2}(t_2) M \dots a_{k_n}(t_n) M g(s)\, dt_1 \dots dt_n\, ds$$

with $\mathbf{k} = (k_1, \dots, k_n)$. For each $\mathbf{k}$ we will be able to show the desired estimate

$$\sup_{t\in[0,1]} \|J_{n,\mathbf{k}}(t)\|_{L^1(\mathbb{R}^2)} \le C N(g).$$

The constant C above depends on the $L^1([0,1]; H^1)$ norms of a_{k_i} and grows with n. As a consequence we face two major summation problems: first with respect to a given multi-index $\mathbf{k}$ followed by summation in n. Difficulties with summation over $\mathbf{k}$ are connected with the fact that a no longer has Besov regularity $B^1_{2,1}$. This lack of regularity is due to the term $\partial_t b$ in the decomposition of a. We notice however that upon substitution into $J_n(t)$ the term $\partial_t b_{k_j}$ can be integrated by parts, which results in a gain of 1/2 derivative[8] or, alternatively, a factor of $2^{-k_j/2}$. The problem however is that this gain needs to be spread across all remaining $n-1$ terms in $J_n(t)$, which leads us to choose k_j to be the highest frequency among all k_i. If the highest frequency is occupied by a Besov term c_{k_j} appearing in the decomposition of a, we select the second highest frequency and continue the process, which in the end ensures summability with respect to $\mathbf{k}$. This analysis may potentially lead to violent growth of the constant C with respect to n and extreme care is needed. We ensure that C decays exponentially in n by imposing smallness conditions on the space-time norms of the coefficients b and c.

We now state our result precisely. Consider the transport equation

$$\partial_t u - a(t,x) M u = g(t,x), \qquad u(0,x) = 0.$$

We assume that for the coefficient a,

$$\|a\|_1 := \|a\|_{L^2_t H^1} = \left(\int_0^1 \|a(t)\|^2_{H^1(\mathbb{R}^2)}\, dt \right)^{1/2} \le \Delta_0. \tag{8}$$

[8] The fact that the gain is only 1/2 derivative rather than the whole derivative is due to the L^2 in time integrability assumption on b.

In addition a can be decomposed as follows:

$$a = \partial_t b + c, \tag{9}$$

where

$$\|b\|_2 := \left(\int_0^1 \|b(t)\|^2_{H^2(\mathbb{R}^2)} \, dt + \int_0^1 \|\partial_t b(t)\|^2_{H^1(\mathbb{R}^d)} \, dt \right)^{1/2} \le \Delta_0, \tag{10}$$

$$\|c\|_3 := \int_0^1 \|c(t)\|_{B^1_{2,1}(\mathbb{R}^2)} \, dt \le \Delta_0, \tag{11}$$

with $B^1_{2,1}(\mathbb{R}^2)$ the classical inhomogeneous Besov space defined by the norm

$$\|v\|_{B^1_{2,1}(\mathbb{R}^2)} = \|P_{\le 0} v\|_{L^2} + \sum_{k \in \mathbb{Z}_+} 2^k \|P_k v\|_{L^2(\mathbb{R}^2)}.$$

The operator M is the classical translation invariant Calderón–Zygmund operator on $\mathbb{R}^2$, given by the symbol $m(\xi)$ satisfying

$$|\partial^\alpha m(\xi)| \le c(1 + |\xi|)^{-|\alpha|}, \quad \forall \xi \in \mathbb{R}^2. \tag{12}$$

We prove the following theorem,

Theorem 1.1 (Main Theorem). *Under the above assumptions, if Δ_0 is sufficiently small, we have the estimate*

$$\sup_{t \in [0,1]} \|u(t)\|_{L^1(\mathbb{R}^2)} \lesssim C N(g) \tag{13}$$

where

$$N(g) = \|g\|_{L^1([0,1]\times\mathbb{R}^2)} \log^+ \{ \|\langle x \rangle^3 g\|_{L^\infty([0,1]\times\mathbb{R}^2)} \} + 1. \tag{14}$$

Remark 1.2. For a function g of compact support the expression $N(g)$ can be controlled as follows:

$$N(g) \lesssim \|g\|_{L^1([0,1]\times\mathbb{R}^2)} \log^+ \|g\|_{L^\infty([0,1]\times\mathbb{R}^2)} + 1. \tag{15}$$

Remark 1.3. Condition (12) implies that the symbol of the operator M is smooth at the origin, which in principle eliminates a large class of Calderón–Zygmund operators from our considerations. We argue however that this condition is not particularly restrictive and can be replaced with assumptions of additional spatial decay on the coefficients $a(t, x)$. Moreover, in our application (see the paragraph below) we consider the corresponding transport equation on a compact manifold (2-sphere) instead of $\mathbb{R}^2$, where the existence of a spectral gap ensures that condition (12) holds. In that context a prototype for M is the operator $(-\Delta)^{-1}\nabla^2$. Moreover, in that case $N(g)$ can be replaced by the $L \log L$ type expression (15).

The above theorem is a vastly simplified model case for the type of result we need in [Kl-Ro6] to prove a conditional regularity result for the Einstein vacuum equations. The main assumption in [Kl-Ro6], concerning the pointwise boundedness of the deformation tensor of the unit, future, normal vector field to a space-like foliation, allows us to bound the flux of the space-time curvature through the boundary $\mathcal{N}^-(p)$ of the causal past of any point p of the space-time under consideration. In [Kl-Ro1]–[Kl-Ro4] (see also [Q]), we were able to show that the boundedness of the flux of curvature through $\mathcal{N}^-(p)$ suffices to control the radius of injectivity of $\mathcal{N}^-(p)$. This result, together with the construction of a first order parametrix in [Kl-Ro5], is used in [Kl-Ro6] to derive pointwise bounds for the curvature tensor of the corresponding space-time. To control the main error term generated by the parametrix one needs however to bound the L^1 norm of the first two tangential derivatives of $\operatorname{tr}\chi$ along $\mathcal{N}^-(p)$, with $\operatorname{tr}\chi$ the trace of the null second fundamental form of $\mathcal{N}^-(p)$. One can show that the second tangential derivative of $\operatorname{tr}\chi$ satisfies a transport equation along the null geodesic generators of $\mathcal{N}^-(p)$ which can be modeled, very roughly, by (1), with g a term whose L^1 norm along $\mathcal{N}^-(p)$ is bounded by the flux of curvature. In fact, a more realistic model would be to consider a transport, similar to (1), along the null geodesics of a past null cone $\mathcal{N}^-(p)$ in Minkowski space $\mathbb{R}^{3+1}$ with t the value of the standard afine parameter along null geodesics and $x = (x^1, x^2)$ denoting the standard sperical coordinates on the 2-spheres S_t, corresponding to constant value of t along $\mathcal{N}^-(p)$. Thus the singular integral operator M would act on S_t.

Finally, we believe that our result, or rather our proof of the result, can be applied to other situations where one needs to make L^1 or L^∞ estimates for singular transport equations, where a simple logarithmic loss is unavoidable.

2. Preliminary results

We recall briefly the classical Littlewood–Paley decomposition of functions defined on $\mathbb{R}^d$,

$$f = f_0 + \sum_{k\in\mathbb{Z}_+} f_k$$

with frequency localized components f_k, i.e. $\widehat{f_k}(\xi) = 0$ for all values of ξ outside the annulus $2^{k-1} \le |\xi| \le 2^{k+1}$ and a function f_0 with frequency localized in the ball $|\xi| \le 1$. Such a decomposition can be easily achieved by choosing a test function $\chi = \chi(|\xi|)$ in Fourier space, supported in $1/2 \le |\xi| \le 2$, and such that, for all $\xi \ne 0$, $\sum_{k\in\mathbb{Z}} \chi(2^{-k}\xi) = 1$. Then for $k > 0$ set $\widehat{f_k}(\xi) = \chi(2^k\xi)\hat{f}(\xi)$ or, in physical space,

$$P_k f = f_k = p_k * f,$$

where $p_k(x) = 2^{nk}p(2^k x)$ and $p(x)$ is the inverse Fourier transform of χ, while

$$\hat{f}_0(\xi) = \Big(1 - \sum_{k\in\mathbb{Z}_+} \chi(2^{-k}\xi)\Big)\hat{f}(\xi)$$

and $f_0 = P_0 f$. The operators P_k are called *cut-off operators* or, somewhat improperly, *Littlewood–Paley projections*.

Let M be a Calderón–Zygmund operator with multiplier m, i.e.,

$$\widehat{Mf}(\xi) = m(\xi)\hat{f}(\xi), \tag{16}$$

Here m is a smooth function satisfying

$$|\partial_\xi^\alpha m(\xi)| \le c(1+|\xi|)^{-|\alpha|}, \quad \forall \xi \in \mathbb{R}^d, \tag{17}$$

for all multi-indices α with $|\alpha| \le d+6$ and a fixed constant $c > 0$. According to the Mikhlin–Hörmander theorem we have

$$|m(x)| \le c|x|^{-d}, \quad |\partial_x m(x)| \le c|x|^{-d-1}. \tag{18}$$

Due to the smoothness of the symbol of M at the origin we can also add the estimate

$$|m(x)| \le c(1+|x|)^{-d-6}. \tag{19}$$

We shall make use of the standard Calderón–Zygmund estimates in L^p, $1 < p < \infty$,

$$\|Mf\|_{L^p} \le C_p \|f\|_{L^p}$$

as well as the weak-L^1 estimate

$$|\{x : |Mf(x)| > \lambda\} \le C\lambda^{-1}\|f\|_{L^1}.$$

Our first result is a global version of the standard local L^1 estimate for a multiplier M. The local estimate in a ball B_R does not require the condition (19) and takes the form

$$\|Mf\|_{L^1(B_R)} \le C_R(\|f\|_{L^1} \log^+ \|f\|_{L^\infty} + 1).$$

We have the following

Lemma 2.1. *Let M be a multiplier satisfying* (19)*. Fix an $L^1(\mathbb{R}^d)$ positive function β and a constant $\mu > 0$. Then for any smooth function f of compact support,*

$$\|Mf\|_{L^1} \le C N_{\mu,\beta}(f),$$

where

$$N_{\mu,\beta}(f) = \mu\|\beta\|_{L^1} + \|f\|_{L^1} \log^+\left\{\sup_{\mathbf{a}\in\mathbb{Z}^d} \frac{\sum_{|\mathbf{b}-\mathbf{a}|\le 3} \|\chi_{\mathbf{b}} f\|_{L^\infty}}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\right\},$$

$\chi_{\mathbf{a}}$ is a partition of unity adapted to the balls of radius one with centers at integer lattice points $\mathbf{a}$, and $\log^+ x = \log(2+|x|)$.

Proof. We first note that the problem can be reduced to the case when the kernel of M, given by the function $m(x)$, has compact support. This follows since

$$Mf(x) = M_0 f(x) + M_1 f(x), \quad M_1 f(x) = \int \chi(x-y) m(x-y) f(y)\, dy,$$

where χ is a smooth cut-off function vanishing on the ball of radius one. Assumption (19) guarantees that $\chi(x)m(x)$ is integrable. As a consequence,

$$\|M_1 f\|_{L^1} \le C\|f\|_{L^1}.$$

To deal with M_0 we proceed in the usual fashion by writing

$$\begin{aligned}\|M_0 f\|_{L^1} &= \int_0^\infty |\{x : |M_0 f(x)| > \lambda\}|\, d\lambda \\ &\le \int_0^\infty |\{x : |M_0 f_{<\lambda}(x)| > \lambda\}|\, d\lambda + \int_0^\infty |\{x : |M_0 f_{\ge\lambda}(x)| > \lambda\}|\, d\lambda,\end{aligned}$$

where $f_{<\lambda}(x)$ is the function coinciding with $f(x)$ on the set where $|f(x)| < \lambda$ and vanishing on its complement, and $f_{\ge\lambda} = f - f_{<\lambda}$. To estimate the term with $f_{<\lambda}$ we use the weak-L^2 estimate

$$\begin{aligned}\int_0^\infty |\{x : |M_0 f_{<\lambda}(x)| > \lambda\}|\, d\lambda &\le C \int_0^\infty \frac{\|f_{<\lambda}\|_{L^2}^2}{\lambda^2}\, d\lambda = C \int \int_{|f(x)|}^\infty \lambda^{-2} |f(x)|^2\, d\lambda\, dx \\ &= C \int |f(x)|\, dx\end{aligned}$$

To estimate the term with $f_{\ge\lambda}$ we decompose $f_{\ge\lambda}$ into the sum of functions $f_{\ge\lambda}^{\mathbf{a}} = \chi_{\mathbf{a}} f_{\ge\lambda}$,

$$f_{\ge\lambda} = \sum_{\mathbf{a}\in\mathbb{Z}^d} \chi_{\mathbf{a}} f_{\ge\lambda},$$

where $\chi_{\mathbf{a}}$ is a partition of unity, parametrized by integer lattice points in $\mathbb{R}^d$ with the property that the support of $\chi_{\mathbf{a}}$ is contained in the ball of radius two around the point $\mathbf{a} \in \mathbb{R}^d$. Since the kernel of M_0 is supported in a ball of radius one, the support of $M_0 f_{\ge\lambda}^{\mathbf{a}}$ is contained in the ball of radius three around k. As a consequence, there are at most $3^d C$ functions $M_0 f_{\ge\lambda}^{\mathbf{a}}$ containing any given point x in their support. Therefore,

$$|\{x : |M_0 f_{\ge\lambda}(x)| > \lambda\}| \le \sum_{\mathbf{a}\in\mathbb{Z}^d} |\{x : |M_0 f_{\ge\lambda}^{\mathbf{a}}(x)| > \lambda(3^d C)^{-1}\}|.$$

We also have the trivial estimate, with another constant still denoted C,

$$|\{x : |M_0 f_{\ge\lambda}^{\mathbf{a}}(x)| > \lambda(3^d C)^{-1}\}| \le 3^d C.$$

Thus, using a weak-L^1 estimate we obtain

$$\begin{aligned}J_{\mathbf{a}} &:= \int_0^\infty |\{x : |M_0 f_{\ge\lambda}^{\mathbf{a}}(x)| > \lambda(3^d C)^{-1}\}|\, d\lambda \\ &\le \int_0^{\lambda_0} 3^d C + 3^d C \int \int_{\lambda_0}^\infty \lambda^{-1} \|\chi_\alpha f_{\ge\lambda}\|_{L^1}\, d\lambda \\ &\le 3^d C\lambda_0 + 3^d C \int_{\lambda_0}^\infty \int_{|f(x)|\ge\lambda} \lambda^{-1} |\chi_{\mathbf{a}} f(x)|\, dx\, d\lambda \\ &\le 3^d C\lambda_0 + 3^d C \int \chi_{\mathbf{a}}(x)|f(x)| \left|\log \frac{|f(x)|}{\lambda_0}\right| dx \\ &\lesssim 3^d C\lambda_0 + 3^d C \int_{|f(x)|\ge\lambda_0} \chi_{\mathbf{a}}(x)|f(x)| \log \frac{|f(x)|}{\lambda_0}\, dx \\ &\lesssim 3^d C\lambda_0 + 3^d C \int \chi_{\mathbf{a}}(x)|f(x)| \log^+ \frac{|f(x)|}{\lambda_0}\, dx\end{aligned}$$

for some $\lambda_0 > 0$. We now choose $\lambda_0 = \mu \int \chi_{\mathbf{a}}(x)\beta(x)\,dx$. The above estimate then becomes

$$\begin{aligned}
J_{\mathbf{a}} &\le 3^d C\bigg(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \int \chi_{\mathbf{a}}(x)|f(x)|\log^+ \frac{|f(x)|}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\,dx\bigg)\\
&\lesssim 3^d C\bigg(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \int |f(x)|\chi_{\mathbf{a}}(x)|\log^+ \sum_{\mathbf{b}\in\mathbb{Z}^d} \frac{\chi_{\mathbf{b}}(x)|f(x)|}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\,dx\bigg)\\
&\lesssim 3^d C\bigg(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \int |f(x)|\chi_{\mathbf{a}}(x)\log^+ \sum_{|\mathbf{b}-\mathbf{a}|\le 3} \frac{\chi_{\mathbf{b}}(x)|f(x)|}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\,dx\bigg)\\
&\lesssim 3^d C\bigg(\mu\|\chi_{\mathbf{a}}\beta\|_{L^1} + \|f\chi_{\mathbf{a}}\|_{L^1}\log^+ \sup_{\mathbf{a}\in\mathbb{Z}^d} \sum_{|\mathbf{b}-\mathbf{a}|\le 3} \frac{\|\chi_{\mathbf{b}} f\|_{L^\infty}}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\bigg)
\end{aligned}$$

Now,

$$\begin{aligned}
\|M_0 f\|_{L^1} &\lesssim \int_0^\infty |\{x : |M_0 f_{<\lambda}(x)| > \lambda\}|\,d\lambda + \int_0^\infty |\{x : |M_0 f_{\ge\lambda}(x)| > \lambda\}|\,d\lambda\\
&\lesssim C\|f\|_{L^1} + \sum_{\mathbf{a}\in\mathbb{Z}^d} J_{\mathbf{a}}\\
&\lesssim C\|f\|_{L^1} + 3^d C\bigg(\mu\|\beta\|_{L^1} + \|f\|_{L^1}\log^+ \sup_{\mathbf{a}\in\mathbb{Z}^d} \sum_{|\mathbf{b}-\mathbf{a}|\le 3} \frac{\|\chi_{\mathbf{b}} f\|_{L^\infty}}{\mu\|\chi_{\mathbf{a}}\beta\|_{L^1}}\bigg)
\end{aligned}$$

as desired. □

We also need to consider powers of M^n of M with multipliers $m^{(n)}(\xi) = m(\xi)^n$. Clearly, there exists a constant $C > 0$ depending only on c and d such that

$$|m^{(n)}(x)| \le C^n|x|^{-d},\ |\partial_x m^{(n)}(x)| \le C^n|x|^{-d-1},\ |m^{(n)}(x)| \le C^n(1+|x|)^{-d-6}. \tag{20}$$

Thus, for a similar $C > 0$,

$$\|M^n f\|_{L^1} \le C^n N_{\mu,\beta}(f). \tag{21}$$

Let $m_k(\xi) = \chi(2^k\xi)m(\xi)$ and denote by M_k the operator defined by the multiplier m_k. Clearly $M_k f = P_k(Mf)$. We shall also denote by M_J the operator $P_J M$ with multiplier $m_J = \sum_{k\in J} m_k$ for any interval $J \subset \mathbb{Z}$. In physical space,

$$M_k f(x) = \int_{\mathbb{R}^d} m_k(x-y)f(y)\,dy, \qquad M_{\ge k} f = \int_{\mathbb{R}^d} m_{\ge k}(x-y)f(y)\,dy.$$

We have the following:

Lemma 2.2. *Let $k \in \mathbb{Z}_+ \cup \{0\}$ and assume that a_k is a function whose frequency is supported in the band $2^{k-1} \le |\xi| \le 2^{k+1}$, or in the case $k = 0$ in the ball $|\xi| \le 1$. Then there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\|[(M^n)_{\ge k}, a_k]f\|_{L^1} \le C^n\|a_k\|_{L^\infty}\|f\|_{L^1}.$$

Proof. We have

$$
\begin{aligned}
C(a_k)f &:= (M^n)_{\geq k}(a_k f)(x) - a_k(x)(M^n)_{\geq k} f(x) \\
&= \int (m^{(n)})_{\geq k}(x-y)(a_k(y) - a_k(x)) f(y)\, dy.
\end{aligned}
$$

To show that the integral operator $C(a_k)$ maps L^1 into L^1 it suffices to show that

$$
\begin{aligned}
I &= \sup_y I(y), \\
I(y) &= \int |(m^{(n)})_{\geq k}(x-y)|\, |a_k(y) - a_k(x)|\, dx \leq C^n \|\alpha_k\|_{L^\infty}.
\end{aligned}
$$

We write

$$
\begin{aligned}
I(y) &\leq I_1(y) + I_2(y), \\
I_1(y) &= \int_{|x-y|\geq 2^{-k}} |(m^{(n)})_{\geq k}(x-y)|\, |a_k(y) - a_k(x)|\, dx, \\
I_2(y) &= \int_{|x-y|\leq 2^{-k}} |(m^{(n)})_{\geq k}(x-y)|\, |a_k(y) - a_k(x)|\, dx.
\end{aligned}
$$

We have

$$
|a_k(y) - a_k(x)| \leq |x-y| \sup_{z\in[x,y]} |\partial a_k(z)| \lesssim 2^k |x-y|\, \|a_k\|_{L^\infty}.
$$

We also have

$$
|(m^{(n)})_{\geq k}(x)| \leq C^n |x|^{-d}.
$$

Thus,

$$
I_2(y) \leq C^n \|a_k\|_{L^\infty} \int_{|x-y|\leq 2^{-k}} |x-y|^{-d} 2^k |x-y|\, dx \lesssim C^n \|a_k\|_{L^\infty}.
$$

Also, since $|(m^{(n)})_{\geq k}(x)| \leq C^n 2^{-k} |x|^{-d-1}$, we have

$$
I_1(y) \leq C^n \|a_k\|_{L^\infty} \int_{|x-y|\geq 2^{-k}} 2^{-k} |x-y|^{-d-1}\, dx \lesssim C^n \|a_k\|_{L^\infty}
$$

as desired. □

We shall now prove the following,

Proposition 2.3. *Let M be a Calderón–Zygmund operator on $\mathbb{R}^2$ with the symbol satisfying* (17) *and $a = a(x)$ a smooth function satisfying the bound*

$$
\|a\|_{B^1_{2,1}(\mathbb{R}^2)} \leq A. \tag{22}
$$

Then for every positive integer n we have

$$
\|(aM)^n f\|_{L^1} \leq C^n A^n N(f) \tag{23}
$$

with $N(f)$ defined by (14).

Remark 2.4. Observe that the proposition remains valid if we replace $(aM)^n$ by $a_{(1)}M_{(1)}a_{(2)}M_{(2)}\dots a_{(n)}M_{(n)}$ with

$$\|a_{(i)}\|_{B^1_{2,1}(\mathbb{R}^2)} \le A, \quad i = 1, \dots, n,$$

and $M_1, \dots, M_n$ translation invariant Calderón–Zygmund operators with symbols which are uniformly bounded by the same constant c (see (17)).

The proof follows immediately from the following lemma.

Lemma 2.5. *Let $(k_1, \dots, k_n)$ be an n-tuple of nonnegative integers and assume that the functions a_{k_i} with $0 \le i \le n$ have frequencies supported in the dyadic shells $[2^{k_i-1}, 2^{k_i+1}]$, or in the case $k_i = 0$ in the ball $|\xi| \le 1$. Then for some positive constant B,*

$$\|Ma_{k_1}M\dots a_{k_n}Mf\|_{L^1} \lesssim B^n A_{k_1\dots k_n} N(f) \tag{24}$$

where

$$A_{k_1\dots k_n} = \|a_{k_1}\|_{H^1}\dots\|a_{k_n}\|_{H^1}. \tag{25}$$

Proof. We prove by induction on n the following stronger version of (24):

$$\|M^l a_{k_1}M\dots a_{k_n}Mf\|_{L^1} \lesssim B_1^{n+l}B_2^n A_{k_1\dots k_n}N(f) \tag{26}$$

with appropriately chosen constants B_1, B_2. Assume that the estimate has been proved for $n-1$ and any $l \in \mathbb{N}$. Splitting $\bar{M} := M^l = \bar{M}_{<k_1} + \bar{M}_{\ge k_1}$ we need to prove

$$\|\bar{M}_{\ge k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} \lesssim B_1^{n+l}B_2^n A_{k_1\dots k_n}N(f), \tag{27}$$

$$\|\bar{M}_{<k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} \lesssim B_1^{n+l}B_2^n A_{k_1\dots k_n}N(f). \tag{28}$$

To deal with the first inequality we write

$$\begin{aligned}\bar{M}_{\ge k_1}a_{k_1}Ma_{k_2}\dots a_{k_n}M &= a_{k_1}\bar{M}_{\ge k_1}Ma_{k_2}\dots a_{k_n}M\\ &\quad + [\bar{M}_{\ge k_1}, a_{k_1}]Ma_{k_2}\dots a_{k_n}M.\end{aligned}$$

According to Lemma 2.2 and the Bernstein inequality $\|a_k\|_{L^\infty} \lesssim \|a_k\|_{H^1}$, we have

$$\|[\bar{M}_{\ge k_1}, a_{k_1}]Ma_{k_2}\dots a_{k_n}Mf\|_{L^1} \lesssim C^l\|a_{k_1}\|_{H^1}\|Ma_{k_2}\dots a_{k_n}Mf\|_{L^1}.$$

Also,

$$\|a_{k_1}\bar{M}_{\ge k_1}Ma_{k_2}\dots a_{k_n}Mf\|_{L^1} \lesssim \|a_{k_1}\|_{L^\infty}\|M^{l+1}a_{k_2}\dots a_{k_n}Mf\|_{L^1}. \tag{29}$$

Thus, taking into account our induction hypothesis,

$$\begin{aligned}\|M_{\ge k_1}(a_{k_1}Ma_{k_2}\dots a_{k_n}M)f\|_{L^1} &\lesssim C^l\|a_{k_1}\|_{H^1}\|Ma_{k_2}M\dots a_{k_n}Mf\|_{L^1}\\ &\quad + \|a_{k_1}\|_{H^1}\|M^{l+1}a_{k_2}\dots a_{k_n}Mf\|_{L^1}\\ &\lesssim (C^lB_1^nB_2^{n-1} + B_1^{n+l}B_2^{n-1})A_{k_1\dots k_n}N(f)\\ &\lesssim B_1^{n+l}B_2^nA_{k_1\dots k_n}N(f)\end{aligned}$$

as desired, provided that the constants B_1, B_2 are sufficiently large; in fact, we need $B_1 \geq C$ and $B_2 \geq 1$.

We now consider the more difficult term

$$\bar{M}_{<k_1}(a_{k_1} M a_{k_2} \dots a_{k_n} M) f = \bar{M}_{<k_1}(a_{k_1} M(g)) = \bar{M}_{<k_1}(a_{k_1} M_{k_1}(g))$$

with $g = (a_{k_2} M a_{k_3} \dots a_{k_n} M) f$. Note that if $k_1 = 0$ the operator $\bar{M}_{<k_1}$ is a multiplier with a smooth symbol of compact support. As a consequence it is bounded on L^1 and, with $a_0 = a_{k_1}$,

$$\begin{aligned}\|\bar{M}_{<0}(a_0 M a_{k_2} \dots a_{k_n} M) f\|_{L^1} &\leq C^l \|a_{k_1}\|_{H^1} \|M a_{k_2} \dots a_{k_n} M f\|_{L^1} \\ &\lesssim C^l B_1^n B_2^{n-1} A_{k_1 \dots k_n} N(f).\end{aligned}$$

Therefore to prove (28) we need to consider the case $k_1 > 0$ and estimate

$$\|\bar{M}_{<k_1}(a_{k_1} M a_{k_2} \dots a_{k_n} M f)\|_{L^1}.$$

We further decompose as follows:

$$\begin{aligned}\bar{M}_{<k_1}(a_{k_1} M a_{k_2} \dots a_{k_n} M f) &= \sum_{[l]_n} \bar{M}_{<k_1} M_{[k]_n,[l]_n}(f), \\ M_{[k]_n,[l]_n}(f) &= a_{k_1} M_{l_1} a_{k_2} \dots M_{l_{n-1}} a_{k_n} M_{l_n} f\end{aligned} \tag{30}$$

with $[l]_n$ denoting an arbitrary integer n-tuple $(l_1, \dots, l_n) \in (\mathbb{Z}_+ \cup \{0\})^n$ and $[k]_n = (k_1, \dots, k_n)$. Whenever there is no possibility of confusion we shall drop the index n and write simply $[k]$, $[l]$. By the triangle inequality,

$$\|\bar{M}_{<k_1}(a_{k_1} M a_{k_2} \dots a_{k_n} M f)\|_{L^1} \leq \sum_{[l]_n} \|\bar{M}_{<k_1} M_{[k]_n,[l]_n}(f)\|_{L^1}.$$

We note that in the expression $\bar{M}_{<k_1} a_{k_1} M_{l_1}(a_{k_2} \dots a_{k_n} M_{l_n} f)$ the frequency l_1 is forced to be of the order of k_1. This allows us to insert a factor of $2^{-|k_1 - l_1|}$ in the above expression. Using (21) we then derive

$$\|\bar{M}_{<k_1} M_{[k],[l]}(f)\|_{L^1} \lesssim 2^{-|k_1 - l_1|} B_1^l B_2 N_{\mu([l]),\beta}(M_{[k],[l]}(f)). \tag{31}$$

Here, the notation $\mu([l])$ indicates that the scalar μ will be chosen dependent on the multi-index $[l] = [l]_n$. Recall that[9]

$$N_{\mu,\beta}(g) = \mu \|\beta\|_{L^1} + \|g\|_{L^1} \log^+ \left\{ \sup_{\mathbf{a} \in \mathbb{Z}^d} \frac{\|\chi_{\mathbf{a}} g\|_{L^\infty}}{\mu \|\chi_{\mathbf{a}} \beta\|_{L^1}} \right\}.$$

[9] For simplicity of notation we drop the summation $\sum_{|\mathbf{b}-\mathbf{a}| \leq 3}$ which will only add a finite number of terms of the same type.

We now make the following choice for the scalar μ, to be justified in the lemmas below:

$$\mu([l]) = A_{k_1\ldots k_n} 2^{-\alpha([l]_n)}, \quad \alpha([l]) = \frac{1}{2}\sum_{m=2}^{n} \min(|l_m - l_{m-1}|, |l_m - k_m|).$$

We also choose the function

$$\beta = (1 + |x|)^{-3}.$$

Observe that

$$\left(\frac{\langle \mathbf{b}\rangle}{\langle \mathbf{a}\rangle}\right)^{-3} \|\chi_{\mathbf{b}}\beta\|_{L^1} \le \|\chi_{\mathbf{a}}\beta\|_{L^1} \le \left(\frac{\langle \mathbf{b}\rangle}{\langle \mathbf{a}\rangle}\right)^{3} \|\chi_{\mathbf{b}}\beta\|_{L^1}. \tag{32}$$

We will make use of the following:

Lemma 2.6. *The expression*

$$M_{[k],[l]}(f) = a_{k_1} M_{l_1} a_{k_2} \ldots a_{k_n} M_{l_n} f$$

satisfies

$$\|M_{[k],[l]}(f)\|_{L^1} \lesssim C^n 2^{-2\alpha([l]_n)} A_{k_1\ldots k_n} \|f\|_{L^1}, \tag{33}$$

$$\|\chi_{\mathbf{a}} M_{[k],[l]}(f)\|_{L^\infty} \lesssim C^n A_{k_1\ldots k_n} \sum_{\mathbf{b}\in\mathbb{Z}^2} \langle |\mathbf{b}-\mathbf{a}|\rangle^{-3} \|\chi_{\mathbf{b}} f\|_{L^\infty}. \tag{34}$$

We postpone the proof of the lemma to the end of this section.

Now, by (31),

$$\begin{aligned}\|\bar{M}_{<k_1}(a_{k_1} M a_{k_2} \ldots a_{k_n} M f)\|_{L^1} &\le \sum_{[l]_n} \|\bar{M}_{<k_1} M_{[k],[l]}(f)\|_{L^1} \\ &\lesssim \sum_{[l]} 2^{-|k_1-l_1|}\left(\mu([l])\|\beta\|_{L^1} + \|M_{[k],[l]}(f)\|_{L^1} \log^+\left\{\sup_{\mathbf{a}\in\mathbb{Z}^d} \frac{\|\chi_{\mathbf{a}} M_{[k],[l]}(f)\|_{L^\infty}}{\mu([l])\|\chi_{\mathbf{a}}\beta\|_{L^1}}\right\}\right).\end{aligned}$$

Given our choice of $\mu([l])$ we have

$$\begin{aligned}&\sum_{[l]} 2^{-|k_1-l_1|}\mu([l]) = A_{k_1\ldots k_n} \sum_{[l]} 2^{-|k_1-l_1|} 2^{-\alpha([l])} \\ &= A_{k_1\ldots k_n} \sum_{[l]} (2^{-|k_1-l_1|} \cdot 2^{-\frac{1}{2}\min(|l_2-l_1|,|l_2-k_2|)} \cdot \ldots \cdot 2^{-\frac{1}{2}\min(|l_n-l_{n-1}|,|l_n-k_n|)}). \lesssim A_{k_1\ldots k_n}.\end{aligned}$$

Thus, to end the proof of (28) it suffices to show that

$$\sum_{[l]} 2^{-|k_1-l_1|} \|M_{[k],[l]}(f)\|_{L^1} \log^+\left\{\sup_{\mathbf{a}\in\mathbb{Z}^d} \frac{\|\chi_{\mathbf{a}} M_{[k],[l]}(f)\|_{L^\infty}}{\mu([l])\|\chi_{\mathbf{a}}\beta\|_{L^1}}\right\} \lesssim C^n A_{k_1\ldots k_n} N(f). \tag{35}$$

Using (33) and (34) and recalling the definition of $\mu([l])$ and $\beta(x)$, we obtain

$$\begin{aligned}
&\sum_{[l]} 2^{-|k_1-l_1|}\|M_{[k],[l]}(f)\|_{L^1}\log^+\left\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\frac{\|\chi_{\mathbf{a}}M_{[k],[l]}(f)\|_{L^\infty}}{\mu([l])\|\chi_{\mathbf{a}}\beta\|_{L^1}}\right\}\\
&\qquad\lesssim C^n A_{k_1\ldots k_n}\sum_{[l]}2^{-|k_1-l_1|}2^{-2\alpha([l])}\|f\|_{L^1}\\
&\qquad\qquad\times\log^+\left\{C^{n-1}\sup_{\mathbf{a}\in\mathbb{Z}^d}\sum_{\mathbf{b}\neq\mathbf{a}}\langle|\mathbf{b}-\mathbf{a}|\rangle^{-3}\frac{2^{\alpha([l])}\|\chi_{\mathbf{b}}f\|_{L^\infty}}{\|\chi_{\mathbf{a}}\beta\|_{L^1}}\right\}\\
&\qquad\lesssim C^{2n}A_{k_1\ldots k_n}\sum_{[l]}2^{-|k_1-l_1|}2^{-\alpha([l])}\|f\|_{L^1}\log^+\left\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\frac{\|\chi_{\mathbf{a}}f\|_{L^\infty}}{\|\chi_{\mathbf{a}}\beta\|_{L^1}}\right\}\\
&\qquad\lesssim C^{2n}A_{k_1\ldots k_n}\|f\|_{L^1}\log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^d}\langle|\mathbf{a}|\rangle^3\|\chi_{\mathbf{a}}f\|_{L^\infty}\}\\
&\qquad\lesssim C^{2n}A_{k_1\ldots k_n}N(f),
\end{aligned}$$

as desired. Here we have used

$$(1+|\mathbf{a}|)^3\lesssim(1+|\mathbf{b}-\mathbf{a}|)^3(1+|\mathbf{b}|)^3$$

and the finiteness of the sum

$$\begin{aligned}
&\sum_{[l]}2^{-|k_1-l_1|}2^{-\alpha([l])}\\
&\qquad=\sum_{[l]}(2^{-|k_1-l_1|}2^{-\frac{1}{2}\min(|l_2-l_1|,|l_2-k_2|)}\cdot\ldots\cdot2^{-\frac{1}{2}\min(|l_n-l_{n-1}|,|l_n-k_n|)}).
\end{aligned}$$

It remains to prove Lemma 2.6. Estimate (33) follows recursively provided that we can establish the following:

$$\|M_{l_{m-1}}a_{k_m}P_{l_m}h\|_{L^1}\lesssim\|a_{k_m}\|_{H^1}2^{-\min(|l_m-l_{m-1}|,|l_m-k_m|)}\|h\|_{L^1}. \tag{36}$$

In fact, since $M_{l_{m-1}}$ is bounded in L^1, it suffices to prove

$$\|P_{l_{m-1}}a_{k_m}P_{l_m}h\|_{L^1}\lesssim\|a_{k_m}\|_{H^1}2^{-\min(|l_m-l_{m-1}|,|l_m-k_m|)}\|h\|_{L^1}. \tag{37}$$

On the other hand, estimate (34) is a localized version of the trivial estimate

$$\|a_{k_1}M_{l_1}a_{k_2}\ldots a_{k_n}M_{l_n}f\|_{L^\infty}\lesssim C^nA_{k_1\ldots k_n}\|f\|_{L^\infty},$$

which holds since each of the frequency localized Calderón–Zygmund operators M_l is bounded on L^p, including $p=1,\infty$. Its localized version follows inductively from the estimate

$$\|\chi_{\mathbf{a}}M_l\chi_{\mathbf{b}}g\|_{L^\infty}\le C(1+|\mathbf{b}-\mathbf{a}|)^{-3}\|g\|_{L^\infty},\qquad l\ge0, \tag{38}$$

which holds true on account of the sharp localization of the kernel of M_l, in physical space, due to the smoothness of the symbol of M at zero. Indeed, the kernel $m(x-y)$ of the operator $\chi_{\mathbf{a}} M_l \chi_{\mathbf{b}}$ satisfies

$$|m(x-y)| \le C\chi_{\mathbf{a}}(x)(1+|x-y|)^{-6}\chi_{\mathbf{b}}(y) \le C(1+|\mathbf{b}-\mathbf{a}|)^{-3} m_1(x-y)$$

with $m_1(x-y) = (1+|x-y|)^{-3}$ in L^1.

To prove (37) we distinguish the following cases.

(1) Assume $l_{m-1} < k_m$. Observe that $P_{l_{m-1}}(a_{k_m} P_{l_m} h) = 0$ unless $|l_m - k_m| \le 2$. Therefore, since

$$\min(|l_m - l_{m-1}|, |l_m - k_m|) \approx 1$$

we have

$$\|P_{l_{m-1}}(a_{k_m} P_{k_m} h)\|_{L^1} \lesssim \|a_{k_m}\|_{H^1}\|h\|_{L^1} \lesssim 2^{-\min(|l_m - l_{m-1}|, |l_m - k_m|)}\|a_{k_m}\|_{H^1}\|h\|_{L^1}$$

as desired.

(2) Assume $l_{m-1} > k_m$. In this case $P_{l_{m-1}}(a_{k_m} P_{l_m} h) = 0$ unless $|l_{m-1} - l_m| \le 2$. Therefore we have again $\min(|l_m - l_{m-1}|, |l_m - k_m|) \approx 1$ and

$$\|P_{l_{m-1}}(a_{k_m} P_{l_{m-1}} h)\|_{L^1} \lesssim \|a_{k_m}\|_{H^1}\|h\|_{L^1} \lesssim 2^{-\min(|l_m - l_{m-1}|, |l_m - k_m|)}\|a_{k_m}\|_{H^1}\|h\|_{L^1}.$$

(3) If $l_{m-1} = k_m$, then $P_{l_{m-1}}(a_{k_m} P_{l_m} h) = 0$ unless $l_m \le k_m$. Then, using the Bernstein inequality $\|P_{l_m} h\|_{L^2} \lesssim 2^{l_m}\|h\|_{L^2}$ we derive

$$\begin{aligned}\|P_{l_{m-1}}(a_{k_m} P_{l_m} h)\|_{L^1} &\lesssim \|(a_{k_m} P_{l_m} h)\|_{L^1} \lesssim \|a_{k_m}\|_{L^2}\|P_{l_m} h)\|_{L^2}\\ &\lesssim 2^{-k_m}\|a_{k_m}\|_{H^1}\|P_{l_m} h\|_{L^2} \lesssim 2^{-k_m + l_m}\|a_k\|_{H^1}\|h\|_{L^1}.\end{aligned}$$

Since in this case $l_m \le k_m = l_{m-1}$, we have

$$\min(|l_m - l_{m-1}|, |l_m - k_m|) = k_m - l_m.$$

Therefore,

$$\|P_{l_{m-1}}(a_{k_m} P_{l_m} h)\|_{L^1} \lesssim 2^{-\min(|l_m - l_{m-1}|, |l_m - k_m|)}\|a_{k_m}\|_{H^1}\|h\|_{L^1}$$

as desired.

Thus in all cases inequality (37) is verified. □

3. Proof of the main theorem

We need to prove the estimate

$$\sup_{t\in[0,1]} \|u(t)\|_{L^1(\mathbb{R}^d)} \lesssim CN(g)$$

where $d = 2$ and

$$N(g) = \|g\|_{L^1([0,1]\times\mathbb{R}^2)} \log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^2} |\mathbf{a}|^2 \|\chi_{\mathbf{a}} g\|_{L^\infty([0,1]\times\mathbb{R}^2)}\} + 1$$

for a solution u to (1), i.e.

$$\partial_t u - a(t,x)Mu = g, \qquad u(0,x) = 0,$$

where the coefficient a admits the decomposition

$$a = \partial_t b + c \tag{39}$$

with a, b and c satisfying the conditions (8), (10) and (11).

We also define the following auxiliary norm:

$$N(g)(t) = \|g(t)\|_{L^1(\mathbb{R}^2)} \log^+\{\sup_{\mathbf{a}\in\mathbb{Z}^2} |\mathbf{a}|^2 \|\chi_{\mathbf{a}} g(t)\|_{L^\infty(\mathbb{R}^2)}\} + 1.$$

We define the iterates $u^0 = 0, u^1, \dots, u^n, u^{n+1}$ according to the recursive formula

$$\partial_t u^{(n+1)}(t,x) = a(t_0,x)Mu^{(n)}(t,x) + g(t,x), \quad u^{(n+1)}(0) = 0. \tag{40}$$

3.1. First iterates

To illustrate our method consider first the case of the iterate

$$u^{(2)}(t_0) = \int_0^{t_0} g(t_1)\,dt_1 + \int_0^{t_0} a(t_1)\,dt_1\, M \int_0^{t_1} g(t_2)\,dt_2.$$

Thus,

$$\begin{aligned}\|\sup_{t_0\in[0,1]} u^{(2)}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim \left\|\sup_{t_0\in[0,1]} \int_0^{t_0} g(t_1)\,dt_1\right\|_{L^1} + \|\sup_{t_0\in[0,1]} I(t_0)\|_{L^1},\\ I(t_0) &= \int_0^{t_0} a(t_1)\,dt_1\, M\int_0^{t_1} g(t_2)\,dt_2.\end{aligned}$$

The first term is trivial. To estimate the second term we need to make use of the decomposition (39). Thus,

$$\begin{aligned} I(t_0) &= I_b(t_0) + I_c(t_0),\\ I_c(t_0) &= \int_0^{t_0} c(t_1)\,dt_1 \int_0^{t_1} Mg(t_2)\,dt_2,\\ I_b(t_0) &= \int_0^{t_0} \partial_{t_1} b(t_1)\,dt_1 \int_0^{t_1} Mg(t_2)\,dt_2\\ &= b(t_0)\int_0^{t_0} Mg(t_2)\,dt_2 - \int_0^{t_0} b(t_1)Mg(t_1)\,dt_1 =: I_{b,1}(t_0) + I_{b,2}(t_0).\end{aligned}$$

To estimate I_c we use the fact that, for $d = 2$, the Besov space $B^1_{2,1}(\mathbb{R}^d)$ embeds in $L^\infty(\mathbb{R}^d)$, and the estimate

$$\|Mg(t)\|_{L^1(\mathbb{R}^d)} \lesssim \|g(t)\|_{L^1(\mathbb{R}^d)} \log^+ \|g(t)\|_{L^\infty(\mathbb{R}^d)} + 1 \lesssim N(g)(t).$$

Thus,

$$\begin{aligned}\|\sup_{t_0\in[0,1]} I_c(t_0)\|_{L^1} &\lesssim \int_0^1 \|c(t_1)\|_{L^\infty}\,dt_1 \int_0^{t_1} \|Mg(t_2)\|_{L^1(\mathbb{R}^d)}\,dt_2\\ &\lesssim \int_0^1 \|c(t_1)\|_{B^1_{2,1}(\mathbb{R}^d)}\,dt_1 \int_0^{t_1} N(g)(t_2)\,dt_2 \lesssim \|c\|_3 N(g).\end{aligned}$$

On the other hand, decomposing $b = b_0 + \sum_{k\in\mathbb{Z}_+} b_k$, we obtain

$$\begin{aligned}\|\sup_{t_0\in[0,1]} I_{b,1}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim \|\sup_{t_0\in[0,1]} b(t_0)\|_{L^\infty(\mathbb{R}^d)} \int_0^{t_0} \|Mg(t_2)\|_{L^1(\mathbb{R}^d)}\,dt_2\\ &\lesssim N(g)\|\sup_{t_0\in[0,1]} b(t_0)\|_{L^\infty(\mathbb{R}^d)}\\ &\lesssim N(g)\sum_{k\in\mathbb{Z}_+\cup\{0\}} \|\sup_{t_0\in[0,1]} b_k(t_0)\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

We now appeal to the following straightforward lemma:

Lemma 3.2. *The following calculus inequality holds true (see* (10)*) for $k \geq 0$:*

$$\sup_{t\in[0,1]} \|b_k(t)\|_{H^1(\mathbb{R}^d)} \lesssim \|\partial_t b_k\|^{1/2}_{L^2_t H^1} \|b_k\|^{1/2}_{L^2_t H^1} \lesssim 2^{-k/2}\|b_k\|_2.$$

Also,

$$\|\sup_{t\in[0,1]} b_k(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\partial_t b_k\|^{1/2}_{L^2_t H^1} \|b_k\|^{1/2}_{L^2_t H^1} \lesssim 2^{-k/2}\|b_k\|_2.$$

In view of the lemma we deduce

$$\begin{aligned}\|\sup_{t_0\in[0,1]} I_{b,1}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim N(g)\sum_{k\in\mathbb{Z}_+\cup\{0\}} \|b_k\|_{L^2_t H^1}\\ &\lesssim N(g)\sum_{k\in\mathbb{Z}_+\cup\{0\}} 2^{-k/2}\|b_k\|_2 \lesssim N(g)\|b\|_2.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\sup_{t_0\in[0,1]} I_{b,2}(t_0)\|_{L^1(\mathbb{R}^d)} &\lesssim \left\|\int_0^1 b(t_1) Mg(t_1)\,dt_1\right\|_{L^1(\mathbb{R}^d)}\\ &\lesssim N(g)\sup_{t_1\in[0,1]} \|b(t_1)\|_{L^\infty} \lesssim N(g)\|b\|_2.\end{aligned}$$

Therefore,

$$\|\sup_{t_0\in[0,1]} u^{(2)}(t_0)\|_{L^1(\mathbb{R}^d)} \lesssim N(g)(\|b\|_2 + \|c\|_3).$$

Remark 3.3. Observe that there is room of a $1/2$ derivative in the estimates for I_b. This room will play an important role for treating the general iterates $u^{(n+1)}$.

Consider now the more difficult case of the iterate $u^{(3)}$:

$$\begin{aligned}
u^{(3)}(t_0) &= \int_0^{t_0} g(t_1)\,dt_1 + \int_0^{t_0} a(t_1) M u^{(2)}(t_1)\,dt_1 \\
&= \int_0^{t_0} g(t_1)\,dt_1 + \int_0^{t_0} a(t_1)\,dt_1\, M\Big(\int_0^{t_1} g(t_2)\,dt_2\Big) \\
&\quad + \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} a(t_1) M a(t_2) M g(t_3)\,dt_1\,dt_2\,dt_3.
\end{aligned}$$

We concentrate our attention on the last term,

$$I(t_0) = \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} a(t_1) M a(t_2) M g(t_3)\,dt_1\,dt_2\,dt_3.$$

As we decompose each $a(t_i) = \partial_t b(t_i) + c(t_i)$ with $i = 1, 2$ we notice that we can integrate by parts only one of the potentially two terms containing $\partial_t b(t_i)$. We need to make that choice judiciously, based on the relative strength of the terms. We begin by decomposing $a(t_1), a(t_2)$ into their Littlewood–Paley pieces and write

$$\begin{aligned}
I(t_0) &= \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} \sum_{k_1,k_2\in\mathbb{Z}_+\cup\{0\}} a_{k_1}(t_1) M a_{k_2}(t_2) M g(t_3)\,dt_1\,dt_2\,dt_3 \\
&= \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} \sum_{0\le k_1<k_2} + \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} \sum_{0\le k_1=k_2} + \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} \sum_{k_1>k_2\ge 0}.
\end{aligned}$$

In what follows we will tacitly assume that all the integer indices k_i take values in the set of nonnegative integers and will not write this constraint explicitly. Consider the last term,

$$J(t_0) = \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} \sum_{k_1>k_2} a_{k_1}(t_1) M a_{k_2}(t_2) M g(t_3)\,dt_1\,dt_2\,dt_3.$$

We further decompose

$$a_{k_1}(t_1) = \partial_t b_{k_1}(t_1) + c_{k_1}(t_1)$$

and concentrate on the term

$$\begin{aligned}
J_b(t_0) &= \int_0^{t_0}\int_0^{t_1}\int_0^{t_2} \sum_{k_1>k_2} \partial_{t_1} b_{k_1}(t_1) M a_{k_2}(t_2) M g(t_3)\,dt_1\,dt_2\,dt_3 \\
&= \sum_{k_1>k_2} b_{k_1}(t_0) \int_0^{t_0}\int_0^{t_2} M a_{k_2}(t_2) M g(t_3)\,dt_2\,dt_3 \\
&\quad - \sum_{k_1>k_2} \int_0^{t_0}\int_0^{t_1} b_{k_1}(t_1) M a_{k_2}(t_1) M g(t_3)\,dt_1\,dt_3.
\end{aligned}$$

Let

$$J_{b1}(t_0) = \sum_{k_1>k_2} b_{k_1}(t_0) \int_0^{t_0} \int_0^{t_2} Ma_{k_2}(t_2) Mg(t_3)\, dt_2\, dt_3$$

and estimate

$$\|J_{b1}(t_0)\|_{L^1} \lesssim \sum_{k_1>k_2} \|b_{k_1}(t_0)\|_{L^\infty} \int_0^{t_0} \int_0^{t_2} \|Ma_{k_2}(t_2) Mg(t_3)\|_{L^1}\, dt_2\, dt_3.$$

Using Lemma 2.6, we have

$$\|Ma_{k_2}(t_2) Mg(t_3)\|_{L^1} \lesssim \|a_{k_2}(t_2)\|_{H^1} N(g)(t_3).$$

Also, according to Lemma 3.2, using the norm $\| \, \|_2$ introduced in (10), we obtain

$$\|b_{k_1}(t_0)\|_{L^\infty} \lesssim 2^{-k_1/2} \|b_{k_1}\|_2.$$

Hence,

$$\begin{aligned} \|J_{b1}(t_0)\|_{L^1} &\lesssim \sum_{k_1>k_2\geq 0} 2^{-k_1/2} \|b_{k_1}\|_2 \int_0^{t_0} \|a_{k_2}(t_2)\|_{H^1}\, dt_2 \int_0^{t_2} N(g)(t_3)\, dt_3 \\ &\lesssim N(g) \sum_{k_1>k_2\geq 0} 2^{-k_1/2} \|b_{k_1}\|_2 \|a_{k_2}\|_1 \lesssim N(g) \|b\|_2 \|a\|_1. \end{aligned}$$

The term $J_{b2}(t_0) = \sum_{k_1>k_2} \int_0^{t_0} \int_0^{t_1} b_{k_1}(t_1) Ma_{k_2}(t_1) Mg(t_3)\, dt_1\, dt_3$ can be treated in exactly the same fashion. Thus,

$$\|J_b(t_0)\|_{L^1} \lesssim N(g) \|b\|_2 \|a\|_1. \tag{41}$$

Consider now the term

$$J_c(t_0) = \int_0^{t_0} \int_0^{t_1} \int_0^{t_2} \sum_{k_1>k_2} c_{k_1}(t_1) Ma_{k_2}(t_2) Mg(t_3)\, dt_1\, dt_2\, dt_3.$$

We further decompose

$$a_{k_2}(t_2) = \partial_t b_{k_2}(t_2) + c_{k_2}(t_2).$$

We show how to treat the term

$$\begin{aligned} J_{cb}(t_0) &= \int_0^{t_0} \int_0^{t_1} \int_0^{t_2} \sum_{k_1>k_2} c_{k_1}(t_1) M\partial_t b_{k_2}(t_2) Mg(t_3)\, dt_1\, dt_2\, dt_3 \\ &= \sum_{k_1>k_2} \int_0^{t_0} \int_0^{t_1} c_{k_1}(t_1) Mb_{k_2}(t_1) Mg(t_3)\, dt_1\, dt_3 \\ &\quad - \sum_{k_1>k_2} \int_0^{t_0} \int_0^{t_1} c_{k_1}(t_1) Mb_{k_2}(t_2) Mg(t_2)\, dt_1\, dt_2. \end{aligned}$$

Hence, using first Lemma 2.6 followed by Lemma 3.2, we obtain

$$
\begin{aligned}
\|J_{cb}(t_0)\|_{L^1} &\lesssim \sum_{k_1>k_2} \int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)Mb_{k_2}(t_1)Mg(t_3)\|_{L^1}\, dt_1\, dt_3 \\
&\quad + \sum_{k_1>k_2} \int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)Mb_{k_2}(t_2)Mg(t_2)\|_{L^1}\, dt_1\, dt_2 \\
&\lesssim \sum_{k_1>k_2} \int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)\|_{H^1}\|b_{k_2}(t_1)\|_{H^1} N(g)(t_3)\, dt_1\, dt_3 \\
&\quad + \sum_{k_1>k_2} \int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)\|_{H^1}\|b_{k_2}(t_2)\|_{H^1} N(g)(t_2)\, dt_1\, dt_2 \\
&\lesssim \sum_{k_1>k_2} \sup_{t\in[0,1]} \|b_{k_2}(t)\|_{H^1} \int_0^{t_0}\int_0^{t_1} \|c_{k_1}(t_1)\|_{H^1} N(g)(t_2)\, dt_1\, dt_2 \\
&\lesssim N(g) \sum_{k_1>k_2\geq 0} 2^{-k_2/2}\|b_{k_2}\|_2 \|c_{k_1}\|_{L^1H^1} \lesssim N(g)\|b\|_2 \sum_{k_1} \|c_{k_1}\|_{L^1H^1} \\
&\lesssim N(g)\|b\|_2\|c\|_3.
\end{aligned}
$$

3.2. General case

Treatment of the general case will follow the scheme laid down for the third iterate $u^{(3)}$. Additional challenge however is to control constants in the estimates, which may grow uncontrollably with respect to the order of the iterates. Recalling (40) we write

$$
\begin{aligned}
u^{(n+1)}(t) &= \int_0^t g(t_1)\, dt_1 + \int_0^t a(t_1)\, dt_1 \int_0^{t_1} Mg(t_2)\, dt_2 + \cdots \\
&\quad + \int_0^t \int_0^{t_1} \cdots \int_0^{t_n} a(t_1)Ma(t_2)M\ldots a(t_n)Mg(t_{n+1})\, dt_1\, dt_2 \ldots dt_{n+1}.
\end{aligned}
$$

To simplify notations introduce the simplex $\Delta_n(t)$ defined by

$$t \geq t_1 \geq \cdots \geq t_n \geq t_{n+1} \geq 0$$

and write

$$u^{(n+1)}(t) = u^{(n)}(t) + J_n(t), \tag{42}$$

where

$$
\begin{aligned}
J_n(t) &= \int_{\Delta_n(t_0)} a(t_1)Ma(t_2)M\ldots a(t_m)Mg(t_{n+1}) \\
&:= \int \cdots \int_{\Delta_n(t_0)} dt_1 \ldots dt_{n+1}\, a(t_1)Ma(t_2)M\ldots a(t_m)Mg(t_{n+1}).
\end{aligned}
$$

To prove (13) it will suffice to show that

$$\sup_{t\in[0,1]} \|J_n(t)\|_{L^1(\mathbb{R}^d)} \lesssim C^n \Delta^n N(g). \tag{43}$$

We decompose each $a(t_i)$ in the expression for J_n into its Littlewood–Paley components according to

$$a(t_i) = \sum_{k\in\mathbb{Z}_+\cup\{0\}} P_k a(t_i) = a_0(t_i) + \sum_{k_i\in\mathbb{Z}_+} a_{k_i}(t_i).$$

Thus, writing $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$ and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$,

$$J_n(t) = J(t) = \sum_{\mathbf{k}\in\mathbb{N}^n} \int_{\Delta_n(t)} a_{k_1}(t_1)M \ldots a_{k_n}(t_n) M g(t_{n+1}). \tag{44}$$

For each $1 \le j \le n$ we define

$$[k_j] = \{(k_1, \ldots, k_n) \in \mathbb{N}^n \mid k_i \le k_j \ \forall i\}, \tag{45}$$

In what follows we will tacitly assume that all indices k_i take values in the set of nonnegative integers and will not write this constraint explicitly. Let

$$J_n^j(t) = J^j(t) = \sum_{\mathbf{k}\in[k_j]} \int_{\Delta_n(t)} a_{k_1}(t_1)M \ldots a_{k_n}(t_n) M g(t_{n+1}). \tag{46}$$

Clearly,

$$\|J_n(t)\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{j=1}^n \|J_n^j(t)\|_{L^1(\mathbb{R}^d)}.$$

We now fix j and decompose in view of (39),

$$a_{k_j}(t_j) = \partial_t b_{k_j}(t_j) + c_{k_j}(t_j). \tag{47}$$

Thus,

$$J^j(t) = J_b^j(t) + J_c^j(t) = \sum_{\mathbf{k}\in[k_j]} J^j_{b,\mathbf{k}}(t) + \sum_{\mathbf{k}\in[k_j]} J^j_{c,\mathbf{k}}(t), \tag{48}$$

$$J^j_{b,\mathbf{k}}(t) = \int_{\Delta_n(t)} a_{k_1}(t_1)M \ldots \partial_t b_{k_j}(t_j) M \ldots a_{k_n}(t_n) M g(t_{n+1})\, dt_1 \ldots dt_{n+1},$$

$$J^j_{c,\mathbf{k}}(t) = \int_{\Delta_n(t)} a_{k_1}(t_1)M \ldots c_{k_j}(t_j) M \ldots a_{k_n}(t_n) M g(t_{n+1})\, dt_1 \ldots dt_{n+1},$$

with the summation convention

$$\sum_{\mathbf{k}\in[k_j]} = \sum_{k_j\in\mathbb{Z}} \sum_{\mathbf{k}'\le k_j}, \qquad \mathbf{k}' = (k_1, \ldots, \widehat{k_j}, \ldots, k_n).$$

We first estimate[10] $J_b = J_b^j$. Integrating by parts gives

$$
\begin{aligned}
&J_{b,\mathbf{k}}(t)\\
&= \int_{\Delta_{n-1}(t)} \dots a_{k_{j-1}}(t_{j-1}) M b_{k_j}(t_{j-1}) M a_{k_{j+1}}(t_{j+1}) \dots M g(t_{n+1})\, dt_1 \dots \widehat{dt_j} \dots dt_{n+1}\\
&\quad - \int_{\Delta_{n-1}(t)} \dots a_{k_{j-1}}(t_{j-1}) M b_{k_j}(t_{j+1}) M a_{k_{j+1}}(t_{j+1}) \dots M g(t_{n+1})\, dt_1 \dots \widehat{dt_j} \dots dt_{n+1}\\
&= J^-_{b,\mathbf{k}}(t) + J^+_{b,\mathbf{k}}(t).
\end{aligned}
$$

Now, with the help of Lemma 2.6, we proceed as in the previous subsection:

$$
\begin{aligned}
&\|J^-_{b,\mathbf{k}}(t)\|_{L^1}\\
&\quad\lesssim C^n \sup_t \|b_{k_j}(t)\|_{H^1} \int_{\Delta_{n-1}(t)} A_{\mathbf{k}}(t_1, \dots, \widehat{t_j}, \dots, t_n) N(g)(t_{n+1})\, dt_1 \dots \widehat{dt_j} \dots dt_{n+1},
\end{aligned}
$$

where

$$
A_{\mathbf{k},j}(\dots, \widehat{t_j}, \dots) = \|a_{k_1}(t_1)\|_{H^1} \dots \widehat{\|a_{k_j}(t_j)\|}_{H^1} \dots \|a_{k_n}(t_n)\|_{H^1}.
$$

Hence, with the help of Lemma 3.2,

$$
\begin{aligned}
&\|J^-_{b,\mathbf{k}}(t)\|_{L^1}\\
&\quad\lesssim C^n N(g) 2^{-k_j/2} \|b_{k_j}\|_2 |\Delta_{n-2}(t)|^{1/2} \Big(\int_{\Delta_{n-2}(t)} A_{\mathbf{k}}(\dots, \widehat{t_j}, \dots)^2\, dt_1 \dots \widehat{dt_j} \dots dt_n \Big)^{1/2},
\end{aligned}
$$

where $|\Delta_{n-2}(t)|$ is the volume of the $(n-2)$ dimensional simplex.[11] Consequently,

$$
\|J^-_{b,\mathbf{k}}(t)\|_{L^1} \lesssim C^n ((n-1)!)^{-1/2} N(g) 2^{-k_j/2} \|b_{k_j}\|_2 \|a_{k_1}\|_1 \dots \widehat{\|a_{k_j}\|_1} \dots \|a_{k_n}\|_1
$$

and, by the triangle inequality and then Cauchy–Schwarz,

$$
\begin{aligned}
&\Big\| \sum_{\mathbf{k}\in[k_j]} J^-_{b,\mathbf{k}}(t) \Big\|_{L^1}\\
&\quad\lesssim C^n ((n-1)!)^{-1/2} N(g) \sum_{\mathbf{k}\in[k_j]} 2^{-k_j/2} \|b_{k_j}\|_2 \|a_{k_1}\|_1 \dots \widehat{\|a_{k_j}\|_1} \dots \|a_{k_n}\|_1\\
&\quad\lesssim C^n ((n-1)!)^{-1/2} N(g) \Big(\sum_{\mathbf{k}\in[k_j]} 2^{-k_j} \Big)^{1/2} \Big(\sum_{\mathbf{k}\in[k_j]} \|b_{k_j}\|_2^2 \|a_{k_1}\|_1^2 \dots \|a_{k_n}\|_1^2 \Big)^{1/2}\\
&\quad\lesssim C^n \Big(\frac{n!}{(n-1)!} \Big)^{1/2} N(g) \|b\|_2 \|a\|_1^{n-1} \lesssim n^{1/2} C^n N(g) \|b\|_2 \|a\|_1^{n-1}.
\end{aligned}
$$

[10] For simplicity, since j is kept fixed we drop the upper index j below.

[11] In our notations it corresponds to an actual $(n-1)$-dimensional simplex.

Proceeding exactly in the same way we derive

$$\Big\|\sum_{\mathbf{k}\in[k_j]} J^{+}_{b,\mathbf{k}}(t)\Big\|_{L^1} \lesssim nC^n N(g)\|b\|_2\|a\|_1^{n-1}.$$

Therefore, recalling that $J_b(t)=\sum_{\mathbf{k}\in[k_j]} J_{b,\mathbf{k}}(t)$,

$$\|J_b^j(t)\|_{L^1(\mathbb{R}^d)} \lesssim nC^n N(g)\|b\|_2\|a\|_1^{n-1}. \tag{49}$$

To estimate $J_c^j(t)=\sum_{\mathbf{k}\in[k_j]} J_{c,\mathbf{k}}(t)$ we have to a further decomposition. We define

$$[k_j,k_l]=\{(k_1,\dots,k_n)\in\mathbb{N}^n \mid k_i\le k_l\le k_j\ \forall i\neq l,j\}. \tag{50}$$

For fixed j we have precisely $n-1$ such regions covering $[k_j]$. Fix $l\neq j$ and consider,

$$J_c^{jl}(t)=\sum_{\mathbf{k}\in[k_j,k_l]} J^{jl}_{c,\mathbf{k}}(t). \tag{51}$$

Clearly,

$$\|J_c^j(t)\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{l\neq j}\|J^{jl}_{c,\mathbf{k}}(t)\|_{L^1(\mathbb{R}^d)}. \tag{52}$$

In view of (39) we decompose

$$a_{k_l}(t_l)=\partial_t b_{k_l}(t_l)+c_{k_l}(t_l). \tag{53}$$

Thus, dropping the upper indices j,l,

$$J_c(t)=J_{cb}(t)+J_{cc}(t)=\sum_{\mathbf{k}\in[k_j,k_l]} J_{cb,\mathbf{k}}(t)+\sum_{\mathbf{k}\in[k_j,k_l]} J_{cc,\mathbf{k}}(t), \tag{54}$$

$$J_{cb,\mathbf{k}}(t)=\int_{\Delta_n(t)} a_{k_1}(t_1)M\dots c_{k_j}(t_j)M\dots\partial_t b_{k_l}(t_l)M\dots a_{k_n}(t_n)Mg(t_{n+1})\,dt_1\dots dt_{n+1},$$

$$J_{cc,\mathbf{k}}(t)=\int_{\Delta_n(t)} a(t_1)_{k_1}M\dots c_{k_j}(t_j)M\dots c_{k_l}(t_l)\dots a_{k_n}(t_n)Mg(t_{n+1})\,dt_1\dots dt_{n+1}.$$

Integrating by parts, and dropping the operators M for a moment, we obtain

$$\begin{aligned}
J_{cb,\mathbf{k}}(t)=&\int_{\Delta_{n-1}(t)}\dots c_{k_j}(t_j)\dots a_{k_{l-1}}(t_{l-1})b_{k_l}(t_{l-1})a_{k_{l+1}}(t_{l+1})\\
&\qquad\qquad\dots g(t_{n+1})\,dt_1\dots\widehat{dt_l}\dots dt_{n+1}\\
&-\int_{\Delta_{n-1}(t)}\dots c_{k_j}(t_j)\dots a_{k_{l-1}}(t_{l-1})b_{k_l}(t_{l+1})a_{k_{l+1}}(t_{l+1})a_{k_{l+2}}(t_{l+2})\\
&\qquad\qquad\dots g(t_{n+1})\,dt_1\dots\widehat{dt_l}\dots dt_{n+1}\\
=&\ J^-_{cb,\mathbf{k}}(t)+J^+_{cb,\mathbf{k}}(t).
\end{aligned}$$

By Lemma 2.6 as before,

$$\|J^{\pm}_{cb,\mathbf{k}}(t)\|_{L^1} \lesssim C^n \sup_t \|b_{k_l}(t)\|_{H^1} \int_{\Delta_{n-1}(t)} B_{\mathbf{k}}(t_1,\ldots,\widehat{t_l},\ldots,t_n)N(g)(t_{n+1})\,dt_1\ldots\widehat{dt_l}\ldots dt_{n+1},$$

where

$$B_{\mathbf{k}}(\ldots,\widehat{t_l},\ldots) = \|a_{k_1}(t_1)\|_{H^1}\ldots\|c_{k_j}(t_j)\|_{H^1}\ldots\widehat{\|a_{k_l}(t_l)\|}_{H^1}\ldots\|a_{k_n}(t_n)\|_{H^1}.$$

Therefore, exactly as before with the help of Lemma 3.2,

$$\|J^{\pm}_{cb,\mathbf{k}}(t)\|_{L^1} \lesssim C^n N(g) 2^{-k_l/2}\|b_{k_l}\|_2 P_{\mathbf{k},n-2}(t),$$
$$P_{\mathbf{k},n-2}(t) = \int_{\Delta_{n-2}(t)} B_{\mathbf{k}}(\ldots,\widehat{t_l},\ldots)\,dt_1\ldots\widehat{dt_l}\ldots dt_n.$$

Observe that

$$P_{\mathbf{k},n-2}(t) \le \int_{\Delta_{n-2}(t)} \|a_{k_1}(t_1)\|_{H^1}\ldots\|c_{k_j}(t_j)\|_{H^1}\ldots\widehat{\|a_{k_l}(t_l)\|}_{H^1}\ldots\|a_{k_n}(t_n)\|_{H^1}\,dt_1\ldots\widehat{dt_l}\ldots dt_n$$

Thus

$$\Big\|\sum_{\mathbf{k}\in[k_j,k_l]} J^{\pm}_{cb,\mathbf{k}}(t)\Big\|_{L^1} \lesssim C^n((n-2)!)^{-1/2}N(g)Q$$

with

$$Q = \sum_{k_l\le k_j} 2^{-k_l/2}\|b_{k_l}\|_2\|c_{k_j}\|_3 \sum_{\mathbf{k}''\le k_l}\|a_{k_1}\|_1\ldots\widehat{\|a_{k_j}\|_1}\ldots\widehat{\|a_{k_l}\|_1}\ldots\|a_{k_n}\|_1$$

with $\mathbf{k}'' = (k_1,\ldots,\widehat{k_j},\ldots,\widehat{k_l},\ldots,k_n)$. Therefore, by Cauchy–Schwarz,

$$\begin{aligned}
Q &\lesssim \sum_{k_l\le k_j} 2^{-k_l/2}k_l^{(n-2)/2}\|b_{k_l}\|_2\|c_{k_j}\|_3\Big(\sum_{\mathbf{k}''\le k_l}\|a_{k_1}\|_1^2\ldots\|a_{k_n}\|_1^2\Big)^{1/2}\\
&\lesssim \|a\|_1^{n-2}\sum_{k_j\in\mathbb{N}}\|c_{k_j}\|_3\sum_{k_l\le k_j}2^{-k_l/2}k_l^{(n-2)/2}\|b_{k_l}\|_2\\
&\lesssim \|a\|_1^{n-2}\|b\|_2\sum_{k_j\in\mathbb{N}}\|c_{k_j}\|_3\Big(\sum_{k_l=0}^{k_j}2^{-k_l}k_l^{n-2}\Big)^{1/2}\\
&\lesssim ((n-1)!)^{1/2}\|a\|_1^{n-2}\|b\|_2\|c\|_3.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Big\|\sum_{\mathbf{k}\le k_l\le k_j} J^{\pm}_{cb,\mathbf{k}}(t)\Big\|_{L^1} &\lesssim C^n\Big(\frac{(n-1)!}{(n-2)!}\Big)^{1/2}N(g)\|a\|_1^{n-2}\|b\|_2\|c\|_3\\
&\lesssim n^{1/2}C^n N(g)\|a\|_1^{n-2}\|b\|_2\|c\|_3.
\end{aligned}$$

Therefore,

$$\sup_{t\in[0,1]} \|J^{jl}_{cb}(t)\|_{L^1(\mathbb{R}^d)} \lesssim n^{1/2} C^n N(g) \|a\|_1^{n-2} \|b\|_2 \|c\|_3. \tag{55}$$

To treat the term $J_{cc,\mathbf{k}}(t)$ we decompose once more. Continuing in the same manner after m steps we arrive at the integral

$$J^{j_1\cdots j_{m-1}}_{c_1\dots c_{m-1}}(t) = \sum_{[k_{j_1},\dots,k_{j_{m-1}}]} \int_{\Delta_n(t)} \cdots \tag{56}$$

with the integrand containing $c_1 = c_{k_{j_1}}, c_2 = c_{k_{j_2}}, \dots, c_{m-1} = c_{k_{j_{m-1}}}$ and

$$[k_{j_1},\dots,k_{j_{m-1}}] = \{(k_1,\dots,k_n) \in \mathbb{N}^n \mid k_i \le k_{j_{m-1}} \le \cdots \le k_{j_1}\ \forall i \ne j_1,\dots,j_{m-1}\}.$$

Clearly $[k_{j_1},\dots,k_{j_{m-1}}]$ can be covered by precisely $n-m+1$ regions of the form $[k_{j_1},\dots,k_{j_m}]$. We have

$$J^{j_1\cdots j_{m-1}}_{c_1\dots c_{m-1}}(t) = \sum_{j_m} J^{j_1\cdots j_m}_{c_1\dots c_{m-1}}(t), \qquad k_{j_m} \le k_{j_{m-1}}, \tag{57}$$

$$J^{j_1\cdots j_m}_{c_1\dots c_{m-1}}(t) = \sum_{[k_{j_1},\dots,k_{j_m}]} \int_{\Delta_n(t)} \cdots . \tag{58}$$

In view of (39) we decompose

$$a_{k_{j_m}}(t_{j_m}) = \partial_t b_{k_{j_m}}(t_{j_m}) + c_{k_{j_m}}(t_{j_m}) \tag{59}$$

and, respectively,

$$J^{j_1\cdots j_m}_{c_1\dots c_{m-1}}(t) = \sum_{\mathbf{k}\in[k_{j_1},\dots,k_{j_m}]} J^{j_1\cdots j_m}_{c_1\dots c_{m-1}b_m,\mathbf{k}}(t) + \sum_{\mathbf{k}\in[k_{j_1},\dots,k_{j_m}]} J^{j_1\cdots j_m}_{c_1\dots c_m,\mathbf{k}}(t),$$

where $b_m = b_{k_{j_m}}$, $c_m = c_{k_{j_m}}$ Proceeding exactly as before, integrating by parts and using Lemma 2.6, we write

$$\|J^{j_1\cdots j_m}_{c_1\dots c_{m-1}b_m,\mathbf{k}}(t)\|_{L^1} \lesssim C^n \sup_t \|b_{k_{j_m}}(t)\|_{H^1} \int_{\Delta_{n-1}(t)} B_{\mathbf{k}}(t_1,\dots,\widehat{t_{j_m}},\dots,t_n) N(g)(t_{n+1}),$$

where

$$\begin{aligned} B_{\mathbf{k}}(\dots,\widehat{t_{j_m}},\dots) = \|c_{k_{j_1}}(t_{j_1})\|_{H^1} \dots \|c_{k_{j_{m-1}}}(t_{j_{m-1}})\|_{H^1} \\ \cdot \|a_{k_{j_{m+1}}}(t_{j_{m+1}})\|_{H^1} \dots \|a_{k_{j_n}}(t_{j_n})\|_{H^1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|J^{j_1\cdots j_m}_{c\dots cb,\mathbf{k}}(t)\|_{L^1} &\lesssim C^n N(g) 2^{-k_{j_m}/2} \|b_{k_{j_m}}\|_2 P_{\mathbf{k},n-2}(t), \\ P_{\mathbf{k},n-2}(t) &= \int_{\Delta_{n-2}(t)} B_{\mathbf{k}}(\dots,\widehat{t_{j_m}},\dots), \end{aligned}$$

where $k_{j_{m+1}},\dots,k_{j_n}$ are the labels for all other frequencies different from $k_{j_1},\dots,k_{j_{m-1}}$.

To estimate $P_{\mathbf{k},n-2}(t)$ we make use of the following obvious lemma.

Lemma 3.3. *Let $f_1, \dots, f_n$ be an ordered sequence of positive, integrable, functions defined on the interval $[0,1] \subset \mathbb{R}$ among which m, say $f_{i_1}, \dots, f_{i_m}$, are in L^1 and $n-m$, say $f_{j_1}, \dots f_{j_{n-m}}$, are in L^2. Then*

$$\int_{\Delta_{n-2}(t)} f_1(t_1)\dots f_n(t_n)\,dt_1\dots dt_n \lesssim \left(\frac{1}{(n-m)!}\right)^{1/2} \|f_{i_1}\|_{L^1}\dots\|f_{i_m}\|_{L^1} \cdot \|f_{j_1}\|_{L^1}\dots\|f_{j_{n-m}}\|_{L^1}.$$

According to Lemma 3.3 we have

$$P_{\mathbf{k},n-2}(t) \lesssim \left(\frac{1}{(n-m-1)!}\right)^{1/2} \|c_{k_{j_1}}\|_{L^1H^1}\dots\|c_{k_{j_{m-1}}}\|_{L^1H^1} \cdot \|a_{k_{j_{m+1}}}\|_1\dots\|a_{k_{j_n}}\|_1.$$

Observe that,

$$\sum_{\mathbf{k}''\le k_{j_m}} \|a_{k_{j_{m+1}}}\|_1\dots\|a_{k_{j_n}}\|_1 \lesssim (k_{j_m})^{(n-1-m)/2}\Big(\sum_{k''\le k_{j_m}} \|a_{k_{j_{m+1}}}\|_1^2\dots\|a_{k_{j_n}}\|_1^2\Big)^{1/2} \lesssim (k_{j_m})^{(n-1-m)/2}\|a\|_1^{m-n},$$

where $\mathbf{k}'' = (k_{j_{m+1}}, \dots, k_{j_n})$. Observe also that

$$\sum_{k_{j_1}\le\cdots\le k_{j_{m-1}}} \|c_{k_{j_1}}\|_{L^1H^1}\dots\|c_{k_{j_{m-1}}}\|_{L^1H^1} \lesssim \frac{1}{(m-1)!}\|c\|_3^{m-1}. \tag{60}$$

Indeed, this follows by symmetry in view of the fact that

$$\sum_{k_{j_1},\dots,k_{j_m}} \|c_{k_{j_1}}\|_{L^1H^1}\dots\|c_{k_{j_{m-1}}}\|_{L^1H^1} \lesssim \|c\|_3^{m-1}.$$

Finally, by Cauchy–Schwarz,

$$\sum_{k_{j_m}\in\mathbb{N}} 2^{-k_{j_m}/2}(k_{j_m})^{(n-1-m)/2}\|b_{k_{j_m}}\|_2 \lesssim ((n-m)!)^{1/2}\|b\|_2.$$

Hence,

$$\sum_{[k_{j_1},\dots,k_{j_m}]} \|J^{j_1\cdots j_m}_{c\dots cb,\mathbf{k}}(t)\|_{L^1} \lesssim C^n\frac{1}{(m-1)!}\left(\frac{(n-m)!}{(n-m-1)!}\right)^{1/2} N(g)\|b\|_2\|a\|_1^{n-m}\|c\|_3^{m-1}.$$

In other words,

$$\sum_{[k_{j_1},\dots,k_{j_m}]} \|J^{j_1\cdots j_m}_{c\dots cb,\mathbf{k}}(t)\|_{L^1} \lesssim n^{1/2}C^n\frac{1}{(m-1)!}\Delta_0^n. \tag{61}$$

We are ready to estimate $J_n(t) = J(t)$ in formula (44). We have

$$\|J(t)\|_{L^1)} \lesssim \sum_{j_1=1}^{n} \|J^{j_1}(t)\|_{L^1}$$

and

$$\|J^{j_1}(t)\|_{L^1} \lesssim \|J^{j_1}_{b_1}(t)\|_{L^1} + \|J^{j_1}_{c_1}(t)\|_{L^1} \lesssim n^{\frac{1}{2}} C^n \Delta_0^n + \|J^{j_1}_{c_1}(t)\|_{L^1}.$$

Hence,

$$\|J(t)\|_{L^1} \lesssim n^{3/2} C^n \Delta_0^n + \sum_{j_1=1}^{n} \|J^{j_1}_{c_1}(t)\|_{L^1}.$$

On the other hand, for each j_1,

$$\|J^{j_1}_{c_1}(t)\|_{L^1} \lesssim \sum_{j_2 \neq j_1}^{n} \|J^{j_1 j_2}_{c_1}(t)\|_{L^1}$$

and

$$\|J^{j_1 j_2}_{c_1}(t)\|_{L^1} \lesssim \|J^{j_1 j_2}_{c_1 b_2}(t)\|_{L^1} + \|J^{j_1 j_2}_{c_1 c_2}(t)\|_{L^1} \lesssim n^{1/2} \frac{C^n \Delta_0^n}{1!} + \|J^{j_1 j_2}_{c_1 c_2}(t)\|_{L^1}.$$

Therefore,

$$\|J(t)\|_{L^1(\mathbb{R}^d)} \lesssim n^{1/2} n C^n \Delta_0^n + n^{1/2} \frac{n(n-1)}{1!} C^n \Delta_0^n + \sum_{j_1 \neq j_2} \|J^{j_1 j_2}_{c_1 c_2}(t)\|_{L^1}.$$

Continuing in this way we derive

$$\begin{aligned}
&\|J_n(t)\|_{L^1} \\
&\lesssim N(g) n^{3/2} \Delta_0^n C^n \left(1 + \frac{n-1}{1!} + \frac{(n-1)(n-2)}{2!} + \cdots + \frac{(n-1)\ldots(n-m)}{m!} + \cdots + 1\right) \\
&\lesssim n^{3/2} \Delta_0^n C^n (1+1)^{n-1} N(g) \lesssim n^{3/2} \Delta_0^n (2C)^n N(g),
\end{aligned}$$

as claimed in (43).

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